SOME PROBLEMS IN THE

MATHEMATICAL THEORY OF ELASTICITY.

by

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SUMMARY

The problems tackled in this thesis fall into two main sections. Part I deals with and develops a method of solving steady-state thermoelastic problems and Part II gives a method of solving crack problems for elastic bodies of cylindrical form.

Part I develops a method first suggested by Lur'e (1955) for the solution of the equations of thermoelastic equilibrium. The first few sections state and explain how this solution is derived. Taking the simplest forms of functions which can be used in Lur'e's solution, we solve the problem of a thick elastic plate having stress free boundaries and deformed by a known temperature distribution on its surfaces. A special case of this solution is shown to be equivalent to a special case of a solution derived by Sneddon and Lockett (1960), who solved the same problem using integral transform methods. It is shown also that elementary solutions may be used to solve the problem of a heat source placed at a point outside an elastic material. Using Fourier transform techniques we show that, using the same basic solution, we can solve a number of problems concerning semi-infinite media, thick plates, and thick plates on rigid foundations, where in each case the exterior bounding surfaces are free from stress and are deformed by known temperature distributions. The solutions derived by Muki (1957) for an unsymmetrical temperature distribution are derived using this simpler method. The basic solution is applied, in the form of double Fourier series to solve a class of problems dealing with rectangular parallelepipeds. We conclude Part I with a discussion of the application of Hankel transforms and Dini series to the basic solution, and show that a number of problems concerning symmetrical elastic bodies, embedded in a rigid material, impervious to the flow of heat and where the free surface of the elastic material is deformed by a known temperature distribution, may be solved in the form of a series solution. Numerical work was carried out for one special case.

In Part II, we deal with the problems of cracks in cylinders. We consider an infinitely long cylinder, of finite radius c, containing a penny shaped crack of radius 1, on the central plane $z = 0$. The crack is assumed to be subjected to an internal pressure $-p(\rho)$ over its surface, and we assume also that the problem is symmetrical about the z axis. The two problems of greatest interest are the cases

in which the walls of the cylinder $\rho = c$, are

(1) free from stress i.e. there is no shear nor normal stress

(2) clamped i.e. there is no shear stress nor normal displacement.

Use is made of two forms of solution given by Sneddon (1954 and 1961), and by combining these, together with the use of integral transform theory, the equations are reduced to the solution, in both cases, of a single Fredholm integral equation of the second kind. In the case of the cylinder with clamped walls, two methods of solution of the integral equation are suggested. Numerical calculations are carried out for the case where the pressure across the crack is constant.

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PREFACE

The problems considered in this thesis fall into two main sections. The first part, contained in paragraphs 9 to 15, is concerned with several problems in the theory of thermoelasticity. A simple solution of the mathematical equations, first derived by the Russian mathematician A.I.Lur'e, is developed in this section. The method is shown to solve, simply, several problems considered by other authors and solved by them using a variety of complicated methods. The method is also shown to be applicable to a number of other problems. In the second part of this work our interest is centered on crack problems. We consider the problems of cracks in a thin elastic strip and in cylinders of finite radius. In each case it is shown that the problem can be reduced to the solution of an integral equation, and for a particular case we solve this numerically.

I should like to express my thanks to Professor I.N.Sneddon who suggested most of the problems considered, and under whose supervision the work was carried out. I am indebted also to Dr. R.P.Srivastav and Professor Sneddon for allowing me to examine their as yet unpublished work on dual series, referred to in paragraph 16 as Sneddon and Srivastav (1962). All the numerical work was carried out on the DEUCE electronic computer at Glasgow.

GLASGOW UNIVERSITY
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I. INTRODUCTION

§1. Analysis of strain.

If we consider a body of length L in some direction, and extend it, in that direction, to length $L + \Delta L$, then we say that the body is under a strain $\Delta L/L$. We wish to generalize this concept and, following Sokolnikoff (1946), we define a body to be strained whenever the relative positions of points in the body are altered.

Let us take a rectangular set of axes (x_1, x_2, x_3) to describe the position of any point P of a body. Let the position of P before deformation be $(x_1, x_2, x_3) = \underset{\sim}{x}$ and let it, in the strained case be $\underset{\sim}{x}' = (x'_1, x'_2, x'_3)$ as shown in Fig.1.1

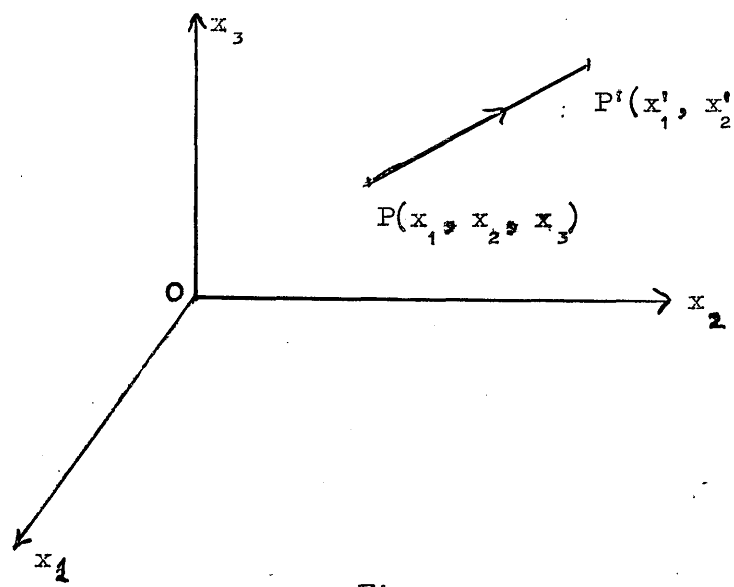


Fig. 1.1

Let us assume that we can connect the new position co-ordinates $\underset{\sim}{x}'$ with the old ones $\underset{\sim}{x}$ by means of continuous functions x'_i such that

$$x'_i = x'_i(x_1, x_2, x_3), (i = 1, 2, 3) \tag{1.1}$$

If this can be done, then physical conditions demand that there is a (1-1) correspondence between P and P' and so there must exist single valued inverse functions x_i such that

$$x_i = x_i(x'_1, x'_2, x'_3), (i = 1, 2, 3) \tag{1.2}$$

Let us make the assumption that these functions are linear. Then, write

$$x'_i = \alpha_{io} + (\delta_{ij} + \alpha_{ij})x_j, (i, j = 1, 2, 3) \tag{1.3}$$

where we have taken the $\alpha_{ij}(i, j = 1, 2, 3)$ to be the constant coefficients and δ_{ij} is the Kronecker delta function which takes the values

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{1.4}$$

We have also adopted the summation convention.

Since we shall deal only with infinitesimal deformations we may assume that the $\alpha_{ij}(i, j = 1, 2, 3)$ are small and that products of them are negligible. If this is so and we apply two successive transformations to the point $\underset{\sim}{x}$ as follows

$$x'_i = \alpha_{io} + (\delta_{ij} + \alpha_{ij})x_j$$

$$x''_k = \gamma_{ko} + (\delta_{ki} + \gamma_{ki})x'_i$$

then

$$x''_k = (\alpha_{ko} + \gamma_{ko}) + (\delta_{kj} + \alpha_{kj} + \gamma_{kj})x_j, \left(k, j = 1, 2, 3\right) \quad (1.5)$$

it follows that we may superimpose any number of successive deformations.

Consider now any quantity in the elastic material which can be written as a vector $\underset{\sim}{A}$ say. Then

$$\underset{\sim}{A} = (A_1, A_2, A_3) = A_i = (x_i - x_i^o)$$

and it follows immediately from Fig.1.2 and equations (1.3) and (1.5) that

$$A' - A = \delta A_i = \alpha_{ij} A_j, \left(i, j = 1, 2, 3\right) \quad (1.6)$$

for a single deformation and

$$\delta A_i = (\alpha_{ij} + \gamma_{ij}) A_j, \left(i, j = 1, 2, 3\right) \quad (1.7)$$

for two successive deformations.

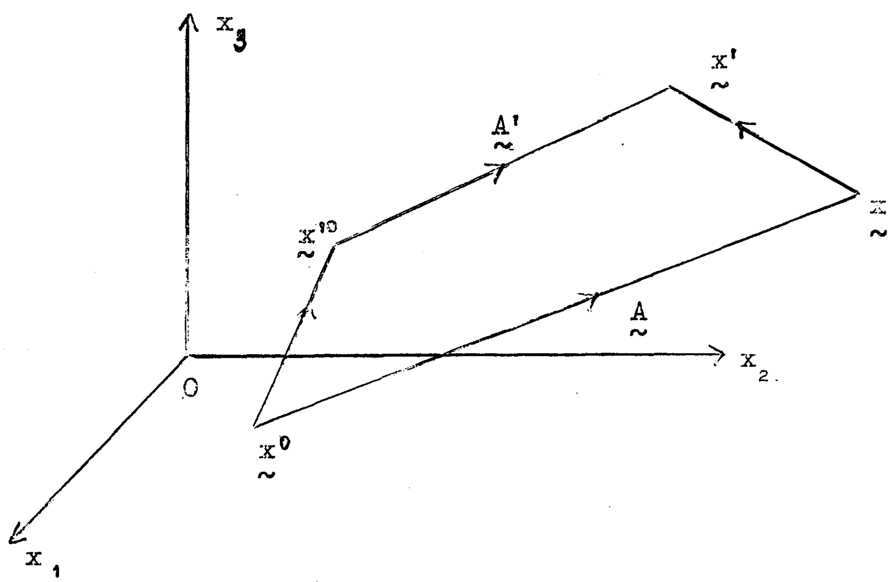


Fig. 1.2

If we consider only rigid body motion $\delta|\underset{\sim}{A}| = 0$ and since

$$\underset{\sim}{A}^2 = A_i \cdot A_i$$

$$|\underset{\sim}{A}|\,\delta\,|\underset{\sim}{A}| = A_i\,\delta A_i = \alpha_{ij} A_i A_j, (i, j = 1, 2, 3).$$

it follows that

$$\alpha_{ij} = -\alpha_{ji}, \left(i, j = 1, 2, 3\right) \quad (1.8)$$

since $\underline{A}$ is arbitrary.

Then the general transformation may be written

$$\delta A_i = \frac{\alpha_{ij} + \alpha_{ji}}{2} A_j + \frac{\alpha_{ij} - \alpha_{ji}}{2} A_j \quad , \left(i,\ j = 1,\ 2,\ 3\right)$$

$$= e_{ij} A_j + w_{ij} A_j \quad , \quad \left(i,\ j = 1,\ 2,\ 3\right)$$

where $e_{ij} = \frac{1}{2}(\alpha_{ij} + \alpha_{ji})$

$W_{ij} = \frac{1}{2}(\alpha_{ij} - \alpha_{ji})$.

It is then easily seen that the $w_{ij},(i,j = 1,\ 2,\ 3)$ represent a rigid body motion and that if we consider pure deformation only

$$\delta A_i = e_{ij} A_j \quad , \left(i,\ j = 1,\ 2,\ 3\right) \qquad (1.9)$$

The nine components $e_{ij},(i,\ j = 1,\ 2,\ 3)$ form a tensor called the <u>stress tensor</u>.

Consider now what we mean by the displacement of a point. Consider the point $P^{0}(\underset{\sim}{x}^{0})$ and after deformation let it take up the position $P'^{0}(\underset{\sim}{x}'^{0})$. We denote the vector $\underset{\sim}{P^{0}P'^{0}}$ by $\underset{\sim}{u}^{0} = (u_1,\ u_2,\ u_3)$ and call this the displacement of P^{0}.

$$u_i = x_i'^{0} - x_i^{0} \qquad (1.10)$$

In Fig. 1.3 consider the points $P^{0}(\underset{\sim}{x}^{0})$ and $P(\underset{\sim}{x})$ joined, before deformation by a vector $\underset{\sim}{A}$.

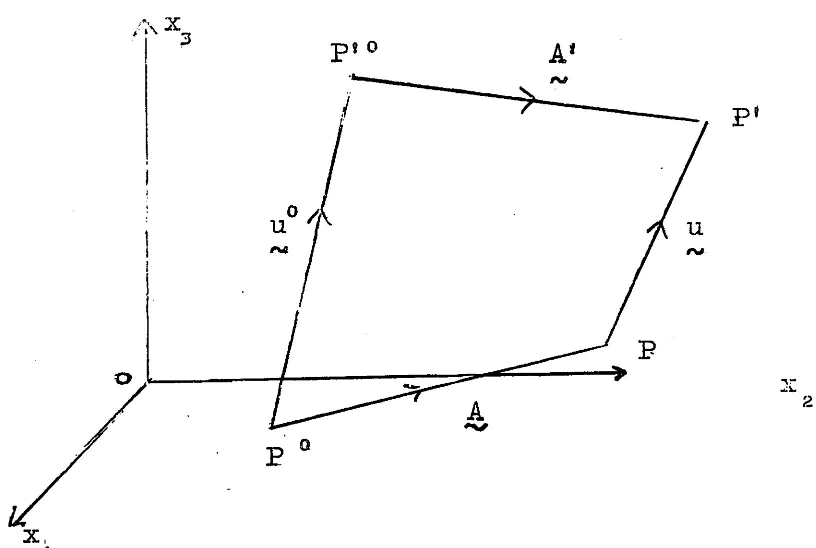


Fig. 1.3

Then, we may write

$$u_i(x_1,\ x_2,\ x_3) = u_i(x_1^{0} + A_1,\ x_2^{0} + A_2,\ x_3^{0} + A_3) = x_i' - x_i$$

$$\delta A_i = (x_i' - x_i'^{0}) - (x_i - x_i^{0})$$

$$= u_i(\underset{\sim}{x}) - u_i(\underset{\sim}{x}^{0}) .$$

Thus,
$$\delta A_i = u_{i,j}\Big|_\alpha A_j \ , \ (i, j = 1, 2, 3) \tag{1.11}$$
where $u_{i,j} = \dfrac{\partial u_i}{\partial x_j}$ and we have used Taylor's expansion.

Thus provided all deformations are infinitesimal, the preceding theory is valid and, comparing equations (1.11), (1.6) and (1.9) we deduce
$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \ , \ (i, j = 1, 2, 3.) \tag{1.12}$$

It remains only to discuss what conditions we require to ensure continuity of displacement. This question is dealt with in Sokolnikoff (1946, pp.24, 27) and he derives the following equations of compatibility
$$\left.\begin{aligned}
e_{11,23} &= (-e_{23,1} + e_{31,2} + e_{12,3})_{,1} \\
e_{22,31} &= (-e_{31,2} + e_{12,3} + e_{23,1})_{,2} \\
e_{33,12} &= (-e_{12,3} + e_{23,1} + e_{31,2})_{,3} \\
2e_{12,12} &= e_{11,22} + e_{22,11} \\
2e_{23,23} &= e_{22,33} + e_{33,22} \\
2e_{31,31} &= e_{33,11} + e_{11,33}
\end{aligned}\right] \ . \tag{1.13}$$

§2. Analysis of stress.

We begin by differentiating between the two types of force which exist in an elastic body.

(a) As the mass is continuously distributed, so is any force stemming from the mass. When referred to the unit of mass this type of force is called the unit-body force.

(b) In stress analysis, we consider the surface force distributed over surfaces drawn in the material, while the body forces are distributed throughout the volume.

These are the two types of force present as given in Prager (1961).

If we write $\underset{\sim}{e}_i$ as the unit vector along the x_i axis and take $\underset{\sim}{F} = \underset{\sim}{e}_i F_i$ to represent the body force per unit volume, the resultant $\underset{\sim}{R} = \underset{\sim}{e}_i R_i$ may be written
$$R_i = \int_\tau F_i \, d\tau \quad , \quad (i = 1, 2, 3) \tag{2.1}$$
and similarly the resultant moment $\underset{\sim}{M} = \underset{\sim}{e}_i M_i$ may be written
$$M_i = \int_\tau \epsilon_{ijk} \, x_j F_k \, d\tau \ , \ (i = 1, 2, 3) \tag{2.2}$$

Consider now an element of material as shown in Fig. 2.1 with volume ΔV and surface area $\Delta\sigma$. Let the surface force per unit area at any point with normal ν be represented by $\overset{(\nu)}{\underset{\sim}{T}}$ or generally $\underset{\sim}{T}$ where $\overset{(\nu)}{\underset{\sim}{T}}$ is not necessarily in the direction ν.

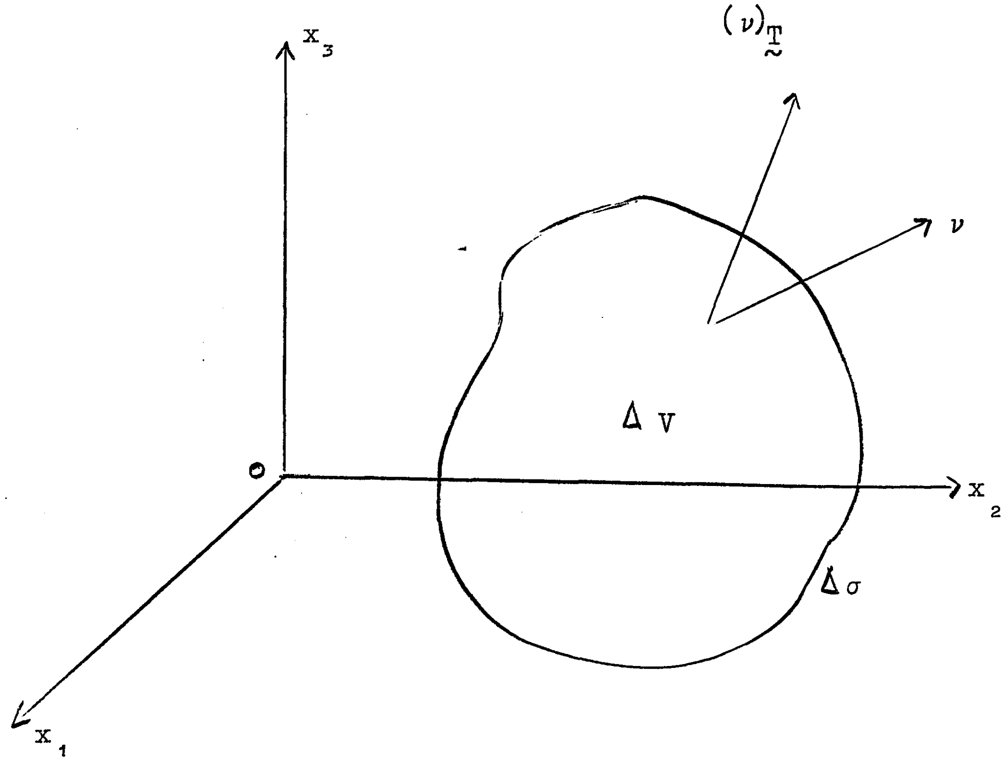


Fig. 2.1

Then $\underset{\sim}{T}$ is called the stress vector.
Consider any point P(x) of the medium and draw a parallelipiped as shown in Fig. 2.2.

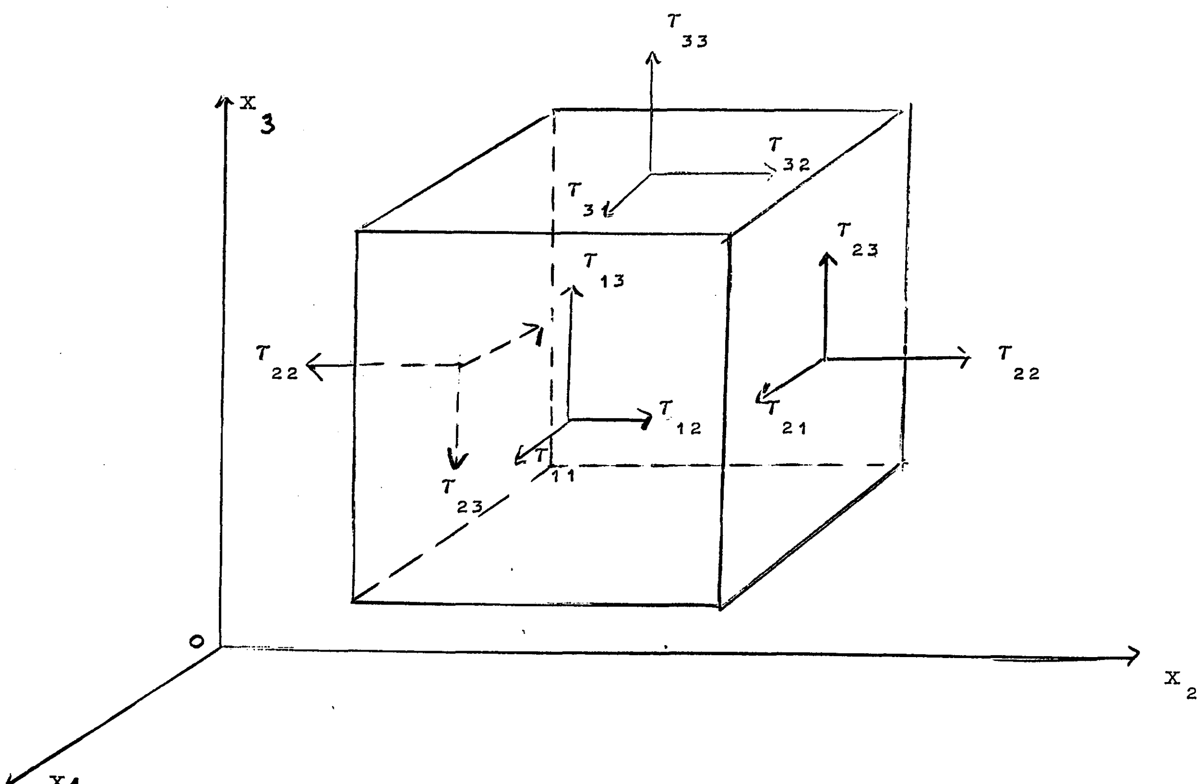


Fig. 2.2

Let ${}^{(i)}\underset{\sim}{T}$ denote the stress vector acting on the face perpendicular to the x_i axis and write

$$ {}^{(i)}\underset{\sim}{T} = \underset{\sim}{e}_j \tau_{ij} \qquad , \quad (i,\ j = 1,\ 2,\ 3) \tag{2.2}$$

where τ_{ij} is the component of ${}^{(i)}\underline{T}$ in the x_j direction. τ_{ij}, $(i,\ j = 1,\ 2,\ 3)$ define nine quantities called the components of <u>stress tensor</u> and when these are known, so is the state of stress at any point in the body.

The arrows in Fig. 2.2 indicating the vectors

$$\underset{\sim}{e}_1 \tau_{11},\ \underset{\sim}{e}_2 \tau_{12},\ \ldots\ldots\ \underset{\sim}{e}_3 \tau_{33}$$

represent the direction of the forces that, for positive τ_{ij}, are exerted by the material exterior to the parallelipiped on the matter within it. Draw normal $\underset{\sim}{\nu}$ to any face. Then if the normal has the direction of positive direction of x_i, the positive τ_{ij} act in the direction of positive x_i, whereas if the normal has direction of negative x_i, the positive τ_{ij} act in the direction opposite to the positive directions of x_i. Tensile stresses are thus positive and compressive ones negative.

τ_{ii} $(i = 1,\ 2,\ 3)$ are called normal components of stress and

τ_{ij} $(i \neq j,\ i,\ j = 1,\ 2,\ 3)$ are called shear components of stress.

Now consider the Fig. 2.3.

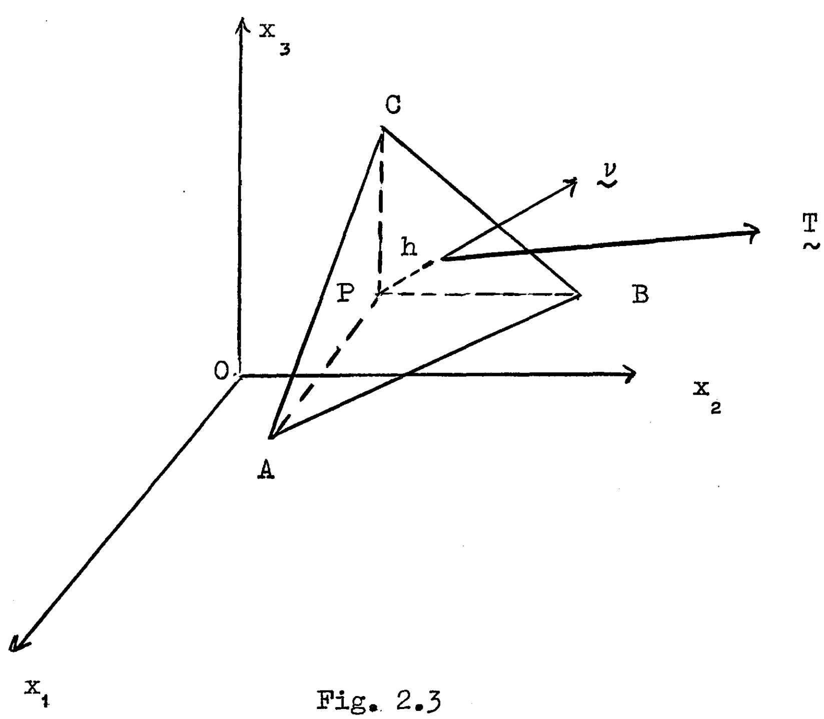


Fig. 2.3

We shall apply the equilibrium principle, which states that: In a continuum, at rest or in motion, the surface forces transmitted onto the continuum inside any volume V are at each instant in equilibrium with the body forces acting on this part of the continuum (provided that, for moving body, inertia forces are included in the body forces).

If we write the area of the face ABC to be σ, it is easily seen that, for the equilibrium principle to apply, in direction x_i $(i = 1, 2, 3)$

$$({}^{(\nu)}T_i + \epsilon_i)\sigma + (-\tau_{ji} + \epsilon_{ij})\sigma\nu_j + (F_i + \epsilon'_i)\tfrac{1}{3}h\sigma = 0$$

where $\nu_i = \cos(x_i, \underset{\sim}{\nu})$ and ϵ_i, ϵ_{ij}, ϵ'_i tend to zero as $h \to 0$. Then as $h \to 0$,

$${}^{(\nu)}T_i = \tau_{ji}\nu_j \quad , (i, j = 1, 2, 3) \qquad (2.3)$$

Thus if $\tau_{ij}(i, j = 1, 2, 3)$ are known, the state of stress at the point P is known. It remains to determine the equations of equilibrium. Suppose we consider an arbitrary volume τ with surface area σ, of a continuous medium which is in equilibrium. Then

$$\int_\tau F_i \, d\tau + \int_\sigma {}^{(\nu)}T_i \, d\sigma = 0, \; i = 1, 2, 3$$

which means, using the Divergence Theorem, that

$$\int_\tau (F_i + \tau_{ji,j}) d\tau = 0$$

and that since the volume τ is arbitrary,

$$F_i + \tau_{ji,j} = 0. \qquad (2.4)$$

Similarly, from a consideration of the moments, we may derive

$$\epsilon_{ijk}\tau_{jk} = 0$$

which means that

$$\tau_{ij} = \tau_{ji} \qquad (2.5)$$

and so the stress tensor is symmetric.

The principle stresses occur when the three normal stresses are finite but all the others are zero. They may be determined from the equation in τ

$$|\tau_{ij} - \delta_{ij}\tau| = 0 \qquad (2.6)$$

The maximum shearing stress, of importance in practical applications is equal to one half the difference between the greatest and least principal stresses.

§3. Stress-strain relations.

It was first noted by Hooke, and enunciated as a law by him, that the extension of a rod was proportional to the applied force. If the tension in the rod T is plotted against its extension e , a graph is obtained similar to Fig. 3.1

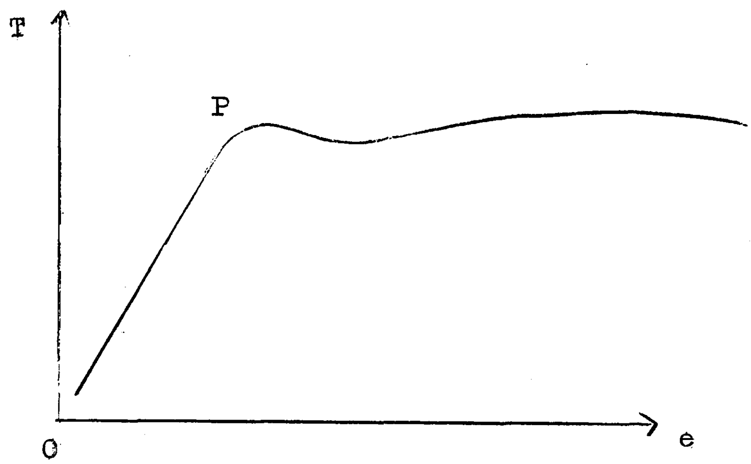


Fig 3.1

In the region denoted by OP, Hooke's law holds and we may write

$$T = E e \tag{3.1}$$

where E is known as Young's Modulus. It is in this region that we are interested and the equations of elasticity which we shall use are based on Cauchy's generalization of this law. If we write

$$\begin{aligned} \tau_1 &= \tau_{11}, \quad \tau_2 = \tau_{22}, \quad \tau_3 = \tau_{33}, \quad \tau_4 = \tau_{23}, \quad \tau_5 = \tau_{31}, \quad \tau_6 = \tau_{12} \\ e_1 &= e_{11}, \quad e_2 = e_{22}, \quad e_3 = e_{33}, \quad e_4 = e_{23}, \quad e_5 = e_{31}, \quad e_6 = e_{12} \end{aligned} \tag{3.2}$$

then Cauchy made the assumption that

$$\tau_i = c_{ij} e_j \quad , (i, j = 1, 2 \ldots 6) \tag{3.3}$$

where the c_{ij} are constants.

If it is further assumed that the elastic material is homogeneous and isotropic, then from considerations of symmetry it may be deduced that

$$c_{12} = c_{13} = c_{23} = c_{32} = c_{31} = c_{21} = \lambda \text{ say}$$

$$c_{33} = c_{22} = c_{11}$$

$$c_{55} = c_{44} = \tfrac{1}{2}(c_{11} - c_{12}) = \mu \text{ say,}$$

and that all the other coefficients are zero. The constants μ and λ are called Lamé's constants after G. Lamé (1852), and since the equations (3.3) must possess a unique inverse we have that

$$\lambda \neq 0 , \quad (3\lambda + 2\mu) \neq 0 . \tag{3.4}$$

The equations (3.3) now reduce to the form

$$\tau_{ij} = \lambda \delta_{ij} \Delta + 2\mu e_{ij} \quad , (i, j = 1, 2, 3) \tag{3.5}$$

where $\Delta = e_{11} + e_{22} + e_{33}$ and δ_{ij} is the Kronecker delta.

We may now collect the various equations governing the state of strain and stress in an elastic body.

From (3.5) above, we have

$$\tau_{ij} = \lambda \delta_{ij} \Delta + 2\mu e_{ij}$$

and from the equations of equilibrium (2.4)

$$\tau_{ij,j} + F_i = 0 \quad , (i, j = 1, 2, 3) \tag{3.6}$$

Further we know from equation (1.12)

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \;, (i, j = 1, 2, 3) \;. \tag{3.7}$$

On the surface of the body we must have

$$\tau_{ij}\,\nu_j = \overset{(\nu)}{T_i} \tag{3.8}$$

and from the equations of compatibility (1.13)

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0 \tag{3.9}$$

It can then be shown that if we find a solution satisfying equations (3.5) and (3.6) subject to equation (3.9), then that solution is unique.

We then have a system of nine equations in τ_{ij}, u_i, $(i, j = 1, 2, 3)$ to solve.

The differential equations of motion of an elastic solid can be obtained from equation (3.6) by adding the forces of motion to the body forces giving

$$\tau_{ij,j} + F_i = \rho \ddot{u}_i \quad , \quad (i, j = 1, 2, 3) \tag{3.10}$$

where ρ is the density of the body.

§4 Basic equations of thermoelasticity

In the previous paragraphs we have considered the deformations and stresses set up in a classical elastic body by surface and body forces. It has been assumed that the elastic bodies being deformed are kept at a constant temperature. However, when an elastic solid is subjected to a non-uniform temperature distribution, neighbouring elements will tend to contract or expand by varying amounts. The elastic properties of the material will cause these extensions or contractions to be reversed, and by expending an amount of energy in doing this, will cause a change of temperature. Thus, the resultant deformation must be considered as the sum of the thermal and elastic effects.

Let us make the assumption that the total strain component e_{ij}, $(i, j = 1, 2, 3)$ is composed of two separate strains e^t_{ij}, $(i, j = 1, 2, 3)$, the thermal strain and e^e_{ij}, the elastic strain. Then

$$e_{ij} = e^t_{ij} + e^e_{ij} \quad , \quad (i, j = 1, 2, 3) \qquad (4.1)$$

We have already, in a previous paragraph, considered the strain tensor e^e_{ij}, $(i, j = 1, 2, 3)$ and its connections with the stress tensor $\tau_{ij} (i, j = 1, 2, 3)$. Consider now the thermal strain tensor e^t_{ij}, $(i, j = 1, 2, 3)$.

Under free thermal expansion an isotropic body experiences the strain e^t_{ij} which, referred to a rectangular set of axes (x_1, x_2, x_3) are given by

$$e^t_{ij} = \alpha\, \theta\, \delta_{ij} \, , (i, j = 1, 2, 3) \qquad (4.2)$$

where θ is the temperature change from T, the temperature of the solid in a state of zero stress and strain, α is the coefficient of linear expansion and δ_{ij} is the Kronecker delta. We assume that θ is sufficiently small for the thermal properties to remain constant throughout the times in which we are interested.

We return now to equation (3.5) which gives the stress-strain relation

$$\tau_{ij} = \lambda\, \Delta\, \delta_{ij} + 2\mu\, e^e_{ij} \; , \; (i, j = 1, 2, 3) \qquad (4.3)$$

where $\Delta = e^e_{ii}$.

Substituting in this equation from equations (4.2) and (4.1) we obtain the relation

$$\tau_{ij} = (\lambda\, \Delta - \gamma\, \theta)\, \delta_{ij} + 2\mu\, e_{ij} \, , (i, j = 1, 2, 3) \qquad (4.4)$$

where λ, μ are Lame's constant, $\gamma = (3\lambda + 2\mu)\alpha$ and

$$\Delta = e_{ii} \; .$$

Equation (4.4) is known as the Duhammel-Neumann Law, and was discovered independently by Neumann (1885) and Duhammel (1838).

Since we have introduced temperature θ to our equations we must add a further equation to our set of equations governing the behaviour of the elastic body viz. the equation describing the behaviour of θ. In the case in which we are most interested viz., the steady state, the equation for θ takes the form

$$K \nabla^2 \theta + Q = 0 \tag{4.5}$$

where K is the conductivity of the material, and Q is related to the quantity of heat generated per unit volume q by the equation

$$Q = q / \rho c \tag{4.6}$$

where ρ is the density, c the specific heat per unit mass of the material, and we have written

$$\nabla^2 \theta = \theta_{,ii} \quad (i = 1, 2, 3) \tag{4.7}$$

where we have used the convention that $\frac{\partial \theta}{\partial x_i} = \theta_{,i}$.

For completeness we consider the equations when time must also be considered.

By the methods of reversible thermodynamics, Biot (1956) has shown that the entropy s per unit volume of the solid is given by

$$s = c\rho \log(1 + \frac{\theta}{T}) + \gamma\Delta \tag{4.8}$$

where the additive constant involved in the definition of entropy has been chosen so that it is zero in the reference state. $(T + \theta)$ is the absolute temperature and θ, T, ρ, c, γ and Δ are as defined above. If θ is small compared with T we may write

$$s = \frac{c\rho\theta}{T} + \gamma\Delta \tag{4.9}$$

for the entropy per unit volume. The quantity of heat absorbed by unit volume of the solid in the course of small deformations and small variations in temperature is given by the relation

$$h = Ts = \rho c \theta + \gamma T \Delta \tag{4.10}$$

From the theory of conduction of heat in solids, it is known that the variation of heat within an isotropic body is governed by the equation

$$\frac{\partial h}{\partial t} = K \nabla^2 \theta + q \tag{4.11}$$

Substituting in equation (4.11) from (4.10) we have

$$\rho c \frac{\partial \theta}{\partial t} + \gamma T \frac{\partial \Delta}{\partial t} = K \nabla^2 \theta + q$$

which may be written

$$\left(\frac{K}{\rho c}\right) \nabla^2 \theta + Q = \dot{\theta} + \gamma' \dot{\Delta} \tag{4.12}$$

Where we have written $\dot{\theta}$ to denote $\frac{\partial \theta}{\partial t}$, K is the conductivity, ρ the density and c the specific heat of the material. γ' has been written for $\gamma T / \rho c$ where $\gamma = (3\lambda + 2\mu)\alpha$.

The set of sixteen equations denoted by equations (4.12), (3.7), (4.4) and (3.8) is sufficient, when taken with the appropriate boundary conditions to determine the temperature variation, and the components of stress and displacement when the heat sources and body forces are prescribed.

§ 5. Dimensionless form of the equations.

In working with the equations of elasticity it is often more convenient to use a dimensionless form of equation. If we take a typical length $\underline{l}$ and typical time $\underline{\tau}$ as units of length and $\underline{T}$ and $\underline{\mu}$ as the units of temperature and stress respectively as suggested by Sneddon and Berry (1958,p.123) we may write the equations in the following form

$$\tau_{ij,j} + X_i = a\ddot{u}_i \tag{5.1}$$

$$\tau_{ij} = \left[(\beta^2 - 2)\Delta - b\theta\right]\delta_{ij} + 2e_{ij} \tag{5.2}$$

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{5.3}$$

$$\underline{\Theta} + \nabla^2\theta = f\dot{\theta} + g\dot{\Delta} \tag{5.4}$$

where

$$\beta^2 = \frac{\lambda + 2\mu}{\mu} = \left(\frac{v_p}{v_s}\right)^2$$

$$X_i = \frac{\rho\, l}{\mu}\cdot F_i$$

$$\underline{\Theta} = \frac{Q l^2}{kT}$$

$$a = \left(\frac{l}{v_s\,\tau}\right)^2, \quad b = \frac{\gamma T}{\mu}, \quad f = \frac{l^2 c\rho}{\tau}, \quad g = \frac{l^2\gamma}{K\tau},$$

It is perhaps of interest to compare the relative sizes of a, b, f, g, as given in the following table from Eason and Sneddon (1959) for $l = 1$ cm., $\tau = 1$ sec., $T = 293^\circ$ K.

	Aluminium	Copper	Iron	Lead
a	1.034×10^{-11}	2.166×10^{-11}	1.532×10^{-11}	2.034×10^{-10}
b	0.0639	0.0417	0.0089	0.2320
f	1.168	0.899	5.208	4.152
g	2.687	1.497	8.035	12.25

§6. Integral transforms.

We have, in the previous paragraphs, derived a set of partial differential equations describing the behaviour of an elastic body. We now wish to solve these equations, subject to certain boundary values. One method of doing this, in certain cases, is by the use of integral transforms.

Suppose that we have a function $\phi(x)$ defined by a differential equation and certain boundary conditions. Then it is often simpler to translate the boundary value problem for $f(x)$ into one for the function $\bar{f}(\xi)$, where

$$\bar{f}(\xi) = \int_a^b f(x)\, K(\xi, x)\,dx \qquad (6.1)$$

$\bar{f}(\xi)$ is obviously a function ξ and is called the integral transform of $f(x)$. $K(\xi, x)$ is known as the kernel of the transform since we wish to find a solution for $f(x)$ we will be interested in kernels $K(\xi, x)$ for which we can find a kernel $H(\xi, x)$ such that

$$f(x) = \int_\alpha^\beta \bar{f}(\xi)\, H(\xi, x)\, d\xi \qquad (6.2)$$

In certain cases the kernels $H(\xi, x)$ and $K(\xi, x)$ take simple forms and we list the particular cases in which we are interested.

(a) Fourier Sine Transform:

If
$$\bar{f}(\xi) = (2/\pi)^{\frac{1}{2}} \int_0^\infty f(x) \sin \xi x\, dx$$
then
$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \bar{f}(\xi) \sin \xi x\, d\xi \qquad (6.3)$$

(b) Fourier Cosine Transform:

If
$$\bar{f}(\xi) = (2/\pi)^{\frac{1}{2}} \int_0^\infty f(x) \cos \xi x\, dx$$
then
$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty f(\xi) \cos \xi x\, d\xi \qquad (6.4)$$

(c) Fourier Complex Transform:

If
$$\bar{f}(\xi) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^\infty f(x) e^{i\xi x}\, dx$$
then
$$f(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^\infty f(\xi)\, e^{-i\xi x}\, d\xi \qquad (6.5)$$

(d) Hankel Transform:

$$\bar{f}(\xi) = \int_0^\infty x\, f(x)\, J_\nu(\xi x)\, dx$$
$$f(x) = \int_0^\infty \xi\, \bar{f}(\xi)\, J_\nu(\xi x)\, d\xi \qquad (6.6)$$

where $J_\nu(z)$ is the Bessel function of the first kind of order ν.

(e) Laplace Transform:

$$\bar{f}(\xi) = \int_0^\infty f(x) e^{-\xi x}\, dx$$
$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{f}(\xi)\, e^{\xi x}\, d\xi \qquad (6.7)$$

where c is greater than the real part of all singularities of $\bar{f}(\xi)$.

The idea of the integral transform may be extended to functions of more than one variable. If we take a function $\phi(x, y)$ in x and y, we may write

$$\bar{\Phi}(\xi, y) = \int_a^b \phi(x, y)\, K(\xi, x)\, dx$$

and

$$\bar{\bar{\Phi}}(\xi, \eta) = \int_\alpha^\beta \bar{\Phi}(\xi, y)\, G(\eta, y)\, dy$$

where $K(\xi, x)$ and $G(\eta, y)$ are two suitable kernels. If we choose the Fourier complex transform we have

$$\bar{\bar{\Phi}}(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy\, \phi(x, y) e^{i(\xi x + \eta y)} \qquad (6.8)$$

To see precisely what is happening, consider the solution of the equation

$$\frac{\partial^2 \phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \phi}{\partial \rho} = \frac{1}{K}\frac{\partial \phi}{\partial t} \qquad (6.9)$$

where $\phi \longrightarrow 0$ as $\rho \longrightarrow \infty$, $\phi(\rho, 0) = \phi_0$ a prescribed function.

Multiplying each side by $\rho J_0(\xi\rho)$ and integrating from 0 to ∞, we find

$$-K \xi^2 \bar{\Phi} = \frac{\partial \bar{\Phi}}{\partial t} \qquad (6.10)$$

where

$$\bar{\Phi} = \int_0^\infty \rho J_0(\xi\rho)\, \phi(\rho, t)\, d\rho \qquad (6.11)$$

Then $\bar{\Phi} = A e^{-\xi^2 K t}$ and at $t = 0$, $\bar{\Phi} = A = \int_0^\infty \phi_0 \rho J_0(\xi\rho)\, d\rho$.

Using the inverse transformation to equation (6.11) we get the result

$$\phi(\rho, t) = \int_0^\infty \phi_0(u)\,du \int_0^\infty u\xi J_0(\xi u) J_0(\xi\rho) e^{-K\xi^2 t}\, d\xi \qquad (6.12)$$

The uses of integral transforms are explained and exploited fully by Sneddon (1957) and Tranter (1951).

§7. Solutions of the equations of equilibrium.

All of the problems considered in this work are steady-state ones, and in order to solve them, we have used in both Part I and Part II, potential solutions together with the theory of integral transforms which we have already discussed. In general, there are a number of methods available for attacking steady-state problems and, in this paragraph, we note a number of them. We begin by writing the equations of equilibrium in vector form.

Equations in vector form.

It follows from equation (3.5), with the quantities e_{ij} $(i, j = 1, 2, 3)$ expressed in terms of the displacement vector $\underset{\sim}{u}$ as in equation (3.7) that the equations of equilibrium of an elastic body may be written in the form

$$\tau_{ij,j} + F_i = 0 \quad ,(i, j = 1, 2, 3) \qquad (7.1)$$

where in the purely elastic case

$$\tau_{ij} = \lambda\delta_{ij}\Delta + \mu(u_{i,j} + u_{j,i}) \qquad (7.2)$$

and in the thermo-elastic case

$$\tau_{ij} = \delta_{ij}(\lambda\Delta - \gamma\theta) + \mu(u_{i,j} + u_{j,i}) \qquad (7.3)$$

and $\Delta = u_{i,i}$.

We recall, from equation (3.8) that on the boundary

$$\tau_{ij}\nu_j = \overset{(\nu)}{T}_i \qquad (7.4)$$

where the ν_j are direction cosines.

If we write

$$P_i = F_i - \gamma\theta_{,i}$$

$$\overset{(\nu)}{B}_i = \overset{(\nu)}{T}_i + \gamma\theta\nu_i \qquad (7.5)$$

then it easily follows that the steady-state thermoelastic problem is equivalent to the elastostatic problem with the body forces P_i and the boundary forces B_i. Thus we can study both thermoelastic and elastic problems simultaneously.

If now we substitute from equation (7.2) into equation (7.1) and re-arrange the terms we see that we have,

$$(\lambda + \mu)\Delta_{,i} + \mu u_{i,jj} + P_i = 0 \qquad (7.6)$$

which may be written vectorially as

$$(\lambda + \mu)\operatorname{grad}\Delta + \mu\nabla^2 \underset{\sim}{u} + \underset{\sim}{P} = 0 \tag{7.7}$$

We can also express equation (7.7) in the alternative form

$$(\lambda + 2\mu)\operatorname{grad}\operatorname{div}\underset{\sim}{u} - \mu\operatorname{curl}\operatorname{curl}\underset{\sim}{u} + \underset{\sim}{P} = 0 \tag{7.8}$$

Kelvin's Solution.

In this form of solution, the displacement vector $\underset{\sim}{u}$ is expressed in terms of a scalar potential ϕ and a vector potential $\underset{\sim}{f}$ in the manner

$$\underset{\sim}{u} = \operatorname{grad}\phi + \operatorname{curl}\underset{\sim}{f} \tag{7.9}$$

We suppose further that the body force $\underset{\sim}{P}$ may be expressed in the form

$$\underset{\sim}{P} = \operatorname{grad}\Phi + \operatorname{curl}\underset{\sim}{F} \tag{7.10}$$

If we now substitute from equations (7.9) and (7.10) into equation (7.8), we find that

$$\operatorname{grad}\left[(\lambda + 2\mu)\nabla^2\phi + \Phi\right] + \operatorname{curl}\left[\mu\nabla^2\underset{\sim}{f} + \underset{\sim}{F}\right] = 0 \tag{7.11}$$

from which it follows that we can obtain a set of particular solutions of the equilibrium equations from particular solutions of the equations in ϕ, Φ, $\underset{\sim}{f}$, $\underset{\sim}{F}$,

$$(\lambda + 2\mu)\nabla^2\phi + \Phi = 0 \tag{7.12}$$

$$\mu\nabla^2\underset{\sim}{f} + \underset{\sim}{F} = 0$$

If we now write

$$\begin{aligned} \Phi(\underset{\sim}{r}) &= -\frac{1}{4\pi}\int_\tau \underset{\sim}{P}(\underset{\sim}{r}') \cdot \operatorname{grad}\frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|}\, d\tau' \\ \underset{\sim}{F}(\underset{\sim}{r}) &= -\frac{1}{4\pi}\int_\tau \underset{\sim}{P}(\underset{\sim}{r}') \times \operatorname{grad}\frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|}\, d\tau' \end{aligned} \tag{7.13}$$

so that equation (7.10) is satisfied, it is known from potential theory that once Φ and $\underset{\sim}{F}$ are determined by means of equations (7.13), we may find ϕ and $\underset{\sim}{f}$ to be

$$\underset{\sim}{\phi}(\underset{\sim}{r}) = \frac{1}{4\pi(\lambda + 2\mu)} \int \frac{\Phi(\underset{\sim}{r}')\, d\tau'}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \tag{7.14}$$

$$\underset{\sim}{f}(\underset{\sim}{r}) = \frac{1}{4\pi\mu} \int \frac{\underset{\sim}{F}(\underset{\sim}{r}')\, d\tau'}{|\underset{\sim}{r} - \underset{\sim}{r}'|}$$

Boussinesq - Papkovitch Potentials.

If in equation (7.8), we write

$$\underset{\sim}{u} = A \text{ grad } (\phi + \underset{\sim}{r}.\underset{\sim}{\psi}) + B\underset{\sim}{\psi} \tag{7.15}$$

where $\underset{\sim}{r}$ is the position vector of a field point and A and B are constants, we find that the equation becomes

$$(\lambda + 2\mu) A \text{ grad } (\nabla^2\phi + \underset{\sim}{r}.\nabla^2\underset{\sim}{\psi}) + [(\lambda + 2\mu)(B + 2A) - \mu B] \text{ grad div } \underset{\sim}{\psi} + B\mu\nabla^2\underset{\sim}{\psi} + \underset{\sim}{P} = 0 \tag{7.16}$$

The scalar quantity ϕ and the vector quantity $\underset{\sim}{\psi}$ are known as the Boussinesq - Papkovitch potentials.

It may be shown that if the constants A and B are chosen such that

$$A = 1, \quad B = \frac{-2(\lambda + 2\mu)}{(\lambda + \mu)}. \tag{7.17}$$

while the potentials ϕ , $\underset{\sim}{\psi}$ satisfy the equations

$$2\mu\frac{(\lambda + 2\mu)}{(\lambda + \mu)}\nabla^2\phi + \underset{\sim}{r}.\underset{\sim}{P} = 0$$

$$2\mu\frac{(\lambda + 2\mu)}{(\lambda + \mu)}\nabla^2\underset{\sim}{\psi} - \underset{\sim}{P} = 0 \tag{7.18}$$

that the displacement $\underset{\sim}{u}$ may be written as

$$\underset{\sim}{u} = \text{grad } (\phi + \underset{\sim}{r}.\underset{\sim}{\psi}) - \frac{2(\lambda + 2\mu)}{(\lambda + \mu)}\underset{\sim}{\psi} \tag{7.19}$$

If now the equations given in (7.18) can be solved for ϕ and $\underset{\sim}{\psi}$, we may solve the elastic problem. Several simple cases of this solution are discussed by Sneddon and Berry (1958).

It follows from equation (7.18) that in the absence of body forces, the elastic problem reduces to the solution of Laplace's equation, since

$$\nabla^2\phi = \nabla^2\underset{\sim}{\psi} = 0$$

Sneddon's Solution.

It has been shown by Sneddon (1961) that a solution of the equations of thermoelastic equilibrium may be obtained in terms of three potential functions ϕ, ψ, and χ, by writing the temperature θ, the displacements (u, v, w) and the stresses in terms of these functions as follows, where we have taken rectangular co-ordinates (x, y, z)

$$
\begin{aligned}
u &= \frac{\partial \chi}{\partial x} + \frac{\partial \phi}{\partial x} + (\beta^2 - 1)\, z \frac{\partial^2 \phi}{\partial x\, \partial z} + z \frac{\partial \psi}{\partial x}, \\
v &= \frac{\partial \chi}{\partial y} + \frac{\partial \phi}{\partial y} + (\beta^2 - 1)\, z \frac{\partial^2 \phi}{\partial y\, \partial z} + z \frac{\partial \psi}{\partial y}, \\
w &= \frac{\partial \chi}{\partial z} - \beta^2 \frac{\partial \phi}{\partial z} + (\beta^2 - 1)\, z \frac{\partial^2 \phi}{\partial z^2} + z \frac{\partial \psi}{\partial z} - \psi,
\end{aligned}
\tag{7.20}
$$

where

$$
\theta = \frac{2}{b} \frac{\partial \psi}{\partial z}, \tag{7.21}
$$

$$
\Delta = -2 \frac{\partial^2 \phi}{\partial z^2}. \tag{7.22}
$$

The components of stress are then given by

$$
\left.
\begin{aligned}
\sigma_x &= 2 \frac{\partial^2 \chi}{\partial x^2} + 2 \frac{\partial^2 \phi}{\partial x^2} - 2(\beta^2 - 2) \frac{\partial^2 \phi}{\partial z^2} + 2(\beta^2 - 1)\, z \frac{\partial^3 \phi}{\partial x^2 \partial z} + 2z \frac{\partial^2 \psi}{\partial x^2} - 2 \frac{\partial \psi}{\partial z} \\
\sigma_y &= 2 \frac{\partial^2 \chi}{\partial y^2} + 2 \frac{\partial^2 \phi}{\partial y^2} - 2(\beta^2 - 2) \frac{\partial^2 \phi}{\partial z^2} + 2(\beta^2 - 1)\, z \frac{\partial^3 \phi}{\partial y^2 \partial z} + 2z \frac{\partial^2 \psi}{\partial y^2} - 2 \frac{\partial \psi}{\partial z} \\
\sigma_z &= 2 \frac{\partial^2 \chi}{\partial z^2} - 2(\beta^2 - 1) \frac{\partial^2 \phi}{\partial z^2} + 2(\beta^2 - 1) \frac{\partial^3 \phi}{\partial z^3} + 2z \frac{\partial^2 \psi}{\partial z^2} - 2 \frac{\partial \psi}{\partial z}
\end{aligned}
\right]
\tag{7.23}
$$

and

$$
\left.
\begin{aligned}
\tau_{yz} &= 2(\beta^2 - 1)\, z \frac{\partial^3 \phi}{\partial y\, \partial z^2} + 2z \frac{\partial^2 \psi}{\partial y\, \partial z} + 2 \frac{\partial^2 \chi}{\partial y\, \partial z} \\
\tau_{xz} &= 2(\beta^2 - 1)\, z \frac{\partial^3 \phi}{\partial x\, \partial z^2} + 2z \frac{\partial^2 \psi}{\partial x\, \partial z} + 2 \frac{\partial^2 \chi}{\partial x\, \partial z} \\
\tau_{xy} &= 2 \frac{\partial^2 \phi}{\partial x\, \partial y} + 2(\beta^2 - 1) \frac{z\, \partial^3 \phi}{\partial x\, \partial y\, \partial z} + 2z \frac{\partial^2 \psi}{\partial x\, \partial y} + 2 \frac{\partial^2 \chi}{\partial x\, \partial y}
\end{aligned}
\right]
\tag{7.24}
$$

Several applications of these solutions are discussed by the author mentioned above.

§8 Literature.

In this paragraph we shall give an indication of the work being carried out in the field of thermoelasticity. We shall leave till later the background literature for the crack problems.

Interest in thermoelastic steady - state problems dates back many years. An explanation of what is meant by a thermoelastic problem and a method of tackling them is described by Love (1944). Goodier (1957) has developed a method of attacking these problems by reducing the thermoelastic problem to an elastic one at constant temperature. In his book Lur'e (1955) derives a method of solution of thermoelastic problems in terms of potential functions and this is the method developed here. The problems of the infinite and semi-infinite mediums together with the problems of the thick plate have attracted attention. Sternberg and McDowell (1955) solved the problem of the semi-infinite medium with the surface free from stress and with a known temperature distribution on the surface. The method used was a combination of the Boussinesq-Papkowitch potential solution and Green's function. Using the same method McDowell (1957) obtained the corresponding solution for the thick plate. The solutions obtained are in the form of elliptic integrals.

The two problems of the semi-infinite solid and thick plate with stress free boundaries and imposed temperature distributions on the surfaces were solved by Muki (1960) in the case in which axial symmetry is not present. The temperature distribution is taken in the form

$$\theta(\rho, z) = \sum_{m=0}^{\infty} \theta_m(\rho) \cos m\phi$$

Knops (1959) was able to determine the solution of some particular problems by taking the difference between two isothermal elastic solutions. Using a direct integration of the governing equations , Sharma (1956) obtained a solution for the plate problem. Nowacki (1957_1 and 1957_2) derived the solutions of the infinite and semi-infinite materials, subject to a known temperature field, by using a thermoelastic displacement potential. Assuming axial symmetry, Sneddon and Lockett(1960_1), used two dimensional integral transform methods to find the solutions of the semi-infinite solid and thick plate problems. The solutions derived by them are given in the form of inverse transforms. Special cases are considered and for these particular solutions, numerical calculations are carried out andthe lines along which the difference in principal stresses is constant, are plotted. These lines correspond to the isochromatic lines used in photoelasticity. The same authors (1960_2) discussed the case of the thick elastic plate lying on a rigid foundation. Olesiak and Sneddon (1960) have found a solution of the problem of determining the thermal stress in an infinite elastic solid containing a penny shaped crack.

The 'classical' equations of time dependent thermoelasticity and the 'fully linked' equations do not concern us in this work. However, there are several text-books which deal with the subject to a much fuller extent. Early work is summarised by Melan and Parkus (1953), while more recent developments are described fully by Nowacki (1960). Gatewood (1957) deals with technical applications in the introduction of his book. The equations of thermoelasticity are also described fully by Boley and Weiner (1960).

II. PROBLEMS IN THERMOELASTICITY

§9 The thermo-elastic equations of equilibrium.

We consider the equations of elasticity derived in part I in the case where there is no time dependence, body forces, or heat sources. Thus for steady state thermo-elastic problems, the equations of equilibrium referred to rectangular co-ordinate axes (x, y, z) in dimensionless form are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \qquad (9.1)$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0$$

where the stresses are related to the displacement vector (u, v, w) or (u_x, u_y, u_z) by the equations

$$(\sigma_x, \sigma_y, \sigma_z) = \left[(\beta^2 - 2)\Delta - b\theta\right] + 2\left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}\right) \qquad (9.2)$$

$$\tau_{xy} = \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}, \qquad \tau_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \qquad \tau_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \qquad (9.3)$$

In these equations μ is the unit of stress and we have written $T(1 + \theta)$ to denote the termperature variation where θ satisfies the equation

$$\nabla^2\theta = \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2} = 0 \qquad (9.4)$$

and T is a constant.

In terms of the Lamé constants λ, μ and the coefficient of linear expansion α, we can express β^2 and b as

$$\beta^2 = \frac{\lambda + 2\mu}{\mu}, \qquad b = \frac{(3\lambda + 2\mu)\alpha T}{\mu} \qquad (9.5)$$

In equation (9.2) we have written Δ for

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad (9.6)$$

In all the above equations we have considered rectangular co-ordinates (x, y, z). If we write $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$, then in terms of the cylindrical polar co-ordinates (ρ, ϕ, z) the equations become

$$\left.\begin{aligned}
&\frac{\partial\sigma_\rho}{\partial\rho} + \frac{1}{\rho}\frac{\partial\tau_{\rho\phi}}{\partial\phi} + \frac{\partial\tau_{\rho z}}{\partial z} + \frac{\sigma_\rho - \sigma_\phi}{\rho} = 0 \\
&\frac{\partial\tau_{\rho\phi}}{\partial\rho} + \frac{1}{\rho}\frac{\partial\sigma_\phi}{\partial\phi} + \frac{\partial\tau_{\phi z}}{\partial z} + \frac{2\tau_{\rho\phi}}{\rho} = 0 \\
&\frac{\partial\tau_{\rho z}}{\partial\rho} + \frac{1}{\rho}\frac{\partial\tau_{\phi z}}{\partial\phi} + \frac{\partial\sigma_z}{\partial z} + \frac{\tau_{\rho z}}{\rho} = 0
\end{aligned}\right\} \tag{9.7}$$

and the relations (9.2) and (9.3) become

$$(\sigma_\rho, \sigma_\phi, \sigma_z) = (\beta^2 - 2)\Delta - b\,\theta + 2\left(\frac{\partial u_\rho}{\partial\rho},\ \frac{u_\rho}{\rho} + \frac{1}{\rho}\frac{\partial u_\phi}{\partial\phi},\ \frac{\partial u_z}{\partial z}\right) \tag{9.8}$$

$$\tau_{\phi z} = \frac{\partial u_\phi}{\partial z} + \frac{1}{\rho}\frac{\partial u_z}{\partial\phi}, \quad \tau_{z\rho} = \frac{\partial u_z}{\partial\rho} + \frac{\partial u_\rho}{\partial z}, \quad \tau_{\rho\phi} = \frac{\partial u_\phi}{\partial\rho} + \frac{1}{\rho}\frac{\partial u_\rho}{\partial\phi} - \frac{u_\phi}{\rho} \tag{9.9}$$

where the temperature θ will satisfy

$$\frac{\partial^2\theta}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\theta}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\theta}{\partial\phi^2} + \frac{\partial^2\theta}{\partial z^2} = 0 \tag{9.10}$$

and the dilatation Δ is given by

$$\Delta = \frac{\partial u_\rho}{\partial\rho} + \frac{u_\rho}{\rho} + \frac{1}{\rho}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z}\,. \tag{9.11}$$

If we wish symmetry about the z-axis then we must have $u_\phi = 0$ and differentiation with respect to ϕ will be a null operator viz. $\frac{\partial}{\partial\phi} = 0$.

§10 The basic solution.

We shall now formulate a solution of the equations of §9 which is a modification of one first used by Lur'e (1955) p.191-199. This solution will be the basic solution and will be used in varying forms in the following sections.

In terms of a potential function $\psi(x, y, z)$ we take a solution for rectangular axes (x, y, z) in the form

$$u = \frac{\partial\psi}{\partial x}, \quad v = \frac{\partial\psi}{\partial y}, \quad w = -\frac{\partial\psi}{\partial z} + \chi(z) \tag{10.1}$$

where (u, v, w) are the displacements of the medium in the (x, y, z) directions respectively. We choose also

$$b\,\theta = -2(\beta^2 - 1)\frac{\partial^2\psi}{\partial z^2} + \beta^2\chi'(z) \tag{10.2}$$

where $\psi(x, y, z)$ is a harmonic function so that

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0 \tag{10.3}$$

and $\chi(z)$ is a quadratic function of z, such that

$$\chi(z) = fz^2 + gz + h \tag{10.4}$$

where f, g, h are constants.

Then it is easily shown that

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0 \tag{10.5}$$

everywhere, and that the stress-strain relations become

$$\sigma_x = -2\left[\frac{\partial^2\psi}{\partial y^2} + \chi'(z)\right]$$

$$\sigma_y = -2\left[\frac{\partial^2\psi}{\partial x^2} + \chi'(z)\right] \tag{10.6}$$

$$\tau_{xy} = 2\,\frac{\partial^2\psi}{\partial x\partial y}$$

It then follows easily that the equations of equilibrium (9.1) are satisfied.

It therefore follows that the solution given by equations (10.1) to (10.4) and (10.6) is appropriate to any problem in which the conditions (10.5) are satisfied at a boundary. It also allows the solution of such problems where we have a given surface temperature distribution or given heat flow distribution.

In terms of cylindrical polar co-ordinates (ρ, ϕ, z) with the corresponding displacement (u_ρ, u_ϕ, u_z) we may take the solution in the form

$$u_\rho = \frac{\partial\psi}{\partial\rho}, \quad u_\phi = \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}, \quad u_z = -\frac{\partial\psi}{\partial z} + \chi(z) \tag{10.7}$$

where χ is given by equation (10.4) and the appropriate form of equation (10.3) to determine ψ becomes

$$\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} = 0 \tag{10.8}$$

The temperature fluxuation is again given by (10.2) and we now have

$$\sigma_z = \tau_{\rho z} = \tau_{\phi z} = 0 \tag{10.9}$$

at every point. The stress-strain relations now assume the forms

$$\sigma_\rho = 2\left(\frac{\partial^2\psi}{\partial\rho^2} + \frac{\partial^2\psi}{\partial z^2}\right) - 2\chi'(z)$$

$$\sigma_\phi = -2\,\frac{\partial^2\psi}{\partial\rho^2} - 2\chi'(z) \tag{10.10}$$

$$\tau_{\rho\phi} = 2\,\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\right)$$

so that the solution given now by equations (10.7), (10.8) and (10.10) is suitable for solving problems in which (10.9) are satisfied at a boundary.

In the case of axial symmetry, we have $\tau_{\rho\phi} = 0$ everywhere and

$$\sigma_\rho - \sigma_\phi = 2\rho \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial \rho}\right) \tag{10.11}$$

§11 Elementary solutions.

In the last paragraph it was shown that if we chose a harmonic function ψ together with a particular type of solution of the thermo-elastic equations we could solve a class of problems with stress free boundaries. Suppose now that ψ is a simple harmonic function. Then in this section, we show that from this simple harmonic function we can construct by summation or integration, solutions of problems of physical interest.

Consider the harmonic function

$$\psi(\rho, z) = \log\left\{\frac{r - (z+\alpha)}{r + (z+\alpha)}\right\} \tag{11.1}$$

where $r^2 = \rho^2 + (z + \alpha)^2$.

By a simple summation procedure we have the following solution

$$\psi(\rho, z) = \frac{-b}{4(\beta^2 - 1)} \sum_n \theta_n \log\left\{\frac{r_n - (z + a_n)}{r_n + (z + a_n)}\right\} \tag{11.2}$$

where now θ_n, a_n are constants and $r_n^2 = \rho^2 + (z + a_n)^2$.

Choose $\chi(z) \equiv 0$, and it follows easily that

$$\theta(\rho, 0) = \sum_n \theta_n a_n/(\rho^2 + a_n^2)^{3/2} \tag{11.3}$$

If we take the solution given above and substitute in (10.11) we find

$$\sigma_\rho - \sigma_\phi = \frac{b}{(\beta^2 - 1)} \sum_n \theta_n (z + a_n)\left\{\frac{1}{r_n^3} + \frac{2}{\rho^2 r_n}\right\} \tag{11.4}$$

As a particular case of the function given by (11.2) we shall consider the solution given by

$$\psi(\rho, z) = -\frac{b}{4(\beta^2 - 1)}\left\{\theta_1 \log\frac{r_1 - (z + a_1)}{r_1 + (z + a_1)} + \theta_2 \log\frac{r_2 - (z + a_2)}{r_2 + (z + a_2)}\right\} \tag{11.5}$$

where $\chi(z) \equiv 0$ and $r_i^2 = \rho^2 + (z + a_i)^2$.

Now if we choose values of the constants as follows

$$a_1 = k + 1, \quad a_2 = -(k-1), \qquad \theta_1 = \theta_2 = -\frac{(k^2-4)^2\theta_0}{8k} \tag{11.6}$$

we have

$$\psi(\rho, z) = \frac{(k^2-4)^2 b\theta_0}{32k(\beta^2-1)}\left[\log\frac{\sqrt{\rho^2+(k+1+z)^2}-(k+1+z)}{\sqrt{\rho^2+(k+1+z)^2}+(k+1+z)} - \log\frac{\sqrt{\rho^2+(k-1-z)^2}-(k-1-z)}{\sqrt{\rho^2+(k-1-z)^2}+(k-1-z)}\right], k > 2 \tag{11.7}$$

It follows that, with the constants given by equation (11.6), equation (11.7) becomes

$$\theta(\rho, z) = \frac{(k^2-4)^2\theta_0}{8k}\left[\frac{k-1-z}{\left[\rho^2+(k-1-z)^2\right]^{3/2}} - \frac{k+1+z}{\left[\rho^2+(k+1+z)^2\right]^{3/2}}\right] \tag{11.8}$$

From equation (11.8) we see that

$$\theta(\rho, 1) = f(\rho), \quad \theta(\rho, -1) = 0 \tag{11.9}$$

where we have taken the function $f(\rho)$ to be

$$f(\rho) = \frac{(k^2-4)^2\theta_0}{8k}\left[\frac{(k-2)}{\left[\rho^2+(k-2)^2\right]^{3/2}} - \frac{(k+2)}{\left[\rho^2+(k+2)^2\right]^{3/2}}\right] \tag{11.10}$$

The solution given by equation (11.7) thus solves the problem of a thick plate $-1 \leqslant z \leqslant 1$ whose plane, stress free surfaces are deformed by means of the temperature distributions given by equation (11.9).

The situation is as shown in Fig.(11.1)

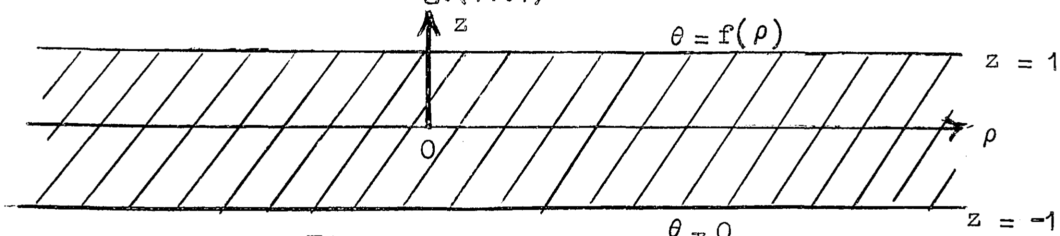


Fig. 11.1

The solution given here was obtained by other methods by Sneddon and Lockett (1960, p.145-153).

The quantities of physical interest in this problem are given as follows

$$(\sigma_\rho - \sigma_\phi) = \frac{-b(k^2-4)^2\theta_0}{8k(\beta^2-1)}\left[(z+k+1)\left\{\frac{1}{r_1^3} + \frac{2}{\rho^2 r_1}\right\} + (z-k+1)\left\{\frac{1}{r_2^3} + \frac{2}{\rho^2 r_2}\right\}\right] \tag{11.11}$$

where $r_1^2 = \rho^2 + (z+k+1)^2$, $r_2^2 = \rho^2 + (z-k+1)^2$.

$$u_\rho = \frac{b(k^2-4)^2\theta_0}{16\rho k(\beta^2-1)}\left[\frac{(z+k+1)}{r_1}+\frac{(z-k+1)}{r_2}\right]. \qquad (11.12')$$

$$u_z = \frac{b(k^2-4)^2\theta_0}{16k(\beta^2-1)}\left[\frac{1}{r_2}+\frac{1}{r_1}\right]. \qquad (11.13')$$

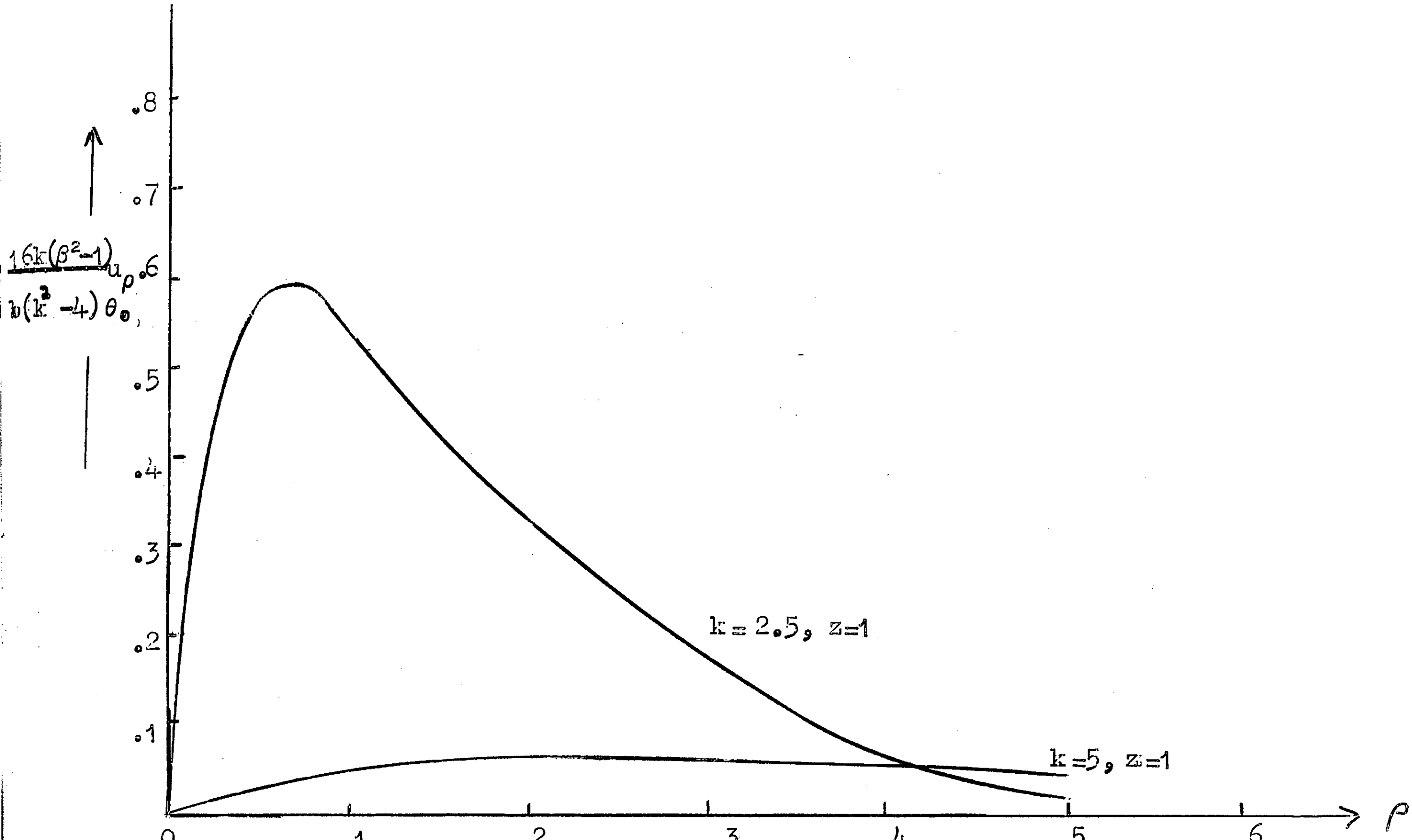
Values of u_ρ and u_z on the faces $z = \pm 1$

$$\frac{16k(\beta^2-1)}{b(k^2-4)^2\theta_0} \cdot u_\rho .$$

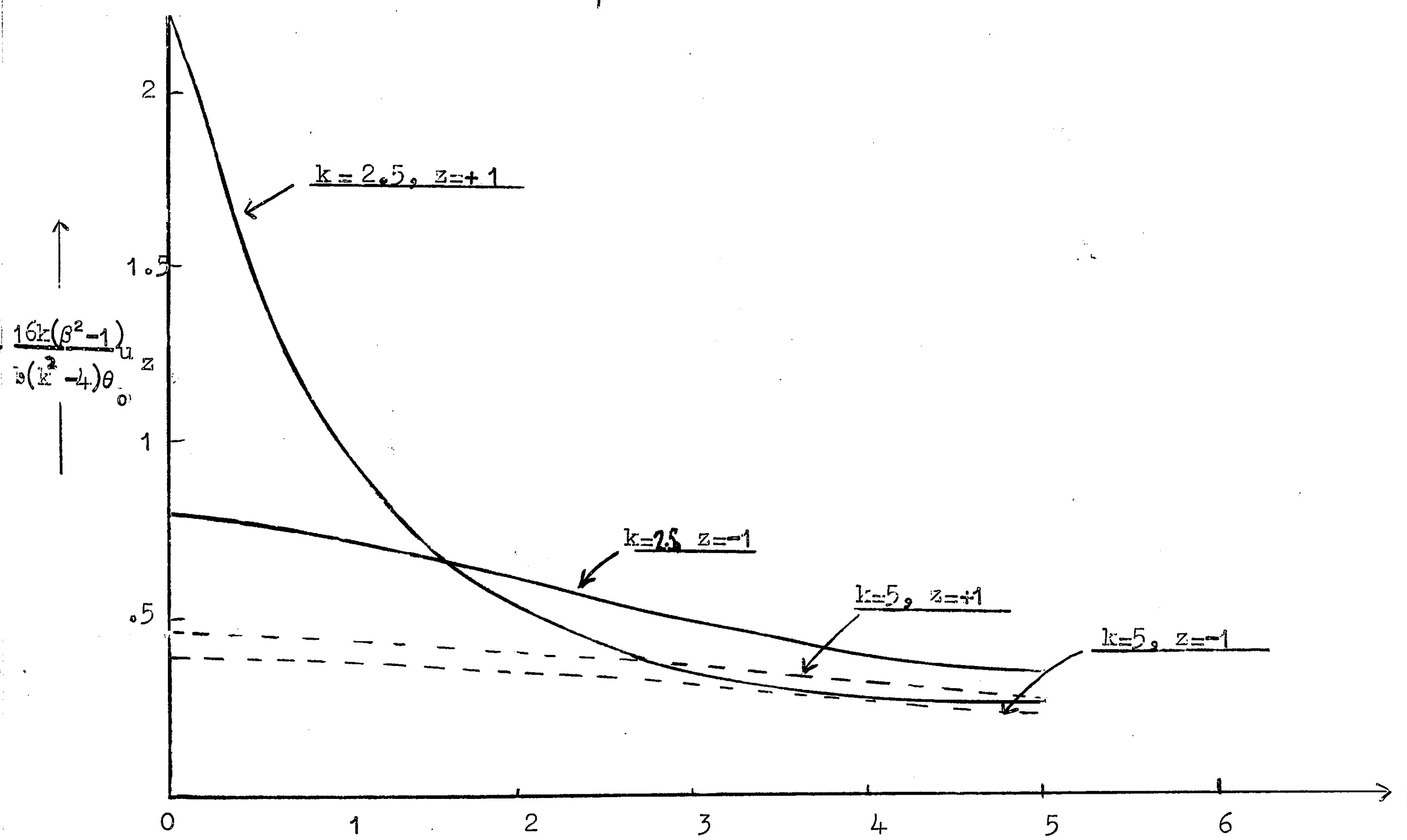
$z \backslash \rho$	0	.2	.4	.6	.8	1	2	5
k = 2.5								
+ 1	0	.351	.535	.585	.568	.529	.340	.106
k = 5								
+ 1	0	.009	.018	.029	.032	.040	.064	.060

$$\frac{16k(\beta^2-1)u_z}{b(k^2-4)^2\theta_0}$$

$z \backslash \rho$	0	.2	.4	.6	.8	1	2	5
k = 2.5								
z = + 1	2.222	1.968	1.673	1.390	1.169	1.002	.537	.280
z = − 1	.8	.797	.791	.778	.765	.743	.639	.358
k = 5								
z = + 1	.476	.475	.474	.469	.465	.458	.416	.2
z = − 1	.4	.399	.399	.398	.396	.392	.372	.283



Graph of u_ρ on faces of plate for k = 2.5, 5.



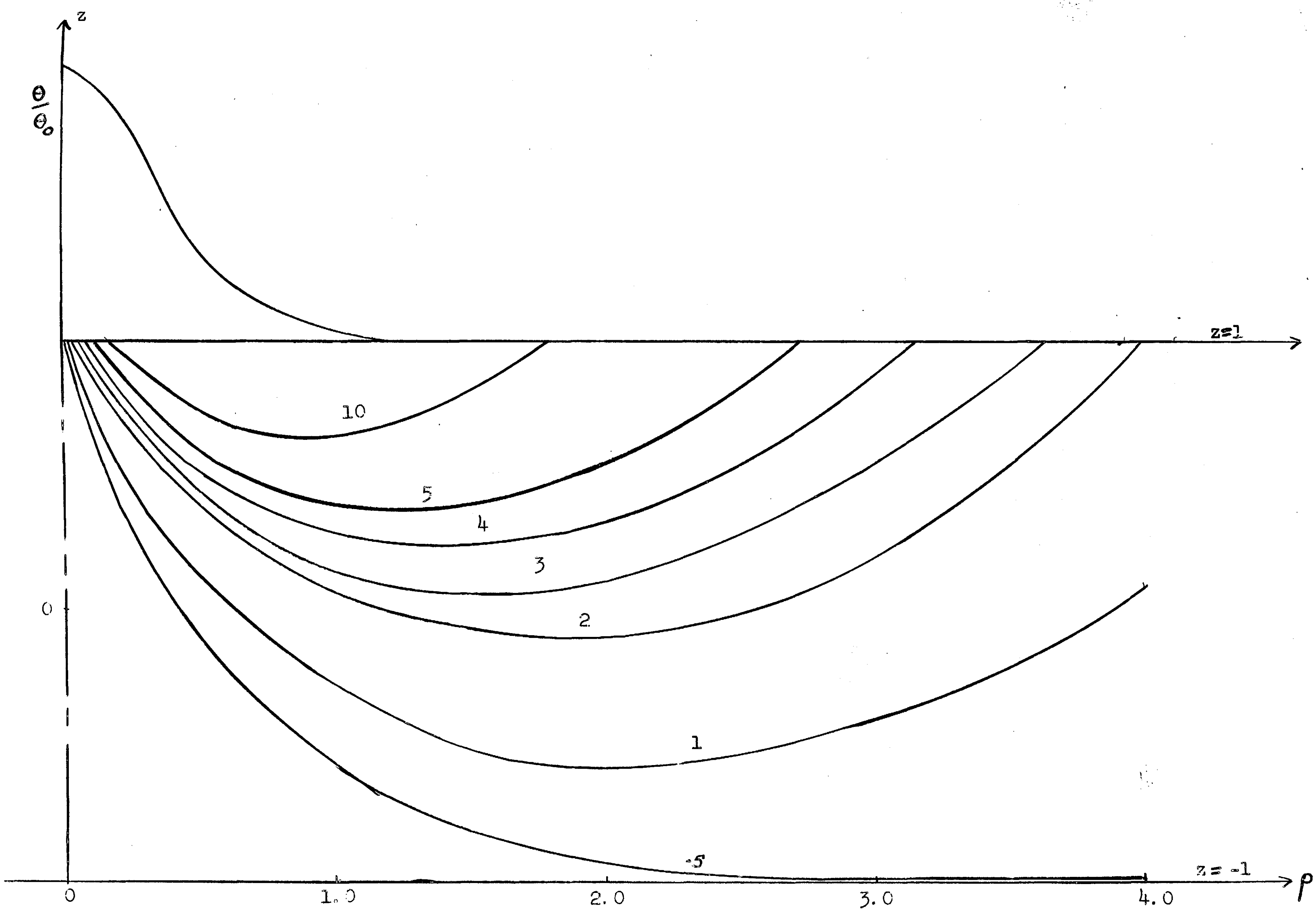
Graph of u_z on faces of plate for k = 2.5, 5.

ρ \ z	1.0	0.8	0.6	0.4	0.2	0.0	−0.2	−0.4	−0.6	−0.8	−1.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.2	0.3915	0.2979	0.2287	0.1760	0.1355	0.1033	0.0767	0.0544	0.0348	0.0170	0.0
0.4	1.5312	1.1680	0.8983	0.6934	0.5340	0.4070	0.3082	0.2148	0.1376	0.0672	0.0
0.6	3.3198	2.5424	1.9617	1.5183	1.1718	0.8945	0.6665	0.4732	0.3034	0.1482	0.0
0.8	5.6099	4.3193	3.3474	2.5999	2.0124	1.5398	1.1494	0.8170	0.5244	0.2563	0.0
1.0	8.2280	6.3762	4.9677	3.8751	3.0101	2.3097	1.7279	1.2304	0.7906	0.3867	0.0
1.2	10.9972	8.5847	6.7288	5.2752	4.1143	3.1673	2.3756	1.6949	1.0905	0.5338	0.0
1.4	13.7559	10.8236	8.5398	6.7316	5.2738	4.0746	3.0649	2.1914	1.4120	0.6918	0.0
1.6	16.3711	12.9884	10.3192	8.1815	6.4405	4.9953	3.7690	2.7011	1.7433	0.8549	0.0
1.8	18.7438	14.9966	12.0000	9.5711	7.5720	5.8968	4.4634	3.2067	2.0732	1.0177	0.0
2.0	20.8096	16.7898	13.5313	10.8581	8.6336	6.7515	5.1273	3.6928	2.3917	1.1753	0.0
2.4	23.9115	19.6067	16.0234	13.0118	10.4503	8.2400	6.2993	4.5600	2.9639	1.4596	0.0
2.8	25.6724	21.3631	17.6855	14.5225	11.7748	9.3579	7.1997	5.2374	3.4160	1.6857	0.0
3.2	26.2990	22.1703	18.5660	15.3995	12.5940	10.0817	7.8023	5.7016	3.7308	1.8446	0.0
3.6	26.0655	22.2200	18.7945	15.7270	12.9603	10.4420	8.1237	5.9608	3.9118	1.9374	0.0
4.0	25.2334	21.7154	18.5250	15.6193	12.9568	10.4978	8.2047	6.0419	3.9752	1.9718	0.0

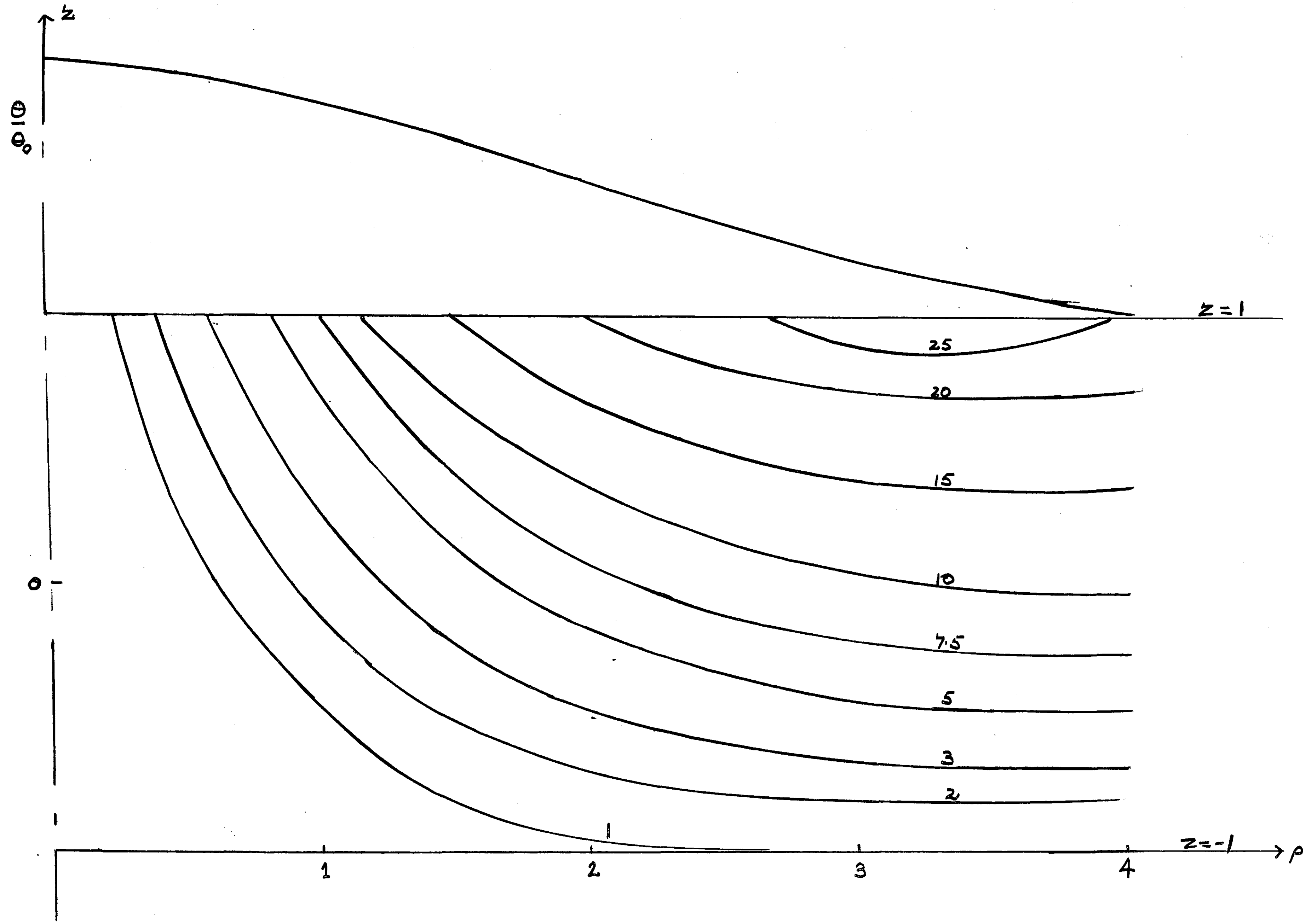
Values of $100 \left| \sigma_\rho - \sigma_\phi \right| \frac{\beta^2 - 1}{b\theta_0}$ = constant, for the thick plate $-1 \leq z \leq 1$ in the case $k = 5.0$

ρ \ z	1.0	0.8	0.6	0.4	0.2	0.0	−0.2	−0.4	−0.6	−0.8	−1.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.2	9.4782	2.7722	1.0653	0.4880	0.2517	0.1406	0.0825	0.0490	0.0278	0.0126	0.0
0.4	21.1180	7.8685	3.4137	1.6730	0.8985	0.5152	0.3074	0.1848	0.1056	0.0482	0.0
0.6	24.0179	11.2250	5.5935	2.9886	1.6972	1.0099	0.6180	0.3778	0.2182	0.1001	0.0
0.8	22.0760	12.2284	6.8925	4.0148	2.4218	1.5035	0.9477	0.5915	0.3461	0.1601	0.0
1.0	18.8860	11.8005	7.3423	4.6168	2.9499	1.9111	1.2427	0.7929	0.4709	0.2196	0.0
1.2	15.8145	10.7631	7.2282	4.8443	3.2575	2.1960	1.4715	0.9597	0.5785	0.2722	0.0
1.4	13.2118	9.5655	6.8108	4.8067	3.3760	2.3577	1.6242	1.0814	0.6612	0.3137	0.0
1.6	11.0929	8.4081	6.2622	4.6063	3.3548	2.4152	1.7050	1.1566	0.7165	0.3426	0.0
1.8	9.3848	7.3648	5.6801	4.3188	3.2406	2.3938	1.7260	1.1902	0.7460	0.3592	0.0
2.0	8.0052	6.4528	5.1151	3.9936	3.0706	2.3174	1.7013	1.1899	0.7535	0.3651	0.0
2.4	5.9622	4.9911	4.1141	3.3382	2.6612	2.0747	1.5658	1.1198	0.7206	0.3526	0.0
2.8	4.5625	3.9137	3.3090	2.7536	2.2488	1.7922	1.3787	1.0014	0.6518	0.3212	0.0
3.2	3.5699	3.1123	2.6763	2.2651	1.8801	1.5209	1.1853	0.8701	0.5709	0.2826	0.0
3.6	2.8444	2.5075	2.1810	1.8671	1.5669	1.2804	1.0068	0.7444	0.4911	0.2439	0.0
4.0	2.3011	2.0440	1.7919	1.5461	1.3072	1.0756	0.8509	0.6323	0.4186	0.2084	0.0

Values of $100 \left| \sigma_\rho - \sigma_\phi \right| \frac{\beta^2 - 1}{b\theta_0}$ = constant, for the thick plate $-1 \leq z \leq 1$ in the case $k = 2.5$



Values of $100 \left| \sigma_\rho - \sigma_\phi \right| (\beta^2-1)/b\theta_0$ = constant , for the value of $k = 2.5$



Values of $100|\sigma_\rho - \sigma_\phi| \cdot (\beta^2 - 1)/b\Theta_0$ = constant, for the value of $k = 5$

Suppose now that we return to the simple harmonic function which we considered in equations (11.1). Now, instead of a summation procedure we apply an integration one.

If we integrate over the parameter α , we obtain a potential function of the form

$$\psi(\rho, z) = \frac{-b}{4(\beta^2 - 1)} \int \Psi(\alpha) \log \left[\frac{r(\alpha) - (z + \alpha)}{r(\alpha) + (z + \alpha)} \right] d\alpha \qquad (11.11)$$

where $r(\alpha) = \left[\rho^2 + (z + \alpha)^2\right]^{1/2}$ and the function Ψ is arbitrary. It follows that, for this potential function, the temperature distribution will be given by

$$\theta(\rho, z) = \int \Psi(\alpha) \frac{z + \alpha}{r(\alpha)^3} d\alpha \qquad (11.12)$$

The difference in stresses (the quantity of physical interest) is given by

$$(\sigma_\rho - \sigma_\phi) = \frac{-b}{(\beta^2 - 1)} \int \Psi(\alpha)\ (z + \alpha) \left[\frac{1}{(r(\alpha))^3} + \frac{2}{\rho^2 r(\alpha)} \right] d\alpha \qquad (11.13)$$

The components of the displacements are given by

$$u_\rho = \frac{-b}{2(\beta^2 - 1)\rho} \int \Psi(\alpha) \frac{(z + \alpha)}{r} d\alpha \qquad (11.14)$$

$$u_z = \frac{-b}{2(\beta^2 - 1)} \int \Psi(\alpha) \frac{1}{r} d\alpha \qquad (11.15)$$

We now consider a particular case. If we take $\Psi(\alpha)$ to be given by

$$\Psi(\alpha) = \begin{cases} 0 & \alpha < -(k + 1) \\ -\tfrac{1}{2}(k^2 - 1)\theta_0 & -(k + 1) < \alpha < k - 1 \\ 0 & \alpha > (k - 1) \end{cases} \qquad (11.16)$$

then it follows from equation (11.12) that

$$\theta(\rho, z) = -\tfrac{1}{2}(k^2 - 1)\theta_0 \left[\frac{1}{\left[\rho^2 + (k + 1 - z)^2\right]^{1/2}} - \frac{1}{\left[\rho^2 + (k - 1 + z)^2\right]^{3/2}} \right] \qquad (11.17)$$

For this solution it is easily seen that

$$\theta(\rho, 1) = 0$$

$$\theta(\rho, 0) = g(\rho)$$

where

$$g(\rho) = \tfrac{1}{2}(k^2 - 1)\theta_0 \left[\frac{1}{\left[\rho^2 + (k-1)^2\right]^{1/2}} - \frac{1}{\left[\rho^2 + (k+1)^2\right]^{1/2}} \right] \tag{11.18}$$

If we now substitute the value of $\Psi(\alpha)$ given by equation (11.16) into equation (11.13) we find that the difference in principal stresses in a plate $0 \leqslant z \leqslant 1$ whose surfaces are free from applied stress and whose surface temperatures are defined by the equations $\theta(\rho, 1) = 0$, $\theta(\rho, 0) = g(\rho)$, where we have defined $g(\rho)$ in equation (11.18), is given by the equation

$$\sigma_\rho - \sigma_\phi = \frac{b(k^2 - 1)\theta_0}{2(\beta^2 - 1)} (\rho_2 - \rho_1) \left[\frac{1}{\rho_1 \rho_2} + \frac{2}{\rho^2} \right] \tag{11.19}$$

where $\rho_1^2 = \rho^2 + (k - 1 + z)^2$ and $\rho_2^2 = \rho^2 + (k + 1 - z)^2$.

Similarly the displacements are given by

$$u_\rho = \frac{b(k^2 - 1)\theta_0}{2(\beta^2 - 1)\rho} \left[\sqrt{\rho^2 + (z + k - 1)^2} - \sqrt{\rho^2 + (z - k - 1)^2} \right] \tag{11.20}$$

$$u_z = \frac{b(k^2 - 1)\theta_0}{4(\beta^2 - 1)} \log \left[\frac{(z + k - 1) + \rho_1}{(z - k - 1) + \rho_2} \right] \tag{11.21}$$

The situation is as shown in Fig.11.2.

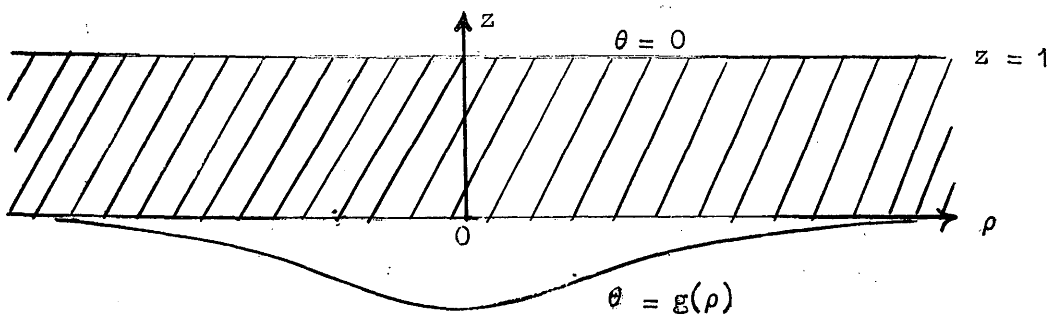


Fig.11.2

We now take a second particular case of $\Psi(\alpha)$, viz.

$$\Psi(\alpha) = \frac{Q}{4\pi\kappa} e^{-h(\alpha - \delta)} \tag{11.22}$$

where $\delta > 0$ and we choose the limits of integration to be (δ, ∞).

Then, in this case, we have the potential function

$$\psi(\rho, z) = \frac{-bQ}{16\pi\kappa(\beta^2 - 1)} \int_\delta^\infty e^{-(\alpha - \delta)h} \log\left[\frac{r(\alpha) - (z + \alpha)}{r(\alpha) + (z + \alpha)}\right] d\alpha \tag{11.23}$$

which immediately leads to the temperature distribution

$$\theta(\rho, z) = \frac{Qe^{h\delta}}{4\pi\kappa} \int_\delta^\infty \frac{(z + \alpha)e^{-h\alpha} d\alpha}{\left[(z + \alpha)^2 + \rho^2\right]^{3/2}} \tag{11.24}$$

The value of the temperature on the surface $z = 0$, is given by

$$\theta(\rho, 0) = \frac{Qe^{h\delta}}{4\pi\kappa} \int_\delta^\infty \frac{\alpha e^{-h\alpha}}{(\alpha^2 + \rho^2)^{3/2}} d\alpha \tag{11.25}$$

Now, it has been shown by Thomas (1957, p.482-493) that the temperature field given by (11.25) is just the surface temperature distribution produced by a heat source of strength Q placed at the point $(0, 0, -\delta)$ above the semi-infinite solid $z \geq 0$ with the boundary condition

$$h\theta - \frac{\partial\theta}{\partial z} = \frac{Q\delta}{4\pi\kappa(\delta^2 + \rho^2)^{3/2}}, \quad \text{on } z = 0. \qquad (11.26$$

The situation is as shown in Fig.11.3.

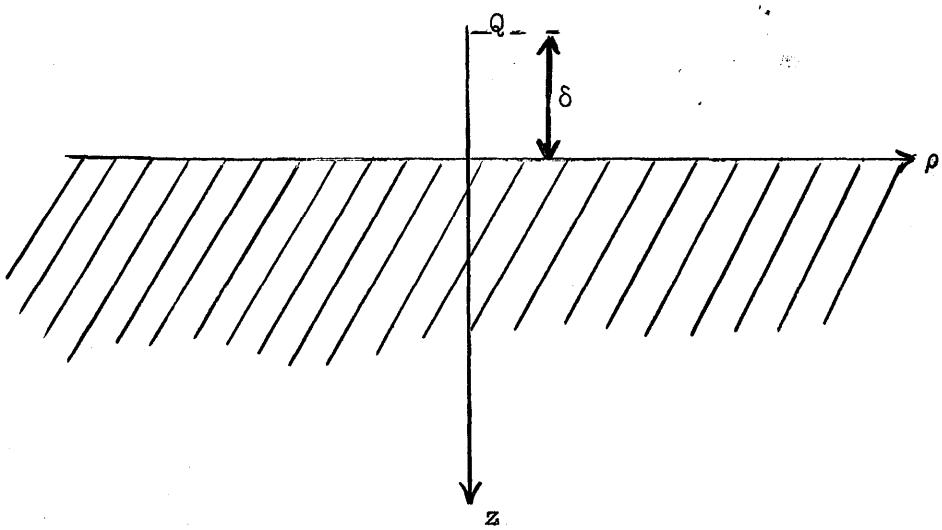


Fig.11.3

§12 Application of Fourier techniques.

In this section we shall make use of the techniques of Fourier analysis to derive solutions of certain problems for the semi-infinite solid and thick plates. As usual we shall denote the Fourier transform of a function $f(x, y)$ by $\overline{f}(\xi, \eta)$. We consider, in this paragraph only rectangular co-ordinates and write

$$2\pi \overline{f}(\xi, \eta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x, y)\, e^{i(\xi x + \eta y)}\, dx\, dy \tag{12.1}$$

so that

$$f(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f}(\xi, \eta)\, e^{-i(\xi x + \eta y)}\, d\xi\, d\eta \tag{12.2}$$

If we consult the equations of paragraph 10 we see that the potential function $\psi(x, y, z)$ must satisfy the equation

$$\nabla^2 \psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0$$

so that an elementary form of ψ would be

$$\psi(x, y, z) = e^{-i(\xi x + \eta y)}\, e^{-\zeta z} \tag{12.3}$$

where we must take $\zeta^2 = \xi^2 + \eta^2$.

If we apply an integration procedure to the simple solution (12.3) we have the function

$$\psi(x, y, z) = \frac{-b}{4\pi(\beta^2 - 1)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\overline{f}(\xi, \eta)}{\zeta^2}\, e^{-i(\xi x + \eta y)}\, e^{-\zeta z}\, d\xi\, d\eta \tag{12.4}$$

which gives immediately from §10 that $\sigma_3 = \tau_{x3} = \tau_{y3} = 0$. In equation (12.4) $\overline{f}(\xi, \eta)$ is the Fourier transform of an arbitrary function $f(x, y)$.

For this potential function it is easily seen, by substituting from (12.4) into equation (10.2) that the temperature distribution is given by

$$\theta(x, y, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f}(\xi, \eta)\, e^{-i(\xi x + \eta y)}\, e^{-\zeta z}\, d\xi\, d\eta \tag{12.5}$$

so that, if we choose $\chi(z) = 0$, we find the following conditions on θ

$$\left.\begin{aligned} &\theta \to 0 \qquad \text{as } z \to \infty \\ &\theta(x, y, 0) = f(x, y) \end{aligned}\right] \tag{12.6}$$

These functions lead to the consideration of the problem shown in Fig. 12.1

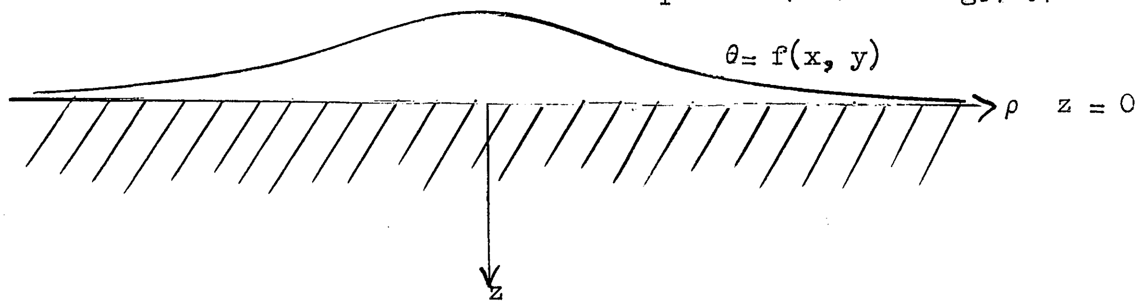


Fig. 12.1

with boundary conditions

$$\sigma_3 = \tau_{x3} = \tau_{y3} = 0. \qquad \theta = f(x, y) \quad \text{on } z = 0$$
$$\theta \to 0 \text{ as } z \to \infty \tag{12.7}$$

All physical quantities tend to 0 as $z \to \infty$, $x \to \pm\infty$, $y \to \pm\infty$

The physical quantities of interest are then given by

$$u_x = \frac{b}{4\pi(\beta^2 - 1)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{i\xi \bar{f}(\xi, \eta)}{\zeta^2} e^{-i(\xi x + \eta y) - \zeta z} d\xi d\eta \tag{12.8}$$

$$u_y = \frac{b}{4\pi(\beta^2 - 1)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{i\eta \bar{f}(\xi, \eta)}{\zeta^2} e^{-i(\xi x + \eta y) - \zeta z} d\xi d\eta \tag{12.9}$$

$$u_3 = \frac{-b}{4\pi(\beta^2 - 1)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\bar{f}(\xi, \eta)}{\zeta} e^{-i(\xi x + \eta y) - \zeta z} d\xi d\eta \tag{12.10}$$

$$\sigma_x = \frac{-b}{2\pi(\beta^2 - 1)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\eta^2 \bar{f}(\xi, \eta)}{\zeta^2} e^{-i(\xi x + \quad y) - \zeta z} d\xi d\eta \tag{12.11}$$

$$\sigma_y = \frac{-b}{2\pi(\beta^2 - 1)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\xi^2 \bar{f}(\xi, \eta)}{\zeta^2} e^{-i(\xi x + \eta y) - \zeta z} d\xi d\eta \tag{12.12}$$

$$\tau_{xy} = \frac{b}{2\pi(\beta^2 - 1)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\xi\eta \bar{f}(\xi, \eta)}{\zeta^2} e^{-i(\xi x + \eta y) - \zeta z} d\xi d\eta \tag{12.13}$$

These are the physical quantities required for the discussion of the semi-infinite solid. Now let us consider the problem of a thick plate $-1 \leqslant z \leqslant 1$ whose traction free surfaces are deformed by the temperature distributions $\theta = f(x, y)$ on $z = 1$ and $\theta = g(x, y)$ on $z = -1$.

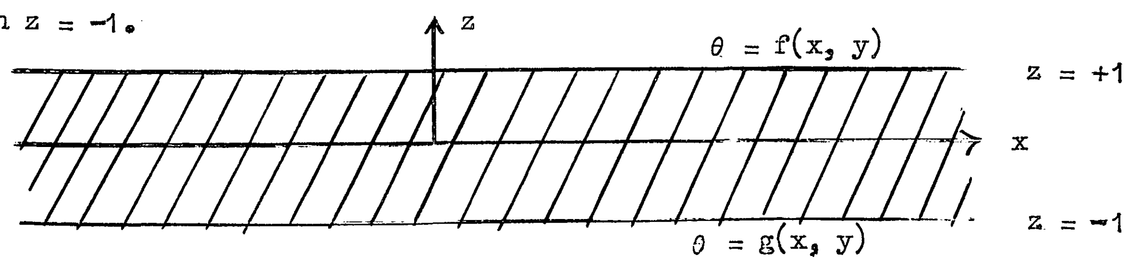


Fig. 12.2

Choose the potential function $\psi(x, y, z)$ to be

$$\psi(x, y, z) = \frac{-b}{4\pi(\beta^2 - 1)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\overline{f}(\xi, \eta)\sinh(1 + z)\zeta + \overline{g}(\xi, \eta)\sinh(1 - z)\zeta}{\zeta^2 \sinh 2\zeta} e^{-i(\xi x + \eta y)} d\xi d\eta \qquad (12.14)$$

and $\chi(z) = 0$.

Then it follows that the temperature distribution is given by

$$\theta(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\overline{f}(\xi, \eta)\sinh(1 + z)\zeta + \overline{g}(\xi, \eta)\sinh(1 - z)\zeta}{\sinh 2\zeta} e^{-i(\xi x + \eta y)} d\xi d\eta \qquad (12.15)$$

Since $\overline{f}(\xi, \eta)$, $\overline{g}(\xi, \eta)$ are the Fourier transforms of $f(x, y)$ and $g(x, y)$ respectively, it follows that

$$\left.\begin{aligned} \theta &= f(x, y) \quad \text{on} \quad z = 1 \\ \theta &= g(x, y) \quad \text{on} \quad z = -1 \end{aligned}\right] \qquad (12.16)$$

The following boundary conditions are thus satisfied by the function given in (12.14)

$$\left.\begin{aligned} \sigma_3 = \tau_{x3} = \tau_{y3} = 0, \quad \theta = f(x, y) \quad &\text{on } z = 1 \\ \sigma_3 = \tau_{x3} = \tau_{y3} = 0, \quad \theta = g(x, y) \quad &\text{on } z = -1 \end{aligned}\right] \qquad (12.17)$$

Expressions for the physical quantities involved, similar to these given in equations (12.8) to (12.13) are then easily derived.

Consider a third problem as shown in Fig.12.3

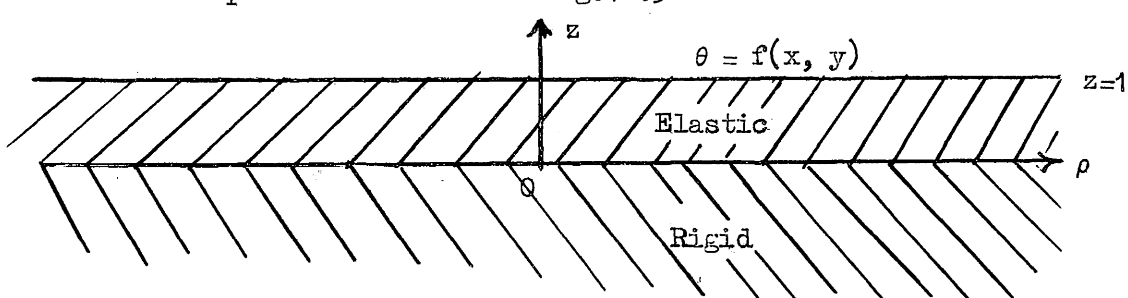


Fig.12.3

with boundary conditions

$$\left.\begin{aligned} \sigma_3 = \tau_{x3} = \tau_{y3} = 0, \quad \theta = f(x, y) \quad &\text{on } z = 1 \\ \frac{\partial \theta}{\partial z} = w = 0 \quad &\text{on } z = 0 \\ \text{all quantities finite as } x, y \to \pm\infty. & \end{aligned}\right] \qquad (12.18)$$

The third condition given will be satisfied, in most cases, provided $f(x, y)$ is a suitably chosen function.

A suitable potential function is

$$\psi(x, y, z) = \frac{-b}{4\pi(\beta^2 - 1)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\overline{f}(\xi, \eta)\cosh \zeta z}{\zeta^2 \cosh \zeta} e^{-i(\xi x + \eta y)}\, d\xi d\eta \qquad (12.19)$$

and we take $\chi(z) = 0$.

With the usual procedure we find that

$$\theta(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\overline{f}(\xi, \eta)\cosh \zeta z}{\cosh \zeta} e^{-i(\xi x + \eta y)}\, d\xi d\eta \qquad (12.20)$$

from which it is easy to find the following expressions

$$u_x = \frac{b}{4\pi(\beta^2 - 1)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{i\xi\overline{f}(\xi, \eta)\cosh \zeta z}{\zeta^2 \cosh \zeta} e^{-i(\xi x + \eta y)}\, d\xi d\eta \qquad (12.21)$$

$$u_y = \frac{b}{4\pi(\beta^2 - 1)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{i\eta\overline{f}(\xi, \eta)\cosh \zeta z}{\zeta^2 \cosh \zeta} e^{-i(\xi x + \eta y)}\, d\xi d\eta \qquad (12.22)$$

$$u_z = \frac{b}{4\pi(\beta^2 - 1)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\overline{f}(\xi, \eta)\sinh \zeta z}{\zeta \cosh \zeta} e^{-i(\xi x + \eta y)}\, d\xi d\eta \qquad (12.23)$$

$$\sigma_x = \frac{-b}{2\pi(\beta^2 - 1)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\eta^2\overline{f}(\xi, \eta)\cosh \zeta z}{\zeta^2 \cosh \zeta} e^{-i(\xi x + \eta y)}\, d\xi d\eta \qquad (12.24)$$

$$\sigma_y = \frac{-b}{2\pi(\beta^2 - 1)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\xi^2\overline{f}(\xi, \eta)\cosh \zeta z}{\zeta^2 \cosh \zeta} e^{-i(\xi x + \eta y)}\, d\xi d\eta \qquad (12.25)$$

$$\tau_{xy} = \frac{b}{2\pi(\beta^2 - 1)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\xi\eta\overline{f}(\xi, \eta)\cosh \zeta z}{\zeta^2 \cosh \zeta} e^{-i(\xi x + \eta y)}\, d\xi d\eta \qquad (12.26)$$

$$\frac{\partial\theta}{\partial z} = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\zeta\overline{f}(\xi, \eta)\sinh \zeta z}{\cosh \zeta} e^{-i(\xi x + \eta y)}\, d\xi d\eta \qquad (12.27)$$

from which the conditions (12.18) follow.

Thus the function given by equation (12.19) gives the stress field in an elastic layer $0 \leq z \leq 1$ which is resting on a rigid foundation $z \leq 0$, whose boundary $z = 0$ is impervious to the flow of heat. The layer is deformed by the application of a temperature field $\theta = f(x, y)$ on the stress free boundary $z = 1$.

§13 Application of Fourier techniques (cont.).

In §12 it was shown that by using the basic solution in conjunction with the theory of Fourier techniques that we could solve a certain class of problems in rectangular coordinates. The question now arises as to whether we can apply the basic solution to the analogous set of problems in cylindrical co-ordinates (ρ, ϕ, z) when we have symmetry about the z-axis. In this case we make use of Hankel transform techniques. If we denote the Bessel function of order ν by $J_\nu(z)$ and denote the Hankel transform of the function $f(\rho)$ by $\overline{f}(\xi)$ then we have, from the Hankel inversion theory (Sneddon, 1951) that if

$$\overline{f}(\xi) = \int_0^\infty \rho\, f(\rho)\, J_\nu(\xi\rho)\, d\rho \tag{13.1}$$

then

$$f(\rho) = \int_0^\infty \xi\, \overline{f}(\xi)\, J_\nu(\xi\rho)\, d\rho \tag{13.2}$$

The case with which we are mainly concerned here is $\nu = 0$.

To return to the equations for the basic solutions in the axially symmetric case as given in §10 we see that a particular harmonic function is

$$\psi(\rho, z) = e^{-\alpha z} J_0(\alpha\rho)$$

It follows that a potential function satisfying the equations in §10 and also suitable for fitting boundary conditions in this case is

$$\psi(\rho, z) = \frac{-b}{2(\beta^2 - 1)} \int_0^\infty \frac{\overline{f}(\xi)}{\xi} J_0(\xi\rho) e^{-\xi z}\, d\xi \tag{13.3}$$

where we have again taken $\chi(z) = 0$.

From equation (10.2) it is easily shown that the temperature distribution is given by

$$\theta(\rho, z) = \int_0^\infty \xi\, \overline{f}(\xi)\, J_0(\xi\rho) e^{-\xi z}\, d\xi \tag{13.4}$$

from which it follows that $\theta \longrightarrow 0$ as $z \longrightarrow \infty$

$$\text{and} \quad \theta(\rho, 0) = f(\rho) \quad \text{on} \quad z = 0 \tag{13.5}$$

Thus the function given in (13.3) gives the solution to the problem for a semi-infinite solid $z \geqslant 0$ whose stress free surface $z = 0$ is deformed by the temperature distribution $\theta(\rho, 0) = f(\rho)$.

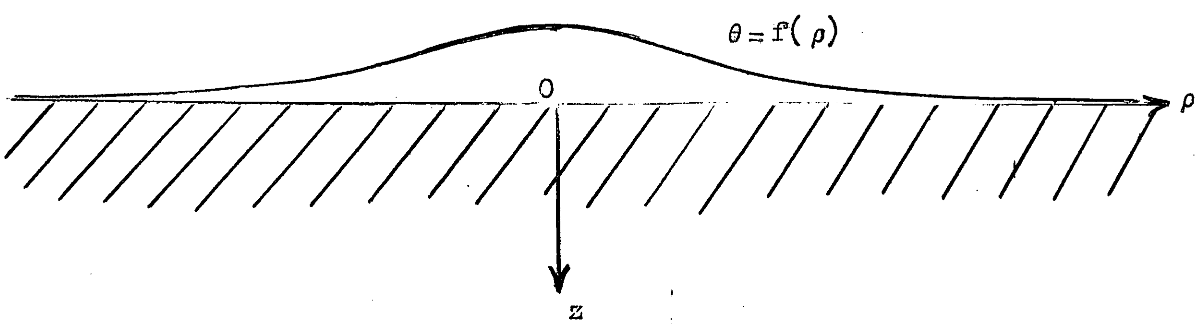


Fig. 13.1

The quantities of physical interest are then given by

$$u_\rho = \frac{b}{2(\beta^2 - 1)} \int_0^\infty \bar{f}(\xi)\, J_1(\xi \rho) e^{-\xi z}\, d\xi \tag{13.6}$$

$$u_3 = - \frac{b}{2(\beta^2 - 1)} \int_0^\infty \bar{f}(\xi)\, J_0(\xi \rho) e^{-\xi z}\, d\xi \tag{13.7}$$

$$\sigma_\rho - \sigma_\phi = \frac{b}{\beta^2 - 1} \int_0^\infty \xi\, \bar{f}(\xi) \left[J_0(\rho \xi) - 2J_1(\rho\xi) / (\rho\xi) \right] e^{-\xi z}\, d\xi \tag{13.8}$$

These are the solutions derived by Lockett and Sneddon (Q.Appl.M. July, 1960) who consider the stress distribution in detail and give some particular problems.

Another suitable potential function solution is

$$\psi(\rho, z) = - \frac{b}{2(\beta^2 - 1)} \int_0^\infty \frac{\bar{f}(\xi) \sinh\xi(1 + z) + \bar{g}(\xi) \sinh\xi(1 - z)}{\xi \sinh 2\xi} J_0(\xi \rho)\, d\xi \tag{13.9}$$

with $\chi(z) = 0$.

The temperature distribution in this case is given by

$$\theta(\rho, z) = \int_0^\infty \frac{\xi \bar{f}(\xi) \sinh\xi(1 + z) + \xi\bar{g}(\xi) \sinh\xi(1 - z)}{\sinh 2\xi} J_0(\xi \rho)\, d\xi \tag{13.10}$$

so that on the boundaries $z = \pm 1$ we have the conditions

$$\begin{aligned} \theta(\rho, 1) &= f(\rho) \\ \theta(\rho, -1) &= g(\rho) \end{aligned} \tag{13.11}$$

It follows that the function given by equation (13.9) provides the solution to the problem for a thick plate $-1 \leqslant z \leqslant 1$ to whose stress free boundaries $z = \pm 1$, are applied the temperature distributions given by equations (13.11). The quantities of physical interest viz.,

$$u_\rho = \frac{b}{2(\beta^2 - 1)} \int_0^\infty \frac{\bar{f}(\xi) \sinh\xi(1 + z) + \bar{g}(\xi) \sinh\xi(1 - z)}{\sinh 2\xi} J_1(\xi\rho)\, d\xi \tag{13.12}$$

$$u_z = \frac{b}{2(\beta^2 - 1)} \int_0^\infty \frac{\bar{f}(\xi) \cosh\xi(1 + z) - \bar{g}(\xi) \cosh\xi(1 - z)}{\sinh 2\xi} J_0(\xi \rho)\, d\xi \tag{13.13}$$

$$\sigma_\rho - \sigma_\phi = \frac{b}{(\beta^2 - 1)} \int_0^\infty \xi \frac{\bar{f}(\xi) \sinh\xi(1 + z) + \bar{g}(\xi) \sinh\xi(1 - z)}{\sinh 2\xi} \left[J_0(\xi\rho) - \frac{2J_1(\xi\rho)}{(\xi\rho)} \right] d\xi \tag{13.14}$$

are in agreement with the values given by Sneddon and Lockett (Q.A.M., July 1960) when $g(\rho)$ is taken to be zero to agree with the temperature distribution taken by them.

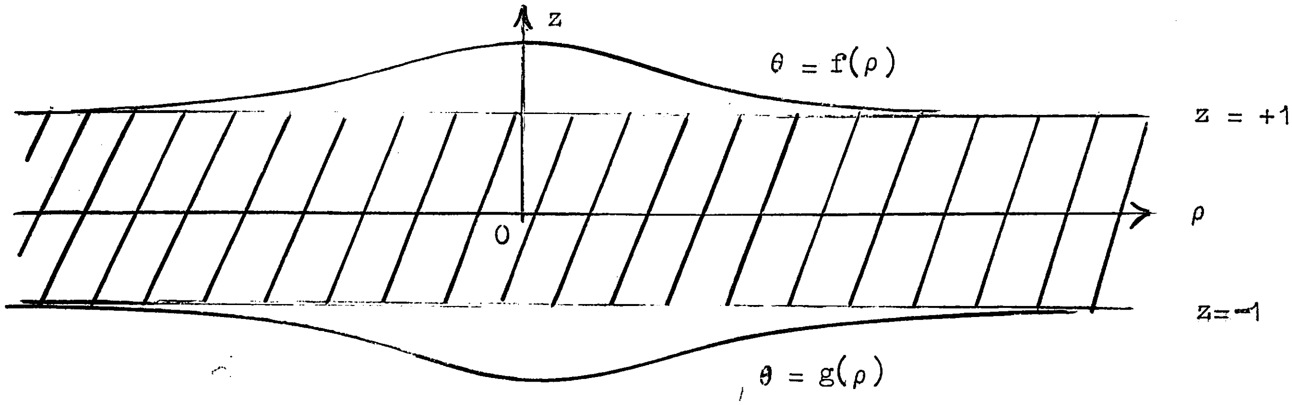


Fig. 13.2

Another problem which is of interest is that of an elastic layer $0 \leqslant z \leqslant 1$ resting on a rigid foundation $z \leqslant 0$ whose boundary $z = 0$ is impervious to flow of heat.

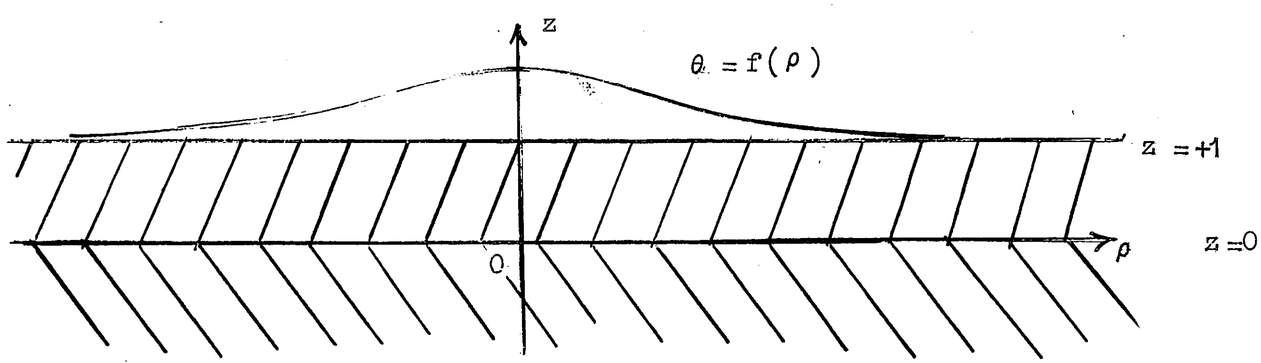


Fig. 13.3

To solve this problem we choose our potential function

$$\psi(\rho, z) = -\frac{b}{2(\beta^2 - 1)}\int_0^\infty \bar{f}(\xi)\frac{\cosh \xi z}{\xi \cosh \xi} J_0(\rho\xi)\, d\xi \qquad (13.15)$$

and $\chi(z) \equiv 0$.

With the solution (13.15) we find the following quantities on substituting in the equations of §10

$$\theta(\rho, z) = \int_0^\infty \xi \bar{f}(\xi)\frac{\cosh \xi z}{\cosh \xi} J_0(\rho\xi)\, d\xi \qquad (13.16)$$

$$\frac{\partial\theta}{\partial z} = \int_0^\infty \xi^2 \bar{f}(\xi)\frac{\sinh \xi z}{\cosh \xi} J_0(\rho\xi)\, d\xi \qquad (13.17)$$

$$u_z(\rho, z) = \frac{b}{2(\beta^2 - 1)}\int_0^\infty \bar{f}(\xi)\frac{\sinh \xi z}{\cosh \xi} J_0(\rho\xi)\, d\xi \qquad (13.18)$$

$$u_\rho(\rho, z) = \frac{b}{2(\beta^2 - 1)}\int_0^\infty \bar{f}(\xi)\frac{\cosh \xi z}{\cosh \xi} J_1(\xi\rho)\, d\xi \qquad (13.19)$$

so that

$$\theta(\rho\,,\,1) = f(\rho)$$
$$w = \frac{\partial\theta}{\partial z} = 0 \quad \text{on } z = 0 \tag{13.20}$$

The conditions (13.20) are those required for the solution of the problem.

The solution given above was first derived by Sneddon and Lockett (1960, p.309-317) using a different method, and they discuss it in some detail.

In the three problems considered in this paragraph we have demanded only a known temperature distribution $\theta(\rho\,,\,z)$ on the stress free surfaces. Suppose we prescribe the heat flux instead. Let us begin with the solution discussed by Lur'e (1955) who considered in some detail the following problem

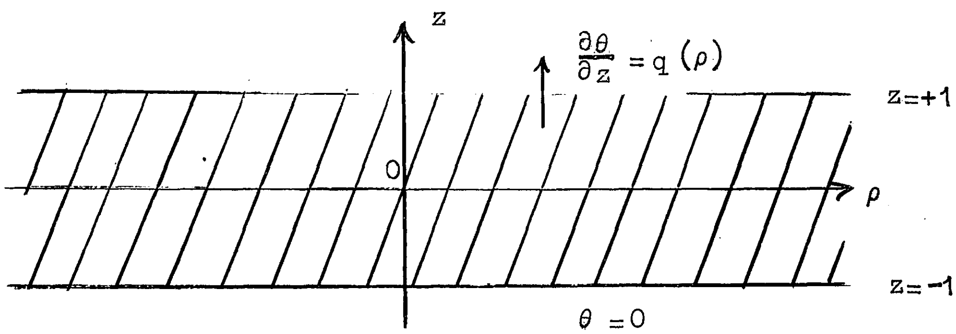


Fig.13.4

where $q(\rho) = (2\pi\rho)^{-1} Q_0 \delta(\rho)$.

Let us take our potential function $\psi(\rho\,,\,z)$ to be

$$\psi(\rho\,,\,z) = -\frac{b}{2(\beta^2 - 1)}\int_0^\infty \frac{\overline{Q}(\xi)\sinh\xi(1+z)}{\xi^2\cosh 2\xi} J_0(\xi\rho)\,d\xi \tag{13.21}$$

and $\chi(z) \equiv 0$.

Then the temperature distribution is given by

$$\theta(\rho\,,\,z) = \int_0^\infty \frac{\overline{Q}(\xi)\sinh\xi(1+z)}{\cosh 2\xi} J_0(\xi\rho)\,d\xi \tag{13.22}$$

so that we have

$$\theta(\rho\,,\,-1) = 0$$

$$\frac{\partial\theta}{\partial z}(\rho\,,\,1) = q(\rho)$$

where we have written $\overline{Q}(\xi)$ to be the Hankel transform of $q(\rho)$.

Further if we consider a linear combination of $\theta(\rho, z)$ and its normal derivative with respect to z to give us our boundary condition we may consider problems of the type considered in §11 and solved by equations (11.22) to (11.28) of a heat source placed at a point above the semi-infinite solid z = 0.

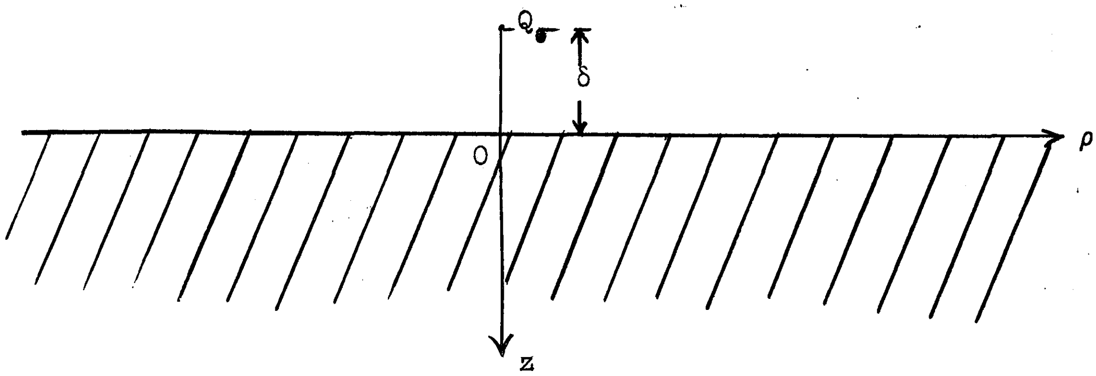


Fig. 13.5

The boundary condition on the stress free force z = 0 is

$$h\,\theta - \frac{\partial\theta}{\partial z} = \frac{Q\,\delta}{4\pi\kappa(\delta^2+\rho^2)^{3/2}} \tag{13.23}$$

A suitable potential function for this problem would be $\psi(\rho, z)$ where

$$\frac{\partial\psi}{\partial z} = \frac{+b\,Q_\sigma}{8(\beta^2-1)\pi\kappa}\int_0^\infty \frac{e^{-\alpha(z+\delta)}}{(h+\alpha)} J_0(\alpha\rho)\,d\alpha \tag{13.24}$$

which reduces to the solution given in (11.24) for the temperature field if we write

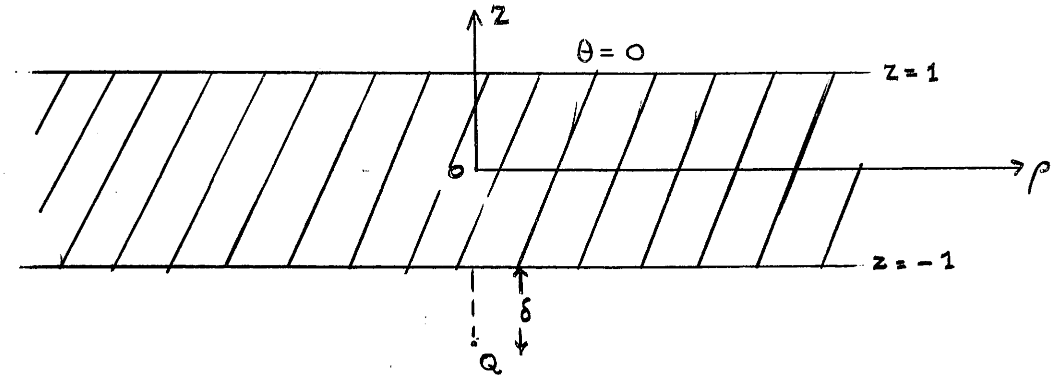
$$\frac{1}{h+\alpha} = \int_0^\infty e^{-(h+\alpha)t}\,dt$$

and interchange the order of integration.

Let us now consider the effect of placing a heat source Q at a distance δ above the stress free surface z = −1 of a plate −1 ⩽ z ⩽ 1, whose other surface z = + 1 is maintained at a temperature $g(\rho)$. To facilitate the expressions used, let us take $g(\rho) = 0$ and choose

$$\psi(\rho, z) = -\frac{b\,Q}{8\pi\kappa(\beta^2-1)}\int_0^\infty \frac{e^{-h\xi}\sinh\xi(1-z)}{\xi\left\{h\sinh 2\xi + \xi\cosh 2\xi\right\}}\cdot J_0(\xi\rho)\,d\xi \tag{13.25}$$

$$\chi(z) \equiv 0\,.$$



Then we have the following expressions for the quantities of physical interest.

$$\theta(\rho, z) = \frac{Q}{4\pi k}\int_0^\infty \frac{\xi e^{-h\xi}\sinh\xi(1-z)}{h\sinh 2\xi + \xi\cosh 2\xi} \cdot J_0(\xi\rho)\, d\xi \qquad (13.26)$$

$$u_\rho = \frac{bQ}{8\pi\kappa(\beta^2-1)}\int_0^\infty \frac{e^{-h\xi}\sinh\xi(1-z)}{h\sinh 2\xi + \xi\cosh 2\xi} J_1(\xi\rho)\, d\xi \qquad (13.27)$$

$$u_z = \frac{-bQ}{8\pi\kappa(\beta^2-1)}\int_0^\infty \frac{e^{-h\xi}\cosh\xi(1-z)}{h\sinh 2\xi + \xi\cosh 2\xi} J_0(\xi\rho)\, d\xi \qquad (13.28)$$

and

$$(\sigma_\rho - \sigma_\phi) = \frac{-bQ}{4\pi\kappa(\beta^2-1)}\int_0^\infty \frac{e^{-h\xi}\sinh\xi(1-z)}{h\sinh 2\xi + \xi\cosh 2\xi} J_2(\xi\rho)\, d\xi \qquad (13.29)$$

A further problem of interest is that of a point source place outside an elastic layer $0 \geq z \geq -1$ lying on a rigid foundation $z \geq 0$ whose boundary $z = 0$ is impervious to heat. A suitable potential function in this case would be

$$\psi(\rho, z) = -\frac{bQ}{8(\beta^2-1)\pi\kappa}\int_0^\infty \frac{e^{-h\xi}\cosh\xi z}{\xi(h\cosh\xi + \xi\sinh\xi)} \cdot J_0(\xi\rho)\, d\xi \qquad (13.30)$$

$$\chi(z) \equiv 0.$$

The temperature field set up in this case would be

$$\theta(\rho, z) = \frac{Q}{4\pi\kappa}\int_0^\infty \frac{\xi e^{-h\xi}\cosh\xi z}{h\cosh\xi + \xi\sinh\xi} J_0(\xi\rho)\, d\xi \qquad (13.31)$$

so that on $z = 0$, $\frac{\partial\theta}{\partial z}(\rho, 0) = 0$, and on $z = -1$, $h\theta - \frac{\partial\theta}{\partial z} = \frac{Q\delta}{4\pi\kappa(\delta^2 + \rho^2)^{3/2}}$

It follows from the equation (13.30) that the quantities of physical interest are given by

$$u_\rho = \frac{bQ}{8\pi\kappa(\beta^2-1)} \int_0^\infty \frac{e^{-h\xi}\cosh\xi z}{h\cosh\xi + \xi\sinh\xi} J_1(\xi\rho)\,d\xi \tag{13.32}$$

$$u_z = \frac{bQ}{8\pi\kappa(\beta^2-1)} \int_0^\infty \frac{e^{-h\xi}\sinh\xi z}{h\cosh\xi + \xi\sinh\xi} J_0(\xi\rho)\,d\xi \tag{13.33}$$

$$\sigma_\rho - \sigma_\phi = \frac{bQ}{4\pi\kappa(\beta^2-1)} \int_0^\infty \frac{e^{-h\xi}\cosh\xi z}{h\cosh\xi + \xi\sinh\xi} J_2(\xi\rho)\,d\xi \tag{13.34}$$

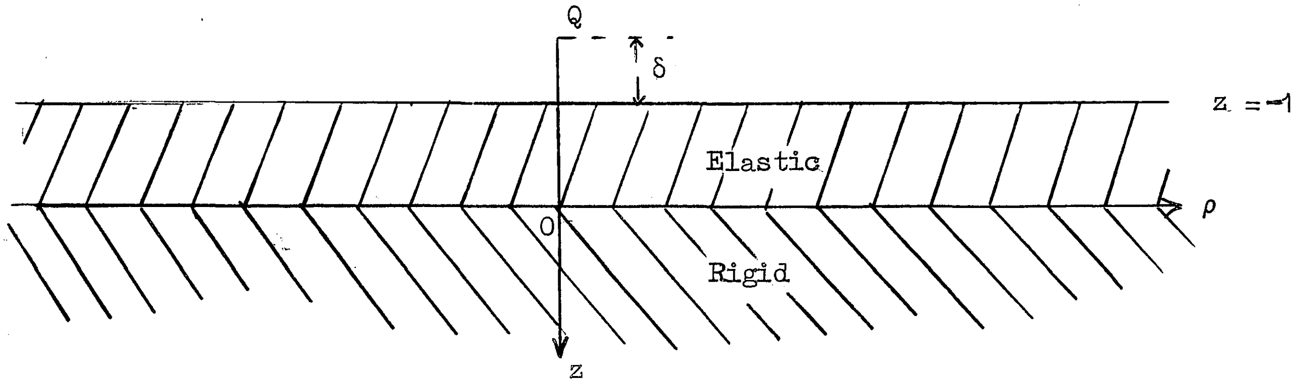


Fig. 13.7

In this paragraph, up till now, we have discussed only problems with symmetry about the z-axis. The basic solution can be extended to cope with problems which do not possess this symmetry. We shall now use co-ordinates (ρ, ϕ, z). The relations in which we are interested will now be given by equations (10.7) to (10.11) where $\psi(\rho, \phi, z)$ will now satisfy the equation

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} = 0 \tag{13.35}$$

A simple solution of this equation is

$$\psi = \cos k\phi\, J_k(\xi\rho)\, e^{\pm\xi z} \tag{13.36}$$

so that we can apply a summation and integration procedure to it to yield the potential function suitable for our purpose

$$\psi(\rho, \phi, z) = -\frac{b}{2(\beta^2-1)} \sum_{m=0}^{\infty} \cos m\phi \int_0^\infty \frac{a_m(\xi)\, J_m(\xi\rho)}{\xi} e^{-\xi z}\, d\xi \tag{13.37}$$

with $\chi(z) = 0$.

If we substitute (13.37) into the equations of §10 we find

$$\theta(\rho, \phi, z) = \sum_{m=0}^{\infty} \cos m\phi \int_0^{\infty} \xi\, a_m(\xi)\, J_m(\xi\rho) e^{-\xi z}\, d\xi \qquad (13.38)$$

From (13.38) it follows that $\theta \to 0$ as $z \to \infty$ and that on the boundary $z = 0$,

$$\theta(\rho, \phi, 0) = \sum_{m=0}^{\infty} \cos m\phi \int_0^{\infty} \xi\, a_m(\xi)\, J_m(\xi\rho)\, d\xi \qquad (13.39)$$

If we now consider the problem of a semi-infinite solid $z \geqslant 0$, with its boundary $z = 0$, free from applied stress and with an asymetric temperature distribution on the surface which can be expanded in a series of cosines as follows

$$\theta(\rho, \phi, 0) = \sum_{m=0}^{\infty} \theta_m(\rho) \cos m\phi \qquad (13.40)$$

then (13.37) gives the solution provided that, when we compare the coefficients of the series (13.39) and (13.40) we take

$$a_m(\xi) = \int_0^{\infty} \rho\, \theta_m(\rho)\, J_m(\rho\xi)\, d\rho = \bar{\theta}_m(\xi) \qquad (13.41)$$

where we are using the m^{th} order Hankel transform.

Let us now write

$$I_{m\pm q}(\rho, z) = \int_0^{\infty} \bar{\theta}_m\, J_{m\pm q}(\rho\xi)\, e^{-\xi z}\, d\xi \qquad (13.42)$$

Then on substituting from equation (13.37) into the equations given by §10 we have, for the problem shown, in Fig. 13.8

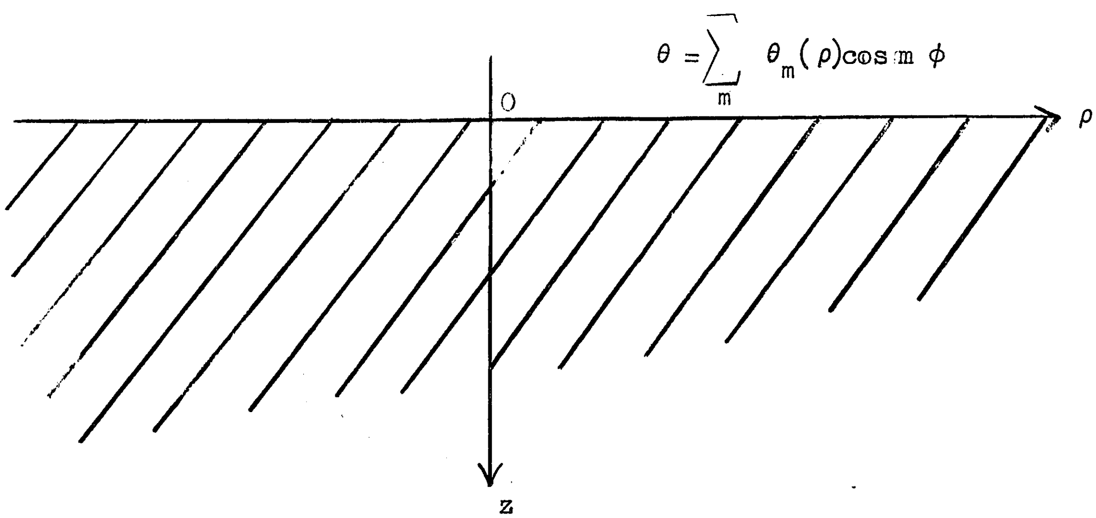


Fig. 13.8

$$u_\rho = \frac{b}{4(\beta^2 - 1)} \sum_{m=0}^{\infty} \cos m\phi \left[I_{m+1} - I_{m-1} \right] \tag{13.43}$$

$$u_\phi = \frac{b}{4(\beta^2 - 1)} \sum_{m=0}^{\infty} \sin m\phi \left[I_{m+1} + I_{m-1} \right] \tag{13.44}$$

$$u_z = \frac{-b}{2(\beta^2 - 1)} \sum_{m=0}^{\infty} \cos m\phi \, I_m \tag{13.45}$$

$$\sigma_\rho = \frac{-b}{(\beta^2 - 1)} \sum_{m=0}^{\infty} \cos m\phi \left\{ \frac{m-1}{2\rho} I_{m-1} + \frac{m+1}{2\rho} I_{m+1} \right\} \tag{13.46}$$

after suitable arrangement of the Bessel functions involved using relations given in Watson (1944). The above results are in agreement with those obtained by Muki (1957, p.42-54) using different methods.

From the above analysis, the solutions of §10 are applicable to the problem of a semi-infinite solid deformed by an asymmetrical temperature distribution. It is of interest to see whether we can apply them to the problems of thick plates.

We consider firstly the thick plate $-1 \leq z \leq 1$ with the stress free surfaces $z = 1$, deformed by the application of asymmetrical temperature distributions $f(\rho, \phi)$ and $g(\rho, \phi)$ respectively, where we assume that there may be expanded in cosine series as follows

$$\begin{aligned} f(\rho, \phi) &= \sum_{m=0}^{\infty} \theta_{1,m}(\rho) \cos m\phi \\ g(\rho, \phi) &= \sum_{m=0}^{\infty} \theta_{2,m}(\rho) \cos m\phi \end{aligned} \tag{13.47}$$

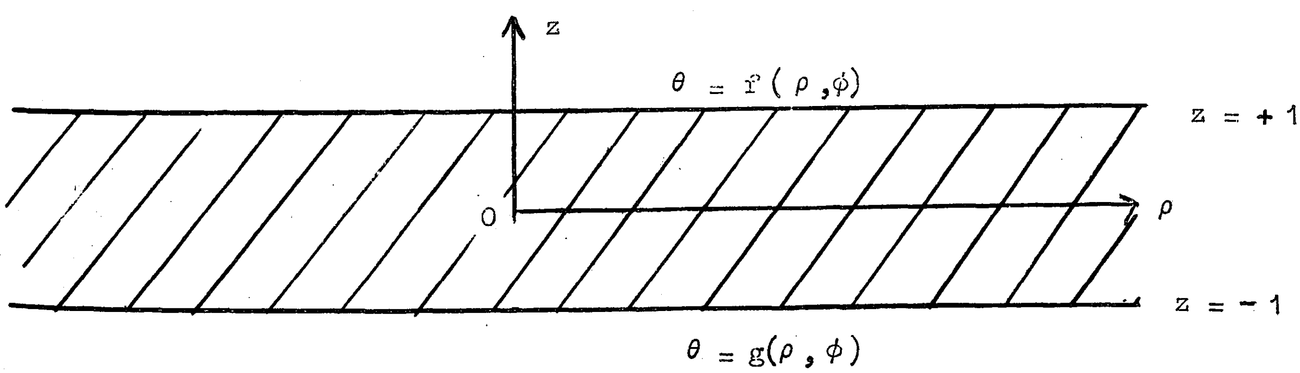


Fig. 13.9

A suitable solution in this case would be

$$\psi(\rho, \phi, z) = \frac{-b}{2(\beta^2 - 1)} \sum_{m=0}^{\infty} \cos m\phi \int_0^{\infty} \frac{\bar{\theta}_{1,m}(\xi)\sinh\xi(1+z) + \bar{\theta}_{2,m}(\xi)\sinh\xi(1-z)}{\xi \sinh 2\xi} J_m(\rho\xi)\,d\xi \quad (13.48)$$

$$\chi(z) \equiv 0 ,$$

where we have denoted the m^{th} order Hankel transforms of $\theta_{1,m}(\rho)$ and $\theta_{2,m}(\rho)$ by $\bar{\theta}_{1,m}(\xi)$, $\bar{\theta}_{2,m}(\xi)$ respectively.

It follows that the temperature distribution in the plate is given by

$$\theta(\rho, \phi, z) = \sum_{m=0}^{\infty} \cos m\phi \int_0^{1} \frac{\bar{\theta}_{1,m}(\xi)\sinh\xi(1+z) + \bar{\theta}_{2,m}(\xi)\sinh\xi(1-z)}{\sinh 2\xi} \xi J_m(\xi\rho)\,d\xi \quad (13.49)$$

The temperature conditions (13.47) are then easily seen to be satisfied, on the boundaries $z = \pm 1$,

If we substitute from equation (13.48) in equation (10.7) we find that the first two components of the displacement vector u_ρ, u_ϕ are again given by (13.43) and (13.44) respectively provided that we now write

$$I_{m+q}(\rho, z) = \int_0^{\infty} \left[\bar{\theta}_{1,m}(\xi)\sinh\xi(1+z) + \bar{\theta}_{2,m}(\xi)\sinh\xi(1-z)\right] \frac{J_m(\rho, \xi)}{\sinh 2\xi}\,d\xi \quad (13.50)$$

while the third component u_z is given by

$$u_z = \frac{-b}{2(\beta^2 - 1)} \sum_{m=0}^{\infty} \cos m\phi \int_0^{\infty} \frac{\bar{\theta}_{1,m}(\xi)\cosh\xi(1+z) - \bar{\theta}_{2,m}(\xi)\cosh\xi(1-z)}{\sinh 2\xi} J_m(\rho\xi)\,d\xi \quad (13.51)$$

These results are again in agreement with those stated by Muki.

Now, consider the problem of an elastic layer $0 \leqslant z \leqslant 1$ resting on an impervious rigid foundation $z \leqslant 0$ and being deformed by the application of an asymetric temperature distribution on the surface $z = 1$, given by

$$\theta(\rho, \phi, 1) = \sum_{m=0}^{\infty} \theta_m(\rho)\cos m\phi \quad (13.52)$$

where the boundary $z = 1$ is stress free.

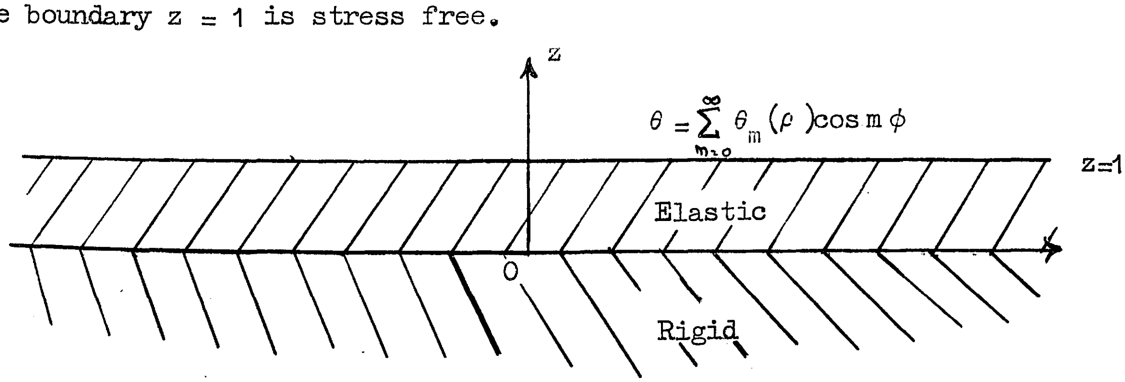


Fig. 13.10

In (13.48) write $\bar{\theta}_{1,m}(\xi) = \bar{\theta}_{2,m}(\xi) = \bar{\theta}_m(\xi)$ so that equation (13.48) becomes

$$\psi(\rho, \phi, z) = \frac{-b}{2(\beta^2 - 1)} \sum_{m=0}^{\infty} \cos m\phi \int_0^{\infty} \bar{\theta}_m(\xi) \frac{\cosh \xi z}{\xi \cosh \xi} J_m(\rho\xi) d\xi \tag{13.53}$$

$$\chi(z) \equiv 0.$$

When we substitute from equation (13.53) into equation (10.7) we find the following quantities

$$\theta(\rho, \phi, z) = \sum_{m=0}^{\infty} \cos m\phi \int_0^{\infty} \xi \bar{\theta}_m(\xi) \frac{\cosh \xi z}{\cosh \xi} J_m(\rho\xi) d\xi \tag{13.54}$$

while the first two components of the displacement vector (u_ρ, u_ϕ) are given by equations (13.43) and (13.44) respectively again providing that we now take

$$I_{m+q}(\rho, z) = \int_0^{\infty} \bar{\theta}_m(\xi) \frac{\cosh \xi z}{\cosh \xi} J_{m+q}(\rho\xi) d\xi \tag{13.55}$$

and the third component u_z is given by

$$u_z = \frac{b}{2(\beta^2 - 1)} \sum_{m=0}^{\infty} \cos m\phi \int_0^{\infty} \bar{\theta}_m(\xi) \frac{\sinh \xi z}{\cosh \xi} J_m(\rho\xi) d\xi \tag{13.56}$$

It immediately follows from equation (13.54) that on the stress free boundary $z = 1$,

$$\theta(\rho, \phi, 1) = \sum_{m=0}^{\infty} \theta_m(\rho) \cos m\phi$$

as required and also that on the boundary $z = 0$

$$w = \frac{\partial \theta}{\partial z} = 0.$$

The potential function given by equation (13.53) thus gives a solution of this problem.

§14 Solutions for rectangular parallelepipeds.

We now consider the application of the basic solution of §10 to problems which are bounded in the x and y directions as well as in the z direction. For the cases in which the medium was unbounded in directions perpendicular to the z axis we found that Fourier and Hankel transform techniques were suitable. It therefore seems possible to apply either Fourier series (when we are dealing with rectangular co-ordinates) or Dini series (when we are using polar co-ordinates) to solve the problems for regions bounded in the x and y or ρ directions.

Consider the problem of a rectangular parallelepiped. In order to keep the formal calculations as simple as possible we shall assume that the parallelepiped has one pair of faces in the form of a square, the length of whose edge is twice our unit of length. We assume that the faces of the parallelepiped, perpendicular to these faces are impervious to heat and that the normal displacement on these faces vanishes. The solutions we derive therefore give the stresses in an elastic plug embedded in either a rigid semi-infinite medium o in a rigid plate. We employ double Fourier series.

Firstly consider the problem shown.

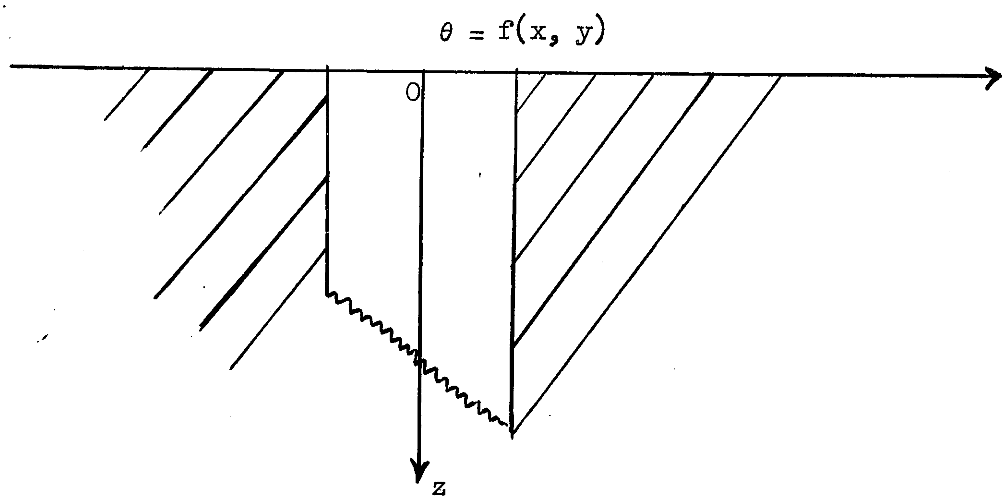


Fig. 14.1

with boundary conditions

$$\left.\begin{array}{l} \sigma_z = \tau_{xz} = \tau_{yz} = 0\ , \quad \theta = f(x, y) \quad \text{on } z = 0\ . \\ \theta \longrightarrow 0 \quad \text{as } z \to \infty \\ u_x = 0\ ,\ x = \pm 1\ , \quad u_y = 0\ ,\ y = \pm 1 \\ \dfrac{\partial \theta}{\partial x} = 0\ ,\ x = \pm 1\ , \quad \dfrac{\partial \theta}{\partial y} = 0\ ,\ y = \pm 1 \end{array}\right] \tag{14.1}$$

From equation (10.3)

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0$$

a suitable simple solution is

$$\psi(x, y, z) = e^{\pm pz} \cos \alpha x \cos \beta y \tag{14.2}$$

where $p^2 = \alpha^2 + \beta^2$.

If we apply a summation procedure to this solution we obtain the potential function

$$\psi(x, y, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty}{}' \frac{a_{mn} \cos m\pi x \cos n\pi y}{p_{mn}^2} e^{-p_{mn} z} \tag{14.3}$$

$$\chi(z) \equiv 0$$

where now $p_{mn}^2 = \pi^2(m^2 + n^2)$ and $(m, n) \neq (0, 0)$.

Then it follows that

$$\theta(x, y, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \cos m\pi x \cos n\pi y \; e^{-p_{mn} z} \qquad (14.4)$$

which tends to 0 as z tends to ∞.

Using the equations of §10 it follows easily that the boundary conditions (14.1) are all satisfied provided that $\theta(x, y, 0) = f(x, y)$ can be expanded in a series as follows,

$$\theta(x, y, 0) = f(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \cos m\pi x \cos n\pi y \qquad -1 \leqslant x \leqslant 1, \quad -1 \leqslant y \leqslant 1 .$$

This will in general be possible, for even functions of temperature, to within an a constant. There is no term a_o. However, this is a matter of deciding on the scale for temperature. From the theory of double Fourier series we may find the constants $a_{m,o}$; $a_{o,n}$; a_{mn} as follows

$$\begin{aligned} a_{m,o} &= \tfrac{1}{2}\iint f(x, y) \cos m\pi x \, dx\,dy \\ a_{o,n} &= \tfrac{1}{2}\iint f(x, y) \cos n\pi y \, dx\,dy \\ a_{m,n} &= \iint f(x, y) \cos m\pi x \cos n\pi y \, dx\,dy \qquad m \neq 0, \; n \neq 0 \end{aligned} \qquad (14.5)$$

where the field of integration is the square $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$.

Equation (14.3) thus gives the formal solution of the problem as expressed by the boundary conditions (14.1).

If we wish to solve the corresponding problem for a thick plate $-\delta \leqslant z \leqslant \delta$ where the temperature distribution on the stress free faces $z = +\delta$ and $z = -\delta$ are given by $f(x, y)$ and $g(x, y)$ respectively then the situation is as shown in Fig. 14.2

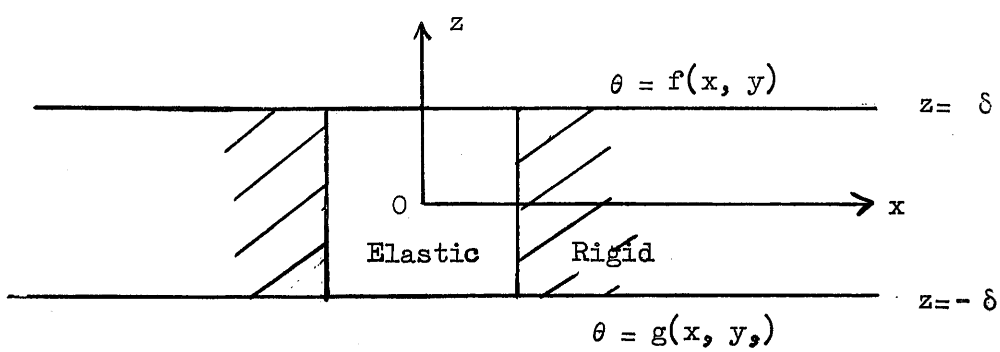


Fig. 14.2

with the boundary conditions

$$\begin{aligned}
&\sigma_z = \tau_{xz} = \tau_{yz} = 0\,; \quad && \theta(x, y, \delta) = f(x, y) \quad && \text{on} \quad z = \delta \\
&\text{~~}u = v = 0,\ \tfrac{\partial\theta}{\partial n} = 0\text{~~} \quad && u = 0,\ \frac{\partial\theta}{\partial x} = 0 \quad && \text{on} \quad x = \pm 1 \\
& && v = 0,\ \frac{\partial\theta}{\partial y} = 0 \quad && \text{on} \quad y = \pm 1 \\
&\text{~~}\sigma_z = \tau_{xz} = \tau_{yz} = 0\text{~~}\,; \quad && \theta(x, y, -\delta) = g(x, y) \quad && \text{on} \quad z = -\delta
\end{aligned} \tag{14.6}$$

We now consider a potential function

$$\psi(x, y, z) = \frac{-b}{2(\beta^2 - 1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_{mn}\sinh(\delta + z)p_{mn} + b_{mn}\sinh(\delta - z)p_{mn}}{p_{mn}^2 \sinh 2\delta p_{mn}} \cos m\pi x \cos n\pi y \tag{14.7}$$

where we now take the quadratic function χ to be

$$\chi(z) = \frac{ba_0}{4\beta^2\delta}(\delta + z)^2 - \frac{bb_0}{4\beta^2\delta}(\delta - z)^2 \tag{14.8}$$

It follows immediately from §10 that

$$\begin{aligned}
\theta(x, y, z) &= \tfrac{1}{2} a_0\left(1 + \frac{z}{\delta}\right) + \tfrac{1}{2} b_0\left(1 - \frac{z}{\delta}\right) \\
&+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_{mn}\sinh(\delta + z)p_{mn} + b_{mn}\sinh(\delta - z)p_{mn}}{\sinh 2\delta p_{mn}} \cos m\pi x \cos n\pi y
\end{aligned} \tag{14.9}$$

It is then easily seen that the conditions given by equation (14.6) are satisfied provided that we choose a_{mn} such that

$$\begin{aligned}
f(x, y) &= a_0 + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \cos m\pi x \cos n\pi y \\
g(x, y) &= b_0 + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{mn} \cos m\pi x \cos n\pi y
\end{aligned} \tag{14.10}$$

The a_{mn} are given by equation (14.5) if $m \neq 0$, $n \neq 0$ and

$$a_0 = \tfrac{1}{4} \iint f(x, y)\, dx\, dy \tag{14.11}$$

The b_n are given by similar expression with $g(x, y)$ written for $f(x, y)$.

The stresses induced are as follows

$$\begin{aligned}
\sigma_x = -\Bigg[&\frac{ba_0}{\beta^2\delta}(\delta + z) + \frac{bb_0}{\beta^2\delta}(\delta - z) \\
&+ \frac{b}{(\beta^2 - 1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} n^2\pi^2 \frac{a_{mn}\sinh(\delta + z)p_{mn} + b_{mn}\sinh(\delta - z)p_{mn}}{p_{mn}^2 \sinh 2\delta p_{mn}} \cos m\pi x \cos n\pi y\Bigg]
\end{aligned} \tag{14.12}$$

$$\sigma_y = -\Bigg[\frac{ba_0}{\beta^2\delta}(\delta + z) + \frac{bb_0}{\beta^2\delta}(\delta - z)$$
$$+ \frac{b}{(\beta^2 - 1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} m^2\pi^2 \frac{a_{mn}\sinh(\delta + z)p_{mn} + b_{mn}\sinh(\delta - z)p_{mn}}{p_{mn}^2 \sinh 2\delta p_{mn}} \cos m\pi x \cos n\pi y\Bigg] \tag{14.13}$$

$$\tau_{xy} = -\frac{b}{(\beta^2 - 1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} mn\pi^2 \frac{a_{mn}\sinh(\delta + z)p_{mn} + b_{mn}\sinh(\delta - z)p_{mn}}{p_{mn}^2 \sinh 2\delta p_{mn}} \sin m\pi x \sin n\pi y \tag{14.14}$$

As a very simple case let us assume that on the face $\underline{z = 1}$, $\theta = \theta_0(1 - x^2)$ and that on $\underline{z = -1}$, $\theta = 0$. Then we have

$$\left.\begin{aligned} a_0 &= \frac{2}{3}\theta_0, \\ a_{m,0} &= \frac{4\theta_0}{m^2\pi^2}(-1)^{m+1}, \\ a_{n,0} &= 0, \\ a_{m,n} &= 0, \\ b_0 &= 0, \quad b_{m,n} = 0. \end{aligned}\right]$$

for the values of the constant terms.

Let us now consider the problem shown below

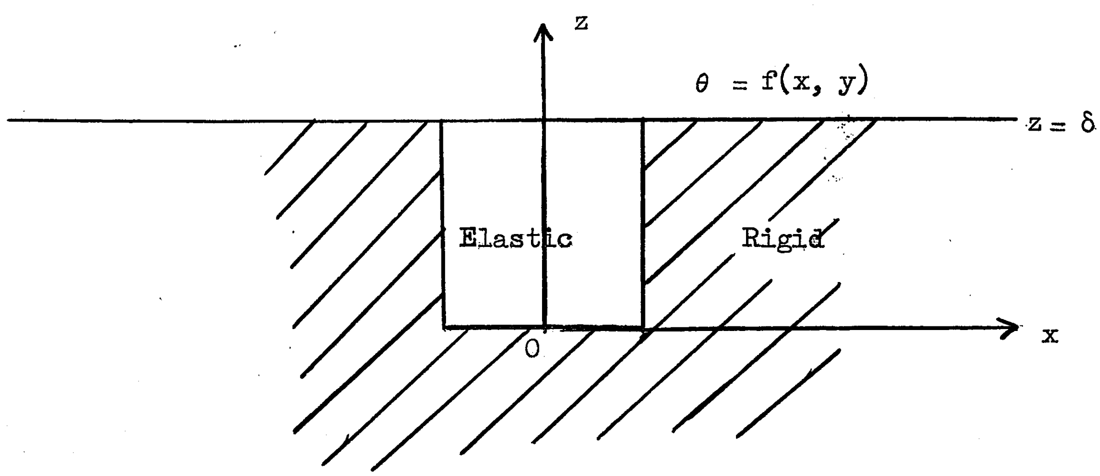


Fig. 14.3

with the boundary conditions

$$\left.\begin{array}{lr} \sigma_x = \tau_{xz} = \tau_{yz} = 0; \quad \theta(x, y, \delta) = f(x, y) & \text{on } z = \delta \\ u = 0, \quad \dfrac{\partial\theta}{\partial x} = 0 \quad \text{on } x = \pm 1; \quad v = 0, \dfrac{\partial\theta}{\partial y} = 0 & \text{on } y = \pm 1 \\ w = \dfrac{\partial\theta}{\partial z} = 0 & \text{on } z = 0 . \end{array}\right] \tag{14.15}$$

The potential function appropriate to this problem is

$$\psi(x, y, z) = -\frac{b}{2(\beta^2 - 1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_{mn} \cosh p_{mn} z}{p_{mn}^2 \cosh p_{mn} \delta} \cos m\pi x \cos n\pi y \tag{14.16}$$

The function given by (14.16) in conjunction with a value of

$$\chi(z) = \frac{b a_0 z}{\beta^2} \tag{14.17}$$

give, with the use of the equations of §10

$$\theta(x, y, z) = a_0 + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \frac{\cosh p_{mn} z}{\cosh p_{mn} \delta} \cos m\pi x \cos n\pi y \tag{14.18}$$

It is then easily seen from the equations of §10, (14.17) and (14.18) that the conditions (14.15) are satisfied provided that

$$f(x, y) = a_0 + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \cos m\pi x \cos n\pi y \tag{14.19}$$

The coefficients a_{mn} are then found from equations (14.5) and (14.11),

The solution as we see above refers to an elastic parallelepiped inserted into a rigid material, impervious to heat, with a given temperature distribution on its surface.

The displacements and stresses are given by

$$u_x = +\frac{b}{2(\beta^2 - 1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} m\pi \frac{a_{mn} \cosh p_{mn} z}{p_{mn}^2 \cosh p_{mn} \delta} \sin m\pi x \cos n\pi y \tag{14.20}$$

$$u_y = \frac{b}{2(\beta^2 - 1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} n\pi \frac{a_{mn} \cosh p_{mn} z}{p_{mn}^2 \cosh p_{mn} \delta} \cos m\pi x \sin n\pi y \tag{14.21}$$

$$u_z = \frac{b a_0}{\beta^2} z + \frac{b}{2(\beta^2 - 1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_{mn} \sinh p_{mn} z}{p_{mn} \cosh p_{mn} \delta} \cos m\pi x \cos n\pi y \tag{14.22}$$

$$\sigma_x = -\frac{2ba_0}{\beta^2} - \frac{b}{(\beta^2-1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} n^2\pi^2 \frac{a_{mn}\cosh p_{mn} z}{p_{mn}^2 \cosh p_{mn}\delta} \cos m\pi x \cos n\pi y \tag{14.23}$$

$$\sigma_y = -\frac{2ba_0}{\beta^2} - \frac{b}{(\beta^2-1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} m^2\pi^2 \frac{a_{mn}\cosh p_{mn} z}{p_{mn}^2 \cosh p_{mn}\delta} \cos m\pi x \cos n\pi y \tag{14.24}$$

$$\tau_{xy} = -\frac{b}{(\beta^2-1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} mn\pi^2 \frac{a_{mn}\cosh p_{mn} z}{p_{mn}^2 \cosh p_{mn}\delta} \sin m\pi x \sin n\pi y . \tag{14.25}$$

§15. Solutions for finite cylinders.

In this section we shall consider several problems which are similar to those considered in §14, but we shall now attempt to solve several problems in cylindrical polars, (ρ, ϕ, z) and we shall assume that there is symmetry about the z-axis. In §14 it was shown how the basic solution could be extended by the use of double Fourier series. In this section we shall use Dini series.

The Dini expansion of a function $f(t)$ is given by

$$f(t) = \sum_{m=1}^{\infty} a_m J_\nu(\lambda_m t) \tag{15.1}$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive zeros (arranged in ascending order of magnitude) of

$$zJ_\nu'(z) + HJ_\nu(z) \tag{15.2}$$

where H, ν are real constants and $\nu + \frac{1}{2} \geq 0$.

In the cases we consider we choose $H = 0$ and $\nu = 0$, so that $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive zeros of $J_1(z)$. The coefficients in equation (15.1) are then given by

$$\left.\begin{aligned} a_0 &= 2\int_0^1 t\, f(t)\, dt \\ a_m &= \frac{2}{J_0^2(\lambda_m)} \int_0^1 t f(t)\, J_0(\lambda_m t)\, dt, \quad m \geq 1 \end{aligned}\right\} \tag{15.3}$$

where we have included a term a_0 because this is the special case quoted in Watson (1944) of $H + \nu = 0$.

A suitable solution of equation (10.3) in the case when we are considering axial symmetry is

$$\psi(\rho, z) = e^{\pm\lambda z} J_0(\lambda\rho).$$

If we apply a summation procedure to this we obtain a potential function of the form

$$\psi(\rho, z) = \frac{-b}{2(\beta^2 - 1)} \sum_{m=1}^{\infty} \frac{a_m \sinh(\delta + z)\lambda_m + b_m \sinh(\delta - z)\lambda_m}{\lambda_m^2 \sinh(2\lambda_m \delta)} J_0(\lambda_m \rho) \tag{15.4}$$

where the sum has been taken over the positive zeros $\lambda_1, \lambda_2, \lambda_3, \ldots$ of the function $J_1(z)$.

For $\chi(z)$ we take the quadratic function

$$\chi(z) = \frac{b}{4\beta^2 \delta} \left[a_0(\delta + z)^2 - b_0(\delta - z)^2 \right]. \tag{15.5}$$

From the equations given in §10 and using equations (15.4) and (15.5) we find that the temperature distribution is given by

$$\theta(\rho, z) = \frac{a_0}{2}\left[1 + \frac{z}{\delta}\right] + \frac{b_0}{2}\left[1 - \frac{z}{\delta}\right] + \sum_{m=1}^{\infty} \left[a_m \sinh(\delta + z)\lambda_m + b_m \sinh(\delta - z)\lambda_m\right] \frac{J_0(\lambda_m \rho)}{\sinh 2\lambda_m \delta} \tag{15.6}$$

while the components of the displacement vector are given by

$$u_\rho = \frac{b}{2(\beta^2 - 1)} \sum_{m=1}^{\infty} \frac{a_m \sinh(\delta + z)\lambda_m + b_m \sinh(\delta - z)\lambda_m}{\lambda_m \sinh(2\lambda_m \delta)} J_1(\lambda_m \rho) \tag{15.7}$$

$$u_\phi = 0$$

$$u_z = \frac{b}{4\beta^2\delta}\left[a_0(\delta + z)^2 - b_0(\delta - z)^2\right] + \frac{b}{2(\beta^2 - 1)} \sum_{m=1}^{\infty} \frac{a_m \cosh(\delta + z)\lambda_m - b_m \cosh(\delta - z)\lambda_m}{\lambda_m \sinh(2\lambda_m \delta)} J_0(\lambda_m \rho) \tag{15.8}$$

Now consider the problem as drawn below

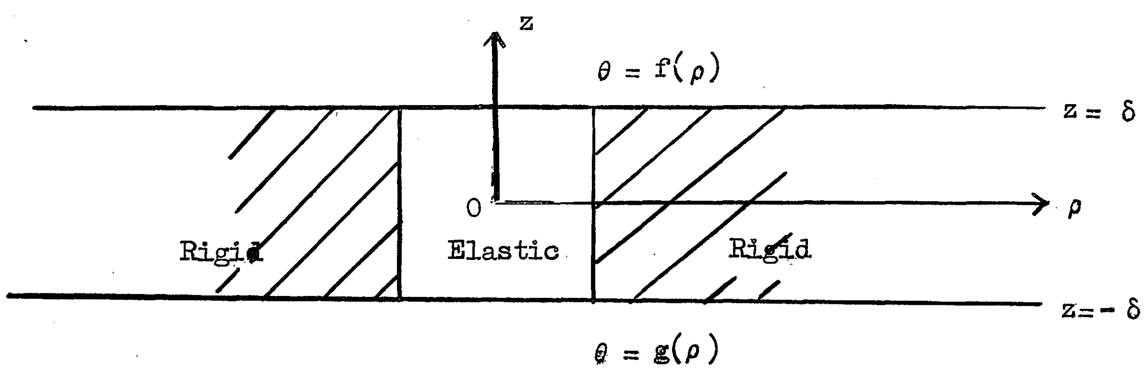


Fig. 15.1

with boundary conditions

$$\begin{aligned} &\sigma_z = \tau_{\rho z} = 0 \;; \quad \theta(\rho, \delta) = f(\rho) \quad &&\text{on } z = \delta \,. \\ &\sigma_z = \tau_{\rho z} = 0 \,, \quad \theta(\rho, -\delta) = g(\rho) \quad &&\text{on } z = -\delta \\ &u_\rho = \frac{\partial \theta}{\partial \rho} = 0 \,, &&\text{on } \rho = 1 \,. \end{aligned} \tag{15.9}$$

These conditions are easily satisfied as is seen from equations (15.6), ... (15.8) provided that we choose the constants a_o, a_m, b_o, b_m of equations (15.6) such that

$$\begin{aligned} \theta(\rho, \delta) &= a_o + \sum_{m=1}^{\infty} a_m J_o(\lambda_m \rho) = f(\rho) \\ \theta(\rho, -\delta) &= b_o + \sum_{m=1}^{\infty} b_m J_o(\lambda_m \rho) = g(\rho) \,. \end{aligned} \tag{15.10}$$

Now from the theory of Dini series and equation (15.3) we must have

$$\begin{aligned} &a_o = 2\int_0^1 t f(t)\, dt \,, \qquad b_o = 2\int_0^1 t g(t)\, dt \\ &a_m = \frac{2}{J_o^2(\lambda_m)} \int_0^1 t f(t)\, J_o(\lambda_m t)\, dt, \quad b_m = \frac{2}{J_o(\lambda_m)} \int_0^1 t g(t)\, J_o(\lambda_m t)\, dt \,, \quad m \geq 1 \end{aligned} \tag{15.11}$$

The expression for the differences in stresses

$$\sigma_\phi - \sigma_\rho = \frac{b}{(\beta^2 - 1)} \sum_{m=1}^{\infty} \frac{a_m \sinh(\delta + z)\lambda_m + b_m \sinh(\delta - z)\lambda_m}{\sinh(2\lambda_m \delta)} \cdot J_2(\lambda_m \rho) \tag{15.12}$$

Consider now a situation as shown

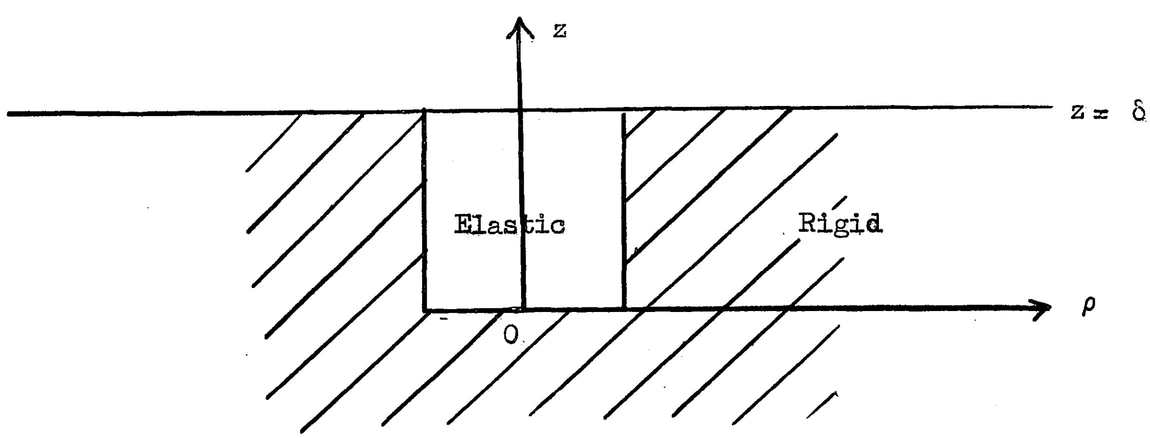


Fig. 15.2

where the boundary conditions will now be

$$\sigma_z = \tau_{\rho z} = 0 ; \quad \theta = f(\rho) \qquad \text{on } z = \delta$$
$$u_z = \frac{\partial \theta}{\partial z} = 0 \qquad z = 0 \tag{15.13}$$
$$u_\rho = \frac{\partial \theta}{\partial \rho} = 0 \qquad \rho = 1$$

The situation then is of a plug of elastic material embedded in a rigid foundation which is impervious to the flow of heat. To solve this problem we choose a potential function

$$\psi(\rho, z) = \frac{-b}{2(\beta^2 - 1)} \sum_{m=1}^{\infty} \frac{a_m \cosh \lambda_m z}{\lambda_m^2 \cosh \lambda_m \delta} J_0(\lambda_m \rho) \tag{15.14}$$

with $\chi(z) = \frac{a_0 z}{\beta^2 \delta}$, where $a_0, a_1, a_2 \ldots$ are defined by equations (15.3). The quantities of physical interest listed below are easily obtained by substituting from (15.14) in the equations of §10

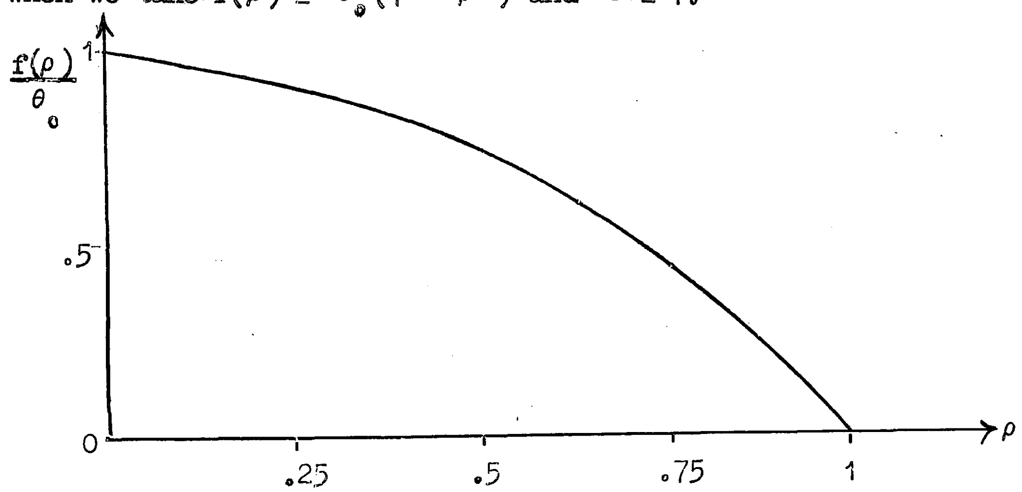
$$\theta(\rho, z) = \sum_{m=1}^{\infty} \frac{a_m \cosh \lambda_m z}{\cosh \lambda_m \delta} J_0(\lambda_m \rho) + \frac{a_0}{b\delta} \tag{15.15}$$

$$u_\rho = \frac{b}{2(\beta^2 - 1)} \sum_{m=1}^{\infty} \frac{a_m \cosh \lambda_m z}{\lambda_m \cosh \lambda_m \delta} J_1(\lambda_m \rho) \tag{15.16}$$

$$u_z = \frac{a_0 z}{\beta^2 \delta} + \frac{b}{2(\beta^2 - 1)} \sum_{m=1}^{\infty} \frac{a_m \sinh \lambda_m z}{\lambda_m \cosh \lambda_m \delta} J_0(\lambda_m \rho) \tag{15.17}$$

$$\sigma_\phi - \sigma_\rho = \frac{b}{(\beta^2 - 1)} \sum_{m=1}^{\infty} \frac{a_m \cosh \lambda_m z}{\cosh \lambda_m \delta} J_2(\lambda_m \rho) \tag{15.18}$$

It is then easily seen that conditions (15.13) are satisfied. As a special case consider what happens when we take $f(\rho) = \theta_0(1 - \rho^2)$ and $\delta = 1$.



Then

$$a_0 = 2\,\theta_0 \int_0^1 t(1 - t^2)\,dt = \frac{\theta_0}{2}$$

$$a_m = \frac{4}{\lambda_m^2 J_0(\lambda_m)}\,, \tag{15.19}$$

so that we may write

$$u_\rho = \frac{-2b\,\theta_0}{\beta^2 - 1} \sum_{m=1}^{\infty} \frac{\cosh \lambda_m z}{\cosh \lambda_m \delta} \; \frac{J_1(\lambda_m \rho)}{\lambda_m^3 J_0(\lambda_m)} \tag{15.20}$$

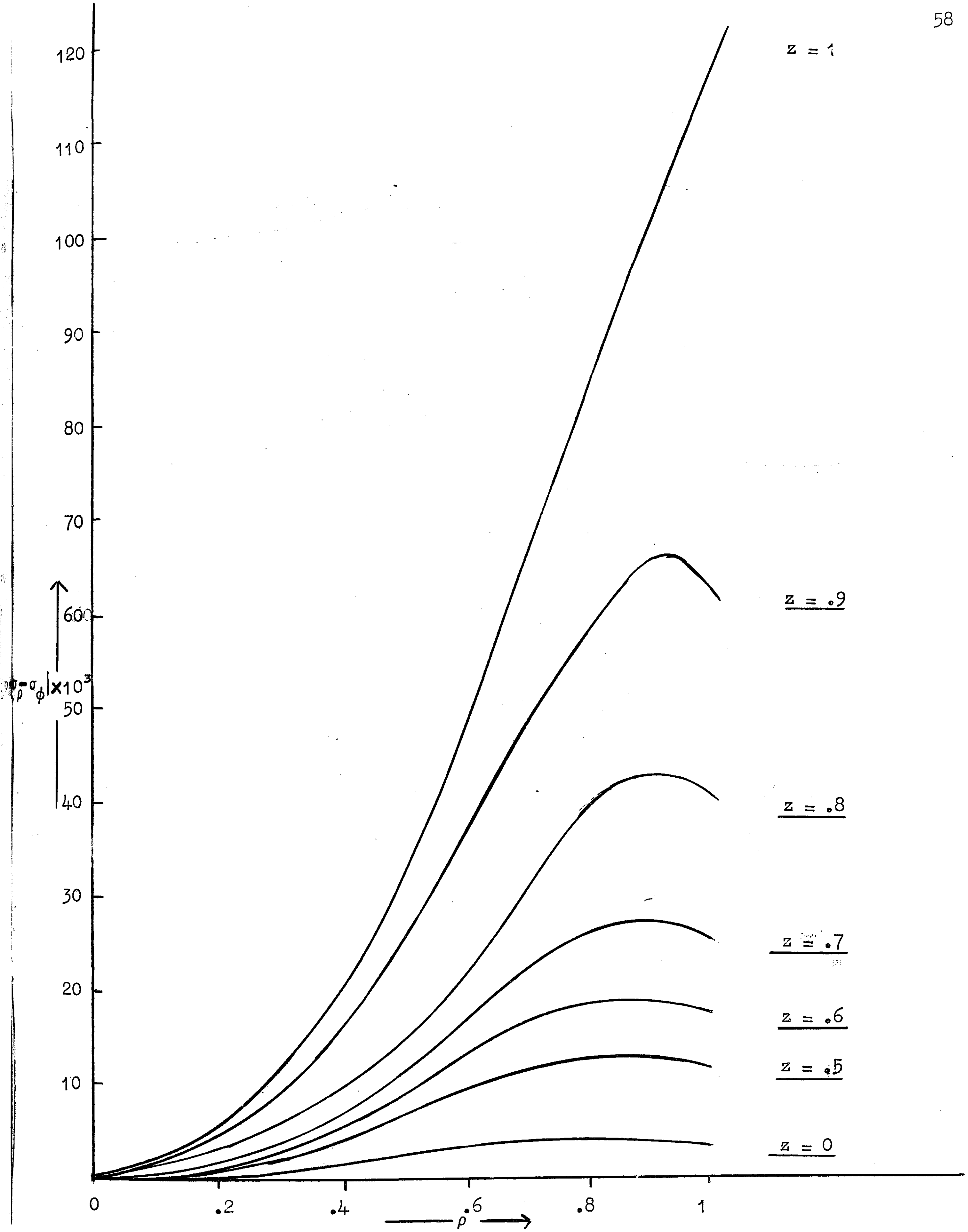
$$u_z = \frac{\theta_0 z}{2\beta^2 \delta} - \frac{2b\,\theta_0}{(\beta^2 - 1)} \sum_{m=1}^{\infty} \frac{\sinh \lambda_m z}{\cosh \lambda_m \delta} \cdot \frac{J_0(\lambda_m \rho)}{\lambda_m^3 J_0(\lambda_m)} \tag{15.21}$$

$$\tfrac{1}{2}(\sigma_\rho - \sigma_\phi) = \frac{2b\,\theta_0}{\beta^2 - 1} \sum_{m=1}^{\infty} \frac{\cosh \lambda_m z}{\cosh \lambda_m \delta} \; \frac{J_2(\lambda_m \rho)}{\lambda_m^2 J_0(\lambda_m)} \tag{15.22}$$

The series given by equation (15.22) was summed over the first fifty terms for various values of z ranging from 0 to 1. The graph is as shown

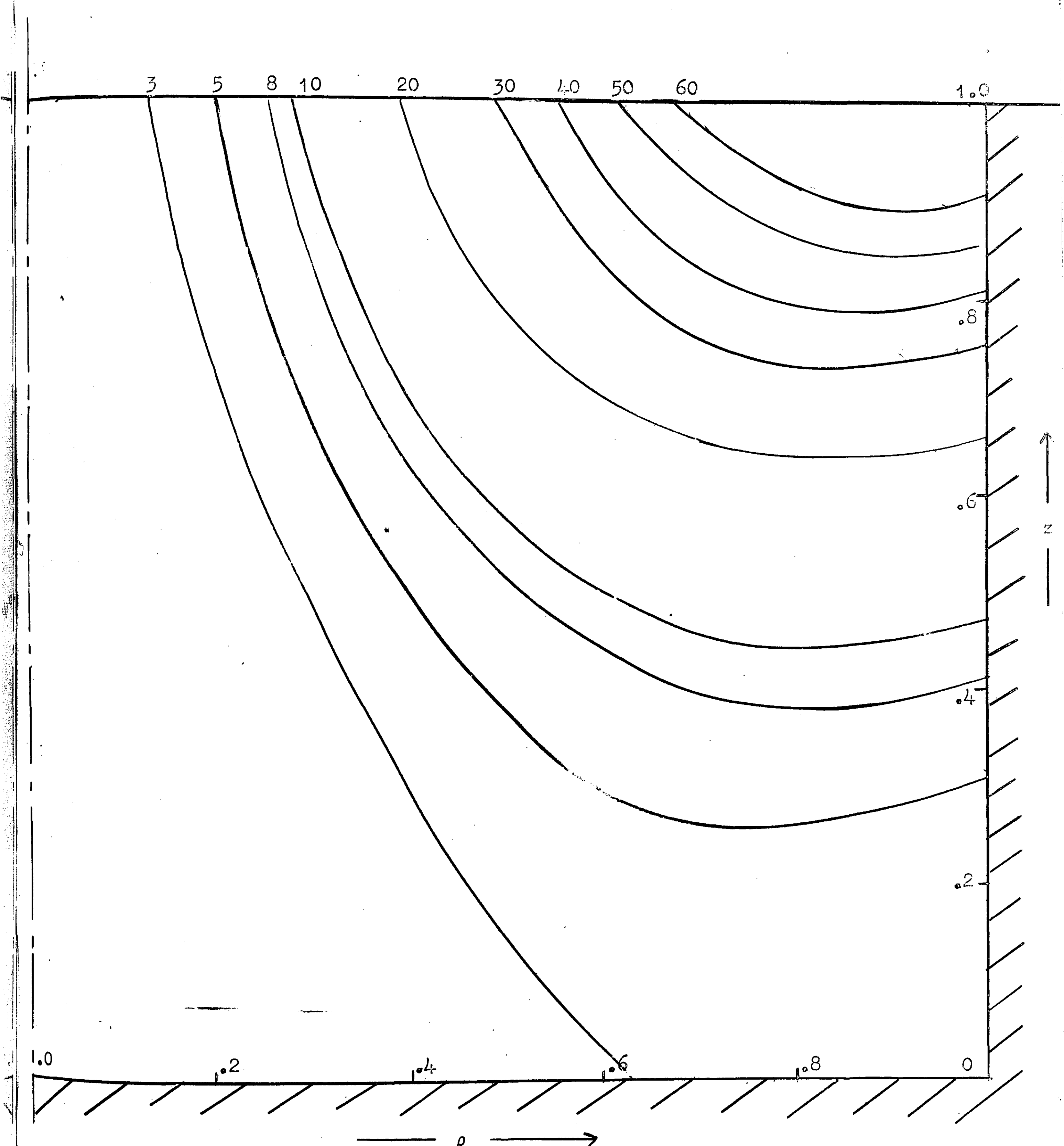
z/ρ	0	.05	.1	.15	.2	.25	.3	.35	.4	.45	.5
1.0	0	.0005	.0012	.0028	.0050	.0078	.0100	.0153	.0200	.0254	.0313
.9	0	.0004	.0011	.0024	.0041	.0063	.0092	.0124	.0165	.0205	.0250
.8	0	.0003	.0009	.0018	.0034	.0051	.0073	.0098	.0129	.0158	.0192
.7	0	.0003	.0008	.0018	.0025	.0034	.0051	.0073	.0097	.0118	.0144
.6	0	.0002	.0007	.0013	.0019	.0030	.0043	.0056	.0072	.0087	.0105
.5	0	.0001	.0004	.0009	.0014	.0022	.0031	.0041	.0052	.0064	.0076
0.0	0	.0000	.0001	.0002	.0004	.0008	.0010	.0013	.0018	.0021	.0024

z \ ρ	.55	.6	.65	.7	.75	.8	.85	.9	.95	1
	.0378	.0450	.0527	.0613	.0706	.0800	.0901	.1112	.1132	.1250
	.0300	.0349	.0405	.0466	.0521	.0577	.0623	.0658	.0665	.0632
	.0228	.0266	.0304	.0338	.0371	.0398	.0415	.0422	.0415	.0378
	.0168	.0194	.0218	.0239	.0259	.0271	.0277	.0276	.0266	.0247
	.0122	.0138	.0154	.0167	.0179	.0187	.0188	.0183	.0175	.0163
	.0087	.0099	.0109	.0118	.0125	.0127	.0127	.0123	.0117	.0108
	.0028	.0030	.0031	.0032	.0034	.0035	.0033	.0032	.0031	.0028



Graph of $|\sigma_{\rho} - \sigma_{\phi}| \times 10^3$ for various values of z.

If on the other hand we plot a graph for $|\sigma_\rho - \sigma_\phi| = \text{constant}$, i.e. the isochromatic lines we obtain a solution as shown.



Graph of $|\sigma_\rho - \sigma_\phi| = \text{constant}$.

So far we have dealt with problems in which the temperature field was defined on the stress free surfaces. Suppose now that we consider the problem of cylindrical shaped elastic material enclosed by a rigid material impervious to the flow of heat, where the heat flow across the stress free faces is prescribed. Then we have the situation shown.

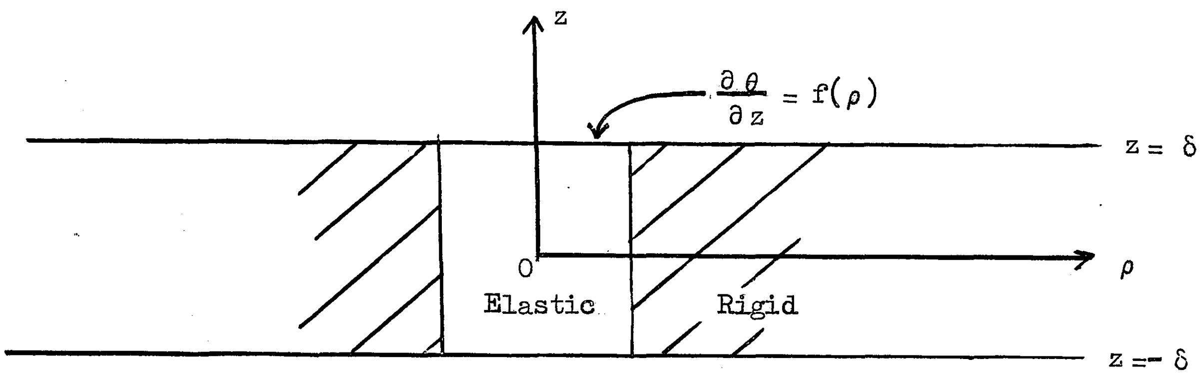


Fig. 15.4

with boundary conditions

$$\sigma_z = \tau_{\rho z} = 0 \; ; \quad \frac{\partial \theta}{\partial z} = f(\rho) \qquad \text{on } z = \delta$$

$$\text{~~}\sigma_\rho = \tau_{\rho z} = 0\text{~~} , \quad \frac{\partial \theta}{\partial z} = g(\rho) \qquad \text{on } z = -\delta \qquad (15.23)$$

$$u_\rho = \frac{\partial \theta}{\partial \rho} = 0 \qquad \text{on } \rho = 1$$

If we take a potential function

$$\psi(\rho, z) = \frac{-b}{2(\beta^2 - 1)} \sum_{m=1}^{\infty} \frac{a_m \cosh \lambda_m (\delta + z) - b_m \cosh \lambda_m (\delta - z)}{\lambda_m^3 \sinh 2\lambda_m \delta} J_0(\lambda_m \rho) \qquad (15.24)$$

and $\chi(z) = \dfrac{b a_0 z^2}{2\beta^2}$.

In the usual way we obtain

$$\theta(\rho, z) = a_0 z + \sum_{m=1}^{\infty} \frac{a_m \cosh \lambda_m (\delta + z) - b_m \cosh \lambda_m (\delta - z)}{\sinh 2\lambda_m \delta} J_0(\lambda_m \rho) \qquad (15.26)$$

$$\frac{\partial \theta}{\partial z} = a_0 + \sum_{m=1}^{\infty} \frac{a_m \sinh \lambda_m (\delta + z) + b_m \sinh \lambda_m (\delta - z)}{\sinh 2\lambda_m \delta} J_0(\lambda_m \rho) . \qquad (15.27)$$

When we attempt to equate the derivative of $\theta(\rho, z)$ to the functions $f(\rho)$ and $g(\rho)$ we find the following equations

$$\frac{\partial\theta}{\partial z} = a_0 + \sum_{m=1}^{\infty} a_m J_0(\lambda_m \rho) = f(\rho) \qquad \text{on } z = \delta \tag{15.28}$$

$$\frac{\partial\theta}{\partial z} = a_0 + \sum_{m=1}^{\infty} b_m J_0(\lambda_m \rho) = g(\rho) \qquad \text{on } z = -\delta. \tag{15.29}$$

If we find the coefficients a_0, a_m, b_0, b_m in the usual way by using equations (15.3) we have

$$a_0 = 2\int_0^1 t f(t)\, dt = 2\int_0^1 t g(t)\, dt\,. \tag{15.30}$$

The solution described by equation (15.24) is therefore only applicable to problems in which (15.30) holds.

The total flux or flow of heat passing <u>out</u> through the boundary ($z = \delta$) is easily shown to be

$$Q_1 = 2\pi\kappa \int_0^1 \rho f(\rho)\, d\rho$$

and the total flux of heat flowing <u>into</u> the elastic material through the face $z = -\delta$ to be

$$Q_2 = 2\pi\kappa \int_0^1 \rho g(\rho)\, d\rho\,.$$

Thus since no heat can pass into the rigid material, it follows that the total amount of heat entering through one face of the plug must flow out through the other face. This is the physical fact expressed by equation (15.30).

Suppose on the other hand that we take a radiation condition over one stress free surface and prescribe the temperature distribution over the other. To make the problem slightly less complicated we assume that on $z = +0$, $\theta(\rho, 0) = 0$. While on $z = -1$,

$$h\theta - \frac{\partial\theta}{\partial z} = \frac{Q\delta}{4\pi\kappa(\delta^2 + \rho^2)^{3/2}}\,.$$

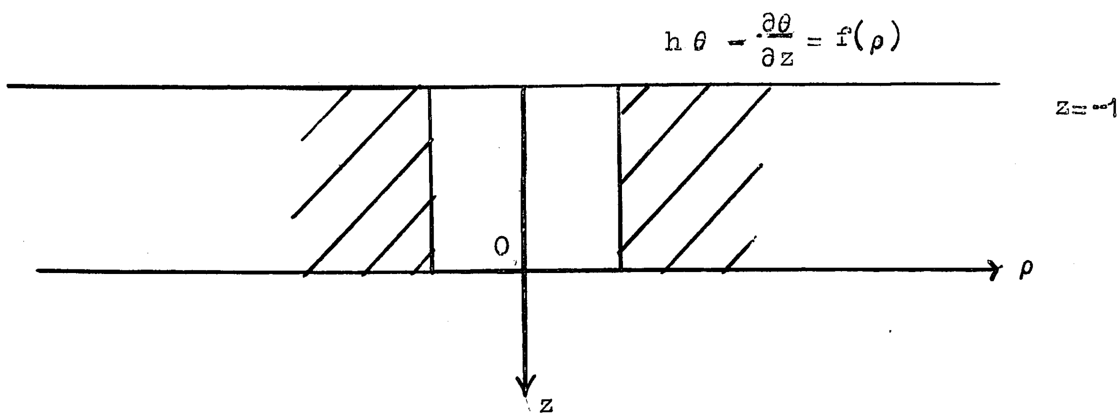


Fig. 15.5

We choose our potential function in this case to be

$$\psi(\rho\ ,\ z) = \frac{b}{2(\beta^2 - 1)} \cdot \sum_{m=1}^{\infty} \frac{a_m \sinh\lambda_m z}{\lambda_m^2 (h \sinh\lambda_m + \lambda_m \cosh\lambda_m)} \cdot J_0(\lambda_m \rho) \tag{15.31}$$

and
$$\chi(z) = - \frac{b a_0 z^2}{2\beta^2(h+1)} \tag{15.32}$$

It follows immediately that

$$\theta(\rho\ ,\ z) = - \frac{a_0 z}{(h+1)} - \sum_{m=1}^{\infty} \frac{a_m \sinh\lambda_m z}{(h \sinh\lambda_m + \lambda_m \cosh\lambda_m)} \cdot J_0(\lambda_m \rho) \tag{15.33}$$

$$\frac{\partial \theta}{\partial z} = - \frac{a_0}{h+1} - \sum_{m=1}^{\infty} \frac{\lambda_m a_m \cosh\lambda_m z}{(h \sinh\lambda_m + \lambda \cosh\lambda_m)} \cdot J_0(\lambda_m \rho) \tag{15.34}$$

Thus on the surface $\underline{z = -1}$ we have,

$$h\theta - \frac{\partial \theta}{\partial z} = a_0 + \sum_{m=1}^{\infty} a_m \cdot J_0(\lambda_m \rho) \tag{15.35}$$

and $\theta(\rho\ ,\ 0) = 0$, on $z = 0$.

The values of a_0, a_m are given by equations (15.3) to be

$$a_0 = 2\int_0^1 t \cdot \frac{Q\delta}{4\pi\kappa(\delta^2 + t^2)^{3/2}}\, dt$$

$$a_m = \frac{2}{J_0^2(\lambda_m)} \int_0^1 t\, \frac{Q\delta}{4\pi\kappa(\delta^2 + t^2)^{3/2}} \cdot J_0(\lambda_m t)\, dt.$$

i.e.
$$a_0 = \frac{Q}{2\pi\kappa}\left[\frac{1}{\delta^3} - \frac{1}{(1+\delta^2)^{3/2}}\right] \tag{15.36}$$

$$a_m = \frac{Q\delta}{2\pi\kappa J_0^2(\lambda_m)} \int_0^1 \frac{t\, J_0(\lambda_m \tau)\, dt}{(\delta^2 + t^2)^{3/2}} . \tag{15.37}$$

III. CRACK PROBLEMS

A great deal of work has been carried out in the investigation of stresses and strains in the vicinity of cracks and holes of various shapes in elastic media. The earliest calculations of the distribution of stress in the neighbourhood of cracks appears to be due to Inglis (1913), who considered the case of an elliptic crack in an infinite elastic plate. Interest in the problems of crack theory seems to stem, however, from the paper by Griffith (1921) in which he discusses the fracture strength of glass and introduces the idea of the dependence of this strength on the prescence of a crack. In the study of the two dimensional theory of elasticity, the major development has been Muskhelishvili's work on the equations of Kolosov (1909). Using complex variable theory, he solves (1953) many problems of cracks and holes in an elastic material. The solution obtained by Westergaard (1939) , in which the author showed that a special class of problems can be solved by the introduction of a complex variable function, can be shown to be a particular case of Muskhelishvili's solution. The methods of complex variable theory have also been used by Wigglesworth (1957) to examine the problem of a notched plate. Using Fourier transform methods, Sneddon and Elliot (1946) have also discussed problems of this type. Koiter (1960) has considered the problem of the infinite elastic sheet containing a doubly- periodic set of holes. A full account of the work carried out in this field has been given by Savin (1951) and by Green and Zerna (1954).

In three dimensional analysis, the analogue of the Griffith crack is the penny shaped crack. The majority of the papers on this type of problem, assume that the crack is circular in shape, and that it deformed by an internal pressure $p(\rho)$. Sack (1946), using oblate spheroidal co-ordinates and a solution given by Neuber (1934), solved the equations of elastic equilibrium. An exhaustive treatment of the penny-shaped crack in an infinite medium has been carried out by Sneddon (1946), using the methods of integral transforms. The quantities of physical interest are evaluated in this paper and graphs showing the variations of these quantities in the vicinity of the crack are plotted. A solution of the same problem has been given by Green (1949). Collins (1961) using a method of the same type as Green's has solved the problem of the crack in an infinite medium, when not only do we consider a variable normal pressure across the crack but also a shearing stress. Payne (1953) has also reduced the problem of the crack in an infinite medium to the determination of a harmonic function, which he determines in terms of an integral. The problem of finding the thermal stresses in the neighbourhood of a Griffith crack has been solved by Olesiak and Sneddon (1960), while Lowengrub (1961) has determined the elastic stresses for the case of a crack in a thick plate. The plane of the crack is taken parallel to the surfaces of the plate. The problem of the external circular crack has been tackled by Ufliand (1959).

A full review of the problems in crack theory is given by Sneddon (1961) and also by Green and Zerna (1954).

§16. A discussion of a two dimensional crack problem.

We begin the consideration of crack problems by considering the problem of a crack in an infinitely long elastic strip. In terms of rectangular co-ordinate axes (x, y) we write the displacement components in the x and y directions as u_x and u_y respectively. We consider a strip of material, $-\infty \leqslant x \leqslant \infty$, $-\pi \leqslant y \leqslant \pi$, with a crack in the interior of the material on the line $x = 0$, stretching from $-c$ to $+c$. The pressure exerted by the crack is taken to be $p(y)$ and we assume that on the edges of the strip the shear stress and normal displacement vanish

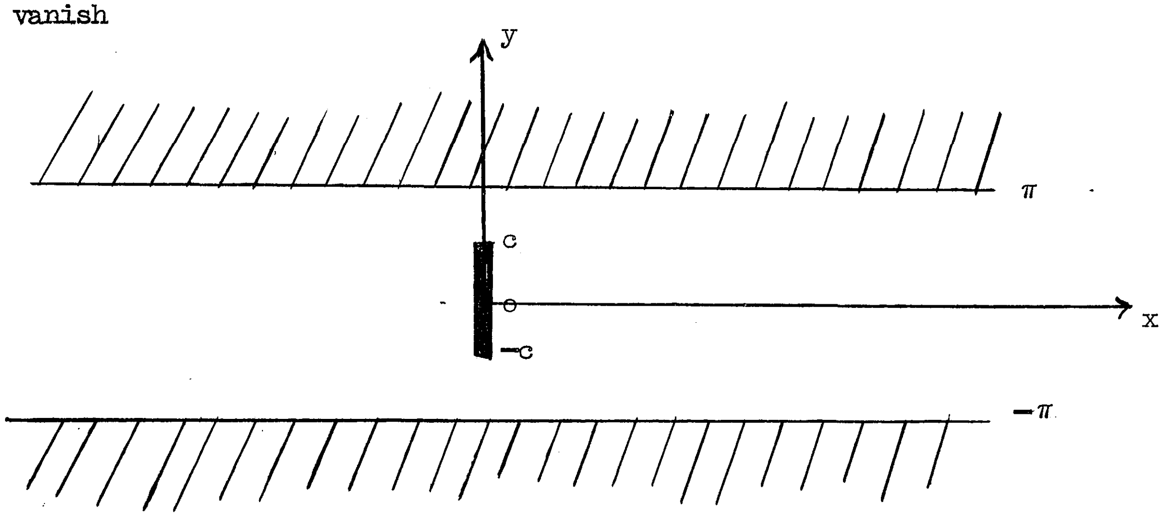


Fig 16.1

Since we have symmetry about the line $y = 0$, we may take the boundary conditions in the form

$$\left.\begin{array}{lll} \tau_{xy} = u_y = 0, & & \text{on } y = \pi, \\ \left.\begin{array}{ll} \tau_{xy} = 0, & \\ \sigma_x = -p(y), & 0 < y < c, \\ u_x = 0, & c < y < \pi, \end{array}\right\} & \text{on } x = 0, & \end{array}\right\} \qquad (16.1)$$

If we consider the solutions of the equations of elastic equilibrium as described in §7, given by Sneddon (1961), we see that a suitable choice of the potential functions would be

$$\chi = \frac{\beta^2}{4(\beta^2 - 1)} a_0 x \quad , \tag{16.2}$$

$$\phi = \frac{1}{2(\beta^2 - 1)} \sum_{n=1}^{\infty} a_n \, n^{-1} \cos ny \; e^{-nx} \tag{16.3}$$

It then follows easily that

$$u_x = \frac{1}{2(\beta^2 - 1)} \sum_{n=1}^{\infty} a_n \left[\beta^2 + (\beta^2 - 1) nx \right] \cos ny \; e^{-nx} + \frac{\beta^2}{4(\beta^2 - 1)} a_0 \tag{16.4}$$

$$u_y = \frac{-1}{2(\beta^2 - 1)} \sum_{n=1}^{\infty} a_n \left[1 - (\beta^2 - 1) nx \right] \sin ny \; e^{-nx} \tag{16.5}$$

while

$$\sigma_x = - \sum_{n=1}^{\infty} n \, a_n (1 + nx) \cos ny \; e^{-nx} \tag{16.6}$$

$$\sigma_y = - \sum_{n=1}^{\infty} n \, a_n (1 - nx) \cos ny \; e^{-nx} \tag{16.7}$$

$$\tau_{xy} = - x \sum_{n=1}^{\infty} n^2 a_n \sin ny \; e^{-nx} \tag{16.8}$$

The boundary conditions (16.1) are then satisfied, provided that we choose the a_n such that

$$\sum_{n=1}^{\infty} n \, a_n \cos ny = p(y) , \quad 0 < y < c , \tag{16.9}$$

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_m \cos ny = 0 , \quad c < y < \pi \tag{16.10}$$

Before attempting to solve these dual trigonometric series it is of interest to notice the importance of the inclusion of the term $\frac{1}{2}a_0$. Suppose that we omit it and consider the dual series

$$\sum_{n=1}^{\infty} n \, a_n \cos ny = p(y), \quad 0 < y < c , \tag{16.11}$$

$$\sum_{n=1}^{\infty} a_n \cos ny = 0 , \quad c < y < \pi \tag{16.12}$$

On integrating these equations, the first from 0 to y and the second from y to π, we obtain the dual series (where we have written $A_n = a_n / n$),

$$\sum_{n=1}^{\infty} nA_n \sin ny = \int_0^y p(y)dy = P(y) , \qquad 0 < y < c \tag{16.13}$$

$$\sum_{n=1}^{\infty} A_n \sin ny = 0 \qquad c < y < \pi \tag{16.14}$$

the solution of which has been given by Tranter (1959) to be

$$a_n = \frac{8}{\pi} n^2 \sin^3 \frac{1}{2} c \int_0^1 s\chi(s)\,{}_2F_1(1 + n, 1 - n; 2; s^2 \sin^2 \tfrac{1}{2}c)ds \tag{16.15}$$

where

$$\chi(s) = \int_0^s \frac{t\, P\left[2\sin^{-1}(t \sin \frac{1}{2}c)\right]}{\sqrt{(s^2 - t^2)(1 - t^2\sin^2\frac{1}{2}c)}} dt$$

If we take

$$p(y) = \cos y \tag{16.16}$$

then

$$\frac{a_n}{n} = \sin^4\tfrac{1}{2}c.n.\ {}_2F_1(1 + n, 1 - n; 3 ; \sin^2\tfrac{c}{2}) \tag{16.17}$$

substituting from equation (16.17) into equations (16.11) and (16.12), we obtain the expressions

$$\sigma_x\Big|_{x=0} = -\sin^4\tfrac{1}{2}c \sum_{n=1}^{\infty} n^3\,{}_2F_1(1 + n, 1 - n; 3 ; \sin^2\tfrac{1}{2}c) \cos ny \tag{16.18}$$

$$u_x\Big|_{x=0} = \sin^4\tfrac{1}{2}c \sum_{n=1}^{\infty} n^2\,{}_2F_1(1 + n, 1 - n; 3 ; \sin^2\tfrac{1}{2}c) \cos ny \times\left(\frac{\beta^2}{2\beta^2-1}\right) \tag{16.19}$$

Consider equation (16.18). Writing

$$\sin^2\tfrac{1}{2}c.n.\ {}_2F_1(1 + n, 1 - n; 3 ; \sin^2\tfrac{1}{2} c) = 4\int_0^{\infty} \frac{J_{2n}(t).J_2(t \sin {}^1/2c)}{t} dt$$

from Watson, p.401, where $0 < c < \pi$ and using the fact that

$$\sum_{n=1}^{\infty} n^2 J_{2n}(t) \cos ny = \frac{t}{4}\left[-\frac{1}{2} \sin \frac{y}{2} \sin(t \sin \frac{y}{2}) + \frac{t}{2} \cos^2\frac{y}{2} \cos(t \sin \frac{y}{2})\right]$$

where we have used the series given in Watson, p.22

we may, on changing the order of integration, write equation (16.18) as

$$\sigma_x\Big|_{x=0} = -\sin^2\frac{c}{2}\left[\frac{1}{2}\cos^2\frac{y}{2}\int_0^\infty t\, J_2(t\sin\frac{c}{2})\cos(t\sin\frac{y}{2})\,dt - \frac{1}{2}\sin\frac{y}{2}\int_0^\infty J_2(t\sin\frac{c}{2})\sin(t\sin\frac{y}{2})\,dt\right] \quad (16.20)$$

Again using Watson, p.405, we have

$$\int_0^\infty t\, J_2(t\sin\frac{c}{2})\cos(t\sin\frac{y}{2})\,dt = \frac{2}{\sin^2\frac{c}{2}}, \qquad 0 < y < c$$

$$= \frac{-2\sin^2\frac{c}{2}}{\cos\frac{y}{2}}\left\{\frac{d}{dy}\left[\frac{\sin\frac{y}{2} + (\sin^2\frac{y}{2} - \sin^2\frac{c}{2})^{\frac{1}{2}}}{\sqrt{(\sin^2\frac{y}{2} - \sin^2\frac{c}{2})}}\right]\right\}, \qquad c < y < \pi$$

and (16.21)

$$\int_0^\infty J_2(t\sin\frac{c}{2})\sin(t\sin\frac{y}{2})\,dt = \frac{2\sin\frac{y}{2}}{\sin^2\frac{c}{2}}, \qquad 0 < y < c$$

$$= \frac{-\sin^2\frac{c}{2}}{\sqrt{(\sin^2\frac{x}{2} - \sin^2\frac{c}{2})}\left[\sin\frac{y}{2} + \sqrt{(\sin^2\frac{y}{2} - \sin^2\frac{c}{2})}\right]}, \qquad c < y < \pi$$

(16.22)

It then easily follows that we obtain

$$\sigma_x\Big|_{x=0} = -\cos y, \qquad 0 < y < c \qquad (16.23)$$

as required.

Using a similar procedure on equation (16.19) we find, using the facts, given in Watson that

$$\int_0^\infty J_2(t\sin\frac{c}{2})\cos(t\sin\frac{y}{2}) = 0, \qquad c < y < \pi$$

$$= \frac{\cos\left\{2\sin^{-1}\left[\sin\frac{y}{2} / \sin\frac{c}{2}\right]\right\}}{\sqrt{\left(\sin^2\frac{c}{2} - \sin^2\frac{y}{2}\right)}}, \qquad 0 < y < c$$

(16.24)

$$\sum_{n=1}^{\infty} n\, J_{2n}(t) \cos ny = \frac{t \cos \frac{y}{2}}{4} \cos(t \sin \frac{y}{2}) , \tag{16.25}$$

where we have used the series given in Watson, p. 327, (12)

$$e^{iz \sin\theta}(2 \cos\theta)^k = \sum_{n=0}^{\infty} \epsilon_{2n} J_{2n,k} \cos(2n\theta) + i \sum_{n=0}^{\infty} \epsilon_{2n+1} J_{2n+1,k} \sin(2n+1)\theta \tag{16.26}$$

then it follows that

$$\begin{aligned} \hat{u}_x\Big|_{x=0} &= \frac{\cos\frac{y}{2}\,(\sin^2\frac{c}{2} - 1 + \cos y)}{\sqrt{(\cos y - \cos c)}}, && 0 < y < c \\ &= 0 , && c < y < \pi \end{aligned} \tag{16.27}$$

Thus we have recovered the required boundary conditions. However, in the region $0 < y < c$, it is seen that the displacement changes sign, which is physically impossible. We must therefore include the term a_0 to satisfy the physical conditions.

Equations of the form of (16.9) and (16.10) have been solved by Sneddon and Srivastav (1962).

Following the authors mentioned above, we set

$$\frac{1}{2} a_o + \sum_{n=1}^{\infty} a_n \cos ny = \int_y^c g(u)\,(u^2 - y^2)^{-1/2}\, du , \qquad 0 < y < c \tag{16.28}$$

where we have introduced a function $g(u)$ as yet undetermined. From the theory of Fourier series, it follows immediately that, in terms of the function $g(u)$, the coefficients are given by

$$\left.\begin{aligned} a_n &= \int_o^c g(u)\, J_o(nu)\, du , \qquad n \geq 1 \\ a_o &= \int_o^c g(u)\, du \end{aligned}\right\} \tag{16.29}$$

Now, let us return to equation (16.9), which we may write in the form

$$\sum_{n=1}^{\infty} a_n \sin ny = \int_o^y p(y)\, dy = \psi(y) \quad \text{say} , \qquad 0 < y < c \tag{16.30}$$

Multiplying each side of this equation by $y(r^2 - y^2)^{\mu-\frac{1}{2}}$ and integrating from 0 to r, we have

$$\sum_{n=1}^{\infty} a_n n^{-\mu} J_{\mu+1}(nr) = \frac{r^{-(\mu+1)}}{2^{\mu-1}\sqrt{\pi}\,\Gamma(\mu+\frac{1}{2})} \int_0^r y(r^2 - y^2)^{\mu-\frac{1}{2}} \psi(y)\, dy \qquad (16.31)$$

where the integrals involved have been evaluated with the use of Erdelyi (1954), p.69, Vol.1.

On substituting in this expression for the values of the coefficients a_n $(n \geqslant 1)$, and interchanging the order of the integration and summation, we have

$$\int_0^c g(u)\, du \left[\sum_{n=1}^{\infty} n^{-\mu} J_{\mu+1}(nr)\, J_0(nu) \right]$$

$$= \frac{r^{-(\mu+1)}}{2^{\mu-1}\sqrt{\pi}\,\Gamma(\mu+\frac{1}{2})} \cdot \int_0^r y(r^2 - y^2)^{\mu-\frac{1}{2}} \psi(y)\, dy, \qquad 0 < r < c \qquad (16.32)$$

We now require the sum of the series contained in the integral on the left hand side.

Let us consider the complex integral,

$$\int_C \frac{z^{-\mu} J_{\mu+1}(zr)\, J_0(zu)\, e^{i\pi z}\, dz}{\sin \pi z} \qquad (16.33)$$

where C is the contour shown.

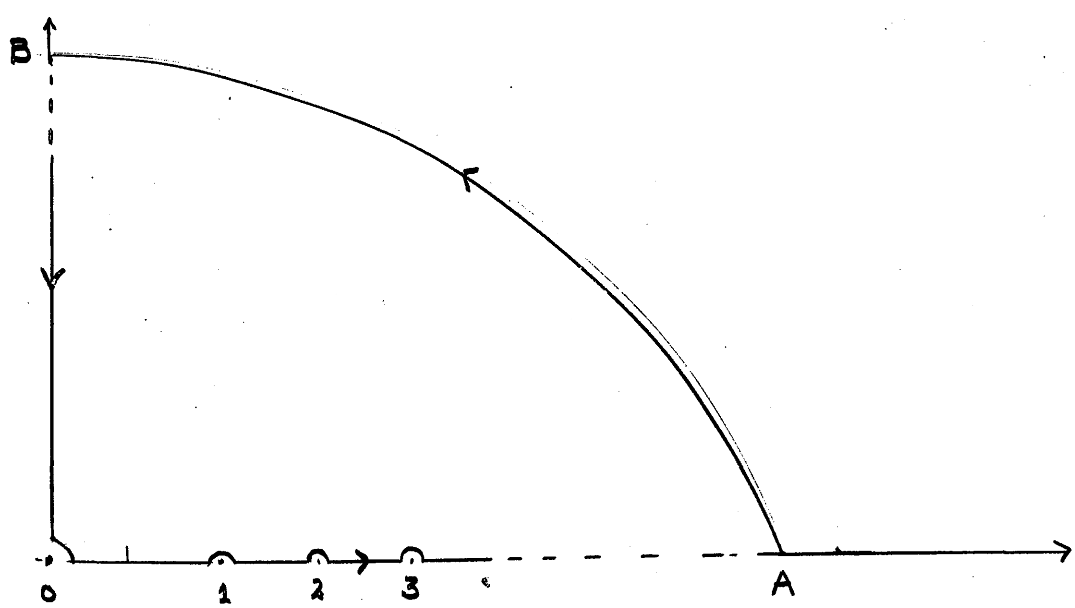


Fig 16.2

It may be shown that

$$\sum_{n=1}^{\infty} n^{-\mu} J_{\mu+1}(nr)\, J_o(nu)$$
$$= \int_o^{\infty} y^{-\mu} J_{\mu+1}(yr)\, J_o(yu)\, dy \;-\; \int_o^{\infty} \frac{e^{-y\pi}}{\sinh \pi y}\, y^{-\mu} I_o(uy)\, I_{\mu+1}(yr)\, dy \tag{16.34}$$

The first integral on the right hand side may be evaluated to give

$$\int_0^{\infty} y^{-\mu} J_{\mu+1}(yr)\, J_0(yu)\, dy = \frac{2^{-\mu}}{(\mu+1) r^{\mu+1}} (r^2 - u^2)^{\mu} \tag{16.35}$$

If we now substitute into equation (16.32), we obtain

$$\int_o^r g(u) \frac{2^{-\mu}}{\Gamma(\mu+1)\, r^{\mu+1}} (r^2 - u^2)^{\mu}\, du = \frac{r^{-(\mu+1)}}{2^{\mu-1}\sqrt{\pi}\,\Gamma(\mu+\frac{1}{2})} \int_o^r y(r^2 - y^2)^{\mu-\frac{1}{2}} \psi(y)\, dy$$
$$+ \int_o^c g(u)\, du \int_o^{\infty} \frac{e^{-y\pi}}{\sinh \pi y}\, y^{-\mu} I_o(uy)\, I_{\mu+1}(yr)\, dy \tag{16.36}$$

where we have again used Erdelyi (1954, p.48, Vol.2)

if now we let $\mu \longrightarrow 0+$ and differentiate with respect to r, we have

$$g(r) = \int_o^c g(u)\, K(u, r)\, du + A(r) \tag{16.37}$$

where we have written

$$A(r) = \frac{2}{\pi} \frac{d}{dr} \int_o^r y(r^2 - y^2)^{-\frac{1}{2}} \psi(y)\, dy \tag{16.38}$$

and

$$K(u, r) = \frac{2}{\pi} r \int_o^{\infty} \frac{ye^{-y\pi}}{\sinh \pi y} I_o(uy)\, I_1(ry)\, dy \tag{16.39}$$

i.e. we have reduced the problem to the solution of a Fredholm integral equation of the second kind. Methods of solving this type of equation are discussed later, when we solve a similar equation with a different kernel.

§17 Cracks in cylinders.

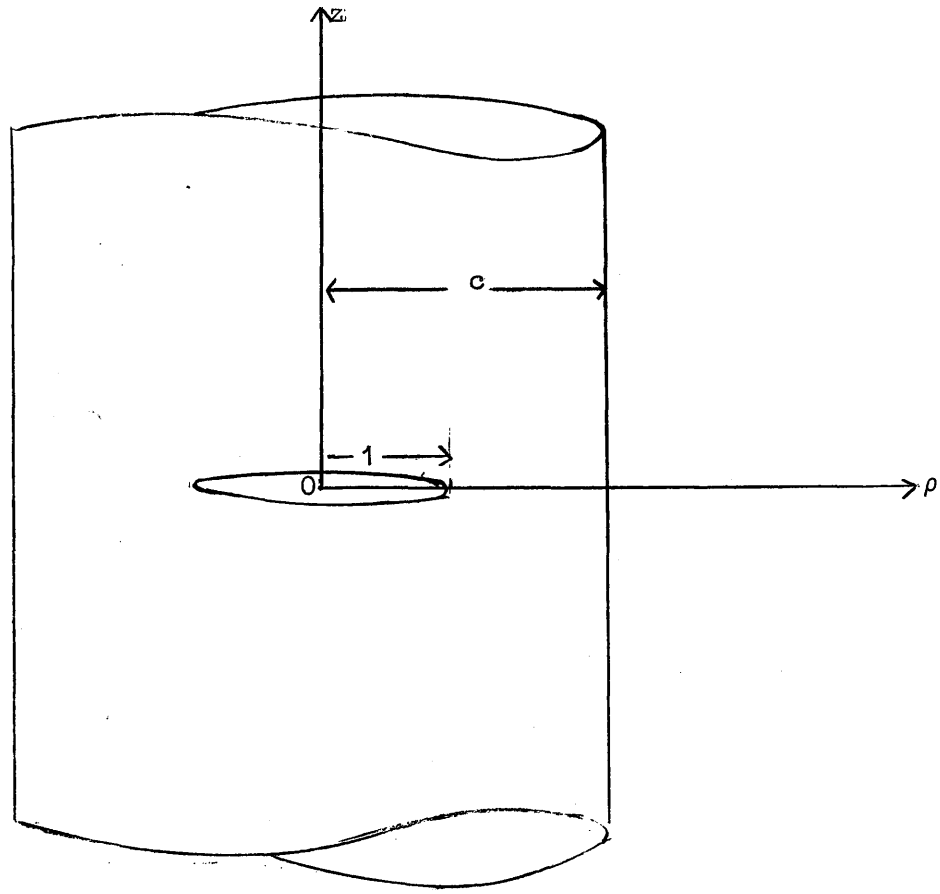


Fig 17.1

We wish to consider the problem of an infinitely long cylinder, of finite radius, with a crack developed inside the material of the cylinder. We assume, in the first case, that the wall of the cylinder is free from applied stress, and that the material is homogeneous. The crack, which we shall take to lie on the plane z = 0, is penny-shaped, and is subjected to an internal pressure $p(\rho)$, so that we shall have an axially symmetric problem. Since we have this symmetry we may employ cylindrical polar co-ordinates (ρ , ϕ , z) with the z axis coinciding with the axis of symmetry. As is usual in crack problems, we require that there should be no shearing stress across the plane of the crack, and that there should be no displacement normal to the plane of the crack, outside it. We shall take the radius of the crack to be our unit of length, and we denote the radius of the cylinder by c where $c > 1$. The conditions for the problem stated above can then be replaced by the conditions for a semi-infinite cylinder as follows.

$$\left.\begin{array}{llll} \sigma_\rho = \tau_{\rho z} = 0 & \text{on } \rho = c, \text{ for all } z \geqslant 0 & & \\ \tau_{\rho z} = 0 & \text{on } 0 \leqslant \rho \leqslant c & & \\ w = 0 & \text{On } c \geqslant \rho > 1 & \multirow{2}{*}{\Big\}} & \text{on } z = 0 \\ \sigma_z = p_0 & \text{on } 0 \leqslant \rho < 1 & & \end{array}\right] \qquad (17.1)$$

This problem is referred to, in the following paragraphs as the first problem.

The second problem we wish to consider is again that of a cylinder with a penny shaped crack developed on the plane $z = 0$ inside the cylinder. However in this case we consider different boundary conditions on the surface $\rho = c$.

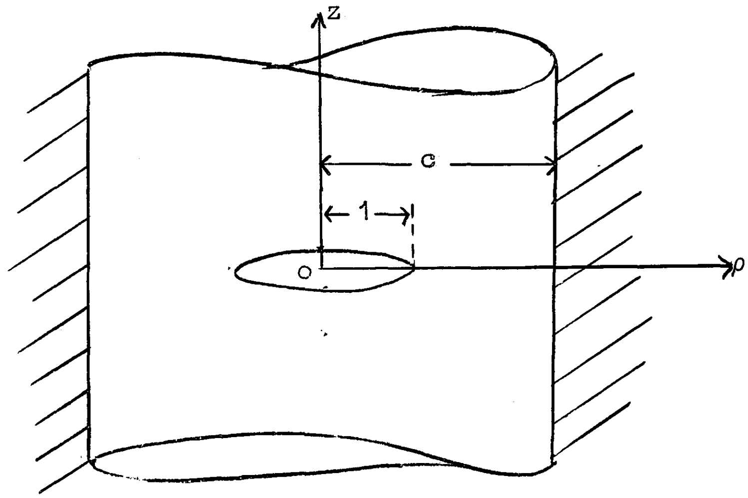


Fig 17.2

In this case we consider the cylinder to have its walls clamped. An alternate view of this problem is to consider the cylinder embedded in a rigid material. The boundary conditions now become

$$\left.\begin{array}{llll} u_\rho = \tau_{\rho z} = 0 & & & \text{on } \rho = c, \\ \sigma_z = -p(\rho), & 0 < \rho < 1 & \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} & \\ w = 0, & c > \rho > 1 & & \text{on } z = 0, \\ \tau_{\rho z} = 0, & \rho \geqslant 0, & & \end{array}\right] \qquad (17.2)$$

We shall, by using a combination of solutions of the equations of elastic equilibrium, reduce these problems to the solution of an integral equation.

The problem above will be referred to as the second problem.

§18. A first solution of the equations of elastic equilibrium.

Since we are dealing with problems concerning cylinders having axial symmetry, we shall choose cylindrical polar co-ordinates in which to work. We take the axis of z to coincide with the axis of the cylinder and employ co-ordinates (ρ, ϕ, z). The components of the displacement vector will then be denoted by (u_ρ, 0, w) in the usual way. The equations of elastic equilibrium, in terms of the normal components of stress (σ_ρ, σ_ϕ, σ_z) and the shear component $\tau_{\rho z}$, then take the form

$$\frac{\partial\sigma_\rho}{\partial\rho} + \frac{\partial\tau_{\rho z}}{\partial z} + \frac{\sigma_\rho - \sigma_\phi}{\rho} = 0 \qquad (18.1)$$

$$\frac{\partial\tau_{\rho z}}{\partial\rho} + \frac{\partial\sigma_z}{\partial z} + \frac{\tau_{\rho z}}{\rho} = 0 \qquad (18.2)$$

where, since we have axial symmetry, we have taken

$$\frac{\partial}{\partial\phi} = 0 \quad , \quad u_\phi = 0$$

where in the usual way

$$(\sigma_\rho, \sigma_\phi, \sigma_z) = \lambda\Delta + 2\mu \left(\frac{\partial u_\rho}{\partial\rho}, \frac{u_\rho}{\rho}, \frac{\partial w}{\partial z} \right) \qquad (18.3)$$

$$\tau_{\rho z} = \mu \left(\frac{\partial u_\rho}{\partial z} + \frac{\partial w}{\partial\rho} \right) \qquad (18.4)$$

and

$$\Delta = \frac{\partial u_\rho}{\partial\rho} + \frac{u_\rho}{\rho} + \frac{\partial w}{\partial z} \qquad (18.5)$$

where λ, μ are Lame's constants and are related to Poisson's ratio ν where normally $\frac{1}{4} < \nu < \frac{1}{2}$ by the equation

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

As a first solution of the equations (18.1) and (18.2) subject to the conditions (18.3) to (18.6) we consider the equations derived by Sneddon (1961). In terms of a potential function

$$\psi = \psi(\rho, z) \qquad (18.6)$$

the components of displacement and stress may be written in the form

$$u_\rho^1 = \frac{\partial\psi}{\partial\rho} + (\beta^2 - 1)\, z\, \frac{\partial^2\psi}{\partial\rho\partial z} \tag{18.7}$$

$$w^1 = -\beta^2 \frac{\partial\psi}{\partial z} + (\beta^2 - 1) z \frac{\partial^2\psi}{\partial z^2} \tag{18.8}$$

$$\mu^{-1}\sigma_z^1 = -2(\beta^2 - 1)\frac{\partial^2\psi}{\partial z^2} + 2(\beta^2 - 1)\, z \frac{\partial^3\psi}{\partial z^3} \tag{18.9}$$

$$\mu^{-1}\tau_{\rho z}^1 = 2z(\beta^2 - 1)\frac{\partial^3\psi}{\partial\rho\partial z^2} + \cancel{\frac{\partial^2\psi}{\partial\rho\partial z}} \tag{18.10}$$

$$\mu^{-1}\sigma_\rho^1 = -2(\beta^2 - 2)\frac{\partial^2\psi}{\partial z^2} + 2\frac{\partial^2\psi}{\partial\rho^2} + 2(\beta^2 - 1)\, z \frac{\partial^3\psi}{\partial\rho^2\partial z} \tag{18.11}$$

and the hoop stress may be determined from the expression

$$\tfrac{1}{2}(\sigma_\phi^1 + \sigma_\rho^1)\mu^{-1} = (2\beta^2 - 3)\frac{\partial^2\psi}{\partial z^2} - (\beta^2 - 1)\, z \frac{\partial^3\psi}{\partial z^3} \tag{18.12}$$

The choice of the function $\psi(\rho, z)$ most suitable to the problems under consideration is given by

$$\psi(\rho, z) = \frac{1}{2(\beta^2 - 1)}\int_0^\infty \eta^{-2} F(\eta)\, e^{-\eta z} J_0(\eta\rho)\, d\eta \tag{18.13}$$

Having made this choice we now substitute from equation (18.13) into equation (18.7) to (18.11) to derive the quantities in which we are interested. The components of displacement are given by

$$u_\rho^1 = -\int_0^\infty \eta^{-1} F(\eta)(1 - 2\nu - \eta z) J_1(\eta\rho)\, e^{-\eta z}\, d\eta \tag{18.14}$$

$$w^1 = \int_0^\infty \eta^{-1} F(\eta)\left[2(1 - \nu) + \eta z\right] J_0(\eta\rho)\, e^{-\eta z}\, d\eta \tag{18.15}$$

while we have for the components of stress

$$\sigma_\rho^1 = -2\mu\int_0^\infty F(\eta)\, e^{-\eta z}\left[(1 - \eta z) J_0(\eta\rho) - (1 - 2\nu - \eta z)\frac{J_1(\eta\rho)}{\eta\rho}\right] d\eta \tag{18.16}$$

$$\sigma_z^1 = -2\mu\int_0^\infty F(\eta)(1 + \eta z)\, e^{-\eta z} J_0(\eta\rho)\, d\eta \tag{18.17}$$

$$\tau_{\rho z}^1 = -2\mu z\int_0^\infty \eta\, F(\eta)\, e^{-\eta z} J_1(\eta\rho)\, d\eta \tag{18.18}$$

We are now interested in the values taken by these quantities on the plane $z = 0$, and on the cylindrical surface $\rho = c$. On $z = 0$ we have

$$\tau^1_{\rho z} = 0, \qquad \rho > 0 \tag{18.19}$$

$$w^1 = 2(1-\nu)\int_0^\infty \eta^{-1}\, F(\eta)\, J_0(\eta\rho)\, d\eta \tag{18.20}$$

$$\sigma^1_z = -2\mu\int_0^\infty F(\eta)\, J_0(\eta\rho)\, d\eta \tag{18.21}$$

and on the surface $\rho = c$,

$$\tau^1_{\rho z} = -\,2\mu z\int_0^\infty \eta\, F(\eta)\, e^{-\eta z}\, J_1(\eta c)\, d\eta \tag{18.22}$$

$$\begin{aligned}\sigma^1_\rho &= -\,2\mu\int_0^\infty F(\eta)\, e^{-\eta z}\left\{J_0(\eta c) - (1-2\nu)\frac{J_1(\eta c)}{\eta c}\right\} d\eta \\ &\quad + 2\mu\int_0^\infty F(\eta)\,.\, z\eta\, e^{-\eta z}\left\{J_0(\eta c) - \frac{J_1(\eta c)}{\eta c}\right\} d\eta\end{aligned} \tag{18.23}$$

$$u^1_\rho = -\int_0^\infty \eta^{-1}\, F(\eta)(1-2\nu-\eta z)\, J_1(\eta c)\, e^{-\eta z}\, d\eta \tag{18.24}$$

To facilitate later working, it is desirable to obtain these expressions in a slightly different form. Let us denote the Fourier sine transform of a function $f(x)$ with respect to the variable ξ by

$$\mathcal{F}_s\left[f(x)\ ;\xi\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin \xi x\, dx \tag{18.25}$$

and the Fourier cosine transform by

$$\mathcal{F}_c\left[f(x)\ ;\xi\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos \xi x\, dx \tag{18.26}$$

Having done this, it is easily seen that

$$\mathcal{F}_c\left[e^{-\eta z}\ ;\xi\right] = \sqrt{\frac{2}{\pi}}\,\frac{\eta}{(\xi^2+\eta^2)} \tag{18.27}$$

$$\mathcal{F}_c\left[ze^{-\eta z}\ ;\xi\right] = \sqrt{\frac{2}{\pi}}\,\frac{(\eta^2-\xi^2)}{(\eta^2+\xi^2)^2} \qquad \mathcal{F}_s\left[ze^{-\eta z};\xi\right] = \sqrt{\frac{2}{\pi}}\cdot\frac{2\xi\eta}{(\xi^2+\eta^2)^2} \tag{18.28}$$

Using equations (18.25) to (18.28) it follows that

$$\mathcal{F}_s\left\{\left[\tau^1_{\rho z}\right]_{\rho=c}\ ;\xi\right\} = -\,4\mu\sqrt{\frac{2}{\pi}}\,\xi\int_0^\infty \frac{\eta^2\, F(\eta)}{(\xi^2+\eta^2)^2}\, J_1(\eta c)\, d\eta \tag{18.29}$$

$$\mathcal{F}_c\left[\sigma_{\rho\,\rho=c}\,;\,\xi\right] = -4\mu\sqrt{\frac{2}{\pi}}\,\xi^2\int_0^\infty \frac{\eta\;J_0(\eta c)\;F(\eta)}{(\xi^2+\eta^2)^2}\,d\eta \;-\; 4\mu\nu\sqrt{\frac{2}{\pi}}\;\frac{1}{c}\int_0^\infty \frac{J_1(\eta c)\;F(\eta)}{(\eta^2+\xi^2)}\,d\eta$$
$$+\;4\mu\sqrt{\frac{2}{\pi}}\,\frac{1}{c}\,\xi^2\int_0^\infty \frac{J_1(\eta c)\;F(\eta)}{(\xi^2+\eta^2)^2}\,d\eta \qquad (18.30)$$

$$\mathcal{F}_c\left[u_{\rho\;\rho=c}\,;\xi\right] = \sqrt{\frac{2}{\pi}}\,(1-2\nu)\int_0^\infty \frac{J_1(\eta c)\;F(\eta)}{(\xi^2+\eta^2)}\,d\eta \;+\; \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{J_1(\eta c)\;F(\eta)}{(\xi^2+\eta^2)}\,d\eta$$
$$-\sqrt{\frac{2}{\pi}}\;2\xi^2\int_0^\infty \frac{J_1(\eta c)\;F(\eta)}{(\xi^2+\eta^2)^2}\,d\eta \qquad (18.31)$$

The expressions (18.29) to (18.31) may be expressed very simply in terms of three infinite integrals i_1, i_2, and i_3, where

$$i_1 = \frac{4\xi}{\pi}\int_0^\infty \frac{\eta\;F(\eta)\;J_0(\eta c)}{(\xi^2+\eta^2)^2}\,d\eta \qquad (18.32)$$

$$i_2 = \frac{4}{\pi}\int_0^\infty \frac{F(\eta)\;J_1(\eta c)}{(\xi^2+\eta^2)}\,d\eta \qquad (18.33)$$

$$i_3 = \frac{4\xi^2}{\pi}\int_0^\infty \frac{F(\eta)\;J_1(\eta c)}{(\xi^2+\eta^2)^2}\,d\eta \qquad (18.34)$$

We can now express equations (18.29) to (18.31) in the much more compact form,

$$\mathcal{F}_s\left\{\left[\tau'_{\rho z}\right]_{\rho=c};\xi\right\} = -\sqrt{2\pi}\,\mu\xi(i_2 - i_3) \qquad (18.35)$$

$$\mathcal{F}_c\left\{\left[\sigma_\rho\right]_{\rho=c};\xi\right\} = -\sqrt{2\pi}\,\frac{\mu}{c}\,(c\xi i_1 + \nu i_2 - i_3) \qquad (18.36)$$

$$\mathcal{F}_c\left\{\left[u_\rho\right]_{\rho=c};\xi\right\} = -\sqrt{\frac{\pi}{2}}\,(i_3 - \nu i_2) \qquad (18.37)$$

It is easily seen from the form of the solution given above that, if $F(\xi)$ is suitably chosen, we may obtain the solution of the problem of a crack in an infinite elastic medium. This is in fact the solution derived by Sneddon(1946). We now wish to introduce a second solution which will reduce the normal and shear stresses on the walls of the cylinder $\rho = c$ to zero, and maintain our conditions on the plane $z = 0$. This may be thought of as a correction solution, and, as the radius of the crack increases and approaches the radius of the cylinder, the second solution will become increasingly important.

§ 19 A second solution of the equations of elastic equilibrium.

A solution of the equations (18.1) to (18.5) has also been derived by Sneddon (1951) in terms of a biharmonic function χ which satisfies the equation

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}\right)\chi = 0 \qquad (19.1)$$

The components of stress and displacement may be expressed in terms of this function as follows,

$$u_\rho^2 = -\frac{(1+\nu)}{E}\frac{\partial^2\chi}{\partial\rho\partial z} \qquad (19.2)$$

$$w^2 = \frac{(1+\nu)}{E}\left[(1-2\nu)\nabla^2\chi + \frac{\partial^2\chi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\chi}{\partial\rho}\right] \qquad (19.3)$$

where we have written ∇^2 for

$$\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2} \qquad (19.4)$$

and E is Young's Modulus, where $E = (3\lambda + 2\mu)\mu/(\lambda + \mu)$

Using the stress-strain relations we can now calculate the normal and shear stress components

$$\sigma_\rho^2 = \frac{\partial}{\partial z}\left[\nu\nabla^2\chi - \frac{\partial^2\chi}{\partial\rho^2}\right] \qquad (19.5)$$

$$\tau_{\rho z}^2 = \frac{\partial}{\partial\rho}\left[(1-\nu)\nabla^2\chi - \frac{\partial^2\chi}{\partial z^2}\right] \qquad (19.6)$$

$$\sigma_\phi^2 = \frac{\partial}{\partial z}\left[\nu\nabla^2\chi - \frac{1}{\rho}\frac{\partial\chi}{\partial\rho}\right] \qquad (19.7)$$

$$\sigma_z^2 = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\chi - \frac{\partial^2\chi}{\partial z^2}\right] \qquad (19.8)$$

We must now choose the form of $\chi(\rho, z)$. We choose this function in the form

$$\chi(\rho, z) = -2\mu\int_0^\infty \xi^{-2}\left\{\left[A(\xi) + 4(1-\nu)B(\xi)\right]I_0(\rho\xi) - \xi\rho B(\xi) I_1(\xi\rho)\right\}\sin\xi z \, d\xi \qquad (19.9)$$

where $A(\xi)$ and $B(\xi)$ are as yet arbitrary functions of the variable ξ and $I_0(\rho\xi)$, $I_1(\rho\xi)$ are modified Bessel functions of order zero and one respectively.

Having chosen the biharmonic function $\chi(\rho, z)$ in this way, we substitute from equation (19.9) into equations (19.5) to (19.8) to obtain the boundary conditions on the plane $z = 0$, and on the cylinder $\rho = c$. On $z = 0$,

$$w^2 = \tau^2_{\rho z} = 0, \qquad \rho \geq 0 \tag{19.10}$$

$$\sigma^2_z = -2\mu \int_0^\infty \xi \left\{ \left[A(\xi) - 2\nu B(\xi) \right] I_0(\xi\rho) - \xi\rho B(\xi) I_1(\xi\rho) \right\} d\xi \tag{19.11}$$

and on $\rho = c$,

$$\sigma^2_\rho = \frac{2\mu}{c} \int_0^\infty \left\{ A(\xi) \left[c\xi I_0(c\xi) - I_1(c\xi) \right] + B(\xi) \left[(3 - 2\nu) c\xi I_0(c\ ) - 4(1-\nu) I_1(c\xi) - c^2\xi^2 I_1(c\xi) \right] \right\} \cos \xi z \, d\xi \tag{19.12}$$

$$\tau^2_{\rho z} = -2\mu \int_0^\infty \xi \left\{ \left[A(\xi) + 2(1-\nu) B(\xi) \right] I_1(c\xi) - c\xi B(\xi) I_0(c\xi) \right\} \sin \xi z \, d\xi \tag{19.13}$$

$$u^2_\rho = \int_0^\infty \left\{ \left[A(\xi) + 4(1-\nu) B(\xi) \right] I_1(\xi\rho) - \xi\rho B(\xi) I_0(\xi\rho) \right\} \cos(\xi z) \, d\xi \tag{19.14}$$

Following the procedure used at the end of paragraph 18, we now re-write equations (19.12) to (19.14) in the form of Fourier sine and cosine transforms.

$$\mathcal{F}_c\left\{ \left[\sigma^2_\rho\right]_{\rho = c}; \xi \right\} = \frac{2\mu}{c}\sqrt{\frac{\pi}{2}} \left\{ A(\xi)\left[c\xi I_0(c\xi) - I_1(c\xi)\right] + B(\xi)\left[(3 - 2\nu)c\xi I_0(c\xi) - c^2\xi^2 I_1(c\xi) - 4(1-\nu) I_1(c\xi)\right]\right\} \tag{19.15}$$

$$\mathcal{F}_s\left\{ \left[\tau^2_{\rho z}\right]_{\rho = c}; \xi \right\} = -2\mu\sqrt{\frac{\pi}{2}}\, \xi \left\{ \left[A(\xi) + 2(1-\nu) B(\xi)\right] I_1(c\xi) - c\xi B(\xi) I_0(c\xi) \right\} \tag{19.16}$$

$$\mathcal{F}_c\left\{ \left[u^2_\rho\right]_{\rho = c}; \xi \right\} = \sqrt{\frac{\pi}{2}} \left\{ \left[A(\xi) + 4(1-\nu) B(\xi)\right] I_1(c\xi) - c\xi B(\xi) I_0(c\xi) \right\} \tag{19.17}$$

We have now set up two sets of equations, each of which is a solution of the elastic equations of equations of equilibrium. We now wish to combine these two sets to give the solutions to the stated problems.

§ 20. Boundary conditions.

We shall now consider in more detail the boundary conditions discussed in §17.

Let us consider firstly the boundary conditions imposed on the first problem, that of the cylinder having its walls free from stress and a crack developed in its interior on the plane $z = 0$. On $z = 0$

$$\tau_{\rho z} = 0, \qquad \rho \geq 0 \tag{20.1}$$

$$w = 0, \qquad \rho > 1 \tag{20.2}$$

$$\sigma_z = -p(\rho) \qquad 1 > \rho > 0 \tag{20.3}$$

where, on $\rho = c$,

$$\sigma_\rho = \tau_{\rho z} = 0 \tag{20.4}$$

Neither of the solutions given in §18 or §19 are capable, separately, of providing a complete solution to this problem. However, if we combine each of the solutions we are able to do so. Since now the components of stress and strain are represented by a combination of two terms, we may re-write the boundary conditions (20.1) to (20.4) in the following way. On $z = 0$,

$$\tau^1_{\rho z} + \tau^2_{\rho z} = 0, \qquad \rho > 0 \tag{20.5}$$

$$w^1 + w^2 = 0, \qquad \rho > 1 \tag{20.6}$$

$$\sigma^1_z + \sigma^2_z = -p(\rho) \qquad 1 > \rho > 0 \tag{20.7}$$

and on $\rho = c$,

$$\sigma^1_\rho + \sigma^2_\rho = 0 \tag{20.8}$$

$$\tau^1_{\rho z} + \tau^2_{\rho z} = 0 \tag{20.9}$$

Consider also the second problem described in §17, that of a cylinder clamped on the wall in such a way that there is no radial displacement on $\rho = c$, and no shearing stress there. The boundary conditions then become, on $z = 0$

$$\tau^1_{\rho z} + \tau^2_{\rho z} = 0, \qquad \rho > 0 \tag{20.10}$$

$$w^1 + w^2 = 0, \qquad \rho > 1 \tag{20.11}$$

$$\sigma^1_z + \sigma^2_z = -p(\rho) \qquad 1 > \rho > 0 \tag{20.12}$$

and on $\rho = c$

$$u^1_\rho + u^2_\rho = 0 \tag{20.13}$$

$$\tau^1_{\rho z} + \tau^2_{\rho z} = 0 \tag{20.14}$$

In both problems the first condition (20.5) or (20.10) is identically satisfied. We are then left with three arbitrary functions $F(\xi)$, $A(\xi)$, and $B(\xi)$ with which to satisfy the remaining boundary conditions. If we solve the last two conditions in each case i.e. (20.8) and (20.9) in the first case, and (20.13), (20.14) in the second, for $A(\xi)$ and $B(\xi)$ then we will obtain integral equations of the same form, but with different values of $A(\xi)$ and $B(\xi)$, from the third and fourth conditions in each case, to determine $F(\xi)$. We begin by determining $A(\xi)$ and $B(\xi)$ and leave the discussion of the integral equations to the next paragraph. For the first problem, if we substitute in equations (20.8) and (20.9) from equations (18.35), (18.36) and (19.15), (19.16) we obtain the relations

$$A(\xi)\left[c\xi\, I_0(c\xi) - I_1(c\xi)\right] + B(\xi)\left[(3-2\nu)c\xi\, I_0(c\xi) - 4(1-\nu)\, I_1(c\xi) - c^2\xi^2\, I_1(c\xi)\right]$$

$$= c\xi i_1 + \nu i_2 - i_3 \tag{20.15}$$

$$A(\xi)\, I_1(c\xi) \;-\; B(\xi)\left[c\xi\, I_0(c\xi) \;-\; 2(1-\nu)\, I_1(c\xi)\right] \;=\; i_3 \;-\; i_2 \qquad (20.16)$$

On solving these two simultaneous equations for $A(\xi)$ and $B(\xi)$ we obtain the results, which we denote by $A_1(\xi)$, $B_1(\xi)$ respectively.

$$\begin{aligned} G(\xi)\, A_1(\xi) \;=& \left[c\xi\, I_0(c\xi) - 2(1-\nu)\, I_1(c\xi)\right] c\xi\, i_1 \\ & - \left[3(1-\nu)\, c\xi\, I_0(c\xi) - 2(2-\nu)(1-\nu)\, I_1(c\xi) - c^2\xi^2\, I_1(c\xi)\right] i_2 \\ & + \left[2(1-\nu)\, c\xi\, I_0(c\xi) - 2(1-\nu)\, I_1(c\xi) - c^2\xi^2\, I_1(c\xi)\right] i_3 \end{aligned} \qquad (20.17)$$

and

$$G(\xi)\, B_1(\xi) \;=\; c\xi\, I_1(c\xi)\, i_1 \;+\left[c\xi\, I_0(c\xi) - (1-\nu)\, I_1(c\xi)\right] i_2 \;-\; c\xi\, I_0(c\xi)\, i_3 \qquad (20.18)$$

where we have written,

$$G(\xi) \;=\; c^2\xi^2\, I_0^2(c\xi) \;-\; (2 - 2\nu \;+\; c^2\xi^2)\, I_1^2(c\xi) \qquad (20.19)$$

and we define the integrals i_1, i_2, i_3, in terms of the function $F(\xi)$ by equations (18.32), (18.33) and (18.34).

Now let us return to the second case to be considered, that of the cylinder with fixed walls. If we substitute from equations (18.35), (18.37) and (19.15),(19.17) into equations (20.13) and (20.14) we obtain the two simultaneous equations in $A(\xi)$ and $B(\xi)$,

$$A(\xi)\, I_1(c\xi) \;-\; B(\xi)\left[c\xi\, I_0(c\xi) \;-\; 4(1-\nu)\, I_1(c\xi)\right] \;=\; i_3 \;-\; \nu i_2 \qquad (20.20)$$

$$A(\xi)\, I_1(c\xi) \;-\; B(\xi)\left[c\xi\, I_0(c\xi) \;-\; 2(1-\nu)\, I_1(c\xi)\right] \;=\; i_3 \;-\; i_2 \qquad (20.21)$$

We solve for $A(\xi)$ and $B(\xi)$ and denote the solution in this case by $A_2(\xi)$ and $B_2(\xi)$. Then

$$A_2(\xi) \;=\; \frac{1}{I_1(c\xi)}\, i_3 \;+\; \frac{1}{2 I_1^2(c\xi)}\left[c\xi\, I_0(c\xi) \;-\; 4(1-\nu)\, I_1(c\xi)\right] i_2 \qquad (20.22)$$

$$B_2(\xi) \;=\; \frac{1}{2 I_1(c\xi)}\, i_2 \qquad (20.23)$$

Having determined $A(\xi)$ and $B(\xi)$ in each case, let us now return to the second and third conditions to determine the function $F(\xi)$.

§21. Integral equations.

In both the problems we are considering, the boundary conditions on the plane $z = 0$ are the same. In both cases we are considering a penny-shaped crack on the surface $z = 0$. We assume that the internal pressure of the crack may be written in the form

$$\sigma_z = - p(\rho) = -2\mu f(\rho) \qquad 0 < \rho < 1 \tag{21.1}$$

Then on substituting for $\sigma_z^1, \sigma_z^2, w^1$, and w^2 in either equations (20.8) and (20.9) or (20.11) and (20.12) we obtain

$$\int_0^\infty F(\xi) J_0(\xi\rho)\, d\xi + \int_0^\infty \xi\left\{\left[A(\xi) - 2\nu B(\xi)\right] I_0(\xi\rho) - \xi\rho\, B(\xi)\, I_1(\xi\rho)\right\} d\xi = f(\rho), \qquad 0 < \rho < 1 \tag{21.2}$$

$$\int_0^\infty \xi^{-1} F(\xi) J_0(\xi\rho)\, d\xi = 0, \qquad \rho > 1 \tag{21.3}$$

Since $A(\xi)$ and $B(\xi)$ are known, we have obtained a set of dual integral equations for the determination of $F(\xi)$. In order to reduce this set of dual integral equations to a single integral equation, we write

$$F(\xi) = \xi \int_0^1 g(t) \sin(\xi t)\, dt \tag{21.4}$$

where the function $g(t)$ has the value zero at $t = 0$.

The second equation of the set of dual integral equations is then immediately satisfied and since

$$\int_0^\infty J_0(\xi\rho) \cos \xi t\, d\xi = \begin{cases} \dfrac{1}{\sqrt{\rho^2 - t^2}}, & \rho > t \\ 0, & \rho < t \end{cases} \tag{21.5}$$

the first equation of the set of integral equations becomes

$$\int_0^\rho \frac{g'(t)\, dt}{\sqrt{(\rho^2 - t^2)}} + \int_0^\infty \xi\left\{\left[A(\xi) - 2\nu B(\xi)\right] I_0(\xi\rho) - \rho\xi\, B(\xi)\, I_1(\xi\rho)\right\} d\xi = f(\rho), \qquad 0 < \rho < 1 \tag{21.6}$$

Since, it is also true that

$$\int_0^t \frac{\rho\, d\rho}{\sqrt{t^2 - \rho^2}} \int_0^\rho \frac{g'(t)\, dt}{\sqrt{\rho^2 - t^2}} = \frac{\pi}{2} g(t) \tag{21.7}$$

equation (21.6) reduces to

$$g(t) = \frac{2}{\pi}\int_0^t \frac{\rho f(\rho)\,d\rho}{\sqrt{t^2-\rho^2}} - \frac{2}{\pi}\int_0^\infty \xi\left\{\left[A(\xi) - 2\nu B(\xi)\right]\left[\int_0^t \frac{\rho I_0(\xi\rho)\,d\rho}{\sqrt{t^2-\rho^2}}\right] - \xi B(\xi)\int_0^t \frac{\rho^2 I_1(\xi\rho)\,d\rho}{\sqrt{t^2-\rho^2}}\right\} d\xi \tag{21.8}$$

We can evaluate the two inner integrals in equation (21.8) and these take the values

$$\int_0^t \frac{\rho I_0(\xi\rho)\,d\rho}{\sqrt{t^2-\rho^2}} = \frac{\sinh \xi t}{\xi} \tag{21.9}$$

$$\int_0^t \frac{\rho^2 I_1(\xi\rho)\,d\rho}{\sqrt{t^2-\rho^2}} = \frac{1}{\xi^2}\left[\xi t \cosh \xi t - \sinh \xi t\right] \tag{21.10}$$

and so, we may write the integral equation for g(t) in the form

$$g(t) = \frac{2}{\pi}\int_0^t \frac{\rho f(\rho)\,d\rho}{\sqrt{t^2-\rho^2}} - \frac{2}{\pi}\int_0^\infty\left\{\left[A(\xi) + (1-2\nu)\,B(\xi)\right]\sinh \xi t - B(\xi)\,\xi t \cosh \xi t\right\} d\xi \tag{21.11}$$

Before proceeding further, we now require to look at the forms of solutions obtained for $A(\xi)$ and $B(\xi)$. These were given in terms of the integrals i_1, i_2, i_3, which themselves were functions depending on $F(\xi)$ and hence from equation (21.4) on $g(t)$. On substituting

$$F(\xi) = \xi\int_0^1 g(u) \sin \xi u\,du$$

in equations (18.32) to (18.34) we obtain the values of the integrals in the form

$$i_1 = \frac{2}{\pi}\int_0^1 g(u)\left\{\sinh \xi u\left[K_0(c\xi) - \xi c K_1(c\xi)\right] + \xi u \cosh \xi u\, K_1(c\xi)\right\} du \tag{21.12}$$

$$i_2 = \frac{4}{\pi}\int_0^1 g(u) \sinh \xi u\, K_1(c\xi)\,du \tag{21.13}$$

$$i_3 = \frac{2}{\pi}\int_0^1 g(u)\left\{\sinh \xi u\left[c\xi\, K_0(c\xi) + K_1(c\xi)\right] - \xi u \cosh \xi u\, K_1(c\xi)\right\} du \tag{21.14}$$

where we have made use of the integrals

$$\int_0^\infty \frac{\sin \eta t\, J_0(\eta c)\,d\eta}{(\xi^2 + \eta^2)} = \frac{1}{\xi}\sinh \xi t\, K_0(\xi c) \tag{21.15}$$

$$\int_0^\infty \frac{\eta^2 \sin \eta t\, J_0(\eta c)\,d\eta}{(\xi^2 + \eta^2)^2} = \frac{1}{2\xi}\left\{\sinh \xi t\left[K_0(\xi c) - \xi c\, K_1(\xi c)\right] + \xi t \cosh \xi t\, K_0(\xi c)\right\} \tag{21.16}$$

$$\int_0^\infty \frac{\eta \sin \eta t\, J_1(\eta c)\,d\eta}{(\xi^2 + \eta^2)} = \sinh \xi t\, K_1(\xi c) \tag{21.17}$$

$$2\xi\int_0^\infty \frac{\eta \sin \eta t\, J_1(\eta c)\,d\eta}{(\xi^2 + \eta^2)^2} = t \cosh \xi t\, K_1(\xi c) - \xi^{-1}\sinh \xi t\left[K_1(c\xi) + c\xi\, K_0(c\xi)\right] \tag{21.18}$$

Having obtained the integrals i_1, i_2, i_3, in terms of the function g(t) we now determine $A(\xi)$ and $B(\xi)$ in terms of g. For the case of the cylinder with free walls, we obtain, after an amount of reduction,

$$A_1(\xi) = \frac{2\left[H(c\xi) - 2 + 2\nu\right]}{\pi G(c\xi)} \int_0^1 g(u)\, \xi u \cosh \xi u \, du + \frac{2\left[(3-2\nu)H(c\xi) - 4 + 4\nu - c^2\xi^2\right]}{\pi G(c\xi)} \int_0^1 g(u) \sinh \xi u \, du \qquad (21.19)$$

$$B_1(\xi) = \frac{2}{\pi G(c\xi)} \int_0^1 g(u)\, \xi u \cosh \xi u \, du + \frac{2\left[1 - H(c\xi)\right]}{G(c\xi)} \int_0^1 g(u) \sinh \xi u \, du \qquad (21.20)$$

where we have written

$$H(c\xi) = c^2\xi^2 I_0(c\xi) K_0(c\xi) + (2 - 2\nu + c^2\xi^2) I_1(c\xi) K_1(c\xi) \qquad (21.21)$$

and $G(c\xi)$ is as before

If, after some reduction, we substitute for A_1 and B_1 in equation (21.11) we obtain the integral equation to determine g in the form

$$g(t) + \frac{2}{\pi}\int_0^1 g(u) K(t, u)\, du = P(t) \qquad (21.22)$$

where

$$P(t) = \frac{2}{\pi}\int_0^1 \frac{\rho f(\rho)\, d\rho}{\sqrt{t^2 - \rho^2}} \qquad (21.23)$$

and the kernel of the equation is given by

$$K(t, u) = \frac{2}{\pi}\int_0^\infty \frac{1}{G(c\xi)} \Big\{ \left[H(c\xi) - 1\right]\left[\xi u \cosh \xi u \sinh \xi t + \xi t \cosh \xi t \sinh \xi u\right] + \left[2H(c\xi) - 3 + 2\nu - c^2\xi^2\right] \sinh \xi u \sinh \xi t - \xi u\, \xi t \cosh \xi u \cosh \xi t \Big\}\, d\xi \qquad (21.24)$$

Having set up the integral equation to determine g(t) in the first case, let us again turn our attention to the cylinder with clamped walls.

In this case,

$$A_2(\xi) + (1 - 2\nu) B_2(\xi) = \frac{1}{I_1(c\xi)} i_3 + \frac{1}{2I_1^2(c\xi)}\left[c\xi I_0(c\xi) - 3I_1(c\xi)\right] i_2 \qquad (21.25)$$

On substituting into equation (21.11) for $B(\xi)$, i_1, i_2, and using equation (21.25) we derive the appropriate integral equation for g(t).

$$g(t) + \frac{2}{\pi}\int_0^1 g(u)\, K(t, u)\, du = P(t) \tag{21.26}$$

where

$$P(t) = \frac{2}{\pi}\int_0^t \frac{\rho f(\rho)\, d\rho}{\sqrt{t^2 - \rho^2}} \tag{21.27}$$

and

$$K(t, u) = \frac{2}{\pi}\int_0^\infty \Bigg\{ \frac{c\xi\left[I_1(c\xi)K_0(c\xi) + I_0(c\xi)K_1(c\xi)\right] - 2I_1(c\xi)K_1(c\xi)}{I_1^2(c\xi)} \sinh\xi t \sinh\xi u$$

$$- \xi K_1(c\xi)I_1^{-1}(c\xi)\left[u \cosh\xi u \sinh\xi t + t \cosh\xi t \sinh\xi u\right]\Bigg\} d\xi \tag{21.28}$$

Using the facts that

$$I_1(c\xi)K_0(c\xi) + I_0(c\xi)K_1(c\xi) = (c\xi)^{-1} \tag{21.29}$$

and

$$\int_0^\infty I_1^{-2}(x) \sinh \alpha x \sinh \beta x = - \int_0^\infty \frac{d}{dx}\left[\frac{K_1(x)}{I_1(x)}\right] x \sinh \alpha x \sinh \beta x\, dx$$

$$= + \int_0^\infty \left\{ K_1(x) I_1^{-1}(x) \sinh \alpha x \sinh \beta x \right\} dx$$

$$+ \int_0^\infty \left\{ x K_1(x) I_1^{-1}(x)\left[\alpha \cosh\alpha x \sinh \beta x + \beta \cosh \beta x \sinh \alpha x\right]\right\} dx \tag{21.30}$$

it is easily seen that the integral equation may be written in the simpler form

$$g(t) = \frac{2}{\pi}\int_0^1 \frac{\rho f(\rho)\, d\rho}{\sqrt{t^2 - \rho^2}} + \frac{4}{c\pi^2}\int_0^1 g(u)\, K(t, u)\, du \tag{21.31}$$

where,

$$K(t, u) = \int_0^\infty \frac{K_1(\xi)}{I_1(\xi)} \sinh \frac{u}{c}\xi \sinh \frac{t}{c}\xi \, d\xi \tag{21.32}$$

§22. Numerical solution of the integral equation.

In this paragraph, we consider the numerical solution of the integral equation for the problem of the cylinder with clamped walls, and with a crack of unit radius developed inside the material of the cylinder, on the plane z = 0. We take the integral equation in the form

$$g(t) - \int_0^1 g(u)\, K(u, t)\, du = \frac{2}{\pi}\, t \qquad (22.1)$$

where K(u, t) is expressed in the form below, and we have taken $f(\rho) = 1$.

$$K(u, t) = \frac{2}{\pi^2 c} \int_0^\infty \frac{K_1(\xi)}{I_1(\xi)} \left[\cosh\left(\frac{u+t}{c}\right)\xi - \cosh\left(\frac{u-t}{c}\right)\xi \right] d\xi \qquad (22.2)$$

We must now compute the value of this kernel for a range of values of u and t, and having done so, to numerically integrate equation (22.1) to ascertain the values of g(t) for t ranging from zero to one. We shall tackle this problem in two distinct ways. Firstly we shall apply an approximate iterative procedure, by expanding the difference in hyperbolic cosines appearing in equation (22.2). Secondly we will apply a numerical integration procedure in an attempt to gain exact solutions. We begin with the iterative method.

(a) Iterative method.

In equation (22.1) write

$$g(t) = \frac{2}{\pi}\, h(t)\ . \qquad (22.3)$$

Then the integral equation may be written in the simpler form

$$h(t) - \int_0^1 h(u)\, K(u, t)\, du = t \qquad (22.4)$$

where K(u, t) is still given by equation (22.2).

If we now express h(t) in the following way

$$h(t) = t + \sum_{r=1}^{\infty} h_r(t) \qquad (22.5)$$

$$= \sum_{r=0}^{\infty} h_r(t)$$

where we have taken $h_0(t) = t$, and the $h_r(t)$ are polynomials in t, we find, by substitution in equation (22.4) that we have defined the iterative procedure for the determination of h (t)

$$h_0(t) = t$$

$$h_1(t) = \int_0^1 uK(u, t)\, du$$

$$h_2(t) = \int_0^1 h_1(t)\, K(u\,,\ t)\, du \qquad (22.6)$$

....

$$h_{r+1}(t) = \int_0^1 h_r(t)\, K(u\,,\ t)\, du$$

....

To proceed further we require to find an expression for the kernel $K(u, t)$. Expanding the difference in hyperbolic cosines in the interior of the expression for $K(u, t)$, we find that

$$\cosh\left(\frac{u+t}{c}\right)\xi - \cosh\left(\frac{u-t}{c}\right)\xi = \frac{2ut}{c^2}\cdot\xi^2 + \frac{1}{3c^4}(u^3t + ut^3)\xi^4$$

$$+ \frac{1}{180c^6}(3u^5t + 10u^3t^3 + 3ut^5)\xi^6$$

$$+ \frac{1}{2,702c^8}(u^7t + 7u^5t^3 + 7u^3t^5 + ut^7)\xi^8$$

$$+ O(c^{-10}). \qquad (22.7)$$

Thus, we may express $K(u, t)$ in the form

$$K(u, t) = \alpha_1(u)t + \alpha_3(u)t^3 + \alpha_5(u)t^5 + \alpha_7(u)t^7 + \ldots$$

where

$$\alpha_1(u) = \frac{2}{\pi^2 c}\left[\frac{2uT_2}{c^2} + \frac{u^3T_4}{3c^4} + \frac{u^5T_6}{60c^6} + \frac{u^7T_8}{2,520c^8} + \ldots\right] \qquad (22.8)$$

$$\alpha_3(u) = \frac{2}{\pi^2 c}\left[\frac{uT_4}{3c^4} + \frac{u^3T_6}{18c^6} + \frac{u^5T_8}{360c^8} + \ldots\right]$$

$$\alpha_5(u) = \frac{2}{\pi^2 c}\left[\frac{uT_6}{60c^6} + \frac{u^3T_8}{360c^8} + \ldots\right]$$

$$\alpha_7(u) = \frac{2}{\pi^2 c}\left[\frac{uT_8}{2,520c^8} + \ldots\right]$$

where we have written

$$T_i = \int_0^\infty \frac{K_1(\xi)}{I_1(\xi)}\,\xi^i\, d\xi \qquad (22.9)$$

and the values of the T_i are given by Tranter (1959)

From the form now taken by the function $K(u, t)$ it follows easily that the functions $h_r(t)$ may be written in the form

$$h_r(t) = P_1^r . t + P_3^r . t^3 + P_5^r . t^5 + P_7^r . t^7 + \dots \qquad (22.10)$$

where P_i^r is the coefficient of t^i in the expression for $h_r(t)$.

Suppose, for example, that c is sufficiently large to enable us to omit powers of t higher than the seventh. The next iteration $h_{r+1}(t)$ may then be obtained from equation (22.10) where the new coefficients will be determined in the following manner,

$$P_i^{r+1} = \int_0^1 \left[P_1^r . u + P_3^r . u^3 + P_5^r . u^5 + P_7^r . u^7 \right] \alpha_i(u)\,du. \qquad (22.11)$$

In the case of $h(t)$, the initial value, we take

$$P_1^0 = 1, \quad P_i = 0, \quad i \neq 1 \qquad (22.12)$$

Then, to find the coefficients of successive iterations we need only calculate the sixteen quantities

$$a_{s,r} = \int_0^1 u^s \alpha_r(u)\,du \quad , \quad \begin{array}{l} s = 1, 3, 5, 7 \\ r = 1, 3, 5, 7 \end{array} \qquad (22.13)$$

Then the general term $h_r(t)$ will be given by

$$h_r(t) = P_1^r . t + P_3^r . t^3 + P_5^r . t^5 + P_7^r . t^7 \qquad (22.14)$$

where

$$P_i^r = a_{1,i} P_1^{r-1} + a_{3,i} P_3^{r-1} + a_{5,i} P_5^{r-1} + a_{7,i} P_7^{r-1} .$$

We tabulate the sixteen quantities $a_{s,r}$, where we take

$$T_2 = 2.5033, 0 \qquad T_6 = 23.431$$

$$T_4 = 3.7713, 9 \qquad T_8 = 302.29 .$$

$$a_{1,1} = \frac{2}{\pi^2 c}\left[\frac{2T_2}{3c^2} + \frac{T_4}{15c^4} + \frac{T_6}{420c^6} + \frac{T_8}{22{,}680c^8}\right]$$

$$= \frac{.3382}{c^3} + \frac{.0509}{c^5} + \frac{.0013}{c^7} + \frac{.0026}{c^9}$$

$$a_{3,1} = \frac{2}{\pi^2 c}\left[\frac{2T_2}{5c^2} + \frac{T_4}{21c^4} + \frac{T_6}{540c^6} + \frac{T_8}{27{,}720c^8}\right]$$

$$= \frac{.2029}{c^3} + \frac{.0364}{c^5} + \frac{.0088}{c^7} + \frac{.0021}{c^9}$$

$$a_{5,1} = \frac{2}{\pi^2 c}\left[\frac{2T_2}{7c^2} + \frac{T_4}{27c^4} + \frac{T_6}{660c^6} + \frac{T_8}{32,760\,c^8}\right]$$

$$= \frac{.1449}{c^3} + \frac{.0283}{c^5} + \frac{.0072}{c^7} + \frac{.0019}{c^9}$$

$$a_{7,1} = \frac{2}{\pi^2 c}\left[\frac{2T_2}{9c^2} + \frac{T_4}{33c^4} + \frac{3T_6}{780c^6} + \frac{T_8}{37,800c^8}\right]$$

$$= \frac{.1127}{c^3} + \frac{.0232}{c^5} + \frac{.0061}{c^7} + \frac{.0016}{c^9}$$

$$a_{1,3} = \frac{2}{\pi^2 c}\left[\frac{T_4}{9c^4} + \frac{T_6}{90c^6} + \frac{T_8}{2,520c^8}\right]$$

$$= \frac{.0510}{c^5} + \frac{.0528}{c^7} + \frac{.0243}{c^9}$$

$$a_{3,3} = \frac{2}{\pi^2 c}\left[\frac{T_4}{15c^4} + \frac{T_6}{126c^6} + \frac{T_8}{3,240\,c^8}\right]$$

$$= \frac{.0510}{c^5} + \frac{.0377}{c^7} + \frac{.0190}{c^9}$$

$$a_{5,3} = \frac{2}{\pi^2 c}\left[\frac{T_4}{21c^4} + \frac{T_6}{162c^6} + \frac{T_8}{3960\,c^8}\right]$$

$$= \frac{.0364}{c^5} + \frac{.0293}{c^7} + \frac{.0155}{c^9}$$

$$a_{7,3} = \frac{2}{\pi^2 c}\left[\frac{T_4}{27c^4} + \frac{T_6}{198c^6} + \frac{T_8}{4,680c^8}\right]$$

$$= \frac{.0283}{c^5} + \frac{.0240}{c^7} + \frac{.0155}{c^9}$$

$$a_{1,5} = \frac{2}{\pi^2 c}\left[\frac{T_6}{180\,c^6} + \frac{T_8}{1800c^8}\right]$$

$$= \frac{.0264}{c^7} + \frac{.0340}{c^9}$$

$$a_{3,5} = \frac{2}{\pi^2 c}\left[\frac{T_6}{300c^6} + \frac{T_8}{2520c^8}\right]$$

$$= \frac{.0158}{c^7} + \frac{.0243}{c^9}$$

$$a_{5,5} = \frac{2}{\pi^2 c}\left[\frac{T_6}{420c^6} + \frac{T_8}{3240c^8}\right]$$

$$= \frac{.0113}{c^7} + \frac{.0176}{c^9}$$

$$a_{7,5} = \frac{2}{\pi^2 c}\left[\frac{T_6}{540c^6} + \frac{T_8}{3960c^8}\right]$$

$$= \frac{.0088}{c^7} + \frac{.0164}{c^9}$$

$$a_{1,7} = \frac{2}{\pi^2 c}\left[\frac{T_8}{7,560c^8}\right] = \frac{.0081}{c^9}$$

$$a_{3,7} = \frac{2}{\pi^2 c}\left[\frac{T_8}{12,600c^8}\right] = \frac{.0049}{c^9}$$

$$a_{5,7} = \frac{2}{c}\left[\frac{T_8}{17,640c^8}\right] = \frac{.0035}{c^9}$$

$$a_{7,7} = \frac{2}{c}\left[\frac{T_8}{22,680c^8}\right] = \frac{.0027}{c^9} .$$

(b) Numerical integration of the integral equation.

We develop now a numerical integration procedure to evaluate the solution g(t) of the integral equation at a number of points. The integral equation is written as

$$g(t) - \frac{2}{\pi^2 c}\int_0^1 g(u)K(u, t)\, du = \frac{2}{\pi} t \qquad (22.19)$$

where now

$$K(u, t) = \int_0^\infty \frac{K_1(\xi)}{I_1(\xi)}\left\{\cosh\frac{u+t}{c}\xi - \cosh\frac{u-t}{c}\xi\right\}d\xi \qquad (22.20)$$

Before we can integrate the equation (22.19), we must first of all evaluate K(u, t) at a number of values of u and t. Suppose that we wish to find the value at the n points

$$t = t_1,\ t_2,\ \ldots,\ t_n$$
$$u = u_1,\ u_2,\ \ldots,\ u_n$$

Then by the symmetry of K(u, t) in t and u we need calculate it only at $\frac{n(n+1)}{2}$ points. Let us now write

$$H(\lambda) = \int_0^\infty \frac{K_1(\xi)}{I_1(\xi)}\left[\cosh(\lambda\xi) - 1\right]d\xi \qquad (22.21)$$

so that

$$K(u, t) = H\left(\frac{u+t}{c}\right) - H\left(\frac{u-t}{c}\right) \qquad (22.22)$$

If the quantities u and t are spaced at the same interval then we require to calculate H(λ) at the $\frac{n(n-1)}{2}$ points

$$\lambda = 0,\ \frac{u_2+t}{c},\ \ldots\ \frac{u_n+t_n}{c}. \qquad (22.23)$$

If we write the spacing of the values of u and t as δ, then we require

$$\lambda = 0,\ \frac{\delta}{c},\ \ldots,\ \frac{2(n-1)\delta}{c} \qquad (22.24)$$

The functions given by

$$\frac{K_1(\xi)}{I_1(\xi)}\left[\cosh\lambda\xi - 1\right] \qquad (22.25)$$

were tabulated at intervals of (.05) from 0 to 5.1 and at intervals of (.1) from 5.1 onwards.

In the actual integration it was decided, to use Weddle's rule for numerical integration where

$$\int_a^b y\,dx = \frac{b-a}{20}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right] \qquad (22.26)$$

and the y_i are the values of the ordinates for $x = x_i$.

Using this rule the values of $H(\lambda)$ required were computed and are shown in Table A. From this table the values of $K(u, t)$ are easily calculated. It will be noticed from the integration formula that each interval we wish to integrate over is divided into a multiple of six divisions. Because of this it was decided to compute the kernel $K(u, t)$ over the ranges of t and u

$$t = 0 \;,\; \frac{1}{12} \;,\; \ldots \;,\; 1$$

$$u = 0 \;,\; \frac{1}{12} \;,\; \frac{2}{12} \;,\ldots, 1 \;.$$

λ =	.05	.1	.15	.2	.25	.3	.35
$H(\lambda)$ =	.0031,302	.0125,322	.0282,421	.0503,195	.0788,498	.1139,453	.1557,470
	.4	.45	.5	.55	.6	.65	.7
	.2044,248	.2601,858	.3232,728	.3939,712	.4726,147	.5595,919	.6553,551
	.75	.8	.85	.9	.95	1.0	1.05
	.7604,316	.8754,361	1.0010,887	1.1382,353	1.2878,760	1.4511,991	1.6296,279
	1.1	1.15	1.2				
	1.8248,817	2.0390,566	2.2747,370				

Table A.

t =	0	1/12	2/12	3/12	4/12	5/12	6/12
g(t) =	0	.0575,368	.1150,938	.1726,918	.2303,522	.2880,980	.3459,534
		7/12	8/12	9/12	10/12	11/12	1
		.4039,463	.4621,064	.5204,680	.5790,708	.6379,612	.6971,940

Table B.

Since we now have the values of $K(u, t)$ for a specified value of c, we may proceed to the solution of the integral equation. We do this by solving a set of simultaneous equations. Using the integration formula (22.26) and equations (22.19) together with the facts that $g(0) = 0$, $K(0, t) = K(u, 0) = 0$, we have a set of twelve simultaneous equations where the m^{th} equation is given by

$$\begin{aligned} g\left(\frac{m}{12}\right) - \frac{2}{\pi^2 c} \cdot \frac{1}{40} \Big[& 5g\left(\frac{1}{12}\right)K\left(\frac{1}{12}, \frac{m}{12}\right) + g\left(\frac{2}{12}\right)K\left(\frac{2}{12}, \frac{m}{12}\right) + 6g\left(\frac{3}{12}\right)K\left(\frac{3}{12}, \frac{m}{12}\right) \\ & + g\left(\frac{4}{12}\right)K\left(\frac{4}{12}, \frac{m}{12}\right) + 5g\left(\frac{5}{12}\right)K\left(\frac{5}{12}, \frac{m}{12}\right) + 2g\left(\frac{6}{12}\right)K\left(\frac{6}{12}, \frac{m}{12}\right) \\ & + 5g\left(\frac{7}{12}\right)K\left(\frac{7}{12}, \frac{m}{12}\right) + g\left(\frac{8}{12}\right)K\left(\frac{8}{12}, \frac{m}{12}\right) + 6g\left(\frac{9}{12}\right)K\left(\frac{9}{12}, \frac{m}{12}\right) \\ & + g\left(\frac{10}{12}\right)K\left(\frac{10}{12}, \frac{m}{12}\right) + 5g\left(\frac{11}{12}\right)K\left(\frac{11}{12}, \frac{m}{12}\right) + g\left(\frac{12}{12}\right)K\left(\frac{12}{12}, \frac{m}{12}\right) \Big] = \frac{2}{\pi} \frac{m}{12}. \end{aligned} \quad (22.27)$$

These equations were solved for the value $c = 5/3$. The values of $g(t)$ are shown in Table B and the graph of $g(t)$ in Fig.(22.2).

The energy required to form the crack is easily computed using the information already calculated. We have

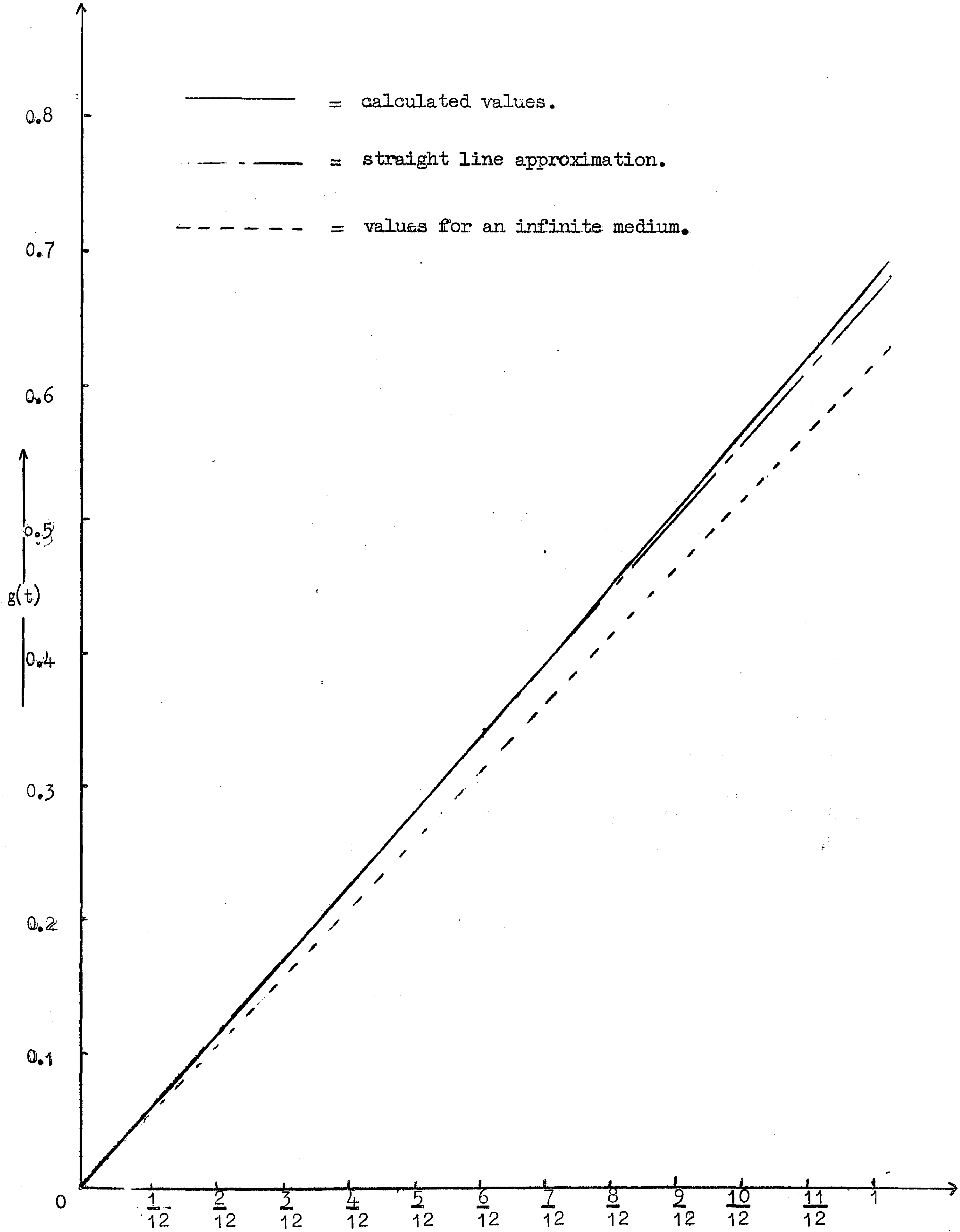
$$\begin{aligned}
\text{Energy} &= 2\pi \int_0^1 \rho \left[w(\rho)\right]_{z=0} d\rho \\
&= 4\pi(1 - \nu)\int_0^1 \rho\, d\rho \int_0^1 g(t)\,dt \int_0^\infty J_0(\eta\rho)\sin \eta t \, d\eta \\
&= 4\pi(1 - \nu)\int_0^1 \rho\, d\rho \int_\rho^1 g(t) \,/\sqrt{t^2 - \rho^2}\, dt
\end{aligned}$$

On changing the order of integration, the integral reduces to

$$\text{Energy} = 4\pi(1 - \nu) \int_0^1 t\, g(t)\, dt$$

From the given table it is easily found that the energy of the crack considered is thus

$$\text{Energy} = 2.9051\,(1 - \nu)$$



Values of the function g(t)

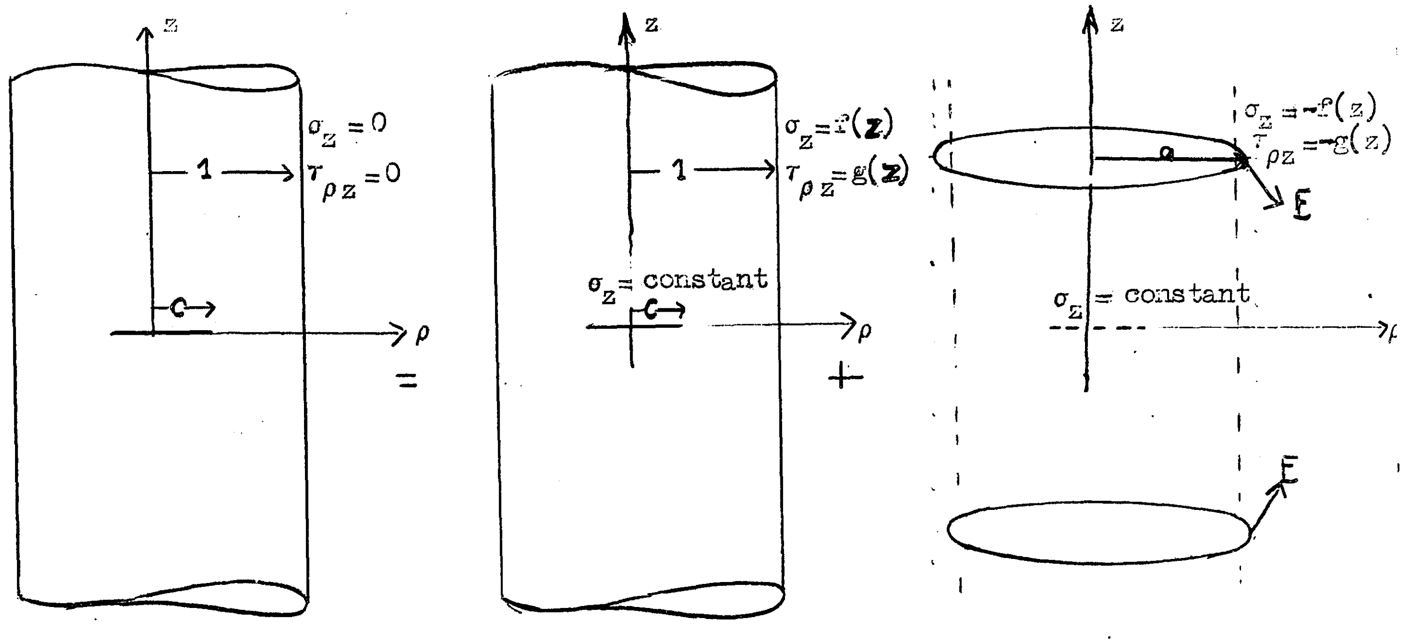
Fig 22.2

IV A suggestion for an approximate approach to the crack problem.

§23 In this paragraph, we consider the problem described in S17, but we now take the radius of the cylinder to be 1 and the radius of the crack to be c.

If the value of c is small compared to 1, the radius of the cylinder, then the problem reduces to that of a crack of radius c placed in an infinite medium, as far as the stresses and displacements induced in the material are concerned. This problem has been solved by Sneddon (1946). We shall attempt to discuss the case when c is of the order of .5 to .8 of the radius of the cylinder. We shall pay particular attention to the stresses in the immediate vicinity of the crack since this is where most interest is focussed.

The method which we shall use is an approximate one. We see from equation (17.1) that the stresses normal and tangential to the wall of the cylinder should be zero. We shall start with a solution for which this is not true, but for which the conditions on the plane $z = 0$ are satisfied. We shall then attempt to introduce suitable forces, such that the boundary conditions on the plane $z = 0$, will remain unaltered to any serious degree and which will cancel out the forces on the cylinder wall. If we can do this then we may add the two solutions to give us a complete solution of the problem, as shown diagrammatically.



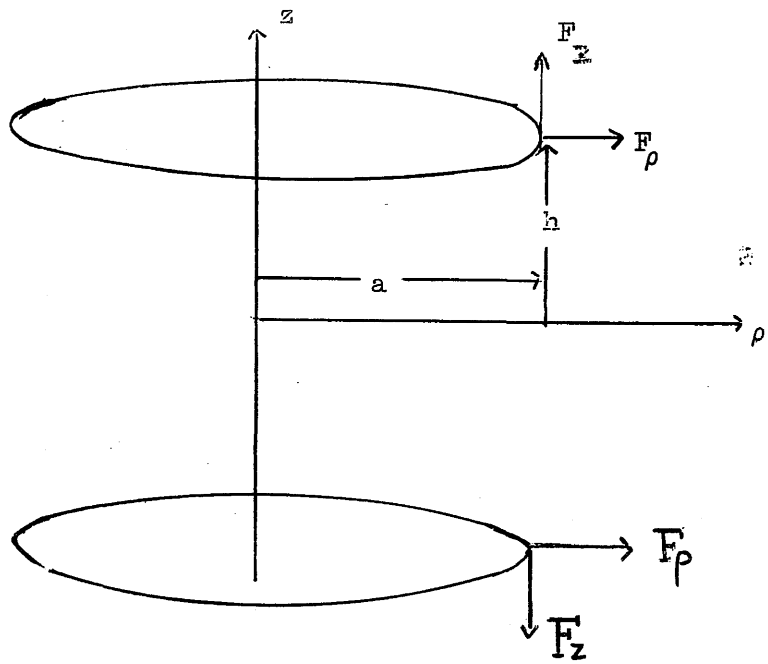
with a > 1.

Any system of forces which we choose to apply must satisfy the two conditions

(a) that it will lie outside the cylinder, so that we may consider the force as an imaginary or virtual one.

(b) that it will be symmetrical, since the stresses on the cylinder walls will be.

The simplest type of suitable force would then be a ring force with radius greater than unity, at a suitable distance from the plane $z = 0$, and with a suitable magnitude. Further, since we wish to maintain our conditions on the plane $z = 0$, it suggests that if we place a force at a height h above the plane $z = 0$, then we should place one at an equal distance below the plane. The type of system of forces we should then consider are as shown.



We see then that we have three parameters with which to work, the radius of the rings a_i, the height from plane $z = 0$, b_j and the magnitude of the forces P_{ij}. To proceed further we must

(i) Find the residual stresses on the cylinder walls from whichever solution we take.

(ii) Calculate the values of σ_z, $\tau_{\rho z}$, w on plane $z = 0$ due to the forces.

(iii) Calculate the values of σ_ρ, $\tau_{\rho z}$ on the cylinder $\rho = 1$ due to the forces.

Let us suppose that it is possible to find ring forces as described such that the conditions on $z = 0$ are not greatly affected. Further let us suppose that we have taken a solution which supplies the approximate solution $z = 0$.

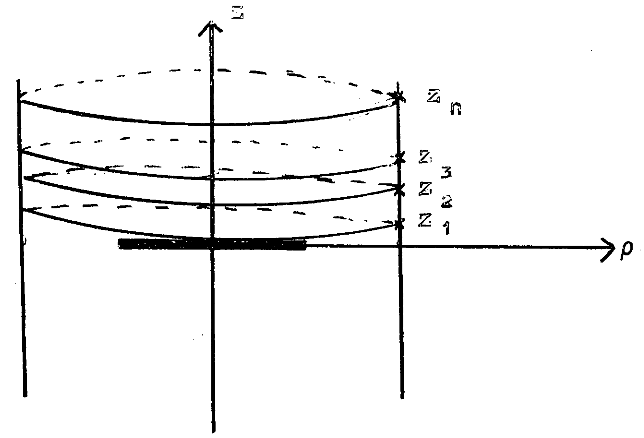
Let us consider the value of the residual stresses on the wall of the cylinder $\rho = 1$ at, say, n points given by the n values of z

$$z = z_1, z_2, \ldots z_n.$$

If we denote the value of the radial stress at the point z_k by σ_ρ^k and similarly let $\tau_{\rho z}^k$ be the value of the shear stress at the same point. Then we will have 2n quantities which we wish to reduce to zero viz.

$$\tau_{\rho z}^1, \quad \tau_{\rho z}^2, \ldots \tau_{\rho z}^n$$
$$\sigma_\rho^1 \quad \sigma_\rho^2 \quad \sigma_\rho^n$$

as illustrated.



Suppose now, that we take a set of ring forces. Let us take the pair of rings of radius a_s and at height b_t from plane $z = 0$. Let us take the magnitude to be P_{st}. We shall leave the actual size of the force P_{st} arbitrary for the moment and proceed to calculate the quantities $\frac{\sigma_\rho}{P_{st}}$ and $\frac{\tau_{\rho z}}{P_{st}}$ due to this particular set of rings. Suppose that at the point z_k, this set gives rise to a radial stress $\frac{\sigma_\rho(z_k)}{P_{st}}$ and let us denote this by $\sigma_{st}(k)$. Similarly we denote the corresponding shear stress at the same point by $\tau_{st}(k)$. Then, at the points $z_1, \ldots z_n$ on the cylinder $\rho = 1$, we shall get values of the radial and shear stress components

$$P_{st}\ \sigma_{st}(1) + P_{st}\ \sigma_{st}(2) \ldots\ ,\ P_{st}\ \sigma_{st}(n)$$
$$P_{st}\ \tau_{st}(1)\ , \ldots\ ,\ P_{st}\ \tau_{st}(n)$$

Suppose now that we set up <u>2n</u> sets of rings like this. We shall then obtain 2n arbitrary constants P_{ij} with which to reduce the resultant stress at any point to zero. We shall then formulate the <u>2n</u> simultaneous equations as follows.

$$\sum_i \sum_j P_{ij}\ \sigma_{ij}(k) + \sigma_\rho^k = 0$$
$$\sum_i \sum_j P_{ij}\ \tau_{ij}(k) + \tau_{\rho z}^k = 0$$

where P_{ij} is the magnitude of the force corresponding to the set of rings of radius a_i and distance from the $z = 0$ plane equal to b_j.

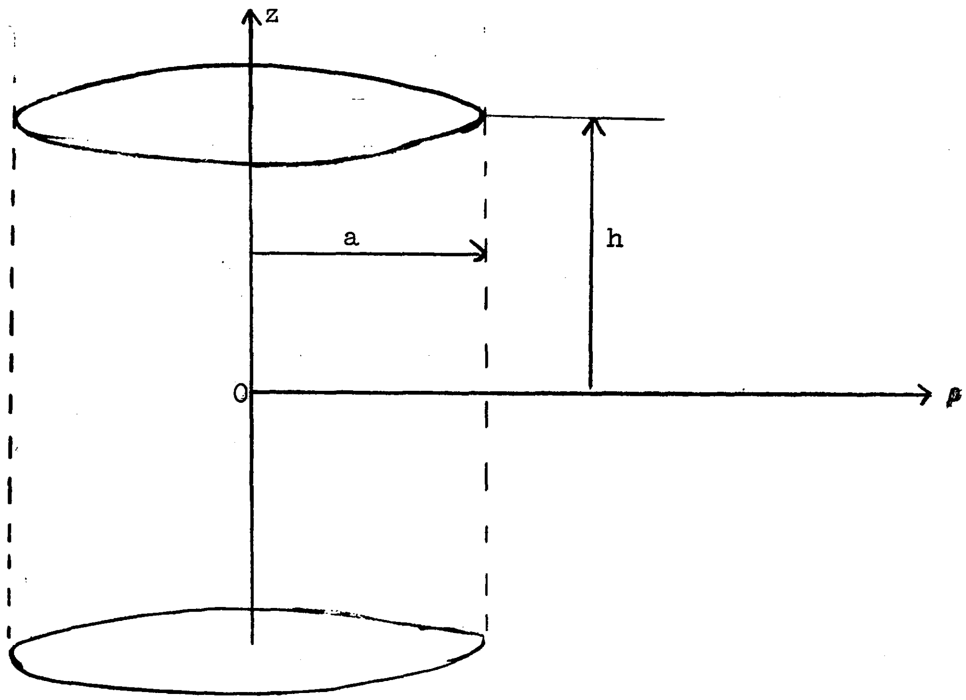
Having solved these equations for the P_{ij} we can redetermine the value of σ_z on $z = 0$. We can determine the degree to which we have approximated by considering whether σ_z is still constant over the area of the crack. We can also recalculate the radial and shear stresses on the cylinder $\rho = 1$ and ensure that we have zero stress there.

In the above argument we assumed that at each point z_k the values of the radial stress σ_ρ and shear stress $\tau_{\rho z}$ were required. In fact we could use the value of σ_ρ at points

$z_1, z_2, \ldots, z_k$ say, and the values of $\tau_{\rho z}$ at $z_{k+1}, \ldots, z_n$.

§ 24 Ring Forces.

We shall consider the effects on displacements and stresses in an infinite medium of two ring forces of radius a and placed symmetrically with respect to the plane $z = 0$ at heights $\pm h$ respectively. the situation is then as shown.



Since we have symmetry, we can choose axes (ρ , ϕ , z) with the z axis coinciding with the axis of symmetry. The four non vanishing components of stress σ_ρ, σ_ϕ, σ_z, $\tau_{\rho z}$, are then related to the displacements u_ρ and u_z by the equations

$$(\sigma_\rho, \sigma_\phi, \sigma_z) = (\beta^2 - 2)\Delta + 2\left(\frac{\partial u_\rho}{\partial \rho}, \frac{u_\rho}{\rho}, \frac{\partial u_z}{\partial z}\right) \tag{24.1}$$

$$\tau_{\rho z} = \left(\frac{\partial u_\rho}{\partial z} + \frac{\partial u_z}{\partial \rho}\right) \tag{24.2}$$

where the dilatation Δ is given by

$$\Delta = \frac{\partial u_\rho}{\sigma \rho} + \frac{\partial u_z}{\partial z} + \frac{u_\rho}{\rho} \tag{24.3}$$

β^2 is a constant, which is dimensionless and in our calculations will be taken to be 3. This corresponds to the case $\lambda = \mu$, i.e. to $\nu = \frac{1}{4}$.

The equations of equilibrium may be written in the form

$$\frac{\partial \sigma_\rho}{\sigma \rho} + \frac{\partial \tau_{\rho z}}{\partial z} + \frac{\sigma_\rho - \sigma_\phi}{\rho} + \sigma F_\rho = 0 \tag{24.4}$$

$$\frac{\partial \tau_{\rho z}}{\sigma \rho} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{\rho z}}{\rho} + \sigma F_z = 0 \tag{24.5}$$

where we have taken F_ρ and F_z to be the body forces acting in the ρ and z directions respectively. σ is the body density. The equations have been written in the form to conform with the solutions given to problems of this type by Eason, Fulton and Sneddon (1956)

The problem of choosing the values of F_ρ and F_z now presents itself. We must have symmetry about the plane $z = 0$ to disturb conditions on $z = 0$ as little as possible and we must have F_ρ, F_z in the form of delta functions to equate them to ring forces.

Choose $F_\rho = 0$, and write $F_z = Z$, where we write

$$Z = \frac{1}{\sigma} f(\rho, z) \tag{24.6}$$

The problem as treated by Eason, Fulton and Sneddon is formulated in terms of Fourier and Hankel transforms. We thus define

$$f(\xi, Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\zeta z}\, dz \int_0^{\infty} \rho \bar{f}(\rho, \zeta) J_0(\xi\rho)\, d\rho \tag{24.7}$$

In the theory quoted in the above paper, Z is a time dependent force and so we must remember when substituting in the equations there that

$$\bar{\bar{Z}}(\xi, \rho, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\omega\tau}\, \bar{Z}(\xi, \rho)\, d\tau$$

$$= \frac{\sqrt{2\pi}}{\sigma}\, \bar{f}(\xi, \zeta)\, \delta(\omega) \tag{24.8}$$

since $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega\tau}\, d\tau = \sqrt{2\pi}\, \delta(\omega)$ (24.9)

With the information given in (24.6)...(24.8) we solve the equations (24.4) and (24.5) following Eason, Fulton and Sneddon. These authors give the following solutions in the case when $F_z = Z = 1/\sigma \;\; f(\rho, z)$

$$u_\rho = \frac{(\beta^2 - 1)}{\sqrt{2\pi}\,\beta^2}\int_{-\infty}^{\infty} i\zeta\, e^{-i\zeta z}\, d\zeta \int_0^\infty \frac{\xi^2 J_1(\xi\rho)\bar{f}(\xi, \zeta)}{(\xi^2 + \zeta^2)^2}\, d\xi \tag{24.10}$$

$$u_z = \frac{1}{\sqrt{2\pi}\,\beta^2}\int_{-\infty}^{\infty} e^{-i\zeta z}\, d\zeta \int_0^\infty \frac{\xi(\beta^2\xi^2 + \zeta^2) J_0(\xi\rho)\bar{f}(\xi, \zeta)}{(\xi^2 + \zeta^2)^2}\, d\xi \tag{24.11}$$

and the stresses are given by

$$\sigma_z = \frac{-1}{\sqrt{2\pi}\,\beta^2}\int_{-\infty}^{\infty} i\zeta\, e^{-i\zeta z}\, d\zeta \int_0^\infty \frac{\xi J_0(\xi\rho)\left[\beta^2\zeta^2 + (3\beta^2 - 2)\xi^2\right]\bar{f}(\xi, \zeta)}{\xi^2 \quad {}^2 \;\; {}^2}\, d\xi \tag{24.12}$$

$$\tau_{\rho z} = \frac{-1}{\sqrt{2\pi}\,\beta^2}\int_{-\infty}^{\infty} e^{-i\zeta z}\, d\zeta \int_0^\infty \frac{\xi^2 J_1(\xi\rho)\left[\beta^2\xi^2 - (\beta^2 - 2)\zeta^2\right]\bar{f}(\xi, \zeta)}{(\xi^2 + \zeta^2)^2}\, d\xi \tag{24.13}$$

$$\sigma_\rho = \frac{-(\beta^2 - 1)}{\rho\,\beta^2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} i\zeta\, e^{-i\zeta z}\, d\zeta \int_0^\infty \frac{\xi^2 J_1(\xi\rho)\,\bar{f}(\xi, \zeta)}{(\xi^2 + \zeta^2)^2}\, d\xi$$

$$- \frac{1}{\sqrt{2\pi}\,\beta^2}\int_{-\infty}^{\infty} i\zeta\, e^{-i\zeta z}\, d\zeta \int_0^\infty \frac{\xi\left[(\beta^2 - 2)\zeta^2 - \beta^2\xi^2\right] J_0(\xi\rho)\,\bar{f}(\xi, \zeta)}{(\xi^2 + \zeta^2)^2}\, d\xi \tag{24.14}$$

Now let us choose the particular form of $f(\rho, z)$

$$f(\rho, z) = \frac{P}{2\pi\rho}\;\delta(\rho - a)\left[\delta(z - h) - \delta(z + h)\right] \tag{24.15}$$

so that

$$\bar{f}(\xi, \zeta) = \frac{P}{2\,\pi^{3/2}}\; J_0(\xi a)\left[e^{i\zeta h} - e^{-i\zeta h}\right] \tag{24.16}$$

Before proceeding any further, it would be as well to evaluate a few integrals which will be needed. Also we need to define a few terms.

$$\left.\begin{aligned} \text{Let}\quad \theta &= z - h \\ \phi &= z + h \\ \operatorname{sgn}\theta &= \frac{\theta}{|\theta|} \end{aligned}\right] \tag{24.17}$$

The integrals which we will require are then given by

$$N_1(\theta) = \int_{-\infty}^{\infty} i\zeta \frac{e^{-i\theta\zeta}}{\xi^2+\zeta^2}\, d\zeta = \pi \operatorname{sgn}\theta\, e^{-\xi|\theta|} \tag{24.18}$$

$$N_2(\theta) = \xi^2 \int_{-\infty}^{\infty} i\zeta \frac{e^{-i\theta\zeta}}{(\xi^2+\zeta^2)}\, d\zeta = \frac{\pi}{2}\, \xi\theta e^{-\xi|\theta|} \tag{24.19}$$

$$N_3(\theta) = \xi \int_{-\infty}^{\infty} \frac{e^{-i\theta}}{(\xi^2+\zeta^2)}\, d \quad = \pi e^{-\xi|\theta|} \tag{24.20}$$

$$N_4(\theta) = \xi^3 \int_{-\infty}^{\infty} \frac{e^{-i\theta\zeta}}{(\xi^2+\zeta^2)^2}\, d\zeta = \frac{\pi}{2}(1 + \xi|\theta|)e^{-\xi|\theta|} \tag{24.21}$$

The quantities in which we are interested may now be calculated.

$$w = u_z = \frac{P}{4\pi^2\beta^2} \int_0^{\infty} \xi J_0(\xi\rho)\, J_0(\xi a)\, .\, I_1\, .\, d\xi$$

where

$$I_1 = \int_{-\infty}^{\infty} (e^{i\zeta h} - e^{-i\zeta h})e^{-i\zeta z} \frac{(\beta^2\xi^2+\zeta^2)}{(\xi^2+\zeta^2)^2}\, d\zeta$$

$$= \int_{-\infty}^{\infty} (e^{-i\zeta\theta} - e^{-i\zeta\phi})\left[\frac{1}{(\xi^2+\zeta^2)} + (\beta^2-1)\,\xi^2 . \frac{1}{(\xi^2+\zeta^2)^2}\right] d\zeta$$

$$= \frac{1}{\xi}\left[N_3(\theta) - N_3(\phi)\right] + \frac{(\beta^2-1)}{\xi}\left[N_4(\theta) - N_4(\phi)\right]$$

$$= \frac{\pi}{\xi}\left[(e^{-\xi|\theta|} - e^{-\xi|\phi|}) + \frac{(\beta^2-1)}{2}\left[(1+\xi|\theta|)e^{-\xi|\theta|} - (1+\xi|\phi|)e^{-\xi|\phi|}\right]\right.$$

$$w = \frac{P}{8\pi\beta^2}\int_0^{\infty} J_0(\xi\rho)J_0(\xi a)\left[\left\{(\beta^2+1) + (\beta^2-1)\xi|\theta|\right\}e^{-\xi|\theta|}\right.$$

$$\left. - \left\{(\beta^2+1) + (\beta^2-1)\xi|\phi|\right\} e^{-\xi|\phi|}\right] d\xi.$$

Thus <u>w = 0</u>, on <u>z = 0</u> (24.22)

while

$$\sigma_z = \frac{-P}{4\pi\beta^2}\int_0^{\infty} \xi J_0(\xi\rho)J_0(\xi a)\left[(\beta^2 \operatorname{sgn}\theta + (\beta^2-1)\xi\theta)e^{-\xi|\theta|}\right.$$
$$\left. -(\beta^2 \operatorname{sgn}\phi + (\beta^2-1)\xi\phi)e^{-\xi|\phi|}\right] d\xi \tag{24.23}$$

$$\tau_{\rho z} = \frac{-P}{4\pi\beta^2}\int_0^{\infty} \xi J_1(\xi\rho)J_0(\xi a)\left[(1 + (\beta^2-1)\xi|\theta|)e^{-\xi|\theta|}\right.$$
$$\left. -(1 + (\beta^2-1)|\phi|)e^{-\xi|\phi|}\right] d\xi \tag{24.24}$$

and

$$\sigma_\rho = -\frac{(\beta^2-1)P}{4\pi\beta^2}\int_0^\infty \xi J_1(\xi\rho)J_0(\xi a)\left[\theta e^{-\xi|\theta|} - \phi e^{-\xi|\phi|}\right]d\xi$$

$$-\frac{P}{4\pi\beta^2}\int_0^\infty \xi J_0(\xi\rho)J_0(\xi a)\left[\left\{(\beta^2-2)\operatorname{sgn}\theta - (\beta^2-1)\xi\theta\right\}e^{-\xi|\theta|} - \left\{(\beta^2-2)\operatorname{sgn}\phi - (\beta^2-1)\xi\phi\right\}e^{-\xi|\phi|}\right]d\xi \qquad (24.25)$$

These are the general stresses and displacements throughout the body. We now wish to make a closer examination of them on the plane $z = 0$ and on the cylinder $\rho = 1$. On $z = 0$,

$$W = \tau_{\rho z} = 0 \qquad (24.26)$$

$$\sigma_z = \frac{P}{2\pi\beta^2}\int_0^\infty \xi J_0(\xi\rho)J_0(\xi a)\left[\beta^2 + (\beta^2-1)\xi h\right]e^{-\xi h}\,d\xi \qquad (24.27)$$

and on $\rho = 1$

$$\tau_{\rho z} = -\frac{P}{4\pi\beta^2}\int_0^\infty \xi J_1(\xi)J_0(\xi a)\left[\left\{1 + (\beta^2-1)\xi|\theta|\right\}e^{-\xi|\theta|} - \left\{1 + (\beta^2-1)|\phi|\right\}e^{-\xi|\phi|}\right]d\xi \qquad (24.28)$$

$$\sigma_\rho = -\frac{(\beta^2-1)P}{4\pi\beta^2}\int_0^\infty \xi J_1(\xi)J_0(\xi a)\left[\theta e^{-\xi|\theta|} - \phi e^{-\xi|\phi|}\right]d\xi$$

$$-\frac{P}{4\pi\beta^2}\int_0^\infty \xi J_0(\xi)J_0(\xi a)\left[\left\{(\beta^2-2)\operatorname{sgn}\theta - (\beta^2-1)\xi\theta\right\}e^{-\xi|\theta|} - \left\{(\beta^2-2)\operatorname{sgn}\phi - (\beta^2-1)\xi\phi\right\}e^{-\xi|\phi|}\right]d\xi. \qquad (24.29)$$

Now change the variable from ξ to (ξa) and write the integrals

$$J_{\mu,\nu}(p, q) = \int_0^\infty s^\mu J_\nu(sp)\,J_0(s)e^{-qs}\,ds \qquad (24.30)$$

Then our conditions become; On $z = 0$:

$$\omega = \tau_{\rho z} = 0$$

$$\sigma_z = \frac{P}{2\pi\beta^2 a^2}\left[\beta^2 J_{1,0}\left(\frac{\rho}{a}, \frac{h}{a}\right) + (\beta^2-1)\frac{h}{a}J_{2,0}\left(\frac{\rho}{a}, \frac{h}{a}\right)\right] \qquad (24.31)$$

while on $\rho = 1$.

$$\sigma_\rho = -\frac{(\beta^2-1)P}{4\pi\beta^2 a^2}\left[\theta J_{1,1}\left(\frac{1}{a}, \frac{|\theta|}{a}\right) - J_{1,1}\left(\frac{1}{a}, \frac{|\phi|}{a}\right)\right]$$

$$- \frac{P}{4\pi\beta^2 a^2}\left[(\beta^2-2)\operatorname{sgn}\theta \; J_{1,0}\left(\frac{1}{a}, \frac{|\theta|}{a}\right) - (\beta^2-1)\frac{\theta}{a} J_{2,0}\left(\frac{1}{a}, \frac{|\theta|}{a}\right)\right.$$

$$\left. - (\beta^2-2)\operatorname{sgn}\phi \; J_{1,0}\left(\frac{1}{a}, \frac{|\phi|}{a}\right) + (\beta^2-1)\frac{\phi}{a} J_{2,0}\left(\frac{1}{a}, \frac{|\phi|}{a}\right)\right] \qquad (24.32)$$

$$\tau_{\rho z} = -\frac{P}{4\pi\beta^2 a^2}\left[J_{1,1}\left(\frac{1}{a}, \frac{|\theta|}{a}\right) + (\beta^2-1)\frac{|\theta|}{a} J_{2,1}\left(\frac{1}{a}, \frac{|\theta|}{a}\right)\right.$$

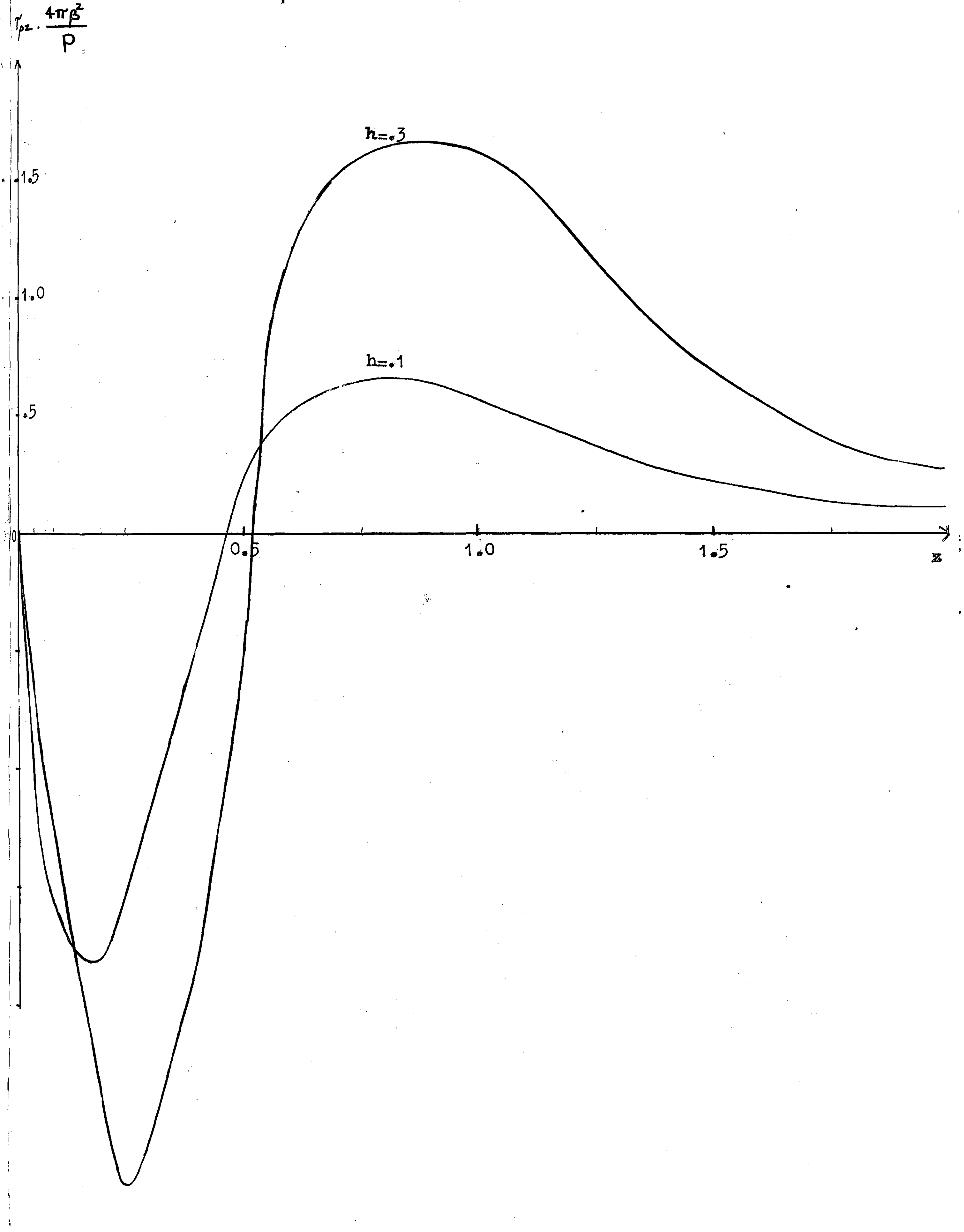
$$\left. - J_{1,1}\left(\frac{1}{a}, \frac{|\phi|}{a}\right) - (\beta^2-1)\frac{|\phi|}{a} J_{2,1}\left(\frac{1}{a}, \frac{|\phi|}{a}\right)\right] \qquad (24.33)$$

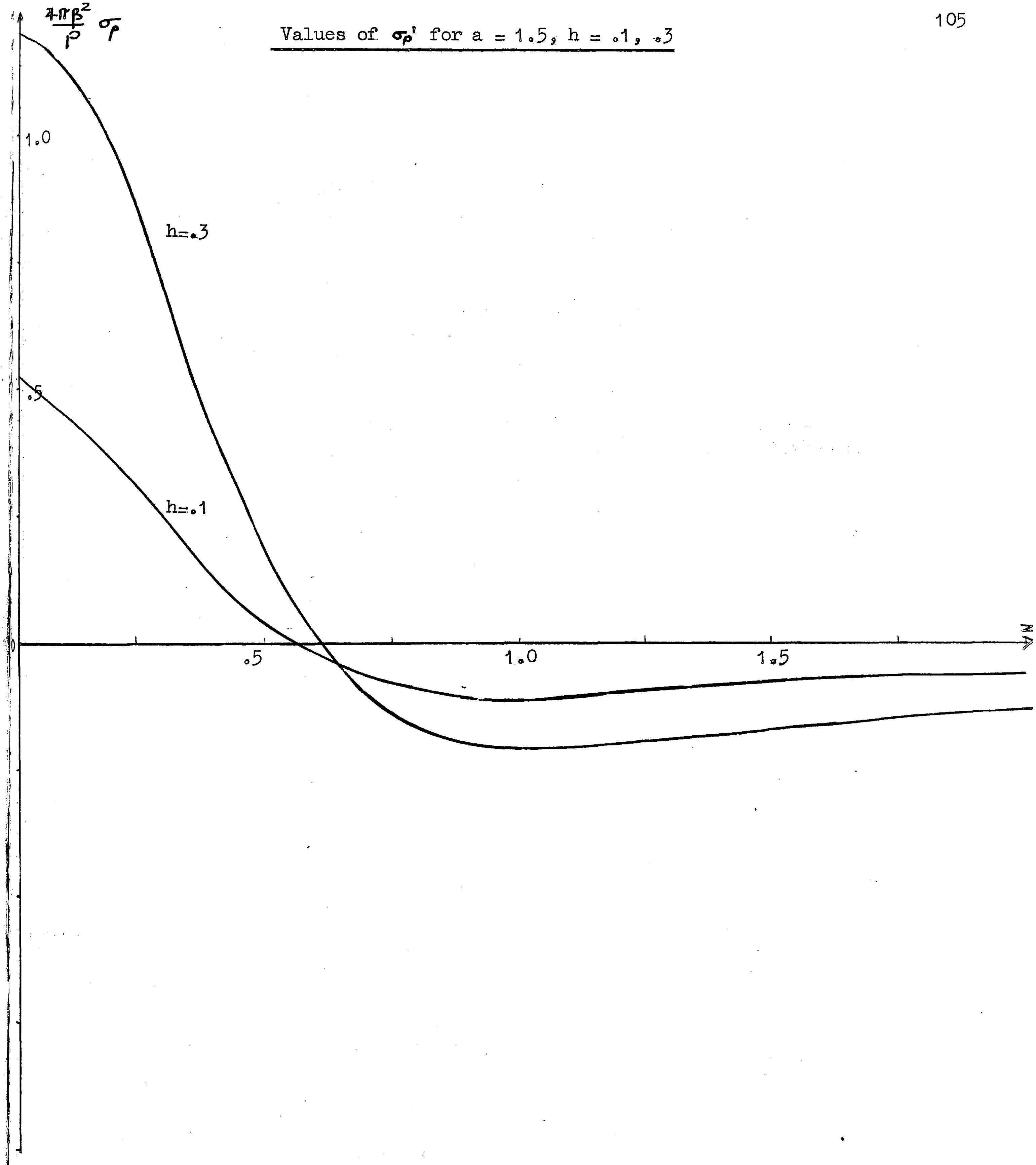
As an example of the type of curves obtained we draw graphs of σ_ρ and $\tau_{\rho z}$ for the values $a = 1.5$, and $h = .1, .3$.

z	0	.05	.10	.15	.20	.25	.30	.35	.40
$\tau_{\rho z}'$	0	−0.7616	−1.6515	−1.7117	−1.7719	−1.5935	−1.2618	−0.8660	−0.4759
σ_ρ'	0.5101	0.4979	0.4631	0.4109	0.3483	0.2820	0.2179	0.1596	0.1091
		.45	.50	.60	.70	.80	.90	1.00	2.00
		−0.1346	0.1389	0.4836	0.6218	0.6393	0.5974	0.5319	0.0981
		0.0668	0.0324	−0.0168	−0.0466	−0.0639	−0.0733	−0.0777	−0.0493
z	0	.05	.10	.15	.20	.25	.30	.35	.40
$\tau_{\rho z}'$	0	−0.6434	−1.2618	−1.8161	−2.2478	−2.4898	−2.7744	−2.2351	−1.7643
σ_ρ'	1.2066	1.1908	1.1441	1.0684	0.9674	0.8467	0.7133	0.5754	0.4405
		.45	.50	.60	.70	.80	.90	1.00	2.00
		−1.1554	−0.5010	0.6470	1.3582	1.6548	1.6804	1.5651	0.3064
		0.3152	0.2037	0.0284	−0.0875	−0.1584	−0.1989	−0.2198	−0.1484

where we have written $\tau_{\rho z}'$, σ_ρ' for $4\pi\beta^2\tau_{\rho z}/P$ and $4\pi\beta^2\sigma_\rho/P$ respectively.

Values of $\tau_{\rho z}'$ for a = 1.5, h = .1, and .3





Values of σ_ρ' for a = 1.5, h = .1, .3

where we have written $\rho_1 = \rho / c$, $\zeta = z/c$ and C_n^m, S_n^m denote the integrals

$$C_n^m = \int_0^\infty \eta^{n-1} e^{-\zeta\eta} J_m(\rho, \eta) \cos\eta \, d\eta \qquad (25.7)$$

$$S_n^m = \int_0^\infty \eta^{n-1} e^{-\zeta\eta} J_m(\rho, \eta) \sin\eta \, d\eta \qquad (25.8)$$

which can be evaluated as shown in the paper.

The values of σ_ρ / p_0 and $\tau_{\rho z} / p_0$ are given as follows, in the case of c = .625.

z	0.0	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.0	1.125	1.250	1.875
σ_ρ / p_0	0.108	0.072	0.005	-0.043	-0.054	-0.054	-0.048	-0.039	-0.032	-0.024	-0.018	-0.005

z	0.0	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.0	1.125	1.250	1.875
$\tau_{\rho z} / p_0$	0.0	0.030	0.020	-0.006	-0.029	-0.046	-0.047	-0.048	-0.044	-0.040	-0.035	-0.017

Having determined these curves, the idea of the suggested method is to solve the equations given in § 23 using the values of a number of curves similar to those given at the end of § 24. This has not yet been carried out.

§25. Solution for a crack in an infinite medium.

We take the solution given by Sneddon (1946) for a penny shaped crack, similar to that described in the last paragraph, of radius c lying in an infinite medium. For a crack of this shape cylindrical co-ordinates (ρ , ϕ , z) were employed since there is symmetry about the z axis. In this co-ordinate system the displacement vector may be written as (u_ρ, 0 , u_z) and the stress in the interior of the medium will be completely specified by the four stress components σ_ρ , σ_ϕ , σ_z, $\tau_{\rho z}$ since the remaining components are identically zero.

The stress produced in the neighbourhood of the crack is the same as that produced in a semi-infinite medium by the boundary conditions

$$\left.\begin{array}{ll} \tau_{\rho z} = 0 & \text{for all } \rho \\ \sigma_z = -p(\rho) & 0 \leqslant \rho < c \\ u_z = 0 & c < \rho < \infty \end{array}\right] \quad \text{on } z = 0 \tag{25.1}$$

If we take $p(\rho) = p_0$, a constant we see that on z = 0, we have the conditions we require for the problem stated in §17. However on the cylindrical surface $\rho = 1$, we shall find finite values of the radial and shear stress. If we follow through the problem as treated by Sneddon for a constant internal pressure we find the following solution

$$u_\rho = \frac{p_0 c}{\pi(\beta^2-1)} \int_0^\infty \left(\frac{1}{(\beta^2-1)} - \zeta\eta \right) \frac{d}{d\eta}\left(\frac{\sin\eta}{\eta}\right) e^{-\zeta\eta} J_1(\rho_1, \eta)\, d\eta \tag{25.2}$$

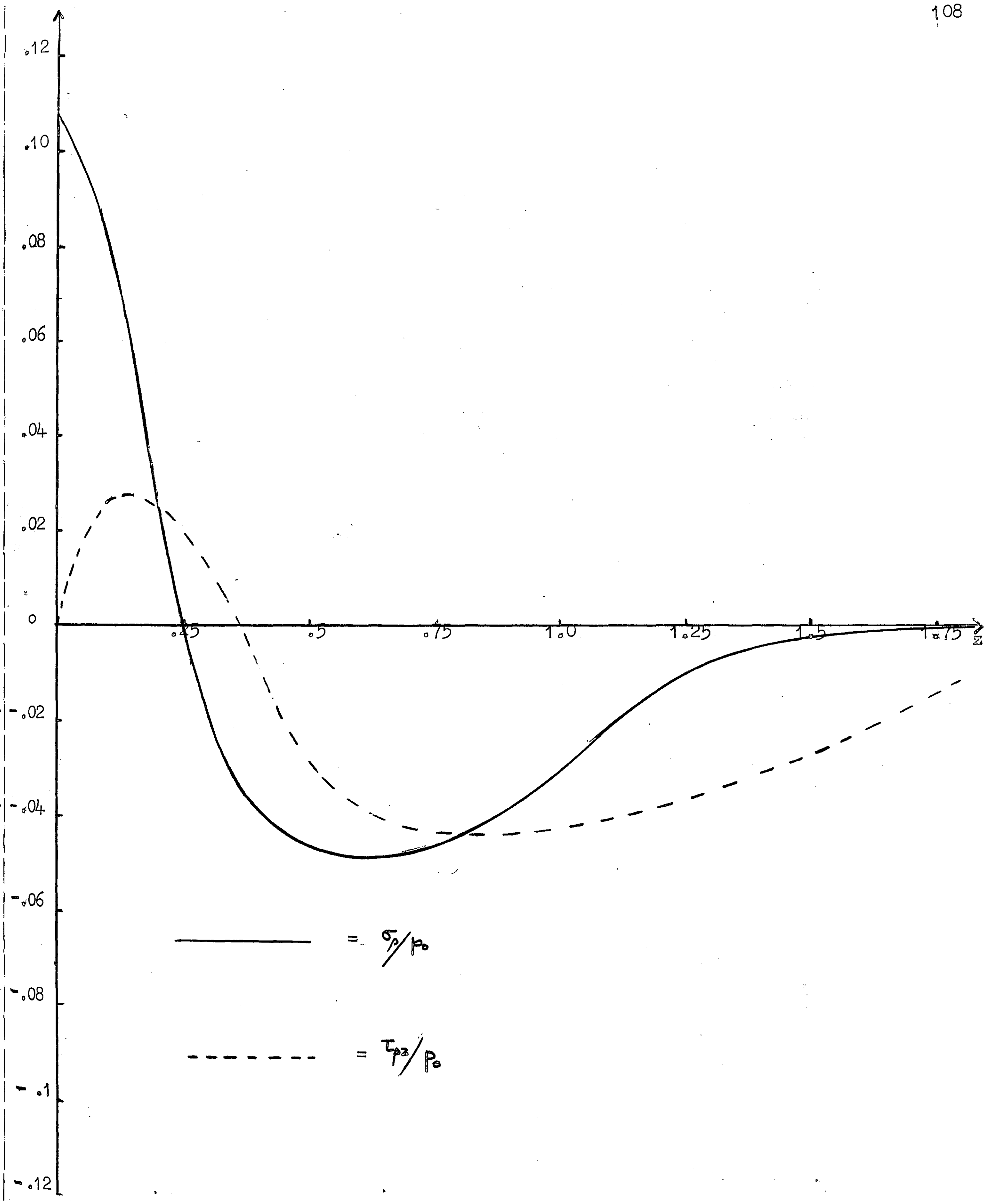
$$u_z = \frac{-p_0 c \beta^2}{\pi(\beta^2-1)} \int_0^\infty \left(1 + \frac{\beta^2-1}{\beta^2}\zeta\eta \right) \frac{d}{d\eta}\left(\frac{\sin\eta}{\eta}\right) e^{-\zeta\eta} J_0(\rho_1, \eta)\, d\eta \tag{25.3}$$

where we take $\beta^2 = 3$ and the components of stress we require are given by

$$\sigma_z = \frac{2p_0}{\pi}\left[C_1^0(\rho_1, \zeta) - S_0^0(\rho_1, \zeta) + \zeta C_2^0(\rho_1, \zeta) - \zeta S_1^0(\rho_1, \zeta) \right] \tag{25.4}$$

$$\tau_{\rho z} = \frac{2p_0\zeta}{\pi}\left[C_2^1(\rho_1, \zeta) - S_1^1(\rho_1, \zeta) \right] \tag{25.5}$$

$$\begin{aligned} \sigma_\rho = \frac{p_0}{\pi}\Bigg[& \frac{3\beta^2-4}{\beta^2-1}\left\{ C_1^0(\rho_1, \zeta) - S_0^0(\rho_1, \zeta) \right\} - \frac{1}{(\beta^2-1)}\left\{ C_1^2(\rho_1, \zeta) - S_0^2(\rho_1, \zeta) \right\} \\ & + \zeta\left\{ C_2^2(\rho_1, \zeta) - S_1^2(\rho_1, \zeta) - C_1^0(\rho_1, \zeta) + S_0^0(\rho_1, \zeta) - C_2^0(\rho_1, \zeta) \right. \\ & \left. + S_1^0(\rho_1, \zeta) \right\} \end{aligned} \tag{25.6}$$



Values of $\frac{\sigma_\rho}{p_0}$ and $\frac{\tau_{\rho z}}{p_0}$ on the wall of the cylinder.

§26. Appendix.

In the previous paragraphs we have made use of the integrals $J_{\mu,\nu}(p, q)$ for values of $\mu = 1, 2$ and $\nu = 0, 1$ for a range of values of (p, q) where

$$J_{\mu,\nu}(p, q) = \int_0^\infty s^\mu J_\nu(sp) J_0(s) e^{-qs}\, ds \tag{26.1}$$

The integral given by equation (26.1) is a particular case of integrals of the type

$$I(\mu, \nu, \lambda) = \int_0^\infty J_\mu(at) J_\nu(bt) e^{-ct} t^\lambda\, dt \tag{26.2}$$

considered by Eason, Noble, and Sneddon (1955). Since we are concerned only with the simpler form of integral (26.1) and since we require many more values of the parameters than are given by Eason, Noble, and Sneddon it would seem as well to streamline their method for this particular problem.

For the values of μ and ν we require, the integral (26.1) converges. Following Eason, Noble, and Sneddon we make a transformation of $J_{\mu,\nu}(p, q)$ in the following way. Write

$$J_\eta(R)\left(\frac{A - Be^{-i\theta}}{A - Be^{+i\theta}}\right)^{\frac{1}{2}\eta} = \sum_{m=-\infty}^{\infty} J_{\eta+m}(A) J_m(B) e^{im\theta} \tag{26.3}$$

where $R^2 = A^2 + B^2 - 2AB\cos\theta = (A - Be^{-i\theta})(A - Be^{+i\theta})$. (Watson, 1944, p.359).

Since η is to be taken as integral, there are no restrictions.

If we now multiply numerator and denominator of the fraction on the left of equation (26.3) by $(A - Be^{-i\theta})$, multiply both sides by $e^{-i\eta\theta}$ and integrate with respect to θ from 0 to 2π we have

$$J_{\eta+n}(A) J_n(B) = \frac{1}{\pi} \mathcal{R} \int_0^\pi \frac{J_\eta(R)}{R^\eta} (A - Be^{-i\theta})^\eta e^{-in\theta}\, d\theta \tag{26.4}$$

If we now write $A = sp$, $B = s$, $n = 0$, $\eta = \nu$ we find that

$$J_\nu(ps) J_0(s) = \frac{1}{\pi} \mathcal{R} \int_0^\pi \frac{J_\nu(rs)}{r^\nu} (p - e^{-i\theta})^\nu\, d\theta \tag{26.5}$$

where now $r^2 = p^2 + 1 - 2p\cos\theta$.

If we now substitute into equation (26.1) from equation (26.3) we have

$$J_{\mu,\nu}(p, q) = \int_0^\infty s^\mu e^{-qs}\, ds \, \frac{1}{\pi} \mathcal{R} \int_0^\pi \frac{J_\nu(rs)}{r^\nu} (p - e^{-i\theta})^\nu\, d\theta$$

$$= \frac{1}{\pi} \mathcal{R} \int_0^\pi (p - e^{-i\theta})^\nu\, d\theta \int_0^\infty \frac{s^\mu J_\nu(rs)}{r^\nu} e^{-qs}\, ds \tag{26.6}$$

Further the inner integral may be evaluated from Watson (1944).

$$\int_0^\infty s^\mu J_\nu (rs)\, e^{-qs}\, ds = \frac{(\frac{1}{2} r/q)^\nu \Gamma(\mu + \nu + 1)}{q^{\mu+1}\Gamma(\nu + 1)}\, {}_2F_1\left(\frac{\mu + \nu + 1}{2}, \frac{\mu + \nu + 2}{2}; \nu+1; -\frac{r^2}{q^2}\right) \quad (26.7)$$

On substituting from equation (26.7) into equation (26.6) and rearranging the terms we have

$$J_{\mu,\nu}(p, q) = \frac{\Gamma(\mu + \nu + 1)}{\Gamma(\nu + 1) 2^\nu \pi q^{\mu+\nu+1}} \Re \int_0 (p - e^{-i\theta})^\nu {}_2F_1\left(\frac{\mu + \nu + 1}{2}, \frac{\mu + \nu + 2}{2}; \nu + 1; -\frac{r^2}{q^2}\right) d\theta \quad (26.8)$$

Before proceeding further, it is as well to note several results which we shall need in the evaluation of the integrals we require. With the usual notation for elliptic integrals of the first and second kinds we may write

$$F(\alpha, \phi) = \int_0^\phi \frac{d\psi}{\Delta \psi}, \quad E(\alpha, \phi) = \int_0^\phi \Delta(\psi)\, d\psi \quad (26.9)$$

where $\Delta(\psi) = (1 - k^2 \sin^2 \psi)^{\frac{1}{2}}$ and $k = \sin \alpha$.

We are interested only in the case where $\phi = \pi/2$, i.e. the complete elliptic integrals. We now list results we shall require.

$$\int_0^{\pi/2} \frac{d\psi}{(1 - k^2 \sin^2 \psi)^{3/2}} = \frac{E}{k'^2}, \quad k'^2 = 1 - k^2 \quad (26.10)$$

$$\int_0^{\pi/2} \frac{\sin^2 \psi\, d\psi}{(1 - k^2 \sin^2 \psi)^{3/2}} = \frac{E - k'^2 F}{k^2 k'^2} \quad (26.11)$$

$$\int_0^{\pi/2} \frac{d\psi}{(1 - k^2 \sin^2 \psi)^{3/2}} = \frac{1}{3(1 - k^2)} (4E - F) \quad (26.12)$$

$$\int_0^{\pi/2} \frac{\sin^2 \psi\, d\psi}{(1 - k^2 \sin^2 \psi)^{3/2}} = \frac{E - F}{3k^2 k^2} + \frac{2E}{3k'^4} \quad (26.13)$$

In the following calculations we shall write

$$k^2 = \frac{4p}{(p + 1)^2 + q^2} \text{ with } k < 1. \quad (26.14)$$

We now evaluate the integrals $J_{1,0}(p, q)$, $J_{2,0}(p, q)$, $J_{1,1}(p, q)$, $J_{2,1}(p, q)$. We start with $J_{1,0}(p, q)$, with $\mu = 1$, $\nu = 0$.

$$J_{1,0}(p, q) = \frac{1}{\pi q^2} \int_0^\pi {}_2F_1(1, 3/2; 1; -r^2/q^2)\, d\theta$$

$$= \frac{1}{\pi q^2} \int_0^{\pi} \frac{d\theta}{(1 + r^2/q^2)^{3/2}}$$

$$= \frac{q}{4\pi p^{3/2}} \cdot \frac{k^3}{k'^2} E(k) \tag{26.15}$$

since $r^2 = p^2 + 1 - 2p\cos\theta$.

$$J_{2,0}(p, q) = \frac{2}{\pi q^3} \int_0^{\pi} {}_2F_1(3/2, 2; 1; -\frac{r^2}{q^2})\, d\theta.$$

Using the fact that

$$(1 + \frac{r^2}{q^2})\, {}_2F_1(3/2, 2; 1; -\frac{r^2}{q^2}) = (1 - \frac{r^2}{2q^2})(1 + \frac{r^2}{q^2})^{-3/2} \tag{26.16}$$

(Erdelyi, Vol.1).
We have

$$J_{2,0}(p, q) = \frac{q^2 k^5}{16\pi p^{3/2} k'^2} (4E - F) - \frac{k^3}{4\pi p^{3/2} k'^2} \cdot E(k). \tag{26.17}$$

Now taking $\mu = 1$, $\nu = 1$ we find that

$$J_{1,1}(p, q) = \frac{1}{\pi q^3} \mathcal{R} \int_0^{\infty} (p - e^{-i\theta})\, {}_2F_1(3/2, 2; 2; -\frac{r^2}{q^2}) d\theta$$

$$= \frac{1}{\pi q^3} \int_0^{\infty} \frac{(p - \cos\theta)}{(1 + r^2/q^2)^{3/2}} d\theta$$

$$= \frac{k}{4\pi p^{3/2}} \left[\frac{(p+1)}{k'^2} E(k) - \frac{2}{k^2 k'^2} \left(E(k) - k'^2 F(k) \right) \right] \tag{26.18}$$

and finally with $\mu = 2$, $\nu = 1$, we have

$$J_{2,1}(p, q) = \frac{3}{\pi q^4} \mathcal{R} \int_0^{\pi} (p - e^{-i\theta})\, {}_2F_1(2, 3/2; 2; -\frac{r^2}{q^2}) d\theta$$

$$= \frac{3}{\pi q^4} \int_0^{\pi} \frac{(p - \cos\theta)}{(1 + r^2/q^2)^{3/2}} d\theta$$

$$= \frac{qk^5}{16\pi p^{3/2}} \left[\frac{(p+1)}{k'^2} \left(4E(k) - F(k)\right) - \frac{2}{k^2 k'^2} \left[E(k) - F(k) \right] - \frac{4}{k^4} E \right] \tag{26.19}$$

We shall calculate the values of $J_{\mu\ \nu}(p, q)$ for $\mu = 1, 2$ and $\nu = 0, 1$ for ranges of values of $p = .025\ (.025)\ .5$, and $q = .025\ (.025)\ .5$. The first column in the following tables gives the values of p, the second the values of $J_{1,0}$, the third the values of $J_{2,0}$, the fourth the values of $J_{1,1}$, and the fifth the values of $J_{2,1}$.

.025	+0.0250	-0.9987	-0.0125	-0.0079
.050	+0.0251	-1.0030	-0.0250	-0.0162
.075	+0.0253	-1.0102	-0.0376	-0.0250
.100	+0.0255	-1.0204	-0.0504	-0.0345
.125	+0.0259	-1.0336	-0.0634	-0.0447
.150	+0.0263	-1.0501	-0.0767	-0.0560
.175	+0.0268	-1.0699	-0.0903	-0.0685
.200	+0.0273	-1.0935	-0.1044	-0.0824
.225	+0.0281	-1.1211	-0.1189	-0.0982
.250	+0.0289	-1.1531	-0.1340	-0.1162
.275	+0.0298	-1.1900	-0.1497	-0.1370
.300	+0.0308	-1.2322	-0.1662	-0.1611
.325	+0.0320	-1.2805	-0.1835	-0.1893
.350	+0.0334	-1.3358	-0.2019	-0.2227
.375	+0.0350	-1.3988	-0.2214	-0.2625
.400	+0.0368	-1.4710	-0.2422	-0.3103
.425	+0.3887	-1.5537	-0.2645	-0.3683
.450	+0.0412	-1.6487	-0.2886	-0.4395
.475	+0.0440	-1.7582	-0.3147	-0.5278
.500	+0.0472	-1.8852	-0.3432	-0.6383

q = .025

.025	+0.0499	-0.9907	-0.0124	-0.0158
.050	+0.0501	-0.9953	-0.0248	-0.0323
.075	+0.0504	-1.0027	-o.0373	-0.0497
.100	+0.0509	-1.0130	-0.0500	-0.0685
.125	+0.0516	-1.0263	-0.0629	-0.0888
.150	+0.0524	-1.0429	-0.0760	-0.1111
.175	+0.0534	-1.0628	-0.0895	-0.1359
.200	+0.0546	-1.0863	-0.1034	-0.1635
.225	+0.0559	-1.1138	-0.1178	-0.1948
.250	+0.0575	-1.1456	-0.1327	-0.2305
.275	+0.0593	-1.1822	-0.1482	-0.2716
.300	+0.0614	-1.2242	-0.1645	-0.3192
.325	+0.0638	-1.2721	-0.1816	-o.3750
.350	+0.0665	-1.3268	-0.1997	-0.4408
.375	+0.0697	-1.3892	-0.2189	-0.5192
.400	+0.0732	-1.4605	-0.2393	-0.6134
.425	+0.0773	-1.5421	-0.2612	-0.7275
.450	+0.0820	-1.6359	-0.2848	-0.8673
.475	+0.0874	-1.7438	-0.3103	-1.0402
.500	+0.0936	-1.8687	-0.3381	-1.2564

q = .050

.025	+0.0745	-0.9774	-0.0122	-0.0234
.050	+0.0748	-0.9825	-0.0244	-0.0478
.075	+0.0753	-0.9903	-0.0368	-0.0737
.100	+0.0760	-1.0008	-0.0493	-0.1015
.125	+0.0770	-1.0144	-0.0620	-0.1316
.150	+0.0782	-1.0310	-0.0749	-0.1646
.175	+0.0797	-1.0509	-0.0882	-0.2012
.200	+0.0814	-1.0744	-0.1019	-0.2421
.225	+0.0834	-1.1017	-0.1160	-0.2883
.250	+0.0857	-1.1333	-0.1306	-0.3409
.275	+0.0884	-1.1695	-0.1458	-0.4014
.300	+0.0915	-1.2109	-0.1617	-0.4715
.325	+0.0950	-1.2581	-0.1785	-0.5535
.350	+0.0991	-1.3120	-0.1961	-0.6500
.375	+0.1036	-1.3733	-0.2148	-0.7648
.400	+0.1089	-1.4433	-0.2346	-0.9024
.425	+0.1149	-1.5232	-0.2558	-1.0688
.450	+0.1217	-1.6148	-0.2786	-1.2721
.475	+0.1296	-1.7201	-0.3032	-1.5229
.500	+0.1387	-1.8417	-0.3298	-1.8356

q = .075

.025	+0.0987	-0.9591	-0.0120	-0.0307
.050	+0.0991	-0.9648	-0.0240	-0.0627
.075	+0.0997	-0.9731	-0.0361	-0.0967
.100	+0.1007	-0.9840	-0.0483	-0.1331
.125	+0.1020	-0.9978	-0.0607	-0.1725
.150	+0.1035	-1.0146	-0.0734	-0.2157
.175	+0.1054	-1.0346	-0.0864	-0.2635
.200	+0.1077	-1.0579	-0.0997	-0.3170
.225	+0.1103	-1.0850	-0.1135	-0.3772
.250	+0.1134	-1.1162	-0.1277	-0.4458
.275	+0.1169	-1.1019	-0.1425	-0.5244
.300	+0.1209	-1.1926	-0.1579	-0.6154
.325	+0.1255	-1.2389	-0.1741	-0.7215
.350	+0.1307	-1.2916	-0.1911	-0.8463
.375	+0.1367	-1.3515	-0.2091	-0.9943
.400	+0.1435	-1.4196	-0.2282	-1.1712
.425	+0.1512	-1.4973	-0.2485	-1.3847
.450	+0.1600	-1.5861	-0.2702	-1.6444
.475	+0.1702	-1.6878	-0.2935	-1.9636
.500	+0.1819	-1.8050	-0.3186	-2.3599

q = .100

.025	+0.1223	-0.9360	-0.0117	-0.0375
.050	+0.1208	-0.9425	-0.0234	-0.0768
.075	+0.1236	-0.9514	-0.0351	-0.1183
.100	+0.1248	-0.9629	-0.0471	-0.1628
.125	+0.1263	-0.9770	-0.0592	-0.2110
.150	+0.1283	-0.9939	-0.0715	-0.2637
.175	+0.1306	-1.0139	-0.0841	-0.3220
.200	+0.1333	-1.0372	-0.0970	-0.3870
.225	+0.1365	-1.0640	-0.1103	-0.4602
.250	+0.1402	-1.0947	-0.1241	-0.5404
.275	+0.1445	-1.1298	-0.1383	-0.6386
.300	+0.1494	-1.1696	-0.1532	-0.7485
.325	+0.1549	-1.2148	-0.1687	-0.8764
.350	+0.1613	-1.2663	-0.1850	-1.0263
.375	+0.1684	-1.3241	-0.2021	-1.2036
.400	+0.1766	-1.3900	-0.2202	-1.4049
.425	+0.1859	-1.4649	-0.2394	-1.6687
.450	+0.1965	-1.5502	-0.2598	-1.9764
.475	+0.2087	-1.6477	-0.2815	-2.3527
.500	+0.2226	-1.7594	-0.3048	-2.8173

q = .125

.025	+0.1453	-0.9085	-0.0113	-0.0439
.050	+0.1458	-0.9159	-0.0226	-0.0898
.075	+0.1468	-0.9256	-0.0341	-0.1384
.100	+0.1482	-0.9376	-0.0456	-0.1904
.125	+0.1500	-0.9521	-0.0573	-0.2466
.150	+0.1522	-0.9692	-0.0692	-0.3081
.175	+0.1549	-0.9893	-0.0813	-0.3759
.200	+0.1581	-1.0124	-0.0938	-0.4514
.225	+0.1618	-1.0389	-0.1066	-0.5363
.250	+0.1661	-1.0691	-0.1198	-0.6325
.275	+0.1710	-1.1034	-0.1334	-0.7424
.300	+0.1767	-1.1422	-0.1476	-0.8690
.325	+0.1832	-1.1861	-0.1623	-1.0157
.350	+0.1904	-1.2356	-0.1777	-1.1872
.375	+0.1987	-1.2916	-0.1939	-1.3894
.400	+0.2081	-1.3549	-0.2108	-1.6294
.425	+0.2188	-1.4266	-0.2287	-1.9164
.450	+0.2309	-1.5080	-0.2476	-2.2625
.475	+0.2447	-1.6006	-0.2676	
.500	+0.2605	-1.7062	-0.2888	

q = .150

.025	+0.1675	-0.8770	-0.0109	-0.0497
.050	+0.1681	-0.8854	-0.0218	-0.1017
.075	+0.1692	-0.8959	+0.0328	-0.1567
.100	+0.1708	-0.9085	-0.0439	-0.2155
.125	+0.1728	-0.9234	-0.0551	-0.2790
.150	+0.1753	-0.9409	-0.0666	-0.3482
.175	+0.1784	-0.9610	-0.0782	-0.4246
.200	+0.1819	-0.9839	-0.0901	-0.5095
.225	+0.1861	-1.0101	-0.1023	-0.6046
.250	+0.1910	-1.0397	-0.1149	-0.7122
.275	+0.1965	-1.0731	-0.1278	-0.8347
.300	+0.2028	-1.1108	-0.1412	-0.9754
.325	+0.2100	-1.1531	-0.1551	-1.1380
.350	+0.2181	-1.2009	-0.1695	-1.3274
.375	+0.2273	-1.2546	-0.1846	-1.5496
.400	+0.2377	-1.3150	-0.2003	-1.8122
.425	+0.2495	-1.3832	-0.2167	-2.1247
.450	+0.2628	-1.4603	-0.2340	-2.4994
.475	+0.2780	-1.5475	-0.2521	-2.9522
.500	+0.2952	-1.1647	-0.2712	-3.5037

q = .175

.025	+0.1888	-0.8419	-0.0104	-0.0549
.050	+0.1895	-0.8514	-0.0209	-0.1104
.075	+0.1907	-0.8627	-0.0314	-0.1730
.100	+0.1924	-0.8760	-0.0420	-0.2378
.125	+0.1947	-0.8915	-0.0528	-0.3077
.150	+0.1974	-0.9092	-0.0631	-0.3839
.175	+0.2008	-0.9293	-0.0747	-0.4676
.200	+0.2046	-0.9521	-0.0860	-0.5606
.225	+0.2093	-0.9778	-0.0976	-0.6645
.250	+0.2146	-1.0069	-0.1094	-0.7816
.275	+0.2206	-1.0394	-0.1216	-0.9146
.300	+0.2275	-1.0758	-0.1342	-1.0668
.325	+0.2353	-1.1166	-0.1471	-1.2421
.350	+0.2441	-1.1623	-0.1605	-1.4454
.375	+0.2540	-1.2135	-0.1744	-1.6828
.400	+0.2653	-1.2710	-0.1888	-1.9620
.425	+0.2779	-1.3354	-0.2037	-2.2923
.450	+0.2922	-1.4079	-0.2193	-2.6860
.475	+0.3084	-1.4895	-0.2354	-3.1584
.500	+0.3267	-1.5816	-0.2522	-3.7294

q = .200

.025	+0.2092	-0.8036	-0.0099	-0.0595
.050	+0.2100	-0.8143	-0.0199	-0.1217
.075	+0.2113	-0.8266	-0.0299	-0.1873
.100	+0.2131	-0.8406	-0.0400	-0.2573
.125	+0.2155	-0.8565	-0.0502	-0.3327
.150	+0.2185	-0.8745	-0.0605	-0.4147
.175	+0.2220	-0.8947	-0.0710	-0.5048
.200	+0.2263	-0.9174	-0.0816	-0.6044
.225	+0.2302	-0.9427	-0.0925	-0.7155
.250	+0.2368	-0.9710	-0.1036	-0.8403
.275	+0.2433	-1.0026	-0.1149	-0.9817
.300	+0.2506	-1.0377	-0.1266	-1.1427
.325	+0.2589	-1.0769	-0.1385	-1.3275
.350	+0.2683	-1.1201	-0.1508	-1.5409
.375	+0.2788	-1.1169	-0.1635	-1.7888
.400	+0.2906	-1.2234	-0.1765	-2.0787
.425	+0.3039	-1.2840	-0.1899	-2.4197
.450	+0.3189	-1.3518	-0.2037	-2.8233
.475	+0.3357	-1.4277	-0.2179	-3.3041
.500	+0.3546	-1.5128	-0.2324	-3.8804

q = .225

.025	+0.2285	-0.7626	-0.0094	-0.0634
.050	+0.2294	-0.7745	-0.0188	-0.1300
.075	+0.2307	-0.7878	-0.0283	-0.1994
.100	+0.2326	-0.8026	-0.0378	-0.2738
.125	+0.2352	-0.8190	-0.0474	-0.3538
.150	+0.2384	-0.8373	-0.0571	-0.4407
.175	+0.2421	-0.8576	-0.0670	-0.5358
.200	+0.2466	-0.8801	-0.0769	-0.6408
.225	+0.2518	-0.9050	-0.0871	-0.7575
.250	+0.2577	-0.9326	-0.0974	-0.8883
.275	+0.2645	-0.9632	-0.1079	-1.0357
.300	+0.2721	-0.9970	-0.1186	-1.2031
.325	+0.2808	-1.0345	-0.1295	-1.3943
.350	+0.2906	-1.0760	-0.1407	-1.6140
.375	+0.3015	-1.1220	-0.1521	-1.8680
.400	+0.3137	-1.1731	-0.1638	-2.1632
.425	+0.3275	-1.2298	-0.1756	-2.5083
.450	+0.3428	-1.2928	-0.1877	-2.9139
.475	+0.3600	-1.3630	-0.1999	-3.3932
.500	+0.3792	-1.4412	-0.2122	-3.9627

q = .250

.025	+0.2468	-0.7195	-0.0088	-0.0666
.050	+0.2476	-0.7326	-0.0177	-0.1360
.075	+0.2491	-0.7469	-0.0266	-0.2093
.100	+0.2511	-0.7624	-0.0355	-0.2872
.125	+0.2538	-0.7794	-0.0445	-0.3710
.150	+0.2570	-0.7980	-0.0536	-0.4616
.175	+0.2610	-0.8184	-0.0628	-0.5607
.200	+0.2656	-0.8407	-0.0720	-0.6697
.225	+0.2709	-0.8652	-0.0814	-0.7905
.250	+0.2771	-0.8921	-0.0909	-0.9254
.275	+0.2841	-0.9217	-0.1005	-1.0770
.300	+0.2920	-0.9542	-0.1103	-1.2483
.325	+0.3009	-0.9900	-0.1202	-1.4431
.350	+0.3109	-1.0294	-0.1303	-1.6657
.375	+0.3221	-1.0727	-0.1404	-1.9217
.400	+0.3346	-1.1206	-0.1506	-2.2174
.425	+0.3485	-1.1734	-0.1610	-2.5608
.450	+0.3640	-1.2318	-0.1713	-2.9614
.475	+0.3813	-1.2964	-0.1816	-3.4311
.500	+0.4005	-1.3679	-0.1918	-3.9840

q = .275

.025	+0.2639	-0.6747	-0.0083	-0.0691
.050	+0.2648	-0.6891	-0.0165	-0.1411
.075	+0.2663	-0.7043	-0.0248	-0.2171
.100	+0.2684	-0.7206	-0.0332	-0.2977
.125	+0.2711	-0.7382	-0.0415	-0.3842
.150	+0.2744	-0.7571	-0.0500	-0.4777
.175	+0.2785	-0.7775	-0.0584	-0.5796
.200	+0.2832	-0.7997	-0.0670	-0.6914
.225	+0.2887	-0.8237	-0.0756	-0.8149
.250	+0.2950	-0.8499	-0.0843	-0.9523
.275	+0.3021	-0.8785	-0.0930	-1.1060
.300	+0.3102	-0.9097	-0.1018	-1.2789
.325	+0.3192	-0.9438	-0.1107	-1.4047
.350	+0.3294	-0.9811	-0.1196	-1.6973
.375	+0.3406	-1.0219	-0.1286	-1.9516
.400	+0.3532	-1.0666	-0.1375	-2.2437
.425	+0.3671	-1.1157	-0.1463	-2.5805
.450	+0.3825	-1.1696	-0.1550	-2.9706
.475	+0.3997	-1.2287	-0.1635	-3.4240
.500	+0.4186	-1.2938	-0.1717	-3.9531

q = .300

.025	+0.2799	-0.6287	-0.0077	-0.0709
.050	+0.2808	-0.6443	-0.0153	-0.1449
.075	+0.2823	-0.6605	-0.0230	-0.2227
.100	+0.2844	-0.6776	-0.0307	-0.3053
.125	+0.2871	-0.6956	-0.0385	-0.3938
.150	+0.2906	-0.7148	-0.0462	-0.4891
.175	+0.2947	-0.7353	-0.0540	-0.5927
.200	+0.2995	-0.7573	-0.0618	-0.7061
.225	+0.3050	-0.7810	-0.0697	-0.8309
.250	+0.3114	-0.8065	-0.0775	-0.9692
.275	+0.3186	-0.8341	-0.0854	-1.1234
.300	+0.3267	-0.8640	-0.0933	-1.2960
.325	+0.3358	-0.8964	-1.0113	-1.4904
.350	+0.3459	-0.9316	-1.0895	-1.7103
.375	+0.3571	-0.9700	-1.1669	-1.9601
.400	+0.3696	-1.0117	-1.2430	-2.2450
.425	+0.3833	-1.0572	-1.3173	-2.5714
.450	+0.3985	-1.1067	-1.3891	-2.9464
.475	+0.4153	-1.1609	-1.4573	-3.3787
.500	+0.4337	-1.2199	-1.5208	-3.8786

q = .325

.025	+0.2946	-0.5820	-0.0071	-0.0721
.050	+0.2955	-0.5987	-0.0141	-0.1473
.075	+0.2970	-0.6159	-0.0212	-0.2264
.100	+0.2992	-0.6336	-0.0283	-0.3102
.125	+0.3019	-0.6522	-0.0354	-0.3997
.150	+0.3054	-0.6717	-0.0425	-0.4960
.175	+0.3095	-0.6922	-0.0496	-0.6004
.200	+0.3143	-0.7141	-0.0566	-0.7143
.225	+0.3199	-0.7373	-0.0637	-0.8392
.250	+0.3263	-0.7622	-0.0708	-0.9771
.275	+0.3334	-0.7888	-0.0778	-1.1301
.300	+0.3415	-0.8175	-0.0847	-1.3006
.325	+0.3505	-0.8483	-0.0916	-1.4917
.350	+0.3605	-0.8816	-0.0983	-1.7067
.375	+0.3716	-0.9175	-0.1049	-1.9494
.400	+0.3838	-0.9564	-0.1113	-2.2245
.425	+0.3973	-0.9985	-0.1174	-2.5373
.450	+0.4121	-1.0440	-0.1232	-2.8941
.475	+0.4283	-1.0933	-0.1285	-3.3020
.500	+0.4460	-1.1468	-0.1331	-3.7693

q = .350

.025	+0.3205	-0.4881	-0.0059	-0.0728
.050	+0.3214	-0.5069	-0.0117	-0.1486
.075	+0.3229	-0.5258	-0.0176	-0.2282
.100	+0.3250	-0.5448	-0.0234	-0.3123
.125	+0.3277	-0.5643	-0.0292	-0.4018
.150	+0.3311	-0.5843	-0.0350	-0.4977
.175	+0.3352	-0.6050	-0.0407	-0.6010
.200	+0.3399	-0.6265	-0.0463	-0.7130
.225	+0.3453	-0.6489	-0.0519	-0.8350
.250	+0.3515	-0.6726	-0.0574	-0.9686
.275	+0.3585	-0.6974	-0.0628	-1.1155
.300	+0.3662	-0.7238	-0.0680	-1.2777
.325	+0.3748	-0.7517	-0.0730	-1.4576
.350	+0.3844	-0.7813	-0.0778	-1.6577
.375	+0.3948	-0.8129	-0.0823	-1.8810
.400	+0.4063	-0.8465	-0.0864	-2.1308
.425	+0.4188	-0.8824	-0.0902	-2.4108
.450	+0.4325	-0.9208	-0.0934	-2.7253
.475	+0.4473	-0.9617	-0.0961	-3.0791
.500	+0.4632	-1.0053	-0.0979	-3.4772

q = .375

.025	+0.3081	-0.5350	-0.0065	-0.0727
.050	+0.3090	-0.5528	-0.0129	-0.1485
.075	+0.3106	-0.5708	-0.0194	-0.2282
.100	+0.3127	-0.5893	-0.0258	-0.3124
.125	+0.3156	-0.6083	-0.0323	-0.4023
.150	+0.3189	-0.6280	-0.0387	-0.4988
.175	+0.3230	-0.6487	-0.0451	-0.6030
.200	+0.3278	-0.6703	-0.0515	-0.7164
.225	+0.3333	-0.6932	-0.0578	-0.8403
.250	+0.3396	-0.7174	-0.0640	-0.9766
.275	+0.3467	-0.7432	-0.0702	-1.1271
.300	+0.3547	-0.7706	-0.0763	-1.2941
.325	+0.3635	-0.8000	-0.0822	-1.4802
.350	+0.3733	-0.8314	-0.0879	-1.6885
.375	+0.3841	-0.8650	-0.0934	-1.9222
.400	+0.3960	-0.9012	-0.0987	-2.1854
.425	+0.4091	-0.9401	-0.1035	-2.4825
.450	+0.4233	-0.9819	-0.1080	-2.8188
.475	+0.4389	-1.0268	-0.1119	-3.2001
.500	+0.4558	-1.0751	-0.1151	-3.6331

q = .400

.025	+0.3316	-0.4417	-0.0053	-0.0723
.050	+0.3325	-0.4614	-0.0105	-0.1477
.075	+0.3340	-0.4810	-0.0158	-0.2267
.100	+0.3360	-0.5007	-0.0210	-0.3100
.125	+0.3387	-0.5205	-0.0262	-0.3986
.150	+0.3420	-0.5407	-0.0313	-0.4932
.175	+0.3460	-0.5614	-0.0363	-0.5949
.200	+0.3506	-0.5828	-0.0413	-0.7047
.225	+0.3559	-0.6049	-0.0462	-0.8239
.250	+0.3620	-0.6279	-0.0509	-0.9540
.275	+0.3687	-0.6520	-0.0555	-1.0964
.300	+0.3763	-0.6772	-0.0599	-1.2529
.325	+0.3846	-0.7038	-0.0641	-1.4255
.350	+0.3938	-0.7318	-0.0680	-1.6165
.375	+0.4038	-0.7614	-0.0715	-1.8283
.400	+0.4148	-0.7927	-0.0747	-2.0637
.425	+0.4267	-0.8259	-0.0774	-2.3258
.450	+0.4397	-0.8611	-0.0796	-2.6181
.475	+0.4536	-0.8984	-0.0811	-2.9441
.500	+0.4686	-0.9378	-0.0819	-3.3078

q = .425

.025	+0.3415	-0.3962	-0.0047	-0.0714
.050	+0.3424	-0.4167	-0.0094	-0.1458
.075	+0.3439	-0.4370	-0.0140	-0.2237
.100	+0.3459	-0.4571	-0.0186	-0.3058
.125	+0.3485	-0.4773	-0.0232	-0.3928
.150	+0.3517	-0.4977	-0.0278	-0.4856
.175	+0.3556	-0.5184	-0.0321	-0.5850
.200	+0.3601	-0.5395	+0.0364	-0.6921
.225	+0.3652	-0.5613	-0.0406	-0.8079
.250	+0.3710	-0.5838	-0.0446	-0.9338
.275	+0.3776	-0.6071	-0.0485	-1.0709
.300	+0.3848	-0.6313	-0.0521	-1.2210
.325	+0.3928	-0.6566	-0.0555	-1.3856
.350	+0.4016	-0.6831	-0.0586	-1.5668
.375	+0.4112	-0.7109	-0.0613	-1.7665
.400	+0.4216	-0.7401	-0.0636	-1.9870
.425	+0.4329	-0.7708	-0.0654	-2.2309
.450	+0.4451	-0.8032	-0.0666	-2.5009
.475	+0.4582	-0.8371	-0.0671	-2.7998
.500	+0.4722	-0.8729	-0.0669	-3.1306

q = .450

.025	+0.3504	-0.3518	-0.0041	-0.0701
.050	+0.3512	-0.3730	-0.0082	-0.1431
.075	+0.3526	-0.3938	-0.0123	-0.2195
.100	+0.3545	-0.4144	-0.0163	-0.2999
.125	+0.3571	-0.4349	-0.0203	-0.3849
.150	+0.3602	-0.4554	-0.0242	-0.4754
.175	+0.3639	-0.4761	-0.0280	-0.5721
.200	+0.3683	-0.4971	-0.0317	-0.6759
.225	+0.3732	-0.5185	-0.0352	-0.7878
.250	+0.3788	-0.5404	-0.0386	-0.9088
.275	+0.3851	-0.5630	-0.0418	-1.0402
.300	+0.3920	-0.5863	-0.0447	-1.1833
.325	+0.3996	-0.6104	-0.0473	-1.3394
.350	+0.4080	-0.6356	-0.0496	-1.5103
.375	+0.4171	-0.6617	-0.0515	-1.6976
.400	+0.4269	-0.6890	-0.0530	-1.9032
.425	+0.4375	-0.7174	-0.0540	-2.1291
.450	+0.4489	-0.7472	-0.0544	-2.3773
.475	+0.4611	-0.7782	-0.0541	-2.6500
.500	+0.4740	-0.8106	-0.0528	-2.9495

q = .475

.025	+0.3580	-0.3087	-0.0036	-0.0685
.050	+0.3588	-0.3305	-0.0071	-0.1397
.075	+0.3602	-0.3518	-0.0107	-0.2142
.100	+0.3621	-0.3728	-0.0141	-0.2925
.125	+0.3645	-0.3935	-0.0175	-0.3752
.150	+0.3675	-0.4141	-0.0209	-0.4630
.175	+0.3711	-0.4347	-0.0241	-0.5565
.200	+0.3752	-0.4555	-0.0272	-0.6566
.225	+0.3800	-0.4766	-0.0301	-0.7641
.250	+0.3853	-0.4980	-0.0328	-0.8799
.275	+0.3913	-0.5199	-0.0353	-1.0052
.300	+0.3979	-0.5424	-0.0376	-1.1409
.325	+0.4051	-0.5655	-0.0395	-1.2884
.350	+0.4130	-0.5893	-0.0411	-1.4488
.375	+0.4215	-0.6139	-0.0424	-1.6237
.400	+0.4308	-0.6394	-0.0431	-1.8145
.425	+0.4407	-0.6659	-0.0434	-2.0227
.450	+0.4513	-0.6933	-0.0430	-2.2500
.475	+0.4626	-0.7217	-0.0420	-2.4980
.500	+0.4745	-0.7512	-0.0402	-2.7682

q = .500

BIBLIOGRAPHY

BIOT, M.A. (1956), "Thermoelasticity and irreversible thermodynamics" (J.App. Phys., Vol.27,No.3, March,240-253)

BOLEY, B.A. and WEINER, J.H. (1960), "Theory of thermal stresses" (John Wiley and Sons)

COLLINS, W.D. (1961), "Axially symmetric stress distribution" (Proc.Roy.Soc.A.,Vol. 266,359-386)

COOKE, J.C. and TRANTER, C.J. (1959) (Quart.J.App.Maths.,Vol.12, 379-386)

DANILOVSKAYA, V.I. (1950) "Thermal stresses in an elastic half space arising after a sudden heating of its boundary" (Prikl.Mat.Mech., Vol.14, May-June, 316-318)

DANILOVSKAYA, V.I. (1952) "On a dynamical problem of thermoelasticity" (Prikl.Mat. Mech., Vol.16, 3, May-June, 341-344)

DUHAMMEL, J.M.C. (1838) "Memoirs ... par divers savants" (Paris)

EASON,G., FULTON, J. and SNEDDON, I.N. (1956) "The generation of waves in an infinite elastic solid by variable body forces" (Phil.Trans.Roy.Soc., A., 955, 575-607)

EASON, G.,NOBLE, B. and SNEDDON, I.N. (1955) "On certain integrals of Lipschitz-Hankel type involving products of Bessel functions" (Proc.Roy.Soc.A.,Vol.247,529-551)

ERDELYI, A. (1954) "Tables of integral transforms" (Vols.1 and 2, McGraw-Hill)

GATEWOOD, B.E. (1957) "Thermal stresses" (McGraw-Hill, New York)

GOODIER, J.N. (1957) "Thermal stresses and deformations" (J.App.Mech., Vol.24,3, Sept.)

GREEN, A.E. (1949) "On Boussinesq's problem and penny-shaped cracks" (Proc.Camb. Phil.Soc.,45,251)

GREEN, A.E. and ZERNA, W. (1954) "Theoretical elasticity" (Oxford Univ. Press)

GRIFFITH, A.A. (1921) "The phenomena of rupture and flow in solids" (Phil.Trans.A., 221,163)

HOWLAND, R.C.J. (1929) "Stress systems in an infinite strip" (Proc.Roy.Soc.A.,124,89)

HOWLAND, R.C.J. and STEVENSON, A.C. (1933) "Biharmonic analysis in a perforated strip " (Proc.Roy.Soc.A., 148,471)

IGNACZAC, J. (1913) "Thermal displacement in an elastic semi-space due to sudden heating of the boundary plane" (Nedb.Arch.Mech.Stos., Vol.9, 395)

INGLIS, C.E. (1913) "Stresses in a plate due to the presence of cracks and sharp corners" (Trans.Inst.Naval Arch., 55,219)

KOITER, W.T. (1960) "Stress distribution in an infinite elastic sheet with a doubly periodic set of equal holes" (Problems in Differential Equations, Univ. of Wisconsin Press, Madison,Wisc.)

KNOPS, R.J. (1959) (J.Mech.Phys.Solids,7,3,182)

KOLOSOV, G.V. (1909) For complete reference, see Muskhelishvili,p.692.

LOVE, A.E.H. (1944) "Mathematical theory of elasticity" (Dover Publications, New York)

LOWENGRUBB, M. (1961) "Stress in the vicinity of a crack in a thick elastic plate" (Quart.App.Math., 19, 119)

LUR'E. A.I. (1955) "Space problems of the theory of elasticity" (Moscow,191-199)

McDOWELL, E.L. (1957) "Thermal stresses in an infinite plate of arbitrary thickness" (Proc.3rd.Midwestern Conf. on Solid Mech.,,72-85)

MELAN,E. and PARKUS, H. (1953) "Warmespannungen" (Springer,Wien)

MINDLIN, R.D. and CHENG, D.H. (1950) "Thermoelastic stress in the semi infinite solid" (J.App. Phys., Vol.21,9,931-933)

MUKI, R. (1960) "Asymmetric problems of the theory of elasticity for a semi infinite solid and a thick plate" (Progress Solid Mech.,I, Amsterdam,399)

MURA, T. (1952) "Thermal strains and stresses in transient state" (Proc.Sec. Jap.Congress App.Mech.)

MUSKHELISHVILI, N.I. (1953) "Some basic problems of the mathematical theory of elasticity" (Trans. by J.R.M.Radok, P.Noordhoff Ltd., Groningen)

NOWACKI, W. (1957_1) (Nodb.Arch.Mech.Stos.,Vol.9,579)

NOWACKI, W. (1957_2) (Nodb.Arch.Mech.Stos.,Vol.9,319)

NOWACKI, W. (1957_3) "A dynamical problem of thermoelasticity" (Nodb.Arch.Mech. Stos.,Vol.9,325)

NOWACKI, W. (1960) "Zagadnienia termosprezystosci" (Warszawa, Panstwowe Wydawnictwo)

OLESIAK, Z. and SNEDDON, I.N. (1960) "The distribution of stress in an infinite elastic solid containing a penny-shaped crack" (Arch.Rat.Mech. and Anal. 4,3,238-254)

PAYNE, L.E. (1953) "On axially symmetric punch, crack, and torsion problems" (Journ.S.I.A.M.,1,53)

PRAGER, W. (1961) "Mechanics of the continua" (Ginn and Co)

SACK, R.A. (1946) "Extension of Griffith's theory of rupture to three dimensions" (Proc.Phys.Soc.,58,149)

SAVIN, G.N. (1951) "Concentration of stress around holes" (German trans. by H.Neuber,V.E.B. Verlag,Berlin)

SHARMA, B. (1956) (J.App.Mech.,23,4,527)

SNEDDON, I.N. (1946) "The distribution of stress in the neighbourhood of a crack in an elastic solid" (Proc.Roy.Soc.A.,187,229)

SNEDDON, I.N. and ELLIOT, H.A. (1946) "The opening of a Griffith crack under internal pressure" (Quart.App.Math.,4,262)

SNEDDON, I.N. (1951) "Fourier transforms" (McGraw-Hill, New York)

SNEDDON, I.N. and BERRY, D.S. (1958) "The classical theory of elasticity" (Handbuch der Physic, 6,123)

SNEDDON, I.N. (1960) "The elementary solution of dual integral equations" (Proc. Glasgow Math.Ass.,4,108)

SNEDDON, I.N. and LOCKETT, F.J. (1960_1) "On the steady state thermoelastic problem for the half space and the thick plate" (Quart.App.Mech.,2,145-153)

SNEDDON, I.N. and LOCKETT, F.J. (1960_2) "The steady state thermoelastic problem for the elastic layer resting on a rigid foundation" (Ann.Mat.Pura ed Appl.,50, 309-317)

SNEDDON, I.N. (1961) "Crack problems in the mathematical theory of elasticity" (North Carolina State College, Raleigh)

SOKOLNIKOFF, I.S. (1946) "Mathematical theory of elasticity" (McGraw-Hill, New York)

STERNBERG, E. and CHAKRAVORTY, J.G. (1958) "On inertia effects in a transient thermoelastic problem" (Tech.Rep.No.2,Contract No.-562(25),Brown Univ.,May)

STERNBERG, E. and McDOWELL, E.L. (1955) "On the steady state thermoelastic problem for the half space" (Dept.of Mech., Univ. of Ill., Chicago, Ill., October)

STEVENSON, A.C. (1943) "Some boundary value problems of two dimensional elasticity" (Phil.Mag.,34,766)

THOMAS, P.M. (1957) "Some conduction problems in the heating of small areas on large solids" (Quart.J.Mech. and App.Math., Oxford,10, No.4, 482-493)

TRANTER, C.J. (1951) "Integral transforms in mathematical physics" (Methuen, London)

TRANTER, C.J. (1959) "Dual trigonometric series" (Proc.Glasgow Math.Soc.,4,49)

UFLIAND, Ia.S. (1959) "Elastic equilibrium of an infinite body weakened by an external circular crack" (App.Math. andMech.,trans. from Russian, Vol.23, 1,134-145)

WATSON, G.N. (1944) "A treatise on the theory of Bessel functions" (Camb. Univ. Press)

WESTERGAARD, H.M. (1939) "Bearing pressure and cracks" (J.App.Mech.,6(2),A-49)

WIGGLESWORTH, L.A. (1957) "Stress distribution in a notched plate" (Mathematics,5,67)