A GEOMETRIC CONTROL SYSTEM WITH APPLICATIONS TO HELICOPTERS

by

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' Academic philosophers, ever since the time of Parmenides, have believed that the world is a unity, ... The most fundamental of my intellectual beliefs is that this is rubbish. I think the universe is all spots and jumps, without unity, without continuity, without coherence of orderliness or any of the other properties that governesses love. Indeed, there is little but prejudice and habit to be said for the view that there is aworld at all...

The external world may be an illusion, but if it exists, it consists of events, short, small and haphazard. Order, unity and continuity are human inventions, just as truly as are catalogues and encyclopedias. '

Bertrand Russel.

" My philosophical development".

This work is dedicated to my parents.

CONTENTS.

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ABSTRACT

This thesis introduces an Automatic Flight Control System for single rotor helicopters which gives a new relevance to the traditional techniques based on the Linear Control Theory. This design was obtained by applying concepts of differential geometry tailored for engineering purposes through the Nonlinear System Theory. The development of this thesis follows the traditional path of applied sciences. First the need to establish techniques for theoretical analysis of flight machanics, where the small disturbance methods are no longer valid, is reviewed. This is followed by a presentation of the nonlinear problem and a survey of the development of the theoretical tools available. At this stage the process, a single rotor helicopter, is modelled. The model is then cast in a form suitable for Nonlinear System Theory techniques. Next, the mathematical theory to be applied is fully developed. It consists of finding the conditions required by a nonlinear system to be transformable under state feedback to a linear canonical form; the construction of the feedback is also presented. A Flight Control System is designed by applying this theory to the helicopter model previously formulated. The above application requires the development of Symbolic Algebraic Manipulation programmes, which are also included. Finally, a set of simulation studies demonstrate the performance of the design.

CHAPTER 1.

INTRODUCTION.

Summary.

The subject developed in this thesis belongs to a branch of applied sciences, namely the theory of Flight Stability and Control. Furthermore, the material presented here is devoted to the control of single rotor helicopters.

This work has borrowed its theoretical principles from another applied science, System Theory. This introductory chapter is partially dedicated to exposing the relationship between these two disciplines and to clarifying the appearance of differential geometry in the development of Flight Stability and Control, which is the essence of the work submitted here. A review of each chapter is also included.

I.1. Introduction.

The study of flight mechanics relieson two different independent sciences, aerodynamics and mechanics. Of course, as pointed out throughout this thesis, the role of mathematics is not a passive auxiliary of an applied science, but rather a tool that suggests and encourages further development of its content.

It is known that the theory of flight stability and control appears from the outset as a mathematically expressed, and therefore essentially deductive, theory. This theory also relies heavily on rigid body mechanics, from where the small disturbance method is obtained which has been used as the main instrument for analysis.

The existence of the small disturbance method implies that the knowledge of aerodynamics is required in order to determine the characteristics of any aircraft. These characteristics are related to the forces and moments acting on the aircraft. The information obtained by applying aerodynamics is summarised in a set of parameters, the so called stability derivatives introduced by G. H. Bryan (Etkin [1972]).

The classical mathematical model used in the flight stability and control theory is obtained by small disturbance rigid body dynamics together with the stability derivatives. This model consists of essentially a set of linear differential equations with constant coefficients. It is precisely through this model that the link between flight stability and control and system theory is carried out. The linear systems theory has provided the results needed to determine the flying qualities of a given aircraft. In fact these two disciplines experienced a concomitant development. This is reflected in the treatises devoted to flying qualities. For instance, the methods of single input single output systems using the Laplace transform, has been expressed by M^{c}Ruer, Ashrenas and Graham in their book "Aircraft Dynamics and Automatic Control" [1970]. The appearance of the state space techniques in system theory (in which linear algebra is the mathematical instrument of analysis) improves the methods of analysis in flight theory. This is reflected in the books by B. Etkin [1972], [1982] and Prouty [1986]. Furthermore most of the recent results published on flight stability and control are obtained by applying this approach.

As is typical of any applied science, the theory of flight stability and control is influenced by the mathematical method used. Thus, the small disturbance method played a striking part in shaping the theory, emphasising the dependence on models expressed in terms of stability derivatives. However, this approach presents serious limitations in the study of some aspects of flight, for instance, those related to agility, which is defined (for rotorcraft) as "The ease with which a helicopter can change its position and state with precision and speed" (Thomson [1987]).

The subject of agility involves situations wherein the small disturbance situation can not be assumed any more, so that a change of model is required and with it a new mathematical treatment.

Agility has been addressed from the practical point of view by Charlton et al [1987] and by Buckingham [1986]. In these heuristic studies the agility of a particular helicopter is determined from the results of flight tests. An agility factor is defined by comparing a "theoretical" maximum performance to an actual performance achieved in the flight tests.

Agility factors for several different manoeuvres have been defined by using this procedure.

The question concerning how easily and quickly a helicopter can change its flight control conditions, obviously depends on the opinion of the pilot. Buckingham et al. [1986] reported the results of a series of piloted simulations which were intended to explore and define control systems for helicopters. Throughout these simulations it is assumed that the helicopter responses are transformed by a control system to an "equivalent" system which is linear. In spite of the fact that the terms and conditions of this equivalence are not defined in this report it is possible to formulate the agility problem in a formal manner from these results.

The first theoretical approach used to analyse the flight mechanics, not relying on the small disturbance model is based on inverse solutions. The first reports on the use of this technique in the solution of flight stability and control problems were presented by Meyer and Cicolani [1975] [1981]. This technique consists of obtaining an inverse solution of the equations of motion involved in a specific flight path, then calculating the control and state histories to fly it. In the method suggested by Meyer and Cicolani, it is shown that it is possible to define an equivalent linear system of the equations of motion, so that the problem could be solved using the standard linear techniques [1981]. In this report an application to a fixed wing aircraft is presented and the application to a helicopter was later reported by Hunt, Su and Meyer [1982] [1984]. Moreover, in these last two references, the mathematical proof of the existence of the linear equivalent system using differential geometry, is also presented. In 1987 Smith and Meyer presented an automatic flight control system concept based on this technique. The difference of these implementations with respect to the one presented here is that the previous works depend on numerical methods for the calculation of the inverse solution, whereas they are presented here inanalytical closed form.

The introduction of differential geometry in the theory of flight stability and control can be considered as the natural effects of the mathematical method on the content of a branch of applied science and can also contribute to its broadening. Because of their degree of abstraction, mathematical theories have a large sphere of potential applications. Even if they were initially called to life or developed on the basis of certain special requirements of physics or engineering, they soon exceed the space of these particular applications. For this reason, the examination of some mathematical theories from the standpoint of possible application combined with the awareness of the physical processes characteristic of a given applied science, may result in improved knowledge about this applied science.

Nevertheless, in this case the path of development followed the opposite direction; it began with the actual necessity and then the search for the tool. In the future this could be inverted and the study of flight mechanics developed using the concept of differential geometry as a foundation.

I.2. Nonlinear Controllability.

In the paper presented by Charlton et al. [1987], an agility factor for helicopters is defined by the ratio of a "theoretical ideal task" time and the actual time required to execute this task.

This problem can be formulated from a mathematical point of view through some concepts and results obtained from the nonlinear system theory. These concepts are very useful when one analyses the above problem. Unfortunately, the proofs of all the statements require an extensive mathematical background which are beyond the scope of this thesis, so that this material (which is fully developed in the references, in particular the book by Boothby [1986]) is presented only as antecedent of the following chapters, especially Chapter III.

Since the equations of flight of the helicopter can be related to a dynamical system H in a known state X_o, the following question can be formulated: "What states can H attain at some future time T under the action of inputs chosen from a specified set Ω ? ". This is a statement of the problem of reachability. A variation of this problem occurs when one wants to transfer H from X_o to a given state X, this problem is known as controllability. It is evident that, the essence of the reachability - controllability question is to decide what can be done with H, considering the control resources available.

Brockett [1973], [1976] explained the nonlinear reachability (controllability) problem using the general system

$$\dot{X}(t) = \sum_{i=1}^{m} u_i \; f_i(X) \; , \qquad X(0)= X_o \in R^n, \qquad (I.1),$$

where the vector functions $f_i(X)$ are C^∞. One can then formulate the question: under what conditions does a smooth p-dimensional manifold M contained in R^n, with $p \leqslant n$ exist, such that the set $\{f_i(X)\}$ spans the tangent space of M at each point ? The connection between this question and the problem of reachability (controllability) is that, if such a manifold M exists, then the state can move anywhere within M but not out of it.

An easy way to understand the problem is to proceed as follows (Brockett [1976]):

Making $u_i(t)= 1$ for a particular i, $1 \leqslant i \leqslant m$ and $u_j= 0$ for $j \neq i$. Then the system (I.1) is reduced to

$$\dot{X}(t)= f_i(X) \; , \qquad X(0)= X_o \qquad (I.2).$$

The solution of the above equation can be denoted by $\varphi(t,X_o)$, that is:

$$X(t)= \varphi(t,X_o).$$

Furthermore, if one considers a time interval $|t| < \epsilon$, such that $\varphi(t,X_o)$ can be expanded in Taylor series around t= 0, then

$$X(t)= \varphi(0+t,X_o),$$

$$= \varphi(0,X_o) + \left.\frac{d\,\varphi(t,X_o)}{d\,t}\right|_{t=0} t + \left.\frac{d^2\,\varphi(t,X_o)}{d\,t^2}\right|_{t=0} \frac{t^2}{2} +$$

H.O.T

Also

$$\varphi(0,X)= X(0)= X_o,$$

$$\left.\frac{d\,\varphi(t,X_o)}{d\,t}\right|_{t=o} = \left.\dot{X}\right|_{t=o} = \left.f_i(X)\right|_{t=o} = f_i(X_o) \quad .$$

And

$$\left.\frac{d^2\,\varphi(t,X_o)}{d\,t^2}\right|_{t=o} = \left.\frac{d}{d\,t}\left[\frac{d\,\varphi(t,X_o)}{d\,t}\right]\right|_{t=o}$$

$$= \left.\frac{d\,f_i(X)}{d\,t}\right|_{t=o}$$

$$= \left.\left(\frac{\partial f_i}{\partial X}\right)\dot{X}\right|_{t=o} \quad .$$

Therefore

$$X(t) = \left(I + t\,f_i + \frac{t^2}{2}\,\frac{\partial f_i}{\partial X}\,f_i + H.O.T \right)(X_o) \quad ,$$

sometimes the above expression is defined as

$$X(t) \equiv (\exp tf_i)\,(X_o) \quad , \qquad (I.3)$$

where $(\exp tf_i)$ denotes the series in the brackets and I is the identity function on R^n. Note that $(\exp tf_i)$ represents a function whose argument is X_o. If one supposes the following control sequence for system (I.1), with $X(0)= X_o$ is applied;

from 0 to t, $u_1= 1$, $u_j= 0$ for all $j\neq 1$;

from t to 2t, $u_2= 1$, $u_j= 0$ for all $j\neq 2$;

from 2t to 3t, $u_1= -1$, $u_j= 0$ for all $j\neq 1$ and

from 3t to 4t, $u_2= -1$, $u_j= 0$ for all $j\neq 2$.

Also X_1, X_2, X_3 and X_4 can also be expressed in a Taylor series. Expanding these terms to the second order one has

$$X_1 \simeq X_0 + t\, f_1(X_0) + \frac{t^2}{2} \frac{\partial f_1}{\partial X} f_1(X_0) \qquad \text{(I.4)},$$

$$X_2 \simeq X_1 + t\, f_2(X_1) + \frac{t^2}{2} \frac{\partial f_2}{\partial X} f_2(X_1) \qquad \text{(I.5)},$$

$$X_3 \simeq X_2 - t\, f_1(X_2) + \frac{t^2}{2} \frac{\partial f_1}{\partial X} f_1(X_2) \qquad \text{(I.6)} \quad \text{and}$$

$$X_4 \simeq X_3 - t\, f_2(X_3) + \frac{t^2}{2} \frac{\partial f_2}{\partial X} f_2(X_3) \qquad \text{(I.7)} .$$

The function f_i can also be developed in Taylor series:

$$f_i = f(X_0+\delta X) = f_i(X_0) + \frac{\partial f_i(X_0)}{\partial X} \delta X + (\text{H.O.T.})\ (X_0) \qquad \text{(I.8)} .$$

These last equations can be manipulated and expressed to the second order, so that X_4 may be written

$$X_4 \simeq \left(I + t^2 \left[\frac{\partial f_2}{\partial X} f_1 - \frac{\partial f_1}{\partial X} f_2 \right] \right) (X_0) \qquad \text{(I.9)} .$$

where

$$\left[\frac{\partial f_2}{\partial X} f_1 - \frac{\partial f_1}{\partial X} f_2 \right] \equiv [f_1, f_2] \qquad \text{(I.10)} .$$

is defined as the Lie bracket of the vector fields f_1 and f_2. Thus, if $[f_1, f_2]$ is not a linear combination of f_i, $i=1,\ldots,m$, then $[f_1, f_2]$ represents a new direction in which the state can move. The problem of finding M, whose tangent space is spanned by f_i, $i=1,\ldots,m$, cannot be solved. With this example Brockett [1973] [1976] showed the central role of the Lie bracket operation in answering the question of reachability-controllability.

From the above example it is easy to consider the definition of involutivity:

Definition. A set of vector fields f_i, $i=1,...m$, is said to be involutive if there exists a set of scalar functions c_{ijk}, such that

$$[f_1, f_2](X) = \sum_{k=1}^{m} c_{ijk} f_k(X) ,$$

for every vector field f_i, $i=1,...m$.

Considering the previous development of system (I.1) persuades one that the property of involutivity is necessary in order to be able to obtain a solution surface (manifold) whose tangent plane is described by the system equation (I.1). The theorem of Frobenius establishes that this condition is necessary and sufficient. This theorem plus the involutive condition plays the central role in the development of the flight control system designed in this thesis.

The results from nonlinear systems theory on which the present work is based are included in the references by Casti [1985], Hirshon [1973], Hermann et. al.[1977], and Sussmann et. al. [1972].

Considering a general nonlinear system

$$\dot{X}(t)= f(X,u) , \qquad X(0)= X_o \qquad (I.11) .$$

where $u \in \Omega$, an admissible set of R^m valued input function; and $X \in M$, a C^∞-connected manifold of dimension n. In order to simplify the notation, it is assumed that M admits globally defined coordinates $X=(x_1,..., x_n)$, allowing one to identify the points of M with their coordinate representations and to write the control system in the usual engineering form given by (I.11).

It is assumed that the vector field $f(.,.)$ is C^∞ with respect to its arguments and that (I.11) is complete, that is, for every bounded measurable control u(t) and every $X_o \in M$, there exists a solution of (I.11) satisfying $X(0)= X_o$ and $X(t) \in M$ for all real positive t.

The controllability of nonlinear systems is expressed by the following definitions and theorems (Hermann and Krenner [1977]):

Definition I.1. Given a point $X^* \in M$, it is said that X^* is reachable from X_o at time T if there exists a bounded measurable input $u \in \Omega$, such that, the trajectory of (I.11) satisfies $X(0)= X_o$, $X(T)= X^*$ and $X(t) \in M$ for all $t \in [0,T]$. The set of reachable states from X_o is denoted by

$$\mathcal{R}(X_o)= \bigcup_{0 \leq T \leq \infty} \{ X: X \text{ reachable from } X_o \text{ at time } T\}$$

It is said that (I.11) is reachable at X_o if $\mathcal{R}(X_o)= M$ and reachable if $\mathcal{R}(X)= M$ for all $X \in M$. ∎

The problem with definition I.1 is that f(X,u) is nonlinear, it may be necessary to travel either a long distance or for a long time to reach points near X_o. Thus, the property of reachability from X_o may not always be of practical use. This motivates the following restriction:

Definition I.2. The system (I.11) is locally reachable at X_o if for every neighbourhood U of X_o, $\mathcal{R}(X_o) \cap U$ is also a neighbourhood of X_o with the trajectory from X_o to $\mathcal{R}(X_o) \cap U$ lying entirely within U. The system (I.11) is locally reachable if it is locally reachable for each $X \in M$. ∎

Given that the reachability given in definitions I.1 and I.2 does not guarantee the condition of symmetry, that is X^* may be reachable from X^{**} but not conversely (in contrast with the case of constant linear systems). A weaker definition is needed:

Definition I.3. Two states X^* and X^{**} are weakly reachable from each other if and only if there exist states

$X_o, X_1, \ldots, X_k \in M$

such that $X_o = X^*$ and $X_k = X^{**}$

and either X_i is reachable from X_{i-1} or X_{i-1} is reachable from X_i, $i=1,\ldots,k$. ∎

The system (I.11) is weakly reachable if it is global reachable, so that it is also possible to define a local version as shown in definition I.2.

The condition of reachability-controllability of a system is characterized by the following theorem.

Theorem I.1. System (I.11) is locally weakly reachable-controllable if and only if for every $X \in M$ and every neighbourhood U of X the interior of $R(x)$ restricted to U is not empty.

Proof. (Hermann and Krenner [1977]).

Assuming that system (I.11) is weakly reachable. Then given any $X_o \in M$ and any neighbourhood U of X_o one can choose $u_1 \in \Omega$ such that

$$f^1(X)= f(X,u_1)$$

does not vanish at X_o.

If $s \Rightarrow \gamma^1{}_s(X)$ denotes the flow on f^1 that is, the solutions of the differential equation

$$\frac{d\,\gamma^1{}_s}{d\,s} = f^1(\gamma^1{}_s(X))$$

that satisfies the initial conditions

$$\gamma^1{}_o(X)= X ,$$

then for some $\delta > 0$, the set

$$V^1 = \{ \gamma^1{}_s(X_o):\ 0 < s < \delta \}$$

is a submanifold of U of dimension one.

Intuitively it is possible to define V^{j-1} as

$$V^{j-1} = \{ \gamma^{j-1}{}_{s_{j-1}} \circ \ldots \circ \gamma^1{}_{s_1} : (s_1, \ldots, s_{j-1}) \text{ in some open subset of the positive orthant } R^{j-1} \} ,$$

where "$\circ$" indicates the composition of functions and $\gamma^i_{s_i}(X)$ is the "flow" of $f^i(x) = f(X, u_i)$ for some $u_i \in \Omega$.

It can be observed that $V^{j-1} \subset R_U(X_o)$ (the space of reachable states from X_o).

If $j \leqslant n$, V^j is constructed choosing an $u_j \in \Omega$ and a $X^{j-1} \in V^{j-1}$. This is always possible as if this is not the case, then every trajectory of (I.11) starting on V^{j-1} would remain on V^{j-1} for a while. This contradicts the local weak controllability of (I.11).

It follows that it is also possible to select an open subset of the positive orthant of R^j, such that the map

$$(s_1, \ldots, s_j) \Rightarrow (\gamma^j_{s_j} \circ, \ldots, \circ\, \gamma^1_{s_1})(X_o)$$

is an imbedding of the subset into U. This subset is called V^j. Continuing in this way until $j = n$, where V^n is a open subset of $R_U(X_o)$, so that the interior of $R_U(X_o)$ is not empty.

As for the converse, one can suppose that $R_U(X_o) \neq \emptyset$, then choosing a control $u(t)$, $t_o \leqslant t \leqslant t_1$, such that the corresponding trajectory $X(t)$ for $t_o \leqslant t \leqslant t_1$, satisfies $X_o(t) = X_o$ and $X(t_1) = X_1$, with X_1 being an interior point of $R_U(X_o)$ and $X(t) \in U$ for all $t \in [t_o, t_1]$.

Letting $\gamma_t(X, t_o)$ be generated by the time dependent vector field

$$f_t(X) = f(X, u(t)),$$

that is

$$\frac{d\, \gamma_t(X, t_o)}{d\, t} = f_t(\gamma_t(X, t_o))$$

with $\gamma_{t_o}(X, t_o) = X$.

Then $\gamma_t(., t_1)$ is a diffeomorphism of a neighbourhood V of X_1 over a neighbouhood of X_o. Morover, one can choose V properly contained in $R_U(X_o)$ and sufficiently small, so that $\gamma_{t_o}(V,t)$ is weakly locally reachable from X_o. ∎

The form of this proof is the commonplace in nonlinear system theory proofs. In Chapter III a similar proof is developed which is more suited to a practical application.

The characterization of the set $R_U(X_o)$, depends on the following definitions.

<u>Definition I.4</u>. (Boothby [1986]). Let $\chi(M)$ be the set of all the C^∞ vector fields over M, this is an infinite dimensional real vector field. This space is a Lie algebra provided that

i) if $\chi(M)$ is a collection of vector fields on the manifold M, then $\chi(M)$ is a real vector space with respect to ordinary vector addition and scalar multiplication

ii) furthermore if f_1 and f_2 belong to $\chi(M)$ then the Lie bracket $[f_1, f_2]$ also belongs to $\chi(M)$. ∎

Let u denote each constant control $u \in \Omega$, then $f(X,u)$ defines a vector field in $\chi(M)$. Assume also that F_o denotes the subset of all such vector fields, for instance, the set of all vector fields generated from $f(X,.)$ through the use of constant inputs. Let F denote the smallest subalgebra of $\chi(M)$ containing F_o, that is, the elements of F are linear combinations of elements of the form:

$[f_1, f_2]$,

$[f_1, [f_2, f_3]]$,

$[f_1, [f_2 [f_3, f_4]]$

.
.
.

$[f_1, [f_2 \ldots [f_i, f_{i-1}] \ldots]]$.

where $f_i \equiv f(X,u)$ for some constant $u_i \in \Omega$.

The representation of the system itself suggests the existance of a space of tangent vectors spanned by the vector fields of F at the point X of the submanifold M. This tangent space is usually denoted by T(X).

<u>Definition. I.5.</u> The system (I.11) is said to satisfy the <u>reachability-controllability rank condition</u> at a point X_o of M if the dimension of $T(X_o)$ is n. If this is true for every $X \in M$, then the system (I.11) satisfies the reachability-controllability rank condition at M. ∎

<u>Theorem I.5</u>. If (I.11) is locally weakly reachable-controllable, then the reachability-controllability rank condition is satisfied on an open subset of M, that is, the <u>rank condition is satisfied generally on M</u>.

The proof of the above theorem, given by Hermann and Krenner [1977] is very similar to theorem I.1, it is constructed by a definition of submanifolds over the different trajectories of the system through a given point. The details are ommited here. ∎

The above results are related to the general system (I.11). In practice it is more common to find systems of the form:

$$\dot{X}(t) = f(x) + \sum_{i=1}^{m} u_i \, g_i(X) , \qquad X(0)= X_o \in M \subset R^n. \qquad (I.12).$$

Hunt presented, in several publications ([1979], [1980], [1982.a] and [1982.b]), a practical characterization of the rank condition for systems given by equation (I.12). This condition can be summarized by the main result obtained in [1982.b]:

If every integral manifold N of the Lie algebra denoted L_g, generated by g_i i= 1,..., m contained in M, contains a point X, where f(.) is tangent to N, then the system (I.12) is controllable if the following conditions hold for at least one such X in each submanifold N:

1) There exists a basis $h_1, \ldots, h_k$ of L_g near X and integers $l_1, \ldots, l_k$, such that the space spanned by

$$\{ h_1(X), \ldots, h_k(X), [f^*, h_1](X), \ldots, [f^*, h_k](X),$$

$$(ad^{l_1} f^*, h_1)(X), \ldots, (ad^{l_k} f^*, h_k)(X) \}$$

has dimension n, where the vector function is defined as:

$$f^* = f - \sum_{k=1}^{k} c_i h_i$$

and c_i, i= 1, ..., k are constants. Also $(ad^2 f^*, h) = [f^*, [f^*, h]]$ and so on.

2) If the vector fields g_i, i= 1 ,..., m are linearly independent and involutive on M, the above condition is replaced by the span of

$$\{ g_1(X), \ldots, g_2(X), [f, g_1], \ldots, [f, g_m], \ldots,$$
$$(ad^{l_1} f, g_1)(X), \ldots, (ad^{l_m}, g_m)(X) \} .$$

This thesis shows the development of a practical application of these results in the design of an Automatic Flight Control System for a single rotor helicopter. The treatment of this theory, in order to obtain an instrument of design, is the topic of Chapter III, after the presentation of a model of the system in Chapter II.

I.3 THE SYSTEM MODEL.

The application of the concepts presented in the previous section require an appropriate representation of the system, in the present case, a single main rotor helicopter. In this thesis a representation derived from the model reported by Padfield [1981] and implemented in a simulation package called "Helistab" has been used. This simulation package was developed in the Royal Aircraft Establishment, Bedford. In general this is considered a very well validated model.

The simulation package "Helistab" is a very helpful instrument in the study of helicopter dynamics, but from the point of view of control system design "Helistab" presents a serious limitation. There is no access to the definition of the command inputs according to a control law defined by the user. Moreover, the inputs can not be defined "on line".

In "Helistab" the helicopter is simulated according to an equation of the form:

$$\dot{X}(t) = f(X) + G(X,u) \qquad (I.14) .$$

That is, the rate of change of the state is the sum of a "drift" term $f(X) \in C^{\infty}$ and a "driven" term $G(X,u)$. This last term represents the forces and moments exerted on the helicopter which depend on the state X and the input command vector u. As shown in section I.1, the "driven" term must be expressed in the form of a sum of the control commands, namely:

$$\sum_{i=1}^{r} u^{*}_{i} \; g_{i}(X) \; , \qquad (I.15) .$$

Where u^{*}_{i} are the input commands and nonlinear relationships among them and in this thesis, the terms g_i are called the control vectors. That is, it is necessary to express the forces and moments in function of the inputs.

The obtaining of such a function has not yet been reported. In chapter two of this thesis a six degree of freedom model of a helicopter, where the "driven" term is expressed according to equation (I.15) is obtained. In this chapter it is shown that the "driven" term has the form:

$$\theta_o\, g_1(X) + \theta_{1s}\, g_2(X) + \theta_{1c}\, g_3(X) + \theta_o^2\, g_4(X) + \theta_o\, \theta_{1s}\, g_5(X) +$$

$$\theta_o\, \theta_{1c}\, g_6(X) + \theta_{1s}^2\, g_7(X) + \theta_{1s}\, \theta_{1c}\, g_8(X) + \theta_{1c}^2\, g_9(X) + \theta_p\, g_{10}(X)\ .$$

Where;

θ_o: is the collective main rotor command;

θ_{1s}: is the longitudinal cyclic main rotor command;

θ_{1c}: is the lateral cyclic main rotor command and

θ_p: is the tail rotor collective command.

The vector fields $g_i(X)$, $i=1,\ldots,10$, depend on the aerodynamic characteristics of the helicopter.

The details of the model used in this thesis are presented in chapter two and in the appendices II.i, i=1,..., 3 .

I.4 NONLINEAR SYSTEM FEEDBACK EQUIVALENCE.

In section I.1 of this introductory chapter the generalization of controllability to "smooth" nonlinear systems was presented briefly. The use of these ideas is treated extensively in chapter three in order to design nonlinear control systems. These ideas were originally introduced by Su [1982] and further developed by Hunt, Su and Meyer [1983]. The same results were independently published by Jackubzyk and Respondec [1980] and Respondec [1985].

A reader unfamiliar with differential geometry and differential manifolds will find the above publications very difficult to follow, between their abstract and the conclusion there is a body of mathematical steps which are not presented in great detail. From the point of view of a possible user of this theory, it is important to grasp the fundamentals of the results to be applied. The purpose of chapter three is to develop in detail the works by the authors mentioned above.

The aim is given a nonlinear system of the form of equation (I.12) find a diffeomorphic transformation in function of its input to a linear canonical form. That is obtaining a nonlinear control law such that the original system behaves like a linear controllable canonical form. In this chapter the nessesary and sufficient conditions required by a nonlinear system in order to obtain the diffeomorphic map mentioned above is presented. A constructive proof of the existance of this map is also shown. All the relevant steps are presented.

I.5 THE FLIGHT CONTROL SYSTEM DESIGN.

The application of Nonlinear Control Systems Theory to flight mechanics is presented in the form of a Flight Control System in chapter four.

The design is composed of several aspects. The complex nature of the system, force one to perform some simplifying assumptions, namely on the input vectors and by considering the helicopter as a two time scale system.

It is shown that the simplification on the input vectors allows one to apply a compensator, which performs a partial linearization and decoupling of the system.

The closed-loop system obtained by introducing this compensator is analysed according to the theory developed in chapter three. A diffeomorphic transformation of the closed-loop system to a linear controllable canonical form is obtained. The development of Symbolic Algebraic Manipulation programmes required in the design are presented. The relationship between the nonlinear system and the linear canonical form as a function of the control inputs of the nonlinear system and the linear one is also presented. Furthermore this relationship is solved for the nonlinear system inputs, so that it is possible to calculate the inputs for the nonlinear system equivalent to the inputs of the linear system. That is, given a input in the linear system one can calculate an input that would drive the nonlinear system in an equivalent way to the linear system.

The next step is to generate the control input of the linear system, the pole assignment technique is used in this thesis.

The Flight Control System is composed of the compensator, the diffeomorphic map of the closed-loop system obtained by applying the compensator to the helicopter and a linear controller (pole placement techniques).

The performance of the Flight Control System is investigated by a series of simulations. According to the diffeomorphic map the nonlinear system is mapped to four decoupled linear systems. The simulations are intended to show that the Flight Control System divides the helicopter state into four sub-systems, each one corresponding to the normal, longitudinal, lateral and heading movements of the vehicle.

CHAPTER II

A HELICOPTER MODEL.

Summary

In this chapter the equations of motion and orientation of a single rotor helicopter are obtained in an (f,g) distribution form.

The interest in such models arises from the necessity to study from the point of view of control systems, rather than design, the dynamical behaviour of the helicopter. The scope of the model presented here is intended to be valid not only around a particular flight condition, but over a set of manoeuvres around a given operating point.

The model derived here is a modification of the equations of motion used in HELISTAB, which is a simulation package developed in the Royal Aircraft Establishment Bedford, for helicopter flight mechanic studies.

II.1 INTRODUCTION

A model is formally defined in terms of the relations between general systems, for instance (Kramer, Smit de [1977]):

" If a system M, epistemologically independent of a system S, is used to obtain information about system S, it is said that M is a model of S ".

It is clear that helicopters require a set of models, each one referring to a particular aspect, for example; design, operation or stability and control. In particular, this thesis is concerned with flight control, a topic that rests completely on formal models. These models are defined as symbolical sets of statements in logical terms about an idealized, relatively simple, situation that represents the structural characteristics of the original factual physical system. Within the scope of the present study, only one kind of formal model is required, namely a mathematical model. Entering more into the subject of flight control, this is considered to be composed of (McRuer, Ashkenas, Graham [1973]):

Guidance: The action of determining the course and speed relative to some reference system to be followed by the vehicle.

Control: The development and application of forces and moments to a vehicle which:

1. Establish some equilibrium state of the vehicle motion (operating point control).
2. Restore a disturbed vehicle to its equilibrium state (operating point) and

regulate, within desired limits, its departures from operating point conditions (stabilization).

With respect to control, two more points could be included, one referring to the improvement of the facility to execute manoeuvres by the pilot (handling qualities) and the other referring to the determination of the set of manoeuvres that the vehicle can execute (agility).

The analysis of guidance, control, agility and handling qualities are theoretically supported by formal models, from which the responses to exogenous and control inputs, as well as stability can be estimated. The importance of having an aircraft model directed to this study has been considered since the begining of aeronautical science (MacRuer, Ashkenas, Graham [1973]), such models according to the classification given by Rosenblueth and Winer (Kramer, Smit de [1977]), had been material and formal. Nowadays it is more appropriate to consider formal models during the first stage of flight mechanic studies. This is due mainly to the developement of new mathematical tools and the enormous power of calculation provided by computers. This is in contrast with the early days of aeronautics in which prototypes of the vehicle (material models) were used to investigate its flying qualities.

Most of the models used in the analysis are mathematical representations obtained via the theory of infinitesimal motions or perturbations around an operating point. This can be seen in any text on stability and control, automatic control applied to aircraft and missiles, and even in recent publications . This procedure involves the neglect of second order terms. It also leads to a system of linear differential equations as a representation of the system obtained, a fact that allows exploitation of its homogenity and additivity features. In these circunstances,assumptions valuable information about the behaviour of an aircraft in a given flight condition can be obtained, in a significant manner.

The approach usually followed in order to analyse the dynamic characteristics of an aircraft is the linear system theory. This theory is based on representations expressed as linear differential equations. The results obtained are valid whenever the relationship between the physical system and the model correspond to a homeomorphism, that is, the relationship of model system entities correspond to an analog relationship of equivalent entities of the physical system. Under the hypothesis of small departures from the flight condition, it is possible to determine, for example, the effect of the longitudinal cyclic or elevator command on the pitch attitude of a helicopter or aeroplane respectively. In this case the system is considered to be composed of Single Input- Single Output (SISO) subsystems, so that the transfer function approach is adequate (McRuer, Ashkenas and Graham [1973]) and (Wanner). Given that the commands of the helicopter are strongly coupled, the physical system is represented better by Multi Input- Multi Output (MIMO) models, a fact that can be confirmed by looking at the rotor dynamic equations (Bramwell [1976]), (Johnson [1980]), (Gessow and Myres [1952]) or a helicopter dynamic model itself (Padfield [1981]). The state space approach establishes the way in which the problem can be reformulated and solved. These ideas were formalized with the controllability and observability concepts in the early 60's by Kalman.

The validity of the results obtained using the linear system theory is restricted by the hypothesis of small perturbations or variations around an operating point. In the case where the restrictions are violated, the physical system and the model will not be related by a homeomorphism, or in extreme conditions not related at all. This latter point can be ilustrated by comparing the linear models of a helicopter at hover and longitudinal flight, for example at 80 knots. This is an important limitation when large perturbations and manoeuvres are involved. One possible solution is the use of several linear models for every operating point. This technique requires an apriori schedule of the manoeuvres and a complicated logic should be incorporated to the flight computer for switching the perturbation control gains and reference control as the aircraft leaves the domain of validity of one perturbation model and enters another. Even the procedure for choosing the set of reference trajectories about which to perturb is unclear at present. This solution is complex in concept and implementation (Meyer, Ciccolani [1975]).

These difficulties mentioned above have motivated the application of nonlinear control theory for the future development of flight control systems.

The object of this chapter is to develop a helicopter model, which will accomplish the fundamental condition of homeomorphism with respect to the physical system, beyond the small perturbation assumption. This model is essential to the development of this research. It will play the role of system "M" in the contest of the definition of "model" given previously.

The following helicopter dynamic model was obtained according to the notation and general development reported by Padfield [1981] and implemented in "Helistab" which is a simulation package developed in the Flight Systems Department of the Royal Aircraft Establishment, Bedford. This package is generally accepted as a well validated model of a six degree of freedom single rotor helicopter.

The model obtained here is restricted to the same assumptions of the six degrees of freedom model used in "Helistab", and therefore its validity can not exceed the limits of these restrictions. Nevertheless, the following model considers the most important nonlinearities.

II.2 NATURE OF THE MODEL.

As Babister [1980] pointed out, flight dynamics deals with the motion of aircraft under the influence of forces, which can be of the five types listed below (inertia "forces" have been eliminated, this can be done if the principle of d'Alembert is not used):

1. Aerodynamic damping forces and moments, depending on the angular velocities of the aircraft.
2. Aerodynamic forces and moments depending on the translational velocities of the aircraft.
3. Aerodynamic forces and moments due to the application of controls.
4. Gravitational forces.
5. Propulsive forces, in the case of the helicopter these forces are strongly related to the aerodynamic forces due to the application of controls.

The flying characteristics of an aircraft depend on its response to the application of its commands which generate the necessary forces of types 3 and 5 (given in the above list). These balance the other forces, making the aircraft execute a desired trajectory. Due to this fact, the problem can be formulated from the system theory point of view. The helicopter will be referred to as the system, in this case it will be associated with a deterministic mathematical representation, specified by five sets; (**T**, **U**, **Y**, **X**, **Ω**) and two functions ˆø and ñ, where:

T is the time set, a subset of the real numbers or natural numbers.

V is the input set, in this case composed of the helicopter commands.

Y is the output set. $\mathbb{R}^P$, $P \in \mathbb{N}$.

X is the state set. $\mathbb{R}^n$, $n \in \mathbb{N}$ $n \geq P$.

Ω is the set of admissible input functions, a subset of the set of all functions T→V, which is closed under splicing, that is, for all u_1 and u_2 in Ω, for all times t_2 in T, there exists a function u^* in Ω such that:

$$u^*(t) = \begin{cases} u_1 & \text{if } t \geq t_2 \\ u_2 & \text{if } t_2 < t \end{cases}$$

$\hat{\phi}: T \times T \times X \times \Omega \rightarrow X$.

The latter is the state transition function and satisfies the conditions of consistency

$\hat{\phi}(t_o, t_o, x, u) = x$

and

$\hat{\phi}(t_2, t_1, \hat{\phi}(t_1, t_o, x, u), u) = \hat{\phi}(t_2, t_o, x, u)$,

for all times t_o, t_1 and t_2 all state x and all admissible input functions u.

The condition of causality is

$\hat{\phi}(t_1, t_o, x, u_1) = \hat{\phi}(t_1, t_o, x, u_2)$

and if $u_1(t) = u_2(t)$ for $t_o \leq t_1$.

$\hat{\eta}: T \times X \rightarrow Y$ is the output function.

The system is assumed to be stationary (constant or time invariant): this implies that T is closed under addition, Ω is closed under the shift operator $z^{-\tau}$ for every τ in T and

$$\hat{\phi}(t_1, t_o, x, u) = \hat{\phi}(t_1+\tau, t_o+\tau, x, z^{\tau}u)$$

for all times t_o, t_1, all delays τ, all states x and all admissible input functions u;

$\hat{n}(t_0,x)= \hat{n}(t_1, x)$

for all times t_0, t_1 and all states x.

The above condition allows one to remove one time variable, replacing $\hat{\varnothing}$ and $\hat{n}$ by simpler $\varnothing$ and n, where:

$\varnothing$: $\mathbf{T} \times \mathbf{X} \times \Omega \to \mathbf{X}$

and

n: $\mathbf{X} \to \mathbf{Y}$

In physics, it is usually not $\varnothing(.)$ that is given, but rather the laws of motion. In other words, some differential equations are given that must be solved in order to find the state transition function. These equations of motion have the form:

$$dX/dt= f(X), \qquad X(0)=X_0 \text{ given}$$

where f is a (possibly time dependent) vector field on $\mathbf{X}$. This last relationship allows the elaboration of a formal structure for the model, for example; the function f can be a vector field on a manifold M, an integral curve of f can be defined at some point m of M, defined as a curve c(.) at m such that its derivative $c'(\lambda)=f(c(\lambda))$ for each λ in a subset of $\mathbf{R}$. The obtaining of the integral curve $c(\lambda)$ leads to a set of ordinary differential equations. Their solution rests on the well-known existence and uniqueness theorems for ordinary differential equations (Padulo, Arbib [1974]), (Thorpe [1985]), (Abraham, Marsden [1981]). This problem will be treated in detail in the next chapter: the present chapter is dedicated to the development of a function f for the helicopter.

II.3 AXES ,SYMBOLS AND EQUATIONS OF MOTION.

The general equations representing the motion of an atmospheric flying vehicle are usually referred to body axes system, which is described as follows.

The body reference frame is represented by **OXYZ**, where the origin **O** lies on the centre of gravity of the helicopter; The axes **OX, OZ**, lie in the plane of symmetry and the axis **OY** is perpendicular to it. The name of the axes are:

OX: Longitudinal axis, positive forward.

OY: Lateral axis, positive starboard.

OZ: Normal axis, positive toward the undercarriage of the vehicle.

The axes will be right handed.

The orientation of the helicopter with respect to the inertial frame, assumed fixed on earth will be specified by the vehicle Euler angles θ, ϕ and ψ:

ψ: Rotation about the axis **OZ**, carries the axes to $OX_\psi Y_\psi Z_\psi$;

θ: Rotation about the axis **OY**, carries the axes to $OX_\theta Y_\theta Z_\theta$ and

ϕ: Rotation about the axis **OX**, carries the axes to their final orientation.

In order to facilitate the analysis the following assumptions are considered:

1) The Earth is a stationary plane in the inertial space.
2) The centripetal acceleration associated with the Earth's rotation is neglected
3) The atmosphere is at rest relative to the Earth.
4) The helicopter is a rigid body.

The symbols used to represent the components of the velocity of the centre of gravity of the helicopter and of its angular velocity, together with the components of the aerodynamic forces, and moments and products of inertia of the helicopter are summarised in the following tables:

VELOCITY AND FORCE COMPONENTS

AXIS	X	Y	Z
VELOCITY	u	v	w
FORCE	X_F	Y_F	Z_F
POSITIVE DIRECTION	forward	starboard	downwards
NAME	longitudinal	lateral	normal

ANGULAR VELOCITY AND MOMENT COMPONENTS

AXIS	X	Y	Z
MOMENT	L_M	M_M	N_M
ANGULAR VELOCITY	p	q	r
POSITIVE DIRECTION	starboard down	nose up	nose to starboard
NAME	rolling	pitching	yawing

MOMENTS AND PRODUCTS OF INERTIA

AXIS	X	Y	Z
MOMENTS OF INERTIA	$I_x=\int(y^2+z^2)\,dm$	$I_y=\int(x^2+z^2)\,dm$	$I_z=\int(x^2+y^2)\,dm$
PRODUCT OF INERTIA	$I_{yz}=\int yz\,dm=0$	$I_{xz}=\int xz\,dm$	$I_{xy}=\int xy\,dm=0$

The general equations representing the motion of a rigid flying vehicle referred to body axes are well known(Etkin [1972]), (Seckel [1964]), (Johnson [1980]), (Babister 1980]), i. e.:

TRANSLATIONAL EQUATIONS

$$\dot{u}= vr - wq - G\ (\sin\theta) + (1/m)\ X_F$$

$$\dot{v} = wp - ur + G\,(\cos\theta) + (1/m)\,Y_F$$

$$\dot{w} = uq - vp + G\,(\cos\theta) + (1/m)\,Z_F$$

ROTATIONAL EQUATIONS

$$I_{xx}\,\dot{p} = (I_{yy} - I_{zz})\,qr + I_{xz}\,(\dot{r} + pq) + L_M$$

$$I_{yy}\,\dot{q} = (I_{zz} - I_{xx})\,rp + I_{xz}\,(r^2 - p^2) + M_M$$

$$I_{zz}\,\dot{r} = (I_{xx} - I_{yy})\,pq + I_{xz}\,(\dot{p} - qr) + N_M$$

ORIENTATION (KINEMATIC EQUATION)

$$\dot{\phi} = p + q\,(\sin\phi)\,(\tan\theta) + r\,(\cos\phi)\,(\tan\theta)$$

$$\dot{\theta} = q\,(\cos\phi) - r\,(\sin\phi)$$

$$\dot{\psi} = q\,(\sin\phi)\,\sec(\theta) + r\,(\cos\phi)\,\sec(\theta)$$

where G is the acceleration due to gravity.

The total forces and moments are the sum of the contribution of each vehicle element. In the case of a single main rotor helicopter the contributions are due to the main rotor, tail rotor, tailplane, fin and fuselage. Considering this, the forces can be expressed as reported by Padfield [1981]:

$$X_F = X_R + X_T + X_{TP} + X_{FN} + X_f$$

$$Y_F = Y_R + Y_T + Y_{TP} + Y_{FN} + Y_f$$

$$Z_F = Z_R + Z_T + Z_{TP} + Z_{FN} + Z_f$$

where the indices from left to right, are the contributions of the main rotor, tail rotor, tail plane, fin and fuselage respectively. If the vertical plane of the helicopter is considered to be a plane of symmetry then the force equations are reduced to:

$$X_F = X_R + X_f$$

$$Y_F = Y_R + Y_T + Y_{FN} + Y_f$$

$$Z_F = Z_R + Z_{TP} + Z_f$$

In appendix A.II.1 it is shown that the rotor forces can be expressed as:

$$X_R = \langle K_{FX}, \theta_{RA} \rangle$$

$$Y_R = \langle K_{FY}, \theta_{RA} \rangle$$

$$Z_R = \langle K_{FZ}, \theta_R \rangle$$

where the symbol $\langle .,\ . \rangle$ indicates the inner product operation, the vector θ_R is a vector whose components are the main rotor commands and a constant,

$$\theta_R = [\theta_0,\ \theta_{1s},\ \theta_{1c},\ 1]^t$$

where

θ_0: is the collective command,

θ_{1s}: is the longitudinal cyclic command,

θ_{1c}: is the lateral cyclic command

and t: indicates transpose.

On the other hand, the vector θ_{RA} is composed of the rotor commands and its coupled and nonlinear terms:

$$\theta_{RA} = [\theta_0,\ \theta_{1s},\ \theta_{1c},\ \theta_0^2,\ \theta_0\ \theta_{1s},\ \theta_0\ \theta_{1c},\ \theta_{1s}^2,\ \theta_{1s}\ \theta_{1c},\ \theta_{1c}^2,\ 1]^t$$

The vectors K_{FX}, K_{FY} and K_{FZ} are vectors whose components are functions of the aerodynamic parameters and the state, these functions are described in Appendix II.1.

The fuselage forces X_f, Y_f and Z_f and moments L_f, M_f and N_f are calculated from semiempirical forms using wind data tunnel (Padfield [1981]). These forces do not depend directly on the helicopter commands, as can be seen in appendix II.3. Their expressions will not be expressed as inner products.

The fin and tailplane forces are described in Appendix II.2. These expressions are not changed to inner products also as they do not depend directly on the input commands.

The tail rotor force Y_T is analysed in appendix II.4. Where it is shown that this force can be expressed as:

$$Y_T = \langle K_{TAIL}, \theta_{TR} \rangle$$

where

$$\theta_{TR} = [\theta_{OT}, 1]^t$$

and θ_{OT} is the tail rotor command.

Therefore the force equations can be expressed as

$$X_F = X_f + \langle K_{FX}, \theta_{RA} \rangle$$

$$Y_F = Y_{FN} + Y_f + \langle K_{TAIL}, \theta_{TR} \rangle + \langle K_{FY}, \theta_{RA} \rangle$$

$$Z_F = Z_{TP} + Z_F + \langle K_{FZ}, \theta_R \rangle \ .$$

The moment equations can be written in a similar way

$$L_M = L_R + L_T + L_{TP} + L_{FN} + L_f$$

$$M_M = M_R + M_T + M_{TP} + M_{FN} + M_f$$

$$N_M = N_R + N_T + N_{TP} + N_{FN} + N_f \quad .$$

If the assumptions of symmetry are considered again then the moment equations can be reduced to:

$$L_M = L_R + L_T + L_{FN}$$

$$M_M = M_R + M_{TP} + M_f$$

$$N_M = N_R + N_T + N_{FN} + N_f \ .$$

In appendix II.1 it is shown that the main rotor moments can be expressed as inner products,

$$L_R = \langle K_{RL}, \ \theta_{RA} \rangle$$

$$M_R = \langle K_{RM}, \ \theta_{RA} \rangle$$

$$N_R = \langle K_{RN}, \ \theta_{RA} \rangle \quad ,$$

where the vector functions K_{RL}, K_{RM}, K_{RN} are described in appendix II.1. In appendix II.2 it is found that the tail rotor moment can be written as

$$L_T = \langle K_{TL}, \ \theta_{TR} \rangle$$

$$N_T = \langle K_{TN}, \ \theta_{TR} \rangle \ .$$

The vector functions K_{TL} and K_{TN} depend on the state and the aerodynamic characteristics of the helicopter.

The description of the tail plane, fuselage and fin moments contribution are described in appendix II.3. As these moments do not depend on the input commands they are not expressed as inner products.

Replacing the above expressions for the moment contributions of the rotors in the moment equations we obtain

$$L_M = L_F + \langle K_{TL}, \ \theta_{TR} \rangle + \langle K_{RM}, \ \theta_{RA} \rangle$$

$$M_M = M_{TP} + M_F + \langle K_{RM}, \ \theta_{RA} \rangle$$

$$N_M = N_{FN} + N_F + \langle K_{TN}, \ \theta_{TR} \rangle + \langle K_{RN}, \ \theta_{RA} \rangle \ .$$

In order to achieve only one vector referring to the helicopter commands, a general vector θ is defined which is formed as follows:

$\theta = [\theta_0,\ \theta_{1s},\ \theta_{1c},\ \theta_0^2,\ \theta_0\ \theta_{1s},\ \theta_0\ \theta_{1c},\ \theta_{1s}^2,\ \theta_{1s}\ \theta_{1c},\ \theta_{1c}^2,\ \theta_{TR},\ 1]^t$

Then the vector θ can be considered as the input command vector of the helicopter, thus permitting the dynamic equations to be expressed as:

FORCE EQUATIONS

$$X_F = X_f + \langle K_X,\ \theta \rangle$$

$$Y_F = Y_{FN} + \langle K_Y,\ \theta \rangle$$

$$Z_F = Z_{TP} + Z_f + \langle K_Z,\ \theta \rangle\ ,$$

where the vectors K_X, K_Y and K_Z are defined as

$$K_X = (1/m)\ (K_{FX1}, \ldots, K_{FX9},\ 0,\ K_{FX10})^t$$

$$K_Y = (1/m)\ (K_{FY1}, \ldots, K_{FY9},\ K_{TAIL1},\ K_{FY10} + K_{TAIL2})^t$$

$$K_Z = (1/m)\ (K_{FZ1}, \ldots, K_{FZ3},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ K_{FZ4})^t\ .$$

MOMENT EQUATIONS

$$L_M = L_{FN} + \langle K_{TL},\ \theta_{TR} \rangle + \langle K_L,\ \theta \rangle$$

$$M_M = M_{TP} + M_F + \langle K_M,\ \theta \rangle$$

$$N_M = N_{FN} + N_F + \langle K_N,\ \theta \rangle\ ,$$

where the vectors K_L, K_M and K_N are defined as

$$K_L = (1/I_{XX})\ (K_{RL1}, \ldots, K_{RL9},\ K_{TL1},\ K_{RL10} + K_{TL2})^t$$

$$K_M = (1/I_{YY})\ (K_{RM1}, \ldots, K_{RM9},\ 0,\ K_{RM10})$$

$$K_N = (K_{RN1}, \ldots, K_{RN9},\ K_{TN1},\ K_{RN10} + K_{TN2})$$

Finally, the equations of motion of the helicopter can be put in terms of of the inner product as follows:

TRANSLATIONAL EQUATIONS

$$\dot{u} = vr - wq - G\ (\sin\theta) + (1/m)\ X_F + \langle K_X,\ \theta \rangle$$

$$\dot{v} = wp - ur - G (\cos\theta)(\sin\phi) + (1/m) Y_{FN} + (1/m) Y_f + \langle K_Y, \boldsymbol{\theta} \rangle$$

$$\dot{w} = uq - vp - G (\cos\theta)(\cos\phi) + (1/m) Z_{TF} + (1/m) Z_f + \langle K_Z, \boldsymbol{\theta} \rangle$$

ROTATIONAL EQUATIONS

$$\dot{p} = (1/I_{XX}) \{ (I_{YY} - I_{XX}) qr + I_{XZ} (\dot{r} + pq) + L_{FN} \} + \langle K_L, \boldsymbol{\theta} \rangle$$

$$\dot{q} = (1/I_{YY}) \{ (I_{ZZ} - I_{XX}) rp + I_{XZ} (r^2 - p^2) + M_{TF} + M_f \} + \langle K_M, \boldsymbol{\theta} \rangle$$

$$\dot{w} = (1/I_{ZZ}) \{ (I_{XX} - I_{YY}) pq + I_{XZ} (\dot{p} - qr) + N_{FN} + N_f \} + \langle K_N, \boldsymbol{\theta} \rangle$$

In this chapter the dynamic equations of the helicopter, in which the input commands are associated in separate terms, have been obtained. This allows the helicopter flight dynamic model to be expressed in the following form

$$\dot{x}(t) = f(x) + \sum_{i=1}^{m} u_i \, g_i(x)$$

where x is in the R^n space, n= 9; u_i are the elements of the vector $\boldsymbol{\theta}$; x= [u, v , w, p, q, r, ϕ, θ, ψ] ^{t}; the function f is

$$
f(x)=\left|\begin{array}{c}
vr - wq - g(\sin\theta) + (1/m)X_f + K_{x11} \\
wp - ur + g(\cos\theta)(\sin\phi) + (1/m)(Y_{FN} + Y_f) + K_{Y11} \\
uq - vp + g(\cos\theta)(\cos\phi) + (1/m)(Z_{TP} + Z_f) + K_{Z11} \\
(1/I_{XX})\{(I_{YY} - I_{ZZ})qr + I_{XZ}(r` + pq) + L_{FN}\} \\
(1/I_{YY})\{(I_{ZZ} - I_{XX})rp + I_{XZ}(r^2 - p^2) + M_{TP} + M_f\} \\
(1/I_{ZZ})\{(I_{XX} - I_{XZ})pq + I_{XZ}(p` - qr) + N_{FN} + N_f\} \\
p + q(\sin\phi)(\tan\theta) + r(\cos\phi)(\tan\theta) \\
q(\cos\phi) - r(\sin\phi) \\
q(\sin\phi)(\sec\theta) + r(\cos\phi)(\sec\theta)
\end{array}\right|
$$

and;

$\theta = \mathbf{u} = [u_1, u_2, \ldots, u_{10}]^t$.

Finally

$$g_i = [K_{Xi}, K_{Yi}, K_{Zi}, K_{Li}, K_{Mi}, K_{Ni}, 0, 0, 0]^t; \quad i = 1,\ldots,m; \quad m = 10.$$

The model presented here has the advantage of enabling the functions f and g_i to be considered as a distribution on a manifold U contained in R^n (for this reason this equation will be referred to as the (f,g) distribution model form). In the following chapter an application of nonlinear control theory based on differential geometry will be presented; in this, it is necessary to have an (f,g) model of the system.

In table 1 the data for a representative transport helicopter are presented. These are the data which are used in throughout thesis.

Chapter III.

LINEAR EQUIVALENCE OF NONLINEAR SYSTEMS.

Summary .

This chapter deals with a direct application of differential geometry to the nonlinear control system problem. The application is presented in detail and some of the more important mathematical tools are discussed. This application rests on the concept of the feedback equivalence theory of nonlinear systems introduced by Respondeck [1985] and Brocket [1978]. However the essence of the method presented here was developed by Su, Hunt and Meyer [1983]. The nonlinear system is assumed to be represented by an (f,g) distribution model.

III.1 INTRODUCTION.

In recent years there has been an increasing interest in the linearization problem of nonlinear control systems. In many circumstances linear models of nonlinear systems do suffice for the design of controllers and observers. However, as pointed out in the previous chapter, in many other situations the intrinsic nonlinearities of the studied system are of interest. To pose the problem in other words: is it possible to transform the system into a linear form using such a transformation as a change of coordinates in the state space, state-input space, output-feedback and input injection?.

When studying smooth systems, differential geometry provides methods and tools like Lie brackets of vector fields, the Lie derivatives of functions, involutive distributions and integral manifolds which help to answer the above questions and to distinguish those systems which may be treated as linear ones.

Among all problems the most natural question is when does a change of coordinates exist?, in other words when is there a diffeomorphism which carries the given nonlinear system into a linear one. Studying such questions, Krener [1973] showed the importance of the Lie algebra of vector fields generated by the system. He also showed when it was possible to obtain such a change of coordinates. Later Brockett in [1978] enlarged the studied class of transformations by also allowing a certain form of feedback to act: his paper has inspired the subsequent research in the field. Among the works originated by Brockett's work are Jakubczyk, Respondec [1980], Su [1982] and Hunt, Su [1981]; in these works, the full feedback group was considered and gave necessary and sufficient conditions for linearization.

In this chapter the system will be assumed to be of the form:

$$\dot{x}(t)= f(x) + \sum_{i=1}^{m} u_i \, g_i(x) \quad ; \qquad x \in R^n \qquad \text{(III.NLS.1)}$$

where f, g_1, g_2,..., g_m are smooth vector fields on R^n. The word smooth will always mean C^∞.

In most cases the assumption of infinite differentiability is not essential, but it is assumed in order to avoid having to count the degree of differentiability needed in some cases.

It is assumed that an initial state $x_o \in R^n$ around the operating point is given. The linear equivalent system is required in the form:

$$\dot{y}(t)= A\, y + \sum_{i=1}^{m} v_i \, b_i(x) \quad ; \qquad y \in R^n \qquad \text{(III.L.1)}$$

Throughout this chapter it is considered that $f(x_o)= 0$. Without this assumption the results still hold, but it is necessary to consider a constant in the vector field Ay. This problem can be avoided if the linearizing transformation maps x_o to 0. Reboulet and Champetier [1984] presented a pseudo-linearization which does not depend on the operating point.

The results presented below are local. This means that the conditions requied need to hold locally around the initial state x_o, and the linearizing transformation exists locally around this point. Nevertheless Hunt, Su and Meyer [1983] gave the necessary conditions for global aspects. The same problem has been studied separately by Respondec [1985].

The spirit of the feedback linearization can be described with the aid of an academic example. The example presented by Su [1982], which has been used in subsequent publications, for clarification and comparation, for example by Reboulet and Champetier [1984], is used here:

Consider the nonlinear system;

$$\begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \end{vmatrix} = \begin{vmatrix} x_2 \\ 0 \end{vmatrix} + \begin{vmatrix} \cos(x_1) \\ 1 \end{vmatrix} u \qquad \text{(III.A)}$$

on R^2. It is straigthforward to observe that if new coordinates of the following form are introduced:

$$z_1 = -x_2 + \ln | \tan(\tfrac{1}{2} x_1 + \tfrac{1}{4} \pi) | \quad ; \qquad \text{(III.B.1)}$$

$$z_2 = x_2/\cos(x_1) \quad ; \qquad \text{(III.B.2)}$$

$$v_3 = x_2^2 \tan(x_1)/\cos(x_1) + u \{x_2 \sin(x_1) + 1\}/\cos(x_1) \qquad \text{(III.B.3)} ,$$

then the above system takes the following linear form:

$$\begin{vmatrix} \dot{z}_1 \\ \dot{z}_2 \end{vmatrix} = \begin{vmatrix} z_2 \\ 0 \end{vmatrix} + \begin{vmatrix} 0 \\ 1 \end{vmatrix} v \qquad \text{(III.C)}$$

This means that the nonlinearity of system (III.A) is not intrinsic and occurs due to an unfortunate choice of coordinates. If the coordinates are replaced according to equations (III.B), the linear system (III.C) is obtained.

The above leads to the problem of finding a change of coordinates of the state-input space, namely linearization by means of a diffeomorphism expressed in terms of the control input, that is by feedback.

Consider a nonlinear control system of the form:

$$\dot{x}(t) = f(x) + \sum_{i=1}^{m} u_i \, g_i(x) \quad ; \qquad x \varepsilon R^n \qquad \text{(III.NLS.1)}$$

Where f, $g_1, \ldots, g_m$ belong to $V^\infty(R^n)$, the family of vector fields on a manifold contained in R^n. This problem leads to the generalization of the linear feedback case, comprehensively studied by Brunovsky [1970]. Namely, consider the linear system;

$$\dot{y}(t) = A\, y(t) + B\, v \quad ; \qquad y \,\varepsilon\, R^n; \quad v \,\varepsilon\, R^m \qquad \text{(III.L.2)}$$

Its dynamics can be modified by the linear feedback $v = F\, y + H\, w$, where F and H are matrices of appropriate dimensions and H is invertible:

$$\dot{y}(t) = (A + B\, F)\, y + (B\, H)\, w \quad .$$

where w is the reference output.

For the nonlinear system (III.NLS.1) where $u = (u_1, \ldots, u_m)^t$, the nonlinear feedback $u = \alpha(x) + \beta(x)\, v$ is applied, where $\alpha(.)$ and $\beta(.)$ are (m×1) and (m×m) valued smooth functions respectively, and β(.) is invertible. This gives the modified system

$$\dot{x}(t) = (f + G\,\alpha)\,x + (G\,\beta)\,u\ ,$$

where $\sum_{i=m}^{m} u_i\, g_i(x) = G\, u$,

The nonlinear decoupling and noniterating control problems are extensively studied by Isidori, Krener, Gori-Giorgi and Monaco [1981], Isidori and Krener [1982], and Isidori [1985] in particular an answer to the question : when can nonlinear systems of the form (III.NLS.1) be transformed to linear systems of the form:

$$\dot{y}(t) = A\,y + \sum_{i=1}^{m} v_i\, b_i(x) \quad ; \qquad y \varepsilon \mathbf{R}^n$$

under a change of coordinates and a feedback? The change of coordinates is given as T= T(x) and the feedback is of the form v= α(x) + β(x) u . Here it is convenient to remark that equation (III.B.3) has this form, can be solved for u, and that then, is obviously a local result. In fact one of the conditions necessary and sufficient for such a transformation to be global is completeness. Equation (III.B.3) is not defined for all $\mathbf{R}^2$. Nevertheless for some practical purposes this condition does not impede the application in a wide operating range of a certain kind of process, as can be seen in the application presented by Liceaga-Castro and Bradley [1987].

The above questions give rise on the following concept. A C^∞ distribution Δ(p) on a manifold M at p, of dimension l= m + k, is a set of m linearly independent vector fields $g_1,\ldots,g_m$ which form a basis and an m-dimensional subspace of the tangent space of M at p. It is said that Δ(p) is a C^∞ n-plane distribution of dimension m on M and $g_1,\ldots,g_m$ is a local basis of Δ. If Δ(.) is defined for all x ϵ M, it is said that Δ(.) is a regular distribution.

Using this definition, the geometrical interpretation of the problem is in finding a manifold M such that its tangent space is generated by the vector fields $\{f, g_1, \ldots, g_m\}$ and that this state can only be defined on this manifold and nowhere else. Then the problem is reduced to finding a coordinate chart for this manifold, where the state-input relationship is linear. Furthermore this change of coordinates can be solved for the input u.

It is intuitively clear that the results and conclusions from systems using the traditional approaches depend on the theorem of existence and uniqueness of the solution of ordinary differential equations. This dependence is very well presented and discussed from the point of view of control systems by Padulo and Arbib [1974]. The Frobenius theorem plays an equivalent role in the geometric approach. This theorem can be expressed in many ways, and at different theoretical levels, as is evident in the literature of the subject, for instance Lang [1962], Abraham and Marsden [1981], Boothby [1987], Choquet-Bruhat, DeWitt-Morette and Dillard-Bleck [1977], Brickell and Clarck [1970]. The version (a crude one) used in Nonlinear Control Systems is usually stated as: a distribution Δ is integrable if and only if it is involutive.

On the other hand, the Frobenius theorem may be considered as the generalization of the existence theorem (of ordinary differential equations) to certain types of partial differential equations, namely Pfaffian systems. Choquet-Bruhat, DeWitt-Morette and Dillard-Bleck [1977] analysed the parametric solution of the Pfaffian equations from this point of view. The same method is applied in this chapter.

In the following sections of this chapter, the procedure proposed by Hunt, Su and Meyer [1983] is presented in detail. This procedure is composed of some of the ideas mentioned in this section, and is synthesized by the concept of linear feedback equivalence of systems which consists of a change of coordinates of the input-output space and feedback.

III.2 Preliminaries.

The purpose of this section is to present the most important concepts and notation essential to the study of nonlinear systems and in particular to the development of this chapter. The source of these concepts is differential geometry. The notation used in this chapter can be considered as standard. On the other hand, only the strictly necessary concepts are presented, nevertheless some proofs are included in the appendices.

Definition. Lie bracket.

Given two C^∞ $f(.)$ and $g(.)$ on R^n, the Lie bracket of f and g is defined by

$$[f,g]= \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g \quad , \qquad \text{(III.LD.1)}$$

where $\frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial x}$ are the Jacobian matrices.

The Lie bracket operation can be applied successively, i. e.,

$(ad^0 f,g)= g$

$(ad^1 f,g)= [f,g]$

$(ad^2 f,g)= [f, [f,g]]$

.

.

.

$(ad^k f,g)= [f, (ad^{k-1} f,g)]$

Let M be a manifold of dimension $l= m + k$ and assume that to each $p \in M$ is assigned an m-dimensional subspace $\Delta(_p)$ of the tangent space of M, denoted $T_p(M)$. Furthermore suppose that in a neighbourhood U of each p there are m linearly independent C^∞-vector fields $f_1, f_2, \dots, f_m$ which form a basis of $\Delta(_p)$ for every $q \in U$. Then $\Delta(_p)$ is said to be a C^∞ m-plane distribution of dimension m on M and $f_1, f_2, \dots, f_m$ form a local basis of $\Delta(_p)$.

A distribution $\Delta(.)$ is said to be <u>involutive</u> if there exists a local basis $f_1, f_2, \dots, f_m$ in a neighbourhood of each point such that

$$[f_i, f_j] = \sum_{k=1}^{m} c^k{}_{ij}\, f_k \,, \qquad 1 \leqslant i,\ j \geqslant m.$$

The Frobenius theorem can be expressed as follows:

Let $\{f_1, f_2, \dots, f_m\}$ be a set of involutive and linearly independent vector fields in R^n and $x_o \in R^n$, then there exists a unique m dimensional C^∞ submanifold S contained in R^n through x_o, with the tangent space generated by $f_1, f_2, \dots, f_m$. The subset S of R^n is defined as the unique integral manifold of $f_1, f_2, \dots, f_m$ through x_o.

The <u>Lie derivative of a function</u> is defined as follows:

Given a C^∞ function h on R^n and a C^∞ vector field f on R^n, the Lie derivative of h with respect to f is expressed by

$$L_f(h) = \langle dh, f \rangle \qquad \text{(III.LD.2)}$$

where $\langle .,. \rangle$ denotes the duality between one-forms and vector fields and dh denotes the gradient of the function f.

<u>Lie derivatives of one-forms</u>:

If dT is a C^∞ one-form on R^n, the Lie derivative of dT with respect to a C^∞ vector field f on R^n is defined by

$$L_f(T) = f^t \frac{\partial\, dT}{\partial\, x} + dT \frac{\partial\, f}{\partial\, x} \qquad \text{(III.LD.3)}$$

where *t* denotes transpose and $\frac{\partial\, dT}{\partial\, x}$ and $\frac{\partial\, f}{\partial\, x}$ are Jacobian matrices.

In Appendix III.1 it is shown that the gradient of the Lie derivative with respect to a C^∞ vector field f on R^n of a C^∞ function h on R^n is equal to the Lie derivative of the gradient of the function with respect to f, i. e.

$$dL_f(h) = L_f(dh) \qquad \text{(III.LD.4)}$$

In the Appendix III.1 above it is also shown that the three Lie derivatives are related by the Leibnitz rule

$$L_f\langle dT,g\rangle = \langle L_f(dT),g\rangle + \langle dT,[f,g]\rangle \qquad \text{(III.LD.5)}$$

with f and T as before and where g is a C^∞ vector field.

The relations defined in this section are the tools used to define a linear equivalent system for smooth nonlinear systems. The way in which these relations can be handled is presented in the next section.

III.3 EQUIVALENCE OF SYSTEMS.

In the introduction to this chapter a philosophy allowing one to tackle the problem of nonlinear control systems was presented. In the next section some mathematical tools were introduced. In the remainder of this chapter, the application of the ideas described in the first section, using the mathematical concepts defined in the second, will be considered. The ideas presented here were originally developed by Su [1982] and Hunt, Su and Meyer [1983]. These ideas are presented in detail in order to display the essence of this relatively new theory and to make easier its general application.

In this section a transformation T is presented, which transforms a nonlinear system into a linear canonical form. The characteristics of such a transformation are obtained, after which, the transformation T between the nonlinear system and the linear canonical form is presented as a function of the control input. The condition needed in order to obtain the transformation T for the control input is also given. Finally, provided that this condition is accomplished a feedback linearization is defined.

The following definitions and theorems give the grounds for the development of this section.

Consider the inverse function theorem (Abraham, Marsden [1981]) which can be expressed as:

<u>Theorem. III.1</u>. Let W and F be a subset of R^{n+m} and let T be a C^∞ mapping such that $T: W \to F$. Let $x_o \in W$, and assuming that $DT(x_o)$ is a linear isomorphism, then T is a C^∞ diffeomorphism on some neighbourhood of x_o onto some neighbourhood of $T(x_o)$. ∎

Now let x_o be the origin of W. By the above theorem, it is clear that if $DT(x_o)$ is an isomorphism, then T maps a neighbourhood of the origin of W to some neighbourhood of $T(x_o)$. Furthermore, if $T(x_o)$ is the origin of the image, then T can be classified according to the following definition:

Definition III.1. Let W be an open neighbourhood of the origin in the R^{n+m} space.

A τ-transformation T with domain w is a diffeomorphism onto an open neighbourhood of the origin in R^{n+m} which is nonsingular and maps the origin to the origin. ∎

It follows that the set W is assumed to be of the form $X \times R^{n+m}$, where is an open neighbourhood of the origin in R^n and will be referred to as the state space, so that

$$T: W= X \times R^{n+m} \rightarrow Y \times R^n \subset R^{n+m} \quad .$$

The nature of $X \times R^n$ and $Y \subset R^n$ will be defined according to the model structure given in the previous chapter, namely

$(x_1, \ldots, x_n; u_1, \ldots, u_m) \in W$ and $(y_1, \ldots, y_n; v_1, \ldots, v_m) \in T(W)$

denote the state-control variable in W and T(W) respectively.

Now let S_1 and S_2 be two different systems such that

S_1 is defined by $\dot{x}= a(x_1, \ldots, x_n; u_1, \ldots, u_m)$, with state transfer function φ and

S_2 is defined by $\dot{y}= b(y_1, \ldots, y_n; v_1, \ldots, v_m)$, with state transfer function ψ.

Definition III.2. Let W be a subset of R^{n+m}. The system S_1 is τ-related to system S_2 if there exists a τ-transformation T on W such that for each state $x_o \in X$, and each admissible control $u(t) \in \Omega$ (the set of admissible control functions), the following conditions are accomplished:

$$y(0)= T(x_o;\ u(0)) \text{ and } T(\varphi(t,x_o;u),\ u(t))= (y(t),\ v(t))$$

whenever

$$\varphi(t;\ x_o,\ u) \in W \quad \text{then} \quad y(t)= \psi(t;\ y_o,\ v) \quad . \ ∎$$

Definition III.3. If S_1 is τ-related to S_2 by the transformation T with domain W it is said that S_1 is T-related to S_2. ∎

Given the above definitions, the problem can be reduced to finding a τ-transformation which maps the nonlinear system

$$\dot{x}(t)= f(x,t) + \sum_{i=1}^{m} u_i\ g_i(x(t)) , \quad f(0)= f_0 \qquad \text{(III.NLS.1)}$$

to a controllable linear system. In particular it can be mapped to a Brunovsky canonical system

$$\dot{y}(t)= A\ y + B\ v, \qquad \text{(III.1.2)}$$

The matrix A is of the following form:

block diag[U_1, ..., U_m]

where the matrices U_1, ..., U_m are matrices of order $K_i \times K_i$, i= 1, ..., m with the unity in their diagonal and zeros elsewhere. And the matrix B is formed as follows:

$$\begin{vmatrix} e_{K_1} & I_{K_1}^t \\ \cdot & \\ \cdot & \\ \cdot & \\ e_{K_m} & I_{K_m}^t \end{vmatrix}$$

where e_i, i= K_1, ..., K_m, are the K_ith standar basis vector of dimension K_i, and I_i, i= K_1, ..., K_m, are the unity matrices of order K_i x K_i (i= 1 ... , m) respectively.

The indices K_i in matrices A and B are the Kronecker indices and σ_1= K_1, σ_2= K_1+K_2, ..., σ_m= $K_1+...+K_m$ are the controllability indices.

In the work published by Hunt, Su and Meyer [1983], it is shown that if a nonlinear system is equivalent to the Brunovsky canonical form then

a) The Jacobian matrix $\frac{\partial T_i}{\partial u_k} = 0$, i= 1, ..., m and k= 1, ..., m .

b) The m × m matrix $\frac{\partial T_j}{\partial u_k}$, j= n+1, ..., n+m and k= 1, ..., m

c) The foolowing partial differential equations hold on a set W contained in R^{n+m}:

$$\langle dT_1,\ g_i\rangle= 0 , \qquad \text{(III.PD.1)}$$

where l= 1,..., σ_1-1, σ_1+1, ..., $\sigma_{m-1}-1$, $\sigma_{m-1}+1$, ..., n-1.

$$\langle dT_{\sigma_1}, f + \sum_{i=1}^{m} u_i(t)\ g_i(x(t))\rangle = T_{n+1}$$

$$\vdots \qquad \text{(III.PD.2)}$$

$$\langle dT_{\sigma_n}, f + \sum_{i=1}^{m} u_i(t)\ g_i(x(t))\rangle = T_{n+m}$$

and

$$\langle dT_1, f\rangle = T_{1+1} \quad ,$$

where l is the defined as in equation (III.PD.1).

Hunt, Su and Meyer showed that if the transformation T exists then conditions a), b) and c) should be satisfied. This is shown in detail in the three points below:

Condition a). Let x_o, u, y_o, φ and φ be defined as in system S_1 and S_2. If $y_o = T(x_o, u(0))$ and

$T(\varphi(t;\ x_o, u)) = (y(t), v(t))$ so that

$y_j(t) = T_j(\varphi(t;\ x_o, u), u(t))$, for j= 1,..., n and

$v_i(t) = T_i(\varphi(t;\ x_o, u), u(t))$, for i= 1,..., n+m .

By this hypothesis $y = (y_1, \ldots, y_n)$ is a state vector of S_2 with control inputs $v(t) = (v_1, \ldots, v_m)$.

If the linear state equation (III.1.2) is expressed in terms of the transformation T and the partial derivatives with respect to time is taken, then the following equation is obtained:

$$\frac{\partial y_i}{\partial t} = \frac{\partial T_i(\varphi(t;\ x_o, u), u(t))}{\partial t} \quad , \quad i = 1, \ldots, n .$$

Using the properties of the state transfer function the above equations can be written as:

$$\frac{\partial y_i}{\partial t} = \frac{\partial T_i(x,u)}{\partial t} \quad , \qquad i = 1, \ldots, n .$$

Applying the chain rule to the last equation one can see that

$$\frac{\partial y_i}{\partial t} = \sum_{i=1}^{n} \frac{\partial T_i}{\partial x_k} \frac{d x_k}{d t} + \sum_{i=1}^{m} \frac{\partial T_i}{\partial u_k} \frac{d u}{d t} \quad ; \quad k= 1,\dots, n$$

The importance of having the transformation of the states T_i, $i= 1, \dots, n$, indepedent of the control u, allows one to solve for the control inputs u, in the transformed state-input space. Liceaga-Castro and Bradley [1987], showed that in some cases it is possible to design a T transformation for nonlinear discrete systems, whenever the transformation of the state does not depend on the control inputs. The importance of this characteristic will become apparent through this section.

The rows of the linear canonical system depend, by hipothesis, on the transformation $y_i= T_i(x,u)$, $i= 1,\dots, n$, and $v_i= T_{n+j}(x,u)$, $j= 1,\dots, m$.

It is obvious that the derivative of y depends on x and u, and not on the derivative of u respect to time, as a consequence, T_i $i= 1,\dots, n$ do not depend on the derivative of u respect to time either, therefore

$$\frac{\partial T_i}{\partial u_k} = 0 , \qquad i= 1,\dots, n .$$

This proves condition a).

Condition b). The T transformation is defined as a diffeomorphism therefore its Jacobian matrix

$$\left| \begin{matrix} \frac{\partial T_1}{\partial x_1} & \cdots & \frac{\partial T_1}{\partial x_n} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial T_n}{\partial x_1} & \cdots & \frac{\partial T_n}{\partial x_n} & 0 & \cdots & 0 \\ \frac{\partial T_{n+1}}{\partial x_1} & \cdots & \frac{\partial T_{n+1}}{\partial x_n} & \frac{\partial T_{n+1}}{\partial u_1} & \cdots & \frac{\partial T_{n+1}}{\partial u_m} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial T_{n+m}}{\partial x_1} & \cdots & \frac{\partial T_{n+m}}{\partial x_n} & \frac{\partial T_{n+m}}{\partial u_1} & \cdots & \frac{\partial T_{n+m}}{\partial u_m} \end{matrix} \right|$$

is nonsingular. Therefore, if T exists, the matrix

$$\begin{vmatrix} \frac{\partial T_{n+1}}{\partial u_1} & \cdots & \frac{\partial T_{n+1}}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial T_{n+m}}{\partial u_1} & \cdots & \frac{\partial T_{n+m}}{\partial u_{n+m}} \end{vmatrix}$$

is also nonsingular. This proves necessity of condition b).

Condition c). The Brunovsky canonical form can be written as

$$\dot{y}_1 = y_2$$

$$\vdots$$

$$\dot{y}_{\sigma_1 - 1} = y_\sigma$$

$$\dot{y}_{\sigma_1} = v_1$$

$$\dot{y}_{\sigma_1 + 1} = y_{\sigma_1 + 2}$$

$$\vdots$$

$$\dot{y}_{\sigma_2 - 1} = y_{\sigma_2}$$

$$\dot{y}_{\sigma_2} = v_2$$

$$\dot{y}_{\sigma_2 + 1} = y_{\sigma_2 + 2}$$

$$\vdots$$

$$\dot{y}_{\sigma_{m-1}-1} = y_{\sigma_{m-1}}$$

$$\dot{y}_{\sigma_{m-1}} = v_{m-1}$$

$$\dot{y}_{\sigma_{m-1}+1} = y_{\sigma_{m-1}+1}$$

$$\vdots$$

$$\dot{y}_{n-1} = y_n$$

$$\dot{y}_n = v_m$$

which can be expressed in terms of T:

$$\dot{T}_1 = T_2$$

$$\vdots$$

$$\dot{T}_{\sigma_1 - 1} = T_\sigma$$

$$\dot{T}_{\sigma_1} = T_{n-1}$$

$$\dot{T}_{\sigma_1+1} = T_{\sigma_1+2}$$

$$\vdots$$

$$\dot{T}_{\sigma_2-1} = T_{\sigma_2}$$

$$\dot{T}_{\sigma_2} = T_{n+2}$$

$$\dot{T}_{\sigma_2+1} = T_{\sigma_2+2}$$

$$\vdots$$

$$\dot{T}_{\sigma_{m-1}-1} = T_{\sigma_{m-1}}$$

$$\dot{T}_{\sigma_{m-1}} = T_{m+n+1}$$

$$\dot{T}_{\sigma_{m-1}+1} = T_{\sigma_{m-1}+2}$$

$$\vdots$$

$$\dot{T}_{n-1} = T_n$$

$$\dot{T}_n = T_{n+m} \quad .$$

From the above equations, one can select the following:

$$\dot{T}_l = T_{l+1} \quad ,$$

where

$l = 1, 2, \ldots, \sigma_1-1, \sigma_1+1, \ldots, \sigma_2-1, \sigma_2+1, \ldots, \sigma_{m-1}-1, \sigma_{m-1}+1, \ldots, n-1$.

On the other hand, if the original nonlinear system (III.NLS.1) is considered, then the above equations can be expressed as the Lie derivative respect $\dot{x}$.

$$\langle dT_l,\ f + \sum_{i=1}^{m} u_i\ g_i \rangle = T_{l+1} \quad ,$$

where l is as defined previously, (the index l will have the same meaning in this chapter) and the arguments have been eliminated for the sake of simplicity.

The last equation can be rewritten as

$$\langle dT,\ f\rangle + \sum_{i=1}^{m} u_i\ \langle dT_i,\ g_i \rangle = T_{l+1} \quad .$$

From condition a) previously shown, which establishes that T_{l+1} terms do not depend on the input u, it is concluded that:

$$\langle dT_l,\ f \rangle = T_{l+1} \quad ,$$

which are the partial differential equations (III.PD.3), and

$$\sum_{i=1}^{m} u_i \langle dT_1, g_i\rangle = 0 \; ; \quad \text{for every admissible } u_i, \text{ so that}$$

$$\sum_{i=1}^{m} \langle dT_1, g_i\rangle = 0$$

which are the partial differential equations (III.PD.1).

On the hand the states with indices σ_1, ..., σ_n are expressed by

$$\dot{T}_{\sigma_1} = T_{n+1} = v_1$$

$$\vdots \qquad \vdots$$

$$\dot{T}_{\sigma_n} = T_{n+m} = v_m \quad .$$

In this case v_i, i= 1, ..., m depends on u_i, i= 1, ..., m then the partial differential equations (III.PD.2) are obtained:

$$T_{n+1} = \langle \, dT_{\sigma_1}, \; f + \sum_{i=1}^{m} u_i \, g_i \rangle$$

$$\vdots$$

$$T_{n+m} = \langle \, dT_n, \; f + \sum_{i=1}^{m} u_i \, g_i \rangle \quad .$$

This proves the condition c). ∎

The three above conditions establish the basic mathematical structure of the T transformation. This structure can be simplified according to its the three Lie derivatives given in section two of this chapter. For example, consider:

$$\langle \, dT_1, \; f\rangle = T_{1+1} \quad ,$$

where

$l = 1, 2, \ldots, \sigma_1 - 1, \sigma_1 + 1, \ldots, \sigma_2 - 1, \sigma_2 + 1, \ldots, \sigma_{m-1} - 1, \sigma_{m-1} + 1, \ldots, n-1$.

The Lie derivative of the one-form dT_1 with respect to f is

$$T_{1+1} = L_f(T_1) \; .$$

This allows the Leibnitz rule to be applied, i.e.

$\langle dT_l, [f,g] \rangle = \langle d\langle dT_l, g_i\rangle, f \rangle - \langle d\langle dT_l, f\rangle, g_i \rangle$,

where i= 1, ..., m.

But as was shown before, the first term of the right side of the last equations is zero so that

$\langle dT_l, [f,g] \rangle = -\langle d\langle dT_l, f\rangle, g\rangle$, i= 1, ..., m.

The last equations can be worked out as follows. Let the set

$L= \{l= 1,2,\ldots, \sigma_1-1, \sigma_1+1,\ldots, \sigma_2-1, \sigma_2+1,\ldots, \sigma_{m-1}-1, \sigma_{m-1}+1,\ldots,n-1\}$,

and from equation (III.PD.3):

$\langle dT_l, [f,g] \rangle = -\langle dT_{l+1}, g_i\rangle$, i= 1 ,...,m .

If $l+1 \in L$ then

$$\langle dT_l, [f,g_i] \rangle = 0 \qquad \text{(III.PD.4)}$$

and if l+1 is a controllability index then

$\langle dT_l, [f,g_i] \neq 0$.

The equations whose index $l+1 \in L$ can be modified as follows:

if $\langle dT_l, [f,g_i] \rangle = 0$.

Then from (III.PD.1) and (III.PD.3) the equation

$\langle dT_{l+1}, [f,g_i] \rangle = \langle \langle dT_{l+1}, g_i\rangle, f \rangle - \langle d\langle dT_{l+1}, f\rangle, g_i\rangle$

can be transformed to

$\langle dT_{l+1}, [f,g_i] \rangle = \langle dT_{l+2}, g_i\rangle$.

If the Leibnitz rule is applied succesively as follows:

$\langle dT_l, [f, [f.g_i]] \rangle = \langle d\langle dT_l, [f,g_i]\rangle, f\rangle - \langle d \langle dT_l, f\rangle, [f,g_i]\rangle$

then

$\langle dT_l, adf^2g_i\rangle = -\langle dT_{l+1}, [f,g_i]\rangle$,

Then it is clear that

$\langle dT_{l+2}, g_i\rangle = \langle dT_l, adf^2g_i\rangle$,

and if $l+2 \in L$ then

$\langle dT_l, adf^2g_i\rangle = 0$.

This procedure can be repeated until the index of T takes the value of a controllability index, for example

$\langle dT_\sigma, [f,g_i] \rangle = \langle d \langle T_{\sigma-1}, g_i\rangle, f\rangle - \langle \langle dT_{\sigma-1}, f\rangle, g_i\rangle$

$$= - \langle d\langle dT_{\sigma-1}, f\rangle, g_i\rangle$$

$$= - \langle dT_{\sigma}, g_i\rangle$$

$$\neq 0 \quad .$$

Then

$$\langle dT_1, (adf^j, g_i)\rangle \neq 0 \quad .$$

where j= 1,..., K_1-2; i= 1,..., m and K_1 is the first Kronecker index.

If the above procdure is applied to the differential equations (III.PD.1), one can replace them by:

$\langle dT_1, (ad_j f, g_i) \rangle = 0$, for j= 0,..., K_1-2 and i= 1,..., m .

$\langle dT_{\sigma_1+1}, (ad_j f, g_i) \rangle = 0$, for j= 0,..., K_2-2 and i= 1,..., m .

.
.
.

$\langle dT_{\sigma_{m+1}+1}, (ad^j f, g_i) \rangle = 0$, for j= 0,..., K_m-2 and i= 1,..., m .

(III.PD.5) .

Equations (III.PD.2) can be transformed in a similar way. For example, they can be written as:

$$T_{n+1} = \langle dT_{\sigma_1}, f\rangle \pm \sum_{i=1}^{m} u_i \langle dT_{\sigma_1}, g_i\rangle$$

.
.
.

$$T_{n+m} = \langle dT_n, f\rangle \pm \sum_{i=1}^{m} u_i \langle dT_n, g_i\rangle$$

The second term on the right hand side of the above equations can be transformed using the Leibnitz rule and the partial differential equations (III.PD.3) and (III.PD.5): the procedure is the same as the one applied to equations (III.PD.1). After this procedure is applied, the partial differential equations are transformed into

$$v_1 = \langle dT_\sigma , f \rangle \pm \sum_{i=1}^{m} u_i \langle dT_1, (ad^k f, g_i) \rangle \quad , \quad k = K_1 - 1 \; , \quad \sigma = \sigma_1 .$$

$$\vdots$$

$$v_m = \langle dT_\sigma , f \rangle \pm \sum_{i=1}^{m} u_i \langle dT_{\sigma 1}, (ad^k f, g_i) \rangle \quad , \quad k = K_m - 1 \; , \; \sigma 1 = \sigma_{m-1} + 1, \quad \sigma = \sigma_n \; ,$$

where "+" applies if K_i is odd and "-" if it is even, for i= 1,..., m.

The partial differential equations (III.PD.6) involve the T transformation of an (f,g) nonlinear system to a linear canonical Brunovsky form. This relationship is a function of the control input u_i, i= 1, ..., m. If these equations are solved for u_i, one can obtain the control input that makes a nonlinear system given by equations (III.NLS.1) behave like a linear system represented by equation (III.L.2); i. e. a feedback linearization.

The condition required to solve for u_i is that the matrix formed by the vector columns

$$\langle dT_1, (ad^{K_1-1}, g_i) \rangle$$

$$\langle dT_{\sigma_1+1}, (ad^{K_2-1}, g_i) \rangle$$

$$\vdots$$

$$\langle dT_{\sigma_{m-1}+1}, (ad^{K_m-1}, g_i) \rangle \qquad i = 1, \ldots, m$$

be nonsingular.

Hence, the solution of the partial differential equations (III.PD.5) and (III.PD.6), with the above matrix nonsingular, defines a transformation from the nonlinear system (III.NLS.1) to the canonical linear system (III.L.2). In the following section the necessary and sufficient conditions for the existence of such a transformation given that the points a), b) and c) given in this section are satisfied are presented.

III.4.- EXISTENCE AND CONSTRUCTION OF THE T-TRANSFORMATION.

In the previous section, a T-transformation that maps a nonlinear system to a linear canonical form was defined. It was also shown that if this transformation exists, it possesses certain properties (namely the conditions a), b) and c)). It remains to determine if this transformation ,T, exists for a nonlinear system and how it can be constructed, given that the conditions established previously are accomplished.

In this section the necessary and sufficient conditions for the existence of a T-transformation for a particular nonlinear system, expressed in an (f,g) form are established. The manner in which this transformation can be constructed is also given.

In order to show the necessary and sufficient conditions for the existence of T, the following sets are defined:

$$C= \{ g_1, [f, g_1], \ldots, (ad^{K_1-1}f, g_1), g_2, [f, g_2], \ldots, (ad^{K_2-1}f, g_2), g_m, [f, g_m], \ldots, (ad^{K_m-1}f, g_m) \} \quad ;$$

and

$$C_j= \{ g_1, [f, g_1], \ldots, (ad^{K_j-2}f, g_1), g_2, [f, g_2], \ldots, (ad^{K_j-2}f, g_2), g_m, [f, g_m], \ldots, (ad^{K_j-2}f, g_m) \} \quad ,$$

for $j= 1, \ldots, m$.

<u>Theorem III.2.</u> The nonlinear system (III.NLS.1) is T-equivalent to the linear canonical system (III.L.2) if and only if the following conditions are satisfied in a neighbourhood about the origin:

1) The set C spans an n-dimensional space, that is, the elements of C are linearly independent. Furthermore the dT_i $i= 1, \ldots, n+m$ gradients are also linearly independent.

2) The sets C_j are involutive for $j= 1, \ldots, m$.

3) The span of C_j is the same as the span of $C \cap C_j$ for $j= 1, \ldots, m$.

The proof of the theorem will be given by parts, first the necessary conditions of point 1), 2) and 3) are checked, followed by the sufficiency conditions which are given in a constructive form.

Nessesary conditions.

Proof of statement 1).

This condition is necessary and sufficient for dT_j j= 1, ..., n+m to be linearly independent. This proof consists of a comparision of the vector fields in C and the dT_k k= 1, ... n one-forms. Assuming that $T_1, \ldots, T_n$ solve the partial differential equations (III.PD.5) and (III.PD.6); and $d_1, d_2, \ldots, d_n \in R$ be n arbitrary constants then the following linear combinations can be obtained:

$$\alpha = d_1\, g_1 + d_2\, (ad^1 f,\ g_1) + \ldots + d_{\sigma_1}\, (ad^{K_1-1} f, g_1) +$$

$$d_{\sigma_1+1}\, g_2 + d_{\sigma_1+2}\, (ad^1 f,\ g_2) + \ldots + d_{\sigma_2}\, (ad^{K_2-1} f,\ g_2) +$$

$$\vdots$$

$$+ d_{\sigma_{m-1}+1}\, g_m + d_{\sigma_{m-1}+2}\, (ad^1 f,\ g_m) + \ldots + d_n\, (ad^{K_m-1} f,\ g_m) \quad .$$

If it can be shown that the constants d_i i= 1,..., n are all zero, it is established that the elements in C are linearly independent:

The vector α is assumed to be zero in a neighbourhood about the origin. If the $\langle \alpha\ ,\ . \rangle$ operation is realized with each dT_i i= 1, ..., n, then the n inner products $\langle \alpha\ ,\ dT_i \rangle$ can be arranged in matrix form

$$M\, D = 0 \quad ,$$

where

$$D = [\ d_1, \ \ldots\ , d_n]^t$$

and the matrix M is comprises of rows of the form

$$[\ \langle dT_k,\ g_1\rangle \ \dots\ \langle dT_k,\ (ad^{K_1-1}f,\ g_1)\rangle\ \ \langle dT_k,\ g_2\rangle\ \dots\ \langle dT_k,\ (ad^{K_2-1}f,\ g_2)\rangle$$

$$\dots\langle dT_k,\ g_m\rangle\ \dots\ \langle dT_k,\ (ad^{K_m-1}f,\ g_m)\rangle\ ]\ ;$$

for $k= 1,\ \dots,\ n$.

It has been shown that if T is a T-transformation, then it satisfies the partial differential equations (III.PD.5) and (III.PD.6), where upon the first row of the M matrix can be transformed to

$$[0\ \dots\ \ \langle dT_1,\ (ad^{K_1-1}f,\ g_1)\rangle\ \ 0\ \dots\ \ \langle dT_1,\ (ad^{K_2-1}f,\ g_2)\rangle\ \ \dots$$

$$0\ \dots\ \ \langle dT_1,\ (ad^{K_m-1}f,\ g_m)\rangle\ \ ]\ ,$$

If $K_1 > K_2 > \dots > K_m$ then the only element different from zero in the above row is the element associated with K_1. Then the inner product of this row and the vector D is:

$$\langle dT_1,\ (ad^{k-1}f,\ g_1)\rangle\ d_\sigma = 0\ \ ;\ \ \text{for } k= K_1\ \text{ and } \sigma= \sigma_1\ .$$

But as shown previously, the above inner product does not vanish, therefore: $d_\sigma= 0$. If K_1 appears s times in the first row, then the inner product will be

$$\langle dT_1,\ (ad^{k-1}f,\ g_1)\rangle\ d_{\sigma_1} + \dots + \langle dT_1,\ (ad^{k-1}f,\ g_s)\rangle\ d_{\sigma_s}= 0$$

In this case the constants d_j $j= \sigma_1,\ \dots\ ,\ \sigma_s$ are all zero. It is then possible to eliminate the first row of matrix M together with the columns multiplied by the vanishing elements of vector D.

Applying the above procedure to the remaining rows, it is found that the matrix M can be reduced to

$$\begin{vmatrix} \langle dT_{\sigma_1},\ g_1\rangle & \langle dT_{\sigma_1},\ g_2\rangle & \dots & \langle dT_{\sigma_1},\ g_m\rangle \\ \langle dT_{\sigma_2},\ g_1\rangle & \langle dT_{\sigma_2},\ g_2\rangle & \dots & \langle dT_{\sigma_2},\ g_m\rangle \\ \vdots & \vdots & \dots & \vdots \\ \langle dT_{\sigma_m},\ g_1\rangle & \langle dT_{\sigma_m},\ g_2\rangle & \dots & \langle dT_{\sigma_m},\ g_m\rangle \end{vmatrix}\ ,$$

and the vector D is reduced to another vector whose components are those

$$\begin{vmatrix} \langle dT_1 , (ad^{K_1-1}f, g_1)\rangle & \dots & \langle dT_1, (ad^{K_1-1}f, g_m)\rangle \\ \langle dT_{\sigma_1+1}, (ad^{K_2-1}f, g_1)\rangle & \dots & \langle dT_{\sigma_1+1}, (ad^{K_2-1}f, g_m)\rangle \\ . & \dots & . \\ . & \dots & . \\ . & \dots & . \\ \langle dT_{\sigma_{m-1}}, (ad^{K_m-1}f, g_1)\rangle & \dots & \langle dT_{\sigma_{m-1}}, (ad^{K_m-1}f, g_m)\rangle \end{vmatrix} \qquad \text{(III.M.1)}$$

is nonsingular.

The linear independence of the one-forms dT_1, ..., dT_n is determined in a similar way by defining the following linear combination:

$$\beta = b_1\ dT_1 + b_2\ dT_2 + \dots + b_n\ dT_n = 0 \quad .$$

where $B = [\ b_1\ \dots\ b_n]^t$ is an arbitrary constant vector.

In this case the inner product $\langle \beta\ ,\ .\ \rangle$ is taken with each element of C. Then it is possible to form the equation:

$$M_\beta\ B = 0\ ,$$

where the matrix M_β is

$$\begin{vmatrix} \langle dT_1,\ g_1\rangle & \langle dT_2,\ g_1\rangle & \cdots & \langle dT_n,\ g_1\rangle \\ \langle dT_1,\ (adf, g_1)\rangle & \langle dT_2,\ (adf,\ g_1)\rangle & \cdots & \langle dT_n,\ (adf,\ g_1\rangle \\ \vdots & \vdots & & \vdots \\ \langle dT_1,\ (ad^{K_1-1}f,\ g_1)\rangle & \langle dT_2,\ (ad^{K_1-1}f,\ g_1)\rangle & \cdots & \langle dT_n,\ (ad^{K_1-1}f,\ g_1)\rangle \\ \langle dT_1,\ g_2\rangle & \langle dT_2,\ g_2\rangle & \cdots & \langle dT_n,\ g_2\rangle \\ \vdots & \vdots & & \vdots \\ \langle dT_1,\ (ad^{K_2-1}f,\ g_2)\rangle & \langle dT_2,\ (ad^{K_2-1}f,\ g_2)\rangle & \cdots & \langle dT_n,\ (ad^{K_2-1}f,\ g_2)\rangle \\ \vdots & \vdots & & \vdots \\ \langle dT_1,\ g_m\rangle & \langle dT_2,\ g_m\rangle & \cdots & \langle dT_n,\ g_m\rangle \\ \vdots & \vdots & & \vdots \\ \langle dT_1,\ (ad^{K_m-1}f,\ g_m)\rangle & \langle dT_2,\ (ad^{K_m-1}f,\ g_m)\rangle & \cdots & \langle dT_n,\ (ad^{K_m-1}f,\ g_m)\rangle \end{vmatrix}$$

Using the same procedure as before, it is possible to show that, if the one-forms dT_1, ..., dT_n satisfy the partial differential equations (III.PD.5) and (III.PD.6) then the above matrix can be transformed to

$$\begin{matrix} \langle dT_1,\ (ad^{K_1-1}f,\ g_1)\rangle & \langle dT_{\sigma_1+1},\ (ad^{K_2-1}f,\ g_1)\rangle & \cdots & \langle dT_{\sigma_{m-1}+1},\ (ad^{K_m-1}f,\ g_1)\rangle \\ \vdots & \vdots & & \vdots \\ \langle dT_1,\ (ad^{K_1-1}f,\ g_m)\rangle & \langle dT_{\sigma_1+1},\ (ad^{K_2-1}f,\ g_m)\rangle & \cdots & \langle dT_{\sigma_{m-1}+1},\ (ad^{K_m-1}f,\ g_m)\rangle \end{matrix}$$

and that β is zero if the above matrix (which is the transpose of (M.1) is nonsingular.

Finally, one can see that if

dT_i i= 1, ..., n are linearly independent T_j j= 1, ..., n+m are the solutions of the partial differential equations (III.PD.5) and (III.PD.6), and the matrix (III.M.1) is nonsingular, then the one-forms dT_1, ...,dT_{n+m}

are linearly independent. This proves point 1). ∎

In the remainder of this chapter the proof of points 2) and 3) are presented together with a method to construct the T-transformation which involves the sufficiency of condition 1), 2) and 3)..

Necessity of conditions 2) and 3).

The conditions 2) and 3) will be analysed under the assumption that the previous results are accomplished. That is, the partial differential equations (III.PD.5) and (III.PD.6) hold and dT_j j= 1, ..., n+m and the elements of C are linearly independent.

Proof of conditions 2) and 3).

The first equations (III.PD.5)

$\langle dT_1, (ad^j f, g_i)\rangle = 0$; j= 0,..., K_1-2 and i= 1,..., m;

imply that dT_1 is perpendicular to all the elements of

$$
\begin{aligned}
C_1 = \{\, & g_1, [f, g_1], \ldots, (ad^{K_1-2} f, g_1), \\
& g_2, [f, g_2], \ldots, (ad^{K_1-2} f, g_2), \\
& \vdots \\
& g_m, [f, g_m], \ldots, (ad^{K_1-2} f, g_m) \,\} .
\end{aligned}
$$

Now it was shown that dT_1 is perpendicular to all the elements of C, except $(ad^x f, g_1)$, for $x= K_1-1$. One can also see that, $(ad^x f, g_1)$ is the only element of C that can not be contained in C_1, provided that $K_1 > K_2 > \ldots > K_m$. Therefore if C spans an n-dimensional space, then C_1 obviously will span at most an n-1 dimensional space. Considering the above facts, it is also clear that the span of C_1 is equal to the span of $C \cap C_1$.

At this point it is possible to outline how to obtain the components of the T-transformation, which are the solutions of the partial differential equations (III.PD.5) and (III.PD.6). Consider in particular the equation

$$\langle dT_1, (ad^j f, g_i)\rangle = 0 \quad ; \quad j = 1,\dots, K_1-2 \text{ and } i = 1,\dots,m .$$

Solving this equation is the same as finding a function T_1 whose gradient is perpendicular to the vector fields in C_1, that is to the space generated by the elements of C_1, which is the same space generated by $C \cap C_1$. It is clear that the solution of the above equations can be reduced to the process of finding a function T_1 whose gradient is perpendicular to $C_1 \cap C$. The answer to the existence of T_1 can be readily aswered by appealing to the famous Frobenius theorem, which when tailored for this problem can be expressed as:

Theorem III.3 (Frobenius) Assuming that vector fields of $C \cap C_1$ pass through a given point $x_o \in R^n$, then the vector fields in $C \cap C_1$, generate the tangent space of a unique sub-manifold N in R^n. The non-vanishing function T_1 whose gradient dT_1 is perpendicular to the space generated by $C \cap C_1$ is defined, if and only if, the vector fields in question are involutive. N is said to be the integral manifold of the vector fields of $C \cap C_1$ through x_o.∎

If K_1 appears s times in C then the following occurs. If $s > 2$ then $K_1 = K_2 = \dots = K_m$, so that there are s terms in C that can not be contained in C_1. In this case C_1 spans, at most, an n-s dimensional space. From the partial differential equations

$$\langle dT_1, (ad^{Kj} f, g_i)\rangle = 0 \ ; \quad j = 0,\dots, K_1-2 \ ; \quad i = 1,\dots, m \quad ,$$
$$\langle dT_{\sigma_1+1}, (ad^{Kj} f, g_i)\rangle = 0 \ ; \quad j = 0,\dots, K_2-2 \ ; \quad i = 1,\dots, m \quad ,$$

$$\vdots$$

$$\langle dT_{\sigma_m+1}, (ad^{Kj} f, g_i)\rangle = 0 \ ; \quad j = 0,\dots, K_m-2 \ ; \quad i = 1,\dots, m \quad ,$$

it is clear that if $dT_1, dT_{\sigma_1+1},\dots, dT_{\sigma_m-1}$ are linearly independent of all the elements of C_1, these one-forms are also linearly independent of all the elements of C except of those terms whose Lie derivative order is

K_1-1. Then C_1 and $C \cap C_1$ span, at most, an n-s dimensional space.

Invoking again the Frobenius theorem, one can see that there exists a unique n-s dimensional integral manifold through some point $x_o \in \mathbf{R}^n$, whose tangent space is generated by $C \cap C_1$, also the T_j, j= 1, $\sigma_1+1, \ldots, \sigma_m-1$ escalar fields, have their gradients perpendicular to the space generated by $C \cap C_1$ at x_o if and only if $C \cap C_1$ is involutive.

The construction of the transformation T will depend on the partial differential equations (III.PD.5),

$$\langle dT_1,\ (ad^j f,\ g_i)\rangle = 0 \quad ; \quad j= 0,\ldots, K_1-2 \ ; \quad i= 1,\ldots, m \ .$$

$$\langle dT_{\sigma_1-1},\ (ad^j f,\ g_i)\rangle = 0 \ ; \ j= 0,\ldots, K_2-2 \ ; \quad i= 1,\ldots, m \ .$$

$$\vdots$$

$$\langle dT_{\sigma_{m-1}+1},\ (ad^j f,\ g_i)\rangle = 0 \quad ; \quad j= 0,\ldots, K_m-2 \quad ; \quad i= 1,\ldots, m \ .$$

These equations can be transformed according to the Leibnitz rule

$$\langle dT_l,\ (ad^j f,\ g_i)\rangle = \langle dT_{l+1},\ (ad^{j-1} f,\ g_i)\rangle$$

for l= 1, ..., σ_1-1, σ_1+1, ..., σ_2-1, σ_2+1, ..., σ_{m-1}, σ_{m+1}, ..., n .

Therefore

$$\langle dT_1,\ (ad^k f,\ g_i)\rangle = 0 \ ; \quad k= K_1-2 \quad ; \quad i= 1,\ldots, m \ .$$

$$\vdots$$

$$\langle dT_{\sigma_1-1},\ g_i\rangle = 0 \quad ; \quad i= 1,\ldots, m \ .$$

$$\vdots$$

$$\langle dT_{\sigma_{m-1}+1},\ (ad^k f,\ g_i)\rangle = 0 \quad ; \quad k= K_m-2 \quad ; \quad i= 1,\ldots, m \ .$$

$$\vdots$$

$$\langle dT_{m-1},\ g_i\rangle = 0 \quad ; \quad i= 1,\ldots, m \ .$$

According to point 1), the sets C and { dT_1, dT_2, ..., dT_n} have n linearly independent elements. Comparing these sets with the above equations, one can see that the number of linearly independent vector fields in $C \cap C_j$, $j= 1,\ldots, m$ is $n-(p/m)$. Where p is the number of equations in (III.PD.5) whose Lie derivative order is equal to or greater than K_j-2. This is also the number of linearly independent vector fields in $C \cap C_j$, $j= 1,\ldots, m$. Therefore there are $n-(n-p/m)= p/m$ elements in the set $\{T_1, T_2, \ldots, T_n\}$ whose gradients are linearly independent of the integral manifold of $C \cap C_j$, $j= 1,\ldots, m$ (or C_j). Note that the validity of this argument depends entirely on the involutivity of the sets $C \cap C_j$, $j= 1,\ldots, m$ (or C_j).

According to the above analysis the solution of the partial differential equations (III.PD.5) is the solution of the partial differential equations

$$\langle dT_1, c_1\rangle = 0; \qquad \forall\, c_1 \in C \cap C_1 ,$$

$$\langle dT_{\sigma_1 -1}, c_2\rangle = 0 \qquad \forall\, c_2 \in C \cap C_2 ,$$

$$\vdots \qquad\qquad\qquad\qquad \text{(III.PD.7)}$$

$$\langle dT_{\sigma_{m+1}+1}, c_m\rangle = 0 \qquad \forall\, c_m \in C \cap C_m .$$

The necessity of conditions 2) and 3) becomes apparent from the above process. ∎

Conditions of sufficiency. Construction of the Transformation.

Hunt, Su and Meyer [1983] presented a constructive proof for the sufficiency of points 1), 2) and 3). This proof consists of the actual construction of the T transformation. From the practical point of view, this proof shows the method to such a transformation may be constructed. The construction described by the above authors is presented as follows. Some comments and a supporting theorem have been included in this proof in order to clarify the procedure.

In the book by Choquet-Bruhat, DeWitt-Morette and Dillard-Bleick

[1977], a treatise of analysis, manifolds and differential equations is developed, in which the relationship among these topics is presented. In fact all are considered as one subject. The solution of the type of partial differential equations discussed above may also be found in this work . According to Choquet-Bruhat et al., equations (III.PD.7) form an exterior differential system, and if the system is related to one-forms then it is called a Pfaffian system.

Individually equations (III.PD.7) are called Pfaffian differential equations.

The relationship between the Pfaffian differential equations and the Frobenius theorem is the same as the relationship between ordinary differential equations and the Existence Theorems for their solutions. One can observe that, in contrast to ordinary differential equations, the solution of the Pfaffian equations is subject to a condition, namely its involutivity. Up to this point, this section has been devoted to establishing this condition before attempting to solve the partial differential equations obtained in the previous section.

The procedure proposed by Hunt et al. for the solution of the partial differential equations (III.PD.7), is now described. It is at once clear that given the solution of these equations one can construct the T-transformation with equations (III.PD.7) and the Leibnitz rule.

First the following constants and sets are defined:

s_1: the number of times that K_1-1 appears in C and

S_1: the subset of C whose elements are the vector fields related to K_1-1th Lie derivative.

s_2: the number of times that K_2-1 appears in C and

S_2: the subset of C whose elements are the vector fields related to the K_{1-2}th Lie derivative.

.

.

s_m: the number of linearly independent vectors in

$S_m = \{g_1, \ldots, g_m\}$, which by hypothesis is m.

(Notice that $s_1 + s_2 + \ldots + s_m = n$) .

Then a parametric mapping

$z(t_1, t_2, \ldots, t_n) = (x_1(t_1, t_2, \ldots, t_n), \ldots, x_n(t_1, t_2, \ldots, t_n))$

that maps the origin to the origin and passes though the s_1 integral curves of the vector fields whose Lie derivative order is K_1-1 and are contained in C is obtained.

The above function is constructed considering that (Choquet-Bruhat et. al. [1977], Thorpe [1985]):

<u>Definition</u>. A parametric curve in R^{n+1} is a C^∞ function $\alpha: I \to R^{n+1}$, where I is some interval of **R** defined by $\alpha(t) = (\alpha_1(t), \ldots, \alpha_{n+1}(t))$ and α_i, i=1 ,...n+1 .

One also needs to state:

Theorem III.4 (Thorpe [1985]). Let X be a C^∞ vector field on an open set $U \subset R^{n+1}$ and $p \in U$. Then there exists an open interval I containing the origin and an integral curve $\alpha: I \to U$ such that

i) $\alpha(0) = q$.

ii) If $\beta: J \to U$ is any other integral curve of X with $\beta(0) = q$, then $J \subset I$ and $\alpha(t) = \beta(t)$ for all $t \in J$. ∎

The above theorem, whose proof can be found in the book by Thorpe [1985], implies that the integral curve of a vector field through a fixed point is unique.

The z mapping is constructed considering that it passes through the integral curve of the $(ad^k f, g_1)$ vector field, which is given by the solution of (Thorpe [1985]):

$$\frac{d\, x(t_1)}{dt_1} = (ad_k f,\ g_1) \qquad ; \qquad k = K_1 - 1 \quad ,$$

with $x_1(0) = 0$.

By the above theorem this integral curve is unique.

If $s_1 \geqslant 2$ then by solving:

$$\frac{d\, x(t_1,\ t_2)}{dt_2} = (ad^r f,\ g_2) \quad ; \quad r= K_2-1$$

with $x(t_1,\ 0)= x(t_1)$, a second integral curve through the origin is found.

This procedure is repeated until the parameter t_{s_1} is introduced by

$$\frac{d\, x(t_1,\ t_2,\ \ldots,\ t_r)}{dt_r} = (ad^\kappa f,\ g_r) \quad ; \text{ where } r= s_1 ; \quad \kappa= K_1-1 \quad ,$$

with $x(t_1,\ t_2,\ \ldots,\ t_{s_1-1},\ 0)= x(t_1,\ t_2,\ \ldots,\ t_{s_1-1})$.

If one can solve the partial differential equations (III.PD.7) with $x(t_1,\ t_2,\ \ldots,\ t_r,\ 0,\ \ldots 0)= x(t_1,\ t_2,\ \ldots,\ t_r)$ as the initial condition, then the solution will be contained in the intersection of the s_1 integral curves obtained above. At this stage one can see that condition 1) is being used to construct a diffeomorphism.

The solution of equations (III.PD.7) can be worked out considering the existence of a function Z such that:

$$\langle\, dZ,\ (ad_k f,\ g_1)\rangle = 0 \quad ; \quad k= K_1-2 \quad .$$

Finding this function Z is equivalent to finding the solution of the linear ordinary differential equations (Choquet-Bruhant [1977], Thorpe [1979], Elsgoltz [1977])

$$\frac{dx}{d\, t_{s_1+1}} = (ad^k f,\ g_1) \ ,\ k= K_1-2 \ ; \qquad \frac{\partial Z}{\partial\, t_{s_1+1}} = 0 \qquad ,$$

with $x(t_1,\ t_2,\ \ldots,\ t_{s_1+1},\ 0)= x(t_1,\ t_2,\ \ldots,\ t_{s_1+1})$.

If $s_2 \geqslant 2$ then the equation

$$\langle dZ,\ (ad^{K_1-2} f,\ g_2)\rangle = 0 \quad ,$$

with $x(t_1,\ t_2,\ \ldots,\ t_{s_1+1},\ 0)= x(t_1,\ t_2,\ \ldots,\ t_{s_1+1})$

If this integration process is repeated s_2 times, the following equation is found:

$$\langle dZ,\ (ad^{K_1-2} f\ ,\ g_{s_2})\rangle = 0 \quad ,$$

which can be solved as follows:

$$\frac{dx}{d\ t_{s_1+s_2}} = (ad^k f,\ g_s)\ ,\ k= K_1-2\ ; \qquad \frac{\partial Z}{\partial\ t_{s_1+s_2}} = 0 \qquad .$$

Next, the s_3 partial differential equations associated with the vector fields contained in S_3 are integrated, starting with equation

$$\langle dZ,\ (ad^{K_1-3}f,\ g_1)\rangle = 0\ ,$$

which is related to the equations

$$\frac{dx}{d\ t_{s_1+s_2+1}} = (ad^k f,\ g_1)\ ,\ k= K_1-3\ ; \qquad \frac{\partial Z}{\partial\ t_{s_1+s_2+1}} = 0$$

and ending with

$$\langle dZ,\ (ad^{K_1-3}f,\ g_{s_3})\rangle = 0\ ,$$

related to

$$\frac{dx}{d\ t_{s_1+s_2+s_3}} = (ad^k f,\ g_s\ )\ ,\ k=K_1-3\ ; \qquad \frac{\partial Z}{\partial\ t_{s_1+s_2+s_3}} = 0 \qquad ,$$

with $x(t_1,\ t_2,\ \ldots,\ t_{s_1+s_2+s_3-1}\ ,\ 0) = x(t_1,\ t_2,\ \ldots,\ t_{s_1+s_2+s_3-1})$

This process ends when the n parameters t_j, $j= 1,\ldots,\ n$ are introduced, that is, when

$$\langle dZ,\ g_m\rangle = 0\ .$$

This equation is associated with

$$\frac{dx}{d\ t_n} = g_m\ ; \qquad \frac{\partial Z}{\partial t_n} = 0$$

satisfying $x(t_1,\ t_2,\ \ldots,\ t_{n-1},\ 0) = x(t_1,\ t_2,\ \ldots,\ t_{n-1})$.

With the above process a map from R^n to R^n which maps the origin to the origin given by

$$(t_1,\ \ldots,\ t_n) \rightarrow (x_1(t_1,\ \ldots,\ t_n),\ \ldots,\ x_n(t_1,\ \ldots,t_n))$$

may be constructed. Furthermore according to the method this map was constructed its Jacobian matrix is

$$\begin{vmatrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_1}{\partial t_n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial x_n}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_n} \end{vmatrix} .$$

Note that when this matrix is evaluated at the origin, its columns are the elements of the set C, and we know this set spans an n-dimensional space, thus the above matrix is nonsingular. Therefore the map is a diffeomorphism on an open neighbourhood W of the origin in R^n, and the image V of W, is also a neighbourhood around the origin in R^n. On the other hand according to the inverse function theorem, one can solve this map for the parameters $t_1, \ldots, t_n$ as functions of $x_1, \ldots, x_n$. This prooves the sufiency of conditions 1), 2) and 3). ∎

At this stage it clear that a diffeomorphism of the state space of the nonlinear system has been obtained. The problem which now arises, is how one can use this diffeomorphism in the definition of the desired functions;

$T_1, T_{\sigma_1+1}, \ldots, T_{\sigma_{m-1}+1}$

By design, each map t_j, j= 1,..., n is C^∞ and takes the origin to the origin on V.

If one surmises that

$T_1 = t_1$

then one has to show that this T_1 satisfies the first set of equations (III.PD.7)

$\langle dT_1, c_1 \rangle = 0 \quad \forall\ c_1 \in C \cap C_1.$

According to the Frobenius theorem and the involutivity of $C \cap C_1$,it is clear that the solution of these equations exists. Furthermore, for fixed $t_1, \ldots, t_{s_1}$ and varying $t_{s_1+1}, \ldots, t_n$ the integral manifold of $C \cap C_1$ is obtained. Therefore the map $T_1 = t_1$ is constant in such a manifold, and it is quickly seen that

$\langle dT_1, c_1\rangle = 0 \quad \forall\ c_1 \in C \cap C_1$.

Note that $(ad^{1}\ f_{,}^{K}\ \bar{g}_1^1)$ is associated with t_1 .

To define the map T_{σ_1+1}, one can proceed in the same way. That

is, one seeks for a $T_{\sigma_1+1} = \hat{t}$ such that:

$\langle dT_{\sigma\ +1}, c_2\rangle = 0 \quad \forall\ c_2 \in C \cap C_2$.

Keeping in mind the elements of C which are not contained in $C \cap C_2$, one can appeal to $(ad^{K_2-1} f, g_2)$. According to the construction of the diffeomorphism, the parameter associated with this vector field originates a constant map in the integral manifold of $C \cap C_2$ (which one knows exists if the vector fields in this set are involutive) by keeping the parameter associated to $(ad^{K_2-1}f, g_2)$ constant while the parameters with higher index vary, as was previously done.

For similar reasons one can define T_{σ_2+1} as the parameter associated

to the vector field $(ad^{K_3-1}f, g_3)$.

The functions $T_{\sigma_3+1}, \ldots, T_{\sigma_{m-1}+1}$ can be defined in the same manner,

defines the solution of the partial differential equations (III.PD.7), which is the same solution of that of equations (III.PD.5).

With this the sufficiency of points 1), 2) and 3) is demostrated ∎.

Finally, as described previously in this section, the rest of the T-transformation components can be evaluated from equations (III.PD.6) and Leibnitz rule, at this stage the construction of $T = (T_1, T_2, \ldots, T_{n+m})$ can be easily obtained.

Before ending this chapter, it seems appropriate to rewrite the main result obtained:

A nonlinear system expressed by equations (III.NLS.1) is T-equivalent with the linear canonical form (III.L.2), where the variables $x_1, \ldots, x_n$ lie on a neighbourhood around the origin of R^n, if and only if:

1) The set C spans an n-dimensional space,

2) The sets C_j j= 1,..., n are involutive and

3) The span of C_j is equal to the span of $C \cap C_j$ for j= 1,..., n .

In the following chapter the theory developed here is applied to a helicopter, represented by the model structure given in chapter II, in order to design a flight control system.

CHAPTER IV.

FLIGHT CONTROL SYSTEM DESIGN.

Summary.

In this chapter the design of an automatic flight control System for helicopters is developed. The helicopter is assumed to be represented by the model defined in Chapter II and the design of the automatic flight control system is realized according to the theory of nonlinear feedback demostrated in Chapter III. A series of simulations is presented in order to investigate the performance of the design.

IV.I. INTRODUCTION.

The object of the present chapter is to concatenate the helicopter model described in Chapter II to the nonlinear feedback described in Chapter III in the form of a flight control system. The link between these two aspects is in principle straightforward, nevertheless the complexity and the high order of the plant in question present enormous difficulties in the implementation of the theory. These obstacles are overcome by

1), making use of the characteristics of the system, which allows one to perform a partial linearization, using a nonlinear compensator, thus reducing the complexity of the original plant substantially,

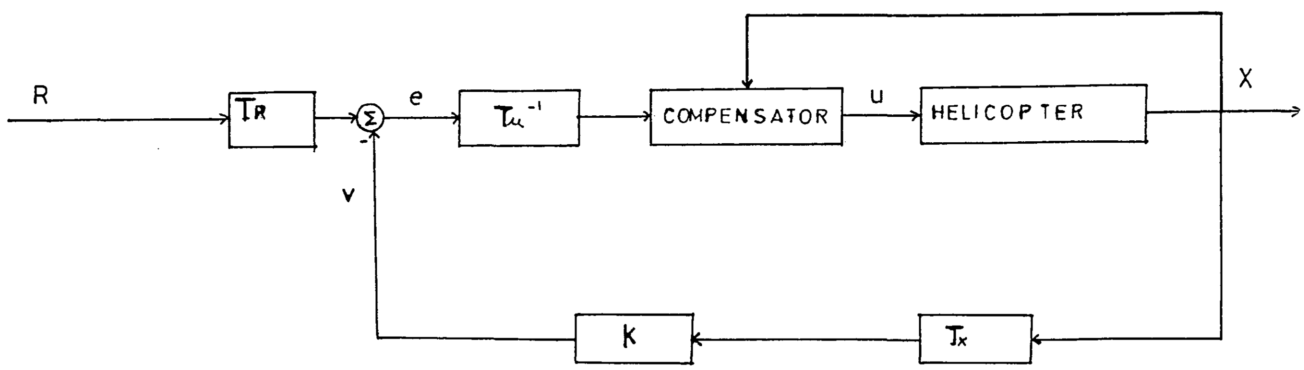
2), the use and development of Symbolic Algebraic Manipulation programs to execute the calculations required in the design of the nonlinear control law, and,

3), by the ubiquitous: simplifying assumptions, in this case they are apropriate in obtaining a solution of the set of partial differential equations involved in the definition of the nonlinear control law.

The three aspects mentioned above are used throughout this chapter to link the helicopter model to the nonlinear control theory, resulting in a flight control system. The process in which these aspects are joined together is explained sequentially in each section of the chapter. First, in section II the nonlinear compensator is introduced. This compensator is the continuous time version of the nonlinear control law presented previously by Liceaga-Castro and Bradley [1987]. In the present application it is shown that, by applying this nonlinear compensator, it is possible to decouple the normal andangular velocities of the helicopter. Furthermore, from the point of view of the feedback linearization theory, the resulting equivalent system has the normal and angular velocity references as inputs, while the state remains unchanged. The problem is reduced to defining the sequence of references in order to control the system. This is much easier than controlling the original plant.

In section III, a set of programmes which determine whether or not a nonlinear system possesses a linear feedback equivalence system is presented. These programmes are developed using the Algebraic Symbolic Manipulation language Reduce. The linearizable helicopter properties are obtained by using these programmes. These properties are also included in section III. In the following section, nonlinear control systems is applied .This is in order to find a diffeomorphic mapping between the Compensator-Helicopter pair to a linear Brunovsky canonical form. In this section, one can see the necessity for considering strong simplifying assumptions in the system in order to find the components of the diffeomorphic map between the plant considered and the linear canonical form. The state mapping and the inverse of the input mapping which, define the control law for the nonlinear system, are obtained in this section. According to the Feedback Equivalence Theory, if a nonlinear system is τ-equivalent to a linear canonical form,the control law designed for the canonical form has an equivalent effect on the nonlinear system. In other words, if the inverse of the τ-transformation is applied to the control inputs of the linear system and then to the nonlinear one, the behaviour of the nonlinear system will be equivalent to that of the canonical linear form.

In section V the Pole Placement Technique is used to set the performance and control of the canonical linear system. It is shown that the effects of the assumptions made in the previous sections are compensated if the poles of the linear system are conveniently placed. This step completes the Flight Control System structure, represented in figure (FIG. IV.I.1) .



R: Reference.

T_R: Transformation of reference.

T_v^{-1}: Inverse transformation of the linear input.

u: Nonlinear input.

x: State.

T_x: Transformation of the state.

Figure (IV.I.1).- Flight control system structure.

Finally, in the last section of the chapter, some simulation results are presented and one can see that, for the manoeuvres considered here, the performance of the flight control system is satisfactory.

IV.2. PARTIAL LINEARIZATION OF THE SYSTEM USING A COMPENSATOR.

According to the model described in Chapter two, the helicopter model can be expressed by an equation of the form

$$\dot{x}(t)= f(x) + \sum_{i=1}^{10} U_i \; b_i(x, \beta, \mathbf{u}) \quad ,$$

where $x \in R^{12}$, $f(.)$ and b_i are C^∞ vector fields in R^{12} and the U_i are components of the input vector

$$[\theta_o, \theta_{1s}, \theta_{1c}, \theta_o^2, \theta_o \theta_{1s}, \theta_o \theta_{1c}, \theta_{1s}^2, \theta_{1s} \theta_{1c}, \theta_c^2, \theta_p]$$

where θ_o, θ_{1s}, θ_{1c} and θ_p are respectively the collective, longitudinal cyclic, lateral cyclic and tail rotor collective pitch angles .β represents the flapping motion and **u** the linear elements of U.

The purpose of this section is to modify the original plant in order to achieve a model which is easier to work with, and to perform a partial linearization and decoupling of the plant by incorporating a compensator.

First, the simplification of the system is referred to the valued vector fields $b_i(.)$, $i = 1, \ldots ,10$, mentioned above. A new set of vector valued input fields are defined as

$$\theta_o [b_1(.) + \theta_o b_4(.)] = \theta_o \hat{g}_1(.)$$
$$\theta_{1s} [b_2(.) + \theta_o b_5(.) + \theta_{1s} b_7(.)] = \theta_{1s} \hat{g}_2(.)$$
$$\theta_{1c} [b_3(.) + \theta_o b_6(.) + \theta_{1s} b_8(.) + \theta_{1c} b_9(.)] = \theta_{1c} \hat{g}_3(.)$$
$$\theta_p [b_4(.)] = \theta_p \hat{g}_{10}(.) \; .$$

If the above vector fields $\hat{g}_i(.)$, $i = 1 , \ldots , 3$ and $\hat{g}_{10}$ are used, the number of vector is evidently reduced from 10 to 4. This simplification is justified due to the fact that the must significant terms in are b_i, $i= 1,\ldots, 3$, . In practice these terms can be identified on line or, as pointed out later, linear robust techniques can be applied. In this thesis the values of input vectors have been updated.

Assuming the above fact the model can be rewritten as;

$$\dot{x}(t) = f(x) + \sum_{i=1}^{4} u_i \ddot{g}_i(.)$$

or

$$\dot{x}(t) = f(x) + G(.)\mathbf{u}$$

Liceaga-Castro and Bradley [1987] showed that it is possible to obtain a feedback linearization for discrete nonlinear systems, in this case the discrete nonlinear input state is mapped to a linear set of first order discrete linear system. Furthermore, this set of linear systems are decoupled with respect to each other and are stable. In foregioig paper it is shown that the condition required for this map to exist and to be a diffeomorphism, is that the input matrix accomplishes the so called "ratio condition", that is that the leading principal minors $\Delta_1, \ldots, \Delta_m$ of the input matrix satisfies the following inequality uniformly;

$$|\det \Delta_1| \geqslant \varepsilon, \quad \frac{|\det \Delta_2|}{|\det \Delta_1|} \geqslant \varepsilon, \quad \ldots, \quad \frac{|\det \Delta_m|}{|\det \Delta_{m-1}|} \geqslant \varepsilon$$

for an arbitrary real number ε.

It should now be clear that the control law obtained from this diffeomorphic map is restricted to systems with the same number of inputs and outputs. The continuous version of this feedback linearization control is summarized as follows, for the nonlinear system described by

$$\dot{x}(t) = f_1(x) + B_1(x)\, w$$

where

x is the state contained in a set A contained properly in $\mathbf{R}^n$,

$f_1(x)$ is a C^∞ m-dimensional vector field,

$B_1(x)$ is an m×m square matrix function and

w represents the system input.

If the matrix function $B_1(x)$ accomplishes the ratio condition around the operating points of the system, then it is possible to obtain a diffeomorphic transformation to the input state space $(x, \mathbf{R}_1)$, by applying the following control law;

$$w = [B_1(x)]^{-1} \{-f_1(x) + \mathbf{R}_1(t)\} \qquad \text{(IV.CM.1)}$$

where $\mathbf{R}_1(t) = (R_1(t), \dots, R_m(t))^t$ is the reference vector.

It is evident that the linear input-state space is given by

$$\dot{x}_i(t) = R_i(t) \qquad i = 1, \dots, m \quad ,$$

by applying this control law as a compensator. The linearization and decoupling of the normal and angular velocities of the helicopter with respect to the commands can also be performed, for example including in the helicopter model three more degrees of freedom, in this case the position coordinate referred to body axes can be incorporated. These three new degrees of freedom are waranties the involutivity of the distribution generated by the system equations

$$
\begin{vmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \\ \dot{p} \\ \dot{q} \\ \dot{r} \\ \\ \dot{\theta} \\ \dot{\phi} \\ \dot{\psi} \\ \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{vmatrix}
=
\begin{vmatrix} vr - wq - G \sin\theta + Fx \\ wp - ur + G \cos\theta \sin\phi + Fx \\ uq - vp + G \cos\theta \cos\phi + Fz \\ \\ I_1 qr + I_2 pq + L_1 \\ I_3 rp + I_4 (r^2 - p^2) + M_1 \\ I_5 pq + I_6 qr + N_1 \\ \\ q \cos\phi - r \sin\phi \\ p + q \sin\phi \tan\theta + r \cos\phi \tan\theta \\ (q \sin\phi + r \cos\phi)/\cos\phi \\ \\ u \\ v \\ w \end{vmatrix}
+
$$

$$
\begin{vmatrix}
b_{11} & b_{12} & b_{13} & b_{14} \\
b_{21} & b_{22} & b_{23} & b_{24} \\
b_{31} & b_{32} & b_{33} & b_{34} \\
b_{41} & b_{42} & b_{43} & b_{44} \\
b_{51} & b_{52} & b_{53} & b_{54} \\
b_{61} & b_{62} & b_{63} & b_{64} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{vmatrix}
\begin{vmatrix}
\theta_O \\
\theta_{1s} \\
\theta_{1c} \\
\theta_P
\end{vmatrix} ,
\qquad \text{(IV.HE.1)}
$$

if the commands are generated according to equation (IV.CM.1) as follows

$$
\begin{vmatrix}
\theta_O \\
\theta_{1s} \\
\theta_{1c} \\
\theta_P
\end{vmatrix}
=
\begin{vmatrix}
b_{31} & b_{32} & b_{33} & b_{34} \\
b_{41} & b_{42} & b_{43} & b_{44} \\
b_{51} & b_{52} & b_{53} & b_{54} \\
b_{61} & b_{62} & b_{63} & b_{64}
\end{vmatrix}^{-1}
\left(
-
\begin{vmatrix}
uq - vp + G \cos\theta \cos\phi + Fz \\
I_1 \, qr + I_2 \, pq + L_1 \\
I_3 \, rp + I_4 \, (r^2 - p^2) + M_1 \\
I_5 \, pq + I_6 \, qr + M_1
\end{vmatrix}
+
\begin{vmatrix}
w_R \\
p_R \\
q_R \\
r_R
\end{vmatrix}
\right) .
$$

Here w_R, p_R, q_R and r_R are the reference values for the normal, rolling pitching and yawing velocities respectively. The closed-loop system resulting from incorporating the compensator is as follows;

$$
\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \\ \dot{p} \\ \dot{q} \\ \dot{r} \\ \\ \dot{\theta} \\ \dot{\phi} \\ \dot{\psi} \\ \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}
=
\begin{bmatrix} vr - wq - G \sin\theta + F_x + h_1(p, q, r, w) \\ wq - ur + G \cos\theta \sin\phi + F_y + h_2(p, q, r, w) \\ 0 \\ \\ 0 \\ 0 \\ 0 \\ \\ q \cos\phi - r \sin\phi \\ p + q \sin\phi \tan\theta + r \cos\phi \tan\theta \\ (q \sin\phi + r \cos\phi) / \cos\theta \\ \\ u \\ v \\ w \end{bmatrix}
+
$$

$$
w_R e_3 + p_R e_4 + q_R e_5 + r_R e_6 \quad , \qquad \text{(CH.1)}
$$

where the functions $h_1(.)$ and $h_2(.)$ arise from the direct effect of the commands in the forward and lateral velocities respectively. The fin, fuselage and tail plane contributions are represented by the terms F_x and F_y.

The unitary vectors e_3, e_4, e_5 and e_6 are the 3th, 4^{th}, 5^{th} and 6^{th} elements of the standard base of R^{12}. In obtaining these vectors it becomes apparent that the G matrix satisfies the ratio condition. It should be noted that the definition of the matrix $B_1(.)$ in the compensator is not unique, in general any combination of four rows of the matrix G form a nonsingular matrix. The selection of the rows which form matrix $B_1(.)$, in the synthesis of the former compensator, are chosen in order to control the angular velocities. It is well known from experience that the flight trajectory control of any aircraft depends on the control of the angular velocities. Buckingham and Padfield [1986] reported this fact through a series of piloted simulations directed to exploring and defining control systems. In the simulations presented in this report, the existence of a control law which decouples and transforms the angular responses into a second order linear system has been assumed.

The control of the angular velocities using the nonlinear compensator equvalent to the singular perturbation pproach applied to a fixed wing aircraft. (Menon et al [1987])

In the work presented by Menon, Badget, Walker and Duke [1987] a singular perturbation technique is applied in order to design a fixed wing aircraft trajectory controller. The advantage presented by this approach is the reduction in the order of the aircraft model in trim conditions and during the the execution of manoeuvres, in order to separate the system model into two sub-systems with different time scales. According to Kokotovic's tutorial [1987], the singular perturbation model for finite dimensional dynamic systems is as an explicit state variable form in which the derivatives of some of the states are multiplied by a small parameter ε, that is,

$$\dot{x} = F_1(x, z, u, \varepsilon, t) \qquad x \in R^{\theta} \qquad \text{(IV.SP.1)}$$

$$\varepsilon\dot{z} = F_2(x, z, u, \varepsilon, t) \qquad z \in R^{s} \qquad \text{(IV.SP.2)}$$

where $u = u(t)$ is the control vector, $F_1(.)$ and F_2 are C^{∞} vector fields with respect to x, z, u, ε and t, x and z form the state and the scalar ε represents the small parameters to be neglected.

In the control and system theory the model given by (IV.SP.1) and (IV.SP2.) is a convenient tool for "Reduced Order Modeling". The order reduction is converted into a parameter perturbation called singular. When the value of ε is set to zero the dimension of the state space of (IV.SP.1) and (VI.SP.2) is reduced from $\theta + s$ to θ because the differential equation (VI.SP.2) degenerates into the algebraic equation

$$0 = F_2(\bar{x}, \bar{z}, u, \varepsilon, t), \qquad \text{(VI.SP.3)}$$

where the bar indicates that the variables belong to the system when $\varepsilon = 0$.

The model given by equations (IV.SP.1) and (IV.SP.2) is referred to as the "standard form" if and only if, the following assumption concerning (IV.SP.3) is satisfied.

Assumption 1. SP.1.

In a domain of interest equation (IV.SP.3) has $k \geqslant 1$ distinct ("isolated") real roots.

$$\bar{z} = \varphi_i(\bar{x}, \bar{u}, t), \qquad i = 1, \ldots, k \quad . \qquad \text{(IV.SP.4)}$$

This assumption assures that a well defined θ-dimentional reduced model (IV.SP.4) is substituted into (IV.SP.1)

$$\dot{\bar{x}} = F_1(\bar{x},\ \varphi_i(\bar{x},\ \bar{u},\ t),\ \bar{u},\ \theta\ t) \qquad \text{(IV.SP.5)}$$

In the sequel the subscript i can be omitted

$$\dot{\bar{x}} = F_1(\bar{x},\ \bar{u},\ t) \qquad \text{(IV.SP.6)}$$

This model is sometimes called the "quasi-steady-state" model, because

z , whose velocity $\dot{z} = g/\varepsilon$ is large when ε is small, may rapidly converge to a root of (IV.SP.3), which is the quasi-steady-state form of (IV.SP.2). This defines the two-time scale property of the system.

The convenience of using a parameter to achieve order reduction, in general, also has a disadvantage; it is not always clear how to choose the parameters which are to be considered small. Fortunately, in many cases, the knowledge of the physical processes and components of the system suffice in the selection of the appropriate parameters. For example the report by Menon et al [1987] considered the fast states for the fixed wing aircrafts as the rotational velocities. This assumption can be considered as the natural choice for flying vehicles. In the case of the helicopter, the experience (Buckingham, Padfield [1986]) shows that this assumption is valid due to the fact that the evolution of the angular velocities is really faster than the other velocities.

Applying the singular perturbation model criterion to the closed-loop system (IV.CH.1) one can include the scale parameters ε_1, ε_2, ε_3 and ε_4 as follows

$$
\begin{vmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{p} \\ \dot{q} \\ \dot{r} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{\psi} \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{vmatrix} = \begin{vmatrix} -G \sin\theta + Fx + \varepsilon_1 (vr - wq) + \varepsilon_2 \, g_1(p, q, r, w) \\ G \cos\theta \sin\phi + Fx + \varepsilon_3 (wq - ur) + \varepsilon_4 \, g_2(p, q, r, w) \\ 0 \\ 0 \\ 0 \\ 0 \\ q \cos\phi - r \sin\phi \\ p + q \sin\phi \tan\theta + r \cos\phi \tan\theta \\ (q \sin\phi + r \cos\phi) / \cos\theta \\ u \\ v \\ w \end{vmatrix} +
$$

$$
w_R \, e_3 + p_R \, e_4 + q_R \, e_5 + r_R \, e_6 \quad ,
$$

$$
\text{or} \quad \dot{x}(t) = f_{c1}(x) + \sum_{i=1}^{4} R_i h_i \quad ,
$$

(IV.C-H.2)

where $R_1 = w_R$, $R_2 = p_R$, $R_3 = q_R$, $R_4 = r_R$, $h_1 = e_3$, $h_2 = e_4$, $h_3 = e_5$, $h_4 = e_6$

The coefficients ε_1, ε_2, ε_3 and ε_4 are zero if the angular velocities are also zero and ε_2 and ε_4 are zero if the helicopter is in trim.

Equation (IV.C-H.2) coincides also with the model used by Meyer, Hunt and Su [1982], in which model inverses are used. Finally, one can assert that in the present case, the ratio condition and assumption (IV.SP.1) are equivalent, i. e., in this two conditions one has to solve for the normal and angular velocities.

In the following section a nonlinear controller is developed using the theory developed in Chapter III for the system given by equation (IV.CH.2).

IV.3. CLOSED-LOOP SYSTEM ANALYSIS.

In the last section a nonlinear compensator was presented. It was designed in order to perform a partial linearization of the helicopter equation of motion. The closed-loop system resulting from the application of this compensator is given by equation (IV.C-H.2). The object of the present section is to analyse whether or not the input-state space of this system can be transformed, through a diffeomorphic map, to a linear Brunovsky canonical form.

According to the results obtained in Chapter III, the nonlinear system

$$\dot{x}(t) = f_{c1}(x) + \sum_{i=1}^{4} R_i \, h_i \qquad \text{(IV.C-H.2)}$$

is linear transformable if and only if the following conditions are satisfied:

1. The set $C = \{R_1, [f_{c1}, R_1], \ldots, (ad^{k_1-1} f_{c1}, R_1),$

$$\vdots$$

$$R_4, [f_{c1}, R_4, \ldots, (ad^{k_4-1} f_{c1}, R_4) \}$$

spans a 12-dimensional space.

2. The sets $C_1 = \{ R_1, [f_{c1}, R_1], \ldots , (ad^{k_1-2}f_{c1}, R_1) \}$

.

.

.

$C_4 = \{ R_4, [f_{c1}, R_4], \ldots , (ad^{k_4-2}f_{c1}, R_4) \}$

are involutive and;

3. The span of C_i, $i = 1, \ldots , 4$ is equal to the span of C_i C.

It is very difficult to investigate whether or not any nonlinear system oforder greater than three satisfies these conditions. Furthermore, during these calculations, errors are readily committed. For plants of the order treated here it is practically impossible to calculate the sets C, C_1, C_2, C_3, and C_4 without the aid of Symbolic Algebraic Manipulation (SAM).

Since the late sixties, it has been known that computers are capable of performing symbolic and algebraic manipulations. Nowadays, there are applications of symbolic algebraic manipulation systems in many fields of scientific research, such as, physics, celestial mechanics, optics, applied mathematics (Marino, Cesaro [1984], Hearn [1985], Fitch [1985]) and now helicopter flight control dynamics.

The basic features of every symbolic algebraic system are

a), the use of integer and rational arithmetics with infinite precision and

b), the manipulation of polynomials, rational and elementary functions.

The symbolic algebraic system used here, Reduce, also offers the following;

c), algebraic manipulation of matrices whose entries are polynomials, rational and elementary functions or combinations of these mathematical entities;

d), calculus: derivatives, partial derivatives and integrals;

e), other types of manipulations, not included in the previous points, new or special rules for specific needs can be added, and

f) the interactive use is also permitted, so that, one can manipulate expressions in the same way as one can use pocket calculators for numbers.

The symbolic algebraic manipulation systems have, in general, very low capabilities for numerical calculations. However the results obtained using Reduce can be given as Fortran instructions and therefore can be directly implemented.

The potential of Reduce in the present application relies mainly on the possibility of performing partial derivatives. This feature makes the computation of Lie brackets and Lie derivatives very easy, so that the calculation of the sets C, C_1, C_2, C_3 and C_4 is facilitated. For example, obtaining the Lie bracket of two vector fields of order n, can be done by executing the procedure presented in figure (IV.RE.1). Using this programme in conjunction with the iterative qualities of Reduce, the set C is easily obtained.

Before presenting the calculation of the set C, the following aspects are considered. Firstly the state is rearanged as follows,

$[x, u, \theta, q, y, v, \phi, p, z, w, \psi, r]^t$

so that the input is also changed to

$q_R e_4$, $p_R e_8$, $w_R e_{10}$ and $r_R e_{12}$,

in order to visualize that the pilot longitudinal stick commands correspond to pitch attitude, lateral stick to roll attitude, collective to normal velocity and pedal position to yaw rate. These associations between the commands and attitudes is the natural way to conceive the helicopter control from the point of view of the pilot. Yue et. al. [1987] assumed also this relationship in the design of control laws using H^∞-optimization and Buckingham et. al. [1986] in the research of advanced control systems for helicopters. Considering this aspect and the rearangement of the state, leads one to suggest that the controllability indices are 4, 4, 2 and 2. Therefore the set C is

$$\{e_4,\ [f_R,\ e_4],\ (ad^2 f_R,\ e_4),\ (ad^3 f_R,\ e_4),$$
$$e_8,\ [f_R,\ e_8],\ (ad^2 f_R,\ e_8),\ (ad^3 f_R,\ e_8),$$
$$e_{10},\ [f_R,\ e_{10}],$$
$$e_{12},\ [f_R,\ e_{12}]\}$$

where f_R is the reordered version of f_{c1} appropriate to the new state. These vector fields are presented in figure (IV.RE.2).

If the selection of the controllability indices is adequate then the elements of C also form a set of linearly independent vector fields. Following the ideas of Marino and Cesareo [1985], [1984], a programme for determining whether or not a set of vector fields are linearly independent has been developed under the following assumptions.

Given a set of m vector fields $\{V_1,\ V_2,\ \ldots\ ,\ V_p\}$, $V_i \in R^n$ $i = 1,\ \ldots\ ,p$.

find the rank r of the corresponding distribution and the spanning set of vector fields. The method used to solve this problem is the triangularization of the n×m matrix formed by the given vector fields. A

Gaussian free fraction algorithm has been implemented, a listing of the programme is presented in figure (IV.RE.3).

Using the programme described above, it is found that the rank of the matrix formed with the elements of C as its columns is 12. It should be noted that this programme is designed to be used for an arbitrary number of vector fields. In this case the matrix formed by the vector fields is square, so that, using the capabilities of Reduce one can also determine whether or not if C spans a twelve dimensional space by calculating the determinant of this matrix. Furthermore, one needs to know if this matrix is nonsingular at the origin. The results which show this matrix is nonsingular are presented in figure (IV.RE.2). This proves the first condition required for the existence of the transformation.

In order to establish if a given set of vector fields is involutive or not, the programme given in figure (IV.RE.4) has been developed. The inputs of this programme are

I., the array G which is provided by the triangularization algorithm,

II., the dimension of the vector fields and

III., the dimension of the space spanned by the vector fields (which is an output of the triangularization algorithm.

The output of the programme is simply "TRUE" if the vector fields are involutive and "FALSE" if otherwise.

The use of the triangularized vector fields G facilitates the determination of the output of this programme, since given a set of vector fields $\{V_1, \ldots, V_p\}$

$[V_i, V_j] \varepsilon$ span $\{V_1, \ldots, V_p\}$ if and only if $[V_i^T, V_j^T]^T \varepsilon$ span $\{V_1^T, \ldots, V_q^T\}^T$ $p \geqslant q$,

where the index T indicates triangularized. The sets are obviously involutive if and only if

$[V_i^T, V_j^T]^T = 0$ for all i, j; i < j

Using the programmes "TRIAN" and "INVOLU" one can readily check that the sets

$$C_1 = C_2 = \{ e_4, [f_{CL}, e_4], (ad^2 f_{CL}, e_4),$$
$$e_8, [f_{CL}, e_8], (ad^2 f_{CL}, e_8),$$
$$e_{10}, [f_{CL}, e_{10}], (ad^2 f_{CL}, e_{10}),$$
$$e_{12} [f_{CL}, e_{12}], (ad^2 f_{CL}, e_{12}) \}$$

are involutive. The sets C_3 and C_4 are trivially involutive. This shows that the closed-loop system satisfies the condition II.

It is easier to confirm that the closed-loop system satisfies the condition III point due to the fact that $(ad\ f_{CL}, e_{10})$ vanishes and that $(ad^2 f_{CL}, e_{10})$ spans the same space as $[f_{CL}, e_{10}]$, so that every C_i C, i= 1, ... , 4 is involutive.

Note that if the position coordinates were not included in equation (IV.HE.1), the set C would not be involutive, and position deviations from a reference state of the helicopter could not be controlled.

In this section it has been proved that if the closed-loop system is in trim condition, that is when ε_i, i = 1, ... , 4 are zero, it can be transformed through a diffeomorphic map to a linear canonical system. The construction of one of these possible maps is presented in the next section.

```
PROCEDURE LIE(P1,P2,NF);
%
% THIS PROCEDURE CALCULATES THE LIE BRACKETT
% OF THE VECTOR FIELDS P1 AND P2.
% THE LIE BRACKETT [P1,P2] IS ASSIIGNED TO THE
% COLUMN MATRIX VET= [P1,P2] .
%
BEGIN
 MATRIX JAF1(12,12);
 FOR I:=1:NF DO
  FOR J:=1:NF DO
   JACB(I,J):=DF(P2(I,1),X(J));
 VET1:=JACB*P1;
 FOR I:=1:NF DO
  FOR J:=1:NF DO
  JAF1(I,J):=DF(P1(I,1),X(J));
  VET2:=JAF1*P2;
 VET:=VET1-VET2;
END;
END;
```

FIGURE IV.RE.1.- PROCEDURE USED TO CALCULATE THE LIE BRACKETT OF TWO VECTOR FIELDS. GIVEN P1 AND P2 THEN VET= [P1,P2].

```
SYSTEM VECTOR FIELD f:
F1(1,1) :=  - (SIN(X7)*GRAV - FX - FXØ)$
F1(2,1) := SIN(X8)*COS(X7)*GRAV + FY + FYØ$
F1(7,1) :=  - (SIN(X8)*X6 - COS(X8)*X5)$
F1(8,1) := ((SIN(X8)*X5 + COS(X8)*X6)*SIN(X7) + COS(X7)*X4)/COS(X7)$
F1(9,1) := (SIN(X8)*X5 + COS(X8)*X6)/COS(X7)$
F1(1Ø,1) := X1$
F1(11,1) := X2$
F1(12,1) := X3$

INPUT VECTOR FIELD g1;
F2(3,1) := 1$
INPUT VECTOR FIELD g2;
F3(4,1) := 1$
INPUT VECTOR FIELD g3;
F4(5,1) := 1$
INPUT VECTOR FIELD g4;
F5(6,1) := 1$


ELEMENTS OF SET C.
THE COLOUMNS OF MATRIX "ORIGEN" ARE THE ELEMENTS OF
THE SET C.

ORIGEN(3,1) := 1 := [ , g1]$
ORIGEN(4,2) := 1 := [ , g2]$
ORIGEN(5,3) := 1 := [ , g3]$
ORIGEN(6,4) := 1 := [ , g4]$

$ THE OPERATION [ ,g] DENOTES THE LIE BRACKETT OF
$ ORDER ZERO.
```

FIGURE IV.RE.2.- CONTINUES NEXT PAGE.

```
ORIGEN(12,5) := (-1) := [f, g1]$

ORIGEN(8,6) := (-1) := [f,g2]$

VECTOR FIELD [f,g3]
ORIGEN(7,7) :=  - COS(X8) $
ORIGEN(8,7) := ( - SIN(X7)*SIN(X8))/COS(X7)$
ORIGEN(9,7) := ( - SIN(X8))/COS(X7)$
VECTOR FIELD [f,g4]
ORIGEN(7,8) := SIN(X8)$
ORIGEN(8,8) := ( - SIN(X7)*COS(X8))/COS(X7)$
ORIGEN(9,8) := ( - COS(X8))/COS(X7)$

VECTOR FIELD [f, [f,g1]]
ORIGEN(2,9) := COS(X7)*COS(X8)*GRAV$
ORIGEN(7,9) :=  - (SIN(X8)*X5 + COS(X8)*X6)$
ORIGEN(8,9) := ( - (SIN(X8)*X6 - COS(X8)*X5)*SIN(X7) ) /COS(X7)$
ORIGEN(9,9) := ( - (SIN(X8)*X6 - COS(X8)*X5))/COS(X7)$

VECTOR FIELD [f,[f,g2]]
ORIGEN(1,10) :=  - COS(X7)*COS(X8)*GRAV$
ORIGEN(7,10) := SIN(X8)*X4$
ORIGEN(8,10) := ( - (SIN(X7)*COS(X8)*X4 - SIN(X8)**2*COS(X7)*X6 - COS(
X7)*COS(X8)**2*X6)*COS(X7))/COS(X7)**2$
ORIGEN(9,10) := ( - COS(X8)*X4)/COS(X7)$
```

FIGURE IV.RE.2.- CONTINUES NEXT PAGE.

```
VECTOR FIELD [f,[f,[f,g1]]]

ORIGEN(1,11) :=  - (SIN(X8)*X5 + COS(X8)*X6)*COS(X7)*GRAV$

ORIGEN(2,11) :=  - (2*SIN(X7)*X5 + SIN(X8)*COS(X7)*X4)*GRAV$

ORIGEN(7,11) := (SIN(X8)*X6 - COS(X8)*X5)*X4$

ORIGEN(8,11) := ( - (((SIN(X8)*X5 + COS(X8)*X6)*SIN(X7) + COS(X7)*X4)*
(SIN(X8)*X5 + COS(X8)*X6)*SIN(X7) + ((SIN(X8)*X6
 - COS(X8)*X5)*SIN(X7) + SIN(X8)*X5 + COS(X8
)*X6)*((SIN(X8)*X6 - COS(X8)*X5)*SIN(X7) - SIN(
X8)*X5 - COS(X8)*X6) - (SIN(X8)*X6 - COS(X8)*X5
)**2))/COS(X7)**2$

ORIGEN(9,11) := ( - (SIN(X8)*X5 + COS(X8)*X6)*X4)/COS(X7)$

ORIGEN(11,11) :=  - COS(X7)*COS(X8)*GRAV$


VECTOR FIELD [f,[f[f,g2]]]

ORIGEN(1,12) := (SIN(X7)*SIN(X8)**2*X5 + SIN(X7)*COS(X8)**2*X5 + 2*SIN
(X8)*COS...7)*X4)*GRAV$

ORIGEN(2,12) := ((SIN(X7)*X4 - SIN(X8)**2*COS(X7)*COS(X8)*X6 - COS(X7)
*COS(X8)**3*X6)*COS(X7)*GRAV)/COS(X7)$

ORIGEN(7,12) := ((SIN(X8)**3*X5*X6 + SIN(X8)**2*COS(X8)*X6**2 + SIN(X8
)*COS(X8)**2*X5*X6 + COS(X8)**3*X6**2 + COS(X8)*X4
**2)*COS(X7)**2)/COS(X7)**2$

ORIGEN(8,12) := ((SIN(X7)*SIN(X8)*X4**2 + SIN(X7)*SIN(X8)*X6**2 - SIN(
X7)*COS(X8)*X5*X6 - SIN(X8)**2*COS(X7)*X4*X5 - COS
(X7)*COS(X8)**2*X4*X5)*COS(X7)**2)/COS(X7)**3$

ORIGEN(9,12) := ((SIN(X8)**3*X6**2 - SIN(X8)**2*COS(X8)*X5*X6 + SIN(X8
)*COS(X8)**2*X6**2 + SIN(X8)*X4**2 - COS(X8)**3*X5
*X6)*COS(X7)**2)/COS(X7)**3$

ORIGEN(10,12) := COS(X7)*COS(X8)*GRAV$


DETERMINANT OF MATRIX "ORIGEN".

DETORIGEN := COS(X7)**3*COS(X8)**4*GRAV**4$
```

FIGURE IV.RE.2 SYSTEM FUNCTION AND ELEMENTS OF THE SET "C".
THE ELEMENTS OF "C" ARE THE COLUMNS OF MATRIX "ORIGEN".
THE VANISHING TERMS ARE NOT SHOWN:

```
 PROCEDURE TRI (RA,NF,MC,SPAN);
%
% THIS PROCEDURE TRIANGULARISES THE MATRIX FUNCTION "RA"
% OF ORDER (NF,MC). THE TRIANGULARIZATION OF MATRIX "RA"
% IS GIVEN BY MATRIX BB.
% THE NUMBER OF INDEPENDENT COLUMNS IS GIVEN BY THE PARAMETER
% "SPAN".
%
 BEGIN  % 1
 INTEGER INE,KM1,J,K,KMM,LL,RR;
   L2:=Ø;
   L:=Ø;
   R:=1;
   KC:=1;
 %
 %
   FOR J:=1:MC DO
    FOR I:=1:NF DO
     GG(R,I,J):=RA(I,J);
   GG(Ø,Ø,Ø):=1;
 %
    WHILE ( KC < MC ) DO
     BEGIN  %2

      L2:=KC-1;;
      L:=L+1;
      WHILE ( MC > L2 ) DO
       BEGIN    %3
        L2:= L2+1;
        IF (GG(R,L,L2) NEQ Ø ) THEN
         BEGIN  %4
            FOR II:=1:NF DO
             BEGIN    %5
              V(II,KC):= GG(R,II,L2):
              V(II,L2):= GG(R,II,KC)
             END:     %5
 %
            FOR JJ:=1:NF DO
             BEGIN   %6
              GG(R,JJ,KC):=V(JJ,KC);
              GG(R,JJ,L2):=V(JJ,L2);
             END;     %6
 %
        INE:=L2;
        L2:=MC;
 %
        LL:=KC+1;
        IF ( KC = MC ) THEN LL:=MC;
        K:=R;
        KM1:=R+1;
        KMM:=R-1;
```

FIGURE IV.RE.3.- CONTINUES NEXT PAGE.

```
%
       IF (GG(KMM,KMM,KMM) NEQ Ø) THEN
       FU:=GG(KMM,KMM,KMM) ELSE FU:=1;
       FU:=1;
%
       FOR I:=LL:MC DO
          BEGIN       %71
          IF (GG(K,L,I) NEQ Ø) THEN
          RAQ:=GG(K,L,K) ELSE RAQ:=1;
          FOR J:=1:NF DO
          BEGIN      %8
      GG(KM1,J,I):= RAQ*GG(K,J,I)-GG(K,J,K)*GG(K,L,I);
          GG(KM1,J,I):=GG(KM1,J,I)/FU
          END;    %8
          END;    %71
%
       R:=R+1;
       KC:=KC+1;
       NON:=Ø;
%
       END     %4
%
      END;      %3
%
    END;     %2
%
  FOR I:=1:MC DO
   FOR J:=1:NF DO
    BB(I,J,I):=Ø;
%
  RR:=Ø;
  FOR I:=1:MC DO
   BEGIN   %1
    FOR JJ:=1:NF DO
      VAX(JJ,1):=GG(I,JJ,I);
      PIN:=Ø;
      FOR IK:=1:NF DO
       PIN:=VAX(IK,1)*VAX(IK,1)+PIN;
      IF (PIN = Ø) THEN <<PON:=3>>
      ELSE <<RR:=RR+1; FOR IL:=1:NF DO BB(RR,IL,RR):=VAX(IL,1);
         SPAN:=RR>>
    END; %1
%
%
END;  % 1
```

FIGURE IV.RE.3.- THIS PROCEDURE TRIANGULARISES A MATRIX FUNCTION USING THE FREE FRACCTION GAUSSIAN ALGORITHM.

```
PROCEDURE INVO (P,SPN,NFI);
%
% THIS PROCEDURE DETERMINES IF "SPN" VECTOR FIELDS
% (COLUMNS OF ARRAY P) OF DIMENSION "NFI" ARE
% INVOLUTIVE.
% THE BOOLEAN (FALSE OR TRUE) ANSWER IS ASSIGNED
% TO THE VARIABLE "INVOLUTIVE" .
%
 BEGIN          %1
  INTEGER I;
  MATRIX VOL(12,2Ø), VACA(12,1);
  I:=2;
  M:=SPN+1;
  MAS:=SPN+2;
  INVOLUTIVE:=TRUE;
%
   WHILE ( I < MAS ) DO
   BEGIN        %2
    H:=1;
%
    WHILE ( H < I ) DO
     BEGIN      %3
      FOR J:=1:NFI DO
       BEGIN    %4
        A1(J,1):= P(H,J,H);
        A2(J,1):= P(I,J,I)
       END;      %4
%
       LIE(A1,A2,NFI);
%
       PINT:=Ø;
       FOR I:=1:NFI DO
       PINT:=PINT+VET(I,1)*VET(I,1);
%
       IF (PINT NEQ Ø) THEN
       BEGIN
        VET1:=VET;
        VACA:=VET1;
        FOR J:=1:NFI DO
        P(M,J,M):=VET(J,1);
        FOR K1:=1:M DO
         FOR K2:=1:NFI DO
          VOL(K2,K1):= P(K1,K2,K1);
%
       TRI(VOL,NFI,M);
        DIN:=SPAN;
%
```

FIGURE IV.RE.4.- CONTINUES NEXT PAGE.

```
        PIN:=Ø;
        FOR LA:=1:NFI DO
        PIN:=PIN+GG(M,LA,M)*GG(M,LA,M)
      END
      ELSE << PIN:=Ø >>;
%
      IF ( PIN=Ø ) THEN
       H:=H+1
       ELSE << COSA:=PIN; INVOLUTIVE:=FALSE; H:=I; I:=MAS >>
     END;          %3
     I:=I+1
    END;       %2
    IF (INVOLUTIVE = FALSE) THEN INVOLUTIVE:=FALSE
    ELSE  INVOLUTIVE:=TRUE;
    RETURN INVOLUTIVE
 END;        %1
```

FIGURE IV.RE.4.- THIS PROCEDURE DETERMINES IF A DISTRIBUTION OF VECTOR FIELDS IS INVOLUITIVE.

IV.4. CONSTRUCTION OF A Z-TRANSFORMATION.

At this point the existence of a diffeomorphic transformation of the closed-loop system has been established. The next step, according to the sequence given at the beginning of this chapter, is the construction of one of these transformations. The way in which this can be done is now considered.

The construction of the diffeomorphic map of a nonlinear system to a linear Brunovsky canonical form, described in Chapter III, where it is shown that the components of the transformation in question are obtained by solving a set of partial differential equations. Given that these equations are linear, they can be reduced to a set of ordinary differential equations, which given the nature of the problem, are obviously nonlinear.

For practical purposes, the problem of finding the diffeomorphic relationship between the nonlinear system to the linear canonical form, is that of obtaining the solution of the nonlinear ordinary differential equations arising from the definition of the transformation itself. In general, the solutions of these of equations tends to increase in complexity with the number of equations. The algebraic problem involved is usually analytically intractable. The previous simplifications performed in the closed-loop system were intended to avoid such problems. This aspect is not characteristic of every application of the theory used here. For instance, R. Marino [1984] presented an application to a synchronus generator connected to an infinite bus. The order of the model of the plant used in was five, with two inputs. In this case the solution was obtained by assuming that the first component of the mapping depended on two variables only.

Unfortunately, in the present case one has to appeal to simplifications, for example the parameters ϵ_1, ϵ_2, ϵ_3 and ϵ_4 in the closed-loop system are considered zero. This is equivalent to the assumption that the helicopter is in a trim condition, posibly with small angular velocities. Under this assumption, it is easy to obtain a solution to the set of partial differential equations that define the map coordinates.

The partial differential equations are

$\langle dT_1, (ad^3\, f_{CL}, e_4)\rangle = 0$,

$\langle dT_1, (ad^2\, f_{CL}, e_4)\rangle = 0$,

$\langle dT_1, [f_{CL}, e_4]\rangle = 0$,

$\langle dT_1, e_4\rangle = 0$;

$\langle dT_5, (ad^3\, f_{CL}, e_8\rangle = 0$,

$\langle dT_5, (ad^2\, f_{CL}, e_8\rangle = 0$,

$\langle dT_5, [f_{CL}, e_8]\rangle = 0$,

$\langle dT_5, e_8\rangle = 0$;

$\langle dT_9, [f_{CL}, e_{10}]\rangle = 0$,

$\langle dT_9, e_{10}\rangle = 0$,

$\langle dT_{11}, [f_{CL}, e_{11}]\rangle = 0$ and

$\langle dT_{11}, e_{11}\rangle = 0$,

where the Lie brackets involved in the equations are presented in figure (IV.RE.2).

One of the possible solutions of the system is, $T_1(x) = x_1$, $T_5(x) = x_5$, $T_9(x) = x_9$ and $T_{11}(x) = x_{11}$ where x is the state vector.

By construction, the remaining components are immediately given by

$T_2(x) = \langle dT_1, f_{CL}\rangle$,

$T_3(x) = \langle dT_2, f_{CL}\rangle$,

$T_4(x) = \langle dT_3, f_{CL}\rangle$;

$T_6(x) = \langle dT_5, f_{CL}\rangle$,

$T_7(x) = \langle dT_6, f_{CL}\rangle$,

$T_8(x) = \langle dT_7, f_{CL}\rangle$;

$T_{10}(x) = \langle dT_9, f_{CL}\rangle$ and

$T_{12}(x) = \langle dT_{11}, f_{CL}\rangle$.

This transformation maps the states of the closed-loop system to a Brunovsky canonical form. Explicit expressions for the components, T_i, are presented in figure (IV.RE.5).

From the point of view of control synthesis, the interest in defining the state transformation is the definition of the inverse of the input mapping. In this case the inverse is

$u = G_T^{-1}\{v - F_v\}$ or

$$
\begin{vmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{vmatrix} = \begin{vmatrix} \langle dT_4, e_4\rangle & \langle dT_4, e_8\rangle & \langle dT_4, e_{10}\rangle & \langle dT_4, e_{12}\rangle \\ \langle dT_8, e_4\rangle & \langle dT_8, e_8\rangle & \langle dT_8, e_{10}\rangle & \langle dT_8, e_{12}\rangle \\ \langle dT_{10}, e_4\rangle & \langle dT_{10}, e_8\rangle & \langle dT_{10}, e_{10}\rangle & \langle dT_{10}, e_{12}\rangle \\ \langle dT_{12}, e_4\rangle & \langle dT_{12}, e_8\rangle & \langle dT_{12}, e_{10}\rangle & \langle dT_{10}, e_{12}\rangle \end{vmatrix}^{-1} \cdot
\left\{ \begin{vmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{vmatrix} - \begin{vmatrix} \langle dT_4, f_{CL}\rangle \\ \langle dT_8, f_{CL}\rangle \\ \langle dT_{10}, f_{CL}\rangle \\ \langle dT_{12}, f_{CL}\rangle \end{vmatrix} \right\} \qquad \text{(IV.C-H.3)},
$$

where v is the control input vector of the linear system. Equation (IV.C-H.3) defines the nonlinear control input of the nonlinear system. This control law will have the same effect in the nonlinear system as the input v has in the linear canonical form. The matrix function G_T^{-1} and the vector field F_v are shown in figure (VI.RE.4).

In general, if v is the input of the linear equivalent system, it can be defined according to any standard linear design. Here it is assumed that the nonlinear system is mapped to a linear Brunovsky canonical form, so that, the pole placement method would appear to be an appropriate approach for the generation of v. This will be dealt with briefly in the next section.

```
T1 := X1Ø$

T2 := X1$

T3 :=  - SIN(X7)*GRAV + FX + FXØ$

T4 := COS(X7)*GRAV*(SIN(X8)*X6 - COS(X8)*X5)$

T5 := X11$

T6 := X2$

T7 := SIN(X8)*COS(X7)*GRAV + FY + FYØ$

T8 := GRAV*(SIN(X7)*SIN(X8)**2*X6 - SIN(X7)*SIN(X8)*COS(X8)*X5 + SIN(
X8)*COS(X7)*COS(X8)*TAN(X7)*X5 + COS(X7)*COS(X8)**2*TAN(X7
)*X6 + COS(X7)*COS(X8)*X4)$

T9 := X12$

T1Ø := X3$

T11 := X9$

T12 := (SIN(X8)*X5 + COS(X8)*X6)/COS(X7)$
```

FIGURE IV.RE.5.- COMPONENTS OF THE DIFFEOMORPHIC TRANSFORMATION.

```
GTINV(1,3) := 1$
GTINV(2,1) := ( - SIN(X7)*SIN(X8))/(COS(X7)**2*COS(X8)*GRAV)$
GTINV(2,2) := 1/(COS(X7)*COS(X8)*GRAV)$
GTINV(2,4) :=  - SIN(X7)$
GTINV(3,1) := ( - COS(X8))/(COS(X7)*GRAV)$
GTINV(3,4) := SIN(X8)*COS(X7)$
GTINV(4,1) := SIN(X8)/(COS(X7)*GRAV)$
GTINV(4,4) := COS(X7)*COS(X8)$
```

FIGURE IV.RE.6.- TERMS RELATED TO NONLINEAR CONTROLLER DESCRIBED BY EQUATION IV.C-H.3 .

$VECTOR FIELD FV OF EQUATION IV.C-H.3 :

```
FV(1,1) := (((SIN(X8)*X5 + COS(X8)*X6)*SIN(X7) + COS(X7)*X4)*(SIN(X8)*
X5 + COS(X8)*X6) + (SIN(X8)*X6 - COS(X8)*X5)**2*SIN(X7)
)*GRAV$

FV(2,1) := ( - ((X4**2 + X6**2)*SIN(X8)*COS(X7) + SIN(X7)*X4*X5 - COS(
X7)*COS(X8)*X5*X6)*COS(X7)*GRAV)/COS(X7)$

FV(4,1) := ( - (2*(SIN(X8)*X5 + COS(X8)*X6)*SIN(X7) + COS(X7)*X4)*(SIN
(X8)*X6 - COS(X8)*X5))/COS(X7)**2$
```


```
FV(1,1) := (((SIN(X8)*X5 + COS(X8)*X6)*SIN(X7) + COS(X7)*X4)*(SIN(X8)*
X5 + COS(X8)*X6) + (SIN(X8)*X6 - COS(X8)*X5)**2*SIN(X7)
)*GRAV$

FV(2,1) := ( - ((X4**2 + X6**2)*SIN(X8)*COS(X7) + SIN(X7)*X4*X5 - COS(
X7)*COS(X8)*X5*X6)*COS(X7)*GRAV)/COS(X7)$

FV(4,1) := ( - (2*(SIN(X8)*X5 + COS(X8)*X6)*SIN(X7) + COS(X7)*X4)*(SIN
(X8)*X6 - COS(X8)*X5))/COS(X7)**2$
```


MATRIX GT OF EQUATION IV.C-H.3 .

```
GT(1,3) :=  - COS(X7)*COS(X8)*GRAV$

GT(1,4) := SIN(X8)*COS(X7)*GRAV$

GT(2,2) := COS(X7)*COS(X8)*GRAV$

GT(2,4) := SIN(X7)*GRAV$

GT(3,1) := 1$

GT(4,3) := SIN(X8)/COS(X7)$

GT(4,4) := COS(X8)/COS(X7)$
```

DETRMINANT OF MATRIX GT.

```
DETGT := COS(X7)*COS(X8)*GRAV**2$
```

FIGURE IV.RE.6.- CONTINUES NEXT PAGE.

INVERSE OF MATRIX GT.

```
GTINV(1,3) := 1$

GTINV(2,1) := ( - SIN(X7)*SIN(X8))/(COS(X7)**2*COS(X8)*GRAV)$

GTINV(2,2) := 1/(COS(X7)*COS(X8)*GRAV)$

GTINV(2,4) :=  - SIN(X7)$

GTINV(3,1) := ( - COS(X8))/(COS(X7)*GRAV)$

GTINV(3,4) := SIN(X8)*COS(X7)$

GTINV(4,1) := SIN(X8)/(COS(X7)*GRAV)$

GTINV(4,4) := COS(X7)*COS(X8)$
```

FIGURE IV.RE.6.- TERMS RELATED TO NONLINEAR CONTROLLEF
DESCRIBED BY EQUATION IV.C-H.3 .

IV.5. LINEAR CONTROLLER.

In this section a possible way to generate the control input of the linear equivalent system is treated briefly. The application of the inverse of the diffeomorphic map to this input, given by equation (IV.C-H.6), will lead the nonlinear system to behave as the linear canonical form, so that, this completes the structure of the Flight Control System.

Nowadays, the analysis and design of linear control systems is very well known, there are extensive and comprehensive treatises about the subject. At this stage of the design presented here the nonlinear control problem has been reduced to the use of very well established procedures. The analysis of the most adequate linear control technique to use is not presented here, this section is restricted to the application of state feedback in order to place the poles of the linear system in the stability region.

The pole placement technique is therefore applied to linear systems of the form;

$$\begin{vmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix} v_1 \quad ,$$

$$\begin{vmatrix} \dot{y}_5 \\ \dot{y}_6 \\ \dot{y}_7 \\ \dot{y}_8 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} y_5 \\ y_6 \\ y_7 \\ y_8 \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix} v_2 \quad ,$$

$$\begin{vmatrix} \dot{y}_9 \\ \dot{y}_{10} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} y_9 \\ y_{10} \end{vmatrix} + \begin{vmatrix} 0 \\ 1 \end{vmatrix} v_3 \quad \text{and}$$

$$\begin{vmatrix} \dot{y}_{11} \\ \dot{y}_{12} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} y_{11} \\ y_{12} \end{vmatrix} + \begin{vmatrix} 0 \\ 1 \end{vmatrix} v_4 \quad .$$

That is the Brunovsky canonical form.

The state feedback that performs the pole placement for the above linear system is given by;

$v_1 = R_1 - K_I Y_I$,

$v_2 = R_2 - K_{II} Y_{II}$,

$v_3 = R_3 - K_{III} Y_{III}$ and

$v_4 = R_4 - K_{IV} Y_{IV}$.

Where;

$Y_I = [y_1, y_2, y_3, y_4]^t$,

$Y_{II} = [y_5, y_6, y_7, y_8]^t$,

$Y_{III} = [y_9, y_{10}]^t$

$Y_{IV} = [y_{11}, y_{12}]^t$;

R_1, R_2, R_3 and R_4 are the references of the linear systems given by the transformation of the reference of the nonlinear system, according to the mapping coordinates presented in figure (IV.RE.6); and

$K_I = [-P_1 P_2 P_3 P_4, P_1 P_2 P_3 + P_1 P_2 P_4 + P_1 P_3 P_4 + P_2 P_3 P_4,$

$-(P_1 P_2 + P_1 P_3 + P_1 P_4 + P_2 P_3 + P_{24} + P_3 P_4), (P_1 + P_2 + P_3 + P_4)]$,

$K_{II} = [-S_1 S_2 S_3 S_4, S_1 S_2 S_3 + S_1 S_2 S_4 + S_1 S_3 S_4 + S_2 S_3 S_4,$

$-(S_1 S_2 + S_1 S_3 + S_1 S_4 + S_2 S_3 + S_2 S_4 + S_{34}), (S_1 + S_2 + S_3 + S_4)]$,

$K_{III} = [-V_1 V_2, V_1 + V_2]$ and

$K_{IV} = [-W_1 W_2, W_1 + W_2]$.

Where P_i, i = 1, ... , 4 ; S_i, i = 1, ... , 4 ; V_i i = 1, 2 and W_i, i = 1, 2 are the desired poles for the linear system and therefore, for the global closed-loop system.

The linear control technique, presented here, is expected to compensate the simplifications performed during the initial stages of the design. Obviously one cannot expect that the transformation presented in the previous section maps the nonlinear system exactly to a linear Brunovsky canonical form. Nevertheless, one can positively expect the transformation to be "near" enough to the canonical form, consequently the linear controller could cope with the discrepances arising from the simplifications.

The conditions which achieve this are shown in the following section by simulating the helicopter together with the flight control system.

IV.6. SYSTEM SIMULATIONS.

In order to visualize the performance of the control law developed in the previous sections, a series of simulations of a helicopter with the flight control system is presented. The results consist of the time responses of the helicopter state during the execution of basic manoeuvres.

The manoeuvres simulated are intended to show that the global closed-loop system is formed by four linear and decoupled sub-systems. It was previously established that this set of sub-systems corresponds exactly to the Brunovsky canonical form if the helicopter is in trim. In other circumstances the sub-systems still correspond to the canonical linear form, but a "noise" is added due to the simplifications made in the nonlinear controller design. One role of the linear controller in the flight control system is to compensate for the effects of the noise added to the system while it is not in trim.

The most elementary properties of the closed-loop system which need to be considered are a),- non-coupling of the four sub-systems and b), whether or not the controller allows the execution of manoeuvres. It should be noted that the Flight Control System was designed referring the position of the helicopter to body axes, for instance $\dot{x}= u$, $\dot{y}= v$ and $\dot{z}= w$, which are of no use in the definition of trajectories or position of the vehicle with respect to Earth. However, in the simulations presented here, the helicopter position has been referred to a reference frame fixed on Earth and expressed by

$$\dot{x}_E = u\,(\cos\theta\,\cos\psi) + v\,(\sin\phi\,\sin\theta\,\cos\psi - \cos\phi\,\sin\psi) + w\,(\cos\phi\,\sin\theta\,\cos\psi + \sin\phi\,\sin\psi),$$

$$\dot{y}_E = u\,(\cos\theta\,\sin\psi) + v\,(\sin\phi\,\sin\theta\,\sin\psi + \cos\phi\,\cos\psi) + w\,(\cos\phi\,\sin\theta\,\sin\psi - \sin\phi\,\cos\psi) \quad \text{and}$$

$$\dot{z}_E = u\,(-\sin\theta) + v\,(\sin\phi\,\cos\theta) + w\,(\cos\phi\,\cos\theta)\ .$$

That when the flight control system is required to drive the helicopter to a certain value of (x_E, y_E, z_E) these values represent the rectangular coordinates of the position with respect to a reference frame fixed on Earth.

The results presented below prove that the above two characteristics a), and b) and also the robustness required to control position are satisfied.

Before presenting the resulting simulations, it is appropriate to recall that the four sub-systems are related to the longitudinal, lateral, normal and heading movements. These sub-systems are:

1. Sub-system 1

   state: $[x_E, u, \theta, q]^T$,

   input: θ_{1s}, longitudinal cyclic command.

2. Sub-system 2

   state: $[y_E, v, \phi, p]^T$,

   input: θ_{1c}, lateral cyclic command.

3. Sub-system 3

   state: $[z_E, w]^T$,

input: θ_o, main rotor collective command.

4. Sub-system 4

state: $[\psi, r]^T$,

input: θ_p. tail rotor collective command.

The following figures show the responses of the sub-systems during the execution of a series of simple manoevres.

DESCRIPTION OF SIMULATIONS

Simulation 1.

The results of the first simulation are presented in figures IV.S1.i, i= 1, ..., 4. These figures show the dynamic characteristics of sub-system 1, which is related to the longitudinal movement of the helicopter, and its influences on the other sub-systems.

In this simulation the fact that it is possible to demand a longitudinal movement from hover to hover by simply defining the following reference to the flight control system is demostrated

$R_1 = T_1\ (X_R)$,

$R_2 = T_5\ (Y_R)$,

$R_3 = T_9\ (Z_R)$ and

$R_4 = T_{11}\ (\psi_R)$,

where T_1, T_5, T_9 and T_{11} are the corresponding components of the state diffeomorphic transformation; X_R, Y_R and Z_R are the desired helicopter position coordinates, referred to a reference frame fixed on Earth, and ψ_R is the heading reference.

The initial flight condition is hover on the origin of the reference frame, which is maintained for one second before the manoeuvre is started.

The responses marked with the symbol "+" correspond to the theoretical equivalent system related to the helicopter position. The poles of the linear equivalent system were set at;

$P_1 = -1.0$, $P_2 = -1.0$, $P_3 = -1.0$, $P_4 = -1.0$;

$S_1 = -2.0$, $S_2 = -2.0$, $S_3 = -1.0$, $S_4 = -1.0$;

$T_1 = -2.0$, $T_2 = -2.0$;

$V_1 = -2.0$, $V_2 = -2.0$.

The value of the reference given for this simulation was:

$X_R = 50$ m., $Y_R = 0$ m., $Z_R = 0$ m., $\psi_R = 0$ rad.

One can see that the helicopter forward displacement is practically overlapped with the theoretical linear equivalent system. This indicates that the linearization is accomplished for this particular operation. When one considers the time history of u, it is clear from the beginning of its response that it does not behave like a third order system with its poles at -1, but that this mismatching does not affect the general performance of the system. On the other hand, the effects of the simplifying assumptions directly affect the three translational velocities.

On the contrary to the forward velocity, the pitch angle and the pitch rate dynamics correspond very closely to second and first order systems respectively. This is particulary obvious for q, as can be seen in its initial response in figure IV.S1.1.

The only one of the remaining sub-systems which is significantly affected is sub-system 3. This is not unexpected, given that the change in attitude and the forward displacement will affect the height. In this case the maximum deviation from the reference is four metres, nevertheless the final error is zero, as can be checked in figure IV.S1.3.

The outputs of sub-systems 2 and 4 are practically unaffected during this manoeuvre, as is shown in the scales of the responses in figures IV.S1.2 and IV.S1.4. In the same figures it can be seen that the decoupling of this sub-system requires some action from its control commands.

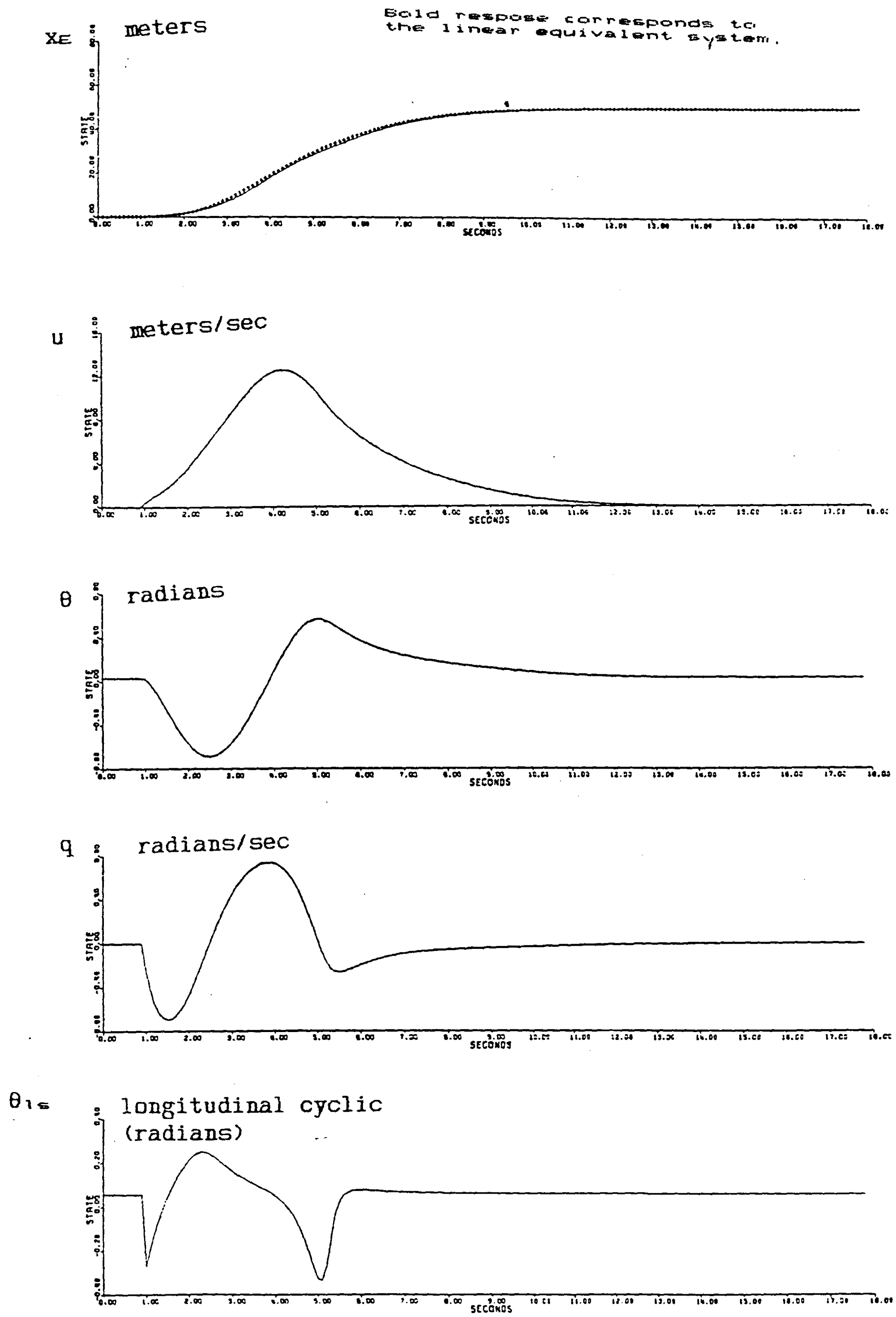


FIGURE IV.S1.1.- SUB-SYSTEM 1. RESPONSE TO $R_1 = T_1$ $(x_R) = 50$ m
$R_i = T_i = 0$ $i = 2, \ldots, 4$. CLOSED-LOOP SYSTEM POLES $P_j = -1$, $j = 1, \ldots, 4$.

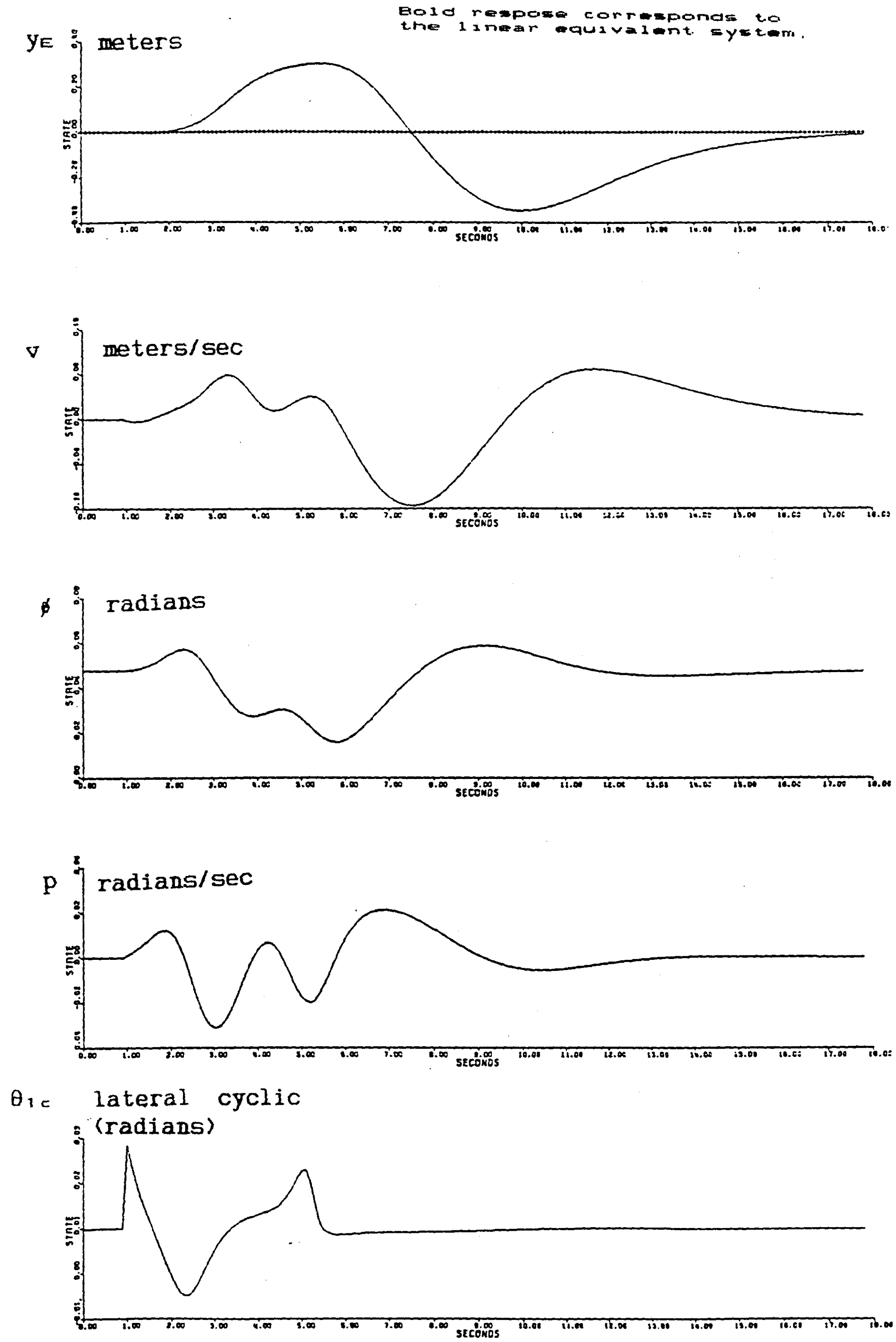


FIGURE IV.S1.2.- EFFECTS ON SUB-SYSTEM 2 BY THE CHANGE IN THE REFERENCE OF SUB-SYSTEM 1 OF SIMULATION 1. CLOSED-LOOP SYSTEM $S_1 = -2$, $S_2 = -2$, $S_3 = -1$, $S_4 = -1$.

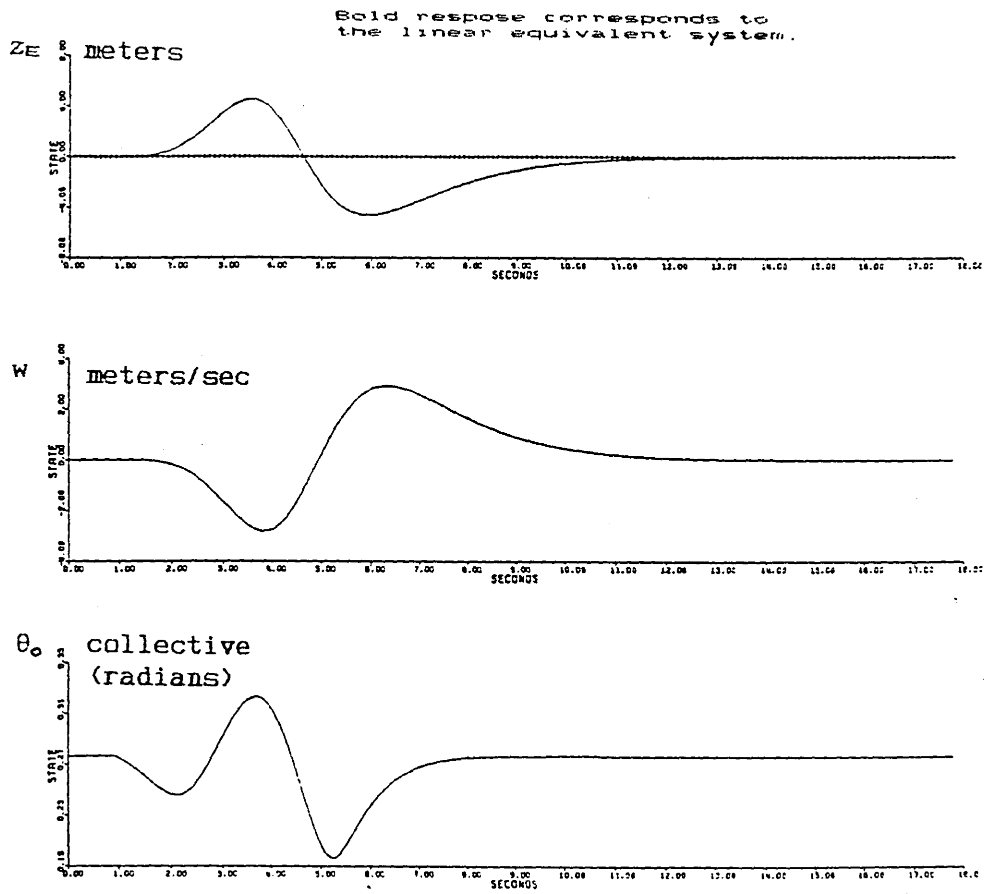


FIGURE IV.S1.3.- EFFECTS ON SUB-SYSTEM 3 BY THE CHANGE IN THE REFERENCE OF SUB-SYSTEM 1 OF SIMULATION 1. CLOSED-LOOP SYSTEM POLES $T_1 = -2$, $T_2 = -2$.

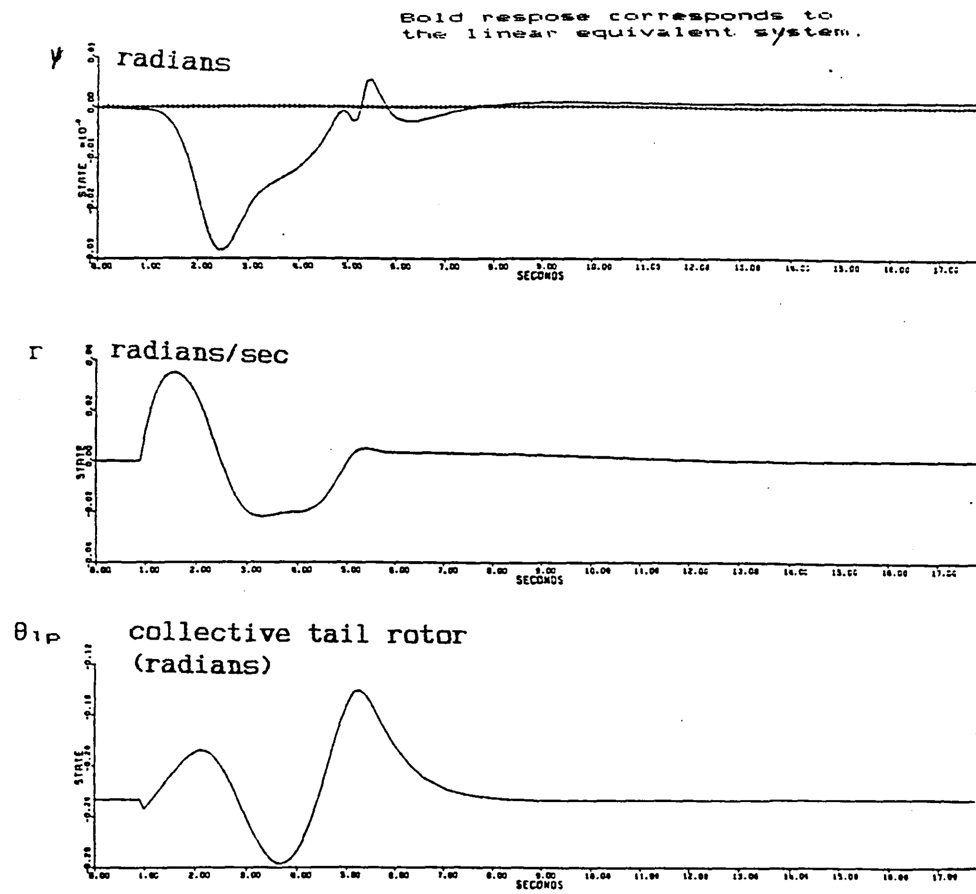


FIGURE IV.S1.4.- EFFECTS ON SUB-SYSTEM 4 BY THE CHANGE IN THE REFERENCE OF SUB-SYSTEM 1 OF SIMULATION 1. CLOSED-LOOP SYSTEM POLES $V_1 = -2$, $V_2 = -2$.

Simulations 2, 3 and 4.

These simulations are intended to show the influence of the selection of poles of the equivalent linear system for sub-system 1.

It is clear that the previous simulation addresses an academic problem rather than a practical one, or at least that the kind of manoeuvres simulated are restricted to small displacements. In fact, a reference larger than X_R= 50 m will generate, in the flight control system, an initial position error such that, in order to compensate it an unrealistic control input is demanded.

In the following simulations the pole of the equivalent linear system corresponding to X_E, Y_E and Z_E is set at the origin and the system output is considered to comprise the translational velocities u, v and w. This is equivalent to defining the reference with

$R_1 = T_2\ (u_R)$,

$R_2 = T_6\ (v_R)$,

$R_3 = T_{10}\ (w_R)$ and

$R_4 = T_{12}\ (\psi_R)$;

$P_1 = 0$, $S_1 = 0$, $T_1 = 0$ and $V_1 = 0$.

where T_2, T_6, T_{10} and T_{12} are the corresponding components of the diffeomorphic transformation of the state, u_R, v_R and w_R the reference translational velocities and P_1, S_1, T_1 and V_1 the poles of the first mode of the linear equivalent systems 1, 2, 3 and 4 respectively.

In order to avoid an unnecessary proliferation of figures, only the responses of sub-system 1 are presented.

The result of simulation 2 is shown in figure IV.S2.1. This simulation is intended to show the response of sub-system 1 in a flight condition other than hover. In this case the forward velocity u is changed from 0 to 20 m/s and then returned to 0, while the other references are kept equal to zero. From the response shown in figure IV.S2.1. it is clear that the system response is symmetric with respect to the increase or decrease of the reference. This does not occur with the input command, the changes from flight conditions other than hover require larger inputs to realize a change from forward flight. Note that the values of the poles are the same as in simulation 1.

The effect on the position of the equivalent system is shown in figures IV.S3.1 and IV.S4.1. In simulation 3 (figure IV.S3.1) the poles were set at P_2= -2,
P_3= -1, P_4= -1 and in simulation 4 at P_2= -0.5, P3= -0.5, P_4= -1.0. The difference between these two responses is obviously the time response and the input command dynamics. This shows that the characteristics of the response of sub-system 1 relies on the selection of the the linear controller and on the magnitude of the reference demand. That is, if the time response of the system is reduced, larger inputs will be required. For instance, in the case of simulation 3, if the reference is changed from 0 m/s to 10 m/s, it will cause a large demand on the longitudinal cyclic resulting in instability in the calculation of the rotor coefficients. On the other hand, by increasing the time response, it is possible to increase the reference to 40 m/s, as shown in figure IV.S4.1.

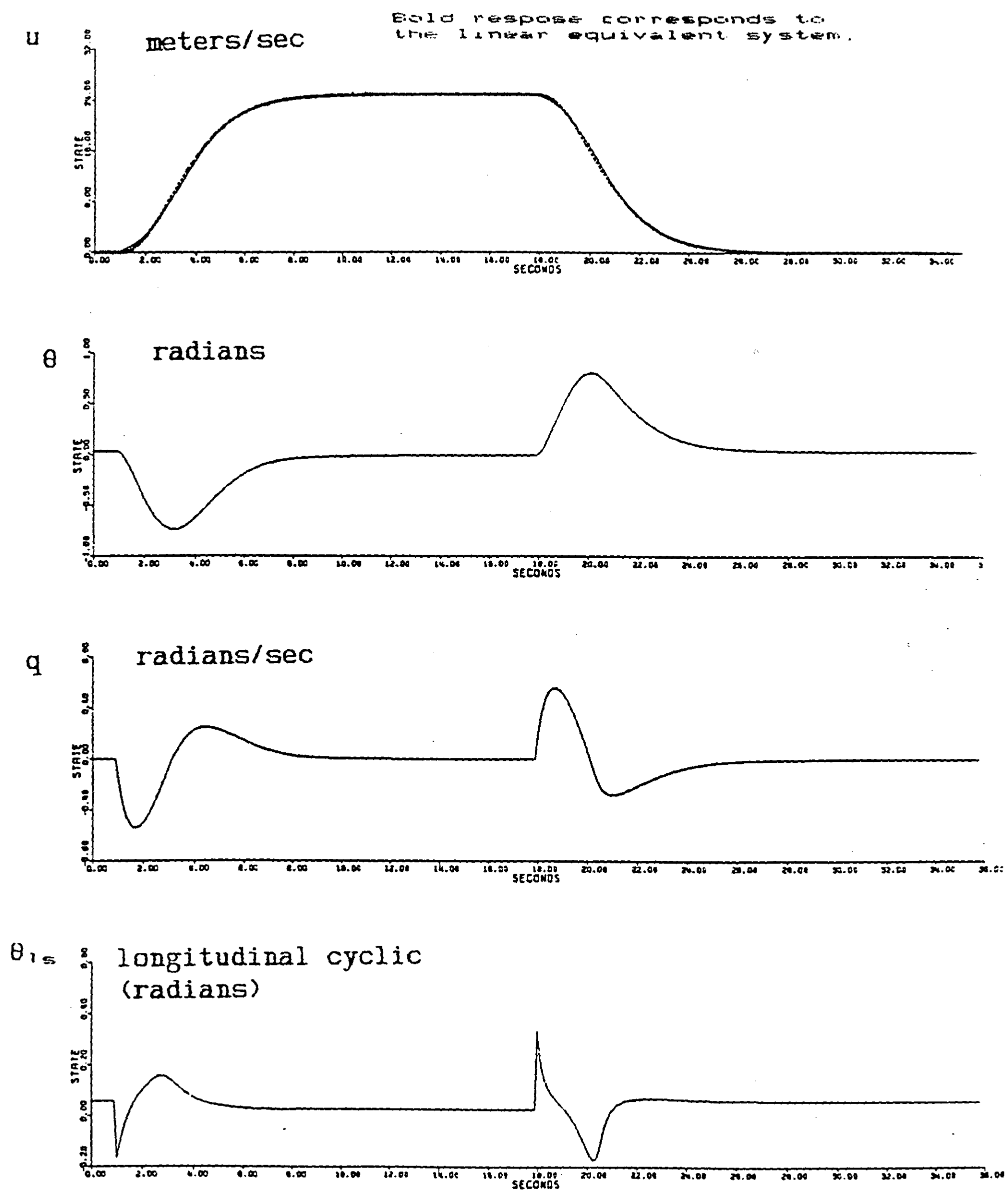


FIGURE IV.S2.1.- SUB-SYSTEM 1 RESPONSE TO $R_1 = T_1(u) = 20$ m/s AND $R_1 = T_1(u) = 0$ m/s. $P_j = -1$, $j =$, ... ,4.

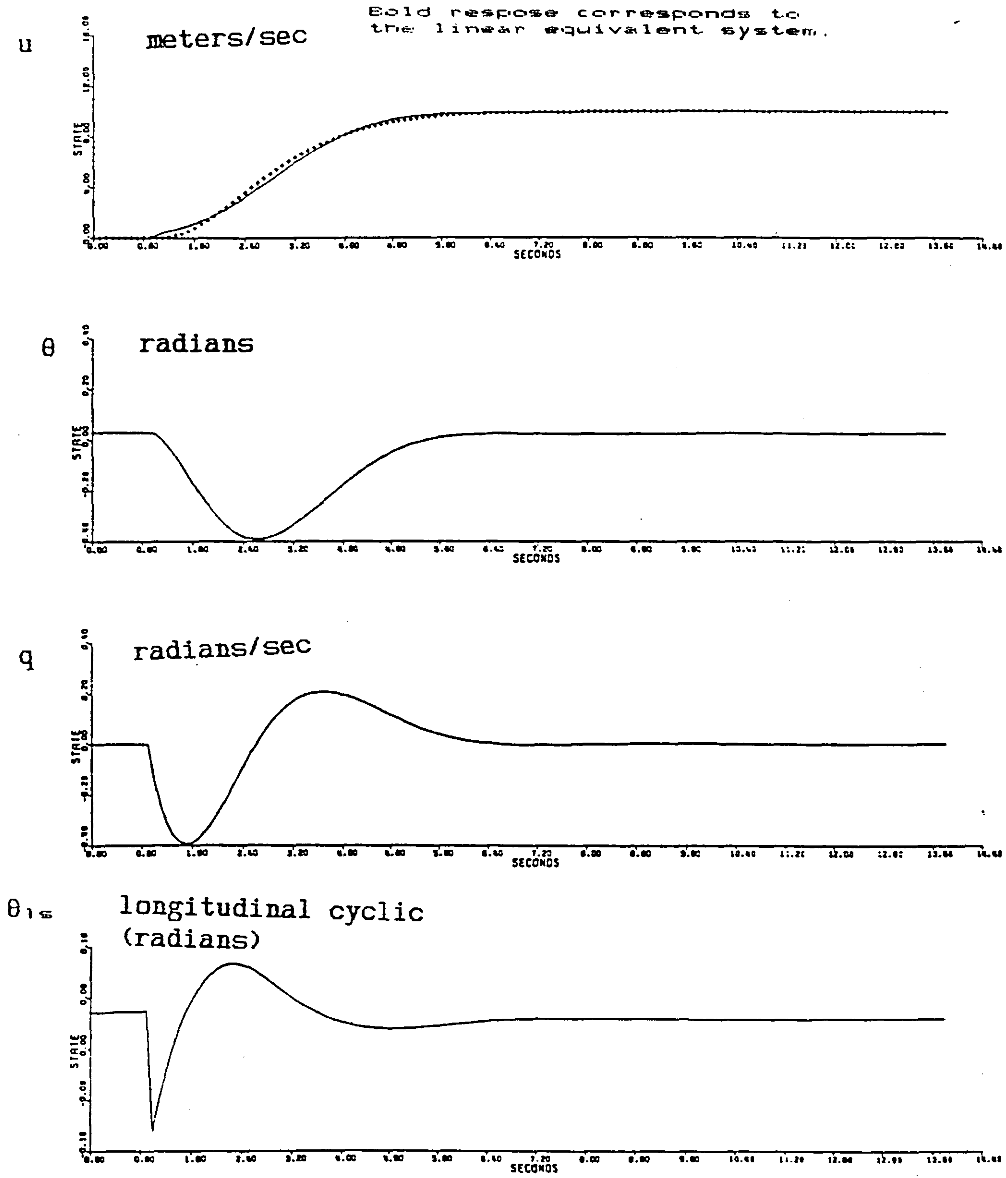


FIGURE IV.S3.1.- SUB-SYSTEM 1. RESPONSE TO $R_1 = T_1(u) = 10$ m/s. CLOSED-LOOP SYSTEM POLES. $P_1 = 0$, $P_2 = -2$, $P_3 = -1$, $P_4 = -1$.

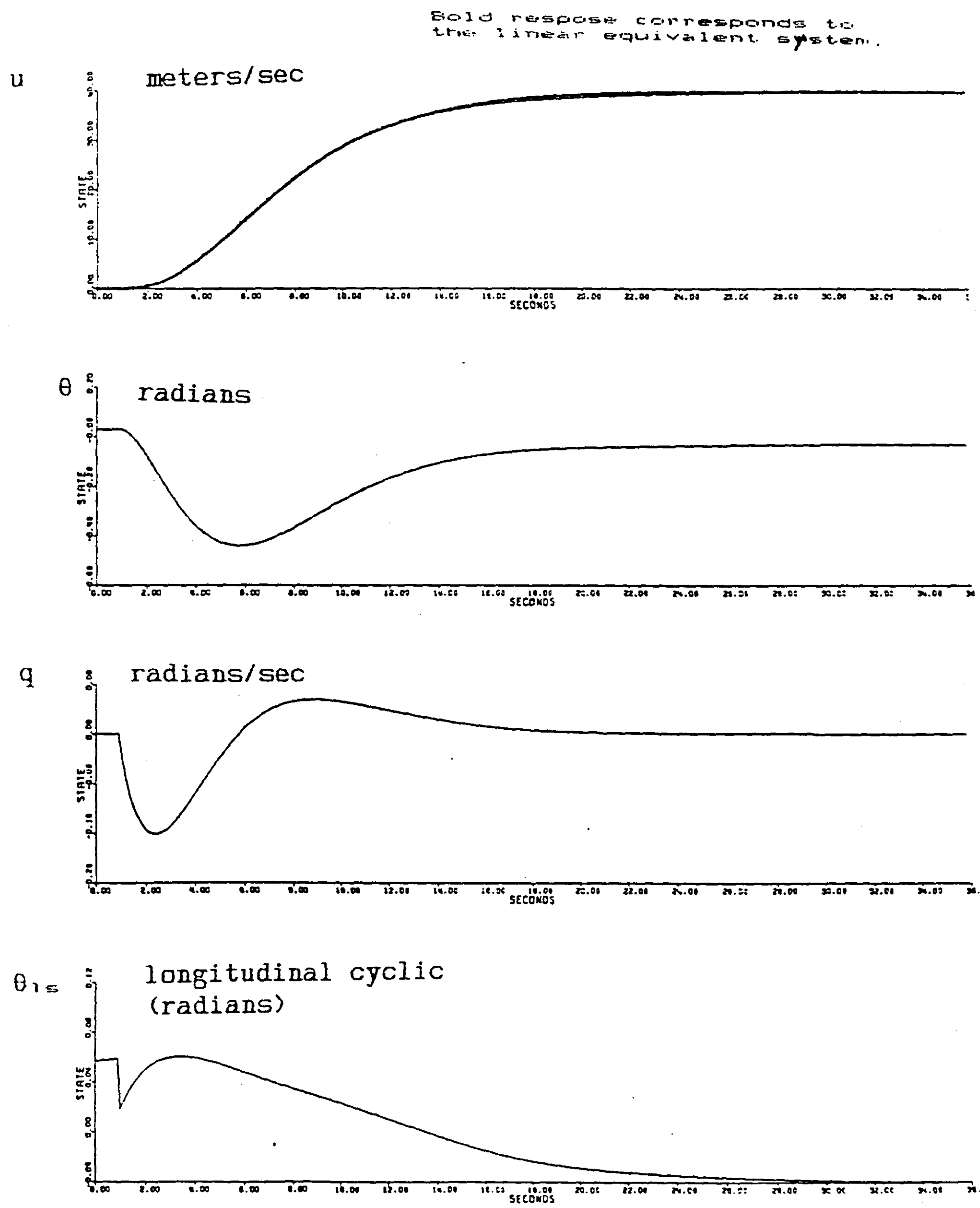


FIGURE IV.S4.1.- SUB-SYSTEM 1 RESPONSE TO $R_1 = T_1(u) = 40$ m/s. $P_1 = 0$, $P_2 = -0.5$, $P_3 = -0.5$, $P_4 = -1.0$.

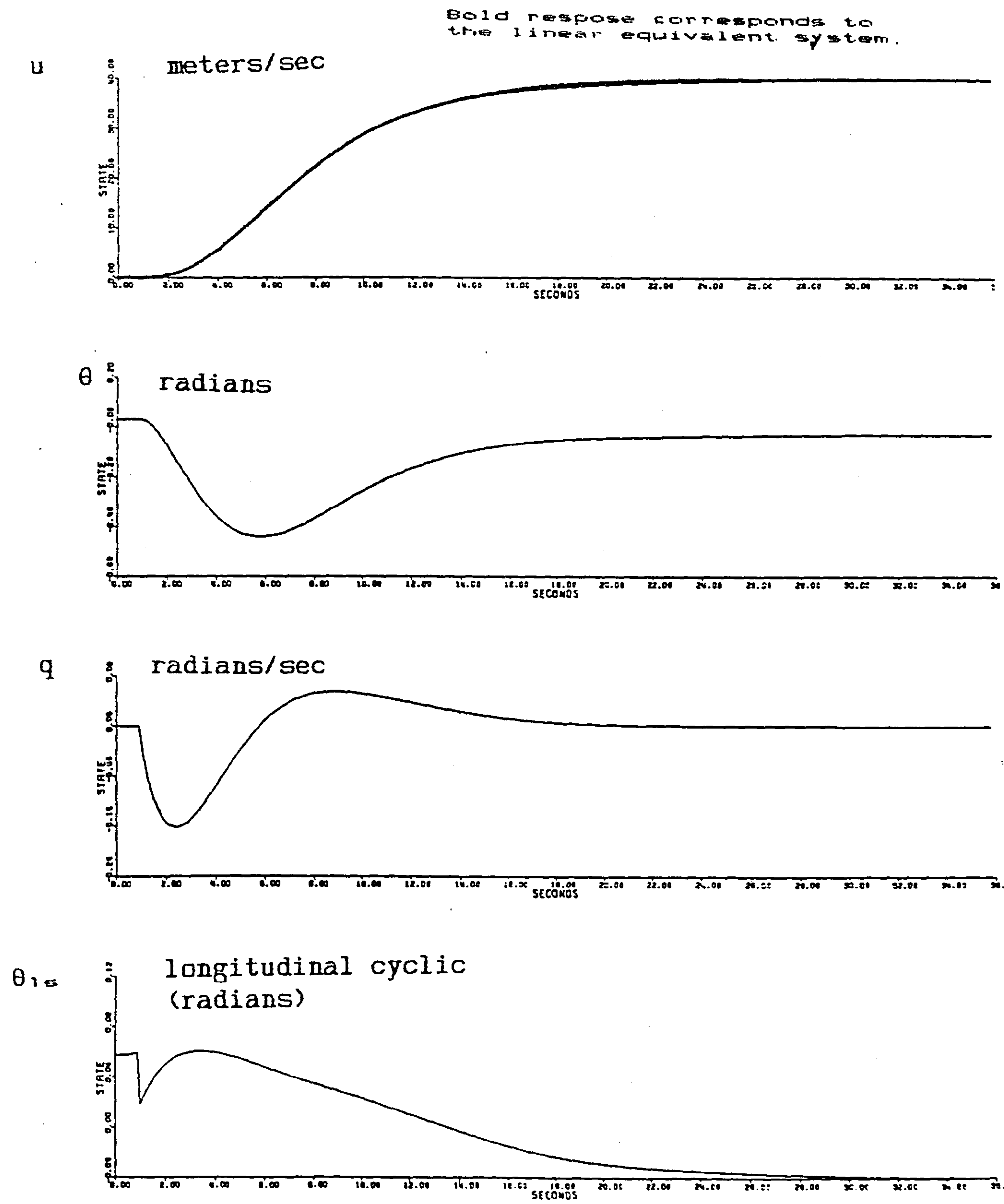


FIGURE IV.S4.1.- SUB-SYSTEM 1 RESPONSE TO $R_1 = T_1(u) = 40$ m/s. $P_1 = 0$, $P_2 = -0.5$, $P_3 = -0.5$, $P_4 = -1.0$.

Simulation 5

This simulation is intended to show that sub-system 2 is linear and decoupled from the remaining sub-systems.

The helicopter is initially in hover flight for one second, after which, from this condition, the reference is generated by

$R_1 = T_1\ (X_R)$,

$R_2 = T_5\ (Y_R)$,

$R_3 = T_9\ (Z_R)$ and

$R_4 = T_{11}\ (\psi_R)$,

with $X_R = 0$, $Z_R = 0$, $\psi_R = 0$ and $Y_R = -30$ m .The manoeuvre thus consists of a side step displacement of 30 m amplitude, keeping the height and the longitudinal displacement constant.

The poles of the sub-systems for this simulation were;

$P_1 = -1$, $P_2 = -1$, $P_3 = -1$, $P_4 = -1$,

$S_1 = -1.5$, $S_2 = -1.5$, $S_3 = -1.5$, $S_4 = -1$,

$T_1 = -1$, $T_2 = -2$,

$V_1 = -1$, $V_2 = -5$.

The system response for simulation 5 is shown by figures IV.S5.1 to IV.S5.4. The response corresponding to sub-system 2 is presented in figure IV.S5.2, which shows that the response of the rolling rate p is a typical response of a first order system and the roll angle ø of a second order system. In spite of the simplifying assumptions in the design, the response of the lateral position is very close to the linear equivalent system as can be confirmed in figure IV.S5.2.

During this manoeuvre, the sub-system affected most, was sub-system 3. It suffered a maximum deviation in height of 4 metres, thus causing a large demand on the collective command. On the other hand, the outputs of sub-systems 1 and 4 were effectively decoupled: nevertheless some action from the longitudinal cyclic and tail rotor collective command are required, as can be seen in figures IV.S5.1 and IV.S5.4.

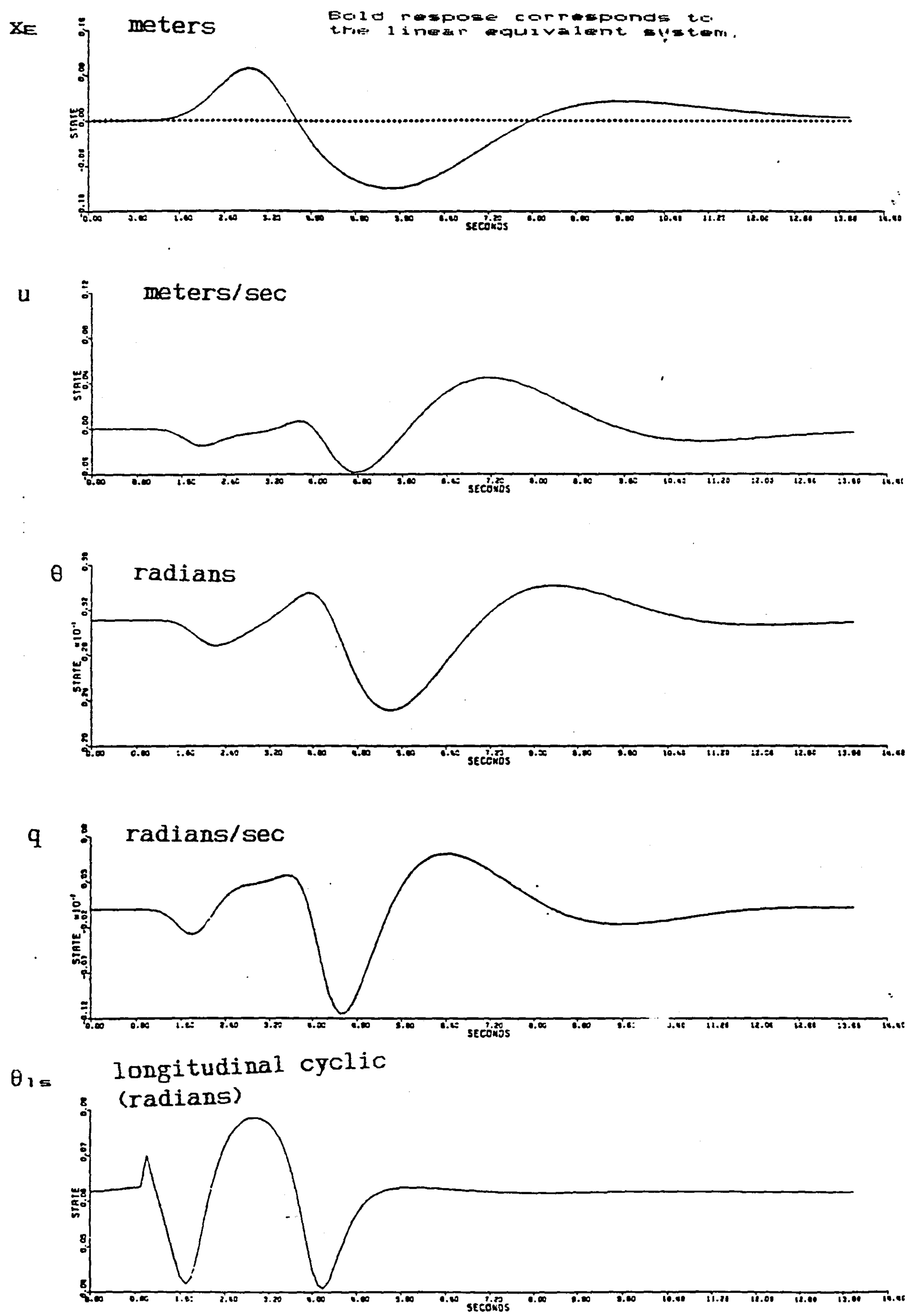


FIGURE IV.S5.1.- EFFECTS ON SUB-SYSTEM 1, BY A CHANGE IN THE REFERENCE OF SUB-SYSTEM 2. CLOSED-LOOP SYSTEM POLES $P_j = -1$, $j = 1, \ldots, 4$.

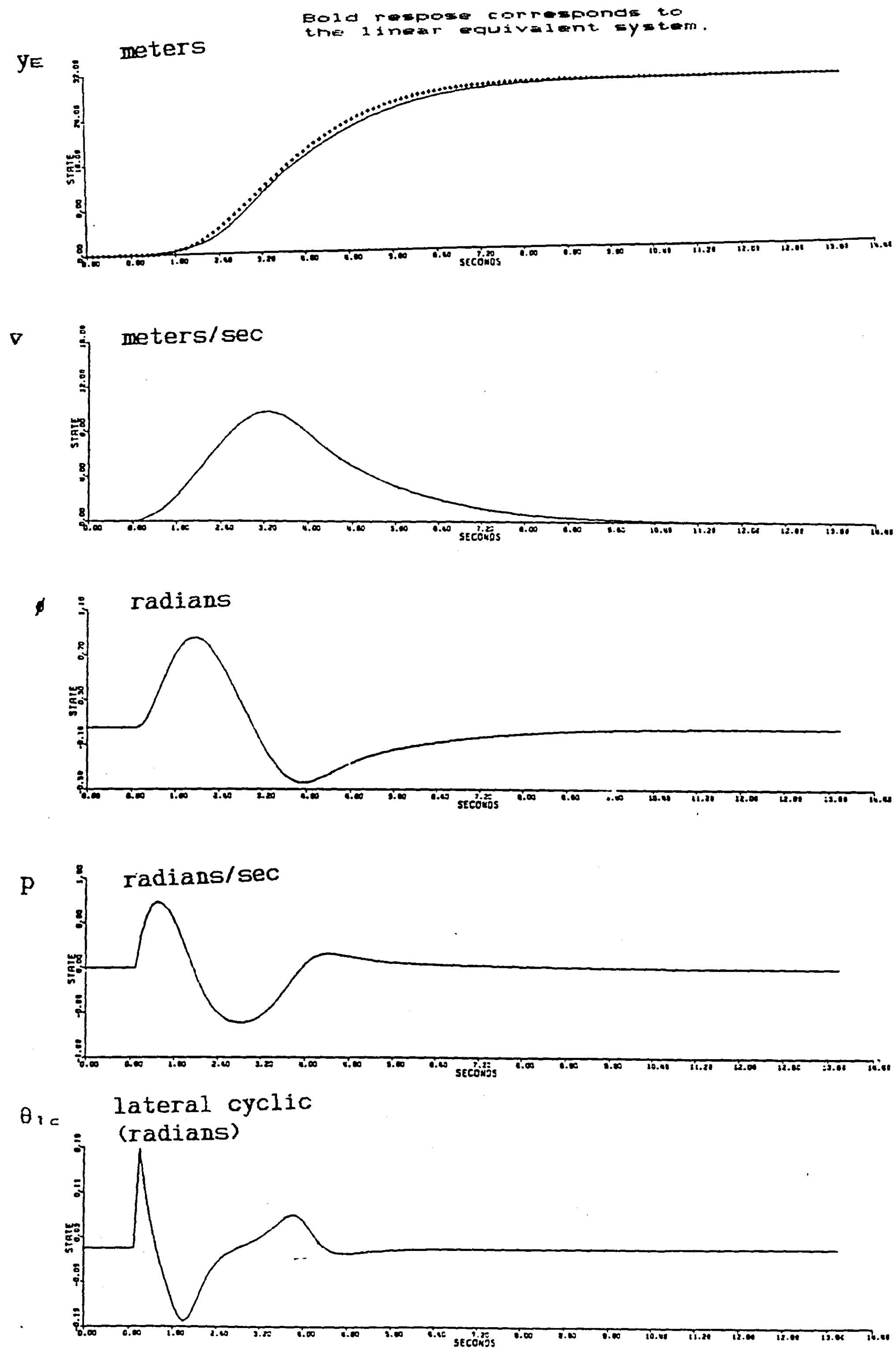


FIGURE IV.S5.2.- SUB-SYSTEM 2. RESPONSE TO $R_2 = T_2(Y_E) = 30$ m. $R_1 = R_3 = R_4 = 0$. CLOSED-LOOP SYSTEM POLES $S^1 = -1.5$, $S_2 = -1.5$, $S_3 = -1.5$, $S_4 = -1$.

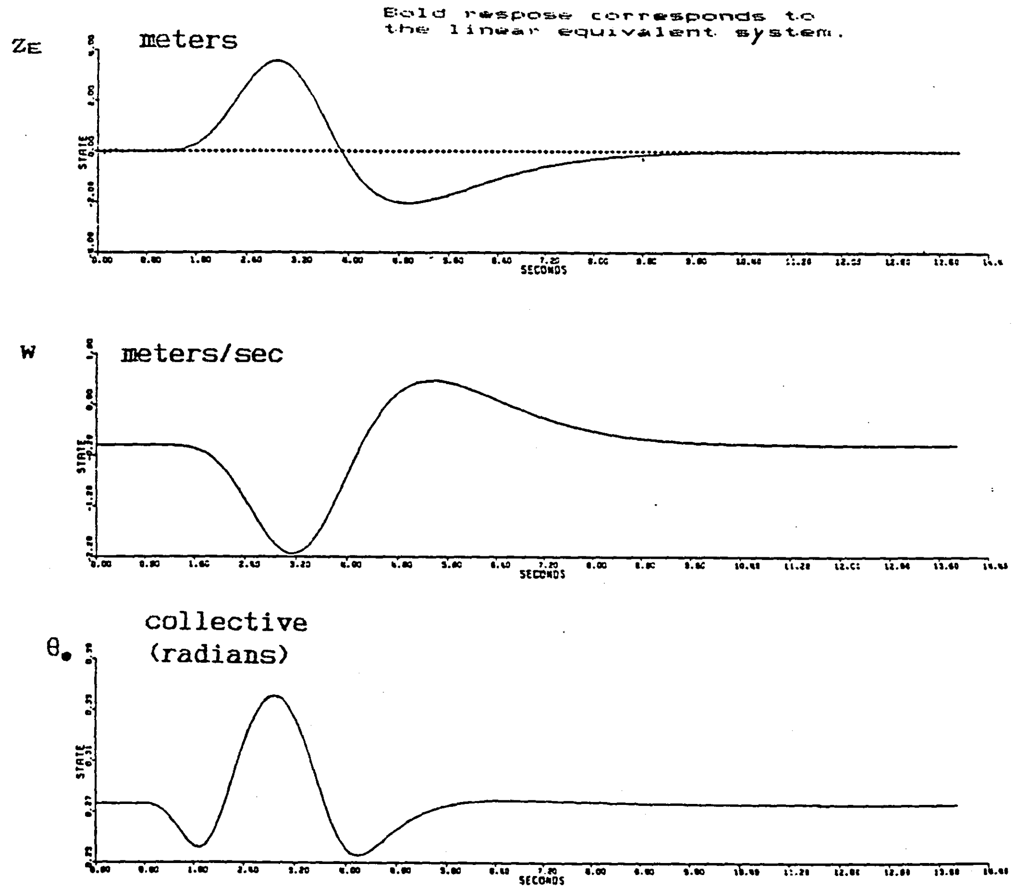


FIGURE IV.S5.3- EFFECTS ON SUB-SYSTEM 3, BY A CHANGE IN THE REFERENCE OF SUB-SYSTEM 2. CLOSED-LOOP SYSTEM POLES $T_1 = -1$, $T_2 = -2$.

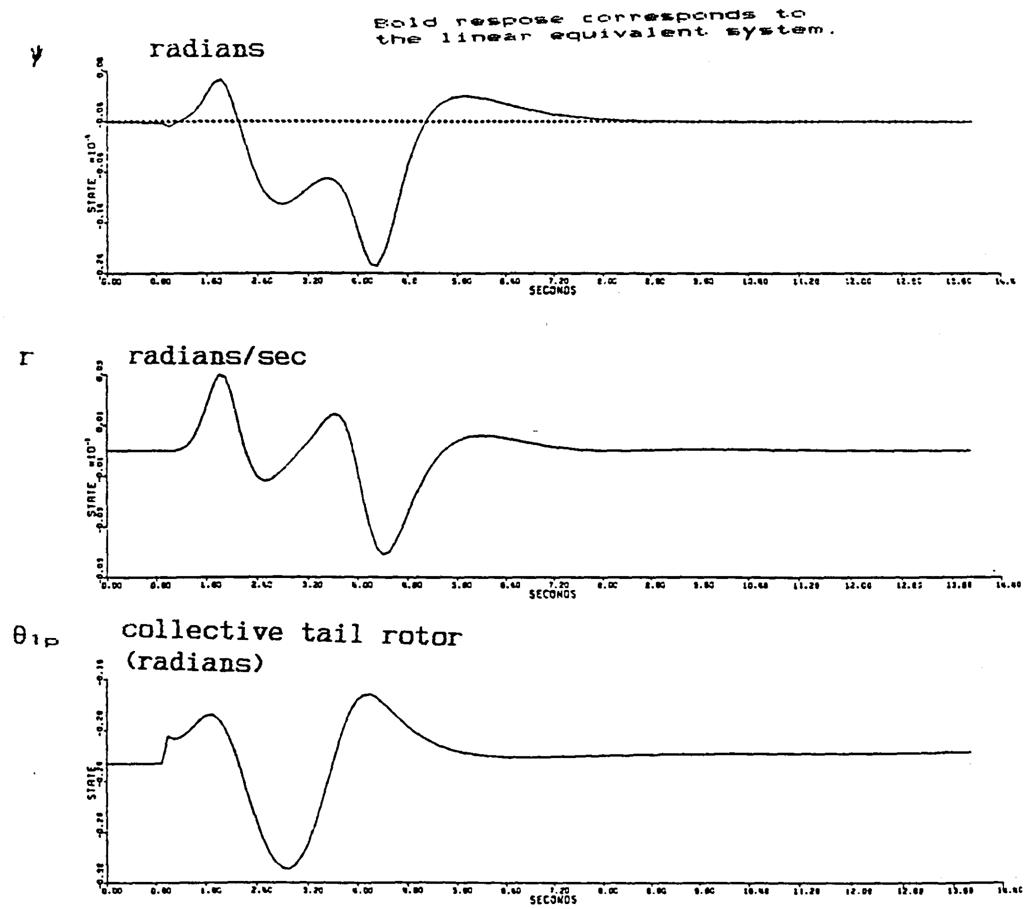


FIGURE IV.S5.4.- EFFECTS ON SUB-SYSTEM 4, BY A CHANGE IN THE REFERENCE OF SUB-SYSTEM 2. CLOSED-LOOP SYSTEM POLES $V_1 = -1$, $V_2 = -5$.

Simulation 6.

The variables related to sub-systems 3 and 4 were obtained without considering simplifying assumptions, so that to avoid a large quantity of figures, the responses of these sub-systems are presented in same simulations . Furthermore, in this simulation, sub-system 1 is also required to respond to a change in the reference, so that the four sub-systems are involved in this manoeuvre.

The reference given to the Flight Control System is defined as follows:

at t= 0 s ,

$R_1 = T_2$ (u_R= 0),

$R_2 = T_6$ (v_R= 0),

$R_3 = T_{10}$ (w_R= 0),

$R_4 = T_{11}$ (ψ_R= 0) ;

at t= 1 s,

$R_1 = T_2$ (u_R= 20 m/s),

$R_2 = T_6$ (v_R= 0 m/s),

$R_3 = T_{10}$ (w_R= -5 m/s),

$R_4 = T_{11}$ (ψ_R= 0);

and at t= 15 s,

$R_1 = T_2$ (u_R= 20 m/s),

$R_2 = T_6$ (v_R= 0 m/s),

$R_3 = T_{10}$ (w_R= -5 m/s),

$R_4 = T_{11}$ ($\dot{\psi}_R$= 8 °/s).

The equivalent poles of the system are P_1= 0, P_2= -1, P_3= -1, P_4= -1, S_1= 0, S_2= -1, S_3= -1, S_4= -1, T_1= 0, T_2= -2, V_1= 0, V_2= -5.

The results of this simulation are presented in figures IV.S6.i i= 1, ..., 4.

From figure IV.S6.3 it is clear that the normal velocity response corresponds to that of the linear equivalent system. Nevertheless, to achieve the time response shown, an initial high collective input is required.

The response of sub-system 4 is shown in figure IV.S6.4. In this figure the bold line corresponds to the rate of change of yaw. From this response it is clear that the yaw rate is sytrongly affected by changes in the translational velocities u and w. Thus a large demand of tail rotor collective is required to maintain the heading angle ψ at a prescribed value. On the other hand a demand on the rate of change of ψ while the vehicle is flying at u= 20 m/s and w= -5 m/s, does not involve such a large tail rotor collective demmand. Here, a substantial demand of the lateral cyclic command is required in order to achieve the heading rate reference. In order to achieve the yaw rate of 8 °/s, sub-system 2 has been substantially modified; for example, note the roll angle change from the hover condition to 17.2 °/s with respect to the response of ψ. It is clear that this corresponds to its linear equivalent system response.

It is obvious from figure IV.S6.1 that sub-system 1 is not completely decoupled from the collective command. The responses of q and θ still correspond to a first and second order system, meanwhile, the response of u is not even similar to that of simulations 2 and 3. Nevertheless the reference is practically reached at the same time as its that of linear equivalent system.

The response of sub-system 2 shown in figure IV.S6.2 is a consequence of the kinematic relationship of the vehicle. In order to maintain the side slip velocity v equal to zero during this manoeuvre the bank angle has to be increased. During the transient of the response of this sub-system, an overshoot of 100% is observed, plus an oscilation around the reference value. The characteristics of this response depend entirely on the value of the poles of the equivalent linear system. For instance in figure IV.S7.2 the response of sub-system 2 is shown for the same manoeuvre, but with the poles of the equivalent linear system changed to $S_1= 0$, $S_2= -0.5$, $S_3= -0.5$ and $S_4= -1.0$, while the other sub-system equivalent system poles values are as in simulation 6.

It is obvious that the change of the pole locations modifies the characteristic of the response, the overshoot has been reduced to 65%.

Again, the effect of the collective command changes on sub-system 2 is shown during the period of 1 to 15 seconds in figures IV.S6.2 and IV.S7.2. These figures show how the value of the poles affect the sub-system dynamics with respect to sub-system 3.

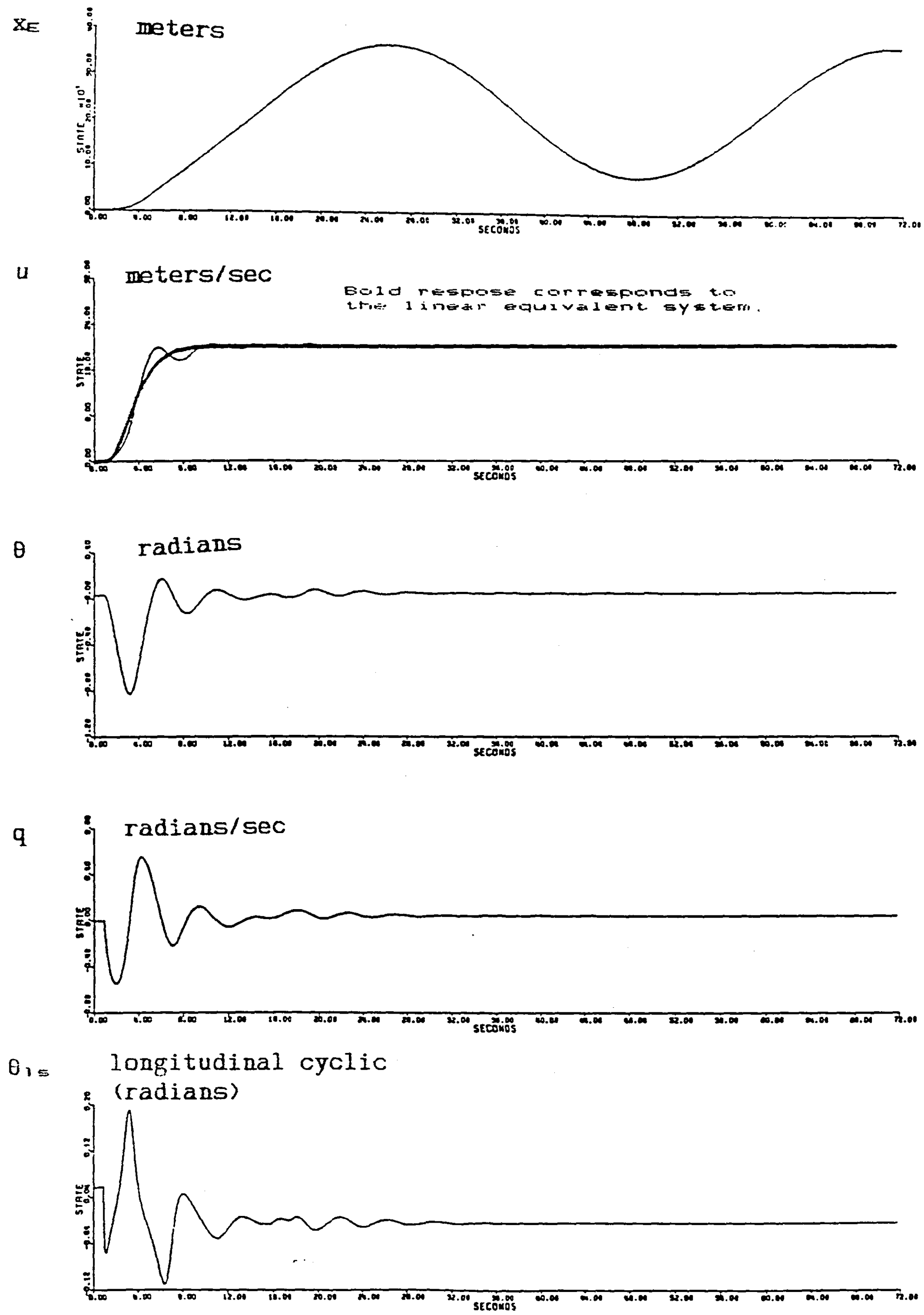


FIGURE IV.S6.1.- SUB-SYSTEM 1. MANOEUVRE DESCRIBED IN SIMULATION 6.

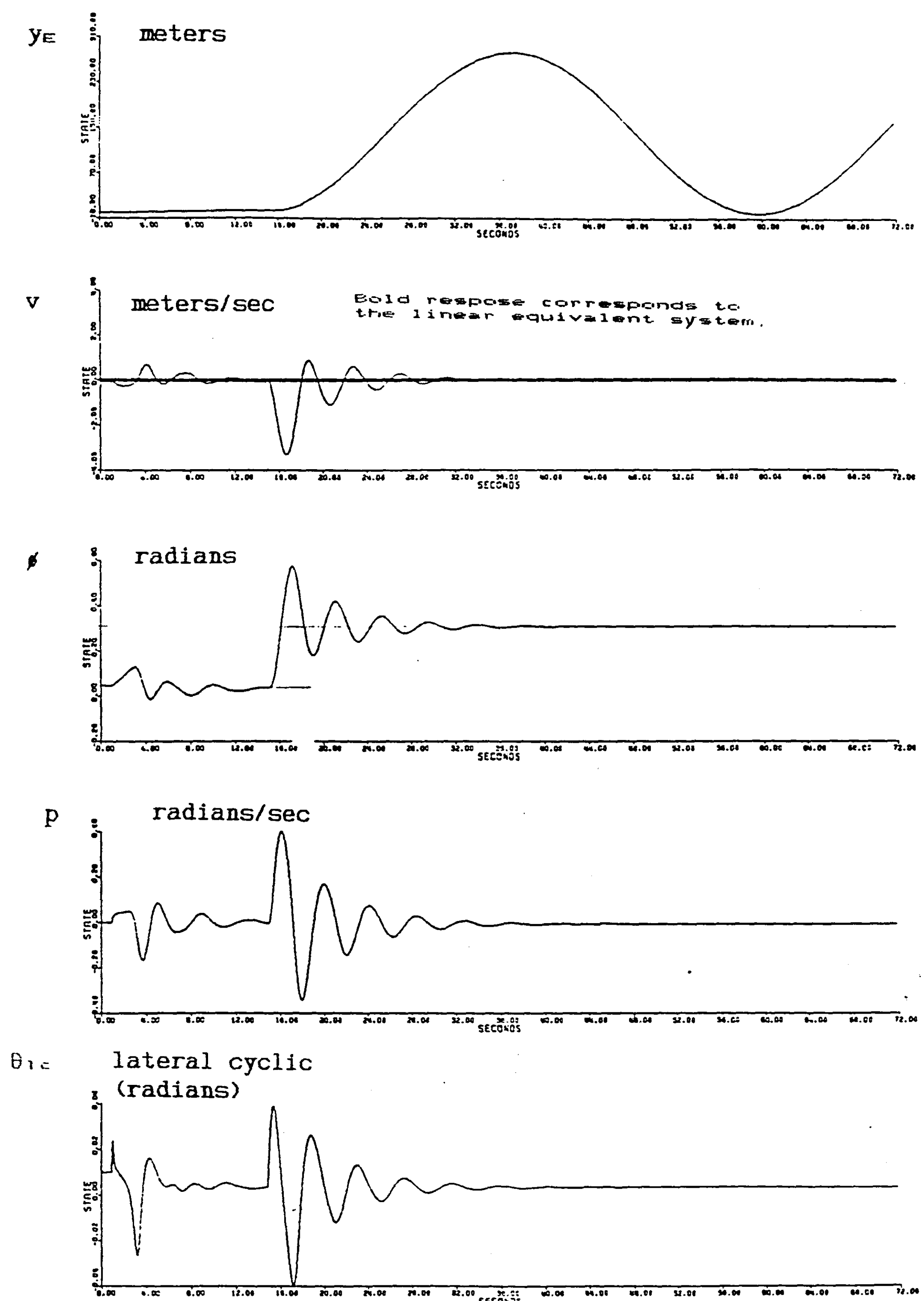


FIGURE IV.S6.2.- SUB-SYSTEM 2. MANOEUVRE DESCRIBED IN SIMULATION 6.

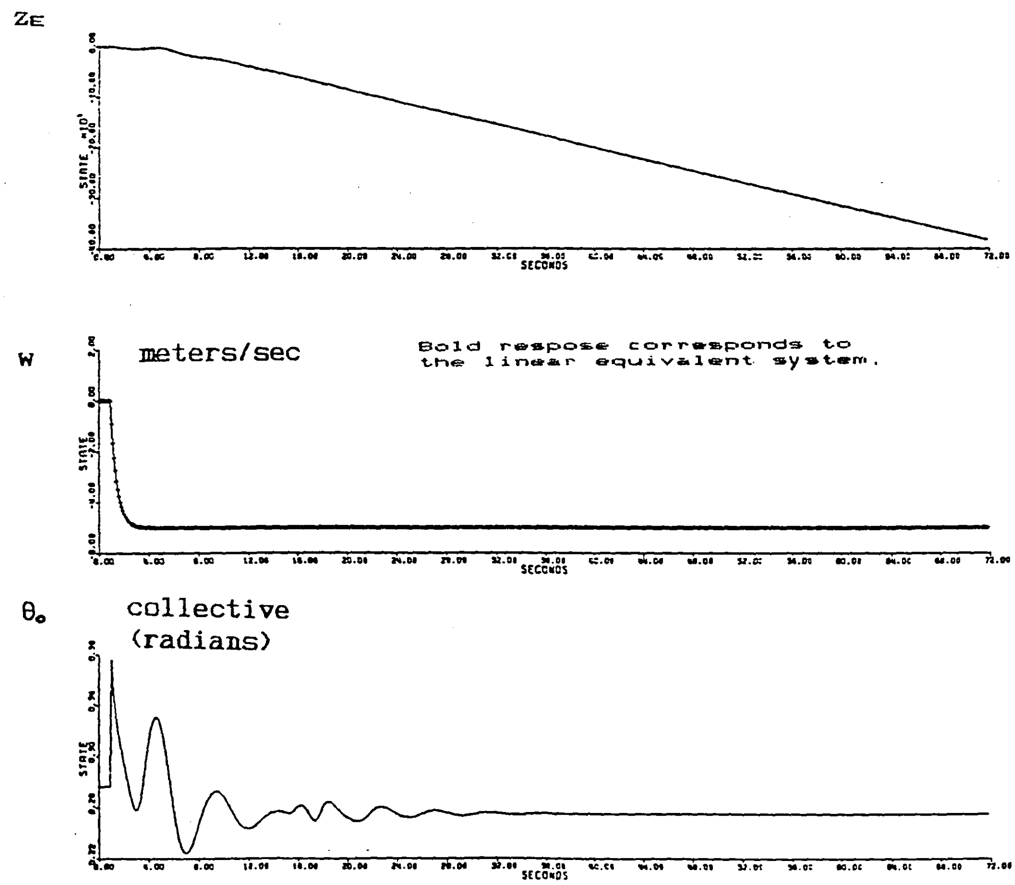


FIGURE IV.S6.3.- SUB-SYSTEM 3. MANOEUVRE DESCRIBED IN SIMULATION 6.

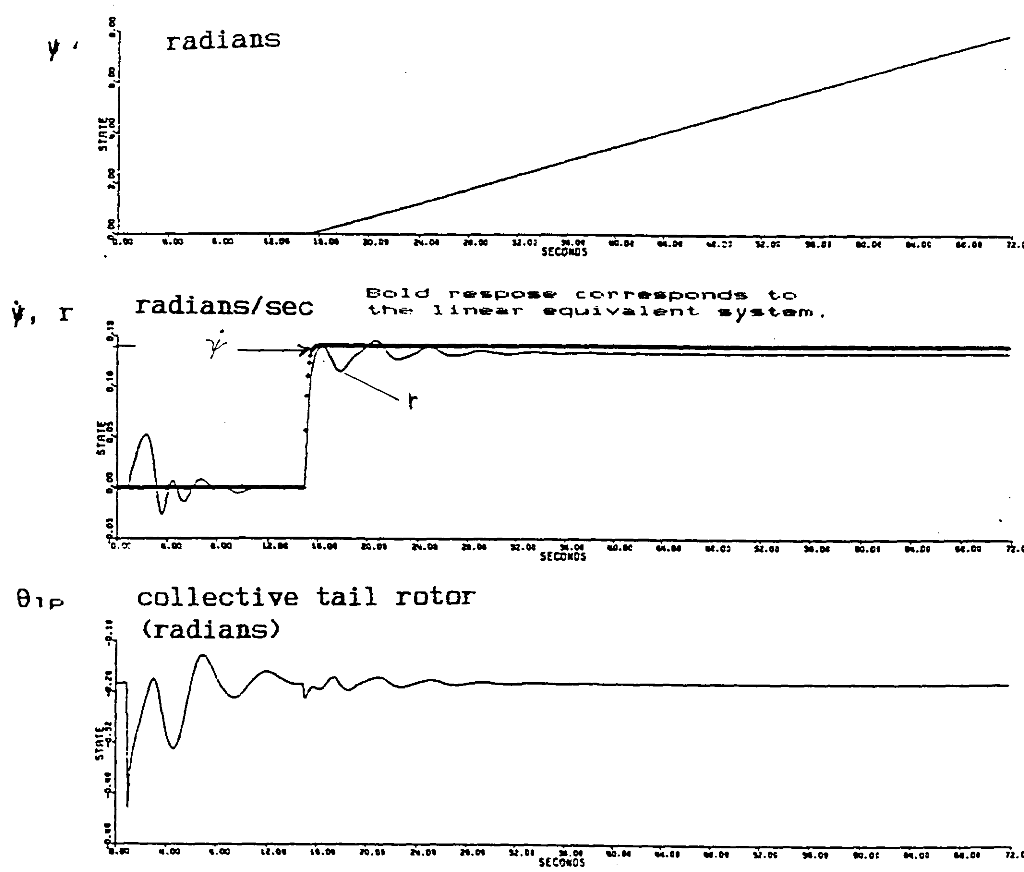


FIGURE IV.S6.4.- SUB-SYSTEM 4. MANOEUVRE DESCRIBED IN SIMULATION 6.

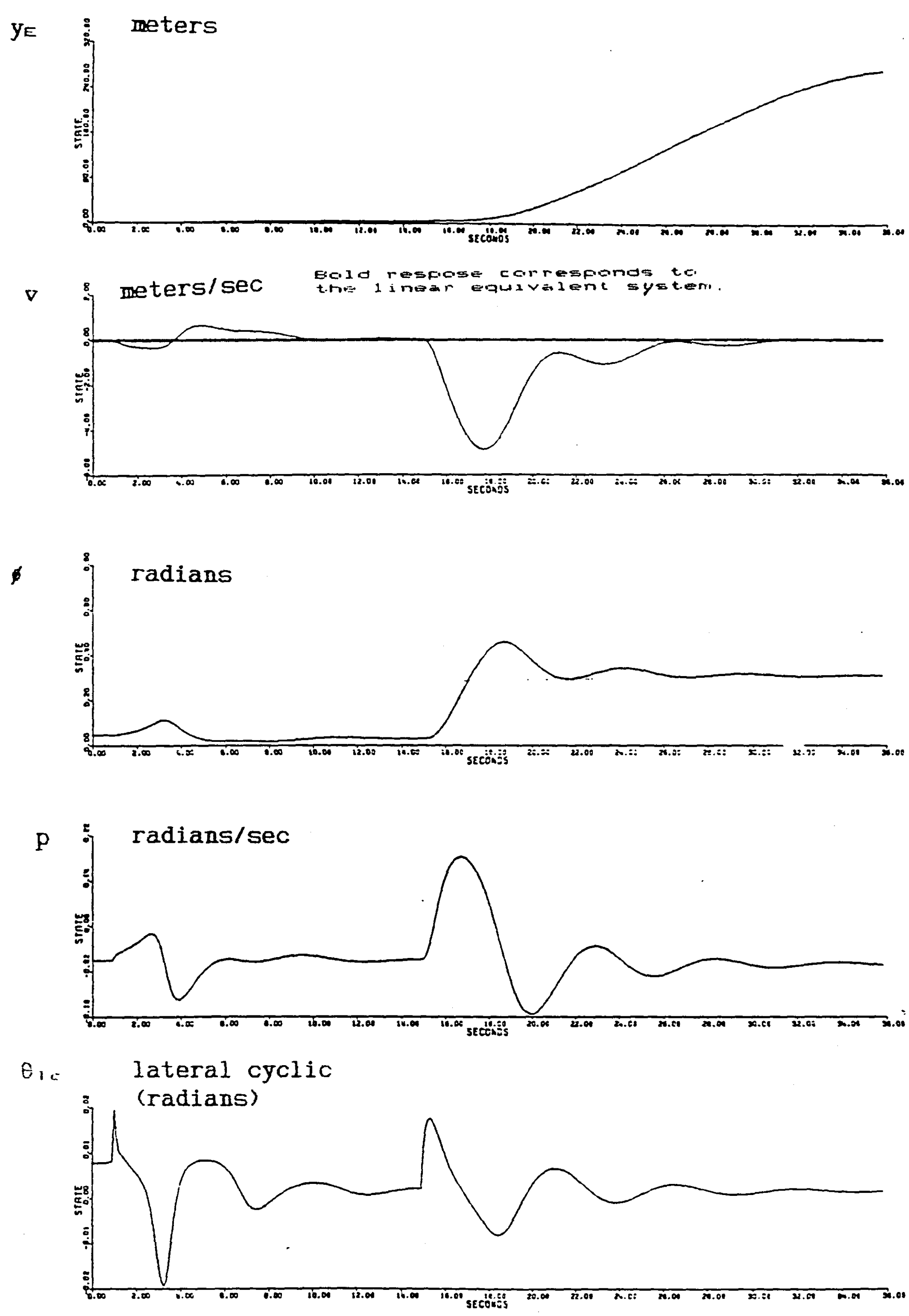


FIGURE IV.S7.2.- SUB-SYSTEM 3. MANOEUVRE DESCRIBED IN SIMULATION 6. WITH CLOSED-LOOP SYSTEM POLES CHANGED TO $S_1 = 0$, $S_2 = -0.5$, $S_3 = -0.5$ $S_4 = -1.0$.

CHAPTER V.

V.1. CONCLUSIONS AND FURTHER DEVELOPMENTS.

In the work comprising this thesis, a flight control system for helicopters has been designed. The relevance of the procedures and results presented here is that they have been obtained through the application of nonlinear system theory, introducing a new approach to the problem of helicopter control.

The advantages of the approach introduced here, over the conventional techniques (linear system theory), is the span of the validity of the results. They are not restricted to a particular operating point thus avoiding the use of several models and scheduled controllers. The design of the control system, using this theory, does not rely on the models obtained by linearizing the helicopter's equations of motion, which introduce a large number of parameters (e.g. aerodynamics derivatives). The latter representations require extensive identification procedures for validation. On the contrary, the model proposed and obtained in this thesis, depends directly on the forces and moments of the rotor in function of the helicopter commands. Unfortunately the relationship of forces and moments cannot be expresed in closed form due to its complexity. Nevertheless, even a representation of a particular flight condition requires fewer parameters than the linearized model: only those coefficients related to the input matrix are required.

The control structure designed here is composed of three different elements, which according to the design, realize the following features:

a) With the nonlinear compensator it is not possible to control the whole system, but it is possible to simplify the model substantially.

b) The direct application of the nonlinear feedback equivalence theory to the original model of the helicopter will be extremely complicated. On the other hand, the application of the present theory to the helicopter-compensator closed-loop system facilitates the design. The diffeomorphic transformation to a linear system of the helicopter-compensator combination is obtained, but not without considering some simplifying assumptions.

c) The effectiveness of the previous two steps is enhanced by a linear controller, in this case pole placement. This shows that the design of the flight control system for helicopters has been reduced to a linear control problem via a diffeomorphic transformation of the state-input space.

The work presented in this thesis can be summarised by the following points:

1) The introduction of the nonlinear system theory as a new and powerful technique for the development of flight control systems for helicopters.

2) The obtaining of the model required in the application of nonlinear system theory for the development of flight control systems.

3) A comprehensive review of the nonlinear feedback linearization Theory, including mathematical tools and proofs.

4) The presentation of a flight control system design using the results of nonlinear system theory. The symbolic algebraic computation facilities required for the application of this theory are also included.

5) The simulation of a helicopter with the flight control system.

From the results of the simulations, it is clear that the behaviour of the global closed-loop system is very close to that of a canonical linear system, showing that the performance of the control system is good.

The results obtained in this thesis suggest that there is a possibility for further applications and developments of flight control systems for helicopters, by use of the approach presented in this thesis. Every aspect of the design presented here is a suitable topic for future study and development, for instance:

I) We can consider the response of the system to wind perturbation, that is adding a perturbation term to the velocity vector.

II) The section related to the linear controller can be developed to further. Given that the equivalent linear system is affected by perturbations inherent in the design, it is reasonable to consider the problem of determination of the "best" linear control approach for improvement the performance of this section of the flight control system. One could venture to consider H^{∞}, model following or optimal control techniques for this purpose.

III) The section concerning the compensator could also be developed further. Given that the control matrix is not constant, it must be calculated on line. This calculation can be avoided if the elements of the control matrix involved in the compensator output are estimated on line with an identification algorithm.

IV) The core of the flight control system design developed in this thesis, relies on the generalization of the concept of controllability from linear control systems to the nonlinear case. The application of this concept to control systems depends on the accessibility of the state which, in many cases is not physically realisable. If this is the case, the natural consequence is an investigation of the dual of controllability; observability.

At present some results on observability and the design of observers for nonlinear systems are available. Bestle and Beitz [1983] presented the dual of the Brunovsky canonical form transformation of single output of nonlinear systems. Research in the development of the multivariable case originated from this work. Its application is substantially more complicated and the conditions required for the system to be transformable to a canonical observer form are more demanding than in the dual case, these facts are established by C. W. Li and L. W. Tao [1986] and Xiao-Hua Xia and Wei-Bin Gao [1988].

V) In future research, stability and robustness of the control system could be studied. This aspect is of great interest, not only for control systems but in a variety of applications, (see for instance Chapter 6 Stability Theory: Singularities, Bifurcations and Catastrophes by Casti [1985]). This topic by itself presents a good intellectual challenge.

An outline for the study of stability and robustness of the flight control system developed in this thesis is given as follows:

Let $\bar{x}$ be an equilibrium state for the system of differential equations

$$\dot{x}(t) = f(x) + \sum_{i=1}^{m} u_i \, g_i(x) = h(x) \qquad \text{V.1.}$$

where h is C^1 in some set W of the state space which contains $\bar{x}$.

Definition V.1.

The point $\bar{x}$ is a state equilibrium point, if for every neighbourhood

U_1 of $\bar{x}$ in U such that every solution x(t) with x(0) in U_1 is defined in U for all t > 0. If U_1 can be chosen so that in addition

$$\lim_{t \to \infty} x(t) = \bar{x}$$

Then $\bar{x}$ is asympotically stable.

Given a compact set k containing $\bar{x}$ in its interior, $\bar{x}$ is said to to beasymotically stable on k if it is stable and every solution starting to beconverges to $\bar{x}$.

Definition V.2.

Let V: U → R be a continuous function defined on a neighbourhood U of $\bar{x}$, differentiable on U - $\{\bar{x}\}$
such that

a) $V(\bar{X}) = 0$ and $V(x) > 0$ if $x \neq \bar{x}$

b) $\dot{V} < 0$ in U - $\{\bar{x}\}$

The function V is defined as a strict Lyapunov function for $\bar{x}$.

Suppose a linear feedback control is applied to the linear equivalent system of V.1. to stabilize (asymtotacally) the system about the origin. In the present content we have

$$\dot{y}(t) = Ay + Bv$$

$$\dot{y}(t) = Ay + B(y_R - Ky)$$

$$\dot{y}(t) = (A - BK)\, y + B_o\, y_R \quad ;$$

one can consider y_R as the origin and $A - BK = C$

$$\dot{y}(t) = Cy$$

where the eigen values of C have a negative real part. Choosing a negative definite matrix Q, the equation $CP^T + PC = Q$ (T denotes transpose) yields a unique, positive, definite solution P, and

$$V(y) = y^T P y$$

is a strict Lyapunov function

Now given that $y_1 = T_1(x)$, $y_2 = T_2(x)$, ... , $y_N = T_N(x)$ then V depends on x so that;

$$\dot{v} = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} \dot{x}_i$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial v}{\partial T_j} \frac{\partial T_i}{\partial x_i} \right) \dot{x}_i$$

$$= \sum_{j=1}^{n} \frac{\partial v}{\partial T_j} T_j$$

$$= \sum_{j=1}^{n} \frac{\partial v}{\partial y_j} \dot{y} .$$

which by condition b) of definition V.2. is known to be negative away from the origin in the y space. Hence, V is a strict Lyapunov function for the origin in the x-space, the system being V.1. with u corresponding to the linear feedback control.

Remembering that the relationship between u_i, i= 1, ... , m and v_i, i=1, ... , m is given by equation (III PD6), it is possible to express u in function of v_i.

Using the theorem of Lyapunov and substituting u_i, i= 1, ... , 4 into system (V.1.), the origin is asympotically stable. Moreover, if k is a compact subset of W containing the origin and the boundary of k is a level set of V(G(k)), then the origin is asympotically stable on k.

This follows because T and T^{-1} map level sets to corresponding level sets, trajectories to corresponding trajectories and the origin to a corresponding origin.

This, if the nonlinear system (V.1.) is mapped according to the diffeomorphic function defined in chapter III, around an open set W containing the origin in the x-space and one uses a linear feedback in the y-space to stabilize asymptotically the linear system (eigenvalues having negative real parts) any strict Lyapunov function v[y(K)] for the linear system is a strict Lyapunov function v[y(k)] for the nonlinear system with the controls corresponding to those of linear feedback. Furthermore, this nonlinear system has its origin as an asymptotically stable equilibrium point on any compact set k whose boundary is a level set of v[y(x)] and with K contained in W (Williems, J. L. [1970]).

As a final comment, one can consider that the object of this research has been achieved. A new approach has been applied to flight mechanics, and given the new advances in symbolic computing, the application of this new approach to the several aspects of helicopter analysis introduces a new powerful tool.

"Zer edo zer esan bearra badaukat".

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APPENDIX II.1 ROTOR FORCES AND MOMENTS.

In this appendix the equations which describe the force and moments produced by a helicopter main rotor are developed . These equations were originally reported by Padfield. They are modified here in order to obtain an (f,g) model, in which the controls u_i are the collective, lateral and longitudinal commands of the helicopter.

The equations presented here, are the same ones as implemented in the six degrees of freedom version of Helistab. The terms appearing in this version are extended to the first harmonic only and blade dynamics is ignored. These assumptions limit the range of validity of the model, nevertheless the most important nonlinearities and coupled terms are not neglected, so that the model obtained is a good starting point from the nonlinear control point of view.

The equations are not presented in the same detail as in the references. Only the part of the theory which is necessary to obtain a suitable model for the development of this thesis, is shown here.

The flapping, forces and moment equations are expressed as an inner product, one of the vectors involved is referred to the aerodynamic and state variables and the other one to the control commands. The elements of the vector referred to the commands can be linear and nonlinear with respect to its elements. In the first case this vector is composed as follows:

$\boldsymbol{\theta}_R = [\theta_0,\ \theta_{1s},\ \theta_{1c},\ 1]^t$,

or

$\boldsymbol{\theta}_w = [\theta_{0w},\ \theta_{1sw},\ \theta_{1cw},\ 1]^t$,

where

θ_0: is the collective command,

θ_{1s}: is the cyclic longitudinal command, and

θ_{1c}: is the cyclic lateral command.

The above commands are expressed with respect to body axes, the subindex "w" denotes that they are expressed with respect to wind axes.

The nonlinear terms in the commands arise from the multiplication of two internal products, for example, let

Let $v=[v_1, v_2, v_3, v_4]^t$ and $w=[w_1, w_2, w_3, w_4]^t$ be two vectors whose elements are functions of the helicopter state and let $\langle . , . \rangle$ denote the inner product operation. In the development of the equations the product

$$(\alpha)\ (\beta)= \langle v,\theta_R\rangle\ \langle w,\theta_R\rangle \qquad (1)$$

is often encountered. If this product is developed, the result can be expressed again as a inner product of the form

$$\langle U,\theta_{RA}\rangle\ , \qquad (2)$$

where:

$$\theta_{RA}=[\theta_0,\ \theta_{1s},\ \theta_{1c},\ \theta_0^2,\ \theta_0\ \theta_{1s},\ \theta_0\ \theta_{1c},\ \theta_{1s}^2,\ \theta_{1s}\ \theta_{1c},\ \theta_{1c}^2,\ 1]^t$$

and

$$U=[v_1\ w_4+v_4\ w_1,\ v_2\ w_4+v_4\ w_2,\ v_4\ w_3+v_3\ w_4, v_1\ w_1,\ v_2\ w_2+v_2\ w_1,\ v_1\ w_3+v_3\ w_1,\ v_2\ w_2,\ v_2\ w_3+v_3\ w_2]^t .$$

In the simulations and model programme used in this Thesis, a subroutine that transforms all terms of type (1) to the terms of type (2) has been implemented. The equations of flapping forces and moments of the main rotor are expressed as internal products, in order to obtain an (f,g) model with respect to the command inputs.

THRUST AND FLAPPING.

The main rotor downwash and thrust coefficients are calculated in exactly the same way as reported in (Padfield [1981]) and are executed as in Helistab. These equations are rewritten in the form of inner products referred to body axes.

It is easier to describe the flapping angles with respect to wind axes, but here it is necessary to change the reference frame to body axes. This are first calculated with respect to wind axes and then transformed to body axes, the relationship between these two different frames is

$$x_w = \begin{vmatrix} \cos \psi_w & (\sin \psi_w) & 0 \\ -(\sin \psi_w) & \cos \psi_w & 0 \\ 0 & 0 & 1 \end{vmatrix} x_b ,$$

where x_w and x_b are vectors referred to wind and body axes respectively; u_H and v_H are the rotor longitudinal and lateral velocities respectively;

$\cos \psi_w = \mu_x/\mu$, $\sin \psi_w = N_{c1} (\mu_y/\mu)$,

$\mu_x = u_H/(\Omega R)$, $\mu_y = v_y/(\Omega R)$, $\mu = (\mu_x^2 + \mu_y^2)$

and N_{c1}: indicates the main rotor rotation sense.

The above matrix which relates the body and wind axes frames will be denoted as:

$$\begin{vmatrix} K_{\psi c} & K_{\psi s} & 0 \\ -K_{\psi s} & K_{\psi c} & 0 \\ 0 & 0 & 1 \end{vmatrix} . \qquad (3)$$

The variables Ω and R are the rotor revolutions per unit of time and rotor radius respectively.

<u>ROTOR THRUST COEFFICIENT.</u>

The rotor thrust coefficient C_T is given by (Padfield [1981], equation (E-10)):

$$C_T = (a_o\ s/2)\ (\theta_o\ (1/3 + \mu^2/2\ ) + ½\ \mu\ (\theta_{1sw} + ½\ p_{nw}\ N_{c1}) + ½\ R_{ML} +$$
$$¼\ \theta_{tw}\ (1 + \mu^2)$$

where:

a_o: is the lift slope of the blade section;

s: is the rotor solidity;

(number of blades)(blade chord)/ $(\pi R) = (b\ c)/(\pi R)$.

p_{nw}: Helicopter normalised roll rate with respect to wind axes;

p_w/Ω ;

p_w: helicopter roll rate with respect to wind axes.

R_{ML}: is equal to $(\mu_z - \lambda_o)$;

where λ_o is the main rotor uniform downwash component;

$\mu_z = w_h/(\Omega R)$ and w_h is the helicopter normal velocity.

θ_{tw}: Is the blade twist.

The thrust coefficient equation can be rewriten as:

$$C_T = \theta_o \{½ (a_o s) (1/3 + \mu^2)\} + \theta_{1sw} \{¼ \mu a_o s\} +$$

$$½ (a_o s) \{½ p_{nw} N_{c1} + ¼ \theta_{tw} (1 + \mu^2)\}$$

If one defines

$C_{Tw1} = ½ (a_o s) (1/3 + ½ \mu^2)$

$C_{Tw2} = ¼ (a_o s) \mu$

$C_{Tw3} = 0$

$C_{Tw4} = ¼ (a_o s) \{p_{nw} N_{c1} + ½ \theta_{tw} (1 + \mu_2)\}$

then it is possible to express the thrust coefficient as:

$C_T = \langle C_{Tw}, \boldsymbol{\theta}_w \rangle$,

where the elements of the vector C_{Tw} are the coefficients C_{Twi}, for i=1,...,4. If the commands are referred to body axes then

$C_T = \langle K_{CT}, \boldsymbol{\theta}_R \rangle$,

where:

$K_{CT} = [C_{Tw1}, C_{Tw2} K_{\psi c}, C_{Tw3} K_{\psi s}, C_{Tw4}]^t$

$= [K_{CT1}, K_{CT2}, K_{CT3}, K_{CT4}]^t$.

Note that the thrust coefficient C_T and the downwash component λ_o are related by a nonlinear algebraic equation. In the simulation programme the values of C_T and λ_0 are calculated according to Padfield [1981] (appendix E).

MAIN ROTOR FLAPPING ANGLES.

The rotor flapping angles are expressed in function of the structural and aerodynamic characteristics of the rotor:

λ_B^2: is the normalised rotating flapping frequency, given by;

$$\lambda_B^2 = 1 + K_B/(I_B\ \Omega_2)$$

where K_B is the spring stiffness of the rotor blade assuming it is a centrally sprung hinge; I_B is the blade moment of inertia, ie.

$$I_B = m_B\ r_B^2\ dr_B$$

n_B: is the inertia number equal to $\gamma/8$, where γ is the Lock number

$$\gamma = (\rho\ c\ a_0\ R^4)/I_B$$

where ρ is the air density and

S_B: is the stiffness number; $S_B = (\lambda_B^2 - 1)/\ n_B$.

In Helistab, the flapping angles are obtained by calculating the terms and coefficients separately due to the complexity of the equations. These terms are redefined here.

The longitudinal and lateral flapping angles are proportional to

$$F_{CF} = -n_B\ /\ \{\ (1 + S_B^2 - ¼\ \mu^4)\ (\lambda_B)\ \}$$

Longitudinal flapping β_{1cw}

The influence of the blades twist is evaluated by:

$$F_{Twc} = 2\ \mu\ \theta_{tw}\ \{\ (1 + ½\ \mu^2)\ \lambda_B/n_B + (8/15)\ S_B\ (1 + (16/9)\ \mu^2)\ \};$$

The downwash effect is calculated by using

$$T_{T\lambda w} = R_{ML}\ \mu\ \{\ (16/9)\ S_B + 2\ \lambda_B/n_B\ (1 + ½\ \mu^2)\ \} + (\lambda_B\ S_B\ \lambda_{1cw})/n_B\ ;$$

where

λ_{1cw}: is the first lateral downwash component in hub wind axes

$$\lambda_{1cw} = \begin{cases} \lambda_o \ (\tan \chi/2), & \text{if } \chi < \tfrac{1}{2}\pi \\ \lambda_o \ (\cot \chi/2), & \text{if } \chi > \tfrac{1}{2}\pi \end{cases}$$

and χ is the wake angle, given by $\chi = \arctan(\ \mu/(\lambda_o - \ \mu_z)$.

The effect of the angular velocity is :

$$F_{wc} = N_{c1}\ p_{nw}\ \{\ (\lambda_B/n_B)\ (1 + \tfrac{1}{2}\ \mu^2 - 2\ S_B/n_B) + (\ (16/18)\ S_B\ \mu^2)\ \} - p_{nw}\ \lambda_B/n_B\ \{\ S_B + (2/n_B)\ (1 + \tfrac{1}{2}\ \mu^2)\}$$

The following term is defined:

$$F_{4w} = F_{CF}\ (F_{Twc} + F_{wc} + F_{\lambda c})$$

The direct effect of the control commands is expressed in a similar way.

Term due to the collective command:

$$F_{1w} = F_{CF}\ \{\ (1 + 2\ \mu^2) + \tfrac{3}{4}\ \mu^4\ \lambda_B/n_B + (16/9)\ S_B\ \mu^2\ \}\ \theta_o \quad .$$

Term due to longitudinal cyclic:

$$F_{2w} = F_{CF}\ \{\ (1 + 2\ \mu^2 + \tfrac{3}{4}\ \mu^4)\ (\lambda_B^2/n_B) + (16/9)\ S_B\ \mu^2\ \}\ \theta_{1sw} \quad .$$

Term due to lateral cyclic:

$$F_{3w} = -\ \{\ F_{CF}\ S_B\ \lambda_B/n_B\ (1 + \tfrac{1}{2}\ \mu^2)\ \}\ \theta_{1cw} \quad .$$

The expression for β_{1cw} can be written as:

$$\beta_{1cw} = \langle F_w,\ \theta_w\ \rangle,$$

where obviously

$$F_w = [F_{1w},\ F_{2w},\ F_{3w},\ F_{4w}]^T \quad .$$

Finally the lateral flapping in terms of θ_{1s} and θ_{1c} can be obtained using the transformation matrix (3):

$$\beta_{1cw} = \langle F\ ,\ \theta_R\rangle$$

where $F = [F_{1w},\ F_{2w}\ K_{\psi c} - F_{3w}\ K_{\psi s},\ F_{2w}\ K_{\psi s} + F_{3w}\ K_{\psi c},\ F_{4w}]^t$

$$= [F_1,\ F_2,\ F_3,\ F_4]^t \quad .$$

<u>Lateral flapping equation β_{1s}</u>

The equations for the lateral flapping are composed of the following terms:

Term related to rotor downwash:

$$G_\lambda = \mu\ R_{ML}\ \{ (16/9)\ (1 - \mu^2) - 2\ S_B/n_B) - (\lambda_B^2/n_B)\ (½\ \mu^2 - 1)\ \}\ \lambda_{1cw} \quad .$$

Term related to the angular velocity:

$$G_\omega = p_{nw}\ N_{c1}\ \{ (\lambda_B^2/n_B)\ (2/n_B)\ (½\ \mu^2 - 1) - S_B) + (16/18)\ \mu^2 +$$

$$q_{nw}\ \lambda_B^2/n_B\ \{ 2\ S_B/n_B + ½\ \mu^2 - 1) \quad .$$

The influence of the twist is calculated with:

$$G_{tw} = 2\ \mu\ \theta_{tw}\ \{ (1 + \mu^2/3 - (5/12)\ \mu^4)\ (8/15 - (S_B/n_B)\ \lambda_B^2\ \}$$

The above terms can be associated as follows:

$$G_{4w} = F_{CF}\ (G_\lambda + G_\omega + G_{tw}).$$

The terms related to the command inputs.

Collective command factor:

$$G_{1w} = (4/3)\ \mu\ F_{CF}\ \{ 1 + ½\ \mu^2\ (1 - \mu^2) - 2\ S_B\ (\lambda_B/n_B)\ \} \quad .$$

Longitudinal cyclic factor:

$$G_{2w} = F_{CF}\ \{ (16/9)\ \mu^2\ (1 - ½\ \mu^2) - (1 + (3/2)\ \mu^2)\ S_B\ (\lambda_B/n_B)\ \} \quad .$$

Lateral cyclic factor:

$$G_{3w} = -\ F_{CF}\ (\lambda_B/n_B)\ (1 - ½\ \mu^4) \quad .$$

Proceeding as before the lateral flapping can be expressed as:

$$\beta_{1sw} = \langle G_w\ ,\ \theta w \rangle\ ,$$

where

$$G_w = [G_{1w},\ G_{2w},\ G_{3w},\ G_{4w}]^t\ ,$$

Finally referring this angle to body axes using the transformation(3):

$$\beta_{1sw} = \langle G_w\ ,\ \theta_R \rangle\ ,$$

where

$$G_w = [G_{1w},\ G_{2w}\ K_{\psi c} - G_{3w}\ K_{\psi s},\ G_{2w}\ K_{\psi s} + G_{3w}\ K_{\psi s},\ G_{4w}] \quad .$$

FLAPPING ANGLES REFERRED TO BODY AXES

The force and moment equations depend on the flapping angles referred to body axes, so that it is necessary to apply the transformation (3) to the flapping angles. The resulting equations are

$\beta_{1c} = \langle K_{\psi c}\, F + K_{\psi s}\, G\ ,\ \theta_R \rangle$ or

$\beta_{1c} = \langle F_B\ ,\ \theta_R \rangle$,

where $F_B = K_{\psi c}\, F + K_{\psi s}\, G$; and

$F_B = [F_{B1},\ F_{B2},\ F_{B3},\ F_{B4}]^t$

and;

$\beta_{1s} = \langle K_{\psi c}\, G - K_{\psi s}\, F,\ \theta_R \rangle$ or

$\beta_{1s} = \langle G_B,\ \theta_R \rangle$,

where $G_B = K_{\psi c}\, G - K_{\psi s}\, F$; and

$G_B = [G_{B1},\ G_{B2},\ G_{B3},\ G_{B4}]^t$,

which can be substituted in the rotor torque equations to obtain a similar expression for the rotor torque. These expressions are obtained as follows.

The main rotor torque coefficient is given by

$C_Q = -\ C_T\ (R_{ML} - \mu\ \beta_{1cw}) + ⅛\ \delta\ s\ (1 + \mu^2)$,

where δ, is the main drag rotor coefficient, (assumed to be of the form $\delta = \delta_0 + \delta_2\ C_T{}^2$). This parameter has not been considered as a function of $\langle K_{CT}\ ,\ \theta_R \rangle$, assuming that $C_T{}^2$ does not have an important influence in the change of the value of δ.

By substituting the expressions of C_T and β_{1cw} in the C_Q equation it is possible to obtain

$C_Q = -\langle R_{ML}\ K_{CT},\ \theta_R \rangle + \langle -\ \mu\ K_{CT},\ \theta_R \rangle\ \langle F,\ \theta_R \rangle + ⅛\ \delta\ s\ (1 + \mu^2)$.

If the vector

$Q_1 = R_{ML}\ [K_{CT1},\ K_{CT2},\ K_{CT3},\ K_{CT4} + \delta\ s\ (1 + \mu^2)/(8\ R_{ML})]^t$

is defined, and if the product of the second term is realized according to the procedure explained at the beginning of this appendix, then the torque coefficient can be expressed as;

$C_Q = \langle Q_1, \theta_{RA} \rangle + \langle Q_2, \theta_{RA} \rangle$,

where

$\langle Q_2, \theta_{RA} \rangle = \langle -\mu K_{CT}, \theta_R \rangle \langle F, \theta_R \rangle$.

Finally if $K_Q = Q_1 + Q_2$ then

$C_Q = \langle K_Q, \theta_{RA} \rangle$.

The main rotor torque is defined as

$Q_R = \rho\ \Omega^2\ R^5\ \pi\ C_Q$.

This equation can be transformed to an inner product

$Q_R = \langle K_{QR}, \theta_{RA} \rangle$,

where $K_{QR} = \rho\ \Omega^2\ R^5\ \pi\ K_Q$.

By extending this procedure the force and moment equations can be expressed in a similar way, as shown below.

MAIN ROTOR FORCES

In order to have a compact notation the variables

$R_{F1} = \rho\ \Omega^2\ R^4\ \pi$;

$R_{F2} = R_{F1}\ R$:

$R_{xy} = -\ ¼\ \delta\ s\ \mu$

are defined.

The longitudinal force component is:

$X_F = F_{F1}\ \{C_T\ (\beta_{1c} + \gamma) - ¼\ \delta\ s\ \mu\}$.

If the flapping and thrust coefficients are expressed as inner products, this equation can be transformed to:

$X_F = \langle K_{CT}, \theta_R \rangle \langle F_{F1}\ F_B\ , \theta_R \rangle + \langle F_{FL}\ \gamma_s\ K_{CT}, \theta_R \rangle + ¼\ \delta\ s\ \mu\ F_{F1}$,

where γ_s is the shaft angle. It is possible to define a vector function K_{Fx} as for the C_Q coefficient, such that

$X_F = \langle K_{FX}, \theta_{RA} \rangle$.

The lateral force is expressed by:

$Y_F = R_{F1} (- C_T \beta_{1s} N_{c1} - ¼ \delta s \mu)$.

If the inner products are substituted then

$Y_F = \langle -R_{F1} K_{CT} , \theta_R \rangle \langle N_{c1} G_B \rangle - ¼ \delta s \mu R_{F1}$.

Proceeding as before this equation can be written as

$Y_F = \langle K_{FY}, \theta_{RA} \rangle$,

for some vector function K_{FY}.

The normal force is:

$Z_F = - R_{F1} C_T$,

which can easily be transformed to:

$Z_F = \langle K_{FZ}, \theta_R \rangle$.

MAIN ROTOR MOMENTS

The rolling moment equation is:

$L_R = L_H + h_R Y_F$,

were h_R is the height of the hub from the centre of gravity and $L_H = - \gamma_s N_{c1} Q_R$ so that the rolling moment can be expressed by

$L_R = \langle - \gamma_s N_{c1} K_{QR}, \theta_{RA} \rangle + \langle h_R K_{FY}, \theta_{RA} \rangle$,

Which can easily be transformed to,

$L_R = \langle K_{RL}, \theta_{RA} \rangle$,

for some vector function K_{RL}.

For the pitching moment it is possible to obtain a similar expression given that

$M_R = R_{F2} (M_H - h_R X_R + x_{cg} Z_R)$,

where x_{cg} is the horizontal distance between the hub and the centre of gravity and

$M_H = - ½ b K_B \beta_{1s}$,

were K_B is the hub stiffness.

So, according to the development outlined here, the pitching moment can be expressed as:

$$M_R = \langle -\tfrac{1}{2} R_{FZ}\, b\, K_B\, G_B,\ \theta_R \rangle + \langle -R_{FZ}\, h_R\, K_{FX},\ \theta_{RA} \rangle + \langle R_{FZ}\, K_{FZ}\, x_{cg},\ \theta_{RA} \rangle$$

and again, this can be reduced to

$$M_R = \langle K_{RM},\ \theta_{RA} \rangle \quad ,$$

for a vector function

$$K_{RM} = (\ -R_{FZ}\, h_R + R_{FZ}\, x_{cg}\ )\ K_{FZ} \ +$$

$$(\ -\tfrac{1}{2} R_{FZ}\, b\, K_B\ )\ \ (\ G_{B1},\ \ldots,\ G_{B3},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1\ )^t \ .$$

Finally the <u>yawing moment</u> is given by:

$$N_R = Q_R\, N_{c1} + \gamma_s\, L_R \ .$$

The terms on the right can also be written as inner products:

$$N_R = \langle N_{c1}\, K_{QR} + \gamma_R\, K_{RL},\ \theta_{RA} \rangle \quad .$$

If $K_{RN} = N_{c1}\, K_{QR} + \gamma_R\, K_{RL}$ then

$$N_R = \langle K_{RN},\ \theta_{RA} \rangle \quad .$$

APPENDIX II.2 TAIL ROTOR FORCE AND MOMENTS.

This appendix is devoted to the equations of tail rotor force and moments which are obtained in such a form that their contributions can be expressed as inner products.

The thrust and power are expressed in terms of normalised velocities

$$\mu_T = (u^2 + (w - K_{\lambda T}\, \lambda_o + q\, (l_T + x_{cg})^2)^{1/2}\, (\Omega\, R_T)^{-1}$$

and

$$\mu_{zT} = (-v + (l_T + x_{cg})\, r - h_T\, p)\, (\Omega\, R_T)^{-1} ,$$

where u, v and w are the aerodynamic velocities at the helicopter centre of gravity; p, q and r are the fuselage angular velocity components, λ_o is the main rotor downwash, and l_T and h_T, the position of the tail rotor aft and above the fuselage reference point, which is the point vertically below the main rotor hub, lying on the fuselage reference line.

The tail rotor thrust coefficient is expressed as

$$C_{TT} = \tfrac{1}{2}\, a_{oT}\, s_T\, (\theta_{oT}^{*}\, (1/3 + \tfrac{1}{2}\, \mu_T^2) + \tfrac{1}{2}\, (\mu_{zT} - \lambda_o)$$

where a_{oT} is the lift curve slope of the tail rotor blades, s_T is the tail rotor solidity, and θ_{oT}^{*} is the collective pitch, given by:

$$\theta_{oT}^{*} = \theta_{oT} + \delta_3\, \beta_{oT}$$

where β_{oT} is the tail rotor coning angle and δ_3 represents the tail rotor collective pitch reduction due to blade flapping and pitch reduction. Finally λ_o is the uniform tail rotor downwash.

If the expression for β_{oT} is substituted in the collective pitch expression, then

$$\theta_{oT}^{*} = \{\theta_{oT} + \delta_3\, (n_{TB}/\lambda_B^2)\, (4/3)\, (\mu_{zT} - \lambda_o)\, \}/\, \{1 + \delta_3\, (n_{TB}/\lambda_B^2)_2\, (1 + \mu_2)\}$$

where $n_{T\beta}$ and $\lambda_{T\beta}$ are the tail rotor inertia number and normalised flapping frequency respectively, assuming that the tail rotor stiffness number is $S_\beta = 0$.

If the following functions

$T_{R1} = \{1 + \delta_3 \; (n_{T\beta}/\lambda_\beta^2)_2 \; (1 + \mu_2)\}^{-1}$,

$T_{R2} = T_{R1} \;\; \delta_3 \; (n_{T\beta}/\lambda_\beta^2) \; (4/3) \; (\mu_{zT} - \lambda_o)$,

$T_{R3} = \; a_{oT} \; s_T \; (1 + 3\mu_T^2/2)/6$ and

$T_{R4} = \; a_{oT} \; s_T \; (\mu_{zT} - \lambda_o)/6$

are defined, then the thrust coefficient can be expressed as:

$C_{TT} = \theta_{oT} \; [T_{R1} \; T_{R3}]^t + [T_{R2} \; T_{R3} + T_{R4}]^t$,

and if $K_{CTT} = [T_{R1} \; T_{R3}, \; T_{R2} \; T_{R3} + T_{R4}]^t = [C_{TT1}, \; C_{TT2}]^t$,

$\theta_{TR} = [\theta_{oT}, \; 1]^T$, this coefficient can be written as

$C_{TT} = \langle K_{CTT}, \; \theta_{TR} \rangle$.

The tail rotor thrust is defined by

$$Y_T = \rho \; \Omega_T^2 \; R_T^4 \; F_T \; C_{TT} \; ,$$

where F_T is the empirical blockage factor given by

$$F_T = 1 - \tfrac{3}{4} \; S_{FN}/(\pi \; R_T^2),$$

where S_{FN} is the fin area.

The tail thrust can be expressed in terms of an inner product,

$$Y_T = \; \rho \; \Omega_T^2 \; R_T^4 \; F_T \; \langle K_{CTT}, \; \theta_{TR} \rangle \; ;$$

and if the vector $K_{TAIL} = \{\rho \; \Omega_T^2 \; R_T^4 \; F_T\} \; K_{CTT} = [K_{1TAIL}, \; K_{2TAIL}]^t$ is defined,

$$Y_T = \langle K_{TAIL}, \; \theta_{TR} \rangle \; .$$

The torque coefficient can be calculated as follows:

$$C_{QT}= (\mu_{zT}-\lambda_o) \langle K_{TAIL}, \theta_{TR}\rangle + ⅛\ \delta_T\ s_T\ (1 + 3\ \mu_T^2)$$

Defining $K_{QT}= (\mu_{zT}-\lambda_o) [K_{1TAIL}, K_{2TAIL} + ⅛\ \delta_T\ s_T\ (1 + 3\ \mu_T^2)]^t$

$$K_{QT}= [K_{1QT}, K_{2QT}]^t$$

One can write the torque coefficient as

$$C_{QT}= \langle K_{QT}, \theta_{TR}\rangle \quad ,$$

The tail rotor torque can be expressed in terms of the above inner

product as

$$Q_T= \rho\ \Omega_T^2\ R_T^3\ F_T\ \langle K_{QT}, \theta_{TR}\rangle \quad \text{and}$$

$$Q_T= \langle K_{QTAIL}, \theta_{TR}\rangle,$$

where $K_{QTAIL}= \rho\ \Omega_T^2\ R_T^3\ F_T\ [K_{QT}]$.

The tail rotor force causes a rolling and yawing moment. The rolling moment is given by

$$L_T= h_T\ Y_T,$$

where h_T is the distance along the longitudinal axis (parallel to the helicopter logitudinal axis) from the tail rotor hub to the centre of gravity of the helicopter. If the force is expressed as an inner product the above equation is transformed to

$L_T = \langle K_{TL}, \theta_{TR} \rangle$

where $K_{LT} = h_T K_{TY}$.

The yawing moment is calculated by

$N_T = -(l_T + x_{cg}) \, Y_T$.

where x_{cg} is the centre of gravity distance, located forward of the fuselage reference point and l_T is the tail rotor location aft of the fuselage reference point.

If the tail rotor force is expressed as an inner product the yawing moment can be written as

$N_T = \langle K_{TN}, \theta_{TR} \rangle$,

where $K_{TN} = -(l_T + x_{cg}) K_{TY}$.

APPENDIX II.3 FUSELAGE, TAIL PLANE AND FIN FORCES AND MOMENTS.

In this appendix the equations used in Helistab to calculate the forces and moments applied to the helicopter by the fuselage, tail plane and fin are described.

Fuselage

Let α_f and β_f denote the fuselage incidence and sideslip respectively. These angles are calculated as follows:

If the rotor downwash λ_o is negative,

$\alpha_f = \arctan(w/u)$

and the fuselage total velocity is

$V_f^2 = w^2 + u^2$;

whereas, if the rotor downwash is positive then

$\alpha_f = \arctan(w_\lambda/u)$

and the fuselage total velocity is;

$V_f^2 = w_\lambda^2 + u^2$.

where u, and w are the longitudinal and normal aircraft total velocities respectively and w_λ is defined as

$w_f = w - K_{\lambda f}\ \Omega\ R\ \lambda_o$

where $K_{\lambda f}$: is a constant that depends on the increase in downwash over the disc.

Ω the rotor rate,

R the rotor radius.

The sideslip β_f is given by

where $\beta_f = \arctan(v/u)$,

and v denotes the lateral vehicle velocity.

Given the fuselage incidence and sideslip, it is possible to calculate the fuselage forces and moments using the following equations

$X_f = ½\ \rho\ (\Omega\ R)^2\ S_p\ V_{fn}^2\ C_{Xf}(\alpha_f)$,

$Y_f = ½\ \rho\ (\Omega\ R)^2\ S_s\ V_{fn}^2\ C_{Yf}(\beta_f)$,

$Z_f = \frac{1}{2} \rho \ (\Omega R)^2 \ S_p \ V_{fn}^2 \ C_{Zf}(\alpha_f)$,

$M_f = \frac{1}{2} \rho \ (\Omega R)^2 \ S_p \ l_{fn} \ V_f^2 \ C_{Mf}(\alpha_f)$ and

$N_f = \frac{1}{2} \rho \ (\Omega R)^2 \ S_s \ l_{fn} \ V_f^2 \ C_{Nf}(\beta_f)$,

where S_p and S_s are are the fuselage plan and side areas respectively, l_f is a reference length and V_{fn} is the velocity normalised to ΩR. The fuselage aerodynamic coefficients are described by the following polynomial functions:

$C_{Xf} = (F_{XO} + F_{X1} \ \alpha_f + F_{X2} \ \alpha_f^2 + F_{X3} \ \alpha_f^3)/(R_{DF} \ P_{RS})$

$C_{Yf} = (F_{YO} + F_{Y1} \ \beta_f + F_{Y2} \ \beta_f^2 + F_{Y3} \ \beta_f^3)/(R_{DF} \ P_{RS})$

$C_{Zf} = (F_{ZO} + F_{Z1} \ \alpha_f + F_{Z2} \ \alpha_f^2 + F_{Z3} \ \alpha_f^3)/(R_{DF} \ P_{RS})$

$C_{Mf} = (F_{MO} + F_{M1} \ \alpha_f + F_{M2} \ \alpha_f^2 + F_{M3} \ \alpha_f^3)/(R_{DF} \ P_{RS})$

$C_{Nf} = (F_{NO} + F_{N1} \ \beta_f + F_{N2} \ \beta_f^2 + F_{N3} \ \beta_f^3)/(R_{DF} \ P_{RS})$,

where the constants F_{Xi}, F_{Yi}, F_Z, F_{Mi} and F_{Ni} for i=1,...,3 are semiempirical constants depending on vehicle geometry.

Tailplane

It is assumed that the tailplane force acts on the vertical plane of the vehicle. The equations that describe the forces and momentsare summarised as follows:

Tailplane force:

$Z_{TP} = \frac{1}{2} \rho \ (\Omega R)^2 \ V_T \ S_{TP} \ C_{ZTP}(\alpha_{TP})$,

where S_{TP} is the tail plane area, α_{TP} is the tail plane incidence. The force coefficients are given by

$C_{ZTP} = -a_{OT} \ \{T_{P1} \ \alpha_{TP} + T_{P3} \ \alpha_{TP}^3 + T_{P5} \ \alpha_{TP}^5\}$,

$a_{OT} = A_{Otpo} + A_{OTP1} \ \beta_f$,

where a_{OT} is the effective lift curve slope for small α_{TP} and where T_{Pi} for i=1,3,5 and A_{OTPi} for i=0,1 are semiempirical coefficients;
S_{TP} is a constant related to the aerodynamic section of the tailplane, for the helicopter; in Helistab it is assumed to be 2.

The effect of the main rotor flow impinging on the tailplane is incorporated in V_T and α_T as follows:

$$V_T^2 = \{ (w - K_{\lambda TF}\ \Omega\ R\ \lambda_o)^2 + u^2 \}\ (\Omega\ R)^{-2}$$

and

$$\alpha_T = \theta_T + \arctan\ (\ \{(w - K_{\lambda TF}\ \Omega\ R\ \lambda_o + (l_{TP} + x_{cg})\ q\ \}\ /u) \quad ,$$

where $K_{\lambda TP}$ is a constant when $\chi_1 < \chi < \chi_2$, where χ is the main rotor wake angle, i. e.:

$$\chi = \arctan\ (\mu/(\lambda_o - \mu_z)) \quad ,$$

$$\chi_1 = \arctan\ ((l_{TF} - R)/(h_R - h_{TF}))\ \text{and}$$

$$\chi_2 = \arctan\ (l_{TF}/(h_R - h_{TF})).$$

Otherwise $K_{\lambda TF} = 0$.

In the above expressions, l_{TP} is the location aft of the fuselage reference point, h_R is the negative coordinate of the rotor hub and h_{TP} is the negative z component of the tail plane centre of pressure.

The moment produced by the tail plane force is

$$M_{TF} = (l_T + x_{cg})\ Z_{TP}\ .$$

Similarly the fin side force can be written as

$$Y_{FN} = ½\ (\Omega\ R)^2\ V_{fn}^2\ S_{fn}\ C_{Yfn}(\beta_{fn}) \quad .$$

where

$$V_{fn}^2 = (u_2 + v^2)/(\Omega\ R)^2;$$

and

$$\beta_{FN} = -\theta_{FN} + \arctan(v - l_{FN}\ r) \quad ,$$

where θ_{FN} is the fin cant angle positive nose starboard, l_{FN} is the location aft of fuselage reference point; u, v and w are the vehicle velocities and r is the vehicle yawing rate. The aerodynamic coefficient is defined by the functions

$$A_{OFN} = A_{OFNO} + A_{OFN1}\ \alpha_f \ ,$$

$$C_{Yfn} = -A_{OFN}\ \{F_{IN1}\ \beta_{fn} + F_{IN3}\ \beta_{fn}^3 + F_{IN5}\ \beta_{fn}^5\} \ ,$$

where A_{OFNi} for i=1,2 and F_{INi} for i=1,3 5 are semiempirical coefficients.

The moments exerted on the helicopter by these forces are

$$L_{FN} = h_{FN}\ Y_{FN}\ \text{and}$$

$$N_{FN} = -(l_{FN} + x_{cg})\, Y_{FN} \,,$$

where x_{cg} is the distance ahead of the hub of the centre of gravity.

APPENDIX III.1

The proof of equation (III.LD.4)

$dL_f(h) = L_f(h)$ (III.LD.4)

encountered in section 2 of chapter III, will be presented in this appendix.

Let h be a C^∞ function on R and f a vector field on R^n. The operation $L_f(h)$ is defined as $\langle dh, f\rangle$ so that:

$dL_f(h) = d\langle dh, f\rangle$;

if $dh = (dh_1, \ldots, dh_n)^t$ and $f = (f_1, \ldots, f_n)^t$ then:

$$dL_f(h) = d(dh_1\, f_1 + dh_2\, f_2 + \ldots + dh_n\, f_n)$$

$$= \Big[(f_1 \frac{\partial\, dh_1}{\partial\, x_1} + dh_1 \frac{\partial\, f_1}{\partial\, x_1}) + \cdots + (f_n \frac{\partial\, dh_n}{\partial\, x_1} + dh_n \frac{\partial\, f_n}{\partial\, x_1}) ,$$

$$(f_1 \frac{\partial\, dh_1}{\partial\, x_2} + dh_1 \frac{\partial\, f_1}{\partial\, x_2} + \cdots + (f_n \frac{\partial\, dh_n}{\partial\, x_2} + dh_n \frac{\partial\, f_n}{\partial\, x_2}) ,$$

$$\vdots$$

$$, (f_1 \frac{\partial\, dh_1}{\partial\, x_n} + dh_n \frac{\partial\, f_1}{\partial\, x_n}) + \cdots + (f_n \frac{\partial\, dh_n}{\partial\, x_n} + dh_n \frac{\partial\, f_n}{\partial\, x_n}) \Big] .$$

Reordering the above expression we have

$$dL_f(h) = (f_1 \frac{\partial\, dh_1}{\partial\, x_1} + \cdots + f_n \frac{\partial\, dh_n}{\partial\, x_1} , \ldots , f_1 \frac{\partial\, dh_1}{\partial\, x_n} + \cdots + f_n \frac{\partial\, dh_n}{\partial\, x_n})$$

$$+ (dh_1 \frac{\partial\, f_1}{\partial\, x_1} + \cdots + dh_n \frac{\partial\, f_n}{\partial\, x_1} + \cdots + dh_1 \frac{\partial\, f_1}{\partial\, x_n} + \cdots + dh_n \frac{\partial\, f_n}{\partial\, x_n})$$

.

The two vectors of the right hand side of this last expression can be factorised as follows :

$$dL_f\ (h) = \begin{vmatrix} \frac{\partial\, dh_1}{\partial\, x_1} & \cdots & \frac{\partial\, dh_n}{\partial\, x_1} \\ \vdots & & \vdots \\ \frac{\partial\, dh_1}{\partial\, x_n} & \cdots & \frac{\partial\, dh_n}{\partial\, x_n} \end{vmatrix} \begin{vmatrix} f_1 \\ \vdots \\ f_n \end{vmatrix} + \begin{vmatrix} \frac{\partial\, f_1}{\partial\, x_1} & \cdots & \frac{\partial\, f_n}{\partial\, x_1} \\ \vdots & & \vdots \\ \frac{\partial\, f_1}{\partial\, x_n} & \cdots & \frac{\partial\, f_n}{\partial\, x_n} \end{vmatrix} \begin{vmatrix} dh_1 \\ \vdots \\ dh_n \end{vmatrix}$$

and this expression can be reduced to :

$$dL_f\ (h) = [\ \frac{\partial\, dh}{\partial\, x}\ ]^t\ f \ +\ [\ \frac{\partial\, f}{\partial\, x}\ ]^t\ dh$$

which is the definition of L_f (dh), so that

$$dL_f\ (h) = L_f\ (dh) \quad .$$

APPENDIX III.1 (Second part)

Proof of equation (III.LD.5)

Given two C^∞ vector fields in R^n, f and g, and a C^∞ one form dh on R^n, then the Lie derivatives defined in section 2 of Chapter III

$L_f(h)$, $[f,g]$ and $L_f(dh)$

are related by

$$L_f < dh, g > = < L_f (dh), g > + < dh, [f, g] > \qquad \text{(III.LD.5)}$$

Proof:

Let $dh = [dh_1 , \dots , dh_n]^t$, $f = [f_1 , \dots , f_n]^t$ and $g = [g_1 , \dots , g_n]^t$.

Then the second term on the right hand side can be written as:

$$< dh, [f, g] > = < (dh_1 , \dots , dh_n), [\frac{\partial g}{\partial x}]f - [\frac{\partial f}{\partial g}]g >$$

$$= < dh, [\frac{\partial g}{\partial x}]f > - < dh, [\frac{\partial f}{\partial x}] g >$$

Extending the right hand term of the above equation, gives the following;

$$< dh, [\frac{\partial g}{\partial x}] f > = (dh)^t \quad [\frac{\partial g}{\partial x}] f$$

which can be extended further to give;

$$< dh, [\frac{\partial f}{\partial x}] g > = (dh)^t \quad [\frac{\partial f}{\partial x}] g$$

thus:

$$< dh, [f,g] > = [\frac{\partial g}{\partial x}]^t \; dh \; f \quad - [\frac{\partial f}{\partial x}]^t \; dh \; g \qquad \text{(A.III.1)}$$

the hand the left hand side term is by definition

$$L_f < dh, g > = < d < dh, g >, f > .$$

Developing this term results in :

$$L_f \langle dh, g \rangle = \langle (g_1 [\frac{\partial\, dh_1}{\partial\, x_1}] + dh_1 [\frac{\partial\, g_1}{\partial\, x_1}]) + \ldots + (g_n [\frac{\partial\, dh_n}{\partial\, x_1}] + dh_n [\frac{\partial\, g_n}{\partial\, x_1}])$$

$$, \ldots , (g_1 [\frac{\partial\, dh_1}{\partial\, x_n}] + dh_1 [\frac{\partial\, g_n}{\partial\, x_n}]) + \cdots + (g_n [\frac{\partial\, dh_n}{\partial\, x_n}] + dh_n [\frac{\partial\, g_n}{\partial\, x_n}]) , f \rangle$$

$$L_f \langle dh, g \rangle = ([\frac{\partial\, dh_1}{\partial\, x_1}] , \ldots , [\frac{\partial\, dh_1}{\partial\, x_n}]) g_1 + \cdots$$

$$+ ([\frac{\partial\, dh_n}{\partial\, x_1}] , \ldots , [\frac{\partial\, dh_n}{\partial\, x_n}]) g_n + ([\frac{\partial\, g_1}{\partial\, x_n}] , \ldots , [\frac{\partial\, g_1}{\partial\, x_n}]) dh_1 +$$

$$\cdots + ([\frac{\partial\, g_n}{\partial\, x_1}] , \ldots , [\frac{\partial\, g_n}{\partial\, x_n}]) dh_n , f \rangle$$

$$= \langle [\frac{\partial\, dh}{\partial\, x}]^t g + [\frac{\partial\, g}{\partial\, x}]^t dh , f \rangle$$, so that

$$L_f \langle dh, g \rangle = ([\frac{\partial\, dh}{\partial\, x}]^t g + [\frac{\partial\, g}{\partial\, x}]^t dh) f \qquad \text{(A.III.2)}$$

With respect to the remaining term of the last equation it can be written as:

$$\langle L_f (dh), g \rangle = \langle d \langle dh, f \rangle, g \rangle$$

and comparing this equation with the previous calculations it is easy to see that:

$$\langle L_f (dh), g \rangle = ([\frac{\partial\, dh}{\partial\, x}]^t f + [\frac{\partial\, f}{\partial\, x}]^t dh) g \qquad \text{(A.III.3)}$$

Finally by substituting equations (A.III.1), (A.III.2) and (A.III.3) into equation (LD.2) the equality is easily checked.

Table 1.

Helicopter data for simulation model.

Main rotor.

a_0	Blade lift slope.	6.0
c	Blade chord.	0.533 m
h_R	Negative Z coordinate of rotor hub.	1.755 m
I_β	Blade flapping moment of inertia.	1281.4 Kg m^2
K_β	Blade flapping stiffness-spring constant.	1037.2 Nw m
R	Blade radius.	7.5 m
s	Rotor solidity.	0.0906 m
X_{cg}	Centre of gravity forward of fuselage reference point.	0.0
γ_s	Rotor shaft forward tilt.	0.087°
δ_0	Blade profile drag coefficient.	0.009
δ_2	Blade lift dependent drag coefficient.	5.333
θ_{tw}	Linear blade twist.	5°

Tail rotor.

a_{0T}	Blade lift curve slope.	6.0
F_T	Fin blockage factor.	0.87
h_T	Negative z coordinate of hub.	0.366
$k_{\lambda T}$	Main rotor downwash factor.	1.5
l_T	Tail rotor location aft of fuselage	9.144

	reference point.	
$\left(\frac{n_B}{\lambda^2_B}\right)_T$	(inertia number)/(flap frequency)^2.	1
R_T	Blade radius.	1.518
S_{ST}	Tail rotor solidity.	0.153
δ_{OT}	Blade profile drag coefficient.	0.009
δ_{2T}	Blade lift dependent drag coefficient.	5.333

Tailplane

a_{OTP}	Lift curve slope at zero incident.	3.5
C_{ZTL}	Maximum normal force coefficient.	2.0
$k_{\lambda TP}$	Main rotor downwash factor.	1.5
l_{TP}	Location aft of fuselage reference point.	9.144m
S_{TP}	Tailplane area.	1.347m^2
θ_T	Tailplane setting (positive nose up relative to fuselage x axis).	3.5°

Fin

h_{FN}	Negative z component of fin centre of pressure.	0.0
l_{FN}	Location aft of fuselage reference point.	$= l_{TP}$
S_{FN}	Fin area.	1.273m^2
θ_{FN}	Fin setting (positive nose starboard).	2.0°

Fuselage

C_{YS}	Aerodynamic sideforce coefficient.	-0.75
$k_{\lambda F}$	Main rotor downwash factor.	1.5
l_f	Fuselage reference length.	13.106m
S_p	Fuselage plan area.	16.723m^2
S_s	Fuselage side area.	23.226m^2

Helicopter inertias

I_{xx}	Moment of inertia.	5695 kg m^2
I_{yy}	"	34578
I_{zz}	"	30239
I_{xz}	Product of inertia.	2068 kg m^2
M_A	Aircraft mass.	5234.6 kg