# On the Equivalence Between Type I Liouville Dynamical Systems in the Plane and the Sphere


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**Abstract** Separable Hamiltonian systems either in sphero-conical coordinates on an $S^2$ sphere or in elliptic coordinates on a $\mathbb{R}^2$ plane are described in a unified way. A back and forth route connecting these Liouville Type I separable systems is unveiled. It is shown how the gnomonic projection and its inverse map allow us to pass from a Liouville Type I separable system with a spherical configuration space to its Liouville Type I partners where the configuration space is a plane and back. Several selected spherical separable systems and their planar cousins are discussed in a classical context.

**Keywords** Separation of variables · Sphero-conical coordinates · Elliptic coordinates · Liouville dynamical systems · Trajectory isomorphism

## 1 Introduction

Hamiltonian systems in $\mathbb{R}^2$ that admit separation of variables were completely determined by Liouville [1] and Morera [2], and can be classified, see [3], in four different types according to the system of coordinates where the separability is manifested: elliptic, polar, parabolic, and Cartesian, respectively. Thus Type I Liouville systems in $\mathbb{R}^2$ are defined by natural Hamiltonians: $H = K + \mathcal{U}$, $K = \frac{m}{2}\left((\frac{dx_1}{dt})^2 + (\frac{dx_2}{dt})^2\right)$, that are separable in elliptic coordinates [3].

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