Faddeev equations with three-nucleon force in momentum space

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Modified Faddeev equations that allow the inclusion of irreducible three-body forces in addition to two-body interactions are formulated and the technical apparatus for their solution in momentum space is described. Results for the triton binding energy are obtained with realistic two-nucleon interactions and the Tucson-Melbourne two-pion exchange three-nucleon force and compared with previous calculations. Excellent agreement with the results of other groups is found confirming that the accuracy of present-day techniques for handling three-nucleon forces is very high indeed.

I. INTRODUCTION

The three-nucleon force is not made by nature. The three-nucleon force is an artifact of theoreticians who view the nucleus as a system of nucleons only: it arises in any theoretical description of nuclear phenomena which freezes microscopic degrees of freedom—in the same way as the two-nucleon force does. Alternatively, sizable contributions to the complicated three-nucleon force may be resolved into reducible processes in a Hilbert space extended for some important nonnucleonic degrees of freedom and may be accounted for there by a simpler interaction. That strategy is pursued in Refs. [1,2] which considers the excitation of a nucleon to a Δ-isobar explicitly. Naturally, not all nonnucleonic degrees of freedom can be treated explicitly in practical calculations. Thus, a residual three-nucleon force will always survive as irreducible and, if expected to remain quantitatively important, has to be included in the Hamiltonian describing many-nucleon systems in nuclear structure and nuclear reactions.

The subject of the present paper is the three-nucleon force in the three-nucleon bound state. The three-nucleon force is used in the two-pion exchange Tucson-Melbourne form [3]. It was previously included in the configuration space calculations of Ref. [4] and in the calculations of Ref. [5] which chose a mixed configuration-space and momentum-space description. Those calculations are our technical standards, to whose results we will be unable to add any further physical insight. Our paper only has technical goals. It presents the first solution of the momentum-space Faddeev equations with the inclusion of a three-nucleon force, but *without* any perturbation expansion. It extends the perturbative treatment of Ref. [6]. For us, the present paper contains the technical basis for a more ambitious calculation which treats the Δ-isobar degree of freedom explicitly as Refs. [1,2] do, but simultaneously takes a residual three-nucleon force into account.

Section II derives the Faddeev equations to be solved and describes the technical apparatus for their solution with respect to three-nucleon binding energy and wave function. Section III discusses our binding-energy results for two realistic two-nucleon potentials when combined with the Tucson-Melbourne three-nucleon force. Our results are compared to those of Refs. [4,5,6] obtained previously. Conclusions are given in Sec. IV.

II. FADDEEV EQUATIONS AND THEIR SOLUTION

A. The inclusion of a three-body force

In order to derive modified Faddeev equations for a bound system of three identical particles that interact both via two-body and genuine three-body forces we start from a Hamiltonian of the form

$$H = H_0 + \sum_{i=1}^{3} (V_i + W_i) \ . \tag{1}$$

For (ijk) being a cyclic permutation of (123), V_i is, according to the usual "odd man out" notation, the two-body potential of the interaction between particles j and k; W_i denotes that part of the three-body potential

$$W = \sum_{i=1}^{3} W_i \ , \tag{2}$$

where particle i interacts simultaneously with particles j and k. In Eq. (1) H_0 is the kinetic energy operator of the three particles. The Schrödinger equation

$$H|\Psi\rangle = E|\Psi\rangle \tag{3}$$

for the Hamiltonian (1) can be rewritten in the case of bound-state problems as

$$|\Psi\rangle = G_0(E) \sum_{i=1}^{3} (V_i + W_i)|\Psi\rangle \ , \tag{4}$$

where

$$G_0(E) = (E - H_0)^{-1} \tag{5}$$

is the free three-body Green's function depending on the total energy E of the bound three-body system.

The following decomposition of the wave function $|\Psi\rangle$ into Faddeev amplitudes $|\psi_i\rangle$ is introduced:

$$|\Psi\rangle = \sum_{i=1}^{3} |\psi_i\rangle \ , \tag{6}$$

$$|\psi_i\rangle = G_0(E)(V_i + W_i)|\Psi\rangle \ . \tag{7}$$

Since all particles are identical, quantities that carry different particle indices, such as operators, states and variables, can be related to each other simply by means of permutations, e.g.,

$$V_2 = P_{123} V_1 P_{123}^{-1} \ , \tag{8}$$

$$V_3 = P_{132} V_1 P_{132}^{-1} \ , \tag{9}$$

where P_{123} and P_{132} are cyclic and anticyclic permutations of three particles defined in terms of transpositions P_{ij} of the two particles i and j through

$$P_{123} = P_{12} P_{23} \ , \tag{10}$$

$$P_{132} = P_{13} P_{23} \ . \tag{11}$$

Relations corresponding to (8) and (9) also hold for W_i. Together with the fact that a wave function of three identical particles is invariant under cyclic permutations they imply that the Faddeev amplitudes transform like

$$|\psi_2\rangle = P_{123}|\psi_1\rangle \ , \tag{12}$$

$$|\psi_3\rangle = P_{132}|\psi_1\rangle \tag{13}$$

into each other. Thus, the wave function $|\Psi\rangle$ takes the form

$$|\Psi\rangle = (1+P)|\psi_i\rangle \ , \tag{14}$$

independent of the label i with

$$P = P_{123} + P_{132} \ . \tag{15}$$

The integral equation

$$|\psi_i\rangle = G_0(E)(V_i + W_i)(1+P)|\psi_i\rangle \tag{16}$$

is obtained after inserting Eq. (14) into Eq. (7). Moving $G_0(E)V_i|\psi_i\rangle$ to the left-hand side and multiplying the resulting equation with $[1 - G_0(E)V_i]^{-1}$ from the left yields

$$|\psi_i\rangle = [1 - G_0(E)V_i]^{-1} G_0(E)[V_i P + W_i(1+P)]|\psi_i\rangle \ . \tag{17}$$

The factor $[1 - G_0(E)V_i]^{-1} G_0(E)$ in Eq. (17) is just the three-body Green's function of channel i with the two-body interaction V_i, but without the three-body contribution W_i, i.e.,

$$G_i(E) = (E - H_0 - V_i)^{-1} \ , \tag{18}$$

which can also be represented as

$$G_i(E) = G_0(E) + G_0(E) T_i(E) G_0(E) \ . \tag{19}$$

In Eq. (19) $T_i(E)$ is the two-body transition matrix embedded in three-particle space, given by the Lippmann-Schwinger equation

$$T_i(E) = V_i + V_i G_0(E) T_i(E) \ . \tag{20}$$

Applying the well-known relation

$$G_i(E) V_i = G_0(E) T_i(E) \tag{21}$$

the two-body potential V_i can be eliminated completely from (17) in favor of $T_i(E)$ with the final result

$$|\psi_i\rangle = G_0(E)\{T_i(E)P + [1 + T_i(E)G_0(E)]W_i(1+P)\}|\psi_i\rangle \ . \tag{22}$$

Equation (22) is a set of three equations for the Faddeev amplitudes $|\psi_i\rangle$ that transform into each other under cyclic permutations. Hence the solution of only one of them is required. Given the solution $|\psi_i\rangle$ the wave function $|\Psi\rangle$ is obtained by means of Eq. (14). Equation (22) is equivalent to Eqs. (3.3) and (3.5) of Ref. [7].

B. Partial wave expansion

To represent (22) in momentum space we introduce partial-wave projected plane-wave states in (Ij)-coupling as basis, i.e.,

$$|pq\nu\rangle_i = |p_i q_i; [(LS)I(ls_i)j]\mathcal{J}\mathcal{J}_z(Tt_i)\mathcal{T}\mathcal{T}_z\rangle \ . \tag{23}$$

The subscript i indicates that particle i is the "spectator" and particles j and k form the so-called "pair". The standard Jacobi momenta are

$$\mathbf{p}_i = \frac{m_k \mathbf{k}_j - m_j \mathbf{k}_k}{m_j + m_k} \ , \tag{24}$$

$$\mathbf{q}_i = \frac{m_i(\mathbf{k}_j + \mathbf{k}_k) - (m_j + m_k)\mathbf{k}_i}{m_i + m_j + m_k} \ , \tag{25}$$

where $\mathbf{k}_i$ is the momentum of the single particle i and m_i is its mass. In the following all particles are nucleons, i.e., $m_i = m_N$. The quantum numbers L, S, I, and T denote the orbital angular momentum, total spin, total angular momentum and isospin of the pair, whereas l is the orbital angular momentum of the spectator relative to the c.m. of the pair, s_i its spin, both coupled to total angular momentum j, and t_i its isospin. In the three-nucleon bound states the total three-particle angular momentum $\mathcal{J}$ and the total isospin $\mathcal{T}$, their z-components being denoted by $\mathcal{J}_z$ and $\mathcal{T}_z$ respectively, are both equal to $\frac{1}{2}$. The whole set of discrete quantum numbers is abbreviated by ν; n will later on be used for the subset of

pair quantum numbers L, S, I, and T. For all continuous and discrete quantum numbers in the basis states of Eq. (23) the differentiating label i will be dropped from now on.

The basis states are normalized

$$ {}_i\langle pq\nu|p'q'\nu'\rangle_i=\delta_{\nu\nu'}\frac{\delta(q-q')}{q^2}\frac{\delta(p-p')}{p^2}\ ; \tag{26}$$

they are complete and are antisymmetrized with respect to the pair particles, provided the condition $(-1)^{L+S+T}=-1$ holds. The last property guarantees that the wave function of Eq. (14) will be antisymmetric under the exchange of any two particles. Since the three-nucleon bound state has positive parity, only basis states with $(-1)^{L+l}=+1$ contribute to its wave function. Provided the two-body potential V_i and the contribution W_i to the three-body force are defined to be nonvanishing in a finite number of partial waves, the partial wave expansion of the Faddeev amplitudes terminates according to Eq. (7). In contrast, the number of partial waves that contribute to the wave function still remains infinite as a consequence of the action of the permutation operator P in Eq. (14).

The matrix elements of all operators needed in the modified Faddeev equations (22) have to be specified in the chosen partial-wave basis. The dynamical input is the complete off-shell two-body transition matrix $T(E)$ and the three-body potential W. The two-body transition matrix in three-particle space $T(E)$ is related to the corresponding transition matrix in two-particle space $t(E)$ by

$$ {}_i\langle pq\nu|T_i(E)|p'q'\nu'\rangle_i=\delta_{\mathcal{J}\mathcal{J}'}\delta_{\mathcal{J}_z\mathcal{J}_z'}\delta_{TT'}\delta_{T_zT_z'}\delta_{II'}\delta_{SS'}\delta_{TT'}\delta_{ll'}\delta_{ss'}\delta_{jj'}\delta_{tt'}\frac{\delta(q-q')}{q^2}t_{nn'}(p,p';E-q^2/2M)\ . \tag{27}$$

The dependence of $t(E)$ on channels, momenta and the available energy is made obvious by the notation; the shorthand n is defined after Eq. (25). The energy at which the two-body transition matrix has to be calculated equals the total energy E minus the kinetic energy of the non-interacting third particle, $M=\frac{3}{4}m_N$ in the case of nucleons being the reduced mass of the spectator i and of the c.m. of the pair (jk).

The momentum-space partial wave decomposition ${}_i\langle pq\nu|W_i|p'q'\nu'\rangle_i=W_{\nu\nu'}(pq,p'q')$ of the three-nucleon potential used, the Tucson-Melbourne force, is given in Ref. [8]. Note that in Ref. [8] the momentum $\mathbf{q}_i$ has the opposite sign compared with the definition of Eq. (25). In general a three-nucleon force as the Tucson-Melbourne one couples all partial waves. A truncation on that coupling is a dynamical approximation.

The free three-particle Green's function is

$$ {}_i\langle pq\nu|G_0(E)|p'q'\nu'\rangle_i=\delta_{\nu\nu'}\frac{\delta(q-q')}{q^2}\frac{\delta(p-p')}{p^2}\frac{1}{E-\frac{p^2}{2\mu}-\frac{q^2}{2M}}\ , \tag{28}$$

where $\mu=\frac{1}{2}m_N$ is the reduced mass of the pair and M the reduced mass already used in Eq. (27).

The matrix elements of the permutation operator can be written in the form

$$ {}_i\langle pq\nu|P|p'q'\nu'\rangle_i=\int_{-1}^{+1}dx\frac{\delta[p-p_1(q,q',x)]}{p^{L+2}}\frac{\delta[p'-p_2(q,q',x)]}{p'^{L'+2}}G_{\nu\nu'}(q,q',x)\ , \tag{29}$$

with

$$ p_1(q,q',x)=\sqrt{\tfrac{1}{4}q^2+q'^2+qq'x}\ , \tag{30}$$

$$ p_2(q,q',x)=\sqrt{q^2+\tfrac{1}{4}q'^2+qq'x}\ , \tag{31}$$

$$ x=\hat{\mathbf{q}}_i\cdot\hat{\mathbf{q}}_i'\ . \tag{32}$$

The expression for $G_{\nu\nu'}(q,q',x)$ is given explicitly in the appendix, whereas details of its derivation can be found in Ref. [9].

In the partial wave expansion the modified Faddeev equations (22) take the form

$$\langle pq\nu|\psi\rangle = \frac{1}{E-p^2/2\mu-q^2/2M}\Bigg[\sum_{\nu'\nu''}\int_0^\infty dq'q'^2\int_{-1}^{+1}dx\, t_{nn''}(p,p_1(q,q',x);E-q^2/2M)$$
$$\times\frac{G_{\nu''\nu'}(q,q',x)}{p_1^{L''}(q,q',x)p_2^{L'}(q,q',x)}\langle p_2(q,q',x)q'\nu'|\psi\rangle$$
$$+\sum_{\nu'}\int_0^\infty dp'p'^2\int_0^\infty dq'q'^2 W_{\nu\nu'}(pq,p'q')\langle p'q'\nu'|\psi\rangle$$
$$+\sum_{\nu'\nu''}\int_0^\infty dq''q''^2\int_0^\infty dq'q'^2\int_{-1}^{+1}dx\, W_{\nu\nu''}[pq,p_1(q'',q',x)q'']$$
$$\times\frac{G_{\nu''\nu'}(q'',q',x)}{p_1^{L''}(q'',q',x)p_2^{L'}(q'',q',x)}\langle p_2(q'',q',x)q'\nu'|\psi\rangle$$
$$+\sum_{\nu'\nu''}\int_0^\infty dp''p''^2\int_0^\infty dp'p'^2 t_{nn''}(p,p'';E-q^2/2M)$$
$$\times\frac{1}{E-p''^2/2\mu-q^2/2M}\int_0^\infty dq'q'^2 W_{\nu''\nu'}(p''q,p'q')\langle p'q'\nu'|\psi\rangle$$
$$+\sum_{\nu'\nu''\nu'''}\int_0^\infty dp''p''^2\int_0^\infty dq'''q'''^2\int_0^\infty dq'q'^2$$
$$\times\int_{-1}^{+1}dx\, t_{nn''}(p,p'';E-q^2/2M)\frac{1}{E-p''^2/2\mu-q^2/2M}$$
$$\times W_{\nu''\nu'''}[p''q,p_1(q''',q',x)q''']\frac{G_{\nu'''\nu'}(q''',q',x)}{p_1^{L'''}(q''',q',x)p_2^{L'}(q''',q',x)}$$
$$\times\langle p_2(q''',q',x)q'\nu'|\psi\rangle\Bigg]\,. \tag{33}$$

The sums over the three-body partial-wave index ν'' imply simultaneous summations over the index n'' (for the two-body quantum numbers) which appears as subscript of the two-body transition matrix in Eq. (33) and labels a subset of quantum numbers contained in ν''. Eq. (33) is actually an integral equation for ${}_i\langle pq\nu|\psi\rangle_i$ as is Eq. (22). However, the integral equation is the same for all three Faddeev amplitudes $|\psi_i\rangle$. Thus, the index i is dropped in Eq. (33) as it was dropped for quantum numbers after Eq. (25). The integral equation couples partial waves. If the two-body transition matrix $t(E)$ and the three-body potential W are assumed to vanish in all partial waves except in the ${}^1S_0$ and ${}^3S_1$-${}^3D_1$ states for the pair, five three-body partial waves are coupled in Eq. (33); if they are assumed to vanish except for pair states with $I\le 2$, 18 three-body partial waves are coupled. We refer to those cases as 5-channel and 18-channel calculations, respectively.

C. Solution of the modified Faddeev equations

Equation (33) is to be solved for the energy value E and the corresponding Faddeev amplitude $\langle pq\nu|\psi\rangle$. Equation (33) has the general structure

$$K(E)|\psi\rangle = |\psi\rangle\,, \tag{34}$$

where $K(E)$ denotes the kernel of the integral Eq. (33). Since E enters the equation in a nonlinear form, it is useful to consider the auxiliary linear eigenvalue problem

$$K(z)|\varphi(z)\rangle = \lambda(z)|\varphi(z)\rangle\,, \tag{35}$$

where the operator $K(z)$ is defined by Eq. (33), $\lambda(z)$ denotes its eigenvalues and $|\varphi(z)\rangle$ the corresponding eigenvectors. The strategy of solving Eq. (34) is to find the value $z=E$ such that in Eq. (35) one of the eigenvalues $\lambda(E)$ becomes one. Then E is the physical binding energy and the corresponding eigenvector $|\varphi(E)\rangle$ is the Faddeev amplitude $|\psi\rangle$ from which the bound-state wave function is derived. In order to find that particular value $z=E$, Eq. (35) has to be solved for a sequence of values z.

The first step for the numerical solution of Eq. (35) is to discretize the integrations over the continuous variables. We employ Gauss-Legendre integration and take $\mathcal{N}_p$ meshpoints for the variable p, $\mathcal{N}_q$ points for q. Then Eq. (35) is transformed into a system of linear homogeneous equations with dimension $D=\mathcal{N}_\nu\mathcal{N}_p\mathcal{N}_q$, where $\mathcal{N}_\nu$ is the number of contributing three-body partial waves. D is typically of the order of several thousands. For example, in the calculation for this paper we choose 22 points for the momentum variable p in the interval between 0 fm^{-1} and 25 fm^{-1} and 14 points for the momentum variable q in the interval between 0 fm^{-1} and 10 fm^{-1}. Thus, the number of coupled linear homogeneous equations is far too large to allow a direct solution of the discretized form of Eq. (34) by matrix inversion or of the discretized form of Eq. (35) by diagonalization with standard numerical techniques.

Malfliet and Tjon [10] successfully used a simple tech-

nique for solving Eq. (35): the ratio method. The method is based on the fact that a sequence of iterations of the kernel $K(z)$ acting on an arbitrary starting vector always converges to the eigenstate with the eigenvalue largest in magnitude. There is no eigenvalue $\lambda(z)$ greater than the one which for $z=E$ corresponds to the physical ground state. However, repulsive parts in the employed force model induce also spurious negative eigenvalues $\lambda(z)$ in the spectrum of $K(z)$, which may happen to be larger in magnitude than the desired "physical" eigenvalue. In that case the iteration sequence of the ratio method will converge to a spurious solution instead of the Faddeev amplitude of the physical ground state. All realistic potentials are partially repulsive, therefore in most interesting cases the problem of spurious solutions has to be faced. One possibility of avoiding spurious solutions is to determine them in a first sequence of iterations and to construct then a starting vector for the ratio method orthogonal to them [7]. An even simpler approach is to add a suitable constant to the operator $K(z)$ such that in the shifted eigenvalue spectrum the physical eigenvalue becomes the largest in magnitude [11]. A complementary way is to increase the weight of the physical eigenstate relative to the spurious ones after each iteration by means of Pade approximation [12]. In this paper we use a further alternative, a method already applied for three-nucleon scattering problems in Ref. [13]: the basic idea is the construction of an optimized basis of small dimension for diagonalizing a large matrix. The alternative has great similarities with the Lanczos algorithm, on which the Los Alamos-Iowa group [4,14] based their trinucleon bound-state calculations. The alternative method of solving Eq. (35) for one z is now described in detail. In the description the dependence of all quantities on z will be dropped.

Let $|\varphi^{(0)}\rangle$ denote an arbitrary initial vector. Repeated multiplication with the kernel K produces a sequence of vectors $|\varphi^{(i)}\rangle$,

$$|\varphi^{(i+1)}\rangle = K|\varphi^{(i)}\rangle \ , \tag{36}$$

$$|\varphi^{(i)}\rangle = K^i|\varphi^{(0)}\rangle \ . \tag{37}$$

The iteration is stopped after N steps. By means of the Schmidt orthogonalization procedure a set of N orthonormal basis vectors $|\bar{\varphi}^{(i)}\rangle$ is constructed from the states $|\varphi^{(i)}\rangle$, i.e.,

$$\begin{aligned}
|\bar{\varphi}^{(0)}\rangle &= C_0|\varphi^{(0)}\rangle \ , \\
|\bar{\varphi}^{(1)}\rangle &= C_1(|\varphi^{(1)}\rangle - |\bar{\varphi}^{(0)}\rangle\langle\bar{\varphi}^{(0)}|\varphi^{(1)}\rangle) \ , \\
|\bar{\varphi}^{(2)}\rangle &= C_2(|\varphi^{(2)}\rangle - |\bar{\varphi}^{(1)}\rangle\langle\bar{\varphi}^{(1)}|\varphi^{(2)}\rangle \\
&\quad - |\bar{\varphi}^{(0)}\rangle\langle\bar{\varphi}^{(0)}|\varphi^{(2)}\rangle) \ , \\
&\vdots \\
|\bar{\varphi}^{(i)}\rangle &= C_i(|\varphi^{(i)}\rangle - \sum_{j=0}^{i-1} |\bar{\varphi}^{(j)}\rangle\langle\bar{\varphi}^{(j)}|\varphi^{(i)}\rangle) \ .
\end{aligned} \tag{38}$$

The normalization constants C_i are determined by the condition

$$\langle\bar{\varphi}^{(i)}|\bar{\varphi}^{(j)}\rangle = \delta_{ij} \ . \tag{39}$$

The original vectors $|\varphi^{(i)}\rangle$ are expanded in that generated basis,

$$|\varphi^{(i)}\rangle = \sum_{j=0}^{i} a_{ij}|\bar{\varphi}^{(j)}\rangle \ , \tag{40}$$

where the coefficients a_{ij} are given by

$$a_{ij} = \langle\bar{\varphi}^{(j)}|\varphi^{(i)}\rangle \ , \quad j \le i \ . \tag{41}$$

Conversely, the inversion of Eq. (40) yields the orthogonal vectors $|\bar{\varphi}^{(i)}\rangle$ as a linear combination of the nonorthogonal vectors $|\varphi^{(i)}\rangle$,

$$|\bar{\varphi}^{(i)}\rangle = \sum_{j=0}^{i} b_{ij}|\varphi^{(j)}\rangle \ . \tag{42}$$

In order to solve the discretized form of Eq. (35), the unknown eigenstate $|\varphi\rangle$ is expanded in terms of the basis states $|\bar{\varphi}^{(i)}\rangle$, i.e.,

$$|\varphi\rangle = \sum_{i=0}^{N} c_i|\bar{\varphi}^{(i)}\rangle \ . \tag{43}$$

Substituting Eq. (43) into (35) yields

$$\begin{aligned}
\sum_{i=0}^{N} c_i K|\bar{\varphi}^{(i)}\rangle &= \sum_{i=0}^{N} c_i \sum_{j=0}^{i} b_{ij} K|\varphi^{(j)}\rangle \\
&= \sum_{i=0}^{N} c_i \sum_{j=0}^{i} b_{ij}|\varphi^{(j+1)}\rangle \\
&= \sum_{i=0}^{N} c_i \sum_{j=0}^{i} b_{ij} \sum_{k=0}^{j+1} a_{j+1,k}|\bar{\varphi}^{(k)}\rangle \\
&= \lambda \sum_{i=0}^{N} c_i|\bar{\varphi}^{(i)}\rangle \ ,
\end{aligned} \tag{44}$$

and after multiplication from the left with $\langle\bar{\varphi}^{(n)}|$ finally

$$\sum_{i=0}^{N} c_i M_{in} = \lambda c_n \ , \tag{45}$$

which is an eigenvalue equation for the matrix M defined by

$$M_{in} = \sum_{j=0}^{i} b_{ij} a_{j+1,n}, \quad 0 < n \le N \ . \tag{46}$$

The eigenvalues of M are approximations to the eigenvalues λ of K. The largest eigenvalues dominate the iterations and therefore the corresponding eigenvectors will have high relative weight in the constructed basis of N states. That is why the largest eigenvalues converge fastest with respect to the number of iterations, i.e., with respect to the dimension N of the basis. It turns out that $N < 10$ is usually sufficient to achieve good accuracy. Clearly, Eq. (45) can be solved without problems for such low dimensions. It should be mentioned that as the number of iterations increases, the generated vectors become more and more collinear and the numerical orthogonalization has to be performed very carefully to avoid large errors.

Among the eigenvalues λ of M for a chosen energy z the one closest to the value one is the best approximation

for the desired physical eigenvalue. The corresponding eigenvector gives the expansion coefficients c_i for $|\varphi\rangle$ according to Eq. (43). The resulting state $|\varphi\rangle$ is considered converged, if Eq. (35) holds *pointwise,* i.e. for all meshpoints, with a desired accuracy. If not, the whole iteration procedure is repeated for the same value of z, $|\varphi\rangle$ taken as an improved initial vector $|\varphi^{(0)}\rangle$ in Eq. (37). During each full step from Eq. (37) to Eq. (45) the components of spurious solutions get suppressed and the resulting state $|\varphi\rangle$ becomes pointwise stable. After having achieved convergence both in the eigenvalue and in the eigenvector for a given energy z, Eq. (35) is solved by the same method for a new value of z. The rate of convergence is increased if the converged eigenvector for the previous energy z is taken as the starting vector for the new energy value, provided the energies are close. We find that the eigenvalue problem Eq. (35) has to be solved for three or four energies z only in order to determine the physical bound-state energy E and the corresponding Faddeev amplitude $|\psi\rangle$: In the vicinity of the physical energy E the dependence of the eigenvalue λ on z is almost linear, thus, the interpolation or extrapolation to the eigenvalue $\lambda=1$ gets simplified.

III. NUMERICAL RESULTS

The modified Faddeev Eqs. (33) are solved for the Reid soft-core (RSC) [15] and the Paris [16] nucleon-nucleon potentials as two-body interaction and the Tucson-Melbourne two-pion exchange three-nucleon force as three-body interaction. The numerical technique of Sec. II is used. Only the triton binding energy will be determined.

We have to restrict ourselves to 5-channel and 18-channel calculations because of practical reasons: In momentum space the three-body force W depends on four independent momentum variables and two three-body partial-wave indices. The computing time and storage requirements for the evaluation of its momentum-space matrix elements increase dramatically with the number of three-body partial waves and meshpoints. With our choice of partial waves and meshpoints we already reach the limit of computer capacity available to us. Increasing the number of channels while leaving the storage space unchanged would require the calculation of the three-nucleon force at fewer meshpoints and the interpolation of its matrix elements to the momenta actually needed. Clearly, such a procedure would yield an undesirable loss of accuracy, since the three-nucleon force is not a really smooth function. In configuration space the Tucson-Melbourne three-nucleon force can be handled easier, since it is local there. Previous calculations in configuration space [4,17] demonstrate that all 34 three-body partial waves with the two-body total angular momentum $I \leq 4$ should be included for a well-converged triton binding energy. However, the 18-channel calculations deviate typically only by less than 200 keV from the converged result, a small deviation compared with differences of several MeV arising from the freedom in choosing the form factor for the πNN vertex in the three-nucleon force.

Given the dynamic problem of Eq. (33) with a chosen number of three-body partial waves, we are interested in the level of numerical accuracy that can be achieved in calculating the triton binding energy with a three-nucleon force. Whereas there is an impressive technical agreement among the results of many different groups as long as only two-nucleon interactions are employed, the differences—with few exceptions—being of the order of only 10 keV, the achieved agreement is far less clear when a three-nucleon force is included. Up to now results obtained by three groups are available for the triton binding energy. The Tucson-Melbourne two-pion exchange three-nucleon force is employed together with different nucleon-nucleon potentials. The groups use different numerical techniques.

(1) The Los Alamos–Iowa group solves the Faddeev equations in configuration space by means of spline techniques and a Lanzcos algorithm [4].

(2) The Sendai group employs a mixed representation, in which the interacting pair is described in configuration space and the spectator particle in momentum space. The resulting equations are solved with the method of continued fractions [5].

(3) The Bochum group works entirely in momentum space as we do and—till now—has treated the three-nucleon interaction in perturbation theory up to fifth order, where finally convergence is achieved [6].

As can be read off from Table I, the agreement between the three groups is very good for the triton binding energy E as long as only a two-nucleon force is employed. Once a three-nucleon force is added, the comparison gets complicated by the fact that in previous calculations the three groups take different values for the cutoff parameter Λ in the πNN dipole form factor of the three-nucleon force. Unfortunately, the binding-energy contribution ΔE arising from the three-nucleon force strongly depends on Λ, i.e., even small differences in that parameter can cause sizable discrepancies in results. A direct comparison between the results of all three groups is possible only for the case of the RSC potential, the only two-nucleon potential used by all groups together with the three-nucleon Tucson-Melbourne force. If those results are interpolated to the same value of $\Lambda=813.6$ MeV, the value taken by the Bochum group [6], the three results for the 18-channel calculation are -8.97 MeV in case of the Los Alamos-Iowa group, -9.08 MeV in case of the Bochum group and -9.23 MeV in case of the Sendai group. Obviously, the differences between the results are more than ten times larger than the differences in the cal-

TABLE I. Triton binding energies in MeV for the RSC and Paris two-nucleon potentials in 5-channel and 18-channel calculations.

	RSC		Paris	
	5	18	5	18
This work	−7.03	−7.23	−7.30	−7.39
Los Alamos [4,18]	−7.02	−7.23	−7.31	−7.39
Bochum [6]		−7.24		−7.33
Sendai [5]	−7.03	−7.24	−7.48	−7.56

TABLE II. Parameters of the Tucson-Melbourne two-pion exchange three-nucleon force.

μa	$\mu^3 b$	$\mu^3 c$	$\mu^3 d$
1.130	−2.580	1.000	−0.753

culations with two-nucleon forces only. Nevertheless, the agreement is usually thought to be rather satisfactory considering the enormous complexity of such calculations. We believe the achieved agreement is even better than it appears, since the force specifications for the apparently identical calculations are in fact different besides the cutoff parameter Λ.

In Table II the parameters a, b, c, and d [3,8] of the Tucson-Melbourne two-pion exchange three-nucleon force are listed, where μ is the pion mass. The Los Alamos-Iowa group takes the mass of the charged pions for μ, i.e., $\mu=139.6$ MeV, whereas the Bochum group uses a charge-averaged pion mass of $\mu=138.03$ MeV and the Sendai group $\mu=138.7$ MeV. Since the dimensionless products of the force parameters with μ or μ^3 respectively are taken to be the same in all three cases, that difference in choice for μ implies that actually each group uses different values of a, b, c, and d. Since the pion mass enters the strength parametrization of the three-nucleon force up to the third power, differences in those parameters up to a few percent arise. In addition, the Sendai group employs a slightly different πNN coupling constant which enters the three-nucleon force as an overall factor. The force parameters a, b, c and d are fixed by pion-nucleon physics and when given in the form of Table II are meant to be expressed in terms of the charged pion mass [8]. One may argue that the experimental constraints are too weak to distinguish between the three employed parameter sets, but this is not our point here. We only want to emphasize that the Hamiltonians underlying the triton calculations have to be the same when one wants to draw conclusions about the numerical accuracy of different techniques for solving Faddeev equations with a three-nucleon force. We therefore perform calculations for *all three different parameter sets adopted by the three groups* such that a direct comparison becomes meaningful. The results are given in Tables III to V, where the cutoff mass Λ is expressed in terms of the charged pion mass $\mu=139.6$ MeV.

Table III shows the excellent agreement of our results with the ones obtained by the Los Alamos group. The deviations are of the same order of magnitude as in the case without a three-nucleon force. At higher values of Λ the dependence of the binding energy on the cutoff mass becomes stronger, a fact which may be the reason for the slight enhancement of the differences in the two calculations at $\Lambda=7.1\mu$. For the sake of completeness our results for the Paris potential are also displayed in Table III and presented graphically in Fig. 1, together with the RSC results. The behavior of the binding energy is very similar to the RSC case, apart from a faster increase of the energy shift with the cutoff mass in the 5-channel calculation. However, it is well known that the Tucson-Melbourne three-nucleon force 5-channel calculations are far from convergence, hence this different feature is of less practical importance.

According to Table IV, our results for the binding-energy E compare equally well with the Bochum group [6] confirming the accuracy of a high-order perturbation-

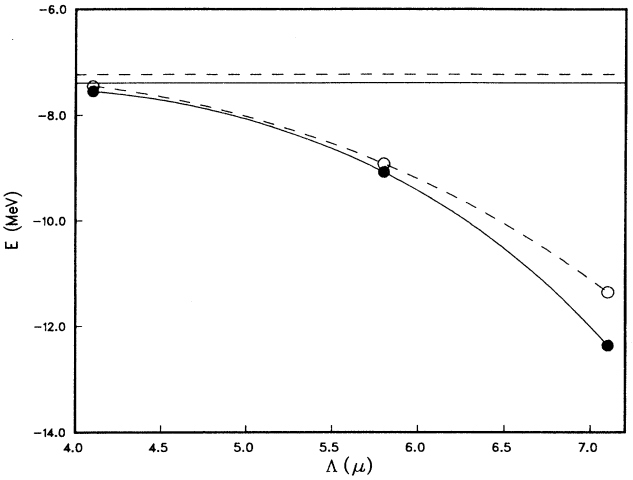

FIG. 1. Triton binding energy E obtained from 18-channel calculations vs the cutoff parameter Λ in the πNN form factor of the Tucson-Melbourne three-nucleon force. The horizontal lines correspond to results without three-nucleon force. The full circles are binding energies calculated actually with the Paris potential and the Tucson-Melbourne three-nucleon force; the empty circles are binding energies actually calculated with the RSC potential and the Tucson-Melbourne three-nucleon force. The solid and dashed curves through the circles are drawn to guide the eye.

TABLE III. Triton binding energy E and the three-nucleon force contribution ΔE in MeV for the RSC and Paris two-nucleon potentials plus the Tucson-Melbourne three-nucleon force with Los Alamos-Iowa parameters at different cutoff masses Λ in 5-channel and 18-channel calculations.

		$\Lambda=4.1\mu$				$\Lambda=5.8\mu$				$\Lambda=7.1\mu$			
		This work		Los Alamos		This work		Los Alamos		This work		Los Alamos	
		E	ΔE	E	ΔE	E	ΔE	E	ΔE	E	ΔE	E	ΔE
RSC	5	−6.95	+0.08	−6.93	+0.09	−7.56	−0.54	−7.55	−0.53	−8.76	−1.74	−8.75	−1.73
RSC	18	−7.45	−0.22	−7.44	−0.21	−8.92	−1.68	−8.93	−1.70	−11.36	−4.13	−11.40	−4.17
Paris	5	−7.24	+0.07			−8.14	−0.83			−11.16	−3.85		
Paris	18	−7.55	−0.16			−9.08	−1.69			−12.37	−4.98		

TABLE IV. Triton binding energy E and three-nucleon force contribution ΔE in MeV for the RSC and Paris two-nucleon potentials plus the Tucson-Melbourne three-nucleon force with Bochum parameters in 5-channel and 18-channel calculations.

		$\Lambda=5.828\mu$			
		This work		Bochum	
		E	ΔE	E	ΔE
RSC	5	−7.63	−0.61		
RSC	18	−9.06	−1.82	−9.08	−1.84
Paris	5	−8.26	−0.95		
Paris	18	−9.24	−1.85	−9.19	−1.86

TABLE V. Triton binding energy E and three-nucleon force contribution ΔE in MeV for the RSC and Paris two-nucleon potentials plus the Tucson-Melbourne three-nucleon force with Sendai parameters in 5-channel and 18-channel calculations.

		$\Lambda=5.730\mu$			
		This work		Sendai	
		E	ΔE	E	ΔE
RSC	5	−7.56	−0.54	−7.56	−0.53
RSC	18	−8.91	−1.68	−9.11	−1.87
Paris	5	−8.14	−0.83	−8.27	−0.79
Paris	18	−9.07	−1.69	−9.49	−1.93

al treatment of the three-nucleon force. For the Paris potential there is a 50 keV difference already at the level of the two-nucleon force alone which carries over to the full calculations for which, however, the energy shifts ΔE are in good agreement. In fact, a recent 18-channel calculation of the Bochum group with the Paris potential, though without three-nucleon force, corrects the older value of −7.33 MeV to −7.38 MeV and thus eliminates also that part of the slight disagreement.

With the exception of the RSC 5-channel calculation we find considerably larger differences between our results and the Sendai results both in the triton binding energy E and in the binding-energy contribution ΔE of the three-nucleon force according to Table V. We do not have an explanation for that disagreement which, in the light of the impressive agreement among the other calculations, appears to be too large to be attributed just to unavoidable numerical errors.

IV. CONCLUSIONS

The paper presents modified Faddeev equations for bound three-body systems that include an irreducible three-body force. The equations are solved for the triton in momentum space. The triton binding energy is determined for Hamiltonians that consist of the RSC or the Paris two-nucleon potentials and the Tucson-Melbourne two-pion exchange three-nucleon force. The results of 5-channel and 18-channel calculations are the first complete solution of momentum-space Faddeev equations with an irreducible three-nucleon force.

We have to emphasize that at first sight unimportant differences in the parametrization of the three-nucleon force can have relatively large influence on the resulting triton binding energy. Once truly the same Hamiltonian is used, we find excellent agreement between our results and those of the Los Alamos-Iowa and the Bochum groups obtained with entirely different techniques. However, there are larger unexplained differences to the Sendai results. The present calculations demonstrate that the accuracy with which the triton binding energy can be numerically determined when a three-nucleon force is taken into account is very high indeed. In fact, the accuracy appears to be of the same quality as for calculations without an irreducible three-nucleon interaction.

For the authors' future work, most importantly, the present calculations demonstrate that Faddeev equations modified by an irreducible three-nucleon force are practically solvable in momentum space. They will therefore form the technical basis for further calculations in which the nucleonic excitation to a Δ-isobar is treated explicitly besides an additional three-nucleon force arising from the non-Δ components of the πN scattering amplitude.

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APPENDIX

The full expression of the function $G_{vv'}(q,q',x)$ which appears in the matrix elements of the permutation operator P in Eq. (29) is given. A detailed derivation can be found in Ref. [9]. Compared with Ref. [9] there is a difference in the phase factor arising from the different definition (25) of the internal momentum $\mathbf{q}$ in terms of single particle momenta.

$$G_{vv'}(q,q',x)=\sum_k P_k(x) \sum_{L_1+L_2=L} \sum_{L_1'+L_2'=L'} q^{L_1+L_1'} q'^{L_2+L_2'} g_{vv'}^{kL_1L_1'L_2L_2'} . \tag{A1}$$

The $P_k(x)$ are the ordinary Legendre Polynomials. The factor $g_{vv'}^{kL_1L_1'L_2L_2'}$ is of purely geometrical nature, having the form

$$
\begin{aligned}
g_{\nu\nu'}^{kL_1L_1'L_2L_2'} = (-)^{l+l'+1}\delta_{\mathcal{J}\mathcal{J}'}\delta_{\mathcal{J}_z\mathcal{J}_z'}\delta_{\mathcal{T}\mathcal{T}'}\delta_{\mathcal{T}_z\mathcal{T}_z'}\hat{I}\hat{I}'\hat{j}\hat{j}'\hat{L}\hat{L}'\hat{S}\hat{S}'\hat{\mathcal{I}}\hat{\mathcal{I}}'\hat{\mathcal{T}}\hat{\mathcal{T}}'\hat{k}^2
\left[\frac{1}{2}\right]^{L_1+L_2'}
\\
\times\left[\frac{(2L+1)!(2L'+1)!}{(2L_1)!(2L_2)!(2L_1')!(2L_2')!}\right]^{1/2}
\begin{Bmatrix} t & t & T \\ t & \mathcal{T} & T' \end{Bmatrix}
\\
\times\sum_{\mathcal{L}\mathcal{S}}(\hat{\mathcal{L}}\hat{\mathcal{S}})^2
\begin{Bmatrix} s & s & S \\ s & \mathcal{S} & S' \end{Bmatrix}
\begin{Bmatrix} L & l & \mathcal{L} \\ S & s & \mathcal{S} \\ I & j & \mathcal{I} \end{Bmatrix}
\begin{Bmatrix} L' & l' & \mathcal{L} \\ S' & s' & \mathcal{S} \\ I' & j' & \mathcal{I} \end{Bmatrix}
\\
\times\sum_{ff'}\langle L_1 0 l 0|f 0\rangle\langle L_2' 0 l' 0|f' 0\rangle\langle k 0 L_2 0|f' 0\rangle\langle k 0 L_1' 0|f 0\rangle
\\
\times
\begin{Bmatrix} L_1 & L_2 & L \\ \mathcal{L} & l & f \end{Bmatrix}
\begin{Bmatrix} L_2' & L_1' & L' \\ \mathcal{L} & l' & f' \end{Bmatrix}
\begin{Bmatrix} f & L_2 & \mathcal{L} \\ f' & L_1' & k \end{Bmatrix},
\end{aligned}
\tag{A2}
$$

where $\hat{x}\equiv\sqrt{2x+1}$. The quantum numbers $\mathcal{L}$ and $\mathcal{S}$ denote the total orbital angular momentum and the total spin of the three-particle system. They appear in the $\mathcal{L}\mathcal{S}$-coupling scheme that is used as an intermediate step when deriving the permutation operator P in Ij-coupling.

[1] Ch. Hajduk and P. U. Sauer, Nucl. Phys. **A322**, 329 (1979).
[2] Ch. Hajduk, P. U. Sauer and W. Strueve, Nucl. Phys. **A405**, 581 (1983).
[3] S. A. Coon, M. D. Scadron, P. C. McNamee, B. R. Barrett, D. W. E. Blatt, and B. H. J. McKellar, Nucl. Phys. **A317**, 242 (1979).
[4] C. R. Chen, G. L. Payne, J. L. Friar, and B. F. Gibson, Phys. Rev. C **33**, 1740 (1986).
[5] T. Sasakawa and S. Ishikawa, Few-Body Syst. **1**, 3 (1986).
[6] A. Bömelburg, Phys. Rev. C **34**, 14 (1986).
[7] W. Glöckle, Nucl. Phys. **A381**, 343 (1982).
[8] S. A. Coon and W. Glöckle, Phys. Rev. C **23**, 1790 (1981).
[9] W. Glöckle, *The Quantum Mechanical Few-Body Problem* (Springer-Verlag, Berlin, 1983).
[10] R. A. Malfliet and J. A. Tjon, Nucl. Phys. **A127**, 161 (1969).
[11] W. Glöckle, in *Models and Methods in Few-Body Physics*, edited by Lidia S. Ferreira, António C. Fonseca, and Ludwig Streit, Lecture Notes in Physics (Springer, Berlin, 1987), p. 3 No. 273.
[12] Ch. Hajduk and P. U. Sauer, Nucl. Phys. **A369**, 321 (1981).
[13] C. R. Chen, G. L. Payne, J. L. Friar, and B. F. Gibson, Phys. Rev. C **44**, 50 (1991).
[14] G. L. Payne, in *Models and Methods in Few-Body Physics* (Ref. [11]), p. 64.
[15] R. V. Reid, Ann. Phys. (N.Y.) **50**, 411 (1968).
[16] M. Lacombe, B. Loiseau, J. M. Richard, R. Vinh Mau, J. Côté, P. Pirès, and R. de Tourreil, Phys. Rev. C **21**, 861 (1980).
[17] S. Ishikawa and T. Sasakawa, Few-Body Syst. **1**, 143 (1986).
[18] J. L. Friar, B. F. Gibson, and G. L. Payne, Phys. Rev. C **37**, 2869 (1988).