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# Cramér-Rao Bound for Wideband DOA Estimation with Uncorrelated Sources

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***Abstract*—In this paper, the closed-form Cramér-Rao bound (CRB) is derived for direction-of-arrival (DOA) estimation under the unconditional model assumption (UMA) for uncorrelated wideband sources. The existence of the CRB is proved based on the rank condition of the introduced augmented co-array manifold (ACM) matrix. The resolution capacity is then investigated and it is found that the number of resolvable sources for the wideband model can exceed the limitation in the narrowband case without requirement of any special array structure.**

## I. Introduction

The Cramér-Rao bound (CRB) is a fundamental and universal statistical metric for evaluating the performance of direction-of-arrival (DOA) estimation algorithms, by providing a lower bound on the variance of unbiased DOA estimation results. In the past few decades, the narrowband CRB for DOA estimation exploiting linear sensor arrays has been systematically studied [1]–[5]. In [1], [2], the authors summarized two typical signal models: the conditional model assumption (CMA) and the unconditional model assumption (UMA), under which the signals are deterministic and stochastic, respectively. Explicit CRB expressions under both CMA and UMA were derived, and comparative studies were conducted. However, almost all of the aforementioned research is focused on the uniform linear array (ULA) structure, which can only resolve fewer sources than the number of physical sensors. Thus, the CRB expressions therein are only applicable to the overdetermined case.

In the underdetermined case, sparse arrays such as nested arrays (NA) [6], coprime arrays [7]–[9], and their extensions [10], [11] have provided increased degrees of freedom (DOFs) to identify more sources than sensors. Recently, several closed-form CRB expressions have been derived pertaining to undetermined DOA estimation problems in the narrowband case [12]–[17]. These derivations commonly adopt UMA and assume the sources are known *a priori* to be uncorrelated. The number of resolvable sources are bounded by the number of unique lags in the virtual difference co-array generated from the underlying sparse array structures [12], [13]. On the other hand, although there are a variety of wideband DOA estimation approaches [18]–[24], the wideband CRB is often evaluated numerically [18], [19], [25]–[27]. In [28], a closed-form wideband CRB expression under UMA was provided, which indicates that the wideband Gaussian model without any prior information cannot identify more sources than the number of sensors. The assumption that the sources are uncorrelated is widely considered in underdetermined DOA estimation algorithms in the wideband scenario with the assistance of sparse arrays [20], [23], [29]. However, the maximum achievable accuracy and number of resolvable sources of these algorithms have not been studied yet. For further performance analysis, it is necessary to derive a closed-form wideband CRB expression for the corresponding signal model, especially in the underdetermined case.

In this paper, we will mainly focus on UMA, since underdetermined DOA estimation has been proved to be infeasible under CMA [12]. We start by exploiting the statistical characteristics of the frequency domain data, and then directly derive the closed-form wideband CRB expression for DOA estimates with the prior information that the sources are uncorrelated. After defining the augmented co-array manifold (ACM) matrix, we prove that the CRB exists if and only if the ACM matrix is of full column rank. According to this rank condition, the proposed CRB expression is applicable to both overdetermined and undetermined cases. The resolution capacity is then investigated and it is found that undetermined DOA estimation can be achieved under the wideband model while no special array structures are needed, which is different from the narrowband scenario. Finally, simulation results are provided to verify our theoretical analysis.

## II. Frequency Domain Signal Model

Assume that there are $K$ uncorrelated signals $\{s_k(t)\}_{k=1}^{K}$ with the same bandwidth impinging from $K$ distinct incident angles $\{\theta_k\}_{k=1}^{K}$ in the far field. These signals are received by a linear array consisting of $M$ sensors. Let $d$ denote the smallest distance between two adjacent sensors and set the sensor locations to be an integer multiple of $d$, i.e., $z_m d$, $m = 1, 2, \cdots, M$. Therefore, the array structure can be described by an integer set such that $\mathbb{S} = \{z_m | z_m \in \mathbb{Z}, 1 \leq m \leq M\}$, where $\mathbb{Z}$ denotes the set of all integers. The sensor locations in the virtual difference co-array are represented by the difference set $\mathbb{D} = \{z_1 - z_2 | z_1, z_2 \in \mathbb{S}\}$.

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The output signal at the $m$-th sensor is sampled into $N$ time snapshots $\{x_m[i]\}_{i=1}^{N}$ with a sampling frequency $f_s$. Then, each received signal is divided into $Q$ non-overlapping groups with the same length $L$, and the time duration of each group is $\Delta t = \frac{L}{f_s}$. The time delay between two sensors (indexed by $m_1$ and $m_2$ respectively) is denoted by $\tau_{m_1 m_2}(\theta), m_1, m_2 \in \{1, 2, \cdots, M\}$. Following the widely adopted wideband model assumptions [19]–[22], [25], [26], [28]–[30], we also assume that:

**A1** The noise is circularly-symmetric Gaussian distributed, and uncorrelated with the source signals.

**A2** $\Delta t$ is sufficiently large, and is much larger than the maximum of $\tau_{m_1 m_2}(\theta)$.

Applying an $L$-point DFT, we can obtain the output signal model in the DFT domain, given by

$$\boldsymbol{X}_l(q) = \boldsymbol{A}_l(\boldsymbol{\theta})\boldsymbol{S}_l(q) + \boldsymbol{N}_l(q),$$

where $\boldsymbol{A}_l(\boldsymbol{\theta})$ denotes the steering matrix for the $l$-th frequency bin. $\boldsymbol{X}_l(q)$, $\boldsymbol{S}_l(q)$, and $\boldsymbol{N}_l(q)$ are column vectors collecting all DFT results of signals $\{x_m[i]\}_{m=1}^{M}$, $\{s_k[i]\}_{k=1}^{K}$, and additive noise $\{n_m[i]\}_{m=1}^{M}$, respectively, in the $q$-th group. $\boldsymbol{A}_l(\boldsymbol{\theta}) = [\boldsymbol{a}_l(\theta_1), \boldsymbol{a}_l(\theta_2), \cdots, \boldsymbol{a}_l(\theta_K)]$, and

$$\boldsymbol{a}_l(\theta_k) = \left[\mathrm{e}^{-\mathrm{j}2\pi\frac{z_1 d}{\lambda(f_l)}\sin\theta_k}, \cdots, \mathrm{e}^{-\mathrm{j}2\pi\frac{z_M d}{\lambda(f_l)}\sin\theta_k}\right]^T,$$

where $\{\cdot\}^T$ denotes the transpose operation, and $\lambda(f_l) = c/f_l$ with $c$ representing the wave speed and $f_l$ denoting the central frequency at the $l$-th frequency bin.

As mentioned in Section I, we focus on UMA instead of CMA in this paper. Denote the covariance matrices of $\{\boldsymbol{X}_l(q)\}_{q=1}^{Q}$, $\{\boldsymbol{S}_l(q)\}_{q=1}^{Q}$, and $\{\boldsymbol{N}_l(q)\}_{q=1}^{Q}$ as $\boldsymbol{R}_{\boldsymbol{X}}(l)$, $\boldsymbol{R}_{\boldsymbol{S}}(l)$, and $\boldsymbol{R}_{\boldsymbol{N}}(l)$, respectively. Since the sources are uncorrelated, we have $\boldsymbol{R}_{\boldsymbol{S}}(l) = \text{blkdiag}[p_1(l), p_2(l), \cdots, p_k(l)]$, where $\text{blkdiag}(\cdot)$ is the block diagonalizing operation and $\{p_k(l)\}_{k=1}^{K}$ denotes the source power at the $l$-th frequency bin. By **A1**, we know that $\{\boldsymbol{R}_{\boldsymbol{N}}(l) = p_N(l)\boldsymbol{I}_M\}_{l=1}^{L}$, where $\{p_N(l)\}_{l=1}^{L}$ denotes the noise power at the $l$-th frequency bin, and $\boldsymbol{I}_M$ denotes the $M$-by-$M$ identity matrix. Furthermore, the source signals are assumed to be zero-mean and stationary, and hence $\{\boldsymbol{X}_l(q)\}_{q=1}^{Q}$ are independent and identically distributed $M$-variate circularly-symmetric Gaussian random vectors with zero-mean. We have

$$\boldsymbol{R}_{\boldsymbol{X}}(l) = \boldsymbol{A}_l(\boldsymbol{\theta})\boldsymbol{R}_{\boldsymbol{S}}(l)\boldsymbol{A}_l^H(\boldsymbol{\theta}) + \boldsymbol{R}_{\boldsymbol{N}}(l), \tag{1}$$

where $\{\cdot\}^H$ denotes the Hermitian transpose operation. If $\Delta t$ is chosen to be large enough as **A2**, then $\{\boldsymbol{X}_l(q)\}_{l=1}^{L}$ are asymptotically uncorrelated across frequency bins.

## III. Derivation and Analysis of the Wideband CRB

### A. Closed-Form Wideband CRB Expression

We are dealing with an overall data vector $\bar{\boldsymbol{\xi}}$ that incorporates $L$ data vectors $\{\boldsymbol{\xi}_l\}_{l=1}^{L}$ from $L$ frequency bins, and each $\boldsymbol{\xi}_l$ contains $Q$ snapshots. If only a certain part of frequency bins are of interest, we can simply remove the uninterested $\boldsymbol{\xi}_l$ from $\bar{\boldsymbol{\xi}}$ and then follow the derivation that comes afterwards. Denote the overall data vector as $\bar{\boldsymbol{\xi}} = \left[\boldsymbol{\xi}_1^T, \boldsymbol{\xi}_2^T, \cdots, \boldsymbol{\xi}_L^T\right]^T$, where $\boldsymbol{\xi}_l = \left[\boldsymbol{X}_l^T(1), \boldsymbol{X}_l^T(2), \cdots, \boldsymbol{X}_l^T(Q)\right]^T$. To proceed with notational convenience, we use $\{\phi_k = \sin\theta_k\}_{k=1}^{K}$ to replace the original DOAs $\{\theta_k\}_{k=1}^{K}$ to be estimated. Besides, we use $\boldsymbol{A}_l$ to represent $\boldsymbol{A}_l(\bar{\boldsymbol{\phi}})$, where $\bar{\boldsymbol{\phi}} = [\phi_1, \phi_2, \cdots, \phi_k]^T$. The received data $\bar{\boldsymbol{\xi}}$ is a function of a $K+KL+L$ dimensional unknown parameter vector

$$\boldsymbol{\alpha} = \left[\bar{\boldsymbol{\phi}}^T, \bar{\boldsymbol{p}}^T, \bar{\boldsymbol{p}}_{\boldsymbol{N}}^T\right]^T,$$

where

$$\bar{\boldsymbol{p}} = \left[\boldsymbol{p}_1^T, \boldsymbol{p}_2^T, \cdots, \boldsymbol{p}_L^T\right]^T,$$
$$\boldsymbol{p}_l = [p_1(l), p_2(l), \cdots, p_K(l)]^T,$$
$$\bar{\boldsymbol{p}}_{\boldsymbol{N}} = [p_N(1), p_N(2), \cdots, p_N(L)]^T.$$

According to the signal model, $\bar{\boldsymbol{\xi}}$ follows a complex normal distribution such that $\bar{\boldsymbol{\xi}} \sim \mathcal{CN}\left[\boldsymbol{0}, \bar{\boldsymbol{\Gamma}}(\boldsymbol{\alpha})\right]$, where the whole covariance matrix is expressed as

$$\bar{\boldsymbol{\Gamma}}(\boldsymbol{\alpha}) = \text{blkdiag}\left[\boldsymbol{\Gamma}_1(\boldsymbol{\alpha}), \cdots, \boldsymbol{\Gamma}_L(\boldsymbol{\alpha})\right],$$

where $\boldsymbol{\Gamma}_l(\boldsymbol{\alpha}) = \boldsymbol{I}_Q \otimes \boldsymbol{R}_{\boldsymbol{X}}(l)$. Denote the CRB matrix as $\boldsymbol{B}(\boldsymbol{\alpha})$. Assume the Fisher information matrix (FIM) is invertible, and then the $(i,j)$-th element of the FIM is given by [4], [31], [32]

$$\left[\boldsymbol{B}^{-1}(\boldsymbol{\alpha})\right]_{i,j} = Q\sum_{l=1}^{L}\left\{\left[\boldsymbol{R}_{\boldsymbol{X}}^T(l)\otimes\boldsymbol{R}_{\boldsymbol{X}}(l)\right]^{-\frac{1}{2}}\frac{\partial\boldsymbol{r}_{\boldsymbol{X}}(l)}{\partial[\boldsymbol{\alpha}]_i}\right\}^H \cdot\left\{\left[\boldsymbol{R}_{\boldsymbol{X}}^T(l)\otimes\boldsymbol{R}_{\boldsymbol{X}}(l)\right]^{-\frac{1}{2}}\frac{\partial\boldsymbol{r}_{\boldsymbol{X}}(l)}{\partial[\boldsymbol{\alpha}]_j}\right\}, \tag{2}$$

where $\partial f(\alpha)/\partial\alpha$ is the partial derivative of a function $f(\alpha)$ with respect to the variable $\alpha$, and $\otimes$ denotes the Kronecker product. $\boldsymbol{r}_{\boldsymbol{X}}(l) = \text{vec}\left[\boldsymbol{R}_{\boldsymbol{X}}(l)\right]$, where $\text{vec}(\cdot)$ is the vectorization operation. According to (1), $\boldsymbol{r}_{\boldsymbol{X}}(l)$ can be expressed as

$$\boldsymbol{r}_{\boldsymbol{X}}(l) = \boldsymbol{A}_d(l)\boldsymbol{p}_l + p_N(l)\boldsymbol{i}_{M^2}, \tag{3}$$

where $\boldsymbol{A}_d(l) = \boldsymbol{A}_l^* \odot \boldsymbol{A}_l$, $\boldsymbol{i}_{M^2} = \text{vec}(\boldsymbol{I}_M)$, and $\odot$ denotes the Khatri-Rao product. Computing the derivatives of $\boldsymbol{r}_{\boldsymbol{X}}(l)$ with respect to $\boldsymbol{\alpha}^T$ yields

$$\frac{\partial\boldsymbol{r}_{\boldsymbol{X}}(l)}{\partial\bar{\boldsymbol{\phi}}^T} = \boldsymbol{A'}_d(l)\boldsymbol{R}_{\boldsymbol{S}}(l), \tag{4a}$$

$$\frac{\partial\boldsymbol{r}_{\boldsymbol{X}}(l)}{\partial\bar{\boldsymbol{p}}^T} = [\boldsymbol{0}, \cdots, \underbrace{\boldsymbol{A}_d(l)}_{\text{the } l\text{-th block}}, \cdots, \boldsymbol{0}], \tag{4b}$$

$$\frac{\partial\boldsymbol{r}_{\boldsymbol{X}}(l)}{\partial\bar{\boldsymbol{p}}_{\boldsymbol{N}}^T} = [\boldsymbol{0}, \cdots, \underbrace{\boldsymbol{i}_{M^2}}_{\text{the } l\text{-th column}}, \cdots, \boldsymbol{0}], \tag{4c}$$

where

$$\boldsymbol{A'}_d(l) = \boldsymbol{A'}_l^* \odot \boldsymbol{A}_l + \boldsymbol{A}_l^* \odot \boldsymbol{A'}_l, \tag{5}$$

$$\boldsymbol{A'}_l = \left[\boldsymbol{a'}_l(1), \boldsymbol{a'}_l(2), \cdots, \boldsymbol{a'}_l(K)\right], \tag{6}$$

$$\boldsymbol{a'}_l(k) = \frac{\partial\boldsymbol{a}_l(\phi_k)}{\partial\phi_k}, \tag{7}$$

and $\{\cdot\}^*$ is the conjugate operation. Substituting (4) into (2), we obtain

$$\boldsymbol{B}^{-1}(\boldsymbol{\alpha}) = Q\begin{bmatrix}\bar{\boldsymbol{G}}_{\bar{\boldsymbol{\phi}}}^H \\ \tilde{\boldsymbol{G}}_{\boldsymbol{p}}^H\end{bmatrix}\left[\bar{\boldsymbol{G}}_{\bar{\boldsymbol{\phi}}}, \tilde{\boldsymbol{G}}_{\boldsymbol{p}}\right], \tag{8}$$

where

$$\bar{\boldsymbol{G}}_{\bar{\boldsymbol{\phi}}} = \tilde{\boldsymbol{W}}\tilde{\boldsymbol{A}}'_d\bar{\boldsymbol{R}}_{\boldsymbol{S}}, \tag{9a}$$

$$\tilde{\boldsymbol{G}}_{\boldsymbol{p}} = \tilde{\boldsymbol{W}}\left[\tilde{\boldsymbol{A}}_d, \tilde{\boldsymbol{i}}\right], \tag{9b}$$

$$\begin{aligned}
\tilde{\boldsymbol{W}} &= \text{blkdiag}\left(\boldsymbol{W}_1, \boldsymbol{W}_2, \cdots, \boldsymbol{W}_L\right), \\
\tilde{\boldsymbol{A}}_d &= \text{blkdiag}\left[\boldsymbol{A}_d\left(1\right), \boldsymbol{A}_d\left(2\right), \cdots, \boldsymbol{A}_d\left(L\right)\right], \qquad (9c) \\
\tilde{\boldsymbol{A}}'_d &= \text{blkdiag}\left[\boldsymbol{A}'_d\left(1\right), \boldsymbol{A}'_d\left(2\right), \cdots, \boldsymbol{A}'_d\left(L\right)\right], \\
\bar{\boldsymbol{R}}_{\boldsymbol{S}} &= \left[\boldsymbol{R}_{\boldsymbol{S}}^T\left(1\right), \boldsymbol{R}_{\boldsymbol{S}}^T\left(2\right), \cdots, \boldsymbol{R}_{\boldsymbol{S}}^T\left(L\right)\right]^T, \\
\tilde{\boldsymbol{i}} &= \boldsymbol{I}_L \otimes \boldsymbol{i}_{M^2}, \\
\boldsymbol{W}_l &= \left[\boldsymbol{R}_{\boldsymbol{X}}^T\left(l\right) \otimes \boldsymbol{R}_{\boldsymbol{X}}\left(l\right)\right]^{-\frac{1}{2}}.
\end{aligned}$$

We are only interested in the principal sub-matrix in $\boldsymbol{B}\left(\boldsymbol{\alpha}\right)$ corresponding to normalized DOAs, denoted by $\boldsymbol{B}\left(\bar{\boldsymbol{\phi}}\right)$. Following the inversion of a partitioned matrix [31], we obtain

$$\boldsymbol{B}\left(\bar{\boldsymbol{\phi}}\right) = \left(Q\bar{\boldsymbol{G}}_{\bar{\boldsymbol{\phi}}}^H \boldsymbol{\Pi}_{\tilde{\boldsymbol{G}}_{\boldsymbol{p}}}^{\perp} \bar{\boldsymbol{G}}_{\bar{\boldsymbol{\phi}}}\right)^{-1}, \tag{10}$$

where $\boldsymbol{\Pi}_{\tilde{\boldsymbol{G}}_{\boldsymbol{p}}}^{\perp} = \boldsymbol{I}_{M^2L} - \tilde{\boldsymbol{G}}_{\boldsymbol{p}}\left(\tilde{\boldsymbol{G}}_{\boldsymbol{p}}^H\tilde{\boldsymbol{G}}_{\boldsymbol{p}}\right)^{-1}\tilde{\boldsymbol{G}}_{\boldsymbol{p}}^H$ is the orthogonal projector onto the null space of $\tilde{\boldsymbol{G}}_{\boldsymbol{p}}^H$. Furthermore, we can rewrite $\tilde{\boldsymbol{G}}_{\boldsymbol{p}}$ as $\tilde{\boldsymbol{G}}_{\boldsymbol{p}} = \left[\tilde{\boldsymbol{V}}, \tilde{\boldsymbol{u}}\right]$, where

$$\begin{aligned}
\tilde{\boldsymbol{V}} &= \text{blkdiag}\left(\boldsymbol{V}_1, \boldsymbol{V}_2, \cdots, \boldsymbol{V}_L\right) = \tilde{\boldsymbol{W}}\tilde{\boldsymbol{A}}_d, \\
\tilde{\boldsymbol{u}} &= \text{blkdiag}\left(\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_L\right) = \tilde{\boldsymbol{W}}\tilde{\boldsymbol{i}}, \\
\boldsymbol{V}_l &= \boldsymbol{W}_l\boldsymbol{A}_d\left(l\right), \ \ \boldsymbol{u}_l = \boldsymbol{W}_l\boldsymbol{i}_{M^2}.
\end{aligned}$$

Since $\tilde{\boldsymbol{G}}_{\boldsymbol{p}}$ shares the same null space with $\left[\tilde{\boldsymbol{V}}, \boldsymbol{\Pi}_{\tilde{\boldsymbol{V}}}^{\perp}\tilde{\boldsymbol{u}}\right]$, we have

$$\begin{aligned}
\boldsymbol{\Pi}_{\tilde{\boldsymbol{G}}_{\boldsymbol{p}}}^{\perp} &= \boldsymbol{I}_{M^2L} - \left[\tilde{\boldsymbol{V}}, \boldsymbol{\Pi}_{\tilde{\boldsymbol{V}}}^{\perp}\tilde{\boldsymbol{u}}\right]\begin{bmatrix}\tilde{\boldsymbol{V}}^H\tilde{\boldsymbol{V}} & \boldsymbol{0} \\ \boldsymbol{0} & \tilde{\boldsymbol{u}}^H\boldsymbol{\Pi}_{\tilde{\boldsymbol{V}}}^{\perp}\tilde{\boldsymbol{u}}\end{bmatrix}^{-1}\begin{bmatrix}\tilde{\boldsymbol{V}}^H \\ \tilde{\boldsymbol{u}}^H\boldsymbol{\Pi}_{\tilde{\boldsymbol{V}}}^{\perp}\end{bmatrix} \\
&= \boldsymbol{\Pi}_{\tilde{\boldsymbol{V}}}^{\perp} - \boldsymbol{\Pi}_{\tilde{\boldsymbol{V}}}^{\perp}\tilde{\boldsymbol{u}}\left(\tilde{\boldsymbol{u}}^H\boldsymbol{\Pi}_{\tilde{\boldsymbol{V}}}^{\perp}\tilde{\boldsymbol{u}}\right)^{-1}\tilde{\boldsymbol{u}}^H\boldsymbol{\Pi}_{\tilde{\boldsymbol{V}}}^{\perp}. \qquad (11)
\end{aligned}$$

Notice that

$$\boldsymbol{\Pi}_{\tilde{\boldsymbol{V}}}^{\perp} = \text{blkdiag}\left(\boldsymbol{\Pi}_{\boldsymbol{V}_1}^{\perp}, \cdots, \boldsymbol{\Pi}_{\boldsymbol{V}_L}^{\perp}\right), \tag{12a}$$

$$\tilde{\boldsymbol{u}}^H\boldsymbol{\Pi}_{\tilde{\boldsymbol{V}}}^{\perp}\tilde{\boldsymbol{u}} = \text{blkdiag}\left(\boldsymbol{u}_1^H\boldsymbol{\Pi}_{\boldsymbol{V}_1}^{\perp}\boldsymbol{u}_1, \cdots, \boldsymbol{u}_L^H\boldsymbol{\Pi}_{\boldsymbol{V}_L}^{\perp}\boldsymbol{u}_L\right) \tag{12b}$$

Substituting (12) into (11) yields

$$\boldsymbol{\Pi}_{\tilde{\boldsymbol{G}}_{\boldsymbol{p}}}^{\perp} = \text{blkdiag}\left[\boldsymbol{\Pi}_{\boldsymbol{G}_{\boldsymbol{p}}(1)}^{\perp}, \boldsymbol{\Pi}_{\boldsymbol{G}_{\boldsymbol{p}}(2)}^{\perp}, \cdots, \boldsymbol{\Pi}_{\boldsymbol{G}_{\boldsymbol{p}}(L)}^{\perp}\right], \tag{13}$$

where

$$\boldsymbol{\Pi}_{\boldsymbol{G}_{\boldsymbol{p}}(l)}^{\perp} = \boldsymbol{\Pi}_{\boldsymbol{V}_l}^{\perp} - \frac{\boldsymbol{\Pi}_{\boldsymbol{V}_l}^{\perp}\boldsymbol{u}_l\boldsymbol{u}_l^H\boldsymbol{\Pi}_{\boldsymbol{V}_l}^{\perp}}{\boldsymbol{u}_l^H\boldsymbol{\Pi}_{\boldsymbol{V}_l}^{\perp}\boldsymbol{u}_l}.$$

Note that $\boldsymbol{\Pi}_{\boldsymbol{G}_{\boldsymbol{p}}(l)}^{\perp}$ is the orthogonal projector onto the null space of $\boldsymbol{G}_{\boldsymbol{p}}^H\left(l\right)$, where $\boldsymbol{G}_{\boldsymbol{p}}\left(l\right) = \boldsymbol{W}_l\left[\boldsymbol{A}_d\left(l\right), \boldsymbol{i}_{M^2}\right]$. Based on these results, (10) can be transformed into

$$\boldsymbol{B}_{\text{u}}\left(\bar{\boldsymbol{\phi}}\right) = \left[Q\sum_{l=1}^{L}\boldsymbol{G}_{\bar{\boldsymbol{\phi}}}^H\left(l\right)\boldsymbol{\Pi}_{\boldsymbol{G}_{\boldsymbol{p}}(l)}^{\perp}\boldsymbol{G}_{\bar{\boldsymbol{\phi}}}\left(l\right)\right]^{-1}, \tag{14}$$

which is the closed-form wideband CRB expression.

### *B. Rank Condition*

In previous derivations, we simply assume that the FIM is nonsingular, but the specific condition under which the CRB exists is not investigated. In this subsection, we first introduce the definition of the augmented co-array manifold (ACM) matrix, and then clarify the rank condition for the existence of the CRB.

*Definition 1:* The augmented co-array manifold matrix containing $L$ frequency bins is defined as

$$\boldsymbol{\Sigma} \triangleq [\tilde{\boldsymbol{A}}'_d\bar{\boldsymbol{R}}_{\boldsymbol{S}}, \tilde{\boldsymbol{A}}_d, \tilde{\boldsymbol{i}}]. \tag{15}$$

*Theorem 1:* The wideband CRB exists if and only if the ACM matrix $\boldsymbol{\Sigma}$ is full column rank, i.e., if and only if

$$\text{rank}(\boldsymbol{\Sigma}) = K + KL + L. \tag{16}$$

*Proof:* According to (8), (9a), (9b), and (15), we have $\boldsymbol{B}^{-1}\left(\boldsymbol{\alpha}\right) = Q\boldsymbol{\Sigma}^H\tilde{\boldsymbol{W}}^H\tilde{\boldsymbol{W}}\boldsymbol{\Sigma}$. Therefore, it is equivalent to proving that $\boldsymbol{\Sigma}^H\tilde{\boldsymbol{W}}^H\tilde{\boldsymbol{W}}\boldsymbol{\Sigma}$ is positive definite if and only if $\boldsymbol{\Sigma}$ has full column rank. To continue, we introduce a vector $\boldsymbol{g} \in \mathbb{C}^{K+KL+L}$.

*(Sufficiency)* Since $\tilde{\boldsymbol{W}}^H\tilde{\boldsymbol{W}}$ is a positive definite and Hermitian matrix, $\boldsymbol{\Sigma}^H(\tilde{\boldsymbol{W}}^H\tilde{\boldsymbol{W}})\boldsymbol{\Sigma}$ is also a Hermitian matrix. If $\boldsymbol{\Sigma}$ is of full column rank, then $\boldsymbol{\Sigma}\boldsymbol{g} = \boldsymbol{0}$ if and only if $\boldsymbol{g} = \boldsymbol{0}$. For any $\boldsymbol{g} \neq \boldsymbol{0}$, we have $\boldsymbol{\Sigma}\boldsymbol{g} \neq \boldsymbol{0}$. Hence $(\boldsymbol{\Sigma}\boldsymbol{g})^H(\tilde{\boldsymbol{W}}^H\tilde{\boldsymbol{W}})(\boldsymbol{\Sigma}\boldsymbol{g}) > 0$, i.e., $\boldsymbol{g}^H(\boldsymbol{\Sigma}^H\tilde{\boldsymbol{W}}^H\tilde{\boldsymbol{W}}\boldsymbol{\Sigma})\boldsymbol{g} > 0$, which means $\boldsymbol{\Sigma}^H\tilde{\boldsymbol{W}}^H\tilde{\boldsymbol{W}}\boldsymbol{\Sigma}$ is positive definite.

*(Necessity)* If $\boldsymbol{\Sigma}^H\tilde{\boldsymbol{W}}^H\tilde{\boldsymbol{W}}\boldsymbol{\Sigma}$ is positive definite, then for any $\boldsymbol{g} \neq \boldsymbol{0}$, we have $(\boldsymbol{\Sigma}\boldsymbol{g})^H(\tilde{\boldsymbol{W}}^H\tilde{\boldsymbol{W}})(\boldsymbol{\Sigma}\boldsymbol{g}) > 0$. Since $\tilde{\boldsymbol{W}}^H\tilde{\boldsymbol{W}}$ is positive definite, we know that $\boldsymbol{\Sigma}\boldsymbol{g} \neq \boldsymbol{0}$. As a result, $\boldsymbol{\Sigma}\boldsymbol{g} = \boldsymbol{0}$ if and only if $\boldsymbol{g} = \boldsymbol{0}$, indicating $\boldsymbol{\Sigma}$ has full column rank.

With these statements, the whole proof is completed. ■

### *C. Resolution Capacity*

Consider an $M$-sensor linear array $\mathbb{S}$, whose difference set is denoted by $\mathbb{D}$. In the narrowband case, the maximum number of resolvable uncorrelated sources $K$ is bounded by $K \leq \frac{|\mathbb{D}|-1}{2}$ [12], [13], where $|\mathbb{D}|$ is the cardinality of $\mathbb{D}$. In the wideband case, however, we have the following proposition concerning the resolution capacity:

*Proposition 1:* Assume that $K \leq \min\{|\mathbb{D}|, \frac{L(M^2-1)}{L+1}\}$. It is possible to identify $K > \frac{|\mathbb{D}|-1}{2}$ sources under the wideband model, which exceeds the limitation in the narrowband case.

*Proof:* For a start, we shall explain the upper bound in this proposition. Since $\boldsymbol{\Sigma}$ has a size of $M^2L$-by-$(K + KL + L)$, the rank condition in *Theorem 1* requires $K \leq \frac{L(M^2-1)}{L+1}$. If we apply the concept of ACM matrix to each frequency bin, we can obtain a group of sub-band ACM matrices:

$$\boldsymbol{\Sigma}_l = \left[\boldsymbol{A}'_d\left(l\right)\boldsymbol{R}_{\boldsymbol{S}}\left(l\right), \boldsymbol{A}_d\left(l\right), \boldsymbol{i}_{M^2}\right], l = 1, 2, \cdots, L.$$

Note that $\boldsymbol{\Sigma}_l$ has a dimension of $M^2$-by-$(2K+1)$, while only $|\mathbb{D}|$ rows in $\boldsymbol{A}'_d\left(l\right)$ and $\boldsymbol{A}_d\left(l\right)$ are linearly independent [12], [13]. A necessary condition for $\boldsymbol{\Sigma}$ to be of full column rank is that its sub-matrix $\tilde{\boldsymbol{A}}_d$ should have full column rank. According to (9c), it is equivalent to $\{\boldsymbol{A}_d\left(l\right)\}_{l=1}^{L}$ all having full column rank, which requires $K \leq |\mathbb{D}|$. On the other hand, $\boldsymbol{A}_d\left(l\right)$

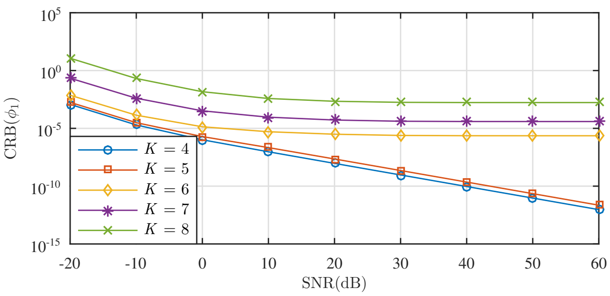


Fig. 1. Wideband CRB versus SNR for different $K$.

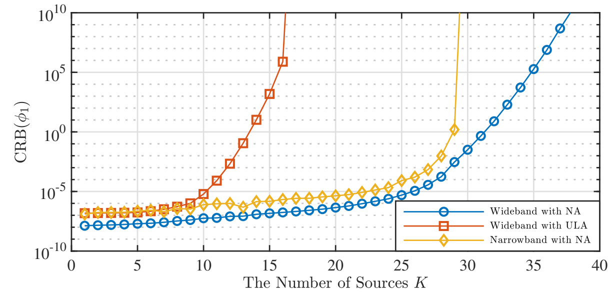


Fig. 2. Wideband CRB versus $K$.

can be treated as the equivalent array manifold matrix linked to $\mathbb{D}$, whose columns are always linearly independent when $K \leq |\mathbb{D}|$. Hence, we assume $K \leq \min\{|\mathbb{D}|, \frac{L(M^2-1)}{L+1}\}$ in the proposition, so that $\boldsymbol{A'}_d(l)\boldsymbol{R_S}(l)$, $\boldsymbol{A}_d(l)$, $\boldsymbol{i}_{M^2}$, $\tilde{\boldsymbol{A}}'_d\bar{\boldsymbol{R}}_{\boldsymbol{S}}$, $\tilde{\boldsymbol{A}}_d$, and $\tilde{\boldsymbol{i}}$ are all of full column rank.

The narrow sub-band property will be definitely inherited directly by the wideband scenario. However, based on the introduced ACM matrix with a larger dimension, we discuss the possibility of increased resolvable source number. Some columns in $\boldsymbol{A'}_d(l)\boldsymbol{R_S}(l)$ and $\boldsymbol{A}_d(l)$ become linearly dependent when $K > \frac{|\mathbb{D}|-1}{2}$. The second sub-matrix in $\boldsymbol{\Sigma}$, namely, $\tilde{\boldsymbol{A}}_d$, is an $M^2L$-by-$KL$ matrix holding all sub-band components on its diagonal. In contrast, the first sub-matrix $\tilde{\boldsymbol{A}}'_d\bar{\boldsymbol{R}}_{\boldsymbol{S}}$ stacks the sub-band components following the column direction, and the number of rows are extended to $M^2L$. Consequently, the number of linearly independent rows in $\boldsymbol{\Sigma}$ will possibly approach $|\mathbb{D}|\,L$, and the linear dependence between the columns in $\{\boldsymbol{A'}_d(l)\boldsymbol{R_S}(l)\}_{l=1}^{L}$ and $\{\boldsymbol{A}_d(l)\}_{l=1}^{L}$ might be eliminated. Therefore, $\boldsymbol{\Sigma}$ might have full column rank for $K > \frac{|\mathbb{D}|-1}{2}$, which completes the proof. ■

*Remark 1:* It has been demonstrated in [28] that the number of resolvable sources is smaller than the sensor number without prior knowledge. Nonetheless, *Proposition 1* indicates that if the sources are known *a priori* to be uncorrelated, it is feasible to identify more sources than the number of sensors by the division of a group of frequency bins, while no special array structure is required. This will be verified by numerical results in Section IV.

## IV. Simulations

In all simulations, we set the sources to be uncorrelated and examine the CRB for the first source $\phi_1$. The number of DFT points is $L = 64$, and the frequency bins of interest cover from 17 to 32. The $K$ sources are uniformly distributed between $-60°$ and $60°$ and have equal powers in each frequency bin. The unit spacing between two sensors is half of the minimum signal-of-interest wavelength with $d = \frac{\lambda_{\min}}{2}$. The central frequency of the $l$-th frequency bin is $f_l = \frac{f_s(l-1)}{L}$, and the number of snapshots is $Q = 500$.

We first focus on the dependence of the CRB on SNR in both overdetermined and determined/underdetermined cases, which is shown in Fig. 1. We use a 6-sensor ULA whose sensor positions are given by $\mathbb{S}_{\text{ULA}} = \{1, 2, 3, 4, 5, 6\}$, and set the number of sources $K$ to vary from 4 to 8. We can see that the CRB decreases monotonically with the increase of SNR in the overdetermined case ($K = 4, 5$). When SNR exceeds 0 dB, the two curves show an inverse logarithmic dependence on SNR. However, in the determined/underdetermined case ($K = 6, 7, 8$), the CRB tends to a constant above some certain SNR threshold values. Note that the wideband CRB exists in the determined/undetermined regime even if a spare array structure is not employed, which verifies *Proposition 1*.

Next, we evaluate the resolution capacity for the wideband model with two array structures. We use a 10-sensor ULA with $\mathbb{S}'_{\text{ULA}} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and a 10-sensor NA with $\mathbb{S}_{\text{nested}} = \{1, 2, 3, 4, 5, 6, 12, 18, 24, 30\}$ [6], whose difference co-array is $\mathbb{D}_{\text{nested}} = \{0, \pm 1, \cdots, \pm 29\}$. We keep SNR at 0 dB and let the number of sources $K$ vary from 1 to 40. As plotted in Fig. 2, the wideband CRB with ULA diverges at $K = 17$, while the wideband CRB with NA stays lower than $10^1$ in the region $K \leq 32$. This not only verifies *Proposition 1*, but also indicates that the resolution capacity is further improved with the assistance of sparse arrays. Moreover, compared with the narrowband CRB curve corresponding to the 32nd frequency bin, the wideband CRB exceed the bound of $K \leq 29$, which validates *Proposition 1*.

## V. Conclusion

The closed-form CRB expression for DOA estimation for wideband uncorrelated sources has been derived. The existence of the CRB was proved by the rank condition of the introduced ACM matrix, and the wideband resolution capacity was then discussed. If the sources are known *a priori* to be uncorrelated, the wideband model is capable of identifying more sources than the number of physical sensors without the assistance of sparse array structures, which overcomes the limitation on the resolution capacity inherited based on individual narrowband frequency bins. Finally, it has been verified by simulations that the derived closed-form CRB expression is valid in both overdetermined and underdetermined cases with more sources being resolved based on a ULA. It has also been shown by simulations that the maximum number of resolvable sources exceeds that employing a single narrowband frequency bin with a lower bound achieved.

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