PRODUCTS AND KUNNETH THEOREMS IN CYCLIC HOMOLOGY AND COHOMOLOGY THEORIES

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Contents.

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"Lord, you establish peace for us,
all that we have accomplished you have done for us"

Isaiah 26:12.

Summary.

This thesis is concerned with the construction of products in cyclic homology and cohomology, by use of the methods of acyclic models. A theory $HC^-_*(A)$ which is dual to cyclic cohomology $HC^*(A)$ over its natural coefficients is introduced, and products are defined in $HC^-_*(A)$ and $HC^*(A)$. The product then induced on cyclic homology $HC_*(A)$ is shown to agree with that defined by Loday and Quillen. By using $HC^-_*(A)$, it is possible to construct a multiplicative chern character $ch : K_i(A) \to HC^-_{i-2\ell}(A)$. Kunneth theorems in the various theories are proved, and some examples are considered.

Introduction

This introduction will take the form of an outline of the thesis, describing its subject matter and its purpose, followed by a discussion of cyclic homology theory, outlining the motivation for its definition, and some of its applications. The content of this second part is taken from the published work of Connes and others, and does not contain any material of my own: it is intended to provide a context for the work following.

This thesis is concerned with the cyclic homology and cohomology theories defined by Alain Connes, and in particular with products in those theories. For an associative algebra A over a field F, the cyclic homology $HC_*(A)$ and cyclic cohomology $HC^*(A)$ may be defined; we give the definitions in Chapter 1. Cyclic cohomology has a periodicity operator $HC^n(A) \to HC^{n+2}(A)$, and so $HC^*(A)$ becomes a module over the polynomial ring $F[\theta]$, where θ is an indeterminate of degree 2, whose action on $HC^*(A)$ is defined by the periodicity operator.

Taking into account the product in cohomology, $HC^*(F)$ is isomorphic to $F[\theta]$, and the module action defined by the periodicity operator agrees with the action defined by the product of $HC^*(F)$ with $HC^*(A)$. Thus it seems reasonable to consider $F[\theta]$ as the natural coefficients for cyclic cohomology. $HC_*(A)$ is the dual of $HC^*(A)$ over F, but not over these coefficients $F[\theta]$. Thus we introduce a further theory $HC^-_*(A)$, which is the $F[\theta]$-dual of $HC^*(A)$.

We wish to construct products by a version of the method of acyclic models. In order to obtain suitable models, we need to look outside the category of algebras. We give the definition of cyclic and cocyclic F-modules, and then the cyclic homology of any cyclic F-module, and the cyclic cohomology of any cocyclic F-module, may be defined.

In Chapter 2, we construct an $F[\theta]$-module product map $HC^-_*(A) \otimes_{F[\theta]} HC^-_*(B) \to HC^-_*(A \otimes B)$, and an $F[\theta]$-module product map $HC^*(A) \otimes_{F[\theta]} HC^*(B) \to HC^*(A \otimes B)$. Further use of the acyclic models method allows us to prove the commutativity and associativity of the products, and an appropriate form of uniqueness. This allows us to identify any product in cohomology constructed by this method with the product defined by Connes in [8]. Loday and Quillen in [23] construct a product in cyclic homology, $HC_n(A) \otimes HC_m(B) \to HC_{n+m+1}(A \otimes B)$. We will show that the relation between $HC^-_*(A)$ and $HC_*(A)$ is analogous to the relation between homology with coefficients in Z and homology with coefficients in $\mathbb{Q}/Z$. Then the product in $\mathbb{Q}/Z$ homology which is induced by the product in Z homology will provide an analogy for the definition of a product on $HC_*(A)$ which is induced by the product on $HC^-_*(\ )$. This product agrees with that given by Loday and Quillen.

In Chapter 3, we discuss the construction of a multiplicative chern character, $ch: K_i(A) \to HC^-_{i+2\ell}(A)$.

In Chapter 4, we prove a variety of Kunneth theorems, by showing that the products which we have constructed give natural chain equivalences of complexes. The Kunneth theorem for $HC_*(A)$ involves more work, since the complex which defines cyclic homology is not a free $F[\theta]$-module.

In Chapter 5, we use the Kunneth theorems and the properties of the products to calculate the cyclic cohomology of a few examples.

We now summarise some of the recent work on cyclic homology, in order to provide a context for the work following. Details are not given, but may be found in the references cited.

Cyclic cohomology was introduced by Alain Connes in 1982 [7,8]. Geometric situations in which his work applies are the actions of a group G on a smooth manifold M, where G may be an infinite discrete group or a Lie group, and a foliation F on a smooth manifold V. These two examples are closely related, since given a foliation, its leaves can be considered as the orbits under the action of a groupoid defined from the foliation, called the holonomy groupoid [6]: a groupoid is a set G with an inverse map defined on G, but with a product map defined only on a certain subset, the composable pairs, of $G \times G$, where the inverse and product maps satisfy the usual conditions.

In the case where the orbit space M/G is locally compact and Hausdorff, the action can be studied via the space M/G, and also via the algebra $C_0(M/G)$ of continuous functions on M/G which tend to zero at infinity. Even in the case where the orbit space does not have the above property, it can be studied via a C^* algebra, written $C_0(M) \times G$ [24]. Connes defines the C^* algebra of a foliation (V,F), which is written $C^*(V,F)$ [6].

The use of C^* algebras is a generalisation of the use of continuous function algebras, since for any commutative C^* algebra A there is a locally compact Hausdorff space $\hat{A}$ such that A is isomorphic to $C_0(\hat{A})$ by a map which preserves the norm and is a $*$-homomorphism for the involution.

In general $C^*(V,F)$ and $C_0(M) \times G$ are noncommutative. However, if the orbit space is locally compact and Hausdorff, the algebra $C_0(M) \times G$ is equivalent to $C_0(M/G)$ in an appropriate sense, that is, the algebras are Morita equivalent. Similarly, if the leaf space V/F is locally compact and Hausdorff, the algebra $C^*(V,F)$ is Morita equivalent to $C_0(V/F)$.

An example of a foliation whose leaf space is not Hausdorff is given by the foliation of a torus by lines of irrational slope θ : since every leaf is dense, the only open sets in the leaf space are the whole set and the empty set, and the leaf space is thus non-Hausdorff.

The C^* algebra of this foliation has as its elements the formal power series $\Sigma\, a_{n,m} U_1^n U_2^m$, where the indeterminates U_1 and U_2 are related by $U_2U_1 = e^{2\pi i\theta} U_2U_1$, and the coefficients $a_{n,m}$ satisfy the condition that $(|n| + |m|)^q\, |a_{n,m}|$ is bounded for each $q \in \mathbb{N}$ [8].

Having obtained the algebras $C_0(M) \times G$ and $C^*(V,F)$, the aim is to use them as we use $C_0(M/G)$ and $C_0(V/F)$, when the orbit space and leaf space are locally compact and Hausdorff, to study the geometry of the situation, for example, to obtain topological invariants. K theory is defined for both algebras and spaces, and has the important property that for a locally compact Hausdorff space X , $K_*(C_0(X)) \cong K^*(X)$; the lower star in $K_*(C_0(X))$ is justified because it is a covariant functor of the algebra $C_0(X)$. Thus for an algebra A , it is $K_*(A)$ which is analogous to the K-cohomology theory of a space, so we consider $K_*(C^*(V,F))$ and $K_*(C_0(M) \times G)$. If two algebras are Morita equivalent, their K-theories are isomorphic; hence if the orbit space is locally compact and Hausdorff, $K_*(C_0(M) \times G) \cong K_*(C_0(M/G)) \cong K^*(M/G)$. Similarly, if the leaf space is locally compact and Hausdorff, $K_*(C^*(V,F) \cong K_*(C_0(V/F)) \cong K^*(V/F)$.

Given a pseudo-differential elliptic operator P on a compact manifold, its analytical index is defined to be the integer given by dimension (kernel P) - dimension (cokernel P) . A topological index may also be defined, and the Atiyah-Singer index theorem proves the equality of these two indices [1]. It is K theory which provides the formulation for index theorems.

Connes and Skandalis in [11] generalise to foliations the Atiyah-Singer index theorem for families [2]. Atiyah and Singer consider families of pseudo-differential elliptic operators on compact manifolds X_y , continuously parametrised by the points of a space Y , and prove the equality of an analytical index and a topological index, both defined in $K^*(Y)$. The definition of the analytical index follows from reducing to a case where the vector spaces $\ker P_y$ and $\operatorname{coker} P_y$ are constant in dimension as y varies, giving vector bundles $\ker P$ and $\operatorname{coker} P$ over Y , and thus the element $[\ker P - \operatorname{coker} P] \in K^*(Y)$. Connes and Skandalis's theorem holds for those pseudo-differential operators on a foliated manifold which are elliptic in the leaf direction, and can thus be thought of as families parametrised by the points of the leaf space. They show how to define an analytical and a topological index lying in $K_*(C^*(V,F))$, and prove that the two are equal. One interesting feature of the theorem is that it holds even when the leaves are non-compact.

Usually information about the indices would be obtained by applying the chern character $ch:K^*(Y) \to H^*(Y:\mathbb{Q})$. Thus we would like to apply a chern character to $K_*(C^*(V,F))$: cyclic homology is defined to act as a receiver for the chern character.

For a manifold M, its K-homology $K_*(M)$ is given in terms of equivalence classes of pseudodifferential operators; see [3]. For an algebra A, $K^*(A)$ can analogously be defined in terms of certain operators on graded Hilbert spaces which have an action of A by bounded operators [21,7].

There is a pairing $K_0(A) \otimes K^0(A) \to \mathbb{C}$ which is given, for a class in $K^0(A)$ represented by such an operator P, and a class in $K_0(A)$ represented by an idempotent e in the matrix algebra of A, by

$$\langle [P], [e] \rangle = \text{index } P_e ,$$

where P_e is a further operator constructed from P and e [7]. For example, if $e \in A$, and P is defined on the Hilbert space H with an action of A, then $P_e : eH \to eH$ is defined by $P_e(ex) = eP(ex)$. This is an extension to the general case of the procedure, in the case $A = C_0(X)$, for twisting an operator by a vector bundle. Here, if F_0 and F_1 are vector bundles on X, given D an elliptic pseudo-differential operator on the smooth sections of the bundle, $D : C^\infty(F_0) \to C^\infty(F_1)$, and given a vector bundle E, a choice of connection on $F_\bullet \otimes E$ allows the construction of an operator $D \otimes 1_E : C^\infty(F_0 \otimes E) \to C^\infty(F_1 \otimes E)$ [3].

A trace on A gives a map $K_0(A) \to \mathbb{C}$. However, there are algebras A for which there exist an operator P , representing a class in $K^0(A)$, such that the map $K_0(A) \to \mathbb{C}$ given by $[e] \to \text{Index } P_e$, is not given by a trace. Thus an appropriate generalisation of a trace must be defined. An n-trace is an (n+1)-linear functional on A , or equivalently a linear functional on the tensor product of (n+1) copies of A , $A^{\otimes n+1}$, with the following additional properties:

(a) $$\tau(a^0 \otimes \ldots \otimes a^n) = (-1)^n \tau(a^1 \otimes \ldots \otimes a^n \otimes a^0)$$

(b) $$\tau(a^0 a^1 \otimes \ldots \otimes a^n) - \tau(a^0 \otimes a^1 a^2 \otimes \ldots \otimes a^n) \ldots + (-1)^{n-1} \tau(a^0 \otimes \ldots \otimes a^{n-1} a^n) + (-1)^n \tau(a^n a^0 \otimes \ldots \otimes a^{n-1}) = 0 .$$

These n-traces are then used to define the cyclic cohomology $HC^*(A)$, and Connes defines a pairing $HC^*(A) \otimes K_0(A) \to \mathbb{C}$ [7,8].

When the algebra has a topology, suitable continuity conditions must also be imposed; these are discussed by Connes in [10].

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Cyclic homology and cohomology can be defined for any associative algebra A , and are written $HC_*(A)$ and $HC^*(A)$ respectively.

Cyclic homology has relations with many other areas of mathematics,

including algebraic K-theory and homotopy-type invariants of manifolds. We shall give a few examples.

Lie algebra homology

The algebra of $(r \times r)$ matrices over an algebra A becomes a Lie algebra, denoted $g\ell_r(A)$, by use of the Lie bracket $[x,y] = xy-yx$. Taking the direct limit under the inclusions $g\ell_r(A) \to g\ell_{r+1}(A)$ gives the algebra $g\ell(A)$. Its Lie algebra homology $H_*(g\ell(A))$ is a Hopf algebra, with a comultiplication Δ induced by the diagonal: elements x such that $\Delta(x) = x \otimes 1 + 1 \otimes x$ are called primitives, and form an algebra. Loday and Quillen [23] prove that for an associative algebra A over a field of characteristic zero,

$$HC_{*-1}(A) \cong \mathrm{Prim}\, H_*(g\ell(A)) .$$

Now the homology of the general linear group GL(A) is also a Hopf algebra, and its primitive part is isomorphic to rational algebraic K theory, $K_*(A) \otimes \mathbb{Q}$. Thus the primitive part of $H_*(g\ell(A))$ is called, by analogy, additive algebraic K-theory. Loday and Quillen have investigated a variety of analogies between algebraic K theory and cyclic homology. One such is to consider what form of periodicity relation might exist for algebraic K-theory emulating the Bott periodicity for topological K-theory of Banach algebras: in cyclic homology there is the periodicity operator S .

Waldhausen's A-space

Waldhausen approaches the problem of calculating the homotopy of the space of pseudo-isotopies $\mathcal{P}(M)$ of a smooth compact manifold M , by defining a space $A(M)$ such that $\pi_i(A(M))\otimes\mathbb{Q} \cong (\pi_{i-2}(\mathcal{P}(M)\otimes\mathbb{Q}))\otimes H_i(M:\mathbb{Q})$ [27]. The definitiion of $A(M)$ is arrived at by using algebraic K-theory and the Quillen + -construction. Following a result of Dwyer, Hsiang and Staffeldt [13] which relates the homotopy of $A(M)$ to a lie algebra homology group, Burghelea [4] has combined this with Loday and Quillen's result to prove that

$$\pi_*(A(M) \otimes \mathbb{Q}) \cong HC_*(C_*(\Omega M:\mathbb{Q}))$$

where $C_*(\Omega M:\mathbb{Q})$ is the differential graded algebra of rational chains on the loop space of M , with product given by the Pontryagin product. This work is summarised by Cartier [5].

Equivariant homology

Closely involved in cyclic homology is Connes' category Λ , an extension of the simplicial category Δ ; its definition and the precise relation will be discussed in Chapter 1. Connes shows [9, Theorem 10] that the classifying space of Λ is the classifying space of the circle S^1, that is, $\mathbb{C}P^\infty$. Jones [17] proves a relationship between cyclic homology and S^1 equivariant cohomology.

Given a space X, the circle acts on $Map(S^1,X) = LX$, the free loop space, by considering S^1 as a multiplicative group, T. The equivariant cohomology of a space Z with circle action is given by $H_T^*(Z) = H^*(ET \times_T Z)$, where ET is a contractible space on which T acts freely. The equivariant cohomology groups are modules over H_T^* (point), which is isomorphic to the polynomial ring $K[u]$, where u is an indeterminate of degree 2. Thus by localising the chain complex with respect to u, a theory $\hat{H}_T^*(\ )$ may be constructed.

Let $S^*(X)$ be the singular cochain complex of X: by using the Alexander Whitney product, it can be made into an associative differential graded algebra. The cyclic homology of this algebra, $HC_*(S^*X)$, is then defined. Cyclic homology is a module over $K[u]$, where the action of u is given by the periodicity operator $S:HC_n(A) \to HC_{n-2}(A)$. The chain complex can then be localised with respect to u to give a theory $\hat{H}C_*(X)$.

The result is then: for a simply connected space X,

$$\hat{H}C_{-n}(S^*(X)) \cong \hat{H}_T^n(LX) .$$

If, for example, X is a smooth manifold, these groups are related to the existence of closed geodesics on X.

The same approach gives a proof of a strengthened version of a result of Goodwillie [15]. The theorem applies to an associative topological monoid with unit, G. BG is the classifying space of G. $S_*(G)$, the singular chain complex of G, is an associative differential graded algebra by using the Eilenberg McLane shuffle product $S_*(G) \otimes S_*(G) \to S_*(G \times G)$ and the map induced by the product $G \times G \to G$. Then

$$\hat{HC}_n(S_*(G)) \cong \hat{H}_n^T(LBG) .$$

<u>The Novikov Conjecture</u>

The Novikov conjecture can be reformulated in terms of cyclic homology: the proof of the conjecture would then follow from as yet unproved properties of cyclic homology.

First we state the Novikov conjecture:

Given M a smooth closed oriented manifold of dimension $4k$, we can define its signature to be the signature of the symmetric bilinear form B on rational cohomology given by $B(\alpha,\beta) = \langle \alpha \cup \beta, [M] \rangle$. This signature is the analytical index of the signature operator D_M^+ on M. The topological index can be expressed as $\langle L(M),[M] \rangle$, where $L(M)$

is the total L-genus of M , a homogeneous polynomial in degree 4k in indeterminates $p_j \in H^{4j}(M:\mathbb{Q})$.

Let $G = \pi_1(M)$, let $f:M \to BG$ be the classifying map of the universal covering of M , then the Novikov conjecture states that for all $a \in H^*(BG:\mathbb{Q})$, $\langle L(M) \cup f^*(a),[M]\rangle$ is an invariant of oriented homotopy type.

A chern character may also be defined in homology, $ch : K_*(X) \to H_*(X:\mathbb{Q})$ [3]. Recall that D_M^+ defines a class $[D_M^+] \in K_0(X)$. Kasparov shows, using the index theorem, that $ch[D_M^+]$ is the Poincare dual of the L-genus, which is written $\mathcal{D}L(M)$ [22, §9]. The higher signature is $\langle L(M) \cup f^*(a),[M]\rangle$ $= \langle f^*(a), L(M) \cap [M]\rangle = \langle f^*(a), \mathcal{D}L(M)\rangle = \langle a, f_*\mathcal{D}L(M)\rangle$, so the Novikov conjecture is implied by the homotopy invariance of $f_*(\mathcal{D}L(M))$. Then, since the chern character is a rational isomorphism, $f_*(\mathcal{D}L(M))$ is homotopy invariant if and only if $f_*[D_M^+]$ is homotopy invariant in $K_0(BG) \otimes \mathbb{Q}$.

Kasparov then constructs a map $\beta:K_0(BG) \to K_0(C_r^*(G))$, where $C_r^*(G)$ is the reduced C^* algebra of the group G [defined in [24], Chapter 7], and shows that $\beta f_*[D_M^+]$ is homotopy invariant. Thus the conjecture is implied by the rational injectivity of β .

For an appropriate definition of the continuous cyclic cohomology of a Banach algebra B , written $HC^*(B)$, there is a pairing between $K_0(B)$ and $HC^*(B)$ [10, Chapter 2]. Thus for $\tau \in HC^*(C_r^*G)$, we have a map $K_0(C_r^*(G)) \to \mathbb{C}$ given by $x \to \langle x,\tau \rangle$. The map $y \to \langle \beta y,\tau \rangle$ from $K_0(BG)$ to $\mathbb{C}$ then factors through $H_*(BG:\mathbb{C})$, by the Chern character. The map $H_*(BG:\mathbb{C}) \to \mathbb{C}$ is given by pairing with an element $\theta\tau \in H^*(BG:\mathbb{C})$. Thus it seems reasonable to look for an explicit map $\theta : HC^*(C_r^*G) \to H^*(BG:\mathbb{C})$, such that the following diagram commutes:

$$
\begin{array}{ccc}
K_0(BG) & \xrightarrow{\ \beta\ } & K_0(C_r^*G) \\
\text{ch} \downarrow & & \downarrow \ \langle\ ,\tau\rangle \\
H_*(BG:\mathbb{C}) & \xrightarrow[\langle\ ,\theta\tau\rangle]{} & \mathbb{C}
\end{array}
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H_*(BG:\mathbb{C}) & \xrightarrow[\langle\ ,\theta\tau\rangle]{} & \mathbb{C}
\end{array}$$

The Novikov conjecture would then by implied by the surjectivity of θ.

§1. Definitions.

Cyclic homology and cohomology were originally defined for algebras [7,8]. However they can be defined for a wider class of objects, and since we will need to use examples which are not algebras, we will give the most general form of the definition [9].

We need first to describe Connes' category Λ, which is an extension of the simplicial category Δ. The objects of Λ are the same as those of Δ, namely ordered sets $\underline{n} = \{0,1,\ldots n\}$, but the morphisms are generated by order preserving maps and cyclic permutations. More precisely, $\Lambda(\underline{n},\underline{m}) = \Delta(\underline{n},\underline{m}) \times K(\underline{n})$, where $K(\underline{n})$ is the group of cyclic permutations of $\underline{n}$, and the composition law is given by the rules for composition of generators given below.

The morphisms of Λ are generated by

a) the face maps $\delta_i \in \Lambda(\underline{n-1}, \underline{n})$, $0 \leq i \leq n$, δ_i the order preserving injection whose image does not contain i.

b) the degeneracy maps $\sigma_i \in \Lambda(\underline{n+1}, \underline{n})$, $0 \leq i \leq n$, σ_i the order preserving surjection such that both $\sigma_i(i) = i$ and $\sigma_i(i+1) = i$

c) the cyclic permutation $\tau_n \in \Lambda(\underline{n},\underline{n})$, $\tau_n(i) = i-1$, modulo $n+1$.

The morphisms satisfy the usual cosimplicial relations [17, §1] together with the following cyclic relations:

(i) $\tau_n \delta_i = \delta_{i-1} \tau_{n-1}$ $\quad 1 \le i \le n$

$\tau_n \delta_o = \delta_n$

(ii) $\tau_n \sigma_i = \sigma_{i-1} \tau_{n+1}$ $\quad 1 \le i \le n$

$\tau_n \sigma_o = \sigma_n \tau_{n+1}^2$

(iii) $\tau_n^{n+1} = 1$.

A cyclic object in a category C is a contravariant functor $\Lambda \to C$, and a cocyclic object is a covariant functor $\Lambda \to C$. Cyclic homology is defined for all cyclic F-modules, cyclic cohomology for all cocyclic F-modules. Since we shall use the composition rules for the structure maps in calculations, we will give explicitly the definition of a cyclic F-module.

A cyclic F-module E consists of a sequence $E(n)$ of F-modules, and structure maps $d_i : E(n) \to E(n-1)$, $s_i : E(n) \to E(n+1)$, $t_n : E(n) \to E(n)$ induced by the cyclic morphisms. These satisfy the following rules for composition:

(1) $d_i d_j = d_{j-1} d_i$ $\quad i < j$

(2) $s_i s_j = s_{j+1} s_i$ $\quad i \le j$

(3) $d_i s_j = s_{j-1} d_i$ $\quad i < j$

$= 1$ $\quad i = j$ or $i = j+1$

$= s_j d_{i-1}$ $\quad i > j+1$

(4) $\quad d_i\, t_n = t_{n-1}\, d_{i-1} \qquad 1 \le i \le n$

$\quad d_0\, t_n = d_n$

(5) $\quad s_i\, t_n = t_{n+1}\, s_{i-1} \qquad 1 \le i \le n$

$\quad s_0\, t_n = t_{n+1}^2\, s_n$

(6) $\quad t_n^{n+1} = 1$.

We give two important examples:

<u>Example 1</u>(a)

An associative algebra A with unit over a field F gives a cyclic F-module $A^{\natural}$, where $A^{\natural}(n)$ is the iterated tensor product $A^{\otimes(n+1)}$, and the structure maps are given by

$$d_i(a_0 \otimes \dots \otimes a_n) = a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n$$

$$d_n(a_0 \otimes \dots \otimes a_n) = a_n a_0 \otimes a_1 \otimes \dots \otimes a_{n-1}$$

$$s_j(a_0 \otimes \dots \otimes a_n) = a_0 \otimes \dots \otimes a_j \otimes 1 \otimes a_{j+1} \dots \otimes a_n$$

$$t_n(a_0 \otimes \dots \otimes a_n) = a_n \otimes a_0 \otimes \dots \otimes a_{n-1} .$$

<u>Example 1</u>(b)

An example of a cyclic F-module which is not obtained from an algebra is C^n , defined as follows. $C^n(m) = F\Lambda(\underline{m},\underline{n})$, the free F-module generated by $\Lambda(\underline{m},\underline{n})$, with the structure maps acting by composition on the left.

A cocyclic F-module P consists of a sequence P(n) of F-modules, and structure maps $\delta_i : P(n) \to P(n+1)$, $\sigma_i : P(n) \to P(n-1)$, $\tau_n : P(n) \to P(n)$, induced by the cyclic morphisms. These satisfy the usual cosimplicial relations and the cyclic relations (i) - (iii) given earlier.

The cyclic homology $HC_*(E)$ of a cyclic F-module E is defined to be the homology of a double complex $C_*(E)$, defined as follows [23]. First we define the maps

$$b:E(n) \to E(n-1) \qquad b = \sum_{i=0}^{n} (-1)^i d_i$$

$$N:E(n) \to E(n) \qquad N = 1 + (-1)^n t_n + (-1)^{2n} t_n^2 \,..\, + (-1)^{n^2} t_n^n$$

$$D:E(n) \to E(n) \qquad D = 1 - (-1)^n t_n$$

$$B:E(n) \to E(n+1) \qquad B = D(t_{n+1} s_n)N \,.$$

These satisfy the relations $b^2 = 0$, $B^2 = 0$, $bB = -Bb$ [23, 1.3 and 1.4]. Then the complex $C_*(E)$ is

$$\begin{array}{ccccc}
b\downarrow & & b\downarrow & & b\downarrow \\
E(2) & \xleftarrow{B} & E(1) & \xleftarrow{B} & E(0) \\
b\downarrow & & b\downarrow & & \\
E(1) & \xleftarrow[B]{} & E(0) & & \\
b\downarrow & & & & \\
E(0) & & & &
\end{array}$$

Observe that b is the usual simplicial or Hochschild boundary. By analogy with the case $E = A^{\natural}$, the complex

$$\ldots . \xrightarrow{b} E(n+1) \xrightarrow{b} E(n) \xrightarrow{b} E(n-1) \xrightarrow{b} \ldots$$

will be called the Hochschild complex and written E_* : its homology will be called the Hochschild homology and written $HH_*(E)$.

This complex can be simplified by replacing each column by its normalisation, that is, by dividing out by the degenerate subcomplex D_* , where $D(n) \subset E(n)$ is spanned by the image of $E(n-1)$ under the degeneracy maps. Since $B(D(n)) \subset D(n+1)$, B induces a map on the normalised complex, given by $t_{n+1}s_nN$. The map induced by b is $\sum_{i=0}^{n} (-1)^i d_i$ as before. Since the degenerate subcomplex is acyclic, the quotient map from the double complex to its normalisation induces an isomorphism in homology.

The complex $C_*(E)$ can itself be thought of as a simplification of a double complex $D_*(E)$ arising from work of Connes giving an invariant description of cyclic homology and cohomology [9]. We define the map

$$b' : E(n) \to E(n-1) , \qquad b' = \sum_{i=0}^{n-1} (-1)^i d_i .$$

Then the complex $D_*(E)$ is

$$\begin{array}{ccccccc}
b\downarrow & & -b'\downarrow & & b\downarrow & & \\
E(2) & \xleftarrow{\;D\;} & E(2) & \xleftarrow{\;N\;} & E(2) & \xleftarrow{\;D\;} & \\
b\downarrow & & -b'\downarrow & & b\downarrow & & \\
E(1) & \xleftarrow{\;D\;} & E(1) & \xleftarrow{\;N\;} & E(1) & \xleftarrow{\;D\;} & \\
b\downarrow & & -b'\downarrow & & b\downarrow & & \\
E(0) & \xleftarrow{\;D\;} & E(0) & \xleftarrow{\;N\;} & E(0) & \xleftarrow{\;D\;} &
\end{array}$$

The simplification arises because the alternate columns are acyclic. A chain homotopy h between the identity map and the zero map, that is, satisfying $(-b')h + h(-b') = 1$, is given by $h = -t_{n+1}s_n : E(n) \to E(n+1)$. Thus eliminating these columns gives a complex with the same homology, and this complex is $C_*(E)$ [23, Proposition 1.5].

Given an algebra A without unit, it does not give a cyclic F-module since there is no action of the degeneracy maps on $A^{\otimes n+1}$. However, since the complex $D_*(E)$ does not involve degeneracies, such a double complex may be defined for A , to give $HC_*(A)$. An alternative definition is given by adjoining a unit to A to obtain an algebra A^+ . The reduced cyclic homology of a cyclic F-module is defined to be the homology of the complex which is obtained from the normalised complex of $C_*(E)$ by replacing $E(0)$ with $E(0)/F$. We can then define $HC_*(A)$ to be the reduced cyclic homology of A^+ , $\overline{HC}_*(A^+)$. Loday and Quillen prove that the two definitions agree [23, Proposition 4.2].

The first column of $C_*(E)$ is a subcomplex E_* of the double complex. The quotient of $C_*(E)$ by E_* is equal to the complex $C_*(E)$ itself, after a degree shift of -2 ; we write $C_*(E)[-2]$ for the complex such that $(C_*(E)[-2])_n = C_{n-2}(E)$. We then have an exact sequence of chain complexes

$$0 \to E_* \to C_*(E) \to C_*(E)[-2] \to 0$$

giving rise to a long exact sequence in homology,

$$\ldots HH_n(E) \xrightarrow{I} HC_n(E) \xrightarrow{S} HC_{n-2}(E) \xrightarrow{B} HH_{n-1}(E) \quad \ldots$$

Note that this sequence includes a periodicity operator $S : HC_n(E) \to HC_{n-2}(E)$, which is given at the chain level by moving a chain diagonally one column to the left and one row down.

The cyclic cohomology of a cocyclic F-module G is similarly given as the homology of a double chain complex $C^*(G)$. This complex is

$$
\begin{array}{ccccc}
b\uparrow & & b\uparrow & & b\uparrow \\
G(2) & \xrightarrow{B} & G(1) & \xrightarrow{B} & G(0) \\
b\uparrow & & b\uparrow & & \\
G(1) & \xrightarrow{B} & G(0) & & \\
b\uparrow & & & & \\
G(0) & & & &
\end{array}
$$

where $b : G(n) \to G(n+1)$, $b = \sum_{i=0}^{n} (-1)^i \delta_i$

$B : G(n) \to G(n-1)$, $B = (1+(-1)^{n-1}\tau_{n-1} \ldots (-1)^{(n-1)^2}\tau_{n-1}^{n-1})$

$(\sigma_{n-1}\tau_n)(1-(-1)^n\tau_n)$

the Hochschild cochain complex G^* is

$$.. \to G(n-1) \xrightarrow{b} G(n) \xrightarrow{b} G(n+1) \xrightarrow{b} ....$$

Note that this sequence includes a periodicity operator $S : HC_n(E) \to HC_{n-2}(E)$, which is given at the chain level by moving a chain diagonally one column to the left and one row down.

The cyclic cohomology of a cocyclic F-module G is similarly given as the homology of a double chain complex $C^*(G)$. This complex is

$$\begin{array}{ccccc}
b\uparrow & & b\uparrow & & b\uparrow \\
G(2) & \xrightarrow{B} & G(1) & \xrightarrow{B} & G(0) \\
b\uparrow & & b\uparrow & & \\
G(1) & \xrightarrow{B} & G(0) & & \\
b\uparrow & & & & \\
G(0) & & & &
\end{array}$$

where $b : G(n) \to G(n+1)$, $b = \sum_{i=0}^{n} (-1)^i \delta_i$

$B : G(n) \to G(n-1)$, $B = (1+(-1)^{n-1}\tau_{n-1} \ldots (-1)^{(n-1)^2}\tau_{n-1}^{n-1})$

$(\sigma_{n-1}\tau_n)(1-(-1)^n\tau_n)$

the Hochschild cochain complex G^* is

$$\ldots \to G(n-1) \xrightarrow{b} G(n) \xrightarrow{b} G(n+1) \xrightarrow{b} \ldots$$

G^* is a quotient complex of $C^*(G)$. There is a short exact sequence of cochain complexes

$$0 \to C^*(G)[-2] \to C^*(G) \to G^* \to 0$$

which gives a long exact sequence in homology

$$\ldots \to HC^{n-2}(G) \xrightarrow{S} HC^n(G) \xrightarrow{I} HH^n(G) \xrightarrow{B} HC^{n-1}(G) \to \ldots$$

Here we have a periodicity operator $S : HC^n(G) \to HC^{n+2}(G)$, which is given at the cochain level by moving a cochain diagonally one column to the right and one row up.

Thus $HC^*(A)$ is a module over the polynomial ring $F[\theta]$, where θ acts by the periodicity operator and has degree 2. As explained in the Introduction, it is appropriate to regard $F[\theta]$ as the natural coefficients, and to look for the $F[\theta]$-module structure at the chain level. This is given by expressing $C^*(G)$ as the graded tensor product $G^* \otimes_F F[\theta]$: so an element of $C^n(G)$ is a sum $\sum_i g_{n-2i} \otimes \theta^i$, where $g_{n-2i} \in G(n-2i)$. The boundary is $b + B\theta$; that is,

$$(b + B\theta)(g \otimes \theta^n) = bg \otimes \theta^n + Bg \otimes \theta^{n+1} .$$

We now wish to define a homology theory $HC_*^-(\ )$, which will be dual to cyclic homology over $F[\theta]$. Let D be the graded integral domain $F[\theta]$, where here θ has degree -2. Define the tensor product $\hat{\otimes}$ of graded F-modules L and M by using the direct product rather than the direct sum,

$$(L \hat{\otimes}_F M)_n = \prod_{i+j=n} L_i \otimes M_j \ .$$

Then for a cyclic F-module E, define $HC_*^-(E)$ to be the homology of the complex $C_*^-(E) = E_* \hat{\otimes}_F D$, with boundary $b + B\theta$. Thus an element of $C_n^-(E)$ is a formal power series $\sum_i e_{n+2i} \otimes \theta^i$, where $e_{n+2i} \in E(n+2i)$.

Let K be the graded field of fractions of D, $F[\theta,\theta^{-1}]$, and set $\hat{C}_*(E) = (E_* \hat{\otimes}_F K\ , b + B\theta)$.

Finally, set $C_*^+(E) = \dfrac{\hat{C}_*(E)}{\theta C_*^-(E)}$, $C_*^+(E) = E_* \otimes \dfrac{F[\theta,\theta^{-1}]}{\theta F[\theta]}$,

and we obtain the cyclic homology complex as defined before, with the action of θ inducing the periodicity operator.

Define the homology theories $HC_*^-(E) = H_*(C_*^-(E))$, $\hat{HC}_*(E) = H_*(\hat{C}(E))$.

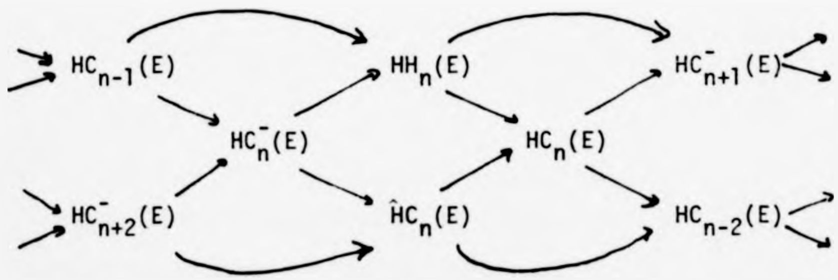
Then the short exact sequence of chain complexes

$$0 \to \theta C_*^-(E) \to \hat{C}_*(E) \to \hat{C}_*(E)/\theta C_*^-(E) \to 0$$

induces a long exact sequence of homology theories

$$\ldots\ HC^-_{n+2}(E) \to \hat{HC}_n(E) \to HC_n(E) \xrightarrow{B} HC^-_{n+1}(E) \to \ldots$$

This can be related to the exact sequences involving Hochschild homology by means of a braid:



We can also express the other theories explicitly in terms of HC_*^- . Given a D-module M , let $\theta^{-1}M$ be the localisation at the multiplicative subset $\{1,\theta,\theta^2, \ldots\}$.

<u>Lemma 1.1</u>.

(a) $\hat{C}_*(E) = \theta^{-1}C_*^-(E)$

(b) $\widehat{HC}_*(E) = \theta^{-1} HC^-_*(E)$.

Proof

(b) follows since taking homology commutes with localisation. □

Lemma 1.2.

(a) $C^+_*(E) = C^-_*(E) \otimes_D K/\theta D$

(b) There is a natural short exact sequence

$$0 \to [HC^-_*(E) \otimes_D K/\theta D]_n \to HC_n(E) \to [\mathrm{Tor}_D(HC^-_*(E), K/\theta D)]_{n-1} \to 0 \quad .$$

Proof

(a) $K/\theta D$ is generated over F by $\{\theta^{-n}, n \geq 0\}$, and the isomorphism is given on the generators by

$$\Big(\sum_{0 \leq i < \infty} a_i \otimes \theta^i\Big) \otimes \theta^{-n} \to \sum_{0 \leq i \leq n} a_i \otimes \theta^{i-n} \quad .$$

(b) This then follows by standard homological algebra, since D is a principal ideal domain [26, Theorem 5.2.8. p.222]. □

Lemma 1.3.

Let E be a cocyclic F-module, $\mathcal{E}$ the dual cyclic F-module.

(a) $C_*^-(E) = \mathrm{Hom}_D(C^*(E), D)$

(b) There is a natural short exact sequence

$$0 \to [\mathrm{Ext}_D(HC^*(E), D)]_{n+1} \to HC_n^-(E) \to [\mathrm{Hom}_D(HC^*(E), D)]_n \to 0 \quad .$$

Proof

If A and B are graded D-modules, $\mathrm{Hom}_D^n(A,B)$ consists of families of homomorphisms $A^m \to B^{m-n}$. We have

$$\mathrm{Hom}^n_{F[\theta]}(E^* \otimes F[\theta], F[\theta]) \cong \mathrm{Hom}^n_F(E^*, F[\theta]) \quad .$$

An element of $\mathrm{Hom}^n_F(E^*, F[\theta])$ consists of a family of homomorphisms $f^m : E^m \to F[\theta]^{m-n}$; then f^m is only non zero if m-n is even, let $m-n = 2p$. Let g_m be the element of E_m given by the map $E^m \to F$. Then the family is equivalent to the power series $\sum_p g_{2p+n} \otimes \theta^p$. Then taking θ to have degree -2 , this is an element of $(E \otimes F[\theta])_n = (C_*^-(E))_n$

(b) follows by standard homological algebra [26, Theorem 5.5.3, p.243]. □

(a) $C^-_*(E) = Hom_D(C^*(E),D)$

(b) There is a natural short exact sequence

$$0 \to [Ext_D(HC^*(E),D)]_{n+1} \to HC^-_n(E) \to [Hom_D(HC^*(E),D)]_n \to 0 \quad .$$

<u>Proof</u>

If A and B are graded D-modules, $Hom^n_D(A,B)$ consists of families of homomorphisms $A^m \to B^{m-n}$. We have

$$Hom^n_{F[\theta]}(E^* \otimes F[\theta], F[\theta]) \cong Hom^n_F(E^*,F[\theta]) \quad .$$

An element of $Hom^n_F(E^*,F[\theta])$ consists of a family of homomorphisms $f^m : E^m \to F[\theta]^{m-n}$; then f^m is only non zero if $m-n$ is even, let $m-n = 2p$. Let g_m be the element of E_m given by the map $E^m \to F$. Then the family is equivalent to the power series $\sum_p g_{2p+n} \otimes \theta^p$. Then taking θ to have degree -2 , this is an element of $(E \otimes F[\theta])_n = (C^-_*(E))_n$

(b) follows by standard homological algebra [26, Theorem 5.5.3, p.243]. □

We have discussed the structure of $HC_*(E)$ as a D-module: as the F-dual of the D-module $HC^*(E)$, it also has a comodule structure over the coalgebra D^*. We shall now give the definition of these terms, from [14]. We will rename the coalgebra D^* as G, for clarity.

A coalgebra Γ over the ring F consists of an F-module Γ, together with a pair of morphisms $\varepsilon:\Gamma \to F$, $\delta:\Gamma \to \Gamma \otimes_F \Gamma$, such that the following equations hold:

(a) $(\varepsilon \otimes 1_\Gamma)\delta = 1_\Gamma = (1_\Gamma \otimes \varepsilon)\delta$

(b) $(\delta \otimes 1_\Gamma)\delta = (1_\Gamma \otimes \delta)\delta$

A Γ-comodule A is an F-module with a structure morphism $\nabla : A \to A \otimes_F \Gamma$ satisfying the following:

(a) $(1_A \otimes \varepsilon)\nabla = 1_A$

(b) $(\nabla \otimes 1_\Gamma)\nabla = (1_A \otimes \delta)\nabla$

The dual of $F[\theta]$ has a coalgebra structure given as follows: It is generated as an F-module by $(\gamma_i)_{i\in \mathbb{N}}$, and the morphisms ε and δ are given by

$$\varepsilon\left(\sum_{i=0}^{\infty} a_i\gamma_i\right) = a_0$$

$$\delta(\gamma_i) = \sum_{j=0}^{i} \gamma_{i-j} \otimes \gamma_j .$$

Then $C_*^+(E) = E_* \otimes_F G$ is a G-comodule, with structure morphisms $\nabla = 1_{E_*} \otimes \delta$.

Any G-comodule A is also a D-module, with an action defined as follows. Given $m \in A$, let $\nabla(m) = \sum_k S^k m \otimes \gamma_k$. Then using condition (b) , $(\nabla \otimes 1_G)\nabla = (1_A \otimes \delta)\nabla$, we obtain $S^i(S^k m) = S^{i+k}(m)$. Thus $\theta^i(m) = S^i m$ is a well defined action of D . On $C_*^+(E)$, since $\nabla(a_i \otimes \gamma_i) = \sum_{j=0}^{i} a_i \otimes \gamma_{i-j} \otimes \gamma_j$, the θ-action is given by $\theta^j(a \otimes \gamma_i) = a \otimes \gamma_{i-j}$. This agrees with the D-module structure already obtained by identifying $C_*^+(E)$ with $E_* \otimes \frac{F[\theta,\theta^{-1}]}{\theta F[\theta]}$.

§2. Products

Given two cyclic F-modules P and Q, we define their product $P \times Q$ by $(P \times Q)(n) = P(n) \otimes_F Q(n)$, with the structure maps acting diagonally. Our aim is then to define a product $HC_*^-(P) \otimes_D HC_*^-(Q) \to HC_*^-(P \times Q)$, by constructing a D-module chain map, $f : C_*^-(P) \otimes_D C_*^-(Q) \to C_*^-(P \times Q)$. The same methods enable us to construct a D-module chain map $g : C_*^-(P \times Q) \to C_*^-(P) \otimes_D C_*^-(Q)$. This gives, by duality, a product in cyclic cohomology, $HC^*(P) \otimes_D HC^*(Q) \to HC^*(P \times Q)$.

The method used is a version of the acyclic models method; we show that it is sufficient to construct a product on certain "universal examples" or models. When this method is used to construct products in simplicial homology, simplices are the models used; here we use the cyclic objects C^n, where $C^n(m) = F\Lambda(\underline{m},\underline{n})$ [Example 1.1]. We will refer to C^n as the models for cyclic homology. These have the appropriate universal property.

Lemma 2.1.

Let P be a cyclic F-module, x an element of $P(n)$, then there exists a unique map of cyclic F-modules $\phi_x : C^n \to P$ such that $\phi_x(i_n) = x$, where i_n is the identity morphism in $C^n(n)$.

Proof

Given $y \in \Lambda(n,m)$, let $\phi_x(y) = y^*x$, where y^* is the map $P(n) \to P(m)$ induced by the cyclic morphism y. Then extend throughout $F\Lambda(n,m)$ by linearity. □

Given Lemma 2.1, once we have defined $f(i_n \otimes i_m)$ for $i_n \in C^n(n)$, $i_m \in C^m(m)$, naturality forces the definition of f on a general element : for $x \in P(n)$, $y \in Q(m)$, define $f(x \otimes y) = (\phi_x \otimes \phi_y) f(i_n \otimes i_m)$.

In order to make it clear which of the complexes is being considered, we will write elements of $(P \times Q)(n)$, lying in $C^-_*(P \times Q)$, in the form (x_n, y_n), while writing elements of $P(r) \otimes Q(s)$, lying in $C^-_*(P) \otimes_D C^-_*(Q)$, in the usual form $x_r \otimes y_s$.

In the remainder of this chapter, we shall construct a D-module chain map $f : C^-_*(P) \otimes_D C^-_*(Q) \to C^-_*(P \times Q)$. Since it is a D-module map, it may be written $\sum_k f_k \theta^k$, where f_0 is a degree-preserving chain map $P_* \otimes Q_* \to (P \times Q)_*$. Then, since the Hochschild complex is a quotient of the cyclic homology chain complex, $P_* \otimes Q_*$ is a quotient of $C^-_*(P) \otimes_D C^-_*(Q)$, and $(P \times Q)_*$ is a quotient of $C^-_*(P \times Q)$.
The map f will fit into the following commutative diagram:

$$\begin{array}{ccc} C^-_*(P) \otimes_D C^-_*(Q) & \xrightarrow{\ f\ } & C^-_*(P \times Q) \\ \downarrow & & \downarrow \\ P_* \otimes Q_* & \xrightarrow{\ f_0\ } & (P \times Q)_* \end{array}$$

We will refer to f as a coextension of f_0 .

We will prove the following:

Theorem A

Given any natural chain equivalence $f_0 : P_* \otimes Q_* \to (P \times Q)_*$, such that in degree 0 , $f_0(x_0 \otimes y_0) = (x_0,y_0)$, there is a coextension to a natural D-module chain map $f : C_*^-(P) \otimes_D C_*^-(Q) \to C_*^-(P \times Q)$.

Theorem B

Given any natural chain equivalence $g_0 : (P \times Q)_* \to P_* \otimes Q_*$, such that in degree 0 , $g_0(x_0,y_0) = x_0 \otimes y_0$, there is a coextension to a natural D-module chain map $g : C_*^-(P \times Q) \to C_*^-(P) \otimes_D C_*^-(Q)$.

Theorem C

(i) The natural D-module chain map f , with the conditions on f_0 given in Theorem A, is a chain equivalence.

(ii) The natural D-module chain map g , with the conditions on g_0 given in Theorem B, is a chain equivalence.

Theorem D

(i) The natural chain equivalence f , with the conditions on f_0 given in Theorem A, is unique up to chain homotopy.

(ii) The natural chain equivalence g, with the conditions on g_0 given in Theorem B, is unique up to chain homotopy.

Theorem E

(i) The product in cyclic homology induced by f is associative and graded commutative.

(ii) The product in cyclic cohomology induced by g is associative and graded commutative.

First we need to calculate the homology of the models, which are in fact not acyclic: this means that the acyclic models method will be supplemented by direct calculation in the low degrees where the homology is non zero.

Lemma 2.2.

(a) The Hochschild homology of the models is given by

$$HH_n(C^k) = F \quad \text{if } n = 0 \text{ or } 1$$
$$= 0 \quad \text{otherwise}$$

(b) $$HC^-_n(C^k) = F \quad \text{if } n = 1$$
$$0 \quad \text{otherwise.}$$

Proof

(a) Define the cyclic sets λ^m by $\lambda^m(n) = \Lambda(n,m)$; then C^m is the free cyclic F-module generated by this cyclic set. Since any cyclic set is a fortiori a simplicial set, it has a geometrical realisation. The Hochschild homology of C^k will then be given by the simplicial homology of the realisation,

$$|\lambda^k| = \coprod_n \frac{\lambda^k(n) \times \Delta^n}{(x,\theta_* y) = (\theta_* x,y)}$$

where θ_* is a map induced by any simplicial morphism. We will show that $|\lambda^k|$ is homeomorphic to $S^1 \times \Delta^k$. This is proved in [12]; we follow the proof given in [17].

First we observe that the simplicial set λ^k is generated using the operations d_i , s_j and t_q on the identity map i_k in $\lambda^k(k)$, where these operations satisfy the usual cyclic relations.

We then construct a simplicial set Σ^k , by giving a triangulation of $\mathbb{R} \times \Delta^k$. Let $\{v_i\}_{i=0,\dots k}$ be the vertices of Δ^k . Let the vertices of the triangulation be (i,v_r) , where i is an integer. The vertices are given an ordering by $(i,v_r) < (j,v_s)$ if either $i < j$, or $i = j$ and $r < s$. The q-simplices of the triangulation are either of the type

$$(i,v_{r_s}),(i,v_{r_{s+1}})\dots(i,v_{r_q}),\ (i+1,v_{r_0})\dots(i+1,\ v_{r_{s-1}})$$

where $r_0 < r_1 \dots < r_q$, or of the type

$$(i,v_{r_s}),(i,v_{r_{s+1}})\dots(i,v_{r_{q-1}}),\ (i+1,v_{r_0})\dots(i+1,\ v_{r_s})$$

where $r_0 < r_1 \dots < r_{q-1}$.

Then Σ^k is the simplicial set generated by this triangulation of $\mathbb{R} \times \Delta^k$.

We define an operation β_q on the q simplices of Σ^k as follows. If the last vertex of σ is (i,v_m) , then the vertices of $\beta_q\sigma$ are the same as those of σ , except that (i,v_m) is replaced by $(i-1,v_m)$, which then becomes the first vertex of $\beta_q\sigma$. It can be checked that β_q satisfies

(i) $$d_i\, \beta_q = \beta_{q-1}\, d_{i-1}$$

$$d_0\, \beta_q = d_q$$

(ii) $$s_i\, \beta_q = \beta_{q+1}\, s_{i-1}$$

$$s_0\, \beta_q = \beta^2_{q+1}\, s_q$$

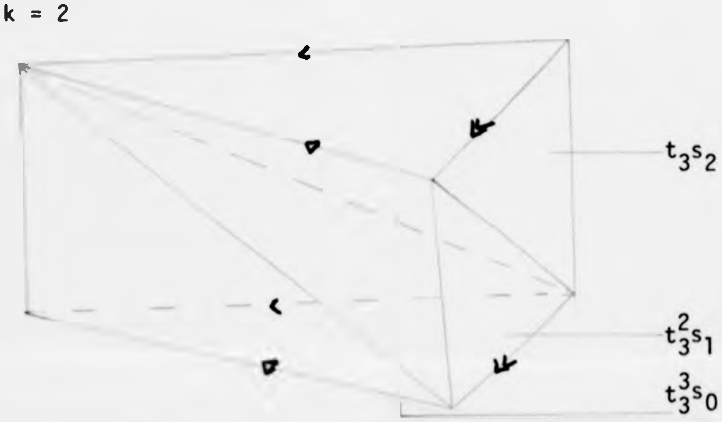
where d_i and s_i are the usual face and degeneracy operations. That is, β_q satisfies all of the usual cyclic relations except $\beta_q^{q+1} = 1$.

Now note that any $(k+1)$ simplex of Σ^k can be expressed as $\beta_{k+1}^i \, s_j \, i_k$, where $i_k = \{0\} \times \Delta^k$, and that every simplex in the triangulation is a face of such a simplex. Thus the simplicial set Σ^k is generated using the operations d_i, s_j and β_q on the simplex i_k , where d_i, s_j and β_q satisfy all the cyclic relations except $\beta_q^{q+1} = 1$.

Thus λ^k is obtained from Σ^k by identifying β_q^{q+1} and 1 . Since β_q^{q+1} translates a q-simplex by -1 , $|\lambda^k|$ is obtained from $|\Sigma^k| = \mathbb{R} \times \Delta^k$ by identifying any two points whose $\mathbb{R}$ coordinates differ by an integer. Thus $|\lambda^k|$ is homeomorphic to $S^1 \times \Delta^k$.

Note that the non-degenerate $(k+1)$ simplices of $|\Sigma^k|$ are $\{\beta_{k+1} s_k i_k , \ldots \beta_{k+1}^i s_{k+1-i} i_k , \ldots \beta_{k+1}^{k+1} s_0 i_k\}$, corresponding in $|\lambda^k|$ to $\{t_{k+1} s_k i_k , \ldots , t_{k+1}^{k+1} s_0 i_k\}$, the non-degenerate terms which occur in the operator B.

e.g. $k = 2$



The proof of part (b) is obtained by substituting this result for $HH_*(C^k)$ in the long exact sequence relating HC_*^- and HH_* . □

We now prove the existence of the chain map.

<u>Lemma 2.3</u>.

There exists a natural D-module chain map $f : C_*^-(C^k) \otimes_D C_*^-(C^\ell) \to C_*^-(C^k \times C^\ell)$ which coextends the simplicial shuffle product.

<u>Proof</u>

We shall refer to the boundary in both complexes as $b + B\theta$. The complex $C_*^-(C^k) \otimes_D C_*^-(C^\ell)$ is naturally isomorphic to $C_*^k \otimes C_*^\ell \otimes D$; $C_*^-(C^k \times C^\ell)$ is isomorphic to $(C^k \times C^\ell)_* \otimes D$. We are constructing a D-module map, so we write $f = \sum_k f_k \theta^k$, where f_k is a map $C_*^k \otimes_F C_*^\ell \to (C^k \times C^\ell)_*$ which raises degree by $2k$. Since the shuffle product descends to the normalised complexes, we will work with these for convenience. [16, p.208]

In order for f to be a chain map we require

$$(b + \theta B)(\sum_k f_k\theta^k)(x \otimes y) = (\sum_k f_k\theta^k)(b + B\theta)(x \otimes y) \quad .$$

Equating coefficients of θ^k , this becomes

(i) $$bf_k = f_k b - Bf_{k-1} + f_{k-1} B$$

where f_{-1} is taken to be 0 . We use (i) to define $f_k(x \otimes y)$ inductively, as follows. Assume by the inductive hypothesis that f_ℓ is defined for all $\ell < k$. We construct $f_k(x \otimes y)$ by induction on the degree of $x \otimes y$. By naturality, it is sufficient to construct f_k on the terms $i_n \otimes i_m$. We assume f_k is defined for all elements of degree < q , and wish to define $f_k(i_n \otimes i_m)$ where $n+m = q$.

The chain $Z = (f_k b - Bf_{k-1} + f_{k-1} B)(i_n \otimes i_m)$ is then well-defined. We check, using equation (i) and the relation $bB = -Bb$, that $bZ = 0$:

$$\begin{aligned} bZ &= [(bf_k)b + B(bf_{k-1}) + (bf_{k-1})B](i_n \otimes i_m) \\ &= [(f_k b - Bf_{k-1} + f_{k-1}B)b + B(f_{k-1}b - Bf_{k-2} + f_{k-2}B) \\ &\qquad + (f_{k-1}b - Bf_{k-2} + f_{k-2}B)B](i_n \otimes i_m) \\ &= [-Bf_{k-1}b - f_{k-1}bB + Bf_{k-1}b + Bf_{k-2}B + f_{k-1}bB - Bf_{k-2}B](i_n \otimes i_m) \\ &= 0 \ . \end{aligned}$$

The Hochschild homology of $(C^n \times C^m)$ is obtained from the Kunneth theorem for simplicial complexes, and Lemma 2.2.

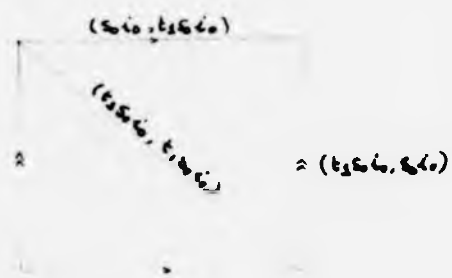
$$HH_k(C^n \times C^m) = F \qquad \text{if } k = 0,2$$
$$= F \oplus F \quad \text{if } k = 1$$
$$= 0 \qquad \text{otherwise.}$$

Thus, since the degree of $Z = (n+m) + 2k - 1$, we have, since $bZ = 0$, W such that $Z = bW$, provided that $(n+m) + 2k - 1 > 2$, and we can then define $f_k(i_n \otimes i_m) = W$. The cases where $n + m + 2k - 1 \leq 2$ must be dealt with directly. Since we have already chosen f_0 satisfying (i), this leaves the construction of f_1 on elements of degree 0 or 1.

(a) $f_1(i_0 \otimes i_0)$: This must satisfy

$$bf_1(i_0 \otimes i_0) = (-Bf_0 + f_0B)(i_0 \otimes i_0)$$
$$= -(t_1s_0i_0, t_1s_0i_0) + (t_1s_0i_0, s_0i_0) + (s_0i_0, t_1s_0i_0) \quad .$$

This expression contains all three non-degenerate 1-simplices in $|\lambda^0 \times \lambda^0| = S^1 \times S^1$:



We see that either of the non-degenerate 2-simplices,

$(t_2s_1s_0i_0, t_2^2s_1s_0i_0)$ and $(t_2^2s_1s_0i_0, t_2s_1s_0i_0)$ is a possible

choice for $f_1(i_0 \otimes i_0)$. We choose $f_1(i_0 \otimes i_0) = (t_2s_1s_0i_0, t_2^2s_1s_0i_0)$.

(b) $f_1(i_0 \otimes i_1)$: This must satisfy

$$bf_1(i_0 \otimes i_1) = (-Bf_0 + f_0B + f_1b)(i_0 \otimes i_1)$$

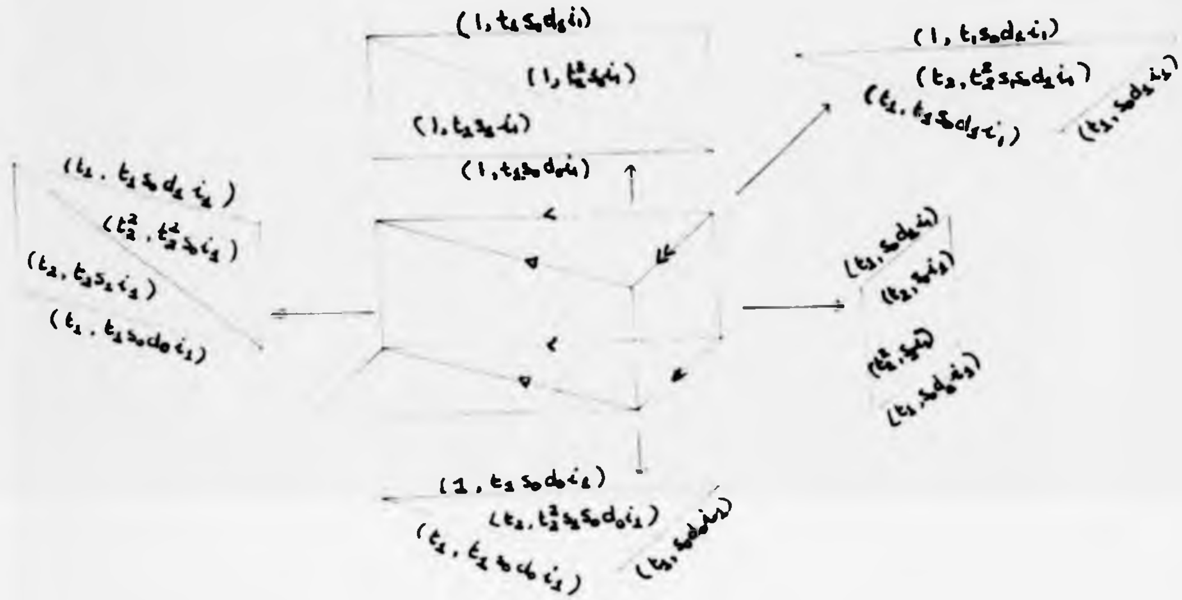
$$= [(s_1s_0i_0, t_2s_1i_1) - (s_1s_0i_0, t_2^2s_0i_1) + (t_2s_1s_0i_0, s_0i_1)$$

$$-(t_2^2s_1s_0i_0, s_1i_1) - (t_2s_1s_0i_0, t_2s_1i_1) + (t_2^2s_1s_0i_0, t_2^2s_0i_1)$$

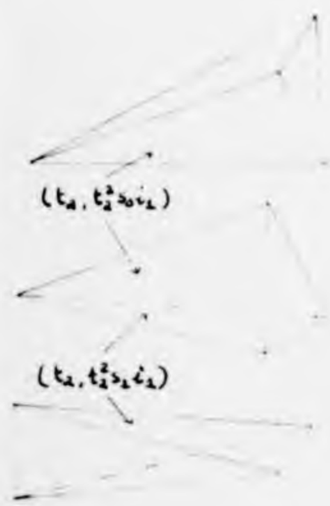
$$+ (t_2s_1s_0i_0, t_2^2s_1s_0d_0i_1) - (t_2s_1s_0i_0, t_2^2s_1s_0d_1i_1)] \ .$$

Consider the non-degenerate 3-simplices in $|\lambda^0 \times \lambda^1| \cong S^1 \times S^1 \times \Delta^1$.

Note that the pair of terms coming from Bf_0 have a common boundary, as do the pair from $f_0(Bi_0 \otimes i_1)$, and the pair from $f_0(i_0 \otimes Bi_1)$, and that these three "squares" form an open prism whose boundary is $f_1(i_0 \otimes bi_1)$. That is, writing t_2^α for $t_2^\alpha s_1s_0i_0$, the above expression consists of the following simplices:



Thus the terms of $(f_0 B - Bf_0 + f_1 b)(i_0 \otimes i_1)$ form the boundary of a solid prism, and give its decomposition into three 3-simplices:



$$(t_3 s_2 s_1 s_0 i_0, t_3^3 s_1 s_0 i_1) \xrightarrow{b} (1, t_2^2 s_0 i_1) - (t_2, t_2^2 s_0 i_1)$$
$$+ (t_2, s_0 i_1) - (t_2, t_2^2 s_1 s_0 d_1 i_1)$$

$$(t_3^2 s_2 s_1 s_0 i_0, t_3^3 s_2 s_0 i_1) \xrightarrow{b} (t_2, t_2^2 s_1 i_1) - (t_2, t_2^2 s_0 i_1)$$
$$+ (t_2^2, t_2^2 s_0 i_1) - (t_2^2, s_1 i_1)$$

$$(t_3 s_2 s_1 s_0 i_0, t_3^2 s_2 s_1 i_1) \xrightarrow{b} (1, t_2 s_1 i_1) - (t_2, t_2 s_1 i_1)$$
$$+ (t_2, t_2^2 s_1 s_0 d_0 i_1) - (t_2, t_2^2 s_1 i_1)$$

so $f_1(i_0 \otimes i_1) = -(t_3s_2s_1s_0i_0, t_3^3s_1s_0i_1) + (t_3^2s_2s_1s_0i_0, t_3^3s_2s_0i_1)$

$$+ (t_3s_2s_1s_0i_0, t_3^2s_2s_1i_1) \quad .$$

(c) $f_1(i_1 \otimes i_0)$: This must satisfy

$$bf_1(i_1 \otimes i_0) = (f_0B - Bf_0 + f_1b)(i_1 \otimes i_0)$$

$$= [(t_2s_1i_1,1) - (t_2^2s_0i_1,1) + (s_0i_1,t_2) - (s_1i_1,t_2^2) - (t_2s_1i_1,t_2)$$

$$+ (t_2^2s_0i_1,t_2^2) + (t_2s_1s_0d_0i_1,t_2^2) - (t_2s_1s_0d_1i_1,t_2^2)] \quad .$$

This is the boundary of a prism in $|\lambda^1 \times \lambda^0|$, with a decomposition into three 3-simplices, whose boundaries are:

$$(t_3^2s_1s_0i_1,t_3^3) \xrightarrow{b} (t_2s_0i_1,t_2^2) - (t_2s_1s_0d_1i_1,t_2^2) + (t_2^2s_0i_1,t_2^2) - (t_2^2s_0i_1,1)$$

$$(t_3s_2s_0i_1,t_3^2) \xrightarrow{b} (s_0i_1,t_2) - (t_2s_1i_1,t_2) + (t_2s_1i_1,t_2^2) - (t_2s_0i_1,t_2^2)$$

$$(t_3s_2s_1i_1,t_3^3) \xrightarrow{b} (s_1i_1,t_2^2) - (t_2s_1s_0d_0i_1,t_2^2) + (t_2s_1i_1,t_2^2) - (t_2s_1i_1,1) \quad .$$

So we have $f_1(i_1 \otimes i_0) = -(t_3s_2s_1i_1,t_3^3) + (t_3s_2s_0i_1,t_3^2) + (t_3^2s_1s_0i_1,t_3^3)$.

Collecting the formulae together, and applying them to arbitrary zero and one dimensional classes, we have

$$f_1(x_0 \otimes y_0) = (t_2s_1s_0x_0, t_2^2s_1s_0y_0)$$

$$f_1(x_0 \otimes y_1) = -(t_3s_2s_1s_0x_0, t_3^3s_1s_0y_1) + (t_3^2s_2s_1s_0x_0, t_3^3s_2s_0y_1)$$
$$+ (t_3s_2s_1s_0x_0, t_3^2s_2s_1y_1)$$

$$f_1(x_1 \otimes y_0) = -(t_3s_2s_1x_1, t_3^3s_2s_1s_0y_0) + (t_3s_2s_0x_1, t_3^2s_2s_1s_0y_0)$$
$$+ (t_3^2s_1s_0x_1, t_3^3s_2s_1s_0y_1) \quad .$$

If we are working in the non-normalised complex, B contains degenerate terms, and we have to modify the formulae by adding degenerate terms, as follows:

add to $f_1(x_0 \otimes y_0)$ the term $(s_1s_0x_0, s_1s_0y_0)$

add to $f_1(x_1 \otimes y_0)$ the terms $(s_2s_1x_1, s_2s_1s_0y_0) + (s_1s_0x_1, s_2s_1s_0y_0)$
$$+ (s_0t_2s_0x_1, s_0t_2^2s_1s_0y_0)$$

add to $f_1(x_0 \otimes y_1)$ the terms $(s_2s_1s_0y_0, s_2s_1y_1) + (s_2s_1s_0x_0, s_1s_0y_1)$
$$+ (s_0t_2^2s_1s_0x_0, s_0t_2s_0y_1) \quad .$$

□

Lemma 2.4.

Any natural chain equivalence $\phi_0 : (C^k)_* \otimes (C^\ell)_* \to (C^k \times C^\ell)_*$ such that $\phi_0(x_0 \otimes y_0) = (x_0, y_0)$, can be coextended to a natural D-module chain map $\phi : C^-_*(C^k) \otimes_D C^-_*(C^\ell) \to C^-_*(C^k \times C^\ell)$.

Proof

By the same argument by acyclic models as in the previous lemma, it is sufficient to construct $\phi_1(x \otimes y)$, where the degree of $x \otimes y$ is 0 or 1 . Given any two natural chain equivalences $f_0, \phi_0 : E_* \otimes G_* \to (E \times G)_*$, where E and G are simplicial F-modules, and f_0 and ϕ_0 satisfy the given condition, there is a chain homotopy h between them, $bh + hb = \phi_0 - f_0$. Here f_0 is the shuffle product.

We wish to construct $\phi_1(x_0 \otimes y_0)$ such that

$$\begin{aligned} b\phi_1(x_0 \otimes y_0) &= - B\phi_0(x_0 \otimes y_0) + \phi_0 B(x_0 \otimes y_0) \\ &= - B(f_0+bh+hb)(x_0 \otimes y_0) + (f_0+bh+hb)B(x_0 \otimes y_0) \\ &= (-Bf_0+f_0B)(x_0 \otimes y_0) + [bBh-Bhb+bhB-hBb](x_0 \otimes y_0) \\ &= bf_1(x_0 \otimes y_0) + b(Bh+hB)(x_0 \otimes y_0) . \end{aligned}$$

So we can take $\phi_1(x_0 \otimes y_0) = f_1(x_0 \otimes y_0) + (Bh+hB)(x_0 \otimes y_0)$.

Then on a chain $x \otimes y$ of degree 1, we require ϕ_1 to satisfy

$$b\phi_1(x\otimes y) = (-B\phi_0 + \phi_0 B + \phi_1 b)(x\otimes y)$$

$$= [-B(f_0+bh+hb) + (f_0+bh+hb)B + (f_1+Bh+hB)b](x\otimes y)$$

$$= [(-Bf_0+f_0B+f_1b) + bBh-Bhb+bhB-hBb+Bhb+hBb](x\otimes y)$$

$$= [bf_1 + b(Bh+hB)](x\otimes y) .$$

So we can take $\phi_1(x\otimes y) = (f_1 + Bh+ hB)(x \otimes y)$. □

We will consider now the construction of a chain inverse for f . So we construct a natural D-module chain map $g:C_*^-(C^k \times C^\ell) \to C_*^-(C^k) \otimes_D C_*^-(C^\ell)$.

Lemma 2.5.

There exists a natural D-module chain map $g:C_*^-(C^k \times C^\ell) \to C_*^-(C^k) \otimes_D C_*^-(C^\ell)$ which coextends the Alexander-Whitney product.

Proof

Write $g = \sum_k g_k \theta^k$. In order for g to be a chain map we require

(i) $bg_k = -Bg_{k-1} + g_{k-1}B + g_k b$.

As in Lemma 2.3, we construct g by the method of acyclic models.

We can use equation (i) to construct $g_k(i_n \otimes i_m)$ by induction on k and on $n+m$, provided that we can start the induction by constructing $g_1(i_0,i_0)$ and $g_1(i_1,i_1)$.

We require:

$$
\begin{aligned}
bg_1(i_0,i_0) &= (g_0B - Bg_0)(i_0,i_0) \\
&= [t_1s_0i_0\otimes i_0 + i_0\otimes t_1s_0i_0 + s_0i_0\otimes i_0 + i_0\otimes s_0i_0] \\
&\quad -[t_1s_0i_0\otimes i_0 + s_0i_0\otimes i_0 + i_0\otimes t_1s_0i_0 + i_0\otimes s_0i_0] \\
&= 0 \quad .
\end{aligned}
$$

So we can take $g_1(i_0,i_0) = 0$.

We require:

$$
\begin{aligned}
bg_1(i_1,i_1) &= (g_0B - Bg_0 + g_1b)(i_1,i_1) \\
&= [d_1i_1\otimes t_2^2s_0i_1 - i_1\otimes t_1i_1 + t_1i_1\otimes i_1 + t_2s_1i_1\otimes d_0i_1 - d_1i_1\otimes s_0t_1i_1 \\
&\quad - s_0t_1i_1\otimes d_0i_1 - t_1i_1\otimes s_0d_0i_1] \\
&\quad -[t_1s_0d_0i_1\otimes i_1 + s_0d_0i_1\otimes i_1 + d_0i_1\otimes t_2^2s_0i_1 - d_0i_1\otimes s_0t_1i_1 + t_2s_1i_1\otimes d_1i_1 \\
&\quad + s_0t_1i_1\otimes d_1i_1 + i_1\otimes t_1s_0d_0i_1] \\
&= b[i_1\otimes t_2^2s_0i_1 - t_2s_1i_1\otimes i_1 - t_1i_1\otimes s_0t_1i_1 - s_0t_1i_1\otimes i_1] \quad .
\end{aligned}
$$

So we can take $g_1(i_1,i_1) = [i_1\otimes t_2^2s_0i_1 - t_2s_1i_1\otimes i_1 - t_1i_1\otimes s_0t_1i_1 - s_0t_1i_1\otimes i_1]$. $\square$

There is an exact analogue of Lemma 2.4; any natural chain equivalence $\gamma_0:(C^k\times C^\ell)_* \to C_*^k \otimes C_*^\ell$, such that $\gamma_0(x_0,y_0) = x_0 \otimes y_0$, can be coextended to a natural D-module chain map $\gamma:C_*^-(C^k\times C^\ell) \to C_*^-(C^k) \otimes_D C_*^-(C^\ell)$. As before we take $\gamma_1 = g_1 + (Bh+hB)$ to start the induction.

We now wish to show that the maps f and g are chain equivalences. First we prove the following lemma.

Lemma 2.6.

Let f_0 be a natural chain equivalence $(C^k)_* \otimes (C^\ell)_* \to (C^k \times C^\ell)_*$ such that $f_0(x_0 \otimes y_0) = (x_0,y_0)$, g_0 be a natural chain equivalence $(C^k\times C^\ell)_* \to (C_*^k) \otimes (C^\ell)_*$ such that $g_0(x_0,y_0) = x_0 \otimes y_0$. Let f be a natural coextension of f_0, g a natural coextension of g_0. Let $\psi = fg - 1$, a natural chain map, $\psi:C_*^-(C^k\times C^\ell) \to C_*^-(C^k\times C^\ell)$. Then there exists a natural D-module chain map $J = \Sigma j_k\theta^k$, $J : C_*^-(C^k\times C^\ell) \to C_*^-(C^k\times C^\ell)$, such that $j_0 = 0$ and J is chain homotopic to ψ.

Proof

Write $\psi = \Sigma\psi_k\theta^k$; then $\psi_0 = f_0g_0 - 1$. By the Eilenberg-Zilber theorem, there exists a chain homotopy $h:(C^k\times C^\ell)_* \to (C^k\times C^\ell)_*$ such that $\psi_0 = f_0g_0 - 1 = bh + hb$.

To construct J, set $j_0 = 0$, $j_1 = \psi_1 - (Bh+hB)$, $j_n = \psi_n$ for all $n > 1$. Then $\psi - J = \Sigma[\psi_k - j_k]\theta^k$

$$= \psi_0 + [\psi_1 - j_1]\theta$$

$$= bh + hb + (Bh + hB)\theta$$

$$= (b + B\theta)h + h(b + B\theta)$$

$$= \partial h + h\partial \ .$$

So a chain homotopy between ψ and J is given by h, where $h(\theta^k \otimes (x,y)) = \theta^k \otimes h(x,y)$.

In order to prove that J is a chain map, it is sufficient to prove that $\psi - J$ is a chain map, since ψ is given to be a chain map. That is, we need to show that $\partial(\psi-J) = (\psi-J)\partial$. However, $\partial(\psi-J) = \partial(\partial h + h\partial) = \partial h\partial = (\partial h + h\partial)\partial = (\psi-J)\partial$. □

The Eilenberg-Zilber theorem can be used again to prove an exact analogue of Lemma 2.6. Given g and f satisfying the conditions of Lemma 2.6, there is a natural D-module chain map $K = \Sigma k_i\theta^i : C_*^-(C^k)\otimes_D C_*^-(C^\ell) \to C_*^-(C^k) \otimes_D C_*^-(C^\ell)$, such that $k_0 = 0$ and K is chain homotopic to $gf - 1$.

We now prove that these maps are indeed chain equivalences.

Lemma 2.7.

(i) Any natural D-module chain map $f:C^-_*(C^k) \otimes_D C^-_*(C^\ell) \to C^-_*(C^k \times C^\ell)$, which coextends a natural chain equivalence $f_0:C^k_* \otimes C^\ell_* \to (C^k \times C^\ell)_*$ such that $f_0(x_0 \otimes y_0) = (x_0,y_0)$, is a chain equivalence.

(ii) Any natural D-module chain map $g:C^-_*(C^k \times C^\ell) \to C^-_*(C^k) \otimes_D C^-_*(C^\ell)$, which coextends a natural chain equivalence $g_0:(C^k \times C^\ell)_* \to C^k_* \otimes C^\ell_*$ such that $g_0(x_0,y_0) = x_0 \otimes y_0$, is a chain equivalence.

Proof

Let $\psi = gf - 1$; by Lemma 2.6, ψ is chain homotopic to ϕ, where $\phi_0 = 0$. Let $\theta = fg - 1$; again, θ is chain homotopic to χ, where $\chi_0 = 0$. Thus we wish to construct chain homotopies h and j such that $\partial h + h\partial = \phi$, $\partial j + j\partial = \chi$.

We will look for a map h of the form $h(\theta^k \otimes x \otimes y) = \sum_{i=0}^{\infty} \theta^{k+i} \otimes h_{k,i}(x \otimes y)$; note that we do not require h to be a D-module map. Equating θ-coefficients in the equation $\partial h + h\partial = \phi$, we obtain the following:

$$(1) \qquad bh_{k,j} + h_{k,j}b + Bh_{k,j-1} + h_{k+1,j-1}B = \phi_j .$$

We rewrite this as

$$(2) \qquad bh_{k,j} = \phi_j - (h_{k,j}\,b + Bh_{k,j-1} + h_{k+1,j-1}\,B)$$

where $h_{k,-1}$ is taken to be 0.

We wish to construct $h_{k,j}(x \otimes y)$ by induction, using equation (2). First, $h_{k,0}(x_0 \otimes y_0)$ is defined, then equation (2) can be used to construct $h_{k,0}(x \otimes y)$ by induction on the degree of $x \otimes y$. Next, equation (2) is used to define $h_{k,j}(x_0 \otimes y_0)$ by induction on j, and finally to define $h_{k,j}(x \otimes y)$ for all $x \otimes y$ by induction on the degree of $x \otimes y$, and on j.

Thus under the inductive hypothesis that $h_{k,\ell}(x \otimes y)$ is defined for all k, for all $\ell < j$, and for all $x \otimes y$ of degree $< q$, the right hand side of equation (2), Z, is defined on $i_n \otimes i_m$, where $n + m = q$. We now check that this is a cycle for b. Applying b, and using the relations $bB = -Bb$ and $b\phi_j + B\phi_{j-1} = \phi_{j-1}B + \phi_j b$, we obtain

$$\begin{aligned} bZ &= b\phi_j - (bh_{k,j})b + B(bh_{k,j-1}) - (bh_{k+1,j-1})B \\ &= b\phi_j + [-\phi_j + h_{k,j}b + Bh_{k,j-1} + h_{k+1,j-1}B]b + B[\phi_{j-1} - h_{k,j-1}b - Bh_{k,j-2} \\ &\quad - h_{k+1,j-2}B] + [-\phi_{j-1} + h_{k+1,j-1}b + Bh_{k+1,j-2} + h_{k+2,j-2}B]B \\ &= [b\phi_j - \phi_j b + B\phi_{j-1} - \phi_{j-1}B] + Bh_{k,j-1}b - h_{k+1,j-1}bB - Bh_{k,j-1}b \\ &\quad - Bh_{k+1,j-2}B + h_{k+1,j-1}bB + Bh_{k+1,j-2}B \\ &= 0 \ . \end{aligned}$$

Thus, since $bZ = 0$, $Z = bW$ and $h_{k,j}(x \otimes y) = W$, provided that the degree of $bh_{k,j}(x \otimes y)$ is ≥ 3. Thus we need to construct directly $h_{k,0}(x \otimes y)$, for $x \otimes y$ of degree ≤ 2, and $h_{k,1}(x_0 \otimes y_0)$. These must satisfy the equations

(i) $(bh_{k,0} + h_{k,0}b)(x \otimes y) = 0$, where degree $(x \otimes y) \leq 2$.

(ii) $(bh_{k,1}+h_{k,1}b + Bh_{k,0}+h_{k+1,0}B)(\theta^k \otimes i_0 \otimes i_0) = \phi_1(\theta^k \otimes i_0 \otimes i_0)$.

Now $\phi_1(\theta^k \otimes i_0 \otimes i_0)$ lies in $\theta^{k+1} \otimes [(C^0)_* \otimes (C^0)_*](2)$; $HH_2(C^0 \otimes C^0)$ is isomorphic to F, with generating cycle $Bi_0 \otimes Bi_0$. So $\phi_1(\theta^k \otimes i_0 \otimes i_0)$ is a linear combination of $\theta^{k+1} \otimes Bi_0 \otimes Bi_0$, and a hochschild boundary, which can be dealt with in the term $bh_{k,1}(\theta^k \otimes i_0 \otimes i_0)$. Thus it is sufficient to construct $h_{k,0}$ satisfying the following equations:

(i) $(bh_{k,0} + h_{k,0}b)(x \otimes y) = 0$

(iii) $(Bh_{k,0} + h_{k+1,0}B)(\theta^k \otimes i_0 \otimes i_0) = \theta^{k+1} \otimes Bi_0 \otimes Bi_0$.

A solution for this is given by $h_{k,0}(\theta^k \otimes x \otimes y) = k\theta^k \otimes Bx \otimes y$.

We check (i); let $\alpha = (-1)^{\text{degree } x}$

$$(bh_{k,0}+h_{k,0}b)(\theta^k \otimes x \otimes y) = b(k\theta^k \otimes Bx \otimes y) + h_{k,0}(\theta^k \otimes bx \otimes y) + \alpha\theta^k \otimes x \otimes by)$$

$$= k\theta^k \otimes bBx \otimes y - \alpha k\theta^k \otimes Bx \otimes by + k\theta^k \otimes Bbx \otimes y$$

$$+ \alpha k\theta^k \otimes Bx \otimes by$$

$$= 0 \quad .$$

We check(iii):

$$(Bh_{k,0}+h_{k+1,0}B)(\theta^k \otimes i_0 \otimes i_0) = kB(\theta^k \otimes Bi_0 \otimes i_0)+h_{k+1,0}(\theta^{k+1} \otimes Bi_0 \otimes i_0+\theta^{k+1} \otimes i_0 \otimes Bi_0)$$

$$= -k\theta^{k+1} \otimes Bi_0 \otimes Bi_0 + (k+1)\theta^{k+1} \otimes Bi_0 \otimes Bi_0$$

$$= \theta^{k+1} \otimes Bi_0 \otimes Bi_0 \quad .$$

We now construct the chain homotopy j such that $\partial j + j\partial = \chi$. Again we look for a map of the form $j(\theta^k \otimes (x,y)) = \sum_{i=0}^{\infty} \theta^{k+i} \otimes j_{k,i}(x,y)$. The construction by induction goes through as above, again provided that we can construct $j_{k,0}$ to satisfy the following equations:

(iv) $(bj_{k,0} + j_{k,0}b)(\theta^k \otimes (x,y)) = 0$, for $x \otimes y$ such that degree $x \otimes y \leq 2$.

(v) $\quad (Bj_{k,0} + j_{k+1,0}B)(\theta^k \otimes (i_0, i_0)) = \theta^{k+1} \otimes f_0(Bi_0 \otimes Bi_0)$.

Here $f_0(Bi_0 \otimes Bi_0)$ is the generating cycle for the Hochschild homology of $(C^0 \times C^0)$ in degree 2 . These equations are satisfied by setting $j_{k,0} = f_0\, h_{k,0}\, g_0$. Then, using the equations $bf_0 = f_0 b$, $bg_0 = g_0 b$ and $bh_{k,0} + h_{k,0}b = 0$, we check equation (iv):

$$\begin{aligned} b(f_0\, h_{k,0}g_0) + (f_0\, h_{k,0}g_0)b &= f_0(bh_{k,0})g_0 + f_0\, h_{k,0}g_0 b \\ &= -f_0\, h_{k,0}bg_0 + f_0\, h_{k,0}g_0 b \\ &= -f_0\, h_{k,0}g_0 b + f_0\, h_{k,0}g_0 b \\ &= 0 \; . \end{aligned}$$

To check (v), we calculate $j_{k,0}(\theta^k \otimes (i_0, i_0))$ and $j_{k,0}(\theta^k \otimes (i_1, i_1))$ explicitly:

$$j_{k,0}(\theta^k \otimes (i_0, i_0)) = k\theta^k \otimes f_0(Bi_0 \otimes i_0)$$

$$j_{k,0}(\theta^k \otimes (i_1, i_1)) = k\theta^k \otimes [f_0(Bd_0 i_1 \otimes i_1) + f_0(Bi_1 \otimes d_1 i_1)] \; .$$

Thus

$$\begin{aligned} Bj_{k,0}(\theta^k \otimes (i_0, i_0)) &= k\theta^{k+1} \otimes B(f_0(Bi_0 \otimes i_0)) \\ &= k\theta^{k+1} \otimes (t_2 s_1 - t_2^2 s_0)(t_1 s_0 i_0, i_0) \end{aligned}$$

$$= k\theta^{k+1} \otimes ((t_2^2 s_1 s_0 i_0, t_2 s_1 s_0 i_0) - (t_2 s_1 s_0 i_0, t_2^2 s_1 s_0 i_0))$$

$$= -k\theta^{k+1} \otimes f_0(Bi_0 \otimes Bi_0) \quad .$$

$$j_{k+1,0}\, B(\theta^k \otimes (i_0, i_0)) = j_{k+1,0}(\theta^{k+1} \otimes (Bi_0, Bi_0))$$

$$= (k+1)\theta^{k+1} \otimes [f_0(Bd_0Bi_0 \otimes Bi_0) + f_0(B^2 i_0 \otimes d_1 Bi_0)]$$

$$= (k+1)\theta^{k+1} \otimes f_0(Bi_0 \otimes Bi_0)$$

since $d_0 t_1 s_0 = 1$, so $Bd_0Bi_0 = Bi_0$.

Thus $(Bj_{k,0} + j_{k+1,0}B)(\theta^k \otimes (i_0, i_0)) = \theta^{k+1} \otimes f_0(Bi_0 \otimes Bi_0)$, as required.

□

Lemma 2.8.

Any two natural D-module chain maps from $C_*^-(C^k) \otimes_D C_*^-(C^\ell) \to C_*^-(C^k \times C^\ell)$, which are coextensions of natural chain equivalences satisfying the conditions of Lemma 2.6, differ by a chain homotopy.

Proof

The proof is standard, but we give it for completeness. Given two such maps, f and ϕ , we wish to show $f - \phi$ is chain homotopic to zero. However, if k is the chain homotopy between $\phi g - 1$ and 0 , h the chain homotopy between $\gamma f - 1$ and 0 , where g is a chain inverse for ϕ, γ a chain inverse for f ,

$$\begin{aligned} f - \phi &= \phi(gf - 1) - (\partial k + k\partial)f \\ &= \phi(\partial h + h\partial) - (\partial k + k\partial)f \\ &= \partial(\phi h - kf) + (\phi h - kf)\partial \end{aligned}$$

so $\phi h - kf$ is the required chain homotopy. □

There is an obvious analogue of this lemma:

Lemma 2.9.

Any two natural D-module chain maps from $C_*^-(C^k \times C^\ell)$ to $C_*^-(C^k \times C^\ell)$, which are coextension of natural chain equivalences satisfying the conditions of Lemma 2.6, differ by a chain homotopy.

Proof

As for Lemma 2.8. □

Corollary 2.10.

The product in $HC_*^-(\ )$ induced by f is graded commutative.

Proof

Consider the diagram

$$\begin{array}{ccc} C_*^-(P) \otimes_D C_*^-(Q) & \xrightarrow{f} & C_*^-(P \times Q) \\ S\downarrow & & \downarrow T \\ C_*^-(Q) \otimes_D C_*^-(P) & \xrightarrow{f} & C_*^-(Q \times P) \end{array}$$

where $S(a \otimes b) = (-1)^{\text{degree } a \ \text{degree } b}\, b \otimes a$, $T(c,d) = (d,c)$.

In order to show that the product is graded commutative up to chain homotopy, we need to show that there is a chain homotopy h such that $Tf - fS = \partial h + h\partial$. However, $Tf - fS = T(f-TfS)$, and TfS is an alternative choice of product $C^{-}_{*}(P) \otimes_{D} C^{-}_{*}(Q) \to C^{-}_{*}(P\times Q)$, so by Lemma 2.8, there is a chain homotopy j such that $f - TfS = \partial j + j\partial$. Then $Tf - fS = \partial(Tj) + (Tj)\partial$.

Corollary 2.11.

The product induced in cyclic cohomology by g is graded commutative.

Proof

As in Corollary 2.10, the existence of a chain homotopy k such that $gT - Sg = \partial k + k\partial$ follows from Lemma 2.9. □

Lemma 2.12.

The product f is associative.

Proof

To show that the product is associative, we need to construct a chain homotopy h such that

$$f(f(x\otimes y)\otimes z) - f(x \otimes f(y\otimes z)) = (\partial h+h\partial)(x\otimes y\otimes z) \quad .$$

We will look for a chain homotopy of the form $h = \Sigma h_k \theta^k$. Substituting for this, and for $f = \Sigma\theta^k f_k$, in the equation, and writing $f_i(f_j \otimes 1)(x \otimes y \otimes z)$ for $f_i(f_j(x \otimes y) \otimes z)$, we obtain on equating coefficients, the equation

$$\text{(i)} \quad bh_k = -(h_k b + Bh_{k-1} + h_{k-1}B) + \sum_{i=0}^{k} f_{k-i}(f_i \otimes 1) - \sum_{i=0}^{k} f_{k-i}(1 \otimes f_i) .$$

We wish to construct h_k by induction on k and the degree of $x \otimes y \otimes z$. Assume (i) holds for $j < k$, and for elements of degree $< m$. Let the degree of $x \otimes y \otimes z$ equal m ; then the right hand side of (i) is defined, let it be called Z . We check $bZ = 0$:

$$\begin{aligned} bZ &= -(bh_k)b + B(bh_{k-1}) - (bh_{k-1})B + \sum_i (bf_{k-i})(f_i \otimes 1) - \sum_i (bf_{k-i})(1 \otimes f_i) \\ &= [(h_k b + Bh_{k-1} + h_{k-1}B - \sum_i f_{k-i}(f_i \otimes 1)b + \Sigma f_{k-i}(1 \otimes f_i)b] \\ &\quad + [B(-h_{k-1}b - Bh_{k-2} - h_{k-2}B + \sum_i f_{k-1-i}(f_i \otimes 1) - \Sigma f_{k-1-i}(1 \otimes f_i)] \\ &\quad - [(h_{k-1}b + Bh_{k-2} + h_{k-2}B - \sum_i f_{k-1-i}(f_i \otimes 1) + \Sigma f_{k-1-i}(1 \otimes f_i)] \\ &\quad + \sum_i (f_{k-i}b + f_{k-1-i}B - Bf_{k-1-i})(f_i \otimes 1) \\ &\quad - \sum_i (f_{k-i}b + f_{k-1-i}B - Bf_{k-1-i})(1 \otimes f_i) \end{aligned}$$

$$= [Bh_{k-1}b - h_{k-1}bB - Bh_{k-1}b - Bh_{k-2}B + h_{k-1}bB + Bh_{k-2}B]$$

$$+ \Sigma f_{k-i}b(f_i \otimes 1) - \Sigma f_{k-i}b(1 \otimes f_i) + \Sigma f_{k-1-i}B(f_i \otimes 1) - \Sigma f_{k-i-1}B(1 \otimes f_i)$$

$$- \Sigma f_{k-i}(f_i \otimes 1)b + \Sigma f_{k-i}(1 \otimes f_i)b - \Sigma f_{k-i-1}(f_i \otimes 1)B + \Sigma f_{k-i-1}(1 \otimes f_i)B$$

$$= \Sigma f_{k-i}(bf_i \otimes 1) - \Sigma f_{k-i}(1 \otimes bf_i) + \Sigma f_{k-1-i}(Bf_i \otimes 1) - \Sigma f_{k-i-1}(1 \otimes Bf_i)$$

$$- \Sigma f_{k-i}(f_i b \otimes 1) + \Sigma f_{k-i}(1 \otimes f_i b) - \Sigma f_{k-i-1}(f_i B \otimes 1) - \Sigma f_{k-1-i}(1 \otimes f_i B)$$

$$= 0 .$$

Now since we are constructing h in the model complex $C_*^-(C^k \times C^\ell \times C^m)$, where the columns have Hochschild homology equal to the simplicial homology of $|C^k \times C^\ell \times C^m| \simeq S^1 \times S^1 \times S^1 \times \Delta^k \times \Delta^\ell \times \Delta^m$, and are thus acyclic in degrees greater than 3 , the equation $bZ = 0$ implies $Z = bW$ provided that the degree of Z is strictly greater than 3 . So we need to construct h_k directly when $k = 0$, and on elements of degree 0 and 1 for $k = 1$. We do so for the product constructed in Lemma 2.3.

We now show that the shuffle product is associative at the chain level, so we can take $h_0 = 0$. We are considering "shuffles", order-preserving maps $\underline{n} \amalg \underline{m} \to \underline{n+m}$, which are bijections on $(\underline{n} - \{0\}) \amalg (\underline{m} - \{0\})$. Hence for associativity, it is sufficient to show that a product of shuffles $(\underline{n} \sqcup\!\sqcup \underline{m}) \sqcup\!\sqcup \underline{p} \to \underline{n+m+p}$ restricts to a shuffle $\underline{m} \sqcup\!\sqcup \underline{p} \to \underline{m+p}$, and thus occurs as a product of shuffles $\underline{n} \sqcup\!\sqcup (\underline{m} \sqcup\!\sqcup \underline{p}) \to \underline{n+m+p}$.

We now consider the construction of h_1 :

(a) $h_1(i_0 \otimes i_0 \otimes i_0)$. Write t_j^α for $t_j^\alpha s_{j-1} \cdots s_0 i_0$. Recall $f_1(i_0 \otimes i_0) = (t_2, t_2^2)$, $f_0(i_0 \otimes i_0) = (i_0, i_0)$. Then we have

$$f_0(f_1(i_0 \otimes i_0) \otimes i_0) = (t_2, t_2^2, 1)$$

$$f_0(i_0 \otimes f_1(i_0 \otimes i_0)) = (1, t_2, t_2^2)$$

$$f_1(f_0(i_0 \otimes i_0) \otimes i_0) = (t_2, t_2, t_2^2)$$

$$f_1(i_0 \otimes f_0(i_0 \otimes i_0)) = (t_2, t_2^2, t_2^2) .$$

So we require

$$\begin{aligned} bh_1(i_0 \otimes i_0 \otimes i_0) &= (t_2, t_2^2, 1) - (1, t_2, t_2^2) + (t_2, t_2, t_2^2) - (t_2, t_2^2, t_2^2) \\ &= b(-(t_3, t_3^2, t_3^3)) . \end{aligned}$$

Thus we can take $h_1(i_0 \otimes i_0 \otimes i_0) = -(t_3, t_3^2, t_3^3)$.

(b) $h_1(i_1 \otimes i_0 \otimes i_0)$. Recall $f_1(i_1 \otimes i_0) = -(t_3 s_2 s_1 i_1, t_3^2) + (t_3 s_2 s_0 i_1, t_3^2) + (t_3^2 s_1 s_0 i_1, t_3^3)$, $f_0(i_1 \otimes i_0) = i_1 \otimes s_0 i_0$. Then we have

$$f_0(f_1(i_1 \otimes i_0) \otimes i_0) = -(t_3s_2s_1i_1, t_3^3, 1) + (t_3s_2s_0i_1, t_3^2, 1) + (t_3^2s_1s_0i_1, t_3^3, 1)$$

$$f_0(i_1 \otimes f_1(i_0, i_0)) = -(s_2s_1i_1, t_3^2, t_3^3) + (s_2s_0i_1, t_3, t_3^3) - (s_1s_0i_1, t_3, t_3^2)$$

$$f_1(f_0(i_1 \otimes i_0) \otimes i_0) = -(t_3s_2s_1i_1, t_3, t_3^3) + (t_3s_2s_0i_1, t_3, t_3^2) + (t_3^2s_1s_0i_1, t_3^2, t_3^3)$$

$$f_1(i_1 \otimes f_0(i_0, i_0)) = -(t_3s_2s_1i_1, t_3^3, t_3^3) + (t_3s_2s_0i_1, t_3^2, t_3^2) + (t_3^2s_1s_0i_1, t_3^3, t_3^3) \quad .$$

Thus we require

$$\begin{aligned} bh_1(i_1 \otimes i_0 \otimes i_0) = {} & [-(s_2s_1i_1, t_3^2, t_3^3) + (t_3s_2s_1s_0d_0i_1, t_3^2, t_3^3) - (t_3s_2s_1i_1, t_3^2, t_3^3) \\ & \quad + (t_3s_2s_1i_1, t_3^3, t_3^3) - (t_3s_2s_1i_1, t_3^3, 1)] \\ & + [(s_2s_0i_1, t_3, t_3^3) - (t_3s_2s_1i_1, t_3, t_3^3) + (t_3s_2s_1i_1, t_3^2, t_3^3) \\ & \quad - (t_3s_2s_0i_1, t_3^2, t_3^3) + (t_3s_2s_0i_1, t_3^2, 1)] \\ & + [-(s_1s_0i_1, t_3, t_3^2) + (t_3s_2s_0i_1, t_3, t_3^2) - (t_3s_2s_0i_1, t_3^2, t_3^2) \\ & \quad + (t_3s_2s_0i_1, t_3^2, t_3^3) - (t_3s_1s_0i_1, t_3^2, t_3^3)] \\ & + [(t_3s_1s_0i_1, t_3^2, t_3^3) - (t_3s_2s_1s_0d_1i_1, t_3^2, t_3^3) + (t_3^2s_1s_0i_1, t_3^2, t_3^3) \\ & \quad - (t_3^2s_1s_0i_1, t_3^3, t_3^3) + (t_3^2s_1s_0i_1, t_3^3, 1)] \\ = {} & b[-(t_4s_3s_2s_1i_1, t_4^3, t_4^4) + (t_4s_3s_2s_0i_1, t_4^2 . t_4^4) - (t_4s_3s_1s_0i_1, t_4^2 . t_4^3) \\ & \quad + (t_4^2s_2s_1s_0i_1, t_4^3, t_4^4)] \quad . \end{aligned}$$

So $h_1(i_1 \otimes i_0 \otimes i_0) = -(t_4 s_3 s_2 s_1 i_1, t_4^3, t_4^4) + (t_4 s_3 s_2 s_0 i_1, t_4^2, t_4^4)$

$$-(t_4 s_3 s_1 s_0 i_1, t_4^2, t_4^3) + (t_4^2 s_2 s_1 s_0 i_1, t_4^3, t_4^4) \quad .$$

(c) $h_1(i_0 \otimes i_1 \otimes i_0)$

Similarly, we require

$bh_1(i_0 \otimes i_1 \otimes i_0) =$

$$[(t_3, t_3^2 s_2 s_0 i_1, t_3^3) - (t_3, t_3^2 s_1 s_0 i_1, t_3^3) + (t_3^2, t_3^2 s_1 s_0 i_1, t_3^3) - (t_3^2, t_3^3 s_2 s_0 i_1, t_3^3)$$

$$+ (t_3^2, t_3^3 s_2 s_0 i_1, 1)]$$

$$+[(1, t_3 s_2 s_1 i_1, t_3^3) - (t_3, t_3 s_2 s_1 i_1, t_3^3) + (t_3, t_3^2 s_2 s_1 s_0 d_0 i_1, t_3^3) - (t_3, t_3^2 s_2 s_1 i_1, t_3^3)$$

$$+ (t_3, t_3^2 s_2 s_1 i_1, 1)]$$

$$-[(1, t_3 s_2 s_0 i_1, t_3^2) - (t_3, t_3 s_2 s_0 i_1, t_3^2) + (t_3, t_3^2 s_2 s_1 i_1, t_3^2) - (t_3, t_3^2 s_2 s_1 i_1, t_3^3)$$

$$+ (t_3, t_3^2 s_2 s_0 i_1, t_3^3)]$$

$$-[(1, t_3^2 s_1 s_0 i_1, t_3^3) - (t_3, t_3^2 s_1 s_0 i_1, t_3^3) + (t_3, t_3^2 s_1 s_0 i_1, t_3^3) - (t_3, t_3^3 s_2 s_0 i_1, t_3^3)$$

$$+ (t_3, t_3^3 s_1 s_0 i_1, 1)]$$

$$= b[(t_4^2, t_4^3 s_3 s_1 s_0 i_1, t_4^4) + (t_4, t_4^2 s_3 s_2 s_1 i_1, t_4^4) - (t_4, t_4^2 s_3 s_2 s_0 i_1, t_4^3) - (t_4, t_4^3 s_2 s_1 s_0 i_1, t_4^4)].$$

So we take $h_1(i_0 \otimes i_1 \otimes i_0) = (t_4^2, t_4^3 s_3 s_1 s_0 i_1, t_4^4) + (t_4, t_4^2 s_3 s_2 s_1 i_1, t_4^4)$

$$- (t_4, t_4^2 s_3 s_2 s_0 i_1, t_4^3) - (t_4, t_4^3 s_2 s_1 s_0 i_1, t_4^4) .$$

(d) $h_1(i_0 \otimes i_0 \otimes i_1)$.

Finally, we require

$bh_1(i_0 \otimes i_0 \otimes i_1) =$

$$[(1,t_3,t_3^3 s_1 s_0 i_1)-(t_3,t_3,t_3^3 s_1 s_0 i_1)+(t_3,t_3^2,t_3^3 s_1 s_0 i_1)-(t_3,t_3^2,t_3^3 s_2 s_1 s_0 d_1 i_1)$$

$$+(t_3,t_3^2,s_1 s_0 i_1)]$$

$$+[(t_3,t_3^2,t_3^3 s_2 s_1 i_1)-(t_3,t_3^2,t_3^3 s_2 s_0 i_1)+(t_3^2,t_3^2,t_3^3 s_2 s_0 i_1)-(t_3^2,t_3^3,t_3^3 s_2 s_0 i_1)$$

$$+(t_3^2,t_3^3,s_2 s_1 i_1)]$$

$$-[(1,t_3^2,t_3^3 s_2 s_0 i_1)-(t_3,t_3^2,t_3^3 s_2 s_0 i_1)+(t_3,t_3^2,t_3^3 s_1 s_0 i_1)-(t_3,t_3^3,t_3^3 s_1 s_0 i_1)$$

$$+(t_3,t_3^3,s_2 s_0 i_1)]$$

$$-[(1,t_3,t_3^2 s_2 s_1 i_1)-(t_3,t_3,t_3^2 s_2 s_1 i_1)+(t_3,t_3^2,t_3^2 s_2 s_1 i_1)-(t_3,t_3^2,t_3^3 s_2 s_1 s_0 d_0 i_1)$$

$$+(t_3,t_3^2,t_3^3 s_2 s_1 i_1)]$$

$$=b[(t_4,t_4^2,t_4^4s_2s_1s_0i_1)+(t_4^2,t_4^3,t_4^4s_3s_2s_0i_1)-(t_4,t_4^3,t_4^4s_3s_1s_0i_1)$$
$$-(t_4,t_4^2,t_4^3s_3s_2s_1i_1)] \; .$$

So we take $h_1(i_0 \otimes i_0 \otimes i_1) = (t_4,t_4^2,t_4^4s_2s_1s_0i_1)+(t_4^2,t_4^3,t_4^4s_3s_2s_0i_1)$

$$-(t_4,t_4^3,t_4^4s_3s_1s_0i_1)-(t_4,t_4^2,t_4^3s_3s_2s_1i_1) \; . \quad \square$$

Lemma 2.13.

The product induced by g is associative.

Proof

We introduce first further notation; given $(x,y) \in (P \times Q)(n)$, $g_k(x,y) \in \sum_i P(i) \otimes Q(n+2k-i)$, we write this as $g_k(x,y) = \sum_i \gamma_{k,\ell}^{P(i)}(x,y) \otimes \gamma_{k,\ell}^{Q(n+2k-i)}(x,y)$. Then for associativity, we require, given $(x,y,z) \in (P \times Q \times R)(n)$, a chain homotopy h such that

$$(i) \quad \sum_k \theta^k [\sum_{i,\alpha,\beta,r,s} \gamma_{k-i,r}^{P(\beta)}(\gamma_{i,s}^{(P\times Q)(\alpha)}((x,y),z)) \otimes \gamma_{k-i,r}^{Q(\alpha+2(k-i)-\beta)}$$
$$(\gamma_{i,s}^{(P\times Q)(\alpha)}((x,y),z)) \otimes \gamma_{i,s}^{R(n+2i-\alpha)}((x,y),z)$$

$$- \sum_{j,\gamma,\delta,t,u} \gamma_{j,t}^{P(n+2j-\gamma)}(x,(y,z)) \otimes \gamma_{k-j,u}^{Q(\gamma+2(k-j)-\delta)}(\gamma_{j,t}^{(Q\times R)(\gamma)}(x,(y,z))$$

$$\otimes \gamma_{k-j,u}^{R(\delta)}(\gamma_{j,t}^{(Q\times R)(\alpha)}(x,(y,z))]$$

$= \partial h + h\partial$.

As before, we look for a chain homotopy of the form $h = \Sigma h_k\theta^k$, and again we can use the method of acyclic models to construct h inductively, provided we can construct directly h_0 , and h_1 on elements of degree 0 or 1 . We work with the product given in Lemma 2.5.

h_0 is zero since the Alexander-Whitney product is associative at the chain level; given a simplex $\sigma^{p+q+r} \in C^{p+q+r}(p+q+r)$, let $\lambda_i(\sigma^{p+q+r})$ be the front i-face of the simplex, $\rho_j(\sigma^{p+q+r})$ be its back j-face, then the associativity follows from $\rho_q(\lambda_{p+q})\sigma^{p+q+r}$ $= \lambda_q(\rho_{q+r})\sigma^{p+q+r}$.

(a) $h_1(i_0,i_0,i_0)$: since $g_1(i_0,i_0) = 0$, taking the component of equation (i) with θ coefficient θ^0 , we require

$$bh_1(i_0,i_0,i_0) = 0 \quad ,$$

so we take $h_1(i_0,i_0,i_0) = 0$.

(b) $h_1(i_1,i_1,i_1)$.

$$\sum_{i,j,r,s} \gamma_{1,r}^{C^0(j)}(\gamma_{0,s}^{(C^0\times C^0)(i)}((i_1,i_1),i_1)) \otimes \gamma_{1,r}^{C^0(2+i-j)}(\gamma_{0,s}^{(C^0\times C^0)(i)}((i_1,i_1),i_1)) \otimes \gamma_{0,s}^{C^0(1-i)}((i_1,i_1),i_1)$$

$$=[(i_1\otimes t_2^2 s_0 i_1 - t_1 i_1 \otimes s_0 t_1 i_1) + (-t_2 s_1 i_1 \otimes i_1 - s_0 t_1 i_1 \otimes i_1)]\otimes d_1 i_1$$

$$\sum_{i,j,r,s} \gamma_{0,r}^{C^0(j)}(\gamma_{1,s}^{(C^0\times C^0)(i)}((i_1,i_1),i_1)) \otimes \gamma_{0,r}^{C^0(j-i)}(\gamma_{1,s}^{(C^0\times C^0)(i)}((i_1,i_1),i_1)) \otimes \gamma_{1,s}^{C^0(3-i)}((i_1,i_1),i_1)$$

$$= [d_0 i_1 \otimes i_1 + i_1 \otimes d_1 i_1] \otimes t_2^2 s_0 i_1$$

$$+[-(d_0 i_1 \otimes t_2 s_1 i_1 + d_1 i_1 \otimes s_0 t_1 i_1) + (-i_1 \otimes t_1 i_1 + t_1 i_1 \otimes s_0 d_0 i_1) - (t_2 s_1 i_1 \otimes d_0 i_1 + s_0 t_1 i_1 \otimes d_0 i_1)]\otimes i_1$$

$$+[-d_1 i_1 \otimes t_1 i_1 - t_1 i_1 \otimes d_0 i_1]\otimes s_0 t_1 i_1$$

$$\sum_{i,j,r,s} \gamma_{0,r}^{C^0(i)}(i_1,(i_1,i_1)) \otimes \gamma_{1,s}^{C^0(j)}(\gamma_{0,r}^{(C^0\times C^0)(1-i)}(i_1,(i_1,i_1))) \otimes \gamma_{1,s}^{C^0(3-i-j)}(\gamma_{0,r}^{(C^0\times C^0)(1-i)}(i_1,(i_1,i_1)))$$

$$= d_0 i_1 \otimes [(i_1 \otimes t_2^2 s_0 i_1 - t_1 i_1 \otimes s_0 t_1 i_1) + (-t_2 s_1 i_1 \otimes i_1 - s_0 t_1 i_1 \otimes i_1)]$$

$$\sum_{i,j,r,s} \gamma_{1,s}^{C^0(i)}(i_1,(i_1,i_1)) \otimes \gamma_{0,r}^{C^0(j)}(\gamma_{1,s}^{(C^0 \times C^0)(3-i)}(i_1,(i_1,i_1))$$

$$\otimes \gamma_{0,r}^{C^0(3-i-j)}(\gamma_{1,s}^{(C^0 \times C^0)(3-i)}(i_1,(i_1,i_1))$$

$$= i_1 \otimes [d_1 i_1 \otimes t_2^2 s_0 i_1 - t_1 i_1 \otimes i_1 + t_2^2 s_0 i_1 \otimes d_1 i_1] + t_1 i_1 \otimes [-d_1 i_1 \otimes s_0 t_1 i_1 - t_1 i_1 \otimes s_0 d_0 i_1$$
$$- s_0 t_1 i_1 \otimes d_0 i_1]$$

$$+ t_2 s_1 i_1 \otimes [-d_0 i_1 \otimes i_1 - i_1 \otimes d_1 i_1] + s_0 t_1 i_1 \otimes [-d_0 i_1 \otimes i_1 - i_1 \otimes d_1 i_1] \ .$$

Thus, taking the component of equation (i) with θ coefficient θ, we require

$$bh_1(i_1,i_1,i_1) = [t_1 i_1 \otimes s_0 d_0 i_1 \otimes i_1 + t_1 i_1 \otimes s_0 t_1 i_1 \otimes d_0 i_1 - t_1 i_1 \otimes s_0 t_1 i_1 \otimes d_1 i_1$$
$$- d_1 i_1 \otimes s_0 t_1 i_1 \otimes i_1 + d_0 i_1 \otimes s_0 t_1 i_1 \otimes i_1]$$
$$+ [-d_1 i_1 \otimes t_1 i_1 \otimes s_0 t_1 i_1 + d_0 i_1 \otimes t_1 i_1 \otimes s_0 t_1 i_1 + t_1 i_1 \otimes d_1 i_1 \otimes s_0 t_1 i_1$$
$$- t_1 i_1 \otimes d_0 i_1 \otimes s_0 t_1 i_1 - t_1 i_1 \otimes t_1 i_1 \otimes s_0 d_0 i_1]$$
$$= b[-t_1 i_1 \otimes s_0 t_1 i_1 \otimes i_1 - t_1 i_1 \otimes t_1 i_1 \otimes s_0 t_1 i_1] \ .$$

So we take $h_1(i_1,i_1,i_1) = -t_1 i_1 \otimes s_0 t_1 i_1 \otimes i_1 - t_1 i_1 \otimes t_1 i_1 \otimes s_0 t_1 i_1$. □

Products in cyclic cohomology.

Given an algebra A over F, we obtain a cyclic F-module $A^{\natural}$, and a dual cocyclic F-module $(A^{\natural})^*$, where $(A^{\natural})^*(n) = \mathrm{Hom}_F(A^{\natural}(n),F)$. The cyclic cohomology of $A \otimes B$ is obtained from the cocyclic F-module $(A^{\natural}\otimes B^{\natural})^*$, rather than the product of cocyclic F-modules $(A^{\natural})^* \times (B^{\natural})^*$, and the two are not in general isomorphic. We write $C^*(P)$ for $C^*((P^{\natural})^*)$, $C_*^-(P)$ for $C_*^-(P^{\natural})$.

A product in cohomology is thus obtained by dualising over D the map $g:C_*^-(A \otimes B) \to C_*^-(A) \otimes_D C_*^-(B)$, to give a map $C^*(A) \otimes_D C^*(B) \to C^*(A \otimes B)$. This is a natural D-module chain map, inducing a product in cohomology which is associative, graded commutative, and unique as a coextension of the product in Hochschild cohomology.

However, in order to obtain a map $C^*(A \otimes B) \to C^*(A) \otimes_D C^*(B)$ by dualising f, we require that either $C_*^-(A)$ or $C_*^-(B)$ be of finite type over D, for then $\mathrm{Hom}_D(C_*^-(A) \otimes_D C_*^-(B),D) \cong C^*(A) \otimes_D C^*(B)$. For the Kunneth theorems, this conditon may be weakened. We require either $HC_*^-(A)$ or $HC_*^-(B)$ to be of finite type, for then there is a chain complex C of finite type which is chain equivalent to $C_*^-(A)$ (respectively, $C_*^-(B)$) : this is proved in [26, Lemma 5.5.9, p.246].

Products in cyclic homology.

The product in cyclic cohomology $C^*(E) \otimes_D C^*(F) \to C^*(E \otimes F)$ induces a dual coproduct in cyclic homology $C_*(E \otimes F) \to C_*(E) \square_G C_*(F)$.

In looking for a product in cyclic homology, we see that since $C_*(E) = E_* \otimes K/\theta D$, the difficulty arises because K/D is not a ring, that is, there is no well-defined multiplication in the coefficients. There is an analogy here with $\mathbb{Q}/Z$ homology, and we adapt the product used in that case. The product is $x \otimes y \to x \cup \beta y$, where β is the Bockstein homomorphism $\beta: H_n(X : \mathbb{Q}/Z) \to H_{n-1}(X : Z)$, and the product uses the module multiplication $Z \otimes \mathbb{Q}/Z \to \mathbb{Q}/Z$.

Thus, given cycles for cyclic homology represented by $x \in (C_*^-(E) \otimes_D K/D)_n$ $y \in (C_*^-(F) \otimes_D K/D)_m$, we construct a product by first applying B to y to obtain an element of $(C_*^-(F))_{m+1}$, and then multiplying x and By by using the product $f: C_*^-(E) \otimes_D C_*^-(F) \to C_*^-(E \times F)$, and the module action $D \otimes K/D \to K/D$. Then we obtain a product $HC_n^-(E) \otimes_D HC_m^-(F)$ $\to HC_{n+m+1}^-(E \times F)$. We check that this agrees with the product defined by Loday and Quillen in [23, Chapter 3], that is $(x \otimes \theta^{-i}) \otimes (y \otimes \theta^{-j})$ $\to f_0(x \otimes By) \otimes \theta^{-i}$ if $j = 0$, 0 otherwise, where f_0 is the shuffle product.

Take cycles represented by $x = \Sigma x_i \otimes \theta^{-i}$, $y = \Sigma y_j \otimes \theta^{-j}$. Now $B[y] = [By_0] \in H_{m+1}(C_*^-(F))$, and for this class to be a cycle, we require $bBy_0 = 0$. Then our product, $f(x \otimes By)$, gives

$[\sum_i \sum_k f_k(x_i \otimes By_0) \otimes \theta^{-i}]$. However, we can show that this agrees with the product of Loday and Quillen by constructing a chain homotopy. Loday and Quillen prove that for any $\alpha \in E(n)$, $\beta \in F(m)$, $Bf_0(\alpha \otimes B\beta) = f_0(B\alpha \otimes B\beta)$: thus the map ϕ_0 defined by $\phi_0(x \otimes y) = f_0(x \otimes By)$ is a chain map. Similarly, $\phi(x \otimes y) = f(x \otimes By)$ is a chain map, so we require a chain homotopy h satisfying $\partial h + h\partial = \phi - \phi_0$. We look for one of the form $h = \Sigma h_k \theta^k$, then equating coefficients, the equation becomes

$$\text{(i)} \qquad bh_k = -h_k b - h_{k-1}B - Bh_{k-1} + \phi_k$$

if $k > 0$, and

$$\text{(ii)} \qquad bh_0 = -h_0 b$$

if $k = 0$.

Thus we can take $h_0 = 0$, and since the right hand side of equation (i), Z , satisfies $bZ = 0$, we can use the equation to construct $h_k(x \otimes y)$ by induction on k and on the degree of $x \otimes y$, provided that the degree of Z is ≥ 3 . But $\phi_1(i_0 \otimes i_0) = f_1(i_0 \otimes Bi_0)$, of degree 3, thus the induction proceeds. Hence the products are chain homotopic and so agree in homology.

Loday and Quillen prove that this product is associative, and graded commutative provided that the field has characteristic zero. The associativity also follows from the associativity of f, and the graded commutativity can be proved for all fields as follows.

Given cycles $x \in HC_n(A)$, $y \in HC_m(B)$, represented by $\Sigma\, x_i \otimes \theta^{-i}$, $\Sigma\, y_j \otimes \theta^{-j}$, we wish to show that

(i) $f(x \otimes By) - (-1)^{\text{degree } x \ \text{degree } y}\, Tf(y \otimes Bx) = 0$.

Thus we wish to find a chain z such that

(ii) $\sum_i \left(f_0(x_i \otimes By_0) - (-1)^{|x||y|}\, Tf(y_i \otimes Bx_0)\right) \otimes \theta^{-i} = \partial z$.

Let $z = (-1)^{|x|}\, \Sigma\, z_i \otimes \theta^{-i}$

Let $z_0 = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} f_{k+1}(x_\ell \otimes y_{k-\ell})$

Let $z_n = \sum_{k=0}^{\infty} \sum_{i=0}^{k+(n-1)} f_k(x_i \otimes y_{k+(n-1)-i})$ for $n > 0$.

We now wish to calculate ∂z; the term with θ coefficient θ^0 is

$$bz_0 + Bz_1 = \sum_{k=0}^{\infty} \sum_{i=0}^{k} (bf_{k+1} + Bf_k)(x_i \otimes y_{k-i})$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{k} (f_{k+1}b + f_kB)(x_i \otimes y_{k-i})$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} (-f_{k+1}B(x_i \otimes y_{k+1-i}) + f_k B(x_i \otimes y_{k-i}))$$

$$= f_0 B(x_0 \otimes y_0) = f_0(Bx_0 \otimes y_0) - (-1) \quad f_0(x_0 \otimes By_0) \quad .$$

The term with θ coefficient θ^p is

$$bz_p + Bz_{p+1} = \sum_{k=0}^{\infty} \Big(\sum_{i=0}^{k+p-1} bf_k(x \otimes y_{k+p-1-i}) + \sum_{i=0}^{k+p} Bf_k(x_i \otimes y_{k+p-i})$$

$$= \sum_{i=0}^{p-1} bf_0(x_i \otimes y_{p-1-i}) + \sum_{k=0}^{\infty} \sum_{i=0}^{k+p} bf_{k+1}(x_i \otimes y_{k+p-i}) + Bf_k(x_i \otimes y_{k+p-i})$$

$$= \sum_{i=0}^{p-1} bf_0(x_i \otimes y_{p-1-i}) + \sum_{k=0}^{\infty} \sum_{i=0}^{k+p} f_{k+1}b(x_i \otimes y_{k+p-i}) + f_k B(x_i \otimes y_{k+p-i})$$

$$= \sum_{i=0}^{p-1} f_0 b(x_i \otimes y_{p-1-i}) + \sum_{k=0}^{\infty} \sum_{i=0}^{k+p} -f_{k+1}B(x_i \otimes y_{k+1+p-i}) + f_k B(x_i \otimes y_{k+p-i})$$

$$= \sum_{i=0}^{p-1} [-f_0(Bx_{i+1} \otimes y_{p-1-i}) - (-1)^{|x|} f_0(x_i \otimes By_{p-i})] +$$

$$+ \sum_{i=0}^{p} [f_0(Bx_i \otimes y_{p-i}) + (-1)^{|x|} f_0(x_i \otimes By_{p-i})]$$

$$= (-1)^{|x|} f_0(x_p \otimes By_0) - f_0(Bx_0 \otimes y_p) \quad .$$

□

Finally, Loday and Quillen's result that $Bf_0(x \otimes By) = f_0(Bx \otimes By)$ can be used to prove the following Lemma.

Lemma 2.15.

The following diagram is commutative

$$\begin{array}{ccc} [HC^-_*(P) \otimes_D HC^-_*(Q)]_n & \xrightarrow{\quad f \quad} & HC^-_n(P\times Q) \\ \uparrow B\otimes B & & \uparrow B \\ [HC_*(P) \otimes_D HC_*(Q)]_{n-2} & \xrightarrow{\ \alpha\ } & HC_{n-1}(P\times Q) \end{array}$$

where α is the Loday Quillen product.

Proof

This follows from the commutativity of the diagram

$$\begin{array}{ccc} Bx_0 \otimes By_0 & \xrightarrow{\quad f \quad} & f_0(Bx_0 \otimes By_0) \\ \uparrow B\otimes B & & \uparrow B \\ (\Sigma x_i \otimes \theta^{-i}) \otimes (\Sigma y_j \otimes \theta^{-j}) & \xrightarrow{\quad \alpha \quad} & \sum_i f_0(x_i \otimes By_0) \otimes \theta^{-i} \end{array} \quad . \qquad \square$$

Module Structures in Cyclic Theories.

Lemma 2.16.

Given a cyclic F algebra E, $HH_*(E)$ is a module over $HC^-_*(E)$ and IB is a module map.

Proof

We use the inclusion map $I : HC^-_n(E) \to HH_n(E)$ to define the action. Given $[x] \in HH_n(E)$, $[y] \in HC^-_m(E)$, then $[Iy] \in HH_m(E)$, and $[f_0(x \otimes Iy)] \in HH_{n+m}(E \times E)$, and using the multiplication μ in E we obtain $[\mu f_0(x \otimes Iy)] \in HH_{n+m}(E)$. Denote this element $[y(x)]$. Then given $z \in HC^-_p(E)$, $(y \cup z)x = \mu f_0(x \otimes I(y \cup z))$ $= \mu f_0(x_0 \otimes \mu f_0(Iy \otimes Iz)) = \mu f_0(\mu f_0(x_0 \otimes Iy) \otimes Iz) = \mu f_0(y(x) \otimes Iz) = z\,(y(x))$.

To show that IB is a module map, we require $[IB\mu f_0(x \otimes Iy)]$ $= [\mu f_0(IBx \otimes Iy)]$ in $HH_*(E)$. We have constructed f_1 such that

$$bf_1(x \otimes Iy) = f_1 b(x \otimes Iy) - IBf_0(x \otimes Iy) + f_0 IB(x \otimes Iy)$$

Since x and Iy are Hochschild cycles, $b(x \otimes Iy) = 0$. Since $BI = 0$ from the long exact sequence relating HC^-_* and HH_*, $IB(x \otimes Iy) = IBx \otimes Iy$. Thus, in Hochschild homology,

$$[0] = -[IB\mu f_0(x \otimes Iy)] + [\mu f_0(IB\, x \otimes Iy)]\,. \qquad \square$$

By $F[\theta]$-duality we obtain the following lemma:

Lemma 2.17.

$HH^*(E)$ is a module over $HC^*(E)$, and IB is a module map. $\square$

Loday and Quillen prove in [23, Proposition 3.4] the following:

Lemma 2.18.

$HH_*(E)$ is a module over $HC_*(E)$, and I is a module map.

§3. The chern character.

The preceding chapter makes it clear that the theories $HC_*(\ )$ and $HC_*^-(\ )$ have products with very different properties, and that only the theory $HC_*^-(\ )$ has a degree-preserving product

$$HC_i^-(A) \otimes_D HC_j^-(B) \to HC_{i+j}^-(A \otimes B) \ .$$

Thus, since the product in K-theory is similarly degree-preserving, $K_i(A) \otimes K_j(B) \to K_{i+j}(A \otimes B)$, $HC_*^-(\ )$ is the only possible receiver for a multiplicative chern character $ch: K_*(A) \to HC_*^-(A)$.

Karoubi [19] gives a definition for a chern character into cyclic homology $HC_*(\ )$, and this can be modified, using a theorem of Jones [17], to give a chern character $ch: K_*(A) \to HC_*^-(A)$. The definition involves a composition of several maps, so we discuss these first individually.

(A) The Hurewicz homomorphism.

Higher algebraic K-theory is defined by $K_i(A) = \pi_i(BGL(A)^+)$, where $BGL(A)^+$ is described as follows. The group $GL(A)$ is the direct limit $\lim\limits_{k} GL_k(A)$, under the inclusions $GL_k(A) \to GL_{k+1}(A)$

given by $\alpha \to \begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}$, $BGL(A)$ is its classifying space. The Quillen + -construction is then applied, which abelianises the fundamental group of $BGLA$ while leaving its integral homology unchanged, by adjoining 2 cells to kill the generators of $[GL(A), GL(A)]$, and 3 cells to "neutralise" the 2 cells as far as the cohomology is concerned. The space $BGLA^+$ is a homotopy commutative and associative H-space.

The Hurewicz homomorphism $h_i : \pi_i(X) \to H_i(X)$ fits into a commutative diagram with the products in homotopy and homology as follows [28, Lemma 3.18]:

$$\begin{array}{ccc} \pi_i(X) \otimes \pi_j(Y) & \longrightarrow & \pi_{i+j}(X \wedge Y) \\ \downarrow h_i \otimes h_j & & \downarrow h_{i+j} \\ H_i(X) \otimes H_j(Y) & \longrightarrow & H_{i+j}(X \wedge Y) \end{array}$$

The product in K-theory is given by the composition of this smash product with a map $\gamma_* : \pi_i(BGL(A)^+ \wedge BGL(B)^+) \to \pi_i(BGL(A \otimes B)^+)$, defined as follows. The obvious map $GL(A) \times GL(B) \to GL(A \otimes B)$ induces a map $*$. $BGL(A)^+ \times BGL(B)^+ \to BGL(A \otimes B)^+$. Then given a basepoint x_0 for $BGL(A)^+$ and a basepoint y_0 for $BGL(B)^+$, there is a map $\tilde{\gamma}$, $\tilde{\gamma} : BGL(A)^+ \times BGL(B)^+ \to BGL(A \otimes B)^+$, given by

$$\tilde{\gamma}(x,y) = x*y - x_0*y - x*y_0 + x_0*y_0 \quad .$$

which is homotopically trivial on the wedge $BGL(A)^+ \vee BGL(B)^+$ and thus induces a map $\gamma : BGL(A)^+ \wedge BGL(B)^+ \to BGL(A\otimes B)^+$.

Thus we have a commutative diagram

$$\begin{array}{ccccc} \pi_i(BGL(A)^+ \otimes \pi_j(BGL(B)^+) & \to & \pi_{i+j}(BGL(A)^+ \wedge BGL(B)^+) & \xrightarrow{\gamma_*} & \pi_{i+j}(BGL(A\otimes B)^+) \\ \downarrow h_i\otimes h_j & & \downarrow h_{i+j} & & \downarrow h_{i+j} \\ H_i(BGL(A)^+)\otimes H_j(BGL(B)^+) & \to & H_{i+j}(BGL(A)^+ \wedge BGL(B)^+) & \xrightarrow{\gamma_*} & H_{i+j}(BGL(A\otimes B)^+) \end{array}$$

Using the isomorphism $H_i(BGL(A)^+) \cong H_i(BGL(A))$, we have a multiplicative map $K_i(A) \to H_i(BGL(A))$.

(B) A map $\psi : H_i(BG) \to HC^-_{i-2\ell}(k[G])$; here $G = GL(A)$.

This replaces the map in Karoubi's construction defined into $HC_{i+2\ell}(k[G])$.

The construction involves an equivariant homology theory G^T_* , related to the usual theory H^T_* and its localised version $\hat{H}^T_*$ by a long exact sequence

$$.. \quad G^T_n(z) \to \hat{H}^T_n(z) \to H^T_{n-2}(z) \to G^T_{n-1}(z) \to \ldots$$

Let the space BG have the trivial circle action, and let u be an indeterminate of degree - 2 . Then we have an inclusion $H_i(BG) \to (H_*(BG) \otimes k[u])_{i-2\ell} \cong G^T_{i-2\ell}(BG)$, and a map $G^T_{i-2\ell}(BG) \to G^T_{i-2\ell}(LBG)$, induced by the inclusion of BG as the fixed point set in the free loop space LBG , with the usual circle action.

Then, writing $S_*(G)$ for the chain complex of G , made into an associative differential graded algebra by using the Eilenberg McLane shuffle product, we can define $HC^-_*(S_*G)$. Jones constructs in [17] an isomorphism $G^T_{i-2\ell}(LBG) \cong HC^-_{i-2\ell}(S_*G)$. Finally, there is a map of differential graded algebras $k[G] \to S_*(G)$, where $k[G]$ has zero differential, which is a chain homotopy equivalence and induces an isomorphism $HC^-_j(S_*(G)) \cong HC^-_j(k[G])$; see [17, §7]. Thus we obtain the map $\psi : H_i(BG) \to HC^-_{i-2\ell}(k[G])$.

In order to show that ψ is multiplicative, it is sufficient to show the multiplicativity of the map $\xi : G^T_j(LBG) \to HC^-_j(S_*(G))$, that is, to prove the commutativity of the diagram

$$\begin{array}{ccccc} G^T_i(LBG)\otimes_{k[u]}G^T_j(LBG') & \to & G^T_{i+j}(LBG \times LBG') & \to & G^T_{i+j}(LB(G \times G')) \\ \downarrow \xi\otimes\xi & & & & \downarrow \xi \\ HC^-_i(S_*(G))\otimes_D HC^-_j(S_*(G')) & \to & HC^-_{i+j}(S_*(G)\otimes S_*(G')) & \to & HC^-_{i+j}(S_*(G \times G')) \end{array}$$

Recall that $HC^-_*(S_*(G))$ is the homology of the double complex

$C_*(S_*(G)) \otimes k[u]$, with boundary $b + Bu$, where u is an indeterminate of degree -2 , and $C_*(S_*(G))$ is the Hochschild complex of the algebra $S_*(G)$. $G_*^T(Z)$ is the homology of a double complex $S_*(Z) \otimes k[u]$, with boundary $b + Ju$, where J is defined, given the circle action $f : T \times Z \to Z$ and the shuffle product $\theta : S_*(T) \otimes S_*(Z) \to S_*(T \times Z)$, by $J(x) = (-1)^{|x|} f_*\theta(z \otimes x)$, for z the fundamental 1-cycle in $S_1(T)$: see [17, §4].

Jones shows that these two double complexes are naturally chain equivalent, so a natural product defined in one theory induces a natural product in the other. Then, by the uniqueness of the construction of the product proved in Chapter 2, any product defined using the models will agree with the induced product, ensuring the commutativity of the diagram.

(C) A map $\phi_* : HC_j^-(k[G]) \to HC_j^-(\widetilde{MA})$.

Here the infinite matrix algebra MA is the direct limit $\lim_k M_k(A)$ under the inclusion $M_k(A) \to M_{k+1}(A)$ given by $\alpha \to \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$, and $\widetilde{MA}$ is obtained from MA by adjoining a unit.

The map $\phi : k[G] \to \widetilde{MA}$ defined on the generators over k , by $\phi(g) = g$, is an algebra homomorphism.

The induced map ϕ_* is multiplicative as a map between functors of the algebra A, by the naturality of the product and the commutativity of the diagram

$$\begin{array}{ccc} k[GLA] \otimes k[GLB] & \xrightarrow{\phi\otimes\phi} & \widetilde{MA} \otimes \widetilde{MB} \\ \downarrow & & \downarrow \\ k[GL(A\otimes B)] & \xrightarrow{\phi} & \widetilde{M(A\otimes B)} \end{array}$$

(D) The induced map $Tr_* : HC_j^-(\widetilde{MA}) \to HC_j^-(A)$, where $Tr:\widetilde{MA} \to A$ is the trace map.

This is multiplicative as a map between functors of the algebra A, by the naturality of the product and the commutativity of the diagram

$$\begin{array}{ccc} \widetilde{MA} \otimes \widetilde{MB} & \xrightarrow{Tr\otimes Tr} & A \otimes B \\ \downarrow & & \downarrow \\ \widetilde{M(A\otimes B)} & \xrightarrow{Tr} & A \otimes B \end{array}$$

Then by composing the maps (A) - (D) we obtain the following:

Theorem 3.1.

There is a multiplicative chern character $ch:K_i(A) \to HC_{i-2\ell}^-(A)$ defined as the composition

$$K_i(A) = \pi_i(BGLA^+) \xrightarrow{h_i} H_i(BGLA) \xrightarrow{\psi} HC_{i-2\ell}^-(k[GLA]) \xrightarrow{\phi_*} HC_{i-2\ell}^-(\widetilde{MA}) \xrightarrow{Tr_*} HC_{i-2\ell}^-(A) . \qquad \square$$

§4. Kunneth Theorems.

Let P and Q be cyclic F-modules. In Chapter 2, we constructed a natural chain equivalence between $C^-_*(P) \otimes_D C^-_*(Q)$ and $C^-_*(P \times Q)$. The chain equivalences are D-module maps, and hence extend to give natural chain maps between $\hat{C}_*(P) \otimes_D \hat{C}_*(Q)$ and $\hat{C}_*(P \times Q)$. These maps are also chain equivalences, since the chain homotopies constructed in Lemma 2.7 also extend to $\hat{C}_*(P) \otimes_D \hat{C}_*(Q)$ and $\hat{C}_*(P \times Q)$. Then, using standard homological algebra for complexes over a principal ideal domain [26, Lemma 5.3.1, p.228], we obtain

Theorem 4.1.

Given cyclic F-modules P and Q

(i) There is an exact sequence of D-modules

$$0 \to (HC^-_*(P) \otimes_D HC^-_*(Q))_n \xrightarrow{f_*} HC^-_n(P \times Q) \to [\mathrm{Tor}_D(HC^-_*(P), HC^-_*(Q))]_{n-1} \to 0$$

where f_* is the product induced by the natural chain map f.

(ii) There is a D-module isomorphism

$$(\hat{HC}_*(P) \otimes_D \hat{HC}_*(Q))_n \cong \hat{HC}_n(P \times Q) . \qquad \square$$

We obtain a Kunneth theorem for the cyclic cohomology of algebras by dualising the chain equivalences as discussed in Chapter 2.

<u>Theorem 4.2</u>.

Let A and B be associative algebras over F .

(i) If one of $HC^-_*(A)$, $HC^-_*(B)$ is of finite type, there is an exact sequence of D-modules

$$0 \to (HC^*(A) \otimes_D HC^*(B))^n \xrightarrow{g^*} HC^n(A \otimes B) \to [\mathrm{Tor}_D(HC^*(A), HC^*(B))]^{n+1} \to 0$$

where g^* is the product induced by the natural chain map g .

(ii) If one of $\hat{HC}_*(A)$, $\hat{HC}_*(B)$ is of finite type, there is a D-module isomorphism

$$(\hat{HC}^*(A) \otimes_D \hat{HC}^*(B))^n \cong \hat{HC}^n(A \otimes B) . \qquad \square$$

We now consider a Kunneth theorem for cyclic homology. Given cyclic F modules P and Q , we have a chain equivalence between $C^-_*(P) \otimes_D C^-_*(Q)$ and $C^-_*(P \times Q)$, that is, between $P_* \otimes Q_* \otimes D$ and $(P \times Q)_* \otimes D$. This extends to a chain equivalence between $P_* \otimes Q_* \otimes K$ and $(P \times Q)_* \otimes K$. Hence there is a chain equivalence between the quotients, $P_* \otimes Q_* \otimes K/D$ and $(P \times Q)_* \otimes K/D$.

$K/\theta D$ may be given a coalgebra structure, with a coproduct $\theta^{-k} \to \sum_{i=0}^{k} \theta^{-k+i} \otimes \theta^{-i}$, when it is isomorphic to the coalgebra G , which is the dual of the algebra D . Eilenberg and Moore in [14] define an extended comodule over an F-coalgebra Γ, with coproduct δ to be a tensor product $A \otimes_F \Gamma$, where A is an F-module, with structure morphism $\nabla = 1_A \otimes \delta$. They prove in [14, Proposition 2.1] that for an extended comodule $A \otimes_F \Gamma$, and any comodule B ,

$$(A \otimes_F \Gamma)\square_\Gamma B \cong A \otimes_F B .$$

Thus, since $C_*(P)$ is an extended comodule $P_* \otimes_F G$, we have $C_*(P)\square_G C_*(Q) \cong P_* \otimes Q_* \otimes G$. Thus the chain equivalence between the quotients $P_* \otimes Q_* \otimes G$ and $(P \times Q)_* \otimes G$ is a chain equivalence between $C_*(P)\square_G C_*(Q)$ and $C_*(P \times Q)$.

We can now dualise over F the steps of the proof of the Kunneth short exact sequence for complexes which are D-modules, to obtain a dual short exact sequence for complexes C and C' which are comodules over G :

$$0 \to [\mathrm{Cotor}_G(H_*(C),H_*(C'))]_{n+1} \to H_n(C\square_G C') \to (H_*(C)\square_G H_*(C'))_n \to 0 \quad .$$

Combining this with the chain equivalence, we obtain the following theorem:

which is the dual of the algebra D . Eilenberg and Moore in [14] define an extended comodule over an F-coalgebra Γ, with coproduct δ to be a tensor product $A \otimes_F \Gamma$, where A is an F-module, with structure morphism $\nabla = 1_A \otimes \delta$. They prove in [14, Proposition 2.1] that for an extended comodule $A \otimes_F \Gamma$, and any comodule B ,

$$(A \otimes_F \Gamma)\Box_\Gamma B \cong A \otimes_F B .$$

Thus, since $C_*(P)$ is an extended comodule $P_* \otimes_F G$, we have $C_*(P)\Box_G C_*(Q) \cong P_* \otimes Q_* \otimes G$. Thus the chain equivalence between the quotients $P_* \otimes Q_* \otimes G$ and $(P \times Q)_* \otimes G$ is a chain equivalence between $C_*(P)\Box_G C_*(Q)$ and $C_*(P \times Q)$.

We can now dualise over F the steps of the proof of the Kunneth short exact sequence for complexes which are D-modules, to obtain a dual short exact sequence for complexes C and C' which are comodules over G :

$$0 \to [\mathrm{Cotor}_G(H_*(C),H_*(C'))]_{n+1} \to H_n(C\Box_G C') \to (H_*(C)\Box_G H_*(C'))_n \to 0 \quad .$$

Combining this with the chain equivalence, we obtain the following theorem:

Theorem 4.3.

Given cyclic F-modules P and Q, there is a short exact sequence of G-comodules

$$0 \to [\mathrm{Cotor}_G(HC_*(P),HC_*(Q))]_{n+1} \to HC_n(P\times Q) \to (HC_*(P)\square_G HC_*(Q))_n \to 0 \ . \qquad \square$$

The remainder of this chapter will be concerned with producing a re-expression of this sequence in terms of the D-module structure of $HC_*(P)$ and $HC_*(Q)$. We obtain the following short exact sequence:

$$0 \to (HC_*(P)\otimes_D HC_*(Q))_{n-1} \overset{\alpha}{\to} HC_n(P\times Q) \to [\mathrm{Tor}_D(HC_*(P),HC_*(Q))]_{n-2} \to 0 \ ,$$

where α is the Loday Quillen product.

Lemma 4.4.

There is a natural D-module isomorphism

$$\mathrm{Tor}_D(HC_*(P),HC_*(Q))_{n-2} \cong (HC_*(P)\square_G HC_*(Q))_n \ .$$

Proof

From the definition,

$$HC_*(P)\square_G HC_*(Q) = \{\sum_i a_i\otimes b_i \in HC_*(P)\otimes_F HC_*(Q) : \sum_i \nabla a_i\otimes b_i - \sum a_i\otimes\nabla b_i = 0\}$$

Recall the notation $\nabla a_i = \sum_k S^k a_i \otimes \gamma_k$, thus we can rewrite this as

$$\sum_i \nabla a_i \otimes b_i - a_i \otimes \nabla b_i = \sum_i \sum_k S^k a_i \otimes \gamma_k \otimes b_i - \sum_i \sum_k a_i \otimes \gamma_k \otimes S^k b_i \ .$$

Recall that a D-module structure was induced from the G-comodule structure by $\theta^k(a) = S^k a$. So we can rewrite the cotensor product as

$$HC_*(P) \square_G HC_*(Q) = \{\sum_i a_i \otimes b_i \in HC_*(P) \otimes_F HC_*(Q) : \sum_i \theta^k a_i \otimes b_i - a_i \otimes \theta^k b_i = 0 \text{ for all } k\} \ .$$

However, given that $\sum_i \theta a_i \otimes b_i - a_i \otimes \theta b_i = 0$, then $\sum \theta^2 a_i \otimes b_i - a_i \otimes \theta^2 b_i = 0$ if and only if $\sum_i (\theta \otimes 1 - 1 \otimes \theta)(\theta a_i \otimes b_i) = 0$. Similarly, given that $\sum a_i \otimes b_i \in \ker(\theta^j \otimes 1 - 1 \otimes \theta^j)$ for all $j < k$, the equation $\sum \theta^k a_i \otimes b_i - a_i \otimes \theta^k b_i = 0$ holds if and only if $\theta^{k-1} a_i \otimes b_i$ lies in $\ker(\theta \otimes 1 - 1 \otimes \theta)$. Thus we have

$$HC_*(P) \square_G HC_*(Q) = \{\sum_i a_i \otimes b_i \in HC_*(P) \otimes_F HC_*(Q) : \sum_i \theta^k a_i \otimes b_i \in \ker(\theta \otimes 1 - 1 \otimes \theta), \text{ for all } k\}.$$

Now consider $\mathrm{Tor}_D(HC_*(P), HC_*(Q))$; in order to calculate this we need to resolve $HC_*(P)$ as a D-module. Consider the generators of

$HC_*(P)$ as a free F-module. These can be placed in families related by the action of θ, that is, $(\{a_i^\alpha\}_{i\in \mathbb{N}} : \theta a_i^\alpha = a_{i-1}^\alpha)$. Note that the families either contain a single element, a_0^α such that $\theta a_0^\alpha = 0$, or contain a non-zero element for each $i \in \mathbb{N}$, so a_i^α corresponds to $\alpha\theta^{-i}$ for some $\alpha \in F$, and the family $\{a_i^\alpha\}$ corresponds to a set of generators for $K/\theta D$ as an F-module.

Let $HC_*^\infty(P)$ be the free D-module with one generator c_i^α for each of the a_i^α. Define a map $d : HC_*^\infty(P) \to HC_*(P)$ by $d(c_i^\alpha) = a_i^\alpha$. Define a map $\phi : HC_*^\infty(P) \to HC_*^\infty(P)$ by $\phi(c_i^\alpha) = \theta c_i^\alpha - c_{i-1}^\alpha$, putting $\phi(c_i^\alpha) = \theta c_i^\alpha$ if a_i^α satisfies $\theta a_i^\alpha = 0$. Then we have

$$d\phi(c_i^\alpha) = d(\theta c_i^\alpha - c_{i-1}^\alpha) = \theta a_i^\alpha - a_{i-1}^\alpha = a_{i-1}^\alpha - a_{i-1}^\alpha = 0 \quad .$$

We now wish to show that the elements $(\theta c_i^\alpha - c_{i-1}^\alpha)$ generate the kernel of d.

Take $\sum_{j=1}^{n} f_j c_j^\alpha \in \ker d$, so this corresponds to $\sum_{j=1}^{n} f_j \alpha\theta^{-j} \in D$, $\sum_{j=1}^{n} f_j \alpha\theta^{n-j} \in \theta^n D$. We want to show that there exist coefficients g_j such that $\sum_{j=1}^{n} f_j c_j^\alpha = \sum_{j=1}^{n} g_j(\theta c_j^\alpha - c_{j-1}^\alpha)$. In the right hand side of this expression, the coefficient of c_i^α is $g_n\theta$ if $i = n$, $g_i\theta - g_{i+1}$ if $i < n$. Thus we wish to show that there exist g_j such that $f_n = g_n\theta$, $f_i = g_i\theta - g_{i+1}$.

Now we have $\alpha(f_n + f_{n-1}\theta \ldots f_1\theta^{n-1}) \in (\theta^n)$, which implies that $f_n \in (\theta)$, so $f_n = g_n\theta$. Then, since $\alpha(g_n+f_{n-1}+f_{n-2}\theta + \ldots f_1\theta^{n-2}) \in (\theta^{n-1})$, we have $g_n + f_{n-1} \in (\theta)$, so $f_{n-1} = g_{n-1}\theta - g_n$. We can continue in this manner to obtain the required equations for all i , $1 < i < n-1$.

Thus the following sequence is exact:

$$0 \to HC^{\infty}_{*}(P) \xrightarrow{\phi} HC^{\infty}_{*}(P) \xrightarrow{d} HC_{*}(P) \to 0 \quad .$$

Thus we have the following:

$$0 \to Tor_D(HC_*(P),HC_*(Q)) \to HC^{\infty}_*(P)\otimes_D HC_*(Q) \xrightarrow{\phi\otimes 1} HC^{\infty}_*(P)\otimes_D HC_*(Q) \xrightarrow{d\otimes 1} HC_*(P)\otimes_D HC_*(Q) \to 0 \ .$$

Note that in $HC^{\infty}_*(P)\otimes_D HC_*(Q)$, $\theta^k\alpha\otimes\beta = \alpha\otimes\theta^k\beta$, so we can obtain a representative of any term which has no power of θ in the first component, that is, a representative of the form $\Sigma\, a_i\otimes b_i$ where $a_i \in HC_*(P)$, $b_i \in HC_*(Q)$. Thus there is an injection $i : HC^{\infty}_*(P)\otimes_D HC_*(Q) \to HC_*(P)\otimes_F HC_*(Q)$.

We now have a commutative diagram

$$\begin{array}{ccc}
HC^{\infty}_*(P)\otimes_D HC_*(Q) & \xrightarrow{\quad i \quad} & HC_*(P)\otimes_F HC_*(Q) \\
\downarrow \phi\otimes 1 & & \downarrow (1\otimes\theta-\theta\otimes 1) \\
HC^{\infty}_*(P)\otimes_D HC_*(Q) & \xrightarrow{\quad i \quad} & HC_*(P)\otimes_F HC_*(Q)
\end{array}$$

since $i(\phi\otimes 1)\Sigma a_i\otimes b_i = i((\Sigma\theta a_i - a_{i-1})\otimes b_i) = \Sigma a_i\otimes\theta b_i - \theta a_i\otimes b_i = (1\otimes\theta-\theta\otimes 1)\Sigma a_i\otimes b_i$.
From this we see that $\ker(\phi\otimes 1) \cong \ker(1\otimes\theta-\theta\otimes 1)$. Then, since $\phi\otimes 1$ is a D-module map,

$$\ker(\phi\otimes 1) = \{\Sigma a_i\otimes b_i \in HC_*(P)\otimes_F HC_*(Q) : \Sigma\theta^k a_i\otimes b_i \in \ker(\theta\otimes 1 - 1\otimes\theta) \text{ for all } k\} .$$

Thus we have $\mathrm{Tor}_D(HC_*(P),HC_*(Q)) = (HC_*(P)\square_G HC_*(Q))$, and since ϕ increases degree by 2 , we have, as required,

$$\mathrm{Tor}_D(HC_*(P),HC_*(Q))_{n-2} \cong (HC_*(P)\square_G HC_*(Q))_n . \qquad \square$$

There is a similar result for the other term of the short exact sequence.

<u>Lemma 4.5</u>.

There is a natural D-module isomorphism

$$\mathrm{Cotor}_G(HC_*(P),HC_*(Q))_n \cong (HC_*(P)\otimes_D HC_*(Q))_{n-2} .$$

<u>Proof</u>

To calculate Cotor, we need an injective G-comodule resolution of $HC_*(P)$: we use the following

$$0 \to HC_*(P) \xrightarrow{\nabla_p} HC_*(P)\otimes_F G \xrightarrow{\phi} HC_*(P)\otimes_F G \to 0$$

where $\phi([\Sigma(a_k\otimes\gamma_k)]\otimes\gamma_\ell) = [\Sigma a_k\otimes\gamma_k]\otimes\gamma_{\ell-1} - [\Sigma a_k\otimes\gamma_{k-1}]\otimes\gamma_\ell$.

We check exactness: certainly $\mathrm{im}\nabla_p \subset \ker\phi$. Suppose $\sum_j[\sum_k a_{j,k}\otimes\gamma_k]\otimes\gamma_j$

$\in \ker\phi$, that is, $\sum_j([\sum_k a_{j,k}\otimes\gamma_k]\otimes\gamma_{j-1} - [\Sigma a_{j,k}\otimes\gamma_{k-1}]\otimes\gamma_j = 0$.

Then, since $[\Sigma a_{j,k}\otimes\gamma_k] \neq 0$, we have $\sum_j\sum_k a_{j,k}\otimes\gamma_k\otimes\gamma_{j-1} - a_{j,k}\otimes\gamma_{k-1}\otimes\gamma_j = 0$.

Evaluating coefficients of $\gamma_k\otimes\gamma_{j-1}$, $a_{j,k} = a_{j-1,k+1}$.

Evaluating coefficients of $\gamma_{k-1}\otimes\gamma_j$, $a_{j,k} = a_{j+1,k-1}$.

Proceeding like this, we obtain $a_{j,k} = a_{j+i,k-i}$ for all i , $-j \le i \le k$.

Thus we can rewrite $\sum_j[\sum_k a_{j,k}\otimes\gamma_k]\otimes\gamma_j$ as $\sum_m[\sum_{i=0}^m a_m\otimes\gamma_i]\otimes\gamma_{m-i}$.

Then, since $\nabla(\gamma_\ell) = \sum_{i=0}^{\ell}\gamma_{\ell-i}\otimes\gamma_i$, this element lies in the image of ∇_p .

We then have the following exact sequence:

$$0 \to HC_*(P)\square_G HC_*(Q) \xrightarrow{\nabla_p\square 1} HC_*(P)\otimes_F HC_*(Q) \xrightarrow{\phi\square 1} HC_*(P)\otimes_F HC_*(Q) \to \mathrm{Cotor}_G(HC_*(P),HC_*(Q)) \to 0 ,$$

by using Eilenberg-Moore's Proposition 2.1 [14], $(A\otimes_F G)\square_G B \cong A\otimes_F B$.
Then to calculate $\phi\square 1$, we need to use Eilenberg and Moore's isomorphism explicitly. The map $A\otimes_F B \to (A\otimes_F G)\square_G B$ is $1_A \otimes \nabla_B$, and the inverse

map $(A \otimes_F G) \square_G B \to A \otimes_F B$ is $1_A \otimes \varepsilon \otimes 1_B$. Thus the induced map $HC_*(P) \otimes_F HC_*(Q) \to HC_*(P) \otimes_F HC_*(Q)$ is

$$[\sum_k a_k \otimes \gamma_k] \otimes [\sum_m b_m \otimes \gamma_m] \xrightarrow{1_{HC_*(P)} \otimes \nabla_Q} \sum_i [\sum_k a_k \otimes \gamma_k] \otimes \gamma_i \otimes [\sum_m b_m \otimes \gamma_{m-i}]$$

$$\xrightarrow{\phi \square 1} \sum_i [\sum_k a_k \otimes \gamma_k] \otimes \gamma_{i-1} \otimes [\sum_m b_m \otimes \gamma_{m-i}]$$

$$- \sum_i [\sum_k a_k \otimes \gamma_{k-1}] \otimes \gamma_i \otimes [\sum_m b_m \otimes \gamma_{m-i}]$$

$$\xrightarrow{1_{HC_*(P)} \otimes \varepsilon \otimes 1_{HC_*(Q)}} [\sum_k a_k \otimes \gamma_k] \otimes [\sum_m b_m \otimes \gamma_{m-1}] - [\sum_k a_k \otimes \gamma_{k-1}] \otimes [\sum_m b_m \otimes \gamma_m]$$

Thus $\mathrm{Cotor}_G(HC_*(P), HC_*(Q)) \cong \dfrac{HC_*(P) \otimes_F HC_*(Q)}{\mathrm{Im}\,\phi \square 1}$

$$\cong \frac{HC_*(P) \otimes_F HC_*(Q)}{\langle [\sum_k a_k \otimes \gamma_k] \otimes [\sum_m b_m \otimes \gamma_{m-1}] - [\sum_k a_k \otimes \gamma_{k-1}] \otimes [\sum_m b_m \otimes \gamma_m] \rangle}$$

Letting $\alpha = [\sum_k a_k \otimes \gamma_k]$, $\beta = [\sum_m b_m \otimes \gamma_m]$, and rewriting in terms of the D-module structure, using $\theta\gamma_p = \gamma_{p-1}$, we have

$$\mathrm{Cotor}_G(HC_*(P), HC_*(Q)) \cong \frac{HC_*(P) \otimes_F HC_*(Q)}{\langle \alpha \otimes \theta\beta - \theta\alpha \otimes \beta \rangle}$$

$$\cong HC_*(P) \otimes_D HC_*(Q)$$

Since ϕ decreases degree by 2, we have

$$\mathrm{Cotor}_G(HC_*(P),HC_*(Q))_n \cong (HC_*(P)\otimes_D HC_*(Q))_{n-2} \quad . \qquad \square$$

The natural isomorphism $(\mathrm{Tor}_D(HC_*(P),HC_*(Q))_{n-2} \cong (HC_*(P)\square_G HC_*(Q))_n$ can be composed with the coproduct map $HC_n(P\times Q) \to (HC_*(P)\square_G HC_*(Q))_n$ to give a natural D-module map $HC_n(P\times Q) \to (\mathrm{Tor}_D(HC_*(P),HC_*(Q))_{n-2}$. Similarly, the natural isomorphism $(\mathrm{Cotor}_G(HC_*(P),HC_*(Q))_{n+1} \cong (HC_*(P)\otimes_D HC_*(Q))_{n-1}$ can be composed with the map from the Kunneth short exact sequence $\psi:(\mathrm{Cotor}_G(HC_*(P),HC_*(Q))_{n+1} \to HC_n(P\times Q)$ to give a natural D-module map $(HC_*(P)\otimes_D HC_*(Q))_{n-1} \to HC_n(P\times Q)$. We will show that this agrees with the Loday Quillen product map .

Lemma 4.6.

The following diagram commutes

$$\begin{array}{ccc}
\mathrm{Cotor}_G(HC_*(P),HC_*(Q))_{n+1} & \xrightarrow{\ \psi\ } & HC_n(P\times Q) \\
\cong & \nearrow_{\alpha} & \\
(HC_*(P)\otimes_D HC_*(Q))_{n-1} & &
\end{array}$$

Proof

ψ is the map induced in homology by the chain map $x\otimes y \to x\otimes\partial y \to f(x\otimes\partial y)$. Let $x = \Sigma x_i\otimes\theta^{-i}$, $y = \Sigma y_i\otimes\theta^{-i}$, then $[x]\otimes[\partial y] = [\Sigma x_i\otimes\theta^{-i}] \otimes [By_0]$. We showed in Chapter 2 that $x\otimes y \to f(x\otimes By)$ agrees with the Loday Quillen product. $\square$

So, applying Lemmas 4.4 - 4.6 to the short exact sequence

$$0 \to \mathrm{Cotor}_G(HC_*(P),HC_*(Q))_{n+1} \xrightarrow{\psi} HC_n(P\times Q) \to (HC_*(P)\square_G HC_*(Q))_n \to 0$$

we obtain the following theorem.

Theorem 4.7.

Given cyclic F-modules P and Q, there is a short exact sequence of D-modules

$$0 \to (HC_*(P)\otimes_D HC_*(Q))_{n-1} \xrightarrow{\alpha} HC_n(P\times Q) \to \mathrm{Tor}_D(HC_*(P),HC_*(Q))_{n-2} \to 0$$

where α is the Loday Quillen product.

So, applying Lemmas 4.4 - 4.6 to the short exact sequence

$$0 \to \mathrm{Cotor}_G(HC_*(P),HC_*(Q))_{n+1} \xrightarrow{\psi} HC_n(P\times Q) \to (HC_*(P)\square_G HC_*(Q))_n \to 0$$

we obtain the following theorem.

Theorem 4.7.

Given cyclic F-modules P and Q , there is a short exact sequence of D-modules

$$0 \to (HC_*(P)\otimes_D HC_*(Q))_{n-1} \xrightarrow{\alpha} HC_n(P\times Q) \to \mathrm{Tor}_D(HC_*(P),HC_*(Q))_{n-2} \to 0$$

where α is the Loday Quillen product.

§5. The cyclic cohomology of polynomial algebras and their quotients.

In this chapter, we will endeavour to use our knowledge of products to calculate some examples.

(A) $F[x]$, F of characteristic zero

The cyclic homology of this algebra is given by [23, Theorem 2.9], but we include it for completeness, since our method is different.

Lemma 5.1.

The Hochschild homology of $F[x]$ is

$$HH_n(F[x]) \cong F[x] \qquad \text{if } n = 0 \text{ or } 1$$
$$\cong 0 \qquad \text{otherwise.}$$

Proof

Let $R = F[x]$, $A = R\otimes_F R \cong F[s,t]$.
Then a projective resolution of R over A is given by

$$0 \leftarrow R \xleftarrow{\phi} A \xleftarrow{\times(s-t)} A \leftarrow 0$$

where $\phi(s) = x = \phi(t)$.

Applying the functor $R\otimes_A -$ to the resolution, we obtain

$$R \xleftarrow{0} R \leftarrow 0 \quad .$$

Thus the only non zero Hochschild homology groups are $HH_0(R) \cong R$, $HH_1(R) \cong R$. □

Lemma 5.2.

(a) $HC^*(R) = F[\theta] \otimes^{F[x]} /F$

(b) $\hat{HC}^*(R) = F[\theta,\theta^{-1}]$.

Proof

We use the long exact sequence relating Hochschild and cyclic homology:

$$\begin{array}{l} 0 \leftarrow HC_0(R) \leftarrow HH_0(R) \leftarrow 0 \leftarrow HC_1(R) \leftarrow HH_1(R) \xleftarrow{B} HC_0(R) \leftarrow HC_2(R) \leftarrow HH_2(R) \leftarrow HC_1(R) \leftarrow \\ \qquad\quad \| \qquad\quad\;\; \| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\; \| \qquad\qquad\qquad\qquad\; \| \\ \qquad\quad R \qquad\quad\; R \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad R \qquad\qquad\qquad\qquad 0 \end{array}$$

First observe that $HC_0(R) \cong HH_0(R) \cong R$.

Now for all commutative algebras we have

$HC_0(A) \cong A$

$HH_1(A) \cong \Omega^1_A = \{\Sigma a_i dx_i : d(xy) = xdy+ydx\}$, the module of Kähler differentials,

$HC_1(A) \cong \Omega^1_A/dA$

$B : HC_0(A) \to HH_1(A)$, that is, $B: A \to \Omega^1_A$, is the derivative.

[See 23, Example 2, Proposition 1.11.]

Since the map $d(x^i) = ix^{i-1}$ has kernel F on $F[x]$ and is surjective, we obtain $HC_2(R) = F$, $HC_1(R) = 0$.

Then since $HH_n(R) = 0$ for $n > 1$, the long exact sequence becomes a succession of periodicity isomorphisms,

$$\cdots \leftarrow 0 \leftarrow HC_{n-2}(R) \leftarrow HC_n(R) \leftarrow 0 \leftarrow \cdots$$

Thus $HC_{2n+1}(R) = 0$, $n \geq 0$, $HC_{2n}(R) = F$, $n \geq 1$.

By considering the dual of the long exact sequence, we obtain cyclic cohomology:

$$HC^n(R) = \begin{cases} R & \text{if } n = 0 \\ F & \text{if } n = 2m,\ m > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Here $S: HC^{2n}(R) \to HC^{2n+2}(R)$ is $S(a) = a$ if $n > 0$. $S: HC^0(R) \to HC^2(R)$ is $S(a_0+a_1x+\ldots) = a_0$. Note that every element of $HC^{2n}(R)$ lies in the image of $S^{(n)}$.

We will now calculate the cup product in cohomology. Since the cup product $HC^{2m}(R) \otimes_{F[\theta]} HC^{2n}(R) \to HC^{2(m+n)}(R)$ is an $F[\theta]$-module map, $S^{(m)}[a] \cup S^{(n)}[b] = S^{(n+m)}[a \cup b]$. Thus the product is determined by the product $HC^0(R) \otimes_F HC^0(R) \to HC^0(R)$, which is multiplication in the ring R . □

We can then use the Kunneth theorem to calculate the cyclic cohomology of $F[x_1, \ldots x_n]$.

<u>Lemma 5.3</u>.

For F of characteristic zero

(a) $HC^*(F[x_1, \ldots x_n]) = F[\theta] \otimes \overline{F[x_1, \ldots x_n]}$

where $\bar{A}$ denotes A/F .

(b) $\hat{HC}^*(F[x_1 \ldots x_n]) = F[\theta, \theta^{-1}]$.

<u>Proof</u>

Both parts are proved by induction on n . The case $n = 1$ is covered by Lemma 5.2, and the induction step is proved by using the Kunneth Theorem 4.1, since $F[x_1, . x_n] = F[x_1 \ldots, x_{n-1}] \otimes_F F[x_n]$.

Part (b) follows immediately from

$$\hat{HC}^*(F[x_1, .. x_n] \otimes_F F[x_n]) \cong \hat{HC}^*(F[x_1 \ldots x_n]) \otimes_{F[\theta]} \hat{HC}^*(F[x_n])$$

$$\cong F[\theta, \theta^{-1}] \otimes_{F[\theta]} F[\theta, \theta^{-1}] .$$

For (a), the Kunneth short exact sequence is

$$0 \to [(HC^*(F[x_1..x_{n-1}]) \otimes_{F[\theta]} HC^*(F[x])]^m \to HC^m(F[x_1,..x_n]) \to [\mathrm{Tor}_{F[\theta]}(HC^*(F[x_1,..x_{n-1}]), HC^*(F[x_n])]^{m-1} \to 0$$

The Tor term gives $\overline{F[x_1 \ldots, x_n]}$ in degree 0, which hence doesn't contribute, and the tensor product term gives $(F[\theta] \otimes \overline{F[x_1 \ldots x_n]})^m$ as required. □

We can also use the Kunneth theorem to calculate the cyclic cohomology of $A[x_1, \ldots x_n]$, where A is any algebra over F.

Lemma 5.4.

Given A an algebra over F, writing $R = F[x_1, \ldots, x_n]$

(a) There is a short exact sequence

$$0 \to HC^m(A) \otimes \left(\frac{HC^m(A)}{SHC^{m-2}(A)} \otimes_F \bar{R}\right) \to HC^m(A[x_1 \ldots x_n]) \to \ker S_{m-1} \otimes \bar{R} \to 0$$

where $S_{m-1} = S : HC^{m-1}(A) \to HC^{m+1}(A)$.

(b) $\bar{HC}^*(A[x_1, \ldots x_n]) \cong \bar{HC}^*(A)$.

Proof

These results follow from the Kunneth theorem since $A[x_1, \ldots x_n] \cong A \otimes_F F[x_1 .. x_n]$. Part (b) follows immediately from

$$\hat{HC}^*(A[x_1,\dots x_n]) \cong \hat{HC}^*(A) \otimes_{F[\theta]} \hat{HC}^*(F[x_1,\dots x_n])$$
$$\cong \hat{HC}^*(A) \otimes_{F[\theta]} F[\theta,\theta^{-1}] \ .$$

The Kunneth theorem gives a short exact sequence:

$$0 \to (HC^*(A)\otimes_{F[\theta]}(F[\theta]\otimes\bar{R}))^m \to HC^m(A[x_1,\dots x_n]) \to [Tor_{F[\theta]}(HC^*(A),F[\theta]\otimes\bar{R})]^{m+1} \to 0 \ .$$

Since $\theta\bar{R} = 0$, the terms in the tensor product represented by $[a] \otimes r \in HC^m(A) \otimes \bar{R}$ are zero if $[a]$ lies in the image of S, so $[a] = \theta[a']$. Thus this term contributes

$$HC^m(A) \otimes \left(\frac{HC^m(A)}{SHC^{m-2}(A)} \otimes \bar{R}\right) \ .$$

To calculate $Tor_{F[\theta]}(HC^*(A),\bar{R})$, we use the following projective resolution of $\bar{R}$ over $F[\theta]$:

$$0 \to F[\theta] \otimes \bar{R} \xrightarrow{\theta\otimes 1} F[\theta] \otimes \bar{R} \xrightarrow{\phi} \bar{R} \to 0$$

where $\phi(\theta^n \otimes r) = \theta^n(r)$. We then have the exact sequence

$$0 \to Tor_{F[\theta]}(HC^*(A),\bar{R}) \to HC^*(A)\otimes_{F[\theta]}(F[\theta]\otimes\bar{R}) \to HC^*(A)\otimes_{F[\theta]}(F[\theta]\otimes\bar{R}) \to HC^*(A)\otimes_{F[\theta]}\bar{R} \to 0 \ .$$

Thus $Tor_{F[\theta]}(HC^*(A),\bar{R})$ consists of terms $[a] \otimes (1 \otimes r)$ such that the equivalence class of $[a] \otimes (\theta \otimes r)$ contains 0 ; but this class is also represented by $[S(a)] \otimes (1 \otimes r)$. Thus the Tor term is isomorphic to $(\ker S \otimes \bar{R})^{m-1}$. □

(B) $F[x,x^{-1}]$

This is the field of fractions of the graded algebra $F[x]$. Again, F is of characteristic zero.

<u>Lemma 5.5.</u>

The Hochschild homology of $F[x,x^{-1}]$ is

$$HH_n(F[x,x^{-1}]) = \begin{cases} F[x,x^{-1}] & \text{if } n = 0 \text{ or } 1 \\ 0 & \text{otherwise.} \end{cases}$$

<u>Proof</u>

Let $R = F[x,x^{-1}]$, $A = R \otimes_F R$.
A projective resolution of R over A is given by

$$0 \leftarrow R \xleftarrow{\phi} A \xleftarrow{x(s-t)} A \leftarrow 0$$

where $\phi(s) = x = \phi(t)$. For $s^{-1}-t^{-1} \in \ker \phi$, $(s^{-1}-t^{-1}) = (s-t)(s^{-2}+t^{-2}+s^{-3}t+st^{-3}+ \ldots .)$. Then applying $R \otimes_A -$, we obtain $R \xleftarrow{0} R \leftarrow 0$, thus the result follows. □

<u>Lemma 5.6.</u>

(a) $HC^*(F[x,x^{-1}]) = F[\theta] \otimes E(y) \otimes \bar{R}$, where degree of $y = 1$

(b) $\hat{HC}^*(F[x,x^{-1}]) \cong F[\theta,\theta^{-1}] \otimes E(y)$.

<u>Proof</u>

We use the long exact sequence

$$\begin{array}{ccccccc} 0 \leftarrow HC_0(R) \leftarrow & HH_0(R) \leftarrow 0 \leftarrow & HC_1(R) \leftarrow HH_1(R) & \xleftarrow{B} & HC_0(R) \leftarrow HC_2(R) \leftarrow & HH_2(R) & \leftarrow \dots \\ \| & \| & \| & & \| & \| & \\ R & R & R & & R & 0 & \end{array}$$

Again, B is the derivative. Thus $HC_2(R) \cong \ker B \cong F$, $HC_1(R) \cong \operatorname{coker} B \cong x^{-1}F \cong F$. Then since $HH_n(R) = 0$ for $n > 1$, the long exact sequence becomes a succession of periodicity isomorphisms,

$$0 \leftarrow HC_n(R) \leftarrow HC_{n+2}(R) \leftarrow 0 .$$

Thus $HC_{2n+2}(R) \cong F$, $HC_{2n+1}(R) \cong F$, for all n .

We will now calculate the cup product in $HC^*(R)$, Since every element of $HC^{2n}(R)$ lies in the image of $HC^0(R)$ under $S^{(n)}$, the product

$HC^{2n}(R) \otimes_{F[\theta]} HC^{2m}(R) \to HC^{2(n+m)}(R)$ is given by $S^n[a] \cup S^m[b] = S^{n+m}[a \cup b]$.

Thus it is determined by the product $HC^0(R) \otimes HC^0(R) \to HC^0(R)$, which is multiplication in R .

Since $HC^{2n+1}(R) = S^{(n)}(HC^1(R))$, the product $HC^{2n+1}(R)\otimes_{F[\theta]} HC^{2m+1}(R) \to HC^{2(n+m+1)}(R)$ is determined by the product $HC^1(R)\otimes_{F[\theta]} HC^1(R) \to HC^2(R)$. By using the explicit form of the cup product given by Connes in [8, Chapter 1], we see that given $\tau, \phi \in HC^1(R)$,

$$\tau \cup \phi = \tau \vee \phi + T(\phi \vee \tau)$$

where T is a map switching factors. Then, since $\tau \vee \phi$ and $\phi \vee \tau$ lie in $HH^2(R)$, and $HH^2(R) = 0$, $\tau \cup \phi = 0$.

Similarly, the product $HC^{2n}(R)\otimes_{F[\theta]}HC^{2m+1}(R) \to HC^{2(n+m)+1}(R)$ is determined by the product $HC^0(R) \otimes HC^1(R) \to HC^1(R)$. The cup product is always a coextension of the Hochschild product: in this case, since $I : HC^0(R) \to HH^0(R)$ and $I : HC^1(R) \to HH^1(R)$ are both injections, it is equal to the Hochschild product. This is the F-dual of the shuffle product in homology, which can be calculated by observing that the generators over F for $HH_1(R) = R$, $\{x^i\}_{i=0,1,\ldots}$, can be represented by the cycles $[x^i \otimes x]$ in the standard bar resolution form of the Hochschild complex. Then the shuffle product of $[x^i] \in HH_0(R)$ and $[x^j \otimes x] \in HH_1(R)$ is $[x^{i+j} \otimes x] \in HH_1(R)$, corresponding to the generator x^{i+j} in R . Thus the required product is multiplication in R .

Thus, setting y to be a generator of $HC^1(R)$, we obtain the stated result. □

Lemma 5.7.

(a) $HC^*(F[x_1,\ldots x_n,x_1^{-1}\ldots x_n^{-1}]) \cong F[\theta] \otimes E(y_1,\ldots y_n) \otimes \overline{F[x_1,\ldots x_n,x_1^{-1}\ldots x_n^{-1}]}$

(b) $\hat{HC}^*(F[x_1,\ldots x_n,x_1^{-1}\ldots x_n^{-1}]) \cong F[\theta,\theta^{-1}] \otimes E(y_1,\ldots y_n)$

where each y_i has degree 1.

Proof

The proof is by induction on n , using Lemma 5.6 for the case $n = 1$, and the Kunneth theorem to prove the inductive step.

(b) follows from $\hat{HC}^*(F[x_1\ldots x_n,\ldots x_n^{-1}]) \cong \hat{HC}^*(F[x_1\ldots x_{n-1},\ldots x_{n-1}^{-1}]) \otimes_{F[\theta]} \hat{HC}^*(F[x_n,x_n^{-1}])$

$$\cong (F[\theta,\theta^{-1}] \otimes E(y_1,\ldots y_{n-1})) \otimes_{F[\theta]} (F[\theta,\theta^{-1}] \otimes E(y_n))$$

$$\cong F[\theta,\theta^{-1}] \otimes E(y_1,\ldots y_n) .$$

(a) follows from the short exact sequence:

$$0 \to (HC^*(F[x_1\ldots x_{n-1}^{-1}]) \otimes_{F[\theta]} HC^*(F[x_n,x_n^{-1}]))^m \to HC^m(F[x_1\ldots x_n^{-1}]) \to$$

$$\mathrm{Tor}_{F[\theta]}(HC^*(F[x_1\ldots x_{n-1}^{-1}]),HC^*(F[x_n,x_n^{-1}]))^{m+1} \to 0 .$$

The Tor term contributes nothing, the tensor product term gives the required answer. □

Lemma 5.8.

For any algebra A over F

(a) $\hat{HC}^*(A[x_1 \ldots x_n, x_1^{-1} .. x_n^{-1}]) \cong \hat{HC}^*(A) \otimes E(y_1, .. y_n)$.

(b) There is a short exact sequence

$$0 \to [HC^*(A) \otimes E(y_1 \ldots y_n)]^m \otimes \left(\frac{HC^m(A)}{SHC^{m-2}(A)} \otimes \bar{R}\right) \to HC^m(A[x_1 .. x_n, .. x_n^{-1}]) \to \mathrm{Ker} S_{m-1} \otimes \bar{R} \to 0$$

where $R = F[x_1 .. x_n, x_1^{-1} .. x_n^{-1}]$, $S_{m-1} = S : HC^{m-1}(A) \to HC^{m+1}(A)$.

Proof

Using the Kunneth theorem and induction on n , as for Lemma 5.4. □

(c) $\frac{F[x]}{(x^n)}$, F of characteristic zero.

Lemma 5.9.

The Hochschild homology of $R = F[x]/(x^n)$ is given by

$$\begin{aligned} HH_n(R) &= R \quad \text{if } n = 0 \\ &= xR \quad \text{if } n = 2m+2 \\ &= R/x^{n-1} \quad \text{if } n = 2m+1 . \end{aligned}$$

Proof

Let $R = F[x]/(x^n)$, $A = R \otimes_F R$, so $A \cong \dfrac{F[s,t]}{(s^n)+(t^n)}$.

A projective resolution of R over A is given by

$$0 \leftarrow R \xleftarrow{\phi} A \xleftarrow{\times(s-t)} A \xleftarrow{\times N} A \xleftarrow{\times(s-t)} A \xleftarrow{\times N} A \leftarrow \ldots$$

where $\phi(s) = x = \phi(t)$, $N = s^{n-1} + s^{n-2}t + \ldots . st^{n-2} + t^{n-1}$.

Taking $R \otimes_A -$, we obtain

$$0 \leftarrow R \xleftarrow{\times nx^{n-1}} R \xleftarrow{0} R \xleftarrow{\times nx^{n-1}} R \xleftarrow{0} R \leftarrow \ldots$$

Thus $HH_0(R) = R$, $HH_{2n+2}(R) = xR$, $HH_{2n+1}(R) = R/x^{n-1}$. □

In order to calculate the shuffle product in Hochschild homology, which we will use later, we need a chain map from the resolution given above to the usual bar resolution.

$$\begin{array}{ccccccccccccc}
0 \leftarrow & R & \xleftarrow{\phi} & R^{\otimes 2} & \xleftarrow{\times(s-t)} & R^{\otimes 2} & \xleftarrow{\times N} & R^{\otimes 2} & \xleftarrow{\times(s-t)} & R^{\otimes 2} & \leftarrow \ldots \\
 & \| & & \| & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & \\
0 \leftarrow & R & \xleftarrow{b'} & R^{\otimes 2} & \xleftarrow{b'} & R^{\otimes 3} & \xleftarrow{b'} & R^{\otimes 4} & \xleftarrow{b'} & R^{\otimes 5} & \leftarrow \ldots
\end{array}$$

<u>Proof</u>

Let $R = F[x]/(x^n)$, $A = R \otimes_F R$, so $A \cong \dfrac{F[s,t]}{(s^n)+(t^n)}$.

A projective resolution of R over A is given by

$$0 \leftarrow R \xleftarrow{\ \phi\ } A \xleftarrow{\times(s-t)} A \xleftarrow{\ \times N\ } A \xleftarrow{\times(s-t)} A \xleftarrow{\ \times N\ } A \leftarrow \,..$$

where $\phi(s) = x = \phi(t)$, $N = s^{n-1} + s^{n-2}t + \ldots.\ st^{n-2} + t^{n-1}$.

Taking $R\otimes_A -$, we obtain

$$0 \leftarrow R \xleftarrow{\times nx^{n-1}} R \xleftarrow{\ 0\ } R \xleftarrow{\times nx^{n-1}} R \xleftarrow{\ 0\ } R \leftarrow \,..$$

Thus $HH_0(R) = R$, $HH_{2n+2}(R) = xR$, $HH_{2n+1}(R) = R/x^{n-1}$. □

In order to calculate the shuffle product in Hochschild homology, which we will use later, we need a chain map from the resolution given above to the usual bar resolution.

$$\begin{array}{ccccccccccc}
0 \leftarrow R & \xleftarrow{\ \phi\ } & R^{\otimes 2} & \xleftarrow{\times(s-t)} & R^{\otimes 2} & \xleftarrow{\ \times N\ } & R^{\otimes 2} & \xleftarrow{\times(s-t)} & R^{\otimes 2} & \leftarrow & .. \\
\| & & \| & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\
0 \leftarrow R & \xleftarrow{\ b'\ } & R^{\otimes 2} & \xleftarrow{\ b'\ } & R^{\otimes 3} & \xleftarrow{\ b'\ } & R^{\otimes 4} & \xleftarrow{\ b'\ } & R^{\otimes 5} & \leftarrow & ..
\end{array}$$

Such a chain map is given by

$$f_{2m}(x^i \otimes x^j) = \sum_{\alpha_i=1}^{n-1} x^i \otimes x^{\alpha_1} \otimes x \otimes x^{\alpha_2} \otimes x \ldots \otimes x^{\alpha_m} \otimes x \otimes x^{j+n-1+2(m-1)-\Sigma\alpha_i}$$

$$f_{2m+1}(x^i \otimes x^j) = \sum_{\alpha_i=1}^{n-1} x^i \otimes x \otimes x^{\alpha_1} \otimes x \ldots \otimes^{\alpha_m} \otimes x \otimes x^{j+n-1+2(m-1)-\Sigma\alpha_i}$$

and this induces a map of generators of Hochschild homology

$$HH_{2m}(R) : [x^p] \rightarrow [\sum_{\alpha_i=1}^{n-1} x^{p+n-1+2(m-1)-\sum_{i=0}^{n}\alpha_i} \otimes x^{\alpha_1} \otimes x \ldots \otimes x^{\alpha_m} \otimes x]$$

$$HH_{2m+1}(R) : [x^p] \rightarrow [\sum_{\alpha_i=1}^{n-1} x^{p+n-1+2(m-1)-\Sigma\alpha_i} \otimes x \otimes x^{\alpha_1} \ldots \otimes x^{\alpha_m} \otimes x]$$

We can now calculate the shuffle product on generators in the standard bar resolution form, which gives the following product on the generators $[x^i] \in HH_n(R)$:

$$[x^i] \otimes [x^j] \rightarrow 0 \qquad \text{if } [x^i] \in HH_{2m+1}(R) \text{ , } [x^j] \in HH_{2n+1}(R)$$
$$\rightarrow [x^{i+j}] \quad \text{otherwise.}$$

Now recall Lemma 2.15: if $\vee$ is the product in Hochschild cohomology, $HH^*(A)$ is a module over $HC^*(A)$ by the action $\tau(\phi) = \phi \vee I\tau$, for $\tau \in HC^*(A)$, $\phi \in HH^*(A)$, and $IB(\phi \vee I\tau) = IB\phi \vee I\tau$.

Lemma 5.10.

$$HC^n\left(\frac{F[x]}{(x^n)}\right) = \frac{F[x]}{(x^n)} \quad \text{if } n = 0$$

$$= 0 \quad \text{if } n = 2p+1$$

$$= V \quad \text{if } n = 2p+2 \text{, where } V \text{ is an}$$

n-dimensional vector space over F.

Proof

Label the generators of V by $\{x_i\}_{i=0,\dots,n-1}$. The proof is by induction on the degree of the cohomology group. We will take the following as the inductive hypothesis: $HC^n(R) = 0$ if $n = 2p+1$, $HC^n(R) = V$ if $n = 2p+2$, and $B: HH^{2p+1}(R) \to HC^{2p}(R)$ satisfies $B\langle x^i\rangle = \langle x^{i+1}\rangle$, for all $p < p'$.

Consider the long exact sequence:

$$0 \to HC^0(R) \to HH^0(R) \to 0 \to HC^1(R) \to HH^1(R) \xrightarrow{B} HC^0(R) \to HC^2(R) \to HH^2(R) \to \quad .$$

We have $HC^0(R) \cong HH^0(R) = R$. Here $B: HH^1(R) \to HC^0(R)$ is the F-dual of the derivative, so $B\langle x^i\rangle = \langle x^{i+1}\rangle$. B is injective, and so $HC^1(R) = 0$. This starts the induction.

Now consider $HC^{2p}(R)$: by the inductive hypothesis, $HC^{2p-1}(R) = 0$, so we obtain from the long exact sequence relating Hochschild and cyclic cohomology, the following short exact sequence:

$$0 \to HH^{2p-1}(R) \xrightarrow{B} HC^{2p-2}(R) \xrightarrow{S} HC^{2p}(R) \xrightarrow{I} HH^{2p}(R) \to 0$$

$$\begin{array}{ccc} \wr\wr & \wr\wr & \wr\wr \\ \langle 1,x,..x^{n-2}\rangle & \langle 1,x,..x^{n-1}\rangle & \langle x,x^2,..x^{n-1}\rangle \end{array} .$$

Now since I surjective, $HC^{2p}(R)$ contains a subspace isomorphic to $\langle x,...x^{n-1}\rangle$, and since $B\langle x^i\rangle = \langle x^{i+1}\rangle$ by the inductive hypothesis, $\ker S \cong \langle x,...,x^{n-1}\rangle$, $\operatorname{Im} S \cong \langle 1\rangle$, so $HC^{2p}(R) \cong \langle 1,x,..x^{n-1}\rangle$.

Now consider $HC^{2p+1}(R)$: by using the inductive hypothesis, we obtain the exactness of the following:

$$0 \to HC^{2p+1}(R) \xrightarrow{I} HH^{2p+1}(R) \xrightarrow{B} HC^{2p}(R) .$$

Thus $HC^{2p+1}(R) = 0$ would be implied by B injective, and it is sufficient to prove that $B\langle x^i\rangle = \langle x^{i+1}\rangle$ for all i . Consider those elements of $I\phi \vee \tau$, where $\tau \in HH^1(R)$, $\phi \in HC^{2p}(R)$. Recall $IB(I\phi \vee \tau) = I\phi \vee IB\tau$. Since $B\tau \neq 0$ in $HC^0(R)$ and $I : HC^0(R) \to HH^0(R)$ is an isomorphism, $IB\tau \neq 0$.

Take ϕ such that $I\phi \subset \langle x \rangle$, and $\tau \subset \langle x^i \rangle$, for $0 \leq i \leq n-3$, so $IB\tau \subset \langle x^{i+1} \rangle$. Since the Hochschild product is given by multiplication in R, $I\phi \vee \tau \subset \langle x^{i+1} \rangle$, $I\phi \vee IB\tau \subset \langle x^{i+2} \rangle$, and since $i \leq n-3$, both $I\phi \vee \tau$ and $I\phi \vee IB\tau$ are non-zero. Then since $I\phi \vee IB\tau = IB(I\phi \vee \tau)$, we have $B\langle x^{i+1} \rangle = \langle x^{i+2} \rangle$, for all $i \in \{0,1,\ldots n-3\}$, that is, $i+1 \in \{1,2,\ldots n-2\}$. Hence it remains to show that $B\langle 1 \rangle = \langle x \rangle$.

Take $1 \in HH^{2p+1}(R)$, $x \in HC^0(R)$, then Ix in $HH^0(R)$ is non zero. Then $1 \vee Ix \neq 0$, $1 \vee Ix = \langle x \rangle$. Now $IB(1) \vee Ix = IB(1 \vee Ix) \subset IB\langle x \rangle \subset \langle x^2 \rangle$, and $IB(1) \vee Ix$ is non zero. So $IB(1) \subset \langle x \rangle$, $B\langle 1 \rangle = \langle x \rangle$. □

Lemma 5.11.

(a) $\hat{HC}^*\left(\frac{F[x]}{(x^n)}\right) \cong F[\theta,\theta^{-1}]$

(b) $HC^*\left(\frac{F[x]}{(x^n)}\right) = F[\theta] \oplus [\bigoplus_{p=0}^{\infty} \bar{R}_{2p}]$

where $\bar{R}_{2p}$ denotes a copy of R/F in degree $2p$. The θ action is zero on each of these terms, and the product $\bar{R}_{2p} \otimes \bar{R}_{2m} \to \bar{R}_{2(p+m)}$ is multiplication in R.

Proof

Note from the previous proof that $S:HC^{2p-2}(R) \to HC^{2p}(R)$ has kernel

$<x, \ldots x^{n-1}>$, and $1 \in HC^{2p}(R)$ is in the image of $HC^0(R)$ under $S^{(p)}$. Write 1_q for $1 \in HC^q(R)$; given 1_{2m} , 1_{2p} , $1_{2m} \cup 1_{2p} = S^m(1_0) \cup S^p(1_0) = S^{m+p}(1_0 \cup 1_0) = S^{m+p}(1_0) = 1_{2(m+p)}$, since the product $HC^0(R) \otimes HC^0(R) \to HC^0(R)$ is given by multiplication in R . Given 1_{2m} , $x^i \in HC^{2p}(R)$, $1_{2m} \cup x^i = S^m(1_0) \cup x^i = 1_0 \cup S^m(x^i) = 0$.

Recall that we have a coextension diagram for products

$$\begin{array}{ccc} HC^{2m}(R) \otimes HC^{2p}(R) & \xrightarrow{\cup} & HC^{2(m+p)}(R) \\ \downarrow I \otimes I & & \downarrow I \\ HH^{2m}(R) \otimes HH^{2p}(R) & \xrightarrow{V} & HH^{2(m+p)}(R) \end{array}$$

Given $x^i \in HC^{2m}(R)$, $x^i \in HC^{2p}(R)$, with neither i or $j = 0$, then from the proof of Lemma 5.10, $I(x^i) = x^i$, $I(x^j) = x^j$. We require $I(x^i \cup x^j) = x^{i+j}$, thus $x^i \cup x^j = x^{i+j}$. □

<u>Lemma 5.12</u>.

Given any algebra A over F ,

(a) $\hat{HC}^m(\frac{A[x]}{(x^n)}) \cong \hat{HC}^m(A)$

(b) There is a short exact sequence

$$0 \to HC^m(A) \oplus \left(\bigoplus_{i=0}^{\infty} \frac{HC^{m-2i}(A)}{SHC^{m-2i-2}(A)} \otimes \bar{R}_{2i} \right) \to HC^m\left(\frac{A[x]}{(x^n)}\right) \to \bigoplus_{i=0}^{\infty} \operatorname{Ker} S_{m-2i-1} \otimes \bar{R}_{2i} \to 0$$

where $S_{m-2i-1} = S : HC^{m-2i-1}(A) \to HC^{m-2i+1}(A)$.

Proof

(a) follows from the Kunneth theorem,

$$\hat{HC}^*\left(\frac{A[x]}{(x^n)}\right) \cong (HC^*(A) \otimes_{F[\theta]} \hat{HC}^*\left(\frac{F[x]}{(x^n)}\right)) \cong \hat{HC}^*(A) \otimes_{F[\theta]} F[\theta,\theta^{-1}] \ .$$

For (b), the Kunneth theorem gives a short exact sequence

$$0 \to (HC^*(A) \otimes_{F[\theta]} (F[\theta] \otimes (\overset{\infty}{\oplus} \bar{R}_{2i})))^m \to HC^m\left(\frac{A[x]}{(x^n)}\right) \to [\mathrm{Tor}_{F[\theta]}(HC^*(A), F[\theta] \otimes (\overset{\infty}{\oplus} \bar{R}_{2i}))]^{m+1} \to 0 \ .$$

Since $\theta\bar{R}_{2n} = 0$, the terms in the tensor product represented by $[a] \otimes r \in HC^{m-2p}(A) \otimes \bar{R}_{2p}$ are zero if $[a] = [Sa']$. To calculate $\mathrm{Tor}_{F[\theta]}(HC^*(A), \bar{R}_{2i})$ we use the usual projective resolution of $\bar{R}$ over $F[\theta]$,

$$0 \to F[\theta] \otimes \bar{R} \xrightarrow{\theta \otimes 1} F[\theta] \otimes \bar{R} \xrightarrow{\phi} \bar{R} \to 0$$

where $\phi(\theta^n \otimes r) = \theta^n(r)$. Thus $\mathrm{Tor}_{F[\theta]}(HC^*(A), \bar{R}_{2i})$ consists of terms $[a] \otimes (1 \otimes r)$ such that the equivalence class of $[a] \otimes (\theta \otimes r)$ is zero, that is, $[Sa] \otimes (1 \otimes r)$ is zero, so $\mathrm{Tor}_{F[\theta]}(HC^*(A), \bar{R}_{2i})^{m+1} = \bigoplus_i \ker S_{m-2i-1} \otimes \bar{R}_{2i}$.

(D) $\dfrac{\mathbb{C}[x]}{(f)}$, f a polynomial.

First we factorise $f(x)$ into irreducible factors, $f(x) = (x-\alpha_i)^{m_i}$. Then we use $R/I \cap J \cong R/I \oplus R/J$ for any ideals I and J of a ring R such that $I + J = R$, to write

$$\frac{\mathbb{C}[x]}{(f)} = \bigoplus_i \frac{\mathbb{C}[x]}{((x-\alpha_i)^{m_i})} .$$

Lemma 5.13.

For algebras A and B over a field of characteristic zero,

$$HC_n(A \otimes B) = HC_n(A) \otimes HC_n(B) .$$

Proof

We will use the chain complex for cyclic homology with boundaries b , $-b'$, N and D ; this complex is defined in Chapter 1.

$C_*(A) \otimes C_*(B)$ is a subcomplex of $C_*(A \otimes B)$ with quotient complex P =

$$\begin{array}{ccccc}
\downarrow & & \downarrow & & \downarrow \\
P_2 & \xleftarrow{D} & P_2 & \xleftarrow{N} & P_2 \leftarrow \\
b\downarrow & & -b'\downarrow & & b\downarrow \\
P_1 & \xleftarrow{D} & P_1 & \xleftarrow{N} & P_1 \leftarrow \\
b\downarrow & & -b'\downarrow & & b\downarrow \\
P_0 & \xleftarrow{D} & P_0 & \xleftarrow{N} & P_0 \leftarrow
\end{array}$$

where $P_n = \dfrac{(A \otimes B)^{\otimes n+1}}{A^{\otimes n+1} \oplus B^{\otimes n+1}}$. For example $P_0 = 0$, $P_1 = A \otimes B \oplus B \otimes A$.

The boundary maps are induced from those on $C_*(A \oplus B)$. By inserting an extra column $R_* =$

$$\begin{array}{c} \downarrow \\ P_2/\mathrm{Im}D \\ b\downarrow \\ P_1/\mathrm{Im}D \\ b\downarrow \\ 0 \end{array}$$

to the left of P , we form a new complex Q whose rows are acyclic

$$Q = \begin{array}{ccccccc} \downarrow & & \downarrow & & \downarrow & & \\ P_2/\mathrm{Im}D & \longleftarrow & P_2 & \xleftarrow{D} & P_2 & \xleftarrow{N} & \\ b\downarrow & & b\downarrow & & -b'\downarrow & & \\ P_1/\mathrm{Im}D & \longleftarrow & P_1 & \xleftarrow{D} & P_1 & \xleftarrow{N} & \\ b\downarrow & & b\downarrow & & -b'\downarrow & & \\ 0 & \longleftarrow & P_0 & \xleftarrow{D} & P_0 & \xleftarrow{N} & \end{array}$$

Q contains a subcomplex $P[+1]$, where $(P[+1])_{n,m} = P_{n+1,m}$, and the quotient complex is R . Thus we have a short exact sequence

$$0 \to P[+1] \to Q \to R \to 0$$

giving a long exact sequence in homology where $P_n = \dfrac{(A\otimes B)^{\otimes n+1}}{A^{\otimes n+1} \oplus B^{\otimes n+1}}$. For example $P_0 = 0$, $P_1 = A\otimes B \oplus B\otimes A$.

The boundary maps are induced from those on $C_*(A \oplus B)$. By inserting an extra column

$$R_* = \begin{array}{c} \downarrow \\ P_2/\mathrm{Im}D \\ b\downarrow \\ P_1/\mathrm{Im}D \\ b\downarrow \\ 0 \end{array}$$

to the left of P , we form a new complex Q whose rows are acyclic

$$Q = \begin{array}{ccccccc} \downarrow & & \downarrow & & \downarrow & & \\ P_2/\mathrm{Im}D & \longleftarrow & P_2 & \xleftarrow{D} & P_2 & \xleftarrow{N} & \\ b\downarrow & & b\downarrow & & -b'\downarrow & & \\ P_1/\mathrm{Im}D & \longleftarrow & P_1 & \xleftarrow{D} & P_1 & \xleftarrow{N} & \\ b\downarrow & & b\downarrow & & -b'\downarrow & & \\ 0 & \longleftarrow & P_0 & \xleftarrow{D} & P_0 & \xleftarrow{N} & \end{array}$$

Q contains a subcomplex $P[+1]$, where $(P[+1])_{n,m} = P_{n+1,m}$, and the quotient complex is R . Thus we have a short exact sequence

$$0 \to P[+1] \to Q \to R \to 0$$

giving a long exact sequence in homology

$$\ldots H_n(P[+1]) \to H_n(Q) \to H_n(R) \to H_{n-1}(P[+1]) \to \ldots \quad .$$

Thus, since Q is acyclic, $H_n(R) \cong H_n(P)$.

The cyclic permutations act on P_n by rotating the factors, so the complex R , with $R_n = P_n/\mathrm{ImD}$, is

$$0 \leftarrow A\otimes B \xleftarrow{b} A\otimes A\otimes B \oplus A\otimes B\otimes B \xleftarrow{b} A\otimes A\otimes A\otimes B \oplus A\otimes A\otimes B\otimes B \oplus A\otimes B\otimes B\otimes B \xleftarrow{b}$$

with $R(n) = \sum_{i=0}^{n-1} A^{\otimes i+1} \otimes B^{n-i}$. The induced boundary b is given on an element of $A^{\otimes i+1} \otimes B^{n-i}$ by $\sum_{i\in I}(-1)^i d_i$, $I = \{0,..i,i+2,..n-1\}$: that is, any face map which would involve multiplying an element of A with one of B is omitted.

We will construct a chain homotopy $s:R(n) \to R(n+1)$ such that $bs + sb = 1$.

Let $s(a_0 \otimes .. a_i \otimes b_{i+1}..\otimes b_n) = 1_A \otimes a_0...\otimes a_i \otimes b_{i+1}..\otimes b_n$.

Then, since d_{n+1} does not occur in bs ,

$$bs(a_0.\otimes a_i \otimes b_{i+1}..\otimes b_n) = a_0\otimes..a_i\otimes b_{i+1}..\otimes b_n - 1_A \otimes b(a_0\otimes...b_n) \quad .$$

Also, $sb(a_0..\otimes a_i\otimes b_{i+1}..b_n) = 1_A \otimes b(a_0...a_n)$. Thus $sb + bs = 1$ as required, and we have a chain homotopy between the identity map and the zero map on R . Hence $H_*(R) = 0$, and so $H_*(P) = 0$. Then from the short exact sequence

$$0 \to C_*(A) \otimes C_*(B) \to C_*(A \otimes B) \to P \to 0$$

we obtain the long exact sequence in homology

$$\to H_n(C_*(A) \otimes C_*(B)) \to H_n(C_*(A \otimes B)) \to H_n(P) \to H_{n-1}(C_*(A) \otimes C_*(B)) \to$$

So P acyclic $\Rightarrow HC_n(A \otimes B) = HC_n(A) \otimes HC_n(B)$. □

Then by applying Lemma 5.13 to the algebra $\frac{\mathbb{C}[x]}{(f)} = \bigoplus_i \frac{\mathbb{C}[x]}{((x-\alpha_i)^{m_i})}$,

and using induction on i , we obtain $HC_n\left(\bigoplus_i \frac{\mathbb{C}[x]}{((x-\alpha_i)^{m_i})}\right) = \bigoplus_i HC_n\left(\frac{\mathbb{C}[x]}{((x-\alpha_i)^{m_i})}\right)$.

Thus writing $\frac{\mathbb{C}[x]}{(f)} = R$, we have $HC_{2n}(R) = R$, $HC_{2n+1}(R) = 0$.

We could repeat the proof of Lemma 5.13 in the cohomology complexes, to obtain $HC^n(A) \otimes HC^n(B) \cong HC^n(A \otimes B)$, with the isomorphism induced by the inclusion of complexes $C^*(A) \otimes C^*(B) \to C^*(A \otimes B)$. Thus the product obtained on $HC^*(A) \otimes HC^*(B)$ is that induced by inclusion from the product on $HC^*(A \otimes B)$. So in this case, for $R = \frac{\mathbb{C}[x]}{(f)}$, we have

$$HC^{*}(R) = \mathbb{C}[\theta] \oplus \left(\bigoplus_{i=0}^{\infty} \bar{R}_{2i} \right)$$

$$\hat{HC}^{*}(R) = \mathbb{C}[\theta, \theta^{-1}] \quad .$$

We also have

Lemma 5.14.

For any algebra A over $\mathbb{C}$,

(a) $$\hat{HC}^{m}\left(\frac{A[x]}{(f)}\right) \cong \hat{HC}^{m}(A)$$

(b) There is a short exact sequence

$$0 \to HC^{m}(A) \oplus \left(\bigoplus_{i=0}^{\infty} \frac{HC^{m-2i}(A)}{SHC^{m-2i-2}(A)} \otimes \bar{R} \right) \to HC^{m}\left(\frac{A[x]}{(f)}\right) \to \bigoplus_{i=0}^{\infty} \ker S_{m-2i-1} \otimes \bar{R} \to 0$$

where $S_{m-2i-1} = S : HC^{m-2i-1}(A) \to HC^{m-2i+1}(A)$.

Proof

Exactly as for Lemma 5.12. □

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