SOME TOPICS IN THE THEORY OF SUPERMANIFOLDS.

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Summary.

We introduce a new type of supermanifold modelled on the whole exterior algebra. The structure of the resulting objects is more simple than supermanifolds previously studied yet still reflects the typical problems in the subject. An analysis of the structure of these objects leads to a greater understanding of the structure of supermanifolds in general. In particular, we are able to give criteria for a supermanifold to admit a vector bundle over its core manifold as a covering manifold. We are also able to identify the vector bundle underlying the z-thickening of a C^∞ manifold as a tensor power of the tangent bundle and to prove that compact supermanifolds admit no embeddings into their model spaces. The structure of those supermanifolds defined by the vanishing of G^∞ functions is also analyzed in depth. Finally, we are able to exhibit supermanifolds that do not admit vector bundles as covering manifolds and we point to the need to investigate the existence of compact simply connected supermanifolds.

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Introduction.

The theory of supermanifolds has arisen from the desire to provide a global formulation for physical theories involving the local notion of supersymmetry. (See [30], [37], [38], [45] and [46]). Two general approaches have been developed to achieve this aim. Firstly, the theory of graded manifolds due to Kostant [27], Berezin & Kac [5] and Berezin & Leites [6] in which the sheaf of C^∞ functions is enlarged to include anticommuting elements and secondly the theory of supermanifolds due to Rogers [34], [35], [36], deWitt [47] and Batchelor [3] in which a manifold is modelled on the local structure of an exterior algebra. The structure of graded manifolds has been investigated in some detail (see [2], [7], [16], [18] and [29]) with the conclusion that these objects are basically given by the sheaf of sections of the exterior bundle of some vector bundle over a C^∞ manifold. Recently, attention has been turned to the topological structure of supermanifolds, since it is not at all clear when a C^∞ manifold can admit the structure of a supermanifold. It is fairly easily established that Rogers' supermanifolds [34], are multifoliate in the sense of Kodaira & Spencer [26], (see [10] or this thesis) but very little has been deduced from this and very few examples of supermanifolds are given in the literature.

The aim of this thesis, then, is to investigate the topological structure of Rogers' supermanifolds, to give some more examples of supermanifolds and finally to point out some of the important open questions in this field. Our first line of attack is to simplify the situation with the introduction of a new class of supermanifolds. Supermanifolds introduced so far are modelled on the cartesian product of the odd part of an exterior algebra with the even part. This allows the topological structure to get quite complicated very quickly and hides the important questions behind a mass of notation. What we do is to model our supermanifolds on the whole exterior algebra, the result of this being a tremendous simplification in the structure allowing us to see what is really happening and to see what is required to "unravel" the supermanifold. Previous examples of supermanifolds have all admitted vector bundles over their core manifolds as covering spaces and it is obvious to conjecture that this is true in general. In fact it is not and we give some counter examples. More importantly, perhaps, we show why it is not true and we give necessary and sufficient conditions for a supermanifold to admit a vector bundle over its core manifold as a covering space. Having dealt with our definition of supermanifolds we turn to the general theory and note that we can apply the same sort of analysis to Rogers supermanifolds. We note, however, that this analysis has to be performed in stages. We remarked above on the multifoliate structure of supermanifolds, what has not been noticed before is the fact that supermanifolds are foliated according to a $\mathbb{Z}$

gradation, one leaf sitting inside another according to the **Z**-degree. What we do is to decompose the supermanifold step by step down the **Z**-gradation applying the analysis we developed for our "restricted" supermanifolds at each step of the gradation. This allows us to give necessary and sufficient conditions for a supermanifold to admit a vector bundle over its core manifold as a covering manifold.

Our approach in this thesis has been to introduce each new complication in the theory one step at a time in order to overcome the difficulties engendered by the notation and to indicate the important points at each step. So in chapter 1 we introduce our simplified notion of restricted supermanifolds and analyze their structure in detail. In chapter 2 we define the notion of an even supermanifold modelled on the even part of the exterior algebra and in chapters 3 and 4 we replace the odd part of the exterior algebra and recover Rogers' original definition of a supermanifold. This approach means that there is a certain degree of repetition and that certain results could be inferred from later, more general ones. Indeed, the results of chapter 4 could be written straight down without developing the first three chapters at all. However we believe that our approach is more helpful and indeed, the preliminary stages can be seen as interesting results in their own right, particularly the results of chapter 1.

In addition to analyzing the general structure of supermanifolds we look at some interesting subsidiary questions, namely the structure of supermanifolds that are given by the vanishing of G^∞ functions and also embeddings of compact supermanifolds.

Our final chapter is, perhaps, a pointer to future research. Having found the conditions required for a supermanifold to look like a vector bundle, (we call these "vectorial supermanifolds"), we give some examples of supermanifolds that do not obey these conditions and we give some non-existence proofs for certain simply connected compact supermanifolds. This leads to questions about the existence of simply connected compact supermanifolds and indeed questions about the structure of "non-vectorial" supermanifolds in general.

For background material on supermanifolds the reader should consult the original paper of Rogers [34] and more recent papers such as [10], [11], [21], [22], [24] and [42].

All the results of this thesis are original research unless otherwise stated, though the reader should note that some of the methods used have been independently developed elsewhere. In particular, the differential equation approach to G^∞ functions is expounded in some detail in [10] as is the G-structure approach to supermanifolds. It has been recently pointed out to us that our result on the structure of the z-thickening of a C^{∞} manifold, (in chapter 2), has been claimed by Rabin & Crane [32].

Chapter 1: Restricted Supermanifolds.

Section 1: Definitions and Elementary results.

Let B_L be the exterior algebra on L odd generators $b_1, b_2, \ldots, b_L$ over $\mathbf{R}$. We shall want to identify B_L with $\mathbf{R}^{2^L}$ as vector spaces and to this end we make the following notational definitions:

$$m = (m_1, m_2, \ldots, m_k) \quad k \leq L \quad \text{shall be a multi-index}$$
$$1 \leq m_1 < m_2 < \ldots < m_k \leq L$$

Thus $b_m = b_{m_1} b_{m_2} \ldots b_{m_k}$ and we define $b_0 = 1 \in \mathbf{R}$, 0 being the empty index. We also define $|m| = k$.

It is to be seen that the b_m, as m ranges over such multi-indices, span B_L as a real vector space. Thus any element of B_L may be written as a real linear combination:

$$x = \sum x^m b_m \qquad x^m \in \mathbf{R}.$$

It is to be noticed that we have maps

$$e : B_L \longrightarrow \mathbf{R} \quad \text{given by}$$
$$e(x) = e(\sum x^m b_m) = x^0$$

and

$$s : B_L \longrightarrow B_L - \mathbf{R} \quad \text{by}$$
$$s(x) = x - e(x).$$

$x^0 = e(x)$ is often referred to as the **Real part** of x. Notice that e extends to a map

$$e : B_L \times B_L \times \ldots \times B_L \longrightarrow \mathbf{R}^n$$

by acting on factors. It will be clear from the context which e we are using.

Examples: (a) Let $x \in B_2$, then x may be written as

$$x = x^0+x^1b_1+x^2b_2+x^{12}b_{12}$$

(b) let $f:B_2 \longrightarrow B_2$ be a smooth map, then f may be decomposed as:

$$\begin{aligned} f(x) &= f(x^0+x^1b_1+x^2b_2+x^{12}b_{12}) \\ &= f(x^0,x^1,x^2,x^{12}) \\ &= f^0(x^0,x^1,x^2,x^{12})+f^1(x^0,x^1,x^2,x^{12})b_1 \\ &\quad +f^2(x^0,x^1,x^2,x^{12})b_2 \\ &\quad +f^{12}(x^0,x^1,x^2,x^{12})b_{12} \end{aligned}$$

that is, we may freely identify f as a map $f:\mathbf{R}^4 \longrightarrow \mathbf{R}^4$. This identification will be used frequently in the sequel without further comment.

A further notational complication is the fact that B_L is a graded algebra:

(i) B_L is graded over $\mathbf{Z}_2$, $B_L = (B_L)_0+(B_L)_1$ where $(B_L)_0 = \{ \sum x^m b_m : |m| \text{ even} \}$ is the **Even part** and $(B_L)_1 = \{ \sum x^m b_m : |m| \text{ odd} \}$ is the **Odd part**.

We shall write $x \in B_L$ as $x = x_0+x_1$ and shall refer to the odd or even parts of x according to this decomposition. We shall also refer to the multi-index m as being odd or even according to whether $|m|$ is an odd or even integer.

(ii) B_L is graded over the integers $\mathbf{Z}$ according to $|m|$. That is

$$x = x^1+x^2+x^3+...+x^L+0+0+...$$

Again, the context will make it clear which notation we are using.

Let $f:B_L \longrightarrow B_L$ be a smooth map.

Definition: f is said to be **Restricted G^1 at x** if and only if

$$f(x+h) = f(x) + Gf(x).h + n(h)\|h\|$$

where $Gf(x) \in B_L$, $\|n(h)\| \longrightarrow 0$ as $\|h\| \longrightarrow 0$, for some $n: B_L \longrightarrow B_L$. If f is restricted G^1 for all x in some open set U, then f is called **Restricted G^1 in U**. If U is all of B_L then we simply say that f is **Restricted G^1**.

This definition implies that if f is restricted G^1 then the ordinary derivative of f, regarded as a linear mapping $\mathbf{R}^{2^L} \rightarrow \mathbf{R}^{2^L}$ is given by:

$Df(x)(h) = Gf(x).h$ (multiplication in the exterior algebra). $Gf: B_L \longrightarrow B_L$ is called the **G-derivative of f**.

For simplicity, we shall frequently drop the use of the word "restricted" in this chapter. We shall be more careful in following chapters as there will be other notions of G-differentiability that we shall be considering.

Notice that if B_L is made into a right B_L module, then f being G^1 implies that Df(x) is a linear map over B_L. Conversely, if Df(x) is linear over B_L, then

$$Df(x)(h) = Df(x)(1.h) = Df(x)(1).h$$

and f is G^1 at x. Thus we have proved:

Proposition 1: $f: B_L \longrightarrow B_L$ is restricted G^1 at x $\Leftrightarrow$ Df(x) is linear over B_L.

In coordinates the second condition is equivalent to the system of partial differential equations:

$(\partial f/\partial x^m) = (\partial f/\partial x^0)b_m$ (derivatives evaluated at x).

We shall use these equations frequently to check whether a given function is G-differentiable.

Notes: (i) These equations bear more than a passing resemblance to the Cauchy-Riemann equations of complex variable theory. We shall be investigating this similarity later in the chapter.

(ii) It is clear from these equations that $Gf(x)=(\partial f/\partial x^0)|_x$.

(iii) This sort of equation has also been obtained by Boyer and Gitler [10].

Proposition 2: f is $G^1 \Rightarrow$ Gf is G^1.

proof: We use proposition 1.

$$\begin{aligned}(\partial/\partial x^m)(Gf) &= (\partial/\partial x^m)(\partial f/\partial x^0)\\ &= (\partial/\partial x^0)(\partial f/\partial x^m)\\ &= (\partial/\partial x^0)(\partial f/\partial x^0)b_m\\ &= (\partial/\partial x^0)(Gf)b_m.\end{aligned}$$

and thus Gf satisfies the condition of proposition 1.

We may phrase this result as "f is $G^1 \Rightarrow$ f is G^∞ ". This is in marked contrast to Rogers' definition of G derivative [34], where there are many functions that are G^1 but not G^∞ . We shall see why this is the case in chapter 3.

Due to the result of proposition 2 we shall frequently refer to G^1 functions as G^∞ functions.

Examples: (a) The identity function and constant functions are G^∞.

(b) Let $f:B_2 \longrightarrow B_2$ be $f(x) = x^2$.

(multiplication).

thus $f(x) = (x^0+x^1b_1+x^2b_2+x^{12}b_{12})^2$

$= (x^0)^2+2x^0(x^1b_1+x^2b_2+x^{12}b_{12})$

so $(\partial f/\partial x^0) = 2(x^0+x^1b_1+x^2b_2+x^{12}b_{12})$

and $(\partial f/\partial x^1) = 2x^0b_1$

so $(\partial f/\partial x^0)b_1 = 2x^0b_1+2x^2b_2b_1$

thus $f(x) = x^2$ is not G^∞. Similarly no non-zero power of x is G^∞, except $f(x) = x$

(c) Let $g:\mathbf{R}\longrightarrow\mathbf{R}$ be any C^∞ function. Define $f:B_2\longrightarrow B_2$ by

$$f(x^0,x^1,x^2,x^{12}) = g(x^0)b_1b_2$$

then f is G^∞, since $(\partial f/\partial x^0) = (dg/dx^0)b_1b_2$

thus $(\partial f/\partial x^0)b_1 = 0 = (\partial f/\partial x^1)$ etc.

(d) $e:B_L\longrightarrow\mathbf{R}$ is not G^∞. Neither is s.

(e) Let $f:B_1\longrightarrow B_1$ be $f(x) = x^2$

then $f(x^0+x^1b_1) = (x^0)^2+2x^0x^1b_1$

thus $(\partial f/\partial x^0) = 2(x^0+x^1b_1)$

and $(\partial f/\partial x^1) = 2x^0b_1 = (\partial f/\partial x^0)b_1$

thus f is G^∞.

(f) If $L>1$ then left multiplication by scalars is G^1, but right multiplication by scalars is G^1 if and only if the scalar is an even element of B_L.

Notice that B_1 is a commutative algebra, it gives many anomalous results, so we shall deal with it as a special case on its own.

With very little effort we may display the most general form of a G^∞ function $f:B_1\longrightarrow B_1$.

Proposition 3: $f:B_1 \longrightarrow B_1$ is G^∞ if and only if

$f(x^0,x^1) = f_0(x^0)+((df_0/dx^0)x^1+g_0(x^0))b_1$

where f_0 and g_0 are C^∞ functions of one real variable.

proof: The system of differential equations derived in proposition 1 reduces to a single equation in this case, namely: $(\partial f/\partial x^1) = (\partial f/\partial x^0)b_1$.

So, if we write $f = f_0+f_1 b_1$ we solve:

$$(\partial f_0/\partial x^1)+(\partial f_1/\partial x^1)b_1 = [(\partial f_0/\partial x^0)+(\partial f_1/\partial x^0)b_1]b_1$$
$$= (\partial f_0/\partial x^0)b_1.$$

Thus equating real and nilpotent parts,

$$(\partial f_0/\partial x^1) = 0 \Rightarrow f_0 = f_0(x^0)$$
$$(\partial f_1/\partial x^1) = (\partial f_0/\partial x^0)$$
$$\Rightarrow \quad f_1(x^0,x^1) = (df_0/dx^0)x^1 + g_0(x^0)$$ as the right hand side is a function of x^0 only.

We shall pursue this special case after we have dealt in a little more detail with the general G^∞ function.

Proposition 4: Let $f:B_L \longrightarrow B_L$ be a G^∞ diffeomorphism, then $e(x) = e(y)$ if and only if $e(f(x)) = e(f(y))$.

proof: $(\partial f/\partial x^m) = (\partial f/\partial x^0)b_m$ implies that $(\partial/\partial x^m)(f_0)=0$ for $|m|>0$. But $f_0=e(f)$ thus $e(f(x))$ is a function of $e(x)$ only.

Now we shall restrict ourselves to the case L>1. We note that proposition 3 may be phrased as "f is G^∞ if and only if

$$f(x^0,x^1) = f(x^0)+(df/dx^0)x^1 \text{ where } f:\mathbf{R} \longrightarrow B_1$$

is C^∞

$$= f(e(x))+(df/dx^0)s(x)".$$

The same is nearly true for L>1, but there is a restriction due to the non-commutativity of B_L.

Definition: Let $h:\mathbf{R}\longrightarrow B_L$ be a C^∞ function, then h is said to be a **G-admissible function** if and only if $h^0,h^1,\ldots,h^{L-2}$ are **R**-affine functions.

Examples: (i) Any affine function $\mathbf{R}\longrightarrow B_L$ is G-admissible.

(ii) $h:\mathbf{R}\longrightarrow B_2$ given by

$$h(x^0) = 2x^0+3+(x^0)^2b_1+(x^0)^3b_2+(\sin(x^0))b_1b_2$$

is G-admissible.

(iii) $h:\mathbf{R}\longrightarrow B_L$ given by

$$h(x^0) = (x^0)^2(1+b_1+b_2+b_1b_2)$$

is not G-admissible.

The point of this definition is the following:

Proposition 5: Let $f:B_L\longrightarrow B_L$ (L>1) be a G^∞ function, then there is a unique G-admissible function $\hat{f}:\mathbf{R}\longrightarrow B_L$ such that

$$f(x) = \hat{f}(e(x))+(d\hat{f}/dx^0)s(x)$$

proof: By inspection of the differential equations derived in proposition 1. Firstly:

$$\begin{aligned}(\partial^2 f/(\partial x^m)^2) &= (\partial/\partial x^m)(\partial f/\partial x^m)\\ &= (\partial/\partial x^m)(\partial f/\partial x^0)b_m\\ &= (\partial/\partial x^m)(Gf)b_m\\ &= (\partial^2 f/(\partial x^0)^2)(b_m)^2\\ &= 0 \quad\quad \text{for } |m|>0\end{aligned}$$

thus f is linear in x^m for $|m|>0$. Next:

$$
\begin{aligned}
(\partial^2 f/\partial x^i \partial x^j) &= (\partial/\partial x^i)(\partial f/\partial x^j) \\
&= (\partial/\partial x^i)(\partial f/\partial x^0)b_j \\
&= (\partial^2 f/(\partial x^0)^2)b_j b_i
\end{aligned}
$$

but

$$
\begin{aligned}
(\partial^2 f/\partial x^i \partial x^j) &= (\partial^2 f/\partial x^j \partial x^i) \\
&= \ldots \\
&= (\partial^2 f/(\partial x^0)^2)b_i b_j \\
&= -(\partial^2 f/(\partial x^0)^2)b_j b_i .
\end{aligned}
$$

Thus we have: $(\partial^2 f/(\partial x^0)^2)b_i b_j = 0$ for $1 \leqslant i,j \leqslant L$.

Thus $f(x^0 + 0.(s(x)))$ is G-admissible. We define this to be $\hat{f}(x^0)$. So we know that f is linear in the x^m for $|m|>0$ and that $\hat{f}$ is admissible; it only remains to find the coefficients of the x^m.

$$(\partial f/\partial x^m) = (\partial f/\partial x^0)b_m = (d\hat{f}/dx^0)b_m$$

thus the coefficient of x^m is $(d\hat{f}/dx^0)b_m$ and so $f(x) = \hat{f}(e(x)) + (d\hat{f}/dx^0)s(x)$.

Example: We exhibit the form of a restricted G^∞ function $f:(B_2) \longrightarrow (B_2)$.

$$
\begin{aligned}
f(x^0,x^1,x^2,x^{12}) = {} & f_0(x^0)+f_1(x^0)b_1+f_2(x^0)b_2+f_{12}(x^0)b_{12}+ \\
& [(df_0/dx^0)+(df_1/dx^0)b_1+(df_2/dx^0)b_2+(df_{12}/dx^0)b_{12}] \times \\
& [x^1 b_1 + x^2 b_2 + x^{12} b_{12}] \\
= {} & f_0(x^0) + \\
& [(df_0/dx^0)x^1 + f_1(x^0)]b_1 + \\
& [(df_0/dx^0)x^2 + f_2(x^0)]b_2 + \\
& [(df_0/dx^0)x^{12} + f_{12}(x^0)]b_{12}
\end{aligned}
$$

, with f_0 affine.

We now aim to define the notion of a restricted G^∞ supermanifold, but before we do that we must make a few standard definitions.

Definition: Let $f:(B_L)^n \longrightarrow B_L$ be a smooth map, then f is said to be **G^1 at x** if and only if

$$f(x_1+h_1, x_2+h_2, \ldots, x_n+h_n) = f(x_1, x_2, \ldots, x_n) + \sum_{j=1}^{n} G_j f(x).h_j + n(h_1, \ldots, h_n) \|(h_1, \ldots, h_n)\|$$

for $G_j f(x) \in B_L$, $\|n\| \rightarrow 0$ as $\|h\| \rightarrow 0$. Then, of course, f is said to be restricted G^1 in an open set U if and only if f is restricted G^1 at all the points of U.

Notes: (a) $(B_L)^n$ is normed as a product of normed vector spaces.

(b) $g:(B_L)^n \longrightarrow (B_L)^m$ is called **Restricted G^1** if and only if all the component functions of g are G^1 in the sense defined above.

(c) As before, any function that is G^1 is G^∞.

(d) The chain rule for restricted G^∞ functions is precisely what one might hope for, that is, the composite of G^∞ functions is G^∞ and the G-derivative of the composite is the composite of the G-derivatives. This is proved in exactly the same way that it is proved for C^∞ functions.

One may wonder at this point whether the inverse function theorem holds for G^∞ functions,

$$f:(B_L)^n \longrightarrow (B_L)^n$$

and if so, what form it takes. In fact the situation is quite simple: Take the case n=1. The equation

$(\partial f/\partial x^m) = (\partial f/\partial x^0)b_m$ forces $(\partial f^p/\partial x^m) = 0$ if m is not a subindex of p (that is, the $m_1,\ldots,m_k$ are not all among the $p_1,\ldots,p_l$). Thus the Jacobean of f considered as a map $f:\mathbf{R}^{2^L}\longrightarrow\mathbf{R}^{2^L}$ is triangular, therefore the determinant is $|(\partial f^0/\partial x^0)|^{2^L}$ and f is invertible if and only if the induced map $f^0:\mathbf{R}\longrightarrow\mathbf{R}$ is invertible. The G-differentiability of f^{-1} follows from the B_L-linearity of Df(x) and hence of $Df(x)^{-1}$.

Now the general case of any positive n follows in an exactly similar manner: $f:(B_L)^n\longrightarrow(B_L)^n$ is locally invertible as a G^∞ map if and only if $f^0:\mathbf{R}^n\longrightarrow\mathbf{R}^n$ is locally invertible as a C^∞ map.

Now suppose that M is a hausdorff, 2nd countable topological space.

Definition: A **Superchart** on M is a pair (U,f) where U is an open subset of M and $f:U\longrightarrow f(U)\subseteq(B_L)^n$ is a homeomorphism. A **Restricted G^∞ atlas**, A, on M is a family of charts $\{(U_i,f_i):i\in I\}$ on M such that:

(i) M is the union of the U_i.

(ii) If (U,f) and (V,g) are elements of A then the restriction $gf^{-1}:f(U\cap V)\longrightarrow g(U\cap V)$ is a G^∞ map.

Two G^∞ atlases are **Equivalent** if and only if their union is a G^∞ atlas. A G^∞ atlas A is **Maximal** if and only if any G^∞ atlas containing A is equal to A.

A **Restricted G^∞ Supermanifold** is a pair (M,A) where A is a maximal restricted G^∞ atlas on M. The integer n is the **Dimension of M over B_L**.

A map between supermanifolds is said to be G^∞ if and only if it is a G^∞ map when expressed in the coordinates of the supermanifolds concerned.

By lifting the result of proposition 4 to the manifold M we obtain an immediate topological obstruction to a C^∞ manifold admitting a G^∞ supermanifold structure. Let x,y be any two points of M. We define a relation R on M as follows:

xRy if and only if there is a superchart (U,f) such that $x,y \in U$ and $e(f(x))=e(f(y))$.
("The real part of x is equal to the real part of y"). This is well defined, by proposition 4. Let ~ be the equivalence relation generated by R (that is, there is a chain of points and coordinate patches joining x to y according to the relation R). The equivalence classes of M under ~ form the leaves of a foliation on M called the **Real Foliation of M**. If the quotient of M by this foliation happens to be hausdorff then the supercharts of M descend to define charts on the quotient and the quotient becomes a C^∞ manifold. This manifold is called the **Core manifold** of M and is denoted by $\tilde{M}$. (In the literature it is sometimes referred to as the **Body of M**). For a general survey of foliation theory see Lawson [28].

So, for example, the only compact surfaces that could possibly admit restricted supermanifold structures are the torus and the Klein bottle, since the superstructure would have to be one-dimensional over B_1.

Hence the foliation would be of codimension one and hence the Euler characteristic would vanish. We shall see later that any restricted supermanifold inherits a natural orientation from its superstructure hence the Klein bottle does not in fact admit such a structure. However the torus does.

Examples: (a) The two dimensional real torus is a supermanifold over B_1 of dimension 1.

Let I^2 be the closed unit square in $\mathbf{R}^2$ and let x,y be the natural coordinates on it. Identifying opposite sides gives us the torus. Define charts on it as follows: (This construction is a generalization of the construction in Rogers [34]).

$f_1(x,y)=(ax+by)1+(cx+dy)b_1 \quad ad-bc \neq 0$ on U_1 where

$U_1=\{(x,y):1/5<x<4/5:1/5<y<4/5\}$

$f_2(x,y)= (ax+by)1+(cx+dy)b_1$ for $y<2/5$

$(ax+b(y-1))1+(cx+d(y-1))b_1$ for $y>3/5$ on U_2

where

$U_2=\{(x,y):1/5<x<4/5:y<2/5\}\cup\{(x,y):1/5<x<4/5:y>3/5\}$

$f_3(x,y)=(ax+by)1+(cx+dy)b_1$ for $x<2/5$

$(a(x-1)+by)1+(c(x-1)+dy)b_1$ for $x>3/5$ on U_3 where

$U_3=\{(x,y):x<2/5:1/5<y<4/5\}\cup\{(x,y):x>3/5:1/5<y<4/5\}$

$f_4(x,y)=(ax+by)1+(cx+dy)b_1$ for $x<2/5$ and $y<2/5$

$(ax+b(y-1))1+(cx+d(y-1))b_1$ for $x<2/5$ and $y>3/5$

$(a(x-1)+by)1+(c(x-1)+dy)b_1$ for $x>3/5$ and $y<2/5$

$(a(x-1)+b(y-1))1+(c(x-1)+d(y-1)b_1$ for $x>3/5$ and $y>3/5$

on U_4 where $U_4=\{(x,y):x<2/5:y<2/5\}\cup\{(x,y):x<2/5:y>3/5\}\cup$

$\{(x,y):x>3/5:y<2/5\}\cup\{(x,y):x>3/5:y>3/5\}$

Pass to the quotient torus, then all transition functions are translations and thus G^∞. Let A be the maximal atlas compatible with this set of charts, then (T,A) is a one dimensional supermanifold over B_1.

Let us inspect the real foliation. If, for example, x,y are elements of U_1 then

$$e(f_1(x,y)) = e(f_1(x',y')) \Rightarrow ax+by = ax'+by'$$
$$\Rightarrow a(x-x')+b(y-y') = 0$$

Inspecting the other charts and passing to the quotient leads us to conclude that the real foliation is composed of lines of slope determined by the coefficients a and b and that any slope may be so obtained. For example, if a=1 and b=0 then the quotient by the foliation is a circle. If a and b are chosen so that the slope is irrational then the quotient by the foliation is not hausdorff and thus not a manifold.

Suppose we define an atlas A' on T by using

$$g_1(x,y) = (a'x+b'y)1+(c'x+d'y)b_1 \quad \text{etc.,etc...}$$

When are A and A' equivalent? We only need to check when gf^{-1} is G-differentiable. It is clear that

$$gf^{-1}(x+yb) = \frac{(ad'-bc')x+(a'b-b'a)y}{(a'd'-b'c')}1 + \frac{(cd'-c'd)x+(a'd-b'c)y}{(a'd'-b'c')}b_1$$

Now, a linear map $f:B_1 \longrightarrow B_1$ given by

$$f(x+yb_1) = (mx+ny)1+(px+qy)b_1$$ is shown to be G^∞ if and only if q=m and n=0. Thus gf^{-1} is G-differentiable if and only if:

$$a'b = b'a \text{ and}$$
$$ad'-bc' = a'd-bc'$$

This condition is an equivalence relation on the set of 2x2 invertible matrices and geometrically means that the real foliations defined by f and g coincide. (Inspect, for example, the case a=c=1, b=d=0). The quadrilateral obtained by a,b,c,d is the quadrilateral obtained from a',b',c',d' sheared parallel to the real foliation, if this is the case.

(b) Any torus of dimension 2^L admits a restricted superstructure over B_L of dimension 1 in the same way as the 2-torus.

(c) The supersphere over B_1.

define a map $q:(B_1)^n \longrightarrow B_1$ by

$$q(a_1,a_2,\ldots,a_n) = (a_1)^2+\ldots+(a_n)^2$$

and define $S = \{x \in (B_1)^n : q(x) = 1\}$. Then S is called the **Supersphere over B_1** of dimension n-1. (n.b. It is not a topological sphere as we shall see below).

S admits the structure of a restricted supermanifold of dimension n-1 over B_1 as follows:

Map $(B_1)^{n-1}$ into $(B_1)^n$ by $z = \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1} \end{pmatrix} \mapsto \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1} \\ 0 \end{pmatrix}$

and let $v = \dfrac{1}{1+q(z)} \begin{pmatrix} 2z \\ 1-q(z) \end{pmatrix} = \dfrac{2z+(1-q(z))e_n}{1+q(z)}$

where $e_n = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix}$ then clearly $v \in S \subset (B_1)^n$

as $q(v) = \dfrac{1}{(1+q(z))^2} [4q(z)+(1-q(z))^2] = 1$

Let $<v_1,v_2>$ denote the scalar product of vectors in $(B_1)^n$ then $<v,e_n> = (1-q(z))/(1+q(z))$, hence

$$1+<v,e_n> = 2/(1+q(z)) \text{ is a unit in } B_1, \text{ hence}$$

$$v = (1+<v,e_n>)z+<v,e_n>e_n \quad \text{and so}$$

$$z = (v-<v,e_n>e_n)/(1+<v,e_n>)$$

So the map defined by $v \longmapsto z$ is a chart on S.

A second chart is obtained by using the map

$$\tilde{z} \qquad \frac{1}{1+q(\tilde{z})} \begin{pmatrix} 2\tilde{z} \\ q(\tilde{z})-1 \end{pmatrix}$$

then similarly $\tilde{z} = (v-<v,e_n>e_n)/(1-<v,e_n>)$

$$v = (1+<v,e_n>)\tilde{z}+<v,e_n>e_n$$

so $\tilde{z} = [(1+<v,e_n>)z+<v,e_n>e_n-<v,e_n>e_n]/(1-<v,e_n>)$

$= z/q(z)$.

One then shows that the map $z \mapsto z/q(z)$ is G-differentiable and that the domains of definition of the above functions cover S. Thus S is a restricted G^∞ supermanifold over B_1. (Note that if this construction was attempted over B_L, for $L>1$, the transition functions would no longer be G-differentiable).

Now notice that $q(z_1,z_2,\ldots,z_n) = 1$

$$\Leftrightarrow q(x_1+y_1b_1,\ldots,x_n+y_nb_1) = 1$$

$$\Leftrightarrow (x_1+y_1b_1)^2+\ldots+(x_n+y_nb_1)^2 = 1$$

$$\Leftrightarrow (x_1)^2+\ldots+(x_n)^2 = 1$$

and $\qquad x_1y_1+\ldots+x_ny_n = 0$ equating real and nilpotent parts. That is, x is an element of the ordinary sphere S^{n-1} and $x.y=0$, implying that y is tangent to the sphere. Thus the supersphere S is topologically the tangent bundle of the ordinary sphere S^{n-1}.

(d) Projective superspace over B_1.

Recall the augmentation, or real, map

$$e:(B_1)^n \longrightarrow \mathbf{R}^n$$

now,

$$(B_1)^n - e^{-1}(0) = \tilde{U}_1 \cup \tilde{U}_2 \cup \ldots \cup \tilde{U}_n$$

where $\tilde{U}_i = \{z \in (B_1)^n : e(z).e_i \neq 0\}$ where e_i is the i^{th} base vector.

Define a relation ~ on $(B_1)^n - e^{-1}(0)$ by $z \sim z'$ if and only if there is a non-zero l in B_1 such that

$$z = lz'$$

then ~ is an equivalence relation.

Define $\mathbf{P}((B_1)^n)$ to be the quotient of $(B_1)^n - e^{-1}(0)$ by this equivalence relation. We shall show that this quotient admits the structure of a restricted supermanifold over B_1 of dimension n-1 by defining charts as follows:

let $\quad f_i : U_i \longrightarrow (B_1)^{n-1} \quad$ by

$$f_i(p(z)) = (z_1/z_i, z_2/z_i, \ldots, \hat{z}_i, \ldots, z_n/z_i)$$

where p is the quotient map

$$p:(B_1)^n - e^{-1}(0) \longrightarrow \mathbf{P}((B_1)^n)$$

$U_i = p(\tilde{U}_i)$ and ^ denotes omission.

It is then straightforward to verify the G-differentiability of the transition functions. The supermanifold obtained in this way is called **Projective superspace over B_1** of dimension n-1.

Let us write out in coordinates the consequences of the definition of ~.

$$z = \begin{pmatrix} x_1+y_1b_1 \\ \vdots \\ x_n+y_nb_1 \end{pmatrix} \sim \begin{pmatrix} x'_1+y'_1b_1 \\ \vdots \\ x'_n+y'_nb_1 \end{pmatrix}$$

if and only if there is a $k = 1+mb_1$ such that

$$\begin{pmatrix} x_1+y_1b_1 \\ \vdots \\ x_n+y_nb_1 \end{pmatrix} = (1+mb_1) \begin{pmatrix} x'_1+y'_1b_1 \\ \vdots \\ x'_n+y'_nb_1 \end{pmatrix}$$

thus

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 1 \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix}$$

and

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 1 \begin{pmatrix} y'_1 \\ \vdots \\ y'_n \end{pmatrix} + m \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix}$$

The first equation gives ordinary projective space while the second implies that two vectors are related only if their difference is a scalar multiple of the vector defining a point of projective space. Such vectors are parameterized by an n-1 dimensional hyperplane perpendicular to the vector defining the point of projective space. Hence $P((B_1)^n)$ is topologically the tangent bundle of ordinary projective space.

We wish to investigate the conditions under which a real C^∞ manifold admits the structure of a restricted supermanifold over B_L for some L. It is noticeably easier to generate examples over B_1 since this is a commutative algebra, so that varieties exist over it. Those that we have defined so far have been, topologically, the tangent bundle of their core manifolds. We shall see to what extent this is true in general.

Our plan of action, then, is to investigate the structure of B_1 supermanifolds in some detail, the structure of "supervarieties" and the structure of supermanifolds over B_L, for $L>1$.

Section 2: Almost Supermanifolds:

Having noticed an analogy with the theory of complex manifolds, we wish to follow this up further by "infinitesimalizing" the definition of supermanifolds to obtain the notion of "almost supermanifolds". A suggested reference for this section is Kobayashi-Nomizu [25].

Definition: Let V be a 2n-dimensional real vector space. Suppose that

$$B: V \longrightarrow V$$

is an endomorphism of V satisfying

$$\text{Kernel } B = \text{Image } B,$$

then B is said to define a **Superstructure on V**.

Notes: (a) The definition implies that $B^2 = 0$ and that dim(Kernel B) = n.

(b) V inherits the structure of a B_1-module via

$$(c+db_1).X = cX+dB(X) \quad \text{for } c,d \in \mathbf{R},\ X \in V.$$

since

$$\begin{aligned}(e+fb_1)[(c+db_1).X] &= (e+fb_1)(cX+dB(X)) \\ &= (ec)X+(ed+fc)B(X) \\ &= [(e+fb_1).(c+db_1)].X\end{aligned}$$

Let $(B_1)^n = \{(b^1,b^2,\ldots,b^n): b^i \in B_1\}$

set $b^i = x^i+y^ib_1$, then the correspondence

$$(B_1)^n \longleftrightarrow \mathbf{R}^{2n} \quad \text{via}$$

$$(b^1,b^2,\ldots,b^n) \longleftrightarrow (x^1,\ldots,x^n,y^1,\ldots,y^n)$$

gives $B(x^1,\ldots,x^n,y^1,\ldots,y^n) = (0,\ldots,0,x^1,\ldots,x^n)$ where B mimics the action of b_1 on $(B_1)^n$.

So, with respect to the usual basis on $\mathbf{R}^{2n}$, B has the matrix

$$[B] = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \quad \text{(in nxn block form)}$$

We call this endomorphism the **Usual Superstructure** on $\mathbf{R}^{2n}$, and denote it by B^0.

Now let $f:(V,B) \longrightarrow (V',B')$ be an **R**-linear map between real vector spaces V and V' equipped with superstructures B and B' respectively.

f is said to be **B -linear** if and only if:

$$f(B(v)) = B'(f(v)) \quad \text{for } v \in V.$$

Let G be the subset of $GL_{2n}(\mathbf{R})$ consisting of B_1-linear maps

$$f:(\mathbf{R}^{2n},B^0) \longrightarrow (\mathbf{R}^{2n},B^0),$$

then $G = \{A \in GL_{2n}(\mathbf{R}) : AB^0 = B^0A\}$

so $A \in G$ if and only if the matrix of A takes the block form

$$\begin{pmatrix} C & 0 \\ D & C \end{pmatrix} \quad \begin{matrix} \text{for } \det C \neq 0 \\ C,D \in M_n(\mathbf{R}) \end{matrix}$$

Notes: (a) If s $GL_{2n}(\mathbf{R})$ and if B is a superstructure on $\mathbf{R}^{2n}$, then so is sBs^{-1}. Thus $GL_{2n}(\mathbf{R})$ acts on the set of superstructures on $\mathbf{R}^{2n}$, which we will denote by $\mathbf{S}$.

If $X_1,X_2,\ldots,X_n$ span V as a B_1-module, then it is easy to see that $X_1,\ldots,X_n,B(X_1),\ldots,B(X_n)$ span V as an **R**-module. Thus one may show that the action of $GL_{2n}(\mathbf{R})$ on $\mathbf{S}$ is transitive and furthermore the isotropy subspace at B_0 is precisely G.

(b) If V admits the superstructure B, then the dual space V^* admits the dual map B^* as a superstructure. Thus any tensor, dual or exterior power of V admits a superstructure.

Our treatment of superstructures, so far, mimics the definition of complex structures on real vector spaces and this analogy is very useful in handling superstructures. However the theories diverge since a superstructure is a nilpotent endomorphism whereas a complex structure is a semisimple endomorphism. We do not get a decomposition of V into eigenspaces since they are not complemented. What we do have, though, is a canonically defined subspace of V, namely the kernel of B. This subspace is clearly B-invariant.

Our next aim is to lift the notion of a superstructure to a manifold.

Definition: An **Almost B_1 Supermanifold** is a smooth manifold M, together with a smooth (1,1) tensor field B on M that satisfies

$$\text{Kernel } B(x) = \text{Image } B(x)$$

where B(x) denotes the evaluation of B at x. B(x) is considered as an element of $Hom(T_xM, T_xM)$ thus B equips each tangent space with a superstructure and as before we have $B(x)^2=0$ and dim Ker B(x)= Dim M/2. Thus M is necessarily even dimensional and orientable.
We can state this definition in an equivalent fashion, namely that an almost B_1 supermanifold is a reduction of the frame bundle of a smooth manifold M to the group G

that we defined above. From this definition and the form of G, we see that in order to admit an almost B_1-structure TM must split as $K \oplus K$ where K is the kernel of B considered as a sub-bundle of TM. Another way to see this is to choose a metric on TM to split the exact sequence:

$$0 \longrightarrow K \longrightarrow TM \longrightarrow TM/K \longrightarrow 0 \quad .$$

If M is an n-dimensional restricted supermanifold over B_1, then it is an almost B_1 supermanifold, for if we choose coordinates $(b^1,..,b^n)$ on $(B_1)^n$ and set $b^i = x^i+y^ib_1$ and define

$$B(\partial/\partial x^i) = (\partial/\partial y^i) \quad 1\leq i\leq n$$

$$B(\partial/\partial y^j) = 0 \qquad 1\leq j\leq n$$

and then pull this up to the manifold via supercharts, then B defines an almost B_1-structure on M.
(Thus we have demonstrated the truth of our earlier assertion that the Klein bottle does not admit the structure of a restricted supermanifold).

Proposition 6: Suppose that $f:(B_1)^m \longrightarrow (B_1)^n$ is a smooth map, then f is restricted G^1 if and only if $f_*B = Bf_*$. (Where B, by abuse of notation, stands for the induced superstructures on both $(B_1)^n$ and $(B_1)^m$).

proof: The proposition follows by realizing that the condition is equivalent to the linearity of the derivative of f over B_1. Explicitly, we mimic the proof of the almost complex case:

f is restricted G^1 if and only if

$$(\partial f^k/\partial x^j)b_1 = (\partial f^k/\partial y^j) \quad \text{for each } k,j \qquad (*)$$

set $f^k = u^k + v^k b_1$

then (*) $\quad (\partial u^k/\partial y^j) + (\partial v^k/\partial y^j) b_1 =$

$$[(\partial u^k/\partial x^j) + (\partial v^k/\partial x^j) b_1] b_1$$

$\Rightarrow \quad (\partial u^k/\partial y^j) = 0 \quad$ and $\quad (\partial v^k/\partial y^j) = (\partial u^k/\partial x^j)$

now, $\quad f_*(\partial/\partial x^j) = \sum_{k=1}^{n} (\partial u^k/\partial x^j)(\partial/\partial u^k) + (\partial v^k/\partial x^j)(\partial/\partial v^k)$

and $\quad f_*(\partial/\partial y^j) = \sum_{k=1}^{n} (\partial u^k/\partial y^j)(\partial/\partial u^k) + (\partial v^k/\partial y^j)(\partial/\partial v^k)$

Now, f is restricted $G^1 \Rightarrow \quad f_*B(\partial/\partial x^j) = f_*(\partial/\partial y^j)$

$$= \sum_{k=1}^{n} (\partial v^k/\partial y^j)(\partial/\partial v^k)$$

and $\quad Bf_*(\partial/\partial x^j) = B\left[\sum_{k=1}^{n} (\partial u^k/\partial x^j)(\partial/\partial u^k) + (\partial v^k/\partial x^j)(\partial/\partial v^k)\right]$

$$= \sum_{k=1}^{n} (\partial u^k/\partial x^j)(\partial/\partial v^k)$$

$$= \sum_{k=1}^{n} (\partial v^k/\partial y^j)(\partial/\partial v^k)$$

$$= f_*B(\partial/\partial x^j)$$

Also, $f_*B(\partial/\partial y^j) = 0 = Bf_*(\partial/\partial y^j)$, hence the result follows by linearity. The converse is similar.

Definition: Let $f:(M,B) \longrightarrow (M',B')$ be a smooth map of almost B_1 supermanifolds. f is said to be **Almost G^∞** if and only if $\quad B'f_* = f_*B$.
Thus a smooth map of restricted supermanifolds is G^∞ if and only if it is almost G^∞ with respect to the induced B_1 structures.

Having defined the notion of an almost supermanifold, we are led to ask whether there any simple criteria for deciding when an almost supermanifold structure has arisen from a genuine supermanifold structure. That is, from the point of view of G-structures, we are asking about the integrability or local flatness of the structure.

We recall a few facts about the integrability problem for G-structures, for it is in this setting that the problem is most amenable. Suggested references are Sternberg [41], Guillemin [19], Singer and Sternberg [39] and Guillemin and Sternberg [20].

Suppose that $P \longrightarrow M$ is a G-structure on M where G is an arbitrary Lie group. (That is, P is a principal bundle over M with structural group G). The G-structure is said to be **Locally Flat** if and only if it is locally equivalent to the standard G-structure on Euclidean space. To be locally flat the structure must satisfy some formal conditions, namely the vanishing of certain structure tensors, c_i, $i \geq 0$, taking their values in the Spencer cohomology groups $H^{i,2}(G)$. (The number of a priori non-zero tensors depends only on the group G). The results of Goldschmidt and Spencer (see Goldschmidt [17]) then tell us that a G-structure satisfying these formal flatness conditions is in fact locally flat.

In our case G is an **Involutive** group, that is $H^{p,q}(G)=0$ for all $p>0$. Thus there is only one formal integrability tensor $c_0 \in H^{0,2}(G)$. (This is because G is defined to be the set of matrices commuting with a given fixed matrix. See Guillemin [19]).

Definition: The **Integrability Tensor** T, for an almost B_1 supermanifold structure is given by

$$T(X,Y) = [BX,BY]-B[BX,Y]-B[X,BY]$$

where X,Y are vector fields on M and [,] denotes the Lie bracket.

As it happens, we have no need to recourse to the general results of Goldschmidt-Spencer for there is a fairly straightforward proof that the vanishing of T implies the integrability of the structure. Compare this with the integrability problem for almost complex structures.

Proposition 7: $T = 0 \Leftrightarrow B$ is locally flat.

proof: ($\Leftarrow$) is trivial.

($\Rightarrow$) We need to show that we can solve the partial differential equations that a change of coordinates would have to satisfy to have B constant in the new coordinates. (In fact, identical to the standard superstructure in the new coordinates).

Firstly, we note that the vanishing of T implies that the image distribution (which is equal to the kernel distribution) is integrable, by Frobenius' theorem. See Warner [43].

That is, there are coordinates $(u^1,\ldots,u^n,v^1,\ldots,v^n) = (u,v)$ such that $(\partial/\partial v^1),\ldots,(\partial/\partial v^n)$ span K at every point.
Thus in this coordinate system, we must have:

$$B(\partial/\partial u^i) = \sum_{j=1}^{n} c^j{}_i(u,v)(\partial/\partial v^j)$$

$$B(\partial/\partial v^j) = 0 \qquad 1\leq j\leq n$$

where c is a matrix with det $c(u,v) \neq 0$ for all u,v.
We require $(x^1,\ldots,x^n,y^1,\ldots,y^n)$ such that

$$B(\partial/\partial x^i) = (\partial/\partial y^i) \qquad 1\leq i\leq n$$

$$B(\partial/\partial y^j) = 0 \qquad 1\leq j\leq n$$

Now, $(\partial/\partial u^i) = \sum_{k=1}^{n}(\partial x^k/\partial u^i)(\partial/\partial x^k) + (\partial y^k/\partial u^i)(\partial/\partial y^k)$

and $(\partial/\partial v^j) = \sum_{k=1}^{n}(\partial x^k/\partial v^j)(\partial/\partial x^k) + (\partial y^k/\partial v^j)(\partial/\partial y^k)$

Thus, $B(\partial/\partial u^i) = \sum_{k=1}^{n}(\partial x^k/\partial u^i)B(\partial/\partial x^k)$

$$= \sum_{k=1}^{n}(\partial x^k/\partial u^i)(\partial/\partial y^k)$$

$$= \sum_{j=1}^{n} c^j_i(u,v)(\partial/\partial v^j)$$

$$\sum_{j=1}^{n} c^j_i \sum_{k=1}^{n}(\partial x^k/\partial v^j)(\partial/\partial x^k) + (\partial y^k/\partial v^j)(\partial/\partial y^k)$$

and $B(\partial/\partial v^j) = \sum_{k=1}^{n}(\partial x^k/\partial v^j)B(\partial/\partial x^k)$

$$= \sum_{k=1}^{n}(\partial x^k/\partial v^j)\quad(\partial/\partial y^k)$$

$= 0$ need to be solved

That is, we need to solve:

$$(\partial x^k/\partial v^i) = 0 \quad 1 \leq i,k \leq n$$

and $$(\partial x^k/\partial u^i) = \sum_{j=1}^{n}(\partial y^k/\partial v^j)c^j_i$$

Thus $x=x(u)$ from the first equation. So try $x=u$ and the second equation becomes:

$$\delta^k_i = \sum_{j=1}^{n} c^j_i\,(\partial y^k/\partial v^j)$$

which may be written $(\partial y^k/\partial v^i) = f^k{}_i(u,v)$, det $f \neq 0$.

$(u,v) \longmapsto (x,y)$ will be 1-1,

since $\det\begin{pmatrix}(\partial x/\partial u) & (\partial x/\partial v)\\ (\partial y/\partial u) & (\partial y/\partial v)\end{pmatrix} = \det\begin{pmatrix} I & 0 \\ 0 & f(u,v)\end{pmatrix} \neq 0$

But this equation clearly has solutions, hence there is the required coordinate change.

Examples: (a) S^{2n} does not admit an almost B_1 supermanifold structure for $n>0$. Let $\chi(S^{2n})$ be the Euler characteristic of S^{2n} and $e(q)$ be the Euler class of q, where we suppose that $TS^{2n} = q \oplus q$, then,

$$0 \neq \chi(S^{2n}) = [e(q) \vee e(q)](S^{2n}) = 0$$

since $e(q)=0$. Thus the hypothesis that $TS^{2n}=q \oplus q$ must be false.

(b) $\mathbf{RP}^{2n}$ does not admit the structure of an almost B_1-supermanifold for if it did, we could lift it locally to the universal cover which is S^{2n}, which we have shown to be impossible.

Section 3: Embeddings of supermanifolds:

It is interesting to note another similarity between G^∞ supermanifolds over B_1 and complex manifolds, namely that neither of them admit embeddings into their model space when they are compact. In the case of compact complex manifolds a maximum principle is proved which bars the existence of global holomorphic funtions on the manifold, which if embedded in $\mathbf{C}^n$ would have to exist. For example, the restriction of the coordinate functions of $\mathbf{C}^n$ to the manifold would be global holomorphic functions.

The idea of the proof for supermanifolds is roughly the same, a few of the details are different though. Let us state and prove the result formally.

Definition: Let M be a restricted G^∞ supermanifold over B_1 and let $i:M \longrightarrow (B_1)^n$ be a continuous map. We say that i is a **G^∞ embedding** if and only if i is both a C^∞ embedding in the usual sense and also i is a G^∞ map.

Proposition 8: Let M be a compact G^∞ supermanifold over B_1, then M does not admit a G^∞ embedding into $(B_1)^n$ for any n.

proof: We prove this by a series of lemmas:

Lemma 1: Let $F:(B_1)^n \longrightarrow B_1$ be G^∞. Then F may be written:

$$F(x,y) = (f(x), \nabla f(x).y + g(x))$$

where x and y are the standard coordinates on $(B_1)^n$

introduced in the previous section.

proof of lemma 1: This is proved in exactly the same way as proposition 3.

Lemma 2: Let M be a compact G^{∞} supermanifold in which all the leaves of the real foliation are compact and let F be a global G^{∞} function on M, then Re F is constant. (We have written

$$F = \text{real part } F + \text{nilpotent part } F$$
$$= \quad \text{Re } F \quad + \quad \text{Nil } F \quad).$$

proof of lemma 2: Choose local standard coordinates (x,y) on M. Locally F may be written as

$$F(x,y) = (f(x), \nabla f(x).y + g(x)) \text{ by lemma 1.}$$

Choose x_0 such that $\nabla f(x_0) \neq 0$ (if this is not possible, then we are done), let M_{x_0} be the leaf of the real foliation through x_0. M_{x_0} is compact and Nil F is continuous thus Nil F attains a maximum on M_{x_0}. By re-choosing coordinates if necessary we see that this is not possible since Nil F takes the form

$$v.y + u$$

where u,v are constant vectors with $v \neq 0$ and y the coordinate on M_{x_0}. Thus the hypothesis that $\nabla f \neq 0$ must be false, hence f = Re F must be constant.

Lemma 3: A G^{∞} supermanifold over B_1 admits no embedding into the nilpotent part of $(B_1)^n$. (That is, the part of $(B_1)^n$ spanned by the y coordinate).

proof of lemma 3: It suffices to prove this for $M = (B_1)^m$. Suppose that

$$i:(B_1)^m \longrightarrow \text{Nil } (B_1)^n$$

is a G^∞ embedding. Then by composing i with projection onto factors, we get G^∞ functions

$$F:(B_1)^n \longrightarrow Nil\ (B_1)$$

and these must take the form

$$F(x,y) = (0\ ,\ g(x))$$

by lemma 1. Hence i cannot be an embedding.

proof of proposition 8: Suppose M is embedded in $(B_1)^n$, for some n. The coordinate functions on $(B_1)^n$ restrict to give globally defined G^∞ functions, F, such that Re F is non constant. In the case that all the leaves of the real foliation of M are compact, this contradicts the result of lemma 2. So we are left with the case in which the real foliation of M admits a non compact leaf. Recall that since i is a G^∞ map, it must preserve real parts, so the image of a non compact leaf under i is a non compact set parallel to the nilpotent axes. Either the image is not bounded in $(B_1)^n$, which immediately contradicts the compactness of i(M), or the image of this leaf does not contain all its accumulation points. By the compactness of i(M), this implies that the image of some other leaf contains such accumulation points. This contradicts the fact that i is an embedding, and we are finished.

It is straightforward to see that this result immediately generalizes to restricted supermanifolds over B_L for L>1, with the same proof. Let us state this as :

Proposition 9: Compact restricted supermanifolds do not admit embeddings into their model spaces .

Section 4: Supervarieties:

In this section we shall be concerned with the structure of supermanifolds defined as being the set of points on which a collection of superpolynomials vanishes. Before we define what we mean by a superpolynomial we shall look at a general G^∞ function.

Let $F:(B_1)^n \longrightarrow B_1$ be a G^∞ function and let M be the set of points:

$$M = \{(x,y) \in (B_1)^n : F(x,y) = c+db_1\}$$

Proposition 10: M may be smoothly identified with the tangent bundle of the core manifold, in the case that this exists.

proof: We may write

$$F(x,y) = f(x) + (\nabla f(x).y + g(x))b_1$$

thus $(x,y) \in M \Rightarrow F(x,y) = c+db_1$

$$\Rightarrow \quad f(x) = c$$

and $\nabla f(x).y+g(x) = d$

the equation $f(x) = c$ defines the core manifold and the equation $\nabla f(x).y+g(x) = d$ has the solution

$$y = \frac{[d-g(x)]\nabla f(x)}{\|\nabla f(x)\|^2} + \tilde{y}$$

where $\tilde{y} \perp \nabla f(x)$. So provided that $\nabla f(x) \neq 0$ (that is, the core manifold is defined), we have

$M = \{(x,y): F(x,y) = c+db_1\}$ is smoothly parameterized by the set $\{(x,\tilde{y}): f(x) = c, \tilde{y} \perp \nabla f(x)\}$, which is the tangent bundle of the core manifold.

Before we consider supervarieties, let us investigate the structure of the intersection of two sets :

$\{(x,y):F(x,y) = c+db_1\}$ and $\{(x,y):G(x,y) = k+lb_1\}$ where F and G are G^∞ functions. We might hope that the intersection of these two sets defines the tangent bundle of the intersection of the core manifolds, but this is not quite true as the identification we have made can upset the behaviour of the intersection.

Example: Let (x^i+y^i) for i=0..2 define coordinates on $(B_1)^3$, and let $x = (x^0,x^1,x^2)$ and $y = (y^0,y^1,y^2)$

define $F(x,y) = x^0 + (y^0+m(x))b_1$

$G(x,y) = x^0 - (x^1)^2 + (y^0-2x^1y^1+n(x))b_1$

where m and n are arbitrary C^∞ functions $\mathbb{R}^3 \rightarrow \mathbb{R}$. Then it is easily checked that F and G are G^∞.

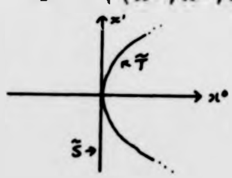
Set $S = \{(x,y):F(x,y) = 0\}$

$T = \{(x,y):G(x,y) = 0\}$

then the core manifolds of these two are

$\tilde{S} = \{(x^0,x^1,x^2): x^0 = 0\}$

$\tilde{T} = \{(x^0,x^1,x^2): x^0 - (x^1)^2\}$



and $\tilde{S}\cap\tilde{T}$ is the x^2 axis

but, $(x,y)\in S \Rightarrow x^0 = 0$ and $y^0+m(x) = 0$ and

$(x,y)\in T \Rightarrow x^0-(x^1)^2 = 0$ and $y^0-2x^1y^1+n(x) = 0$

thus $(x,y)\in S\cap T$ $\quad x^0 = 0 = x^1$ and

$y^0+m(x) = 0$ and

$y^0+n(x) = 0$

so that if you choose $m(x) \neq n(x)$ $\forall x$, as you are free to do, the intersection is empty. Thus the conjecture that the intersection may be identified with the tangent bundle of the intersection of the core manifolds is false. Having established this fact, it is relatively straightforward to establish conditions on F and G that force the conjecture to be true when restricted to functions satisfying such conditions.

What is happening is as follows: In the proof of proposition 10 the equations

$$f(x) = c$$

$$\nabla f(x).y+g(x) = d$$

have the solution in y space

$$y = \frac{[d-g(x)]\nabla f(x)}{\|\nabla f(x)\|^2} + \tilde{y} \qquad \tilde{y} \perp \nabla f(x)$$

That is, the solution space is a hyperplane perpendicular to $\nabla f(x)$. If we now have the equations

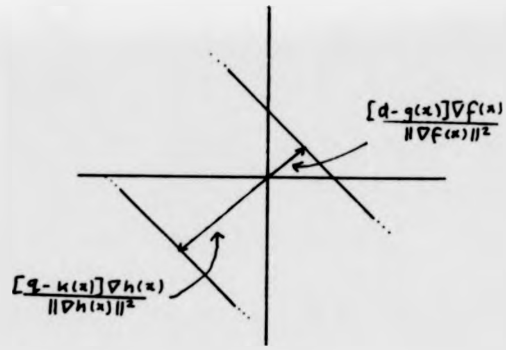
$$h(x) = p$$

$$\nabla h(x).y+k(x) = q$$

arising from the level set of another G^∞ function, then they have the solution in y space

$$y = \frac{[q-k(x)]\nabla h(x)}{\|\nabla h(x)\|^2} + \tilde{y} \qquad \tilde{y} \perp \nabla h(x)$$

Now, if at the point of intersection in x space, we have both $q-k(x) \neq 0$ and $\nabla h(x)$ parallel to $\nabla f(x)$, then the solutions for y may very well not intersect.



However, if $\nabla f(x)$ is not parallel to $\nabla h(x)$ at the points of x intersection then the y planes must intersect. Since translation to the origin is a C^{∞} map we may assume that the intersection is a linear subspace and we do get the result that the intersection is the tangent bundle of the intersection of the core manifolds. Similarly, if we assume that in our G^{∞} function

$$f(x) + (\nabla f(x).y + g(x))b_1 = c + db_1$$

we have $g(x) = 0$ for all x and that $d = 0$ then again, there is no problem, as no translation away from the origin occurs in y space. This leads to the following:

Definition: A **Superpolynomial** over B_1 is a function that is a polynomial in the algebra variables having real valued coefficients.

Thus a superpolynomial satisfies the conditions $g(x) = 0 = d$ derived above. We may summarize our analysis as follows:

Proposition 11: Let F and G be G^{∞} functions and let

$$S = \{(x,y): F(x,y) = 0\}$$

$$T = \{(x,y): G(x,y) = 0\}$$

then S is diffeomorphic to $T\tilde{S}$ and T is diffeomorphic to $T\tilde{T}$. If either

(i) F and G are superpolynomials, or

(ii) $\tilde{S}$ and $\tilde{T}$ meet transversely,

then $S \cap T$ is diffeomorphic to $T(\tilde{S} \cap \tilde{T})$.

Definition: A **Supervariety** is the set of points defined by the vanishing of a set of superpolynomials.

Corollary: Any supervariety is diffeomorphic to the tangent bundle of its core manifold.

Remark: Notice from the above analysis, that Proposition 11 remains true if we drop conditions (i) and (ii) and replace them by the requirement that the intersection of S and T be a manifold.

Section 5: The Structure Theorem:

We may understand the general structure of restricted supermanifold better by pushing the result of proposition 3 to its limits.

Proposition 12: Let $F:(B_1)^n \longrightarrow (B_1)^m$ be a G^∞ map, then F may be written

$$F(x,y) = (f(x),Df(x)(y)+g(x))$$

in our standard coordinates, where f and g are C^∞ maps $\mathbf{R}^n \longrightarrow \mathbf{R}^m$.

proof: Mimic the proof of proposition 3.

If we knew that the B_1 supermanifold M had the structure of a vector bundle over its core manifold then we would recognize these transition functions as being transition functions belonging to the affine tangent bundle of the core manifold. In general, though, the only thing that is clear is that the fibre of the map $M \longrightarrow \tilde{M}$ admits the stucture of an affine manifold, by which we mean that the fibre is a manifold admitting an atlas whose transition functions are affine maps with respect to the coordinates. To go any further we must start to impose some conditions on the supermanifold M.

Definition: The supermanifold M is said to be **Regular** if and only if it has the structure of a fibre bundle over its core manifold.

Definition: The supermanifold M is said to be **Fibre Complete** if the flat affine connection induced in each [illegible] fibre by the affine atlas on that fibre is complete. (Geodesics defined for all the values of an affine parameter).

Proposition 13: Suppose that the flat affine connection induced by an affine atlas on a manifold N is complete, then the universal covering space of N is $\mathbf{R}^n$ where n is the dimension of N.

proof: See Kobayashi-Nomizu [25] and Auslander and Markus [1].

So it is clear that each fibre of a fibre complete supermanifold admits a Euclidean space as a covering manifold. What we want now is a criterion to ensure that we can fit these Euclidean spaces together to form a vector bundle.

Proposition 14: Let M be a regular, fibre complete restricted supermanifold over B_L. Then M admits $T\tilde{\tilde{M}}$ as a covering space if and only if the bundle $b:M \longrightarrow \tilde{\tilde{M}}$ admits a section.

proof: Necessity is clear, for if $p:T\tilde{\tilde{M}} \longrightarrow M$ is a covering then p composed with the zero section $0:\tilde{\tilde{M}} \longrightarrow T\tilde{\tilde{M}}$ is a section of $M \longrightarrow \tilde{\tilde{M}}$.
As for sufficiency, if $M \longrightarrow \tilde{\tilde{M}}$ admits a section, then we may embed $\tilde{\tilde{M}}$ in M via this section. Then we may restrict the vector bundle $b^{-1}T\tilde{\tilde{M}}$ to $\tilde{\tilde{M}}$ (considered as a subset of M) to get a local covering of M that is a vector bundle. Since being a covering is a local matter we may conclude that M admits a vector bundle as a covering manifold.

Hence M admits $T\tilde{M}$ as a covering manifold since locally the transition functions of M may be pulled up to give transition functions on the covering space. The form of these functions clearly shows the vector bundle to be isomorphic to $T\tilde{M}$.

To avoid cumbersome reference we shall refer to fibre complete, regular supermanifolds that do admit such a section as **Vectorial Supermanifolds**. Proposition 14 essentially classifies vectorial supermanifolds up to covering. We shall deal with the question of the existence of non-vectorial supermanifolds in chapter 5, after we have looked at all the types of supermanifold we are to be interested in.

Let us return to the case of restricted supermanifolds over B_L for $L>1$.

Firstly we shall define a restricted class of G^∞ functions.

Definition: Let $F:(B_L)\longrightarrow(B_L)$ be a G^∞ function. If F may be written as

$$F(x) = f(x^0) + (df/dx^0)s(x)$$

where $f:\mathbf{R}\longrightarrow\mathbf{R}$ is an admissible, (thus affine) function, then F is said to be an $\mathbf{H}^\infty$ function. A similar definition is made for H^∞ functions $(B_L)^n\longrightarrow(B_L)^m$.

This definition mimics the definition of H^∞ in Rogers [34]. In later chapters we shall come across an expansion of certain G^∞ functions, (in the sense of Rogers), called the z-expansion. The above definition is the equivalent to this in the case of restricted G^∞ functions.

An **H^∞ Supermanifold** is a G^∞ supermanifold admitting an atlas of charts with H^∞ transition functions. The purpose of this definition is the following

Proposition 15: Let M be a regular H^∞ supermanifold over B_L for L>1, then the core manifold $\tilde{M}$ admits the structure of an affine manifold. If M is vectorial then M admits $T\tilde{M} \otimes (\tilde{B}_L)$ as a covering manifold, where $(\tilde{B}_L) = B_L - \mathbf{R}$.

proof: Simply follow the proof of proposition 13. The assertion about the core manifold follows from the linearity of the admissible functions defining the transition functions.

Note: We freely adopt the terminology developed for restricted supermanifolds over B_1 for all the other types of supermanifolds we are considering, thus we have vectorial supermanifolds over B_L, etc..

So we can conclude that vectorial supermanifolds look, in a natural way, like tangent bundles and indeed it is clear that any tangent bundle admits an H^∞ structure. We shall demonstrate this to be the case for G^∞ supermanifolds over B_L, for L>1 as well. The result will be slightly weaker in this case in that the identification is no longer natural.

Firstly, we must recall some facts from the theory of vector bundles.

Suppose that $0 \longrightarrow E \xrightarrow{i} G \xrightarrow{p} F \longrightarrow 0$ is an exact sequence of vector bundles over a C^∞ manifold M.

Let $\begin{pmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ e_n \end{pmatrix}$ be a frame for E, $\begin{pmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ e_n \\ g_1 \\ \cdot \\ \cdot \\ g_m \end{pmatrix}$ a frame for G using i to identify E as a subbundle of G.

Then $\begin{pmatrix} pg_1 \\ \cdot \\ \cdot \\ pg_m \end{pmatrix} = \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ f_m \end{pmatrix}$ is a frame for F.

We shall abbreviate these frames as (e), $\begin{pmatrix} e \\ g \end{pmatrix}$ and (f)

Now, suppose that we change the frame of G, so

$$\begin{pmatrix} e' \\ g' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} e \\ g \end{pmatrix}$$ where A is nxn and B is mxn etc..

thus $e' = Ae + Bg$

$g' = Ce + Dg$

now $pe'=0$ thus $A(pe) + B(pg) = 0 = B(pg) = B(f)$,

thus B=0 (since the original frame was arbitrary).

We also have $pg' = f' = C(pe) + D(pg) = Df$, thus D is a transition matrix for F and A is a transition matrix for E and B=0. So the transition matrix for G looks like

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$$

Recall, however, that any exact sequence of vector bundles over a paracompact C^∞ manifold admits a splitting, that is, a map $s:F \longrightarrow G$ such that ps=identity.

$$0 \longrightarrow E \longrightarrow G \underset{s}{\overset{r}{\rightleftarrows}} F \longrightarrow 0$$

and via this splitting we have the fact that

$i \oplus s : E \oplus F \longrightarrow G$ is an isomorphism of vector bundles, so that G admits a matrix of transition functions of the form $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$

So, we conclude, over a paracompact C^∞ manifold, any vector bundle whose transition function matrix has lower triangular block form is isomorphic to the vector bundle whose transition functions are the diagonal blocks of the original vector bundle. We shall call this the **Principle of splitting** for vector bundles.

The application of this result to our situation is now fairly clear. Any G^∞ function

$F:(B_L)^n \longrightarrow (B_L)^n$ may be written as

$$F(x) = f(e(x)) + Df(e(x))(s(x))$$

where $f:\mathbf{R}^n \longrightarrow (B_L)^n$ and Df acts on s(x) in the obvious matrix fashion. We may conclude that any matrix of vector bundle transition functions arising from the term

$$Df(e(x))(s(x))$$

is lower triangular when written in matrix form with respect to the basis of $\mathbf{R}^{n2^L}$ arising from the vector space isomorphism

$$(B_L)^n \longrightarrow \mathbf{R}^{n2^L}$$

given by firstly ordering the elements of $(B_L)^n$ according to the cartesian product and then, within this ordering, according to the order

$1, b_1, \ldots, b_L, b_1b_2, \ldots b_1b_L, b_2b_3, \ldots, b_1b_2b_3, \ldots, b_1b_2\ldots b_L$.

of basis elements of B_L considered as a real vector space.

Example: n=1, L=2.

Let $f(x^0) = f^0(x^0)+f^1(x^0)b_1+f^2(x^0)b_2+f^{12}(x^0)b_1b_2$ with f^0 linear.

Then, $F(x) = f(x^0) + (df/dx^0)(s(x))$

$= (df^0/dx^0)(x^1b_1+x^2b_2+x^{12}b_1b_2)+(df^1/dx^0)x^2b_1b_2 - (df^2/dx^0)x^1b_1b_2 + f(x^0)$

thus the associated transition matrix is

$$\begin{pmatrix} (df^0/dx^0) & 0 & 0 \\ 0 & (df^0/dx^0) & 0 \\ (df^1/dx^0) & -(df^2/dx^0) & (df^0/dx^0) \end{pmatrix}$$

with the basis ordered as (x^1,x^2,x^{12}).

We may conclude, then:

Proposition 16: Let M be a vectorial restricted supermanifold over B_L for L>1, then M as a C^∞ manifold admits $(T\tilde{M})^{2^L-1}$ as a covering manifold. (We shall abbreviate this as $T\tilde{M} \otimes (\tilde{B}_L)$ as long as we realize that the identification is not natural). Also M admits the structure of an affine manifold.

Remark: Were we to be working in the holomorphic category, this result would fail, as the exact sequence need not split. In our example above, the off diagonal entries could be anything, so all we could conclude then would be that our covering space was a holomorphic extension of $TM \oplus TM$ by TM. Compare this with the work of Green [18] on holomorphic graded manifolds.

In the case of H^{∞} supermanifolds over B_L, we can exhibit a curious set of criteria for the bundle $M \longrightarrow \tilde{M}$ to admit a section.

Definition: A superchart (U,f) is said to be **Centered** at $x \in U$, if $f(x) = 0 \in (B_L)^n$.

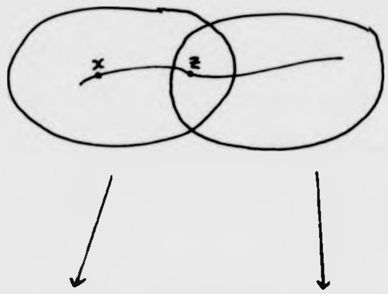
Proposition 17: Let M be a simply connected H^{∞} supermanifold over B_L. If there is a point $x \in M$ that admits a chart centered at x, then $M \xrightarrow{p} \tilde{M}$ admits a section.

proof: Firstly, note that the fibre of the map $M \longrightarrow \tilde{M}$ is connected, by definition. Inspection of the homotopy exact sequence of this fibration then forces $\tilde{M}$ to be simply connected.

Suppose that the coordinate chart (U,f) is centered at x. Then the induced coordinate chart $(\tilde{U},\tilde{f})$ on $\tilde{M}$ is centered (in the usual sense) at p(x). We can now exhibit a local section of $M \longrightarrow \tilde{M}$ around p(x) by using the formula

$$p(y) \longmapsto f^{-1}(fp(y),0) \qquad y \in U$$

Where we have embedded $\mathbf{R}^n \hookrightarrow (B_1)^n$ in the usual fashion This is clearly well defined, p(x) being lifted to x. What we want to do now is to show that we can extend this local lifting to a global lifting. Notice that if z is in the image of this local section then there is a superchart (V,g) centered at z.



$$f(U \cap V) \longrightarrow g(U \cap V)$$

$$gf^{-1}(y,v) = (F(y), DF(y)(v))$$

This follows by simply subtracting off a real constant. It is clear that the local lifting around x agrees with the local lifting around z by because of the form of the transition function between (U,f) and (V,g). Extend this local lifting as far as it will go. Then it is clear that for every point $\tilde{t} \in \tilde{M}$ there is a point t in the image of the lifting such that $p(t) = \tilde{t}$, for, join $\tilde{t}$ to p(x) with a chain of coordinate neigbourhoods induced from supercharts. These supercharts then provide the required lift. It is also clear the maximal connected set produced by this lifting process forms a covering space for $\tilde{M}$, by its very construction. $\tilde{M}$ is simply connected, thus this covering space is, in fact, a copy of $\tilde{M}$ embedded in M and there is a unique point in the lift which projects under p to any given point of $\tilde{M}$. Thus the lifting process has furnished us with a section of $M \longrightarrow \tilde{M}$.

Unfortunately, it is not clear that every H^{∞} supermanifold admits a point that has a coordinate chart centered at it You are simply not free to subtract off non-zero nilpotent parts in an arbitrary fashion, since the nilpotent part is "tied" to the coordinate charts on the core manifold. All we can say is that at any point x on the supermanifold there is a superchart (U,f) with $e(f(x)) = 0$.

Chapter 2: Even Supermanifolds.

Section 1: Definitions and Elementary Results:

In chapter 1 we were concerned with investigating the structure of supermanifolds modelled on open subsets of $(B_L)^n$. In this chapter we shall proceed to the next level of sophistication, namely supermanifolds modelled on open subsets of $(B_L)_0^n$, that is, modelled on the even part of the exterior algebra. We notice that $(B_L)_0$ is a commutative algebra, so we would expect a richer theory to emerge, since any part of algebraic geometry applying to general commutative rings will apply here. To make an analogy with chapter 1, B_1 was a commutative algebra and any C^∞ manifold could appear as the core manifold of a restricted supermanifold over B_1, whereas to be the core manifold of a restricted supermanifold over B_L, $L>1$, it was necessary and sufficient for the C^∞ manifold to admit an affine atlas. The essential generalization of this chapter over the case of B_1 restricted supermanifolds is that there are nilpotent elements of nilpotent degree greater than unity in $(B_L)_0$ for $L>3$ and these become relevant in the general expression for a G^∞ function over $(B_L)_0$. (They were irrelevant over B_L, $L>1$, because of the "linearity" of the G^∞ functions in that case).

As we have remarked in the introduction, we shall be using the results and techniques of chapter 1 throughout the rest of this work. Therefore some of the more elementary propositions that appear in this chapter will be the exact analogues of propositions appearing in chapter 1 and for their proofs we shall refer back to the corresponding proof in that chapter. Conversely, some important definitions applying to restricted supermanifolds apply equally to the material in this chapter. We shall usually repeat these definitions to avoid cumbersome back reference.

Let us start then by the definition of the smooth maps that we are to be concerned with. Let

$$f:(B_L)^n_0 \longrightarrow B_L$$

be a smooth map.

Definition: f is said to be **Even G^1 at x** if and only if

$$f(x+h) = f(x) + \sum_{i=1}^{n} G_i f(x).h_i + n(h)\|h\|$$

where $h = (h_1,\ldots,h_n)$, $G_i f(x) \in B_L$, $n:(B_L)^n_0 \to B_L$ and $\|n(h)\| \longrightarrow 0$ as $\|h\| \to 0$.

If f is even G^1 for all x in some open set U then f is said to be **even G^1 in U**. If U is the whole of $(B_L)^n_0$ then f is simply referred to as being **even G^1**. If f is even G^1 in U, then the $G_i f:U \longrightarrow B_L$ are referred to as the **Partial G-derivatives** of f. In the case n=1 we simply refer to **The partial G-derivative**, Gf.

Notice that we do not restrict ourselves to maps $(B_L)^n_0 \longrightarrow (B_L)_0$. We could do this if we wished, but there is little advantage in making such a restriction at this stage.

Notice that this definition is essentially that given in Rogers [34] in the case n=0. We shall deal with her definition in chapter 4.

Remark: We shall identify $(B_L)^n_o$ with $\mathbf{R}^{n2^{L-1}}$ in the same way that we identified $(B_L)^n$ with $\mathbf{R}^{n2^L}$ in chapter 1, namely by cartesian product first and by ordered degree second.

We shall frequently drop the use of the word "even" from our definition of G^1 as long as the context avoids any confusion.

As we did in chapter 1, we shall restrict our attention for a while to functions $f:(B_L)_0 \longrightarrow B_L$ to gain some insight into the structure of G^1 functions without any significant loss of generality.

We notice that, as with the case of restricted G^1 functions, such a smooth map has its derivative given by

$$Df(x)(h) = Gf(x).h = h.Gf(x)$$

if it is G^1 and as before, if (B_L) is made into a $(B_L)_0$-module, then f is G^1 if and only if Df is linear over $(B_L)_0$. (Notice that $(B_L)_0$ is the centre of B_L). Thus we have proved:

Proposition 1: $f:(B_L)_0 \longrightarrow B_L$ is G^1 at x $\Leftrightarrow$ Df(x) is linear over $(B_L)_0$. In coordinates the second condition is equivalent to the system of partial differential equations:

$$(\partial f/\partial x^m) = (\partial f/\partial x^0)b_m \qquad |m| \text{ even.}$$

As before we see that

$$Gf(x) = (\partial f/\partial x^0)|_x$$

Proposition 2: f is $G^1 \Rightarrow$ Gf is G^1

proof: Exactly the same as for proposition 2 in chapter 1.

So from now on we shall freely use the term G^∞ when referring to G^1 functions.

Examples: (a) The inclusion function

$i:(B_L)_0 \hookrightarrow B_L$

and any constant function are G^1.

(b) $e:(B_L)_0 \longrightarrow \mathbf{R}$ is not G^1. Neither is s. (The functions e and s are defined in obvious analogy to the functions of chapter 1).

(c) Let $f:(B_L)_0 \longrightarrow B_L$ be $f(x) = x^2$, then in marked contrast to the restricted G^∞ case, f is G^∞. This is because of the commutativity of $(B_L)_0$.

$$\begin{aligned} f(x+h) = (x+h)^2 = (x+h)(x+h) &= x^2+xh+hx+h^2 \\ &= x^2+2xh+h^2. \\ &= f(x)+2xh+h^2. \end{aligned}$$

similarly, any power of x is G^∞ wherever it is continuous.

Proposition 3: Let $f:(B_L)_0 \longrightarrow B_L$ be a G^∞ diffeomorphism, then

$e(x) = e(y)$ if and only if $e(f(x)) = e(f(y))$.

proof: same as for the proof of proposition 4 chapter 1.

Remark: We notice that as algebras, $(B_2)_0 \cong B_1$, and that the definitions of G^1 coincide. So we may already conclude that any construction performed over $(B_2)_0$ is the same as the corresponding construction performed over B_1.

Following the pattern of chapter 1, we wish to exhibit in a tractable way the form of the general G^1 function over $(B_L)_0$. Fortunately, in this case, there is a unified method of dealing with all positive L at once. The construction that enables us to do this is the z-expansion of Rogers [34]. This is a tool that produces G^∞ functions from C^∞ functions. So then, let

$$f:\mathbf{R} \longrightarrow B_L$$

be a C^∞ function.

Definition: We define a function

$$z(f):(B_L)_0 \longrightarrow B_L \qquad \text{as follows.}$$

$$\begin{aligned} z(f)(x) = f(x^0) &+ (df/dx^0)s(x) \\ &+ (1/2!)(d^2f/d(x^0)^2)(s(x))^2 \\ &+ (1/3!)(d^3f/d(x^0)^3)(s(x))^3 \\ &+ \ldots \\ = f(x^0) &+ (df/dx^0)(x-x^0) \\ &+ (d^2f/d(x^0)^2)(x-x^0)^2 \\ &+ \ldots \end{aligned}$$

(The series is finite, since the term $(x-x^0)$ is a nilpotent element of $(B_L)_0$).
The function z(f) is called the **z-Extension** of f.

We can immediately apply proposition 1 to z(f) as follows:

$$(\partial/\partial x^m)(z(f)) = 0 + (df/dx^0)b_m$$
$$+ (d^2f/d(x^0)^2)(x-x^0)b_m$$
$$+ (1/2!)(d^3f/d(x^0)^3)(x-x^0)^2b_m$$
$$+ \dots$$
$$= (\partial/\partial x^0)(z(f))b_m$$

and hence we can conclude:

Proposition 4: The z-extension of a C^∞ function is a G^∞ function and we have

$$G(z(f)) = z(\partial f/\partial x^0)$$

Examples: (a) Let $i:\mathbf{R} \longrightarrow B_L$ be the inclusion map.

Then $\quad z(i)(x) = x$

so $\quad z(i) = \text{identity}.$

(b) Let $f:\mathbf{R} \longrightarrow \mathbf{R} \subset B_2$ be $f(x) = x^2$, then

$$z(f)(x) = (x^0)^2 + 2x^0(x^1b_1+x^2b_2+x^{12}b_{12})$$
$$= (x^0+x^1b_1+x^2b_2+x^{12}b_{12})^2$$
$$= x^2$$

If we have a G^∞ function $F:(B_L)_0 \longrightarrow B_L$ then we may construct a C^∞ function $f:\mathbf{R} \longrightarrow B_L$ by setting $f = Fi$ where i is the inclusion $\mathbf{R} \hookrightarrow (B_L)_0$. (Thus f is F restricted to the real part of $(B_L)_0$). What, we may ask, is the connection between f, $z(f)$ and F? The answer is:

Proposition 5: Let $F:(B_L)_0 \longrightarrow B_L$ be a G^∞ function, then there exists a unique C^∞ function $f:\mathbf{R} \longrightarrow B_L$ such that $\qquad F(x) = z(f)(x).$

Moreover, $f = Fi$, thus there is a 1-1 correspondence between the sets $G^\infty((B_L)_0, B_L)$ and $C^\infty(\mathbf{R}, B_L)$.

proof: The quickest proof of this proposition is to apply Taylors theorem to the expression $F(x^0+(x-x^0))$, however there is another interesting proof based on using the $\mathbb{Z}$-grading of B_L and proposition 1:

F is G^∞ if and only if the system

$$(\partial F/\partial x^m) = (\partial F/\partial x^0) b_m \qquad (*)$$

is satisfied, from proposition 1. If we now write

$$F = F^0 + \sum_{|m|>0} F^m b_m$$

and substitute this into (*), we get

$$\begin{aligned}(\partial F/\partial x^n) &= (\partial F^0/\partial x^n) + \sum_{|m|>0} (\partial F^m/\partial x^n) b_m \\ &= (\partial F/\partial x^0) b_n \\ &= [(\partial F^0/\partial x^0) + \sum_{|m|>0} (\partial F^m/\partial x^0) b_m] b_n.\end{aligned}$$

If we now equate terms of $\mathbb{Z}$ degree zero, we must conclude that F^0 is a function of x^0 alone. That is,

$$F^0(x) = f^0(x^0)$$

for some $f^0 : \mathbb{R} \longrightarrow (B_L)^0 = \mathbb{R}$.

(Let us recall that an upper index refers to the $\mathbb{Z}$-grading).

Now let us consider $z(f^0)$. We have shown in proposition 4 that $z(f^0)$ is a G^∞ function, thus it must satisfy the system (*). Therefore so must $F-z(f^0)$, because these equations are linear.

Set $\qquad g = F - z(f^0)$

then g is a G^∞ function with no terms of $\mathbb{Z}$-degree zero, so if we write,

$$g = \sum_{i=1}^{L} g^i b_i + \sum_{|m|>1} g^m b_m \qquad \text{and substitute into } (*),$$

we get:

$$\begin{aligned}(\partial g/\partial x^n) &= \sum_{i=1}^{L} (\partial g^i/\partial x^n) b_i + \sum_{|m|>1} (\partial g^m/\partial x^n) b_m \\ &= (\partial g/\partial x^0) b_n \\ &= [\sum_{i=1}^{L} (\partial g^i/\partial x^0) b_i + \sum_{|m|>1} (\partial g^m/\partial x^0) b_m] b_n.\end{aligned}$$

By equating terms of **Z**-degree 1, we conclude that the g^i are functions of x^0 alone. That is,

$$[F - z(f^0)]^1 = f^1(x^0)$$

for some $f^1 : \mathbf{R} \longrightarrow (B_L)^1$.

We proceed thus, considering in turn the functions

$$F - z(f^0)$$

$$F - z(f^0 + f^1)$$

$$F - z(f^0 + f^1 + f^2)$$

$$\cdots$$

until we get to $F - z(f^0 + \ldots + f^L) = 0$ as there are no terms left. Thus $F = z(f^0 + \ldots + f^L)$ and it is easily checked that $f = f^0 + \ldots + f^L$ is Fi.

(I would like to thank Dr. John Rawnsley for suggesting the above method of proof. For the original proof see Rogers [34]).

Examples: (a) Let $f : \mathbf{R} \longrightarrow \mathbf{R}\ \ B_L$ be $f(x) = x^n$. Then

$$z(f)(b) = b^n$$

because the function $F(b) = b^n$ is G^∞ and $Fi(x) = x^n$, hence the result follows by uniqueness.

(b) Let $f : \mathbf{R} \longrightarrow \mathbf{R} \subset B_L$ be any polynomial with real coefficients. Then $z(f)$ is formally the same polynomial, but with $(B_L)_0$ valued variables.

(c) If f is a power series, then $z(f)$ is the same power series with the real variables replaced by $(B_L)_0$ valued variables. Provided that $z(f)$ converges, and f is a known function, $z(f)$ is a much more convenient expression to handle than the original series. For example, if we define

$$\exp(b) = 1 + b + b^2/2! + b^3/3! + \ldots$$

then we know that this is also

$$\exp(b) = \exp(b^0)[1 + (b-b^0) + (b-b^0)^2/2! + \ldots]$$

The latter is, of course, a finite sum whereas the former was infinite.

Definition: The expression $F(x) = z(f)(x)$ is called the **z-Expansion of F**.

Remarks: (i) The existence of the z-expansion of a G^∞ function allows us to prove several uniqueness of continuation proofs. For example, if U is a connected open set of $(B_L)_0$ and $f: U \longrightarrow B_L$ is G^∞, then f can be uniquely extended over $e^{-1}(e(U))$, since the behaviour of f is determined by its behaviour on e(U). For further details, see [34],[8],[21],[22].

(ii) The results on the z-extension and z-expansion of functions carry over with only notational complication to the case of C^∞ functions $\mathbf{R}^n \longrightarrow B_L$ and G functions $(B_L)_0^n \longrightarrow B_L$. The formula in this case is

$$z(f)(x_1,\ldots,x_n) = f(x_1,\ldots,x_n) +$$
$$\sum_I (1/I!)(\partial^{i_1}/(\partial x_1)^{i_1})\ldots(\partial^{i_n}/(\partial x_n)^{i_n}) f (s(x_1))^{i_1}\ldots(s(x_n))^{i_n}$$

Where $I = (i_1,\ldots,i_n)$ and $I! = i_1!i_2!\ldots i_n!$.

In particular, proposition 5 carries over to give us a 1-1 correspondence between the sets $C^\infty(\mathbf{R}^n, B_L)$ and $G^\infty((B_L)_0^n, B_L)$.

To deal with the z-expansion of functions of many variables we must introduce some more notation. If $f: \mathbf{R}^n \longrightarrow (B_L)^m$ and $F: (B_L)_0^n \longrightarrow (B_L)^m$ and F is the z-extension of f then we write

$F = z(f) = (z(f_1),\ldots,z(f_m))$.

(You can, if you wish, regard this as a definition).

Definition: Let $F:(B_L)^n_o \longrightarrow (B_L)^m_o$ be a G^∞ function. F is said to be an **H^∞ function** if and only if F may be written $F = z(f)$ where $f:\mathbf{R}^n \longrightarrow \mathbf{R}^m$ is a C^∞ function. Thus the set $H^\infty((B_L)^n_o,(B_L)^m_o)$ is in 1-1 correspondence with the set $C^\infty(\mathbf{R}^n,\mathbf{R}^m)$.

(We recall that a simplified notion of H^∞ was introduced in the context of restricted G^∞ functions. Since the analogy of the z-expansion in that case terminated at the second term it is easy to see the consistency).

Definition: An **Even G^∞ supermanifold** is a pair (M,A), where M is a hausdorff, 2nd countable topological space and A is a maximal even G^∞ atlas on M.

(The definition of terms is analogous to those of chapter 1. For completeness we include the definitions here.

A **Superchart** on M is a pair (U,f) where U is an open subset of M and $f:U \longrightarrow f(U) \subseteq (B_L)^n_o$ is a homeomorphism. An **Even G Atlas**, A, on M is a collection of supercharts $\{(U_i,f_i):i\in I\}$ on M such that:

(i) M is the union of the U_i.

(ii) If (U,f) and (V,g) are elements of A then the restriction $gf^{-1}:f(U\cap V) \longrightarrow g(U\cap V)$ is an even G^∞ map.)

A map between even supermanifolds is said to be **Even G^∞** if and only if it is a G^∞ map when expressed in local coordinates.

H^∞ supermanifolds and maps are defined in the obvious fashion.

We recall that in chapter 1 we defined the core manifold of a restricted supermanifold. This construction carries through in the case of even supermanifolds in exactly the same way: xRy if and only if there is a superchart (U,f) such that $x,y \in U$ and $e(f(x)) = e(f(y))$.
("The real part of x is equal to the real part of y"). This is well defined, by proposition 3. Let ~ be the equivalence relation generated by R (that is, there is a chain of points and coordinate patches joining x to y according to the relation R). The equivalence classes of M under ~ form the leaves of a foliation on M called the **Real Foliation of M**. If the quotient of M by this foliation happens to be hausdorff then the supercharts of M descend to define charts on the quotient and the quotient becomes a C^∞ manifold. This manifold is called the **Core manifold** of M and is denoted by $\tilde{M}$.

Examples: (a) $(B_L)^n_o$ is an even H^∞ supermanifold with a single chart, namely the identity.

(b) The 2-dimensional torus admits even supermanifold structures in the same way as it admits restricted supermanifold structures.

(c) Let M be any even supermanifold over $(B_2)_0$ then,

Proposition 6: M is canonically a restricted supermanifold over B_1. Thus if M is vectorial, M admits

$T\tilde{M}$ as a covering manifold.

proof: This simply follows from our earlier observation that $B_1 \cong (B_2)_0$ and that the definitions of differentiability coincide, together with proposition 14 of chapter 1.

(d) The supersphere over $(B_L)_0$.

Define $\quad q:(B_L)^n \longrightarrow B_L \quad$ by

$$q(a_1,a_2,\ldots,a_n) = (a_1)^2 + (a_2)^2 + \ldots + (a_n)^2$$

and let $\quad S = \{v \in (B_L)^n : q(v) = 1\}$.

Then S is called the **Supersphere** of dimension n-1 over $(B_L)_0$. S admits the structure of an H^∞ supermanifold, by using exactly the same form of charts as we used when we defined the supersphere over B_1 in chapter 1. Let us investigate the structure of S. Let < , > be the formal inner product on $(B_L)_0^n$, defined by componentwise multiplication. Thus we may write,

$$\begin{aligned} q(v) &= \langle v,v\rangle \\ &= \langle v^0+v^2+\ldots+v^L, v^0+v^2+\ldots+v^L\rangle \quad \text{L even} \\ &\quad \langle v^0+v^2+\ldots+v^{L-1}, v^0+v^2+\ldots+v^{L-1}\rangle \quad \text{L odd.} \end{aligned}$$

Where we have decomposed v with respect to the $\mathbb{Z}$-grading on $(B_L)_0$ induced from the $\mathbb{Z}$-grading of B_L. Let us assume, without any loss of generality, that L is even, then

$$\begin{aligned} q(v) &= \langle v^0+v^2+\ldots+v^L, v^0+v^2+\ldots+v^L\rangle \\ &= \langle v^0,v^0\rangle + && \text{(degree 0 term)} \\ &\quad 2\langle v^0,v^2\rangle + && \text{(degree 2 term)} \\ &\quad \langle v^2,v^2\rangle + 2\langle v^0,v^4\rangle + && \text{(degree 4 term)} \\ &\quad 2\langle v^0,v^6\rangle + 2\langle v^2,v^4\rangle + && \text{(degree 6 term)} \\ &\quad \ldots\ldots && = 1. \end{aligned}$$

Thus, by equating terms of equal degree:

$$<v^0,v^0> = 1$$

$$<v^0,v^2> = 0$$

$$<v^2,v^2> + 2<v^0,v^4> = 0$$

$$<v^0,v^6> + <v^2,v^4> = 0 \qquad \text{etc...}$$

Now, $<v^0,v^0> = 1 \Rightarrow v^0 \in S^{n-1}$, the usual (n-1)-dimensional sphere.

$<v^0,v^2> = 0 \Rightarrow v^2 \perp v^0$, that is v^2 is tangent to S^{n-1} and any tangent to S^{n-1} could be v^2.

Once v^2 has been fixed, $<v^2,v^2> + 2<v^0,v^4> = 0$

$\Rightarrow v^4 = (-1/2)<v^2,v^2>v^0 + \tilde{v}^4$ with $\tilde{v}^4 \perp v^0$

Once v^2 and v^4 have been fixed, $<v^0,v^6> + <v^2,v^4> = 0$

$\Rightarrow v^6 = -<v^2,v^4>v^0 + \tilde{v}^6$ with $\tilde{v}^6 \perp v^0$.

and so on, since at degree 2m, we have the term $<v^0,v^{2m}>$ expressed purely in terms of the v^{2k}, with k<m. We may conclude from this that,

$v = v^0+v^2+...+v^L$ is determined by

$v^0,v^2,\tilde{v}^4,...,\tilde{v}^L$. Each of the ~ terms is tangent to S^{n-1}, as is v^2, so we conclude that S may be identified with $TS^{n-1} \otimes (\tilde{B}_L)_0$ where $(\tilde{B}_L)_0 = (B_L)_0 - \mathbf{R}$.

(e) Projective superspace over $(B_L)_0$ is defined as in chapter 1 and the analysis applied to the supersphere above extends to it, to conclude that it is diffeomorphic to $T(\mathbf{RP}^{n-1}) \otimes (\tilde{B}_L)_0$.

Section 2: Supervarieties:

Let us return now to one of the themes of chapter 1, to see what happens when we extend our definition of supervarieties to include the algebra $(B_L)_0$.

Definition: Let $f:(B_L)_o^n \longrightarrow B_L$ be a G^∞ function. If f may be expressed as a polynomial in the variables with real coefficients then f is said to be a **Superpolynomial**. An even G^∞ supermanifold M is said to be a **Supervariety** if and only if it can be expressed as the subset of $(B_L)_o^n$ defined by the vanishing of a collection of superpolynomials.
(Recall that we saw the reason for restricting ourselves to polynomials with real coefficients in chapter 1. That analysis remains true in this context).

Proposition 7: Let M be the even supermanifold defined by the vanishing of a single G^∞ function $F:(B_L)_o^n \longrightarrow (B_L)_0$, then M may be smoothly identified with $T\tilde{M} \otimes (\tilde{B}_L)_0$, where $\tilde{M}$ is the core manifold. (Provided this exists).
proof: The idea of the proof mixes the proof of proposition 10 of chapter 1 with the method we used for identifying the supersphere over $(B_L)_0$. Unfortunately, the notation tends to obscure the clarity of the proof, so as an illustration of the methods used here we follow this proposition with an example.

$$M = \{x \in (B_L)^n : F(x) = 0\}.$$

We may write $F(x) = z(f)(x)$ for some C^∞ function

$f:\mathbf{R}^n \longrightarrow \mathbf{R}$, because of proposition 5. That is,

$$0 = z(f)(x_1,x_2,\ldots,x_n)$$
$$= f(x_1,\ldots,x_n) +$$
$$\sum_I (1/I!)(\partial^{i_1}/(\partial x_1)^{i_1})\ldots(\partial^{i_n}/(\partial x_n)^{i_n}) f(s(x_1))^{i_1}\ldots(s(x_n))^{i_n}$$

(Notation: an unbracketted superscript (e.g. $s(x)^1$ refers to the $\mathbf{Z}$-grading, a bracketted supercript (e.g. $(s(x))^i$) refers to a power of the bracketted object).

We may write this as

$$0 = f(x_1,\ldots,x_n) + \nabla f.(s(x)) +$$
$$\sum_{|I|>1} (1/I!)(\partial^{i_1}/(\partial x_1)^{i_1})\ldots(\partial^{i_n}/(\partial x_n)^{i_n}) f(s(x_1))^{i_1}\ldots(s(x_n))^{i_n}$$

but f may be written as

$$f(x_1,\ldots,x_n) = f^0(x_1,\ldots,x_n) + f^2(x_1,\ldots,x_n) + \ldots + f^L(x_1,\ldots,x_n)$$

Where we may assume, without loss of generality, that L is even. We can now substitute this in to the expression above to get:

$$0 = f^0(x_1,\ldots,x_n) + f^2(x_1,\ldots,x_n) + \ldots + f^L(x_1,\ldots,x_n) +$$
$$\nabla f^0.(s(x)^2 + s(x)^4 + \ldots + s(x)^L) + \ldots +$$
$$\nabla f^L.(s(x)^2 + s(x)^4 + \ldots + s(x)^L)$$
+ Higher order terms.

We can now equate terms of equal $\mathbf{Z}$-degree.

degree 0: $0 = f^0(x_1,\ldots,x_n)$ corresponds to the core manifold.

degree 2: $0 = f^2(x_1,\ldots,x_n) + \nabla f^0.(s(x)^2)$ thus we have $s(x)^2 = [-f^2/(\|\nabla f^0\|^2)]\nabla f^0 + \widetilde{s(x)}^2$ where $\widetilde{s(x)}^2 \perp \nabla f^0$, thus $s(x)^2$ is the translate of a vector tangent to the core manifold.

degree 4: $0 = f^4 + \nabla f^0.(s(x)^4)$ + terms in $f^0, f^2, s(x)^2$ and higher derivatives of f^0, f^2.

Thus we may write

$s(x)^4 = [-(f^4+\text{terms})/(\|\nabla f^0\|^2)]\nabla f^0 + \widetilde{s(x)}^4$ where $\widetilde{s(x)}^4 \perp \nabla f^0$. Thus $s(x)^4$ is a translate of a vector tangent to the core manifold.

etc,etc...We cannot explicitly write down all the "terms in $f^0, f^2 \ldots$", but the important thing is that at any stage in this inductive process they are terms in things already defined and fixed. At degree 2m we have the term

$$0 = f^{2m} + \nabla f^0 . s(x)^{2m} + (\text{terms})$$

thus the induction continues until

$$s(x) = s(x)^2 + s(x)^4 + \ldots + s(x)^L$$

is completely determined by $\widetilde{s(x)}^2, \ldots, \widetilde{s(x)}^L$ each of which is a vector tangent to the core manifold. Thus $s(x)$ is composed of terms that are translates of vectors tangent to the core manifold. Thus M may be smoothly identified with $T\tilde{M} \otimes (\tilde{B}_L)_0$ since the translations involved, being defined by products of derivatives, are C^∞.

Example: (a) Let us perform the equation of Z-degree terms on a function $F = z(f)$ where all the third derivatives of f and beyond are zero, over $(B_6)_0^2$.

That is $F:(B_6)_0^2 \longrightarrow (B_6)_0$.

Now, $F(x) = z(f)(x)$

$$= f(x) + \nabla f(x).(s(x)) + \sum_{i_1,i_2} (1/i_1!i_2!)(\partial^2 f/\partial x_i \partial x_j)(s(x_1))^{i_1}(s(x_2))^{i_2}$$

Let us write $x = (x_1, x_2) = (x, y)$

and let $x = x^0 + x^2 + x^4 + x^6$

$y = y^0 + y^2 + y^4 + y^6$

so $s(x) = x^2 + x^4 + x^6$

$$s(y) = y^2 + y^4 + y^6$$

and let $f = f^0 + f^2 + f^4 + f^6$

$$(\partial f/\partial x^0) = M^0 + M^2 + M^4 + M^6$$

$$(\partial f/\partial y^0) = N^0 + N^2 + N^4 + N^6$$

$$(\partial^2 f/(\partial y^0)^2) = A^0 + A^2 + A^4 + A^6$$

$$(\partial^2 f/(\partial x^0)^2) = B^0 + B^2 + B^4 + B^6$$

$$(\partial^2 f/\partial x^0 \partial y^0) = C^0 + C^2 + C^4 + C^6$$

Then the equation $F(x) = 0$ becomes

$$\begin{aligned}0 = {} & (f^0+f^2+f^4+f^6) + (M^0+M^2+M^4+M^6)(x^2+x^4+x^6) + \\ & (N^0+N^2+N^4+N^6)(y^2+y^4+y^6) + \\ & (1/2)(A^0+A^2+A^4+A^6)(x^2+x^4+x^6)^2 + \\ & (1/2)(B^0+B^2+B^4+B^6)(y^2+y^4+y^6)^2 + \\ & (C^0+C^2+C^4+C^6)(x^2+x^4+x^6)(y^2+y^4+y^6) \\ = {} & (f^0+f^2+f^4+f^6)+(M^0x^2+M^0x^4+M^0x^6)+(M^2x^2+M^2x^4)+(M^4x^2)+ \\ & (N^0y^2+N^0y^4+N^0y^6)+(N^2y^2+N^2y^4)+(N^4y^2)+ \\ & (1/2)(A^0x^2x^4+A^0(x^2)^2+A^2(x^2)^2+B^0y^2y^4+B^0(y^2)^2+B^2(y^2)^2)+ \\ & (C^0x^2y^2+C^0x^2y^4+C^0x^4y^2+C^2x^2y^2)\end{aligned}$$

after eliminating terms of $\mathbb{Z}$-degree greater than 6.

We can now equate terms of equal $\mathbb{Z}$-degree:

degree 0: $f^0 = 0$ the core manifold.

degree 2: Let us set $\begin{pmatrix} M^0 \\ N^0 \end{pmatrix} = v$ for convenience.

We have $f^2 + M^0x^2 + N^0y^2 = 0$ thus,

$$\begin{pmatrix} x^2 \\ y^2 \end{pmatrix} = -[f^2/\|v\|^2]v + \begin{pmatrix} \tilde{x}^2 \\ y^2 \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} \tilde{x}^2 \\ y^2 \end{pmatrix} \perp v$$

degree 4: We have

$$f^4 + M^0x^4 + N^0y^4 + M^2x^2 + N^2y^2 + (1/2)A^0(x^2)^2 +$$
$$(1/2)(B^0(y^2)^2 + C^0x^2y^2) = 0 \quad \text{and thus,}$$

$$\begin{pmatrix} x^4 \\ y^4 \end{pmatrix} = -[(f^4 + (..))/\|v\|^2]v + \begin{pmatrix} \tilde{x}^4 \\ y^4 \end{pmatrix} \quad \text{where} \begin{pmatrix} \tilde{x}^4 \\ y^4 \end{pmatrix} \perp v$$

degree 6: We have

$$f^6 + M^0x^6 + N^0y^6 + M^2x^4 + N^2y^4 + M^4x^2 + N^4y^2 +$$
$$(1/2)(A^0x^2x^4 + A^2(x^2)^2 + B^0y^2y^4 + B^2(y^2)^2) +$$
$$C^0x^2y^4 + C^0x^4y^2 + C^2x^2y^2 = 0. \quad \text{Thus,}$$

$$\begin{pmatrix} x^6 \\ y^6 \end{pmatrix} = -[(f^6 + (..))/\|v\|^2]v + \begin{pmatrix} \tilde{x}^6 \\ y^6 \end{pmatrix} \quad \text{where} \begin{pmatrix} \tilde{x}^6 \\ y^6 \end{pmatrix} \perp v$$

as required.

(b) Proposition 7 fails for functions

$F:(B_L)^n_o \longrightarrow B_L$ as stated. For example, let L=2 and n=2, and set

$$F(x,y) = (x^0)^2 + (y^0)^2 + \begin{pmatrix} 2x^0 \\ 2y^0 \end{pmatrix}\begin{pmatrix} s(x) \\ s(y) \end{pmatrix} + x^0b_1 - 1$$

then $F(x,y) = 0$ forces

degree 0: $(x^0)^2 + (y^0)^2 = 1$

degree 1: $x^0 = 0$

degree 2: $2(x^0x^2 + y^0y^2) = 0$.

Thus the degree 1 terms interfere with the degree 0 terms defining the core manifold. One can show, however, that the proposition remains true, provided the odd f^i define the core manifold along with the f^0 term. (Note, of course, that we need not worry about this if we are solely concerned with superpolynomials).

We must now deal with the intersection of two sets

$S = \{ x : F(x) = 0\}$

$T = \{ x : G(x) = 0\}$

for two G^∞ functions F and G. We see from the proof of proposition 7, (or from the example), that it is no longer sufficient to require F and G to be H^{∞} for the intersection to behave, since there are now new terms shifting the planes away from the origin in s(x)-space. (We suggest that the reader refers back to the proof of proposition 11 of chapter 1). However, if as before, we require that the core manifolds intersect transversely, then the problem is avoided and we obtain

Proposition 8: If $\tilde{\tilde{S}}$ and $\tilde{\tilde{T}}$ intersect transversely, then $S \cap T$ is diffeomorphic to $T(\tilde{\tilde{S}} \cap \tilde{\tilde{T}}) \otimes (\tilde{\tilde{B}}_L)_0$

The reader may wish to compare this result with the results of Picken [31] on the structure of matrix supergroups. Those results may be easily recovered from this sort of analysis.

Section 3: The z-Thickening of a manifold

We recall from chapter 1 that any C^∞ manifold could be the core manifold of some H^∞ restricted supermanifold over B_1. Having seen no obvious constraints, so far, on the structure of the core manifold of an even supermanifold we must ask whether the same result is true in this context. In fact it is true and the process for generating such an H^∞ supermanifold, which we describe below, is due to Rogers [34], see also [8]. In this section we shall identify the resulting supermanifold.

Let M be a C^∞ manifold of dimension n over **R** and let $\{(U_i, f_i) : i \in I\}$ be a covering of M by coordinate charts. Form the disjoint union

$$X = \coprod_{i \in I} (U^i \times (B_L)_0^n)$$

and define an equivalence relation on X as follows. Let (x,s) and (y,t) be elements of X, then

(x,s) ~ (y,t) if and only if $x \in U_i$, $y \in U_j$, for some i and j and

$$\begin{cases} y = x \\ t = z(f_j(f_i)^{-1})(s) \end{cases} \quad \& \ \epsilon(s) = f_i(x)$$

then the quotient X/~ is denoted by z(M) and is called the **z Thickening** of M. Thus z(M) is glued together using the z-extension of the transition functions on M. It may be shown that z(M) is an H^∞ supermanifold and that

$$\widetilde{z(M)} = M.$$

It is clear from its construction that z(M), as a C^∞ manifold, has the structure of a vector bundle over its core manifold. What is not clear, however, is the identity of the vector bundle. One might expect, looking at the form of the z-extension, that the z-thickening is some complicated sub bundle of the jet bundle of M, determined by the form of M. It is also clear that z(M) is not canonically a vector bundle even though it is a bundle of vector spaces over M. What we shall do is to produce a diffeomorphism from z(M) to a well known vector bundle, namely $TM \otimes (\tilde{B}_L)_0$.

Note first of all that the constuction of z(M) is achieved by patching together pieces of $\mathbf{R}^n \times (B_L)^n_\circ$ by diffeomorphisms. So if

$$f:U_1 \longrightarrow U_2$$

is a diffeomorphism of open sets of $\mathbf{R}^n$, we patch together $U_1 \times (B_L)^n_\circ$ and $U_2 \times (B_L)^n_\circ$ as follows:

$$(x,s) \sim (y,t) \quad \text{if and only if}$$

$$\begin{cases} y = f(x) \text{ and} \\ t = z(f)(s) \text{ and} \\ e(s) = x. \end{cases}$$

We may write $f:U_1 \longrightarrow U_2$ as

$$f = (f_1, f_2, \ldots, f_n) \quad \text{where the } f_i:U_i \rightarrow \mathbf{R}.$$

(Recall that U_2 is a subset of $\mathbf{R}^n$).

Thus the equation

$$t = z(f)(s) \quad \text{may be written:}$$

$$\begin{pmatrix} t_1 \\ t_2 \\ \cdot \\ \cdot \\ \cdot \\ t_n \end{pmatrix} = \begin{pmatrix} z(f_1)(s) \\ z(f_2)(s) \\ \cdot \\ \cdot \\ \cdot \\ z(f_n)(s) \end{pmatrix}$$

That is, if $t_1, \ldots, t_n$ are given, then s is the intersection of the solutions of the system

$$z(f_i)(s) = t_i \qquad (*)$$

What we shall show is that such s are in smooth 1-1 correspondence with the unique point $\hat{s}$ satisfying the system

$$\nabla f_i . \hat{s} = t_i . \qquad (+)$$

We use the $\mathbb{Z}$-grading again and write $s = s^2 + s^4 + \ldots + s^L$ (and assume without any loss of generality that L is even). Fix x and y = f(x). We have seen in section 2 that the solution to systems of equations like (*) take the form

$$s_i^2 = k_i^2 \nabla f_i + \tilde{s}_i^2 \qquad \text{with } \tilde{s}_i^2 \perp \nabla f_i$$

$$\cdot \qquad \cdot$$

$$\cdot \qquad \cdot$$

$$s_i^L = k_i^L \nabla f_i + \tilde{s}_i^L \qquad \tilde{s}_i^L \perp \nabla f_i$$

and solutions to systems like (+) take the form

$$\hat{s}_i^2 = \hat{k}_i^2 \nabla f_i + \hat{s}_i'^2 \qquad \hat{s}_i'^2 \perp \nabla f_i$$

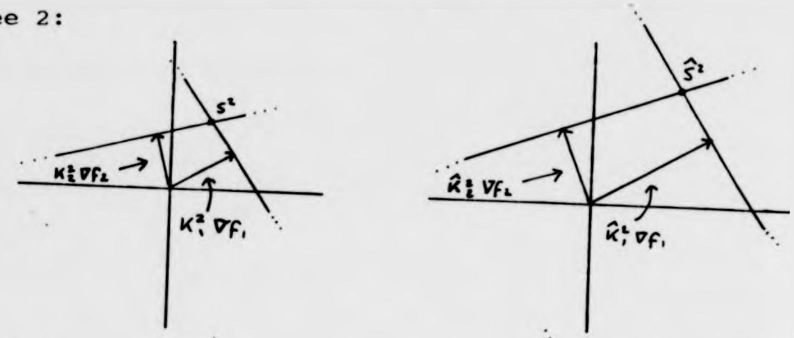
$$\cdot \qquad \cdot$$

$$\cdot \qquad \cdot$$

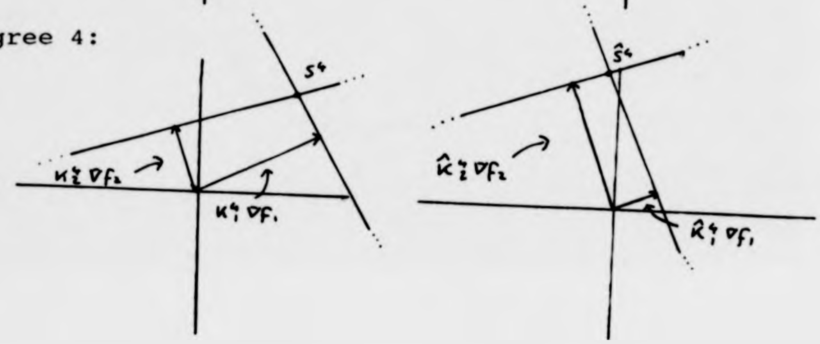
$$\hat{s}_i^L = \hat{k}_i^L \nabla f_i + \hat{s}_i'^L \qquad \hat{s}_i'^L \perp \nabla f_i . \quad \text{for each } i$$

The picture is like this:

degree 2:



degree 4:



and so on up the $\mathbb{Z}$-grading. (We know that they must intersect, since f is a diffeomorphism and thus the ∇f_i must be linearly independent).

Thus to every point $s = s^2 + \ldots + s^L$ there corresponds a unique point $\hat{s} = \hat{s}^2 + \hat{s}^4 + \ldots + \hat{s}^L$ where s satisfies system (*) and $\hat{s}$ satisfies system (+). The map $s \to \hat{s}$ is thus a C^∞ map with C^∞ inverse since the map is a translation that is a polynomial in the coordinates and higher derivatives of f. (The inverse is well defined and C^∞ since f being a diffeomorphism forces z(f) to be a G^∞ diffeomorphism, see [8]). System (+) clearly is a glueing function from $TM \otimes (\tilde{B}_L)_0$ where M is the quotient of $U_1 \amalg U_2$ by the function f (the "clutching function"). Now if M is any C^∞ manifold then it may be expressed as the quotient by clutching functions of

$\amalg U_i \quad i \in I$. So we may apply the above argument to obtain:

Proposition 9: The z-thickening, z(M), of a C^∞ manifold M is diffeomorphic to the total space of $TM \otimes (\tilde{B}_L)_0$.

Section 4: The Structure Theorem:

Having identified the z-thickening of a C^∞ manifold as essentially being the tangent bundle of that manifold and having seen the apparent ubiquitousness of the tangent bundle of the core manifold in our examples of even supermanifolds, it would be attractive to conjecture that, modulo certain conditions, the tangent bundle plays the same role for even supermanifolds as it did for restricted supermanifolds. The truth of this statement is not as clear, however, as it was for restricted supermanifolds. For example, it is not even clear that the fibre of the map $M \longrightarrow \widetilde{M}$ admits an affine structure. The clue for how to proceed comes from our analysis of the structure of the supersphere over $(B_L)_0$. We noticed there that we could "decompose" the manifold according to $\mathbb{Z}$-degree, starting from degree zero and working up. In this section we turn that idea on its head and decompose the structure of the supermanifold from the top degree downwards.

Our first step must to be look at the structure of G^∞ diffeomorphisms. Let

$$F:(B_L)_0^n \longrightarrow (B_L)_0^n$$

be a G^∞ diffeomorphism, then F may be written as

$$F = z(f)$$

where $\quad f:\mathbb{R}^n \longrightarrow (B_L)_0^n$

is a diffeomorphism onto its image. In turn f may be decomposed as

$$f = (f_1, f_2, \ldots, f_n)$$

where each $f_i : \mathbb{R}^n \longrightarrow (B_L)_0$.

Thus we may write:

$$F(x) = z(f)(x) = \begin{pmatrix} z(f_1)(x) \\ z(f_2)(x) \\ \vdots \\ z(f_n)(x) \end{pmatrix}$$

$$= \begin{pmatrix} f_1(x^0) + \nabla f_1 . s(x) + \sum_{|I|>1} (1/I!)(\partial^I f_1/\partial x^I) s(x)^I \\ \vdots \\ f_n(x^0) + \nabla f_n . s(x) + \sum_{|I|>1} (1/I!)(\partial^I f_n/\partial x^I) s(x)^I \end{pmatrix}$$

A close inspection of the components of this equation, or of the generalization of proposition 1, leads us to the following conclusion.

Proposition 10: If m is a multiindex, then the value of $F(x)^m$ depends only on x^n where n is a subindex of m. We may state this as:

If $F(x)^q = F(y)^q$ for all $q \le m$, then

$x^q = y^q$ for all $q \le m$.

(Recall that $(x_1, x_2, \ldots, x_n)^q = ((x_1)^q, \ldots, (x_n)^q)$).
The essential reason for the truth of this proposition is clear. In the z-expansion of F, the x^m for $|m|>0$ only appear in the s(x) and their powers. Multiplication in the exterior algebra only increases $\mathbb{Z}$-degree.

This proposition is the generalization of proposition 3, further up the $\mathbb{Z}$-grading. We can use this proposition to obtain foliations of any supermanifold in an analagous fashion to the real foliation. To this end, let $e_m : (B_L)^n_0 \longrightarrow \mathbb{R}^n$

be the map $e_m(x) = e_m(x_1, \ldots, x_n)$

$$= ((x_1)^m, \ldots, (x_n)^m).$$

Define a relation R_m on M by saying that xR_my if and only if there is a coordinate chart (U,f) such that x and y are in U and

$$e_q(f(x)) = e_q(f(y)) \quad \text{for all } q \leq m.$$

Then let $\sim_m$ be the equivalence relation generated by R_m. The equivalence classes of M under $\sim_m$ form the leaves of a foliation of dimension $n2^{L-1-|m|}$, for each m. Indeed it can be easily shown that M admits a complete lattice of multifoliations, in the sense of Kodaira and Spencer [26] with join and meet defined appropriately. (This is performed in Boyer and Gitlers' paper [10], by considering integrable sub bundles of TM defined by the G-structure on TM imposed by the G^{*} structure. We believe that the above approach "sees" these foliations more directly). A corollary to the above remarks, (or it can be seen directly, in the same fashion as proposition 10), is the fact that M must admit foliation by $\mathbb{Z}$-degree. By this we mean

If $\quad F(x)^k = F(y)^k$ for all $k \leq z$

then $\quad x^k = y^k$ for all $k \leq z$, where k and z are integers and we have decomposed the exterior algebra according to $\mathbb{Z}$-degree. This allows us to define an equivalence relation $\sim_z$ in the same way as above. The leaves of the corresponding foliations are nested according to the order on the integers.

Let k be an integer and let M_k be the quotient of M by the corresponding foliation. (Notice that M_0 is thus the core manifold $\tilde{M}$ of M). To proceed further, barring pathological G^∞ stuctures where the quotient is non hausdorff, we must impose some regularity conditions. Because the foliation of M is $\mathbb{Z}$-graded we have quotient maps

$$M_k \longrightarrow M_{k-2}$$

We also have quotient maps

$$M_k \longrightarrow M_0.$$

(M_L or M_{L-1} is M according to whether L is even or odd).

Definition: A supermanifold M is said to be **Completely Regular** if and only if the maps

$$M_k \longrightarrow M_{k-2}$$

and the maps

$2 \le k \le L$

k even

$$M_k \longrightarrow M_0$$

are fibre bundle maps. Thus M_0, the core manifold, is a hausdorff C^∞ manifold.

For the rest of this section we shall deal exclusively with completely regular supermanifolds. We shall also assume, for notational simplicity only, that L is even. The case L odd is dealt with in analogous fashion.

What we shall do now is to look, step by step, at the structure of M_L over M_{L-2}, M_{L-2} over M_{L-4}, ..., down to M_2 over $M_0 = \tilde{M}$.

First of all then, $M = M_L$ over M_{L-2}. If we look at a typical transition function of the supermanifold M_L we conclude that it has the form

$$\begin{pmatrix} x^0 \\ \vdots \\ x^L \end{pmatrix} \mapsto \begin{pmatrix} \text{(some function of} \\ x^0 \ldots x^{L-2}) \\ \text{(some function of } x^0 \ldots x^{L-2}) + Df^0(x)(x^L) \end{pmatrix}$$

where we have decomposed x according to **Z**-degree.

Example: Let M be a 1-dimensional H^∞ supermanifold over $(B_4)_0$. Then a typical transition function looks like

$$\begin{aligned} F(x) &= f(x^0) + (df/dx^0)s(x) + (1/2)(d^2f/d(x^0)^2)s(x)^2 \\ &= f(x^0) + (df/dx^0)(x^{12}b_{12}+x^{13}b_{13}+x^{14}b_{14}+x^{23}b_{23}+ \\ &\qquad x^{24}b_{24}+x^{34}b_{34}+x^{1234}b_{1234}) \\ &\quad + (d^2f/d(x^0)^2)(x^{12}x^{34}+x^{13}x^{24}+x^{14}x^{23})b_{1234} \end{aligned}$$

So writing this in the form above, we have

$$\begin{pmatrix} x^0 \\ x^{12} \\ x^{13} \\ x^{14} \\ x^{23} \\ x^{24} \\ x^{34} \\ x^{1234} \end{pmatrix} \mapsto \begin{pmatrix} f(x^0) \\ (df/dx^0)x^{12} \\ (df/dx^0)x^{13} \\ (df/dx^0)x^{14} \\ (df/dx^0)x^{23} \\ (df/dx^0)x^{24} \\ (df/dx^0)x^{34} \\ (d^2f/d(x^0)^2)(x^{12}x^{34}+x^{13}x^{24}+x^{14}x^{23}) + \\ (df/dx^0)x^{1234} \end{pmatrix}$$

So here we have $x^L = x^{1234}$

and $Df(x)(x^L) = (df/dx^0)x^{1234}$

(We have chosen an H^∞ function in the example merely for notational convenience).

What we conclude from the above analysis is the following. M_L has the structure of an affine bundle over M_{L-2} and if we knew that the universal cover of a typical fibre was a Euclidean space and that we could glue them together to form a vector bundle covering M_L, then M_L must admit the total space of the vector bundle

$$(q_{L-2})^{-1}TM_0 \otimes (B_L)^L \longrightarrow M_{L-2}$$

as a covering manifold, where

$$q_{L-2}: M_{L-2} \longrightarrow M_0$$

is the quotient map. This is simply because we can identify the transition functions of the vector bundle concerned. (Recall the principle of splitting from chapter 1 and the fact that the affine tangent bundle is diffeomorphic to the tangent bundle). This leads to the following:

Definition: Let M be a completely regular supermanifold over $(B_L)_0$. M is said to be **Fibre complete at degree L** if and only if the affine connection in the fibre of $M_L \longrightarrow M_{L-2}$ induced by the affine transition functions is complete. M is said to be **Vectorial at degree L** if and only if M is fibre complete at degree L and the bundle $M_L \longrightarrow M_{L-2}$ admits a section.

What we have proved so far is:

Proposition 11: Let M be an even supermanifold that is vectorial at degree L,then M admits the total space of the pullback vector bundle

$$\begin{array}{ccc} (q_{L-2})^{-1}TM_0 \otimes (B_L)^L & & \\ \downarrow & & \\ M_{L-2} & \xrightarrow{q_{L-2}} & M_0 \end{array}$$

as a covering manifold.

That is the first stage complete. Now let us look at the structure of M_{L-2} over M_{L-4}. As before the transition functions must take the form

$$\begin{pmatrix} x^0 \\ x^2 \\ \vdots \\ x^{L-2} \end{pmatrix} \mapsto \begin{pmatrix} \text{(some function} \\ \text{of } x^0,\ldots,x^{L-4}) \\ \text{(some function of } x^0,\ldots,x^{L-4}) + \\ Df^0(x)(x^{L-2}) \end{pmatrix}$$

Where $Df^0(x)$ acts on x^{L-2} in the obvious fashion. In the example above, the action of Df^0 is given by

$$\begin{pmatrix} x^{12} \\ x^{13} \\ x^{14} \\ x^{23} \\ x^{24} \\ x^{34} \end{pmatrix} \longmapsto \begin{pmatrix} (df/dx^0)x^{12} \\ (df/dx^0)x^{13} \\ (df/dx^0)x^{14} \\ (df/dx^0)x^{23} \\ (df/dx^0)x^{24} \\ (df/dx^0)x^{34} \end{pmatrix}$$

Thus, we can conclude that M_{L-2} is an affine bundle over M_{L-4}. The definition we make now must be obvious.

Definition: M is said to be **Fibre Complete at degree L-2** if and only if the affine connection induced in the fibre of the bundle $M_{L-2} \longrightarrow M_{L-4}$ is complete. M is said to be **Vectorial at degree L-2** if and only if this bundle admits a section and M is fibre complete at degree L-2.

So by inspection of the transition functions of the above bundle we can infer

Proposition 11': Let M be vectorial at degree L and at degree L-2, then M_{L-2} admits the total space of the pullback vector bundle

$$\begin{array}{ccc} (q_{L-4})^{-1}TM_0 \otimes (B_L)^{L-2} & & \\ \downarrow & & \\ M_{L-4} & \xrightarrow{q_{L-4}} & M_0 \end{array}$$

as a covering manifold.

This process continues on down the **Z**-grading provided we make the following

Definition: Let M be a completely regular even supermanifold. M is said to be **Fibre Complete at degree k** provided the affine connection induced in the fibre of the map $M_k \longrightarrow M_{k-2}$ is complete. M is said to be **Vectorial at degree k** if and only if M is fibre complete at degree k and the bundle $M_k \longrightarrow M_{k-2}$ admits a section. M is said to be **Vectorial** if and only if it is vectorial at degree k for k = L,L-2,...,2.

Let us gather these results together.

Proposition 12: Let M be a vectorial even supermanifold, then
M admits the total space of the pullback vector bundle

$$\begin{array}{ccc} (q_{L-2})^{-1}TM_0 \otimes (B_L)^L & & \\ \downarrow & & \\ M_{L-2} & \xrightarrow{q_{L-2}} & M_0 \end{array}$$

as a covering manifold. Let the covering map be p_L.
M_{L-2} admits the total space of the pullback vector bundle

$$\begin{array}{ccc} (q_{L-4})^{-1}TM_0 \otimes (B_L)^{L-2} & & \\ \downarrow & & \\ M_{L-4} & \xrightarrow{q_{L-4}} & M_0 \end{array}$$

as a covering manifold. Let the covering map be p_{L-2}.
...
down to...

M_4 admits the total space of the pullback vector bundle

$$\begin{array}{ccc} (q_2)^{-1}TM_0 \otimes (B_L)^4 & & \\ \downarrow & & \\ M_2 & \xrightarrow{q_2} & M_0 \end{array}$$

as a covering manifold. Let the covering map be p_4.

and M_2 admits the total space of the bundle

$$\begin{array}{c} TM_0 \otimes (B_L)^2 \\ \downarrow \\ M_0 \end{array}$$

as a covering manifold. Let the covering map be p_2.

Fortunately, we can stick all these covering maps together, due to the following

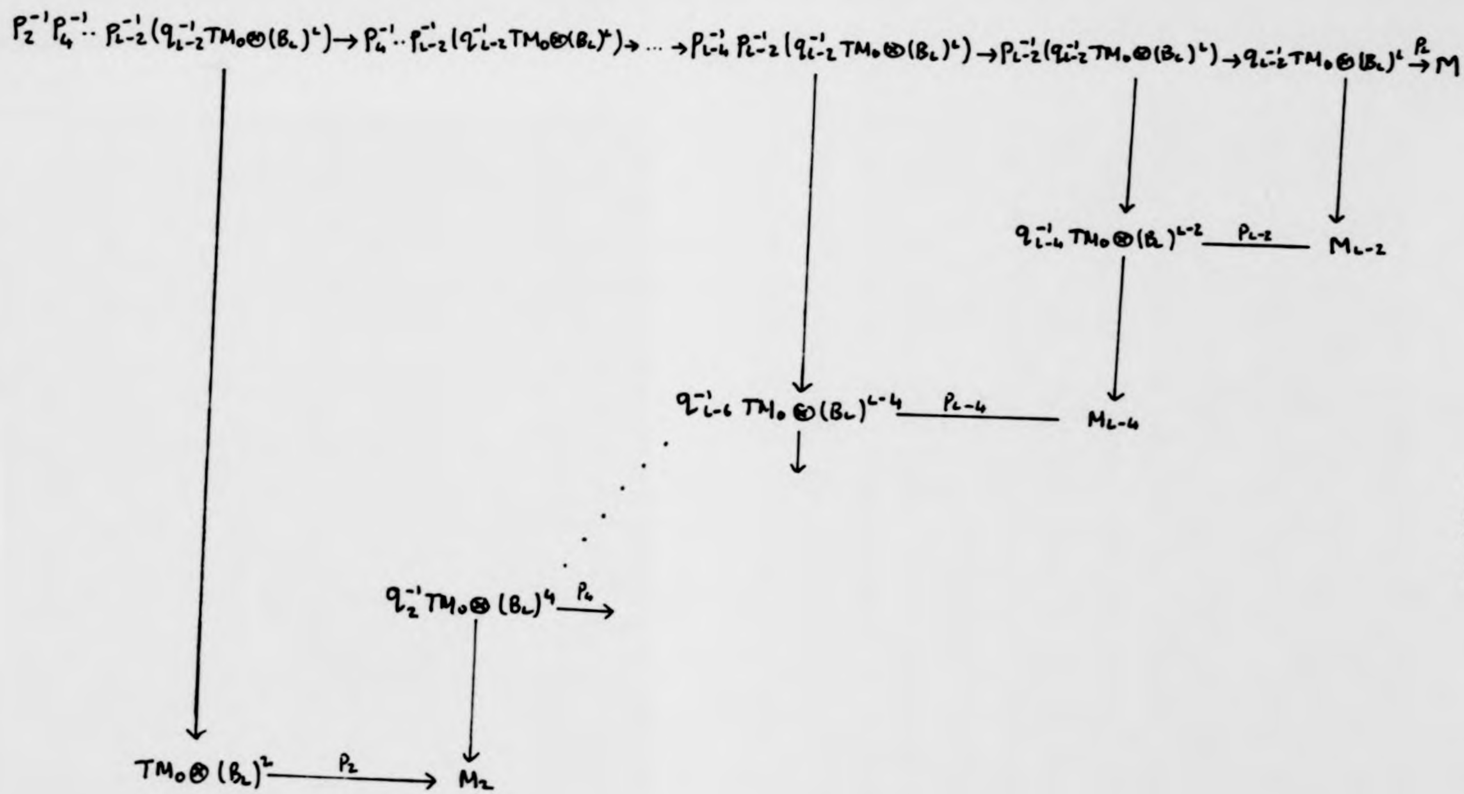
Lemma: In the diagram

$$\begin{array}{ccc} f^{-1}E & \xrightarrow{\hat{f}} & E \\ \downarrow & & \downarrow p \\ \hat{N} & \xrightarrow{f} & N \end{array}$$

if $E \xrightarrow{p} N$ is a fibre bundle and f is a covering map, then $\hat{f}$ is a covering map.

The proof of this is elementary.

We use this lemma to fit all the above diagrams together in the following diagram.


$p_2^{-1}p_4^{-1}\cdots p_{l-2}^{-1}(q_{l-2}^{-1}TM_0\otimes(B_l)^l) \to p_4^{-1}\cdots p_{l-2}^{-1}(q_{l-2}^{-1}TM_0\otimes(B_l)^l) \to \cdots \to p_{l-4}^{-1}p_{l-2}^{-1}(q_{l-2}^{-1}TM_0\otimes(B_l)^l) \to p_{l-2}^{-1}(q_{l-2}^{-1}TM_0\otimes(B_l)^l) \to q_{l-2}^{-1}TM_0\otimes(B_l)^l \xrightarrow{p_l} M$
$q_{l-4}^{-1}TM_0\otimes(B_l)^{l-2} \xrightarrow{p_{l-2}} M_{l-2}$
$q_{l-6}^{-1}TM_0\otimes(B_l)^{l-4} \xrightarrow{p_{l-4}} M_{l-4}$
$q_2^{-1}TM_0\otimes(B_l)^4 \xrightarrow{p_4}$
$TM_0\otimes(B_l)^2 \xrightarrow{p_2} M_2$

Since the p_i are all covering maps and the vertical arrows are fibrations we can conclude that the total space of

$$(p_2)^{-1}\ldots(p_{L-2})^{-1}[(q_{L-2})^{-1}TM_0 \otimes (B_L)^L]$$
$$\downarrow$$
$$M_0$$

is a covering manifold for M. It is now an elementary, though tedious, diagram chase to identify this as $TM_0 \otimes (B_L)_0 \longrightarrow M_0$. What we have shown, in conclusion, is

Proposition 13: Let M be a vectorial even supermanifold over $(B_L)_0$, then M admits $TM_0 \otimes (\tilde{B}_L)_0$ as a covering manifold, where $M_0 = \tilde{M}$ is the core manifold of M.

We must emphasize that the transition functions of $TM_0 \otimes (B_L)_0$ do not give G^∞ transition functions on the supermanifold M. Throughout the above we have identified affine tangent bundles with tangent bundles at every stage and this is unavoidable.
We must also point out that there is no analogue of proposition 16 of Ch 1 for even supermanifolds over $(B_L)_0$ for $L>3$. Being H^∞ is not strong enough to make the local sections join together. If, however $L<4$ or M admits an affine covering the proposition remains valid.

Section 5: Embeddings:

We saw in section 3 of chapter 1 that compact restricted supermanifolds admitted no embeddings into their model spaces. The same result is true for even supermanifolds, the proof being nearly identical. Let us recall the definitions and indicate the alterations required to make the proof work. The reader may wish to re-read section 3 of chapter 1 to motivate this proof.

Definition: Let M be an even supermanifold over $(B_L)_0$ and let $i:M \longrightarrow (B_L)_0^n$ be a continuous map. We say that i is a **G^∞ embedding** if and only if i is both a C^∞ embedding and a G^∞ map.

Proposition 14: Let M be a compact even supermanifold, then M admits no G^∞ embedding into $(B_L)_0^n$ for any n.

proof: By a series of lemmas:

Lemma 1: Let $F:(B_L)_0^n \longrightarrow (B_L)_0$ be a G^∞ map, where L is even, then F may be written as

$$F(x^0,\ldots,x^L) = f^0(x^0) + \nabla f^0.s(x) + (\text{higher terms})$$

where we have decomposed into Z-degree components. (x^L appears only in the second term).

If L is odd, the same expansion is true, replacing L by L-1.

proof: This is simply proposition 5 decomposed as in section 4.

Lemma 2: Let M be a compact even supermanifold in which the leaves of the foliation defined by $\sim_{L-2}$ (if L is even) or $\sim_{L-3}$ (if L is odd) are compact. Let F be a globally defined G^{∞} function on M, then Re F is a constant.

proof: The proof for L even is almost exactly the same as for the case of restricted supermanifolds. One considers an x_0 such that $\nabla f^0(x_0) \neq 0$ and then one defines M_{x_0} to be the leaf of the degree L-2 foliation through x_0. As M_{x_0} is compact the continuous function F^L takes on a maximum value on M_{x_0}. By rechoosing coordinates if necessary we deduce that F^L takes the form

$$v.x^L + u$$

where v is non zero, by hypothesis. Then as before this is a contradiction hence Re F must be constant.

The case L odd is slightly different since now the top degree term of F has more than one component. The proof goes through, though, since the dependence of F^{L-1} on x^{L-1} is still linear.

Lemma 3: An even G^{∞} supermanifold admits no embedding into the nilpotent part of $(B_L)^n_0$.

proof: this is proved in the same way as lemma 3 of proposition 8 of chapter 1 was proved, namely that, by inspection of lemma 1, x^L appears only in the second term where it is multiplied by ∇f^0. If ∇f^0 were zero, then the supposed embedding could not be injective.

The proof of proposition 14 now follows the same lines as proposition 8 of chapter 1.

Chapter 3: Odd G^∞ Functions.

In chapter 1 we were concerned with the notion of G^∞ functions defined on the whole of the exterior algebra B_L, then in chapter 2 we removed the odd part of this algebra and studied the consequences. In this and the next chapter we put the odd part back again, but in a slightly different way, namely as a cartesian product with the even part. The aim of this short chapter is to look in detail at the structure of G^∞ functions between purely the odd part of the exterior algebra. We shall see that the removal of the real part of the exterior algebra, to which even G^∞ functions are "tied" has far-reaching consequences.

Definition: Let $F:(B_L)_1^n \longrightarrow B_L$ be a smooth function. F is said to be **Odd G^1 at x** if and only if

$$F(x+h) = F(x) + \sum_{i=1}^{n} G_i F(x).h_i + n(h)\|h\|,$$

where $G_iF(x) \in B_L$ and $\|n(h)\| \to 0$ as $\|h\| \to 0$. If U is an open set of $(B_L)_1^n$, then F is said to be **G^1 in U** if and only if it is G^1 at all points of U. If U is all of $(B_L)_1^n$ then F is merely called **G^1**.

Remarks: We already have a problem in proceeding further in that if we were to say that the G_iF were the partial G derivatives of F, they would only be defined up to the addition of a constant. Since the variable h is necessarily a nilpotent element of B_L the formula above only defines the G_iF up to the addition of a constant of top degree in B_L. This would be a troublesome problem if we were to go on to try to define a notion of a tangent space of supermanifolds including odd variables. There are various ways of addressing this problem (see, for example, Boyer and Gitler [10]), but as we are only going to be concerned with the structure of the supermanifold itself, the consequences are easily sidestepped. All that we are interested in is a sensible definition of a function being G^∞ and as this is an essentially differential notion it is natural to strip a function of any constants before using the above definition inductively. Notice that the G_iF (stripped of constants) are then well defined objects and we can say F is G^{i+1} if and only if the G_iF (stripped of constants) are G^i. Another possible answer to this problem is to regard taking the G-derivative as naturally decreasing the $\mathbb{Z}$-degree of the function, as no top degree element could possibly arise in the G_iF as the result of a formula for F.

We have to make this inductive definition of G^∞ because, as we have indicated previously, F being G^1 no longer implies that the G_iF are G^1.

Example: Let $F:(B_2)_1 \longrightarrow B_2$ be defined by

$$F(sb_1 + tb_2) = stb_1b_2$$

then

$$F((s+h)b_1 + tb_2) = F(s,t) + htb_1b_2$$

$$F(sb_1 + (t+k)b_2) = F(s,t) + ksb_1b_2$$

thus F is G^1 with $GF = tb_2 - sb_1$.

Then we have

$$GF((s+h)b_1 + tb_2) = GF(s,t) - hb_1 \Rightarrow G^2F = -1$$

$$GF(sb_1 + (t+k)b_2) = GF(s,t) + kb_2 \Rightarrow G^2F = +1$$

which is a contradiction, thus F cannot be G^2.

In fact the behaviour of the function in the example is typical, a function multilinear in the variables being G^1 but only G^2 if it is linear in the variables. We shall formalize this and then see that a function that is G^2 is in fact G^∞.

Proposition 1: Let $F:(B_L)_1^n \longrightarrow B_L$ be G^1, then

(i) $(\partial^2 F/(\partial (x_i)^m)^2) = 0$ and if F is G^2

(ii) $(\partial^2 F/\partial (x_i)^m \partial (x_i)^n) = 0 \quad m \neq n$

proof: Recall that the i subscript refers to the cartesian product and the m and n superscript to the part of the exterior algebra.

(i) Follows by noticing that, as the square of a nilpotent basis element is zero,

$(\partial F/\partial (x_i)^m)^m$ cannot be a function of $(x_i)^m$.

(ii) Note first that $(\partial F^n/\partial (x_i)^m) = 0$ unless $m \leq n$ so we merely have to look at F of the form

$F(\ldots sb_m + tb_n \ldots) = stb_m b_n b_p$ and higher multilinear products, using the result of part (i) that implies that F is linear in its variables separately. Then we have

$$F(\ldots(s+h)b_m + tb_n \ldots) = F(\ldots) + htb_m b_n b_p$$

$$F(\ldots sb_m + (t+k)b_n \ldots) = F(\ldots) + ksb_m b_n b_p$$

and then

$$GF(\ldots(s+h),t,\ldots) = GF(\ldots) - b_m b_p \Rightarrow G^2F = b_p$$

$GF(\ldots s,(t+k),\ldots) = GF(..) + b_n b_p \Rightarrow G^2F = -b_p$
which is a contradiction and the result follows.

Thus if F is a G^2 function it depends only linearly and not multilinearly on the $(x_i)^m$, for any fixed i. This implies, for n=1, that F must simply be multiplication on the left by a scalar. (Inspect the definition of G^1). In fact it simple to deduce the behaviour for n>1 for F "looks like" scalar multiplication to each of its variables separately. That is, F can only be linear in the $(x_i)^m$ for fixed i and it can only be multilinear in the $(x_i)^m$ as i varies. What we have demonstrated is the following.

Proposition 2: Let $F:(B_L)_1^n \longrightarrow B_L$ be a G^2 function, then F may be written as

$$F(x_1,\ldots,x_n) = \sum_q K_q x^q$$

where the K_q are constants and q is a multi-index

$$q = (q_1,q_2,\ldots,q_k)$$

where $1 \leq q_1 < q_2 < \ldots < q_k \leq n$.

Corollary: If F is G^2 then F is G^∞.
proof: Apply the definition of G differentiability to the above formula.

Remarks: (a) We shall only concern ourselves with G^2 functions, which we shall frequently refer to as being G^∞.

(b) It is clear that there is no notion of an odd G^∞ diffeomorphism $(B_L)_1^n \longrightarrow (B_L)_1^n$ since multiplication by a nilpotent scalar shifts the Z-degree of an element upwards. Thus there is no notion of a purely odd G^{∞} supermanifold, though a special case of the supermanifolds defined in chapter 5 will have the "smallest" even part possible.

Chapter 4: Rogers' Supermanifolds.

Section 1: Definitions and elementary results:

As promised in chapter 3 it is time to bring the odd variables back into the picture and to study the structure of supermanifolds modelled on the cartesian product of the odd and even parts of the exterior algebra. This means that we have finally arrived at the definition of a supermanifold given in Rogers' paper [34]. We shall apply all the techniques devised in the earlier chapters to analyse the structure of these objects.

Our first move will be to make a notational simplification that is standard in the literature. Set

$$(B_L)_0^m \times (B_L)_1^n = (B_L)^{m,n}.$$

Let $F:(B_L)^{m,n} \longrightarrow B_L$ be a smooth map.

Definition: F is said to be **G^1 at (x,y)** if and only if

$$\begin{aligned} F(x+h,y+k) = F(x,y) &+ \sum_{i=1}^{m} G_i F(x,y).h_i \\ &+ \sum_{j=1}^{n} G_{m+j} F(x,y).k_j \\ &+ n(h,k)\,\|(h,k)\| \end{aligned}$$

where $(x,y) \in (B_L)^{m,n}$ has $x \in (B_L)_0^m$ and $y \in (B_L)_1^n$ and we have $\|n(h,k)\| \to 0$ as $\|(h,k)\| \to 0$ and the $G_k F(x,y) \in B_L$. If U is an open set of $(B_L)^{m,n}$ then F is said to be **G^1 in U** if F is G^1 at all the points of U. If U is the whole of $(B_L)^{m,n}$ we simply say that F is **G^1**.

The maps $G_i F:(B_L)^{m,n} \longrightarrow B_L$ $1 \le i \le m$ are called the

Even G derivatives of F. The numbers G_jF $m+1\leq j\leq m+n$ are subject to the same sort of ambiguity of definition that we discussed in chapter 3. We recall that in order to discuss the notion of the higher G derivatives of F we have to strip the G_j of any constants, for $m+1\leq j\leq m+n$. Bearing this in mind we are now free to say that F is $\underline{\underline{G}}^{i+1}$ if and only if all the G_kF are G^i.

Definition: Let $f:(B_L)^{m,n}\longrightarrow(B_L)_i$ $i = 0$ or 1. We say that f is $\underline{\underline{G}}^1$ if and only if f is G^1 as a map into B_L. Let $h:(B_L)^{m,n}\longrightarrow(B_L)^{m',n'}$, then h is said to be $\underline{\underline{G}}^1$ if and only if the projection of h onto factors is G^1 in the above sense.

We already have structure theorems for even and for odd G^∞ functions. We can put these together to deduce the following.

Proposition 1: Let $F:(B_L)^{m,n}\longrightarrow B_L$ be a G^2 map and let q be a multi index with

$$1\leq q_1<q_2<\ldots<q_k\leq n,$$

then there exist functions $f_q:\mathbb{R}^m\longrightarrow B_L$ such that

$$F(x,y) = \sum_q z(f_q)(x).y_q$$

moreover these functions are unique modulo constants in $\mathbb{Z}$-degree $L-|q|+1$ upwards.

proof: This is simply proposition 5 of chapter 2, together with proposition 2 of chapter 3. The functional dependence of F is multilinear in the y variables with constants that are G^2 functions of the even variables and hence must be of the stated form. This expansion is called the **z-Expansion of F**.

Remarks: (a) This expansion is due to Rogers [34], where she proves it by using Taylors theorem.

(b) This proposition shows that if F is G^1 in its even variables and G^2 in its odd variables then F is G^∞. We shall only be concerned with G^∞ functions in this work.

(c) F is said to be $\underline{\underline{H}}^\infty$ if and only if all the f_q are maps $\mathbf{R}^m \longrightarrow \mathbf{R}$ in the z-expansion of F.

Examples: (a) Let us write down the general form of a G^∞ function $F:(B_2)^{1,1} \longrightarrow B_2$. By the proposition we have

$$\begin{aligned} F(x,y) &= z(f)(x) + z(g)(x).y \quad \text{for some } f,g:\mathbf{R} \to B_2 \\ &= f^0(x^0)+f^1(x^0)b_1+f^2(x^0)b_2+f^{12}(x^0)b_1b_2 + \\ &\quad (df^0/dx^0)x^{12}b_1b_2 + \\ &\quad [g^0(x^0)+g^1(x^0)b_1+g^2(x^0)b_2+g^{12}(x^0)b_1b_2 + \\ &\quad (dg^0/dx^0)x^{12}b_1b_2][x^1b_1+x^2b_2] \\ &= f^0(x^0) + \\ &\quad (f^1(x^0) + g^0(x^0)x^1)b_1 + \\ &\quad (f^2(x^0) + g^0(x^0)x^2)b_2 + \\ &\quad [f^{12}(x^0) + g^1(x^0)x^2 + g^2(x^0)x^1 + \\ &\qquad (df^0/dx^0)x^{12}]b_1b_2 \end{aligned}$$

where we have written $(x,y) = (x^0+x^{12}b_1b_2, x^1b_1+x^2b_2)$

(b) Let $G:(B_2)^{1,2} \longrightarrow B_2$ be an H^∞ function, then

$$\begin{aligned} G(x,y) &= z(f)(x) + z(g)(x)y_1 + z(h)(x)y_2 + z(k)(x)y_1y_2 \\ &= f(x^0) + (df/dx^0)x^{12}b_1b_2 + \\ &[g(x^0) + (dg/dx^0)x^{12}b_1b_2][pb_1 + qb_2] + \\ &[h(x^0) + (dh/dx^0)x^{12}b_1b_2][sb_1 + tb_2] + \\ &[k(x^0) + (dk/dx^0)x^{12}b_1b_2][pb_1 + qb_2][sb_1 + tb_2] \end{aligned}$$

$$= f(x^0) +$$
$$[g(x^0)p + h(x^0)s]b_1 +$$
$$[g(x^0)q + h(x^0)t]b_2 +$$
$$[(df/dx^0)x^{12} + k(x^0)(pt-qs)]b_1b_2.$$

Definition: An **(m,n) dimensional G^∞ Supermanifold** is a pair (M,A) where M is a hausdorff, 2^{nd} countable topological space and A is a maximal atlas of G^∞ supercharts on M.

Remarks: This definition is formally the same as the definition of an even G^∞ supermanifold given in chapter 2, therefore for the definition of the terms arising in here we suggest the reader refers back to section 1 of chapter 2. The notion of an H^∞ supermanifold is defined in the obvious fashion.

Examples: (a) $(B_L)^{m,n}$ is an H^∞ supermanifold of dimension (m,n) with a single chart, namely the identity.

(b) The 2-dimensional torus and the Klein bottle admit (1,1) dimensional supermanifold structures over B_1. This is shown in the same way that we showed that the torus admits a restricted supermanifold structure. This also demonstrates that a supermanifold, in this new sense, need not be an orientable C^∞ manifold.

(c) The supersphere over $(B_L)^{m,n}$:
We have seen in previous chapters how to define, in a natural way, the supersphere over B_L and over $(B_L)_0$ as being the set of points at which a quadratic function

vanishes. We cannot do quite the same over $(B_L)^{m,n}$ since there are odd variables present and we have seen above that a G^∞ function does not have quadratic terms. There is, however, a fairly canonical way to proceed.

Consider the set of matrices of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{matrix} m \\ 2n \end{matrix}$$
$$\begin{matrix} m & 2n \end{matrix}$$

where A and D take their values in $(B_L)_0$ and C and B take their values in $(B_L)_1$. Define the **Supertranspose** of such a matrix to be

$$\begin{pmatrix} A^T & C^T \\ -B^T & D^T \end{pmatrix}$$ and let

$$Q = \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} \begin{matrix} m \\ 2n \end{matrix} \quad \text{where} \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{matrix} n \\ n \end{matrix}$$
$$\begin{matrix} m & 2n \end{matrix} \qquad\qquad \begin{matrix} n & n \end{matrix}$$

then the set of matrices g such that $g^TQg = Q$ is defined to be the **Supergroup** $Osp_L(m,2n)$. (See Leites [29] for an exposition of superlinear algebra and Rittenberg and Scheunert [33] for the definition of the supergroups).

If $v \in (B_L)^{m,2n}$ then $q(v) = v^TQv$ is a quadratic form. Let $S = \{v \in (B_L)^{m,2n} : q(v) = 1\}$, then S is said to be the **Supersphere** over $(B_L)^{m,2n}$.

We shall see, after we have analysed the structure of S, that S admits the structure of an (m-1,2n) dimensional supermanifold.

Now, $q(v) = \langle v,v\rangle$

$$= \langle v^0+v^1+\ldots+v^L, v^0+v^1+\ldots\ldots+v^L\rangle$$

where we have graded by $\mathbb{Z}$-degree.

$$= \langle v^0,v^0\rangle + \qquad \text{in degree } 0$$

$$2\langle v^0,v^1\rangle + \qquad \text{in degree } 1$$

$$\langle v^1,v^1\rangle + 2\langle v^0,v^2\rangle + \qquad \text{in degree } 2$$

$$2\langle v^0,v^3\rangle + 2\langle v^1,v^2\rangle + \qquad \text{in degree } 3$$

etc, up the $\mathbb{Z}$-grading.

Note now, however, that we have $\langle v^i,v^j\rangle = 0$ if $i \neq j$ mod 2, thus we are left with the equations

$$\langle v^0,v^0\rangle = 1$$

$$\langle v^1,v^1\rangle + 2\langle v^0,v^2\rangle = 0 \quad \text{etc...}$$

to which we may apply the same analysis that we performed on the supersphere over $(B_L)_0$ in chapter 2. There is one slight difference, however, in that there are odd variables present. We see that there is no restriction on their values so we may conclude that S is diffeomorphic to $[TS^{m-1} \otimes (\tilde{B}_L)_0] \times (B_L)_1^{2n}$. It is now clear how to put the structure of an (m-1,2n) dimensional supermanifold on S.

(d) The analysis above may be interpreted as saying that the supersphere is a trivial vector bundle over the z-thickening of the core manifold. Taking this as motivation, we may ask whether it is possible to start with any vector bundle over a C^∞ manifold and produce from it, in a natural way, an (m,n) dimensional supermanifold.

Let $E^n \longrightarrow M^m$ be a rank n vector bundle over the m dimensional C^∞ manifold M. This structure is locally glued together with clutching functions that take the form

$$F: \mathbf{R}^m \times \mathbf{R}^n \longrightarrow \mathbf{R}^m \times \mathbf{R}^n$$

$$F(x,s) \quad = \quad (f(x), A(x)(s))$$

where $f: \mathbf{R}^m \longrightarrow \mathbf{R}^m$ and $A: \mathbf{R}^m \longrightarrow \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ are smooth maps. Define the function

$$\hat{F}: (B_L)^{m,n} \longrightarrow (B_L)^{m,n} \quad \text{by}$$

$$\hat{F}(x,s) \quad = \quad (z(f)(x), A(e(x))(s))$$

where $A(e(x))$ acts on $(B_L)_1^n$ in the obvious matrix fashion. If F is a diffeomorphism then so is $\hat{F}$, hence pieces of $(B_L)^{m,n}$ are glued together by $\hat{F}$ to give a G^∞ supermanifold of dimension (m,n). It is clear that, as C^∞ manifolds, this construction yields

$$\begin{array}{ccc} q^{-1}E \otimes (B_L)_1 & \longrightarrow & E \\ \downarrow & & \downarrow \\ z(M) & \longrightarrow & M \end{array}$$

Remark: We recall from the theory of graded manifolds (Kostant [27]) that any graded manifold may be expressed as the sections of the exterior power of some vector bundle over a C^∞ manifold, by Batchelors' theorem (see Batchelor [2], Blattner and Rawnsley [7]). Thus corresponding to any graded manifold there is a G^∞ supermanifold.

Section 2: The vanishing set of a G^∞ function:

In chapters 1 and 2 we defined the notion of a supervariety over B_1 and $(B_L)_0$, respectively, to be the vanishing set of a collection of superpolynomials and showed that, subject to certain intersection properties, the resulting manifold was essentially the tangent bundle of the core manifold. Recall, however, that this result did not depend on the supermanifolds concerned being supervarieties, but only on them being the vanishing set of a collection of G^∞ functions whose vanishing sets intersected in a manifold. In this chapter we are dealing with G^∞ functions where there are essentially no polynomial terms in the odd variables, so we shall no longer be concerned with the notion of supervarieties. What we can ask about, however, is the structure of the vanishing set of a G^∞ function. The obvious conjecture is that the structure of such a set mirrors the construction given at the end of the last section and that there is a vector bundle other than the tangent bundle involved. The structure of the supersphere would suggest that the vector bundle is trivial, but the supersphere is not typical in that there are no constraints at all on the odd sector. To proceed further, then, we shall have to put some restriction on the G^∞ functions concerned.

Definition: Let $F:(B_L)^{m,n} \longrightarrow B_L$ be a G^∞ function, Then F may be written

$$F(x,y) = \sum_{q} z(f_q)(x) \cdot y_q.$$

F is said to be **Odd non-degenerate** if and only if the functions $(f_1)^0,\ldots,(f_n)^0$ are never zero.

This condition corresponds naturally to the condition that $\nabla(f_0)^0$ is never zero, that is required to ensure that the core manifold is well defined.

Proposition 2: Let M be the zero set of an odd non degenerate G^∞ function $F:(B_L)^{m,n}\longrightarrow B_L$, then M may be smoothly identified with $[T\tilde{M} \otimes (\tilde{B}_L)_0] \times (B_L)_1^{n-1}$, provided that the core manifold $\tilde{M}$ is defined.

proof: The proof of this proposition proceeds in close analogy to the proof of proposition 7 of chapter 2, the difference being in notational complication. Let us, therefore, start this proof with an example.

Example: Let $F:(B_2)^{1,2}\longrightarrow B_2$, then F may be written

$$\begin{aligned}
F(x,y) &= z(f)(x) + z(g)(x)y_1 + z(h)(x)y_2 + z(k)(x)y_1y_2 \\
&= f^0(x^0) + \\
&\quad [f^1(x^0) + g^0(x^0)s + h^0(x^0)u]b_1 + \\
&\quad [f^2(x^0) + g^0(x^0)t + h^0(x^0)v]b_2 + \\
&\quad [(df^0/dx^0)x^{12} + g^1(x^0)t + g^2(x^0)s + \\
&\qquad h^1(x^0)v + h^2(x^0)u + \\
&\qquad k^0(x^0)(ut-vs)]b_1b_2
\end{aligned}$$

Where we have written $x = x^0 + x^{12}b_1b_2$ and

$$\begin{aligned}
y &= (y^1,y^2) \\
&= (sb_1 + tb_2, ub_1 + vb_2)
\end{aligned}$$

Thus equating by degree:

$$0 = f^0(x^0) \qquad \text{The core manifold.}$$

$$0 = \begin{pmatrix} f_1(x^0) \\ f^2(x^0) \end{pmatrix} + g^0(x^0)\begin{pmatrix} s \\ t \end{pmatrix} + h^0(x^0)\begin{pmatrix} u \\ v \end{pmatrix}$$

Thus provided g^0 and h^0 are non zero $\begin{pmatrix} s \\ t \end{pmatrix}$ and $\begin{pmatrix} u \\ v \end{pmatrix}$ are affinely dependent. We see that s and t are free to take on any values and once they are fixed the values of u and v are fixed. The analysis at degree 2 proceeds as before and we conclude that the result is true for this example.

The general case follows in exactly the same fashion. If we write

$$F(x,y) = z(f_0)(x) + \sum_{i=1}^{n} z(f_i)(x)y_i + \text{higher terms},$$

we see that x is determined by $x^0, x^2, \ldots, x^L$ (we assume for convenience that L is even) and that y is determined by $y^1, y^3, \ldots, y^{L-1}$. The x^i are determined in exactly the same fashion as before, since the lowest degree expression containing x^i looks like

$\nabla(f_0)^0 x^i$ + (terms in things already determined) as before. The lowest degree expression containing the y^i looks like

$\sum_{j=1}^{n} (f_j)^0 (y_j)^i$ + (terms in things already determined) and thus provided all the $(f_j)^0$ are non zero, fixing the first n-1 of these determines the last. Hence the conclusion of the theorem follows.

When we consider the intersection of two such sets we can say very little more than the fact that if the intersection of the sets is a manifold then the resulting supermanifold is of the form discussed at the end of the previous section. This is because there is no reason to suspect that the intersection of two trivial bundles is trivial. We can conclude, though, that the resulting supermanifold is homotopy equivalent to the core manifold. (Compare with Picken[31]).

Section 3: The Structure Theorem.

In section 4 of chapter 2 we studied the structure of vectorial supermanifolds, concluding that they admitted covering manifolds that are direct sums of the tangent bundle of the core manifold. In this section we wish to generalize this analysis to Rogers supermanifolds. The procedure for doing this broadly follows the lines of the analysis of vectorial even supermanifolds, the difference being that there are now putative vector bundle transition functions present and these have to be checked for non degeneracy. Let us first of all write out the general form of a G^∞ function.

Proposition 3: Let $F:(B_L)^{m,n} \longrightarrow (B_L)^{m,n}$ be a G^∞ map, then there are functions $f_{j,v}$ with $1 \leq j \leq m+n$ and v a multi-index with $1 \leq v_1 < v_2 < \ldots < v_k \leq n$ such that

$$F(x,y) = \sum_v \begin{pmatrix} z(f_{1,v})(x)y_v \\ z(f_{2,v})(x)y_v \\ \\ z(f_{(m+n),v})(x)y_v \end{pmatrix}$$

where $f_{j,v}:\mathbb{R}^m \longrightarrow (B_L)_k$ with $k = \begin{cases} v & 1 \leq j \leq m \\ v+1 & m+1 \leq j \leq m+n. \end{cases}$

Now, if m is a multi-index with

$$1 \leq m_1 < m_2 < \ldots < m_k \leq L$$

then it still makes sense to talk about $F(x,y)^m$. If we inspect the components of the above expansion then we conclude the following.

Proposition 4: If m is a multiindex, then the value of $F(x,y)^m$ depends only on $(x,y)^n$ where n is a subindex of m. We may state this as: If F is a diffeomorphism ⅃

If $F(x,y)^q = F(x',y')^q$ for all $q \leq m$, then

$(x,y)^q = (x',y')^q$ for all $q \leq m$.

Thus we can conclude that a Rogers supermanifold is multifoliated in the same way as an even supermanifold. (See [26]). Let us repeat the details here.

Let e_m be the map

$$e_m(x) = e_m(x_1,\ldots,x_n) = ((x_1)^m,\ldots,(x_n)^m).$$

Define a relation R_m on M by saying that xR_my if and only if there is a coordinate chart (U,f) such that x and y are in U and

$$e_q(f(x)) = e_q(f(y)) \quad \text{for all } q \leq m.$$

Then let $\sim_m$ be the equivalence relation generated by R_m. The equivalence classes of M under $\sim_m$ form the leaves of a foliation for each m. As before it can be easily shown that M admits a complete lattice of foliations with join and meet defined appropriately. (See Boyer and Gitler [10]). Again, a corollary to the above remarks, (or it can be seen directly, in the same fashion as proposition 4), is the fact that M must admit foliation by $\mathbb{Z}$-degree. By this we mean

If $F(x,y)^k = F(x',y')^k$ for all $k \leq z$

then $(x,y)^k = (x',y')^k$ for all $k \leq z$, where k and z are integers and we have decomposed the exterior algebra according to $\mathbb{Z}$-degree. This allows us to define an equivalence relation $\sim_Z$ in the same way as above. The leaves of the corresponding foliations are nested according to the order on the integers.

As before, we must bar pathological G^∞ structures, so if M_k is the quotient of M by the equivalence relation $\sim_k$, we have quotient maps

$M_k \longrightarrow M_{k-1}$ and quotient maps

$M_k \longrightarrow M_0$ where M_0 is the core manifold.

(We are assuming that m>0, the case m = 0 is dealt with in similar fashion). M_L is, of course, equal to M.

Definition: A Rogers supermanifold M is said to be **Completely Regular** if and only if the quotient maps

$$M_k \longrightarrow M_{k-1}$$

and $$M_k \longrightarrow M_0$$

are fibre bundle maps. Thus the core manifold M_0 is a hausdorff C^∞ manifold.

Suppose that $F:(B_L)^{m,n} \longrightarrow (B_L)^{m,n}$ is a G^∞ map, then we may form the (m+n)x(m+n) real valued matrix

$e(G_i F^j(x,y))$ at any point (x,y). This definition is unambiguous since any ambiguity in the odd derivatives occurs in the top degree and so is eliminated by e. This matrix will have the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where $A \in M_m(\mathbf{R})$ and $B \in M_n(\mathbf{R})$. The **Super-rank** of F is defined to be the pair of integers (rank A, rank B) and is denoted srk F. We recall that F is a G^∞ diffeomorphism if and only if srk F = (m,n) at all points of the domain of F. (See, for example, Boyer and Gitler [10]). If we refer back to proposition 3 we obtain the following.

Proposition 5: F is a G^∞ diffeomorphism if and only if

(a) The map $F^0 : \mathbf{R}^m \longrightarrow \mathbf{R}^m$ defined by

$F^0 = ((f_{1,0})^0, (f_{2,0})^0, \ldots, (f_{m,0})^0)$ is a C^∞ diffeomorphism for all x. and

(b) The map $G^0 : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ defined by

$$G^0(y) = \begin{pmatrix} (f_{m+1,1})^0 & (f_{m+1,2})^0, \ldots, & (f_{m+1,n})^0 \\ (f_{m+2,1})^0 & \ldots & (f_{m+2,n})^0 \\ \cdot & & \\ (f_{m+n,1})^0 & & (f_{m+n,n})^0 \end{pmatrix}$$

is invertible, that is, an element of $GL_n(\mathbf{R})$ for all y.

proof: Simply observe that the matrix given in (b) is the zeroth order part of the odd derivatives and apply the remark about super-rank above.

We must now decompose the structure of M, relative to the M_k, as we did for even supermanifolds. As we go, the same sort of criteria for us to continue the decomposition will arise, so we will make our definitions now, and justify them as we go.

Definition: Let M be a completely regular Rogers supermanifold over B_L, then M is said to be **Fibre Complete at degree k** if and only if the affine connection induced in the fibre of the map $M_k \longrightarrow M_{k-1}$ is complete. M is said to be **Vectorial at degree k** if and only if M is fibre complete at degree k and the bundle

$M_k \longrightarrow M_{k-1}$ admits a section. M is said to be **Vectorial** if and only if it is vectorial at degree k for k = L, L-1, ..., 1.

Let us start our decomposition with an example to guide us. We shall assume, for convenience only, that L is even.

Example: Let $F:(B_2)^{1,2} \longrightarrow (B_2)^{1,2}$ be an H^∞ transition function. Let us write

$$(x, y_1, y_2) = (x^0 + x^{12} b_1 b_2, pb_1 + qb_2, sb_1 + tb_2)$$

then we may write

$$\begin{aligned} F(x,y) = \quad & f(x^0) + [(df/dx^0)x^{12} + k(x^0)(pt-qs)]b_1 b_2 \\ & [g(x^0)p + h(x^0)s]b_1 + [g(x^0)q + h(x^0)t]b_2 \\ & [m(x^0)p + n(x^0)s]b_1 + [m(x^0)q + n(x^0)t]b_2 \quad . \end{aligned}$$

First we consider the structure of M over M_{L-1}. The transition functions of M look like

$$\begin{pmatrix} (x,y)^0 \\ (x,y)^1 \\ \cdot \\ (x,y)^L \end{pmatrix} \mapsto \begin{pmatrix} \text{(Some function of} \\ (x,y)^0 \ldots (x,y)^{L-1}) \\ \\ \text{(Some function of } (x,y)^0 \ldots (x,y)^{L-1} \\ + DF^0(x^0)(x^L) \end{pmatrix}$$

Thus, if M is vectorial at degree L we can conclude that M admits the total space of the pullback bundle

$$\begin{array}{c} (q_{L-1})^{-1} TM_0 \otimes (B_L)^0 \\ \downarrow \\ M_{L-1} \longrightarrow M_0 \end{array}$$

as a covering manifold.

Now consider the structure of $M_{L-1} \longrightarrow M_{L-2}$. The transition functions look like

$$\begin{pmatrix} (x,y)^0 \\ (x,y)^1 \\ \cdot \\ (x,y)^{L-1} \end{pmatrix} \mapsto \begin{pmatrix} \text{(some function of} \\ (x,y)^0 \ldots (x,y)^{L-2}) \\ \\ \text{(some function of } (x,y)^0 \ldots (x,y)^{L-2}) \\ + G^0(x^0)(y^{L-1}) \end{pmatrix}$$

where G^0 is the map defined in proposition 5. Because F is a G^∞ diffeomorphism we can conclude that G^0 is an invertible matrix, hence, provided that M is vectorial at degree L-1, there is a vector bundle $E \to M_0$ defined by the $G^0(x^0) \in GL_n(\mathbf{R})$ such that M_{L-1} admits the pull back bundle

$$\begin{array}{ccc} (q_{L-2})^{-1}E \otimes (B_L)^{L-1} & \longrightarrow & E \\ \downarrow & & \downarrow \\ M_{L-2} & \longrightarrow & M_0 \end{array}$$

as a covering manifold. This decomposition continues down the $\mathbf{Z}$-grading, with the DF^0 and the G^0 alternately appearing. We now follow the proof of proposition 13 of chapter 2 to fit these coverings into a large diagram and conclude.

Proposition 6: Let M be a vectorial Rogers supermanifold over B_L, then M admits the total space of the vector bundle

$$\begin{array}{ccccc} q^{-1}E \otimes (B_L)_1 & = & q^{-1}E \otimes (B_L)_1 & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ TM_0 \otimes (\tilde{B}_L)_0 & = & z(M_0) & \longrightarrow & M_0 \end{array}$$

as a covering manifold.

Example: Lot uc roturn to tho notion of projoctivo superspace, which we have defined in previous chapters over B_1 and over $(B_L)_0$. The definition is easily extended to $(B_L)^{m,n}$. Let $e:(B_L)^{m,n} \longrightarrow \mathbf{R}^m$ be the real map. Then we may write

$$\widetilde{X} = (B_L)^{m,n} - e^{-1}(0) = \widetilde{U}_1 \cup \widetilde{U}_2 \cup \ldots \cup \widetilde{U}_m$$

where $\widetilde{U}_i = \{z \in (B_L)^{m,n} : e(z).e_i \neq 0\}$.

We may define an equivalence relation on $\widetilde{X}$ by saying that $(x,y) \sim (x',y')$ if and only if

$(x,y) = (bx',by')$ for some non-zero $b \in (B_L)_0$.

Let X be the quotient of $\widetilde{X}$ by this equivalence relation and let $p:\widetilde{X} \longrightarrow X$ be the quotient map. We can define supercharts on X as follows. Let $U_i = p(\widetilde{U}_i)$ and define

$$f_i : U_i \longrightarrow (B_L)^{m-1,n} \quad \text{by}$$

$$f_i(p(x,y)) = (x_1/x_i, \ldots, \hat{x}_i, \ldots, x_m/x_i, y_1/x_i, \ldots, y_n/x_i),$$

then it can be checked that the corresponding transition functions are G^∞ maps. We denote X by $\mathbf{P}((B_L)^{m,n})$ and call it **Super Projective space of dimension (m-1,n)**. It is clear from the form of the transition functions that the vector bundle E of proposition 6 is simply the direct sum of n copies of the canonical line bundle over projective m-1 space in this case. (See Wells [44]).

Section 4: Embeddings:

In previous chapters we have seen that compact supermanifolds admit no G^∞ embeddings into their model spaces. The purpose of this section is to extend this result to include Rogers supermanifolds, the proof being of the same form as before, differing only in the exact detail. We suggest that the reader refers back to the relevant sections of chapter 1 and 2 for motivation and for more detail.

Definition: Let M be a Rogers supermanifold over B_L and let $i:M\longrightarrow(B_L)^{m,n}$ be a continuous map. We say that i is a **G^∞ embedding** if and only if i is both a C^∞ embedding and a G^∞ map.

Proposition 7: Let M be a compact Rogers supermanifold, then M admits no G^∞ embedding into $(B_L)^{m,n}$ for any (m,n).

proof: As before, we prove this by a sequence of lemmas:

Lemma 1: Let $F:(B_L)^{m,n}\longrightarrow B_L$ be a G^∞ map, then F may be written as

$$F(x,y) = (f_0)^0(x^0) + \nabla(f_0)^0.s(x) + \sum_{i=1}^{n}(f_i)^0.y_i + \text{(higher terms)}.$$

proof: This is simply the z-expansion of F with the lowest order terms at each degree isolated.

Lemma 2: Let M be a compact Rogers supermanifold in which all the leaves of the foliation defined by $\sim_{L-1}$ are compact and let F be a globally defined G^∞ function on M.

(a) If L is even, then Re F is constant.

(b) If L is odd, then the $(f_i)^0$ in the expansion of lemma 1 are zero.

proof: (a) Suppose not, then find an x_0 such that $\nabla(f_0)^0(x_0)$ is non zero and let M_{x_0} be the leaf of the foliation of $\sim_{L-1}$ passing through x_0. M_{x_0} is compact and F^L is a continuous function, thus F^L takes a maximum value on M_{x_0}. By re-choosing coordinates if necessary, we deduce that F^L takes the form

$$v.x^L + u$$

where v is a non zero vector, by hypothesis. As before, this is a contradiction, thus Re F must be constant.

(b) The proof for L odd is of the same form as for L even, the difference being that the top degree term F^L is now of the form $\sum_{i=1}^{n}(f_i)^0(y_i)^L$ + constant when considered as a function defined on a leaf of the foliation defined by $\sim_{L-1}$. (Choose a leaf where any of the $(f_i)^0$ are non zero).

Lemma 3: Let $F:(B_L)^{m,n} \longrightarrow (B_L)^{m',n'}$ be a G^∞ embedding.

(a) If L is even, then Re F is non-constant.

(b) If L is odd then if F is written $F = (F_1, F_2, \ldots, F_{m+n})$ and each of the F_i are expanded as in lemma 1, then the corresponding $(f_i)^0$ are not all zero.

proof: (a) If L is even, then the only functional dependence of F on the top degree term x^L is through terms like $\nabla(f_0)^0.(x_j)^L$ (j refers to the cartesian product). If all these terms are zero, that is Re F is constant, then F cannot be a function of x^L, which is a contradiction.

(b) Similar to (a) noting that the top degree term is now y^L and its functional dependence is on the $(f_i)^0$.

The proof of proposition 7 now follows the lines of proposition 8 of chapter 1, taking care to split the cases according to L odd or even.

Chapter 5: Non-Vectorial Supermanifolds.

In the previous four chapters we have seen that it is possible to classify all vectorial supermanifolds up to coverings, their structure being essentially that of a vector bundle over the core manifold. The two conditions that a supermanifold has to satisfy to be vectorial are essentially that certain fibres have to be complete affine manifolds and that certain bundles have to admit sections. It is well known that not all bundles admit sections, (for example a principal bundle admits a section if and only if it is trivial), so there would seem to be no a priori reason for all supermanifold bundles to admit sections and similarly there are examples of simply connected incomplete flat affine manifolds, (for example ($\mathbf{R}^n$-0) for n>2), so again, there is no reason to suspect that the fibres associated with a general supermanifold admit a complete flat affine structure. What we wish to do then, in this chapter, is to try and build supermanifolds that are non-vectorial, or to prove that, in certain dimensions, they cannot exist for topological reasons. This desire will not be completely fulfilled, since the construction of non-vectorial supermanifolds seems fraught with difficulties (not least that they are usually of quite high real dimension), however, we shall give several examples and indicate the possible course of future research.

The easiest source of non-vectorial supermanifolds is over $(B_1)^{m,n}$. Let us write down the general form of a G^∞ function $F:(B_1)^{m,n} \longrightarrow B_1$

$$
\begin{aligned}
F(x,y) &= F(x^0, y^1 b_1) \\
&= z(f_0)(x^0) + \sum_{i=1}^{n} z(f_i)(x^0) y_i \\
&= f_0(x^0) + \sum_{i=1}^{n} f_i(x^0) y_i b_1
\end{aligned}
$$

which may be written as

$$F(x,y) = f^0(x^0) + f^1(x^0)b_1 + \sum_{i=1}^{n} f_i(x^0) y_i b_1$$

where we have written

$$f_0(x^0) = f^0(x^0) + f^1(x_0)b_1.$$

From this we may conclude that the general form of a G^∞ function $F:(B_1)^{m,n} \longrightarrow (B_1)^{m,n}$ is $F(x^0,y^1) = (f^0(x^0),[f^1(x^0) + A(x^0)y^1]b_1)$, where $A:\mathbf{R}^m \rightarrow \mathrm{Hom}(\mathbf{R}^n,\mathbf{R}^n)$, $f^0:\mathbf{R}^m \longrightarrow \mathbf{R}^m$ and $f^1:\mathbf{R}^m \longrightarrow \mathbf{R}^n$.

From this we may deduce that any fibre bundle with affine transition maps is a completely regular supermanifold over $(B_1)^{m,n}$ for some (m,n). (A fibre bundle $p:B \longrightarrow M$ has affine transition maps if and only if there is a trivializing cover $\{U_i\}$ of M and trivializing maps f_i such that the diagram

$$
\begin{array}{ccc}
p^{-1}(U_i) & \xrightarrow{f_i} & U_i \times F \\
\downarrow p & \swarrow \pi_1 & \\
U_i & &
\end{array}
$$

commutes and where F is an affine manifold and the transition maps $f_j(f_i)^{-1}$ are affine maps when restricted to fibres). Thus we may write down a pair of non vectorial supermanifolds.

Example 1: $\mathbf{R}^n \times (\mathbf{R}^m - 0)$ is a non vectorial supermanifold of dimension (m,n) over B_1, for $m>2$, with core manifold $\mathbf{R}^n$. This is because ($\mathbf{R}^m$-0) is a simply connected, incomplete, flat affine manifold for m>2, thus cannot admit a Euclidean space as a covering manifold.

Example 2: S^3 is a non vectorial supermanifold of dimension (2,1) over B_1, with core manifold S^2. This is because S^3 is the total space of the Hopf fibration $S^3 \longrightarrow S^2$, the fibre being the affine manifold S^1. The vector field generating the S^1 action has integral curves giving the affine structure and the transition functions are affine maps. It is well known that S^3 is simply connected and compact (Spanier [40]), thus it cannot be covered by a vector bundle. It is non-vectorial because it does not admit a section. (It is a non-trivial principal S^1-bundle).

The previous example suggests that if we want to find non-vectorial supermanifolds and in particular supermanifolds not admitting the required sections, we should look for compact simply connected supermanifolds, for these cannot possibly admit vector bundles as covering manifolds, (see Spanier [40] for the theory of covering spaces). Part of the difficulty in finding non-vectorial supermanifolds arises from the fact that there are topological obstructions in low dimensions to such supermanifolds existing. Let us go through these obstructions in turn. We shall deal only with fibre complete supermanifolds.

(i) Simply connected restricted supermanifolds.

Any compact, fibre complete restricted supermanifold over B_l has the form of an n-torus bundle over an n-manifold. This is because the fibre, being a closed subset of a compact hausdorff space is compact and every complete compact affine manifold is a torus (Kobayashi and Nomizu [25]). So let us assume that

$$F \longrightarrow E \xrightarrow{p} B$$

is an n-torus bundle over an n-manifold and that the total space E is simply connected. (We shall assume that all our spaces are connected). The first section of the exact sequence is

$$\cdots \to \underset{\overset{\|}{0}}{\pi_1(E)} \to \pi_1(B) \to \underset{\overset{\|}{0}}{\pi_0(F)} \to \underset{\overset{\|}{0}}{\pi_0(E)} \to \underset{\overset{\|}{0}}{\pi_0(B)} \to \cdots$$

(This is an exact sequence of sets, where relevant). Thus $\pi_1(B) = 0$. Now let us look at the next section of exact sequence.

$$\cdots \to \underset{\overset{\|}{0}}{\pi_2(F)} \to \pi_2(E) \xrightarrow{p_*} \pi_2(B) \to \underset{\overset{\|}{nZ}}{\pi_1(F)} \to \underset{\overset{\|}{0}}{\pi_1(E)} \to \cdots$$

Recall that π_k(n-torus) = 0 for k>1 since its universal cover is $\mathbf{R}^n$. Now the next section.

$$\cdots \to \underset{\overset{\|}{0}}{\pi_3(F)} \to \pi_3(E) \to \pi_3(B) \to \underset{\overset{\|}{0}}{\pi_2(F)} \to \pi_2(E) \to \cdots$$

and so on up the sequence. From this we can conclude that

$$\begin{cases} \pi_1(E) = \pi_1(B) = 0 \\ \pi_2(B)/p^*\pi_2(E) = nZ \\ \pi_k(E) = \pi_k(B) \qquad k>2. \end{cases}$$

Let us now consider the implications of these relations on core manifolds of low dimension.

Lemma: Let M be a simply connected compact manifold, then M is not 1-dimensional and if M is of dimension 2 it is S^2 and if M is 3-dimensional it is homotopy equivalent to S^3.

proof: For dimensions 1 and 2 this is a consequence of the corresponding classification theorems (Spanier [40]). In dimension 3 we have

(Integer coefficients)

$\pi_1(B) = 0$

$\Rightarrow H_1(B) = 0$ (Hurewicz isomorphism)

$\Rightarrow H^1(B) = 0$ (Canonical duality)

$\Rightarrow H_2(B) = 0$ (Poincare duality)

$\Rightarrow \pi_2(B) = 0$ (Hurewicz isomorphism)

and $H_3(B) = \mathbf{Z}$ since M is an orientable compact 3-manifold so $\pi_3(B) = \mathbf{Z}$ again by the Hurewicz theorem. Whiteheads theorem now implies that M is homotopy equivalent to S^3 (Spanier [40]).

This immediately implies the following:

Proposition 1: There are no fibre complete, simply connected, compact restricted supermanifolds of dimension 1,2 or 3 over B_1.

If we look at fibre complete restricted supermanifolds over B_L for L>1, The same conclusion is reached, since the fibre is now a $2^{L-1}n$-torus. We can, however, deduce even more.

Proposition 2: There are no fibre complete, compact, simply connected restricted supermanifolds over B_L for $L>1$ provided the affine connection induced on the core manifold by the affine transition functions is complete. (This extra condition can be dropped for dimensions 1,2 and 3).

proof: If this is the case then the core manifold is a torus and thus all homotopy groups above the first disappear contradicting the conditions given above.

We can deduce, for example, that the fundamental group of a compact, fibre complete, restricted supermanifold over B_1 must be infinite, lest the simply connected covering space be compact. (In dimensions 1,2,3)

(ii) Simply connected even supermanifolds:

The same sort of analysis can be applied to fibre complete even supermanifolds, the difference being that the fibre of the map $M \rightarrow \tilde{M}$ need not be a torus, but a torus bundle over a torus or a torus bundle over a torus bundle over a torus etc... . Even so it is an easy application of the exact sequence of a fibration that the fibre has the same homotopy type as a torus. From this we can conclude:

Proposition 3: There are no compact, simply connected, fibre complete even supermanifolds of dimension 1,2 or 3 over $(B_L)_0$.

Remark: Similar constraints apply to (m,n) dimensional supermanifolds.

Thus the difficulty with finding supermanifolds that are non-vectorial because they do not admit the required sections is that in low dimensions they are either non-compact or their fundamental group is large. If we drop the requirement of compactness, though, there is an obvious candidate for a fibre complete, simply connected restricted supermanifold not admitting the tangent bundle of the core manifold as a covering, of dimension 2 over B_1, namely $S^3 \times \mathbf{R}$ considered as a cylinder bundle over S^2. This satisfies all the topological requirements to be a supermanifold, including the requirement that the tangent bundle splits as the direct sum of two copies of some vector bundle. (In fact, any line bundle and any 2-plane bundle over S^3 is trivial, see Husemoller [23] for example, thus the tangent line bundle 1 to the Hopf foliation of S^3 is trivial as is any complementary 2-plane bundle. Take the product of 1 with the trivial line bundle over $\mathbf{R}$ to get the required splitting of $T(S^3 \times \mathbf{R})$). It is not clear however that this is sufficient to get a supermanifold structure on $S^3 \times \mathbf{R}$. So we make the following:

Conjecture: $S^3 \times \mathbf{R}$ admits a supermanifold structure.

It is easier to generate examples of non-fibre complete restricted, even and Rogers supermanifolds.

Example 3: $\mathbf{R} \times (\mathbf{R}^{2^{L-1}-1} - 0)$ is an even, non fibre complete supermanifold of dimension 1 for L>0. Simply embed the above as an open subset of $(B_L)_0$ and use the single induced chart. Similar examples exist for restricted supermanifolds and Rogers supermanifolds.

Our final example is of a non-vectorial supermanifold over a non-simply connected core manifold.

Example 4: $\mathbf{T}^2 \times S^1 \times \mathbf{R}$ admits a restricted G^∞ supermanifold structure with the Klein bottle K as core manifold, where T is the 2-torus. This supermanifold cannot be vectorial since there are no injective homomorphisms from the fundamental group of T(K) to the fundamental group of $T^2 \times S^1 \times \mathbf{R}$, thus T(K) cannot cover $T^2 \times S^1 \times \mathbf{R}$. We define the supermanifold structure as follows. Let $T(T^2) = T^2 \times \mathbf{R}^2$ have the canonical supermanifold structure, that is, as the z-thickening of T^2. Take coordinates (x,v_1,v_2) on $T^2 \times \mathbf{R}^2$ and let X be the open subset of $T^2 \times \mathbf{R}^2$ defined by $-2<v_1<2$. Then clearly X is a supermanifold, with core T^2. If we take coordinates (a,b) on the torus then the map $T \rightarrow T$ given by $(a,b) \mapsto (a+180°, b+180°)$ is homotopic to the identity map. (The quotient of T by this map is the Klein bottle). Define a map $F:U_1 \rightarrow U_2$, where U_1 and U_2 are the open subsets of X defined by $1<v_1<2$ and $-2<v_1<-1$ respectively, as follows.

$$F(a,b,v_1,v_2) = (a+180°, b+180°, v_1-3, v_2)$$

Then this map is G^∞ and X glued together by F is a G^∞ supermanifold with core manifold K, (the effect of the map is to identify opposite fibres) and as F is homotopic to the identity, this supermanifold is diffeomorphic to $T^2 \times S^1 \times \mathbf{R}$.

Thus there are non-trivial examples of non-vectorial supermanifolds for both of the reasons stated in the introduction to this chapter. We feel that is would be of interest to know whether there are any compact simply connected supermanifolds and if so, what is the minimum dimension at which they appear. It would also be useful to be able to generate examples of non-vectorial supermanifolds in some systematic fashion, with the same sort of ease with which we can generate vectorial supermanifolds.

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