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Is there a paradox of pledgeability? *

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Abstract

Donaldson, Gromb and Piacentino (2019) suggest that, in the presence of limited commitment, increasing the fraction of a firm's cash flows that can be pledged as collateral might make the firm worse off. We show that, in fact, firms can never be hurt by increased pledgeability of cash flows in their framework. We then show that the first best can always be implemented by non-state contingent collateralized debt contracts that differ from the ones they consider.

Key words: Collateral, Secured debt, Pledgeability

JEL classification: G21, G32, G33, G38

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Donaldson, Gromb and Piacentino (2019) (henceforth DGP) develop a dynamic model in which collateral (i) provides property rights that accrue to a secured creditor upon default, and (ii) gives an initial creditor the right of exclusion, preventing a subsequent creditor from seizing the collateral. DGP make the point that an initial creditor must rely on collateral to secure its claims when enough assets can be pledged as collateral. This is because if the initial creditor does not collateralize at least partially, it is too easy for a future borrower to fund new (possibly negative NPV) projects using collateralized credit that dilutes the claims of the initial creditor.

DGP identify what they term an inefficient *collateral rat race* that ensues when only a fraction of the firm's assets can be pledged as collateral. In this case, they argue that the demand for collateral from the initial creditors can be so high that it encumbers the assets, creating a collateral overhang that may inefficiently constrain future borrowing and investments. Their abstract highlights that: "*Our results suggest that policies aimed at increasing the supply of collateral can backfire, triggering an inefficient collateral rat race*." They also provide a motivating example in which they discuss two scenarios: a low-pledgeability case, in which the first best is implemented, and a high pledgeability case in which—supposedly—it is not. This suggests that, paradoxically, increasing the share of cash flows that a firm can pledge as collateral can make it worse off.

Our paper shows that, in fact, firms can never be hurt by having access to more pledgeable cash flows in DGP's setting. The result is intuitive, it extends beyond their two-state setting, and its validity does not require any of their parametric assumptions. To see the logic, consider the effect of a positive shock to a firm's pledgeable assets. Regardless of whether the firm was investing efficiently before, in the absence of informational asymmetries, having access to more collateral has an option value that cannot hurt. The firm can always fully offset the shock by issuing more secured debt at the outset, so as to keep the amount of collateral available for future investments constant, in which case the allocation implemented will be the same as before. Moreover, the firm might do better, if the availability of collateral previously constrained investment.

We complete our analysis by showing that alternative (non-state-contingent) collateral contracts can implement the first best for all parameter values. Our findings suggest that future investigations of the conditions under which pledgeability might hurt a firm should explicitly consider informational asymmetries between the firm and its investors.

In an extension, DGP relax the equivalence between pledgeable and collateralizable assets assumed in their core model. Specifically, Section 4.7 assumes that a fraction of pledgeable assets cannot be used as collateral. DGP then argue that high collateralizability may be associated with underinvestment. In Appendix 1 we show that this requires the cash flows of negative NPV projects to be more collateralizable than those of positive NPV ones. If the fraction of cash flows that is collateralizable is project-independent—as is assumed to hold for pledgeability—then increased collateralizability can only help a firm.

The motivating example. To provide intuition, DGP first give an example in which a firm requires external debt finance to pursue investment projects at dates 0 and 1, where the date 0 project has a positive NPV, but the date 1 project has a negative NPV. They show that when the fraction θ of cash flows that can be pledged is low then unsecured debt can be used to finance the positive NPV date 0 project, as it does not leave enough pledgeable assets to fund the negative NPV date 1 project.

In this example, the positive NPV date 0 project costs 200, and pays 600 at date 2. The negative NPV date 1 project costs 500, and pays 400 at date 2. When pledgeability is low (e.g., $\theta = \frac{2}{5}$) the date 0 project can be funded with unsecured debt without being concerned that the date 1 project will be undertaken, because there is not enough total pledgeable cash flow to cover its cost: $\frac{2}{5}(600+400) = 400 < 500$. When pledgeability rises to $\theta = \frac{1}{2}$, the date 0 project must be financed using some secured debt, as now the total pledgeable cash flow covers the cost of the date 1 project: $\frac{1}{2}(600+400) = 500$. As a result, a date-1 creditor C_1 would be willing to lend if the date 0 project were funded with unsecured debt. However, DGP observe that if at date 0 the firm issued *fully* secured debt—i.e., debt fully backed by collateral—then at date 1 the inefficient investment is prevented. Because the date 0 debt is riskless, competition in the credit market pushes its face value to 200. Thus, debt can be backed by $\sigma = \frac{2}{3}$ of project 0's pledgeable cash flow, as $\frac{2}{3}\frac{1}{2}600 = 200$. Once the date-0 creditor C_0 has priority, a date-1 creditor C_1 is unwilling to lend.

DGP then ask, "*But what if project 1 were unexpectedly good, with payoff 550?*" This payoff exceeds its 500 cost—can it be financed? The answer is that, when $\theta = \frac{1}{2}$ *and* $\sigma = \frac{2}{3}$, there is not enough pledgeable cash flow net the repayment to a date-1 creditor to cover its cost: $\frac{1}{2}(600 + 550) - 200 = 375 < 500$. By fully collateralizing project 0, the borrower cannot pledge enough to finance its positive NPV project at date 1. That is, the collateral overhang results in an inefficient outcome. As DGP put it: *"By collateralizing its debt to C_0, [the Borrower] B has encumbered its assets and cannot pledge enough to C_1 to finance a positive NPV project. There is a collateral overhang."*

This presentation suggests that inefficient investment might be an equilibrium outcome. In fact, it is not. Full collateralization of $\sigma = \frac{2}{3}$ of the pledgeable cash flows from project 0 is not needed to discourage a date 1 lender from funding the negative NPV project. Indeed, securing any fraction $\sigma' \in (0, \frac{1}{4}]$ achieves the optimum: (i) it prevents investment if the date 1 project has a negative NPV, because $\frac{1}{2}((1-\sigma')600+400) < 500$; and (ii) it enables investment if the date 1 project has a positive NPV, because $\frac{1}{2}((1 - \sigma')600 + 550) \geq 500$. As a result, *the collateral rat race has no effect on efficiency* in this example; in equilibrium, a higher rate of pledgeability θ does not make the firm worse off.

This result reflects that the conditions for an inefficient outcome (see DGP's *Corollary 2*) are not satisfied in the example.[1] So, in this example a paradox in which increasing pledgeability hurts a firm does not arise. Our paper shows that it never does.

Setup. There are three dates ($t = 0, 1, 2$) and one consumption good dubbed *cash*. A borrower B has no cash but has access to two investment projects: one at $t = 0$ and one at $t = 1$. The date 0 project requires investment $I_0 > 0$ at $t = 0$ to generate X_0 for sure at $t = 2$. At date 1, a state $s \in \{H, L\}$ realizes, where $p := \Pr[s = H]$ is the probability of a high state. In state s, B can invest in a project that requires borrowing I_1^s and delivers X_1^s for sure at date 2. The state L project has a negative NPV and is inefficient to fund, while the state H project has a positive NPV and is efficient to fund.

B can raise financing from a competitive credit market at each date. In particular, B can make a take-it-or-leave-it offer to a set of competitive financiers at dates $t = 0, 1$. There is no discounting, all agents are risk neutral, they consume only at date 2 and there are no informational asymmetries at any date. The final cash flow is $X := i_0 X_0 + i_1^s X_1^s$, for $i_0 \in \{0, 1\}$ and $i_1^s \in \{0, 1\}$. Here, $i_0 = 0$ means that there is no investment at date 0, and $i_1^s = 0$ means that there is no investment at date 1 in state s.

There are two frictions. First, a fraction $1 - \theta$ of the final cash flow X is not pledgeable. That is, B can always divert this fraction of the final project payoffs. Second, at date 0, B cannot credibly constrain its future investment and financing actions—a form of limited commitment that DGP term 'non-exclusivity'.

In addition, DGP exogenously constrain the set of admissible contracts. Specifically, DGP assume that if a security is backed by a fraction σ of the pledgeable cash flow θX, so that the value of the collateral is $\sigma\theta X$, then that fraction σ cannot depend on the state of the world s, even though this state is observable and verifiable by all parties ex post. When $\sigma = 1$, all pledgeable cash flows θX are used as collateral.

Assumptions. DGP impose five restrictions on model parameters:

A1. Project 0's pledgeable payoff in state L alone exceeds its investment cost: $I_0 < (1 - p)\theta X_0$, which implies that $I_0 < X_0$. Thus, a creditor is willing to lend at date 0 if she anticipates no dilution in state L.

[1] A necessary condition detailed in the corollary for an inefficient outcome is that the investment required by the positive NPV date 1 project (i.e., 500) must exceed that required by the negative NPV date 1 project (also 500) *plus* θ times the difference in the two projects' cash flows $550 - 400 = 150$. Thus, for an inefficient outcome to arise, one needs $500 \geq 500 + \theta 150$, which is violated by all $\theta > 0$.

A2. Project 1's NPV is positive in state H but not in state L: $X_1^H - I_1^H > 0 > X_1^L - I_1^L$.

A3. The pledgeable cash flow fails to cover the investment needed at date 1 in all states: $\theta(X_0 + X_1^s) < I_0 + I_1^s$, $\forall s$. Thus, B may be unable to fund Project 1 in state H.

A4. Project 1's non-pledgeable payoff is not too small: $(1-\theta)X_1^L > \theta X_0 - I_0$. Thus, B has an incentive to undertake Project 1 even in the negative NPV state L.

A5. Project 1's cost is not too high: $I_1^H < \theta(X_0 + X_1^H)$. That is, there is enough pledgeable total cash to fund Project 1 in the positive NPV state H.

DGP then observe that for a *collateral rat race* to result in a *collateral overhang*—that is, an inefficient outcome—two further conditions need to be met:

A6. The pledgeable cash flows are high enough to fund Project 1 in the negative NPV state L: $\theta \geq \theta^* := \frac{I_1^L}{X_0 + X_1^L}$. Thus, a date-0 creditor is not willing to lend unsecured.

A7. Project 1's cost in state H is large enough: $I_1^H \geq I_1^*(\theta) := I_1^L + \theta(X_1^H - X_1^L)$. Thus, the date 0 collateralization demand makes financing Project 1 in state H impossible.

Two preliminary results. A2 asserts that the project has positive a NPV in state H, but a negative NPV in state L, making the problem interesting. Lemma 1 shows that if A2 holds, then A7 can be satisfied by some θ only if $X_1^H > X_1^L$, which we henceforth assume.

Lemma 1. *If $X_1^H \leq X_1^L$, then A7 and A2 do not simultaneously hold for any $\theta \in [0,1]$.*

Proof. If $X_1^H < X_1^L$, then $I_1^H \geq I_1^* \iff \theta \geq \frac{I_1^L - I_1^H}{X_1^L - X_1^H}$. The condition can be satisfied by some $\theta \in [0,1]$ only if $\frac{I_1^L - I_1^H}{X_1^L - X_1^H} \leq 1$, or, equivalently only if $I_1^L - X_1^L \leq I_1^H - X_1^H$. However, from A2, $I_1^L - X_1^L > 0 > I_1^H - X_1^H$, which yields a contradiction. Finally, if $X_1^H = X_1^L = X_1$ then A7 reads $I_1^H \geq I_1^L$. However, A2 requires $I_1^H < X_1 < I_1^L$, a contradiction. □

By Lemma 1, A7 can be rewritten as an *upper bound* on pledgeability θ required for a collateral overhang (i.e., an inefficient outcome) to arise:

A7′. $\theta \leq \hat{\theta} := \frac{I_1^H - I_1^L}{X_1^H - X_1^L}$.

We next show that Assumptions A1, A3, A4, and A5 can be rewritten as representing an upper and a lower bound on the set of feasible pledgeability levels θ:

Lemma 2. *Conditions A1, A3, A4 and A5 can be rewritten compactly as* $\theta \in (\underline{\theta}, \bar{\theta})$*, where*

$$\underline{\theta} := \max\left\{\frac{I_0}{(1-p)X_0}\,,\;\frac{I_1^H}{X_0 + X_1^H}\right\} > 0 \quad \textit{and} \quad \bar{\theta} := \min\left\{\frac{I_0 + I_1^H}{X_0 + X_1^H}\,,\;\frac{I_0 + X_1^L}{X_0 + X_1^L}\right\} < 1. \tag{1}$$

Proof. From A2, $X_1^L < I_1^L$, so A4 implies that A3 never binds in state $s = L$. We then rewrite A1 as $\theta > \frac{I_0}{(1-p)X_0}$, A3 as $\theta < \frac{I_0+I_1^H}{X_0+X_1^H}$, A4 as $\theta < \frac{I_0+X_1^L}{X_0+X_1^L}$ and A5 as $\theta > \frac{I_1^H}{X_0+X_1^H}$. That $\underline{\theta} > 0$ and $\bar{\theta} < 1$ follows immediately from A1-A5. $\square$

Equilibrium allocation and implementation. DGP summarize the equilibrium outcomes and their implementation using secured and unsecured debt in *Corollary 2*:

> *Corollary 2 (DGP). The equilibrium outcome is as follows.*
>
> *If* $\theta < \theta^*$*, the first best is attained. At Date 0, B borrows unsecured. At Date 1, B borrows secured in state H and does not borrow in state L.*
>
> *If* $\theta \geq \theta^*$ *and* $I_1^H < I_1^*$ *the first best is attained. At Date 0, B borrows partially secured. At Date 1, B borrows secured in state H and does not borrow in state L.*
>
> *If* $\theta \geq \theta^*$ *and* $I_1^H \geq I_1^*$*, the first best is not attained due to the collateral rat race and the collateral overhang. At Date 0, B borrows secured with face value* I_0*. At Date 1, B does not borrow in state H or state L.*

The third case, where a collateral overhang arises, provides the foundation of DGP's contribution, as detailed in their abstract: "*Creditors thus require collateral for protection against possible dilution by collateralized debt. There is a collateral rat race. But collateralized borrowing has a cost: it encumbers assets constraining future borrowing and investment. There is a collateral overhang. Our results suggest that policies aimed at increasing the supply of collateral can backfire, triggering an inefficient collateral rat race.*"

While such statements throughout the paper emphasize the inefficiency of increasing pledgeability, DGP's *Proposition 1* proposes a weaker notion of a paradox: *"If* $\theta < \theta^*$*,* C_0 *[the date-0 creditor] lends unsecured and the first best is attained; [...] if* $\theta \geq \theta^*$ C_0 *does not lend unsecured."* There are two important points to make about this result. First, when $\theta \geq \theta^*$, secured lending may well implement the first best; when this is so, higher pledgeability of cash flows does not hurt the firm and is irrelevant. Second, when $\theta < \theta^*$, creditors do not need to lend unsecured: a continuum of partially-secured loans can implement the same equilibrium outcome—i.e., the first best. To highlight this point, we show that regardless of the degree of pledgeability θ, the *same* partially unsecured debt contract can implement the first best—whenever it can be attained. Adopting this alternative implementation, we rewrite Corollary 2 in terms of $\hat{\theta} = \frac{I_1^H - I_1^L}{X_1^H - X_1^L}$ rather than θ^* and I_1^*:

Corollary 2 (content restated). The equilibrium outcome is as follows.

If $\theta \leq \hat{\theta}$, then the date-0 creditor C_0 lends fully secured and there is no investment at date 1, regardless of whether the state is high or low.

If $\theta > \hat{\theta}$, then C_0 lends partially secured, with collateral σ set such that $\theta(1-\sigma)X_0 = I_1^H - \theta X_1^H$. B borrows (secured) at date 1 in state H and does not borrow in state L, so the first best attains.

Proof. Rewrite A7 as $I_1^H \geq I_1^L + \theta(X_1^H - X_1^L)$, and rewrite A5 as $I_1^H < \theta(X_0 + X_1^H)$. It follows that, A7 can hold only if $\theta(X_0 + X_1^H) > I_1^L + \theta(X_1^H - X_1^L)$. Rewriting this as $\theta > \frac{I_1^L}{X_0 + X_1^L} = \theta^*$, it is clear that A6 always holds. As a result, the case where $\theta \geq \theta^*$ and $I_1^H \geq I_1^*$ corresponds to $\theta \leq \hat{\theta}$. Next, note that a partially-secured debt contract with collateral σ set to equate $\theta(1-\sigma)X_0 = I_1^H - \theta X_1^H$ always implements the first best, when $\theta < \hat{\theta}$. This contract enables the financing of the good project, as the good project requires exactly $\theta(1-\sigma)X_0$, which is the remaining collateral from the date 0 project. Moreover, the contract prevents the financing of the bad project at date 1, as the bad project requires $I_1^L - \theta X_1^L > I_1^H - \theta X_1^H = \theta(1-\sigma)X_0$, where the first inequality follows from the fact that $I_1^L - \theta X_1^L > I_1^H - \theta X_1^H \iff \theta > \hat{\theta}$. It follows from A1 that a date 0 lender breaks even under such a contract, establishing the equivalence argument. □

This restatement of Corollary 2 based on Lemma 1 and A7′ clarifies that borrowers never realize a gain from being able to lend unsecured in DGP's framework.

For a paradox to arise in which increased pledgeability of cash flows results in an inefficient collateral rat race, there must be a firm for which there is a low $\theta < \theta^*$ at which the first best is implemented, while at a higher $\theta' \geq \theta^*$ it is not. Theorem 1 establishes that, in DGP's framework, this is impossible: greater pledgeability can only improve real investment efficiency, thereby helping borrowing firms.

Theorem 1. *There is no 'paradox': firm value weakly increases with pledgeability θ.*

1. *If $\theta^* \geq \bar{\theta}$, then the first best is implemented for every θ;*
2. *If $\theta^* \in (\underline{\theta}, \bar{\theta})$, then $\theta > \hat{\theta}$ for all $\theta \in (\underline{\theta}, \bar{\theta})$, so the first best always obtains;*
3. *If $\theta^* \leq \underline{\theta}$, then there are three sub-cases:*
   (a) *If $\hat{\theta} \leq \underline{\theta}$, then the first best is implemented for every θ;*

(b) *If $\hat{\theta} \geq \bar{\theta}$, then the first best is* never *implemented for any θ;*

(c) *If $\hat{\theta} \in (\underline{\theta}, \bar{\theta})$, the first best is not implemented for $\theta \leq \hat{\theta}$, while it is for $\theta > \hat{\theta}$.*

Proof. Case 1. Follows from *Corollary 2* (DGP): if $\theta^* \geq \bar{\theta}$, then A6 is violated for all θ.

Case 2. Rewrite A7 as: $I_1^H - \theta X_1^H > I_1^L - \theta X_1^L$. If $\theta^* \in (\underline{\theta}, \bar{\theta})$, there must exist some degree of pledgeability $\theta \in (\underline{\theta}, \theta^*)$. From A6, $\theta < \theta^*$ if and only if $\theta < \frac{I_1^L}{X_0 + X_1^L}$. Because $X_0 + X_1^L > 0$, we can rewrite this inequality as $I_1^L - \theta X_1^L > \theta X_0$. Moreover, A5′ requires $\theta > \frac{I_1^H}{X_0 + X_1^H}$, which we rewrite as $\theta X_0 > I_1^H - \theta X_1^H$. Combining the two inequalities yields $I_1^L - \theta X_1^L > \theta X_0 > I_1^H - \theta X_1^H$, which contradicts A7. Finally, if A7 is violated for $\theta < \theta^*$, then it follows that it is also violated for every $\theta' > \theta^*$, concluding the proof for Case (2).

Case 3a. Follows from *Corollary 2* (DGP): if $\hat{\theta} \leq \underline{\theta}$, A7 is violated $\forall\, \theta$.

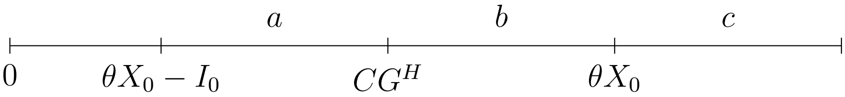
Case 3b. Follows from *Corollary 2* (DGP): if $\hat{\theta} \geq \bar{\theta}$ and $\theta^* \leq \underline{\theta}$, A6 and A7 hold $\forall\, \theta$.

Case 3c. Follows from the fact that when $\hat{\theta}$ is interior and $\theta^* \leq \underline{\theta}$, then A6 and A7 jointly hold for a low $\theta \leq \hat{\theta}$ (in which case we do not get first-best), while A7 is violated for every $\theta > \hat{\theta}$ (in which case we get first-best). □

Theorem 1 shows that one cannot make a firm better off by reducing the pledgeability θ of its cash flows. The key condition used in the proof is A5, which states that there is enough pledgeable cash in the high state at date 1 to invest in the positive NPV project if date-0 creditors lend unsecured. The proof establishes that if the threshold θ^* satisfies conditions A1-A5 (i.e., if $\theta^* \in (\underline{\theta}, \bar{\theta})$), then it is not possible for A7 to hold. Thus, $\theta^* \in (\underline{\theta}, \bar{\theta})$ is incompatible with the final case in Corollary 2, where greater pledgeability can possibly hurt a firm. Relaxing A5 would not change the result. If θ is so low that A5 does not hold, then it would be impossible to finance the positive NPV project regardless of the date 0 contract, rendering the problem uninteresting. Moreover, increasing pledgeability to a level that satisfies A5 could only make the firm better off.

Graphical argument and generalization. We now show that Theorem 1 is driven by the fundamental economic forces of the model, and that it extends beyond DGP's setting. To this end, we present a graphical proof of Theorem 1, which uses the representation of the problem in Figure 1. Define the *collateral-gap* in state s to be: $CG^s := I_1^s - \theta X_1^s$, for $s \in \{L, H\}$. This quantity describes the shortfall of collateral in state s at date 1. We rewrite A3 as $CG^s > \theta X_0 - I_0$, $\forall s$, while A5 reads $CG^H < \theta X_0$. As our proof to the alternative statement of Corollary 2 shows, inefficiencies can arise only if $I_1^H \geq I_1^*$, which we write as $CG^L \leq CG^H$. In such a case, A6 always holds as $\theta \geq \theta^* \iff CG^L \leq \theta X_0$.

Figure 1: The Collateral-Gaps Argument



Given DGP's assumptions, $CG^L \in \{a, b, c\}$ in Figure 1. For a collateral overhang to arise for some $\theta \in [0,1]$, by Lemma 1 it must be that $X_1^H > X_1^L$. It follows that $\frac{\partial CG^H}{\partial \theta} = -X_1^H < \frac{\partial CG^L}{\partial \theta} = -X_1^L$. That is, as pledgeability θ rises, CG^H shifts to the left (i.e., it falls) faster than CG^L. Recall from DGP's characterization that whenever $CG^L \in \{b, c\}$, we are at first-best. If $CG^L \in a$, there is an inefficient collateral overhang—no date-1 project is funded, regardless of its NPV. Because $\frac{\partial CG^H}{\partial \theta} < \frac{\partial CG^L}{\partial \theta}$, if we start from $CG^L \in a$ and increase θ to transition to a different region, then the transition must be to $CG^L \in b$. In this case, we now implement the first best for such a θ. Moreover, $CG^L \in b$ is an absorbing state: once we enter it for some θ, we stay there for every larger θ. Thus, if $CG^L \in a$, then efficiency can only *increase* as θ rises.

This graphical argument suggests that the beneficial role played by greater pledgeability of cash flows should extend beyond DGP's setting. To show this, we relax the structure of assumptions A1-A5, and allow for an arbitrary number of date-1 projects.

We now consider any finite number of date-1 states, indexed by $s \in 1, 2, ..., N$ and characterized by I_1^s and X_1^s. Without loss of generality, order states by NPV so that if $X_1^s - I_1^s > X_1^{s'} - I_1^{s'}$ then $s > s'$. To start, we prove a slight generalization of our Lemma 1:

Lemma 3. *Consider any two projects s and s' with $X_1^s - I_1^s > X_1^{s'} - I_1^{s'}$ and $CG^s > CG^{s'}$ for some $\theta \in [0,1]$. Then $X_1^s > X_1^{s'}$ and $I_1^s > I_1^{s'}$.*

Proof. Rewrite $X_1^s - I_1^s > X_1^{s'} - I_1^{s'}$ as $I_1^{s'} - I_1^s > X_1^{s'} - X_1^s$, and rewrite $CG^s > CG^{s'}$ as $I_1^{s'} - I_1^s < \theta(X_1^{s'} - X_1^s)$. The two inequalities jointly hold only if $X_1^{s'} - X_1^s < \theta(X_1^{s'} - X_1^s)$. Since $\theta \in [0,1]$, both sides of this inequality must be negative, which implies that $X_1^{s'} < X_1^s$. This and $CG^s > CG^{s'}$ further imply that $I_1^{s'} < I_1^s$. □

We now generalize Theorem 1 to show that pledgeability can never hurt a firm in any $N-$state setting, regardless of whether assumptions A1-A5 hold or not.

Theorem 2. *In our $N-$state setting, firm value weakly increases with pledgeability θ.*

Proof. For greater pledgeability of cash flows to reduce the efficiency of the equilibrium allocation, there must exist at least one pair of states $s > s'$ and pledgeability levels $\theta > \theta'$ such that: (1) $CG^s(\theta) > CG^{s'}(\theta)$, and (2) $CG^s(\theta') < CG^{s'}(\theta')$. From Lemma 3, for (2) to hold for some θ and $s > s'$, we must have $X_1^s > X_1^{s'}$. First, $CG^s(\theta) > CG^{s'}(\theta)$ holds if and only if $I_1^s - I_1^{s'} > \theta(X_1^s - X_1^{s'})$. Second, $CG^s(\theta') < CG^{s'}(\theta')$ holds if and only if $I_1^s - I_1^{s'} < \theta'(X_1^s - X_1^{s'})$. The conditions jointly hold only if $\theta' > \theta$, a contradiction. □

Investment at date 1 as an 'excuse to dilute'. In DGP's setting, the possibility of dilution is tightly linked to investment at date 1. In particular, a borrower is not allowed to dilute date-0 creditors unless it invests in a new project.[2] One may wonder whether this assumption is critical in sustaining the beneficial role played by pledgeability. To see that it is not needed, first note that if a borrower can freely dilute the date-0 creditors, then lending at date 0 requires full collateralization. Thus, at date 1, the borrower has access to collateral of $\theta X_0 - I_0$. By A3, $I_1^s - \theta X_1^s > \theta X_0 - I_0 > 0$, so there is never enough collateral left to finance a project at date one. Beyond A3, the borrower never has an incentive to finance negative NPV projects when he is free to dilute, as the option of diluting without 'burning cash' is always more desirable. Moreover, increasing θ so that A3 ceases to hold makes the borrower better off, enabling the financing of positive NPV projects.

Optimal non-state-contingent contracts. Thus far, we have restricted attention to the family of debt contracts considered by DGP. However, DGP note that state-contingent collateralization—i.e., making the fraction of secured output σ a function of the state s—can always implement the first best: "*We have assumed away state-contingent collateralization. Were it possible, it could circumvent the inefficiencies arising in our analysis.*"

We conclude by establishing that state-contingent collateral is not needed to implement the first best. Lemma 1 showed that inefficiencies arise only when $X_1^H > X_1^L$ and $I_1^H > I_1^L$. This strict difference in investment levels across states, which is *needed* to generate inefficiencies, gives the borrower a simple, non-state-contingent instrument to implement the first best. Proposition 1 shows that reducing collateral demands when the firm's rate of investment is sufficiently high at date 1 always implements the first best.

Proposition 1. *Under A1-A7, the first best can be implemented by borrowing (partially) secured at date 0, with a collateral discount if B invests more than $\hat{I} \in [I_1^L, I_1^H)$ at date 1. For instance, B can issue debt with face value D_0 and collateral rate $\sigma_0(I_1)$ at date 0, where $\sigma_0(I_1) = 1$ if $I_1 \leq \hat{I}$, $\sigma_0(I_1) = 0$ if $I_1 > \hat{I}$, for $\hat{I} \in [I_1^L, I_1^H)$.*

Proof. See Appendix 2. □

[2] We thank an anonymous referee for pointing this out to us.

References

Donaldson, J. R., D. Gromb, and G. Piacentino, "The paradox of pledgeability," *Journal of Financial Economics*, 2019.

Appendices

Appendix 1: Pledgeability vs. Collateralizability

In Section 3.2, DGP use the term *'collateralizability'* to describe a property of **all** pledgeable cash flows, as is standard in the literature. Specifically, creditors can secure as collateral any fraction $\sigma \in [0,1]$ of the pledgeable cash flows θX. Later, in Section 4.7, DGP introduce a distinction between *'pledgeable'* and *'collateralizable'* assets, arguing that some pledgeable assets might not be usable as collateral. They redefine collateralizable assets as a fraction $\mu \in [0,1]$ of the pledgeable cash flows θX that can be used as collateral—i.e., only for these assets can property rights be assigned to an individual creditor. In contrast to pledgeable cash flows, which are a given fraction θ of the firm's cash flows, collateralizable assets are introduced with a time-specific index μ_t.

In their Proposition 4, DGP assume that p is small 'enough' and state that *'If $\mu_1 > \mu_1^*$, B does not invest at Date 0 or Date 1'*, where the threshold μ_1^* solves

$$(1+\mu_1^*)\theta X_1^L + (1-\mu_0)\theta X_0 = 2I_1^L. \tag{2}$$

The result suggests that high collateralizability may hurt a firm. One might wonder whether Proposition 4 delivers an alternative 'paradox of collateralizability'. We now clarify that this is not so. To illustrate, suppose that, like pledgeability θ, the fraction of collateralizable assets is independent of the specific project, so that $\mu_t = \mu$ for all t. If $X_1^L > X_0$, then using equation (2), the condition $\mu > \mu_1^*$ can be written as $\mu \geq \frac{2I_1^L - \theta(X_0 + X_1^L)}{\theta(X_1^L - X_0)}$. There exists some μ such that $\mu > \mu_1^*$ only if $\frac{2I_1^L - \theta(X_0+X_1^L)}{\theta(X_1^L - X_0)} < 1$, or, equivalently, if $\theta X_1^L > I_1^L$. However, this violates A2, which requires the bad project to have a negative NPV. Similarly, if $X_1^L = X_0$, re-arranging the condition again yields that $\mu > \mu_1^*$ if and only if $0 > I_1^L - \theta X_1^L$, violating A2.

The remaining case of $X_1^L < X_0$ is more interesting. The condition $\mu > \mu_1^*$ can be written as: $\mu < \frac{\theta(X_0 + X_1^L) - 2I_1^L}{\theta(X_0 - X_1^L)}$, revealing that when $\mu_1 = \mu_0$, contrary to what a 'paradox of collateralizability' would require, the inefficient date-0 under-investment detailed in Proposition 4 arises **only** when collateralizability is sufficiently **low**. To see the intuition, consider equation (11) in DGP with $\sigma_0 = \mu_0 = \mu_1 = \mu$: $(1+\mu)\theta X_1^L + (1-\mu)\theta X_0 \geq 2I_1^L$. This equation details the conditions under which B would borrow at date 1 with a bad project. When μ increases, the right-hand side does not change. However, the derivative of the left-hand side with respect to μ is $\theta(X_1^L - X_0) < 0$. Thus, increasing μ makes this condition **harder** to satisfy. As a result, when collateralizability is not project-specific, there is no paradox, and increasing μ is beneficial.[3] Proposition 4 effectively says that *a disproportionally higher collateralizability of the negative NPV date-1 project, relative to the positive NPV date-0 project, can encumber a firm's assets.* This unsurprising result extends immediately to Corollary 3, where DGP take the analogous derivative with respect to μ_1, leaving μ_0 fixed.

Appendix 2: Proof of Proposition 1

Suppose B offers a contract $(D_0, \sigma_0(I_1))$ at date 0, where $\sigma_0(I_1) = 1$ if $I_1 \leq \hat{I}$, $\sigma_0(I_1) = 0$ otherwise, and $\hat{I} \in [I_1^L, I_1^H)$. For simplicity, we use DGP's formulation that $t = 0$ creditors only secure revenues from Project 0 as collateral. B cannot raise funds when $s = L$, because $\theta X_1^L < I_1^L$ by A1 and A3. In contrast, B can borrow when $s = H$ only if $\theta(X_0 + X_1^H) \geq I_1^H$, which holds by A5. When $s = H$, creditors are willing to lend only if $I_1^H \leq D_1$. Optimization by B means that this constraint binds, i.e., $D_1^* = I_1^H$. Conjecture that the date 0 face value is: $D_0^* := \frac{I_0 + pI_1^H - p\theta(X_0 + X_1^H)}{1-p}$. First, note that $D_0 > 0$ if and only if $I_0 + pI_1^H > p\theta(X_0 + X_1^H)$. But $D_0 > 0$ then follows since $I_0 + pI_1^H > pI_0 + pI_1^H > p\theta(X_0 + X_1^H)$, where the last inequality follows from A3. Moreover, $D_0 \leq \theta X_0$ if and only if $p\theta X_1^H + \theta X_0 \geq I_0 + pI_1^H$. From A1, $I_0 < (1-p)\theta X_0$. Thus, $I_0 + pI_1^H < (1-p)\theta X_0 + pI_1^H$. Moreover, $p\theta X_1^H + \theta X_0 \geq (1-p)\theta X_0 + pI_1^H$ if and only if $\theta(X_0 + X_1^H) \geq I_1^H$, which always holds by A5. As a result, we conclude that $D_0 \in (0, \theta X_0]$. It remains to check that such a D_0 makes the participation constraint for the $t = 0$ creditors just bind, as required by optimality. In the low state, because there is no investment at date 1, cash flow is $\theta X_0 \geq D_0$. In the high state, cash flow is $\theta(X_0 + X_1^H)$, but I_1^H will go to date-1 creditors (who are secured). Thus, date-0 creditors get $\min\{\theta(X_0 + X_1^H) - I_1^H, D_0\}$. The amount of credit available is enough to cover the face value of debt if and only if $\theta(X_0 + X_1^H) - I_1^H \geq D_0$, or equivalently if and only if $\theta(X_0 + X_1^H) \geq I_0 + I_1^H$. However, by A3, this condition is always violated. Thus, date-0 creditors are diluted in the high state, and their participation constraint reads $I_0 \leq p(\theta(X_0 + X_1^H) - I_1^H) + (1-p)D_0$, which just binds at D_0^*. $\square$

[3] Note that, like DGP, we are not explicitly considering the role played by the positive NPV date-1 project. For simplicity, one can think that there isn't one or, as DGP put it, that *'p is not too large'*.