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Advances in the Truncated Euler–Maruyama Method for Stochastic Differential Delay Equations

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ABSTRACT. Guo et al. [GMY17] are the first to study the strong convergence of the explicit numerical method for the highly nonlinear stochastic differential delay equations (SDDEs) under the generalised Khasminskii-type condition. The method used there is the truncated Euler–Maruyama (EM) method. In this paper we will point out that a main condition imposed in [GMY17] is somehow restrictive in the sense that the condition could force the step size to be so small that the truncated EM method would be inapplicable. The key aim of this paper is then to establish the convergence rate without this restriction.

1. Introduction. Stochastic differential delay equations (SDDEs) have been used in many branches of science and industry (see, e.g., [Arn,CLM01,DZ92,Kha,LL]). The classical theory on the existence and uniqueness of the solution to the SDDE requires the coefficients of the SDDE satisfy the local Lipschitz condition and the linear growth condition (see, e.g., [KM,M97,M02,Moh]). The numerical solutions under the linear growth condition plus the local Lipschitz condition have been discussed intensively by many authors (see, e.g., [BB,BB05,CKR06, DFLM, HM05,KloP,KP,MS,Mil,Schurz,WM08]).

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The generalised Khasminskii-type theorems established by [M02, MR05] replace the linear growth condition with the generalised Khasminskii-type condition in terms of Lyapunov functions. The numerical solutions of SDDEs under the generalised Khasminskii-type condition were discussed by Mao [M11], and the theory there showed that the Euler–Maruyama (EM) numerical solutions converge to the true solutions *in probability* Influenced by [M15], Guo et al. [GMY17] were the first to study the *strong convergence* of the truncated EM method for the SDDEs under the generalised Khasminskii-type condition. In this paper, we will explain, via an example, that a main condition imposed in [GMY17] is sometimes so restrictive that the step size would be too small for the truncated EM method to be applicable. The key aim of this paper is to establish the convergence rate without this restrictive condition.

This paper is organised as follows: In Section 2, we will introduce the necessary notation, recall the truncated EM method and review one of the main results of [GMY17] and then point out a restrictive condition imposed in [GMY17] via an example. In Section 3, we will establish the strong convergence theory without this restrictive condition. In Section 4, we will compare our new result with the one in [GMY17] to highlight our significant contribution in this paper. In Section 5, we will establish the stronger convergence theory for the solutions over a finite time interval and this was not discussed in [GMY17]. In Section 6, we will discuss three more examples to illustrate our new theory. Finally, we will conclude our paper in Section 7.

2. Preliminaries. In this section, we will recall the truncated EM method for the SDDEs defined in [GMY17]. We will make some modification in order for the EM method to be more flexible. The main aim of this section is to point out a restrictive condition imposed in [GMY17] via an example. Removing this restrictive condition is the motivation for us to write this paper.

2.1 Notation. Throughout this paper, we will use the same notation as used in [GMY17]. However, for the convenience of the reader, we recall some here. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{trace(A^TA)}$. Let $\mathbb{R}_+ = [0, \infty)$ and $\tau > 0$. Denote by $C([-\tau, 0]; \mathbb{R}^n)$ the family of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau \le \theta \le 0} |\varphi(\theta)|$. Let $(\omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Moreover, for two real numbers a and b, we use $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. If G is a set, its indicator function is denoted by I_G, namely $I_G(x) = 1$ if $x \in G$ and 0 otherwise. If a is a real number, we denote by $\lfloor a \rfloor$ the largest integer which is less or equal to a, e.g., $\lfloor -1.2 \rfloor = -2$ and $\lfloor 2.3 \rfloor = 2$.

2.2 SDDEs. Consider an n-dimensional nonlinear SDDE

$$dx(t) = f(x(t), x(t-\tau))dt + g(x(t), x(t-\tau))dB(t), \quad t \ge 0, \tag{2.1}$$

with the initial data $\{x(\theta) : -\tau \le \theta \le 0\} = \xi \in C([-\tau, 0]; \mathbb{R}^n)$, where

$$f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}.$$

We assume, as a standing hypothesis, that the coefficients f and g obey the local Lipschitz condition, namely, for every positive number R there is a positive constant

K_R such that

$$|f(x,y) - f(\bar{x},\bar{y})|^2 \vee |g(x,y) - g(\bar{x},\bar{y})|^2 \le K_R(|x-\bar{x}|^2 + |y-\bar{y}|^2)$$

for those $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \le R$. As we are concerned with highly nonlinear SDDEs, we will not impose the linear growth condition. Instead, we need the Khasminskii-type condition.

Assumption 2.1. *There are constants $K_1 > 0$ and $\bar{p} > 2$ such that*

$$x^T f(x,y) + \frac{\bar{p}-1}{2}|g(x,y)|^2 \le K_1(1 + |x|^2 + |y|^2) \tag{2.2}$$

for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

It is known (see, e.g., [MR05]) that under Assumption 2.1, the SDDE (2.1) has unique global solution $x(t)$ on $t \in [-\tau, \infty)$ which satisfies

$$\sup_{-\tau \le t \le T} \mathbb{E}|x(t)|^{\bar{p}} < \infty, \quad \forall T > 0. \tag{2.3}$$

2.3 The truncated EM method. Recall the truncated EM numerical scheme defined in [GMY17]. We first choose a strictly increasing continuous function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu(u) \to \infty$ as $u \to \infty$ and

$$\sup_{|x| \vee |y| \le u} \big(|f(x,y)| \vee |g(x,y)|\big) \le \mu(u), \quad \forall u \ge 1. \tag{2.4}$$

Denote by μ^{-1} the inverse function of μ and we see that μ^{-1} is a strictly increasing continuous function from $[\mu(0), \infty)$ to $\mathbb{R}_+$. Choose a constant $\Delta^* \in (0,1]$ and a strictly decreasing function $h : (0, \Delta^*] \to (0, \infty)$ such that

$$h(\Delta^*) \ge \mu(1), \quad \lim_{\Delta \to 0} h(\Delta) = \infty \quad \text{and} \quad \Delta^{1/4} h(\Delta) \le 1, \quad \forall \Delta \in (0, \Delta^*]. \tag{2.5}$$

For a given step size $\Delta \in (0, \Delta^*]$, let us define a mapping π_Δ from $\mathbb{R}^n$ to the closed ball $\{x \in \mathbb{R}^n : |x| \le \mu^{-1}(h(\Delta))\}$ by

$$\pi_\Delta(x) = (|x| \wedge \mu^{-1}(h(\Delta)))\,\frac{x}{|x|},$$

where we set $x/|x| = 0$ when $x = 0$. That is, π_Δ will map x to itself when $|x| \le \mu^{-1}(h(\Delta))$ and to $\mu^{-1}(h(\Delta))x/|x|$ when $|x| > \mu^{-1}(h(\Delta))$. We then define the truncated functions

$$f_\Delta(x,y) = f(\pi_\Delta(x), \pi_\Delta(y)) \quad \text{and} \quad g_\Delta(x,y) = g(\pi_\Delta(x), \pi_\Delta(y)) \tag{2.6}$$

for $x, y \in \mathbb{R}^n$. It is easy to see that

$$|f_\Delta(x,y)| \vee |g_\Delta(x,y)| \le \mu(\mu^{-1}(h(\Delta))) = h(\Delta), \quad \forall x, y \in \mathbb{R}^n. \tag{2.7}$$

That is, both truncated functions f_Δ and g_Δ are bounded although f and g may not.

From now on, we will let the step size Δ be a *fraction* of τ. That is, we will use $\Delta = \tau/M$ for some positive integer M. When we use the terms of a sufficiently small Δ, we mean that we choose M sufficiently large.

Define $t_k = k\Delta$ for $k = -M, -(M-1), \cdots, 0, 1, 2, \cdots$. The discrete-time truncated EM solutions are defined by setting $X_\Delta(t_k) = \xi(t_k)$ for $k = -M, -(M-1), \cdots, 0$ and then forming

$$X_\Delta(t_{k+1}) = X_\Delta(t_k) + f_\Delta(X_\Delta(t_k), X_\Delta(t_{k-M}))\Delta + g_\Delta(X_\Delta(t_k), X_\Delta(t_{k-M}))\Delta B_k \tag{2.8}$$

for $k = 0, 1, 2, \cdots$, where $\Delta B_k = B(t_{k+1}) - B(t_k)$. As in [GMY17], it is more convenient to work on the continuous-time approximations. Recall that there are two continuous-time versions. One is the continuous-time step process $\bar{x}_\Delta(t)$ on $t \in [-\tau, \infty)$ defined by

$$\bar{x}_\Delta(t) = \sum_{k=-M}^{\infty} X_\Delta(t_k) I_{[k\Delta,(k+1)\Delta)}(t), \tag{2.9}$$

where $I_{[k\Delta,(k+1)\Delta)}(t)$ is the indicator function of $[k\Delta, (k+1)\Delta)$ (please recall the notation defined in the beginning of this section). The other one is the continuous-time continuous process $x_\Delta(t)$ on $t \in [-\tau, \infty)$ defined by $x_\Delta(t) = \xi(t)$ for $t \in [-\tau, 0]$ while for $t \ge 0$

$$x_\Delta(t) = \xi(0) + \int_0^t f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))ds + \int_0^t g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))dB(s). \tag{2.10}$$

We see that $x_\Delta(t)$ is an Itô process on $t \ge 0$ with its Itô differential

$$dx_\Delta(t) = f_\Delta(\bar{x}_\Delta(t), \bar{x}_\Delta(t-\tau))dt + g_\Delta(\bar{x}_\Delta(t), \bar{x}_\Delta(t-\tau))dB(t). \tag{2.11}$$

It is useful to know that $X_\Delta(t_k) = \bar{x}_\Delta(t_k) = x_\Delta(t_k)$ for every $k \ge -M$, namely three of them coincide at t_k.

2.4 Review of the main result in [GMY17]**.** We recall one more notation used in [GMY17]. Let $\mathcal{U}$ denote the family of continuous functions $U : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ such that for each $b > 0$, there is a positive constant κ_b for which

$$U(x, \bar{x}) \le \kappa_b |x - \bar{x}|^2, \quad \forall x, \bar{x} \in \mathbb{R}^n \text{ with } |x| \vee |\bar{x}| \le b.$$

Let us state the assumptions imposed in [GMY17] for the strong convergence rate.

Assumption 2.2. *There is a pair of constants $K_2 > 0$ and $\gamma \in (0, 1]$ such that the initial data ξ satisfies*

$$|\xi(u) - \xi(v)| \le K_2 |u - v|^\gamma, \quad -\tau \le v < u \le 0.$$

Assumption 2.3. *Assume that there are positive constants α and K_3 and a function $U \in \mathcal{U}$ such that*

$$\begin{aligned} &(x - \bar{x})^T (f(x, y) - f(\bar{x}, \bar{y})) + \frac{1+\alpha}{2} |g(x, y) - g(\bar{x}, \bar{y})|^2 \\ &\le K_3 (|x - \bar{x}|^2 + |y - \bar{y}|^2) - U(x, \bar{x}) + U(y, \bar{y}) \end{aligned} \tag{2.12}$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$.

Assumption 2.4. *Assume that there is a pair of positive constants* r *and* K_4 *such that*

$$|f(x,y) - f(\bar{x},\bar{y})|^2 \vee |g(x,y) - g(\bar{x},\bar{y})|^2 \\ \le K_4(|x-\bar{x}|^2 + |y-\bar{y}|^2)(1 + |x|^r + |\bar{x}|^r + |y|^r + |\bar{y}|^r) \tag{2.13}$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$.

The following theorem is one of the main results in [GMY17].

Theorem 2.5. *Let Assumptions 2.1-2.4 hold and* $\bar{p} > r$. *Assume that*

$$h(\Delta) \ge \mu\big((\Delta^{2\gamma} \vee [\Delta(h(\Delta))^2])^{-1/(\bar{p}-2)}\big) \tag{2.14}$$

for all sufficiently small $\Delta \in (0,1]$. *Then, for every such small* Δ,

$$\mathbb{E}|x(T) - x_\Delta(T)|^2 \le C(\Delta^{2\gamma} \vee [\Delta(h(\Delta))^2]) \tag{2.15}$$

and

$$\mathbb{E}|x(T) - \bar{x}_\Delta(T)|^2 \le C(\Delta^{2\gamma} \vee [\Delta(h(\Delta))^2]). \tag{2.16}$$

2.5 A motivating example. Let us now point out condition (2.14) is sometimes so restrictive that the step size would be too small for the truncated EM method to be applicable. Indeed, consider a highly nonlinear scalar SDDE

$$dx(t) = [-9x^3(t) + |x(t-\tau)|^{3/2}]dt + x^2(t)dB(t), \quad t \ge 0, \tag{2.17}$$

with the initial data $\{x(\theta) : -\tau \le \theta \le 0\} = \xi \in C([-\tau,0];\mathbb{R})$, where $B(t)$ is a scalar Brownian motion. Assume that ξ satisfies Assumption 2.2 with $K_2 = 1$ and $\gamma = 0.5$. Clearly, the coefficients

$$f(x,y) = -9x^3 + |y|^{3/2} \quad \text{and} \quad g(x,y) = x^2 \quad (x, y \in \mathbb{R})$$

are locally Lipschitz continuous. Moreover, when $\bar{p} = 18.5$, we have, for $x, y \in \mathbb{R}$,

$$xf(x,y) + \frac{\bar{p}-1}{2}|g(x,y)|^2 = -9x^4 + x|y|^{3/2} + 8.75x^4.$$

But, by the well-known Young inequality,

$$x|y|^{3/2} \le (x^4)^{1/4}(y^2)^{3/4} \le 0.25x^4 + 0.75y^2.$$

We therefore have

$$xf(x,y) + \frac{\bar{p}-1}{2}|g(x,y)|^2 \le 0.75y^2.$$

This shows that Assumption 2.1 is satisfied with $\bar{p} = 18.5$ and $K_1 = 0.75$. To verify Assumption 2.3, we note that, for $x, \bar{x}, y, \bar{y} \in \mathbb{R}$,

$$(x-\bar{x})(f(x,y) - f(\bar{x},\bar{y})) = (x-\bar{x})\big[-9(x^3 - \bar{x}^3) + |y|^{3/2} - |\bar{y}|^{3/2}\big] \\ \le -4.5|x-\bar{x}|^2(x^2+\bar{x}^2) + 0.5(x-\bar{x})^2 + 0.5(|y|^{3/2} - |\bar{y}|^{3/2})^2$$

while

$$(x^2-\bar{x}^2)^2=(x-\bar{x})^2(x+\bar{x})^2\le 2|x-\bar{x}|^2(x^2+\bar{x}^2).$$

But, by the mean-valued theorem,

$$\begin{aligned}(|y|^{3/2}-|\bar{y}|^{3/2})^2 &\le 2.25|y-\bar{y}|^2(\sqrt{|y|}+\sqrt{|\bar{y}|})^2\le 4.5|y-\bar{y}|^2(|y|+|\bar{y}|)\\ &\le 4.5|y-\bar{y}|^2(1+0.5y^2+0.5\bar{y}^2)=4.5|y-\bar{y}|^2+2.25|y-\bar{y}|^2(y^2+\bar{y}^2).\end{aligned}$$

Hence, for $\alpha=2.375$,

$$\begin{aligned}&(x-\bar{x})(f(x,y)-f(\bar{x},\bar{y}))+\frac{1+\alpha}{2}|g(x,y)-g(\bar{x},\bar{y})|^2\\ &\le 2.25[|x-\bar{x}|^2+|y-\bar{y}|^2]-U(x,\bar{x})+U(y,\bar{y}),\end{aligned}$$

where

$$U(x,\bar{x})=1.125|x-\bar{x}|^2(x^2+\bar{x}^2).$$

We have hence verified Assumption 2.3 with $\alpha=2.375$, $K_3=2.25$ and U being defined above. It is also straightforward to show that Assumption 2.4 is satisfied with $r=4$ (and some K_4 which is not important).

To apply Theorem 2.5, we still need to design functions μ and h satisfying (2.4) and (2.5), respectively. Note that

$$\sup_{|x|\vee|y|\le u}(|f(x,y)|\vee|g(x,y)|)\le 10u^3,\quad \forall u\ge 1.$$

We can hence have $\mu(u)=10u^3$ and its inverse function $\mu^{-1}(u)=(u/10)^{1/3}$ for $u\ge 0$. We also define $h(\Delta)=\Delta^{-1/5}$ for $\Delta\in(0,\Delta^*]$, where $\Delta^*=10^{-5}$ (so $h(\Delta^*)=10=\mu(1)$ as required). Then condition (2.14) becomes

$$\Delta^{-1/5}\ge 10\Delta^{-9/82.5},\ \text{namely, } \Delta\le 10^{-11}.$$

By Theorem 2.5, we can then conclude that the truncated EM solutions will approximate the true solution $x(t)$ in the sense that

$$\mathbb{E}|x_\Delta(T)-x(T)|^2\vee\mathbb{E}|\bar{x}_\Delta(T)-x(T)|^2\le C\Delta^{3/5} \tag{2.18}$$

for $\Delta\le 10^{-11}$. The problem is that the stepsize needs to be so small that the truncated EM is almost inapplicable.

This example shows that condition (2.14) is too restrictive sometimes. Could we remove this condition and still establish the strong convergence theory? We will give our positive answer in the next section.

3 Main Results

3.1 Lemmas. First of all, we modify the choice of function h to make it more flexible by choosing a constant $\hat{h}\ge 1$ and a strictly decreasing function $h:(0,1]\to[\mu(0),\infty)$ such that

$$\lim_{\Delta\to 0}h(\Delta)=\infty\quad\text{and}\quad \Delta^{1/4}h(\Delta)\le\hat{h},\quad\forall\Delta\in(0,1]. \tag{3.1}$$

From now on, our function h will satisfy this condition instead of (2.5). There are lots of choices for $h(\cdot)$. For example, $h(\Delta)=\hat{h}\Delta^{-\epsilon}$ for some $\epsilon\in(0,1/4]$. Before we proceed, let us make a useful remark.

Remark 3.1. Comparing (3.1) with (2.5), we here simply let $\Delta^* = 1$ and remove condition $h(\Delta^*) \geq \mu(1)$ while we also replace condition $\Delta^{1/4}h(\Delta) \leq 1$ by a weaker one $\Delta^{1/4}h(\Delta) \leq \hat{h}$. In other words, we have made the choice of function h more flexible. We emphasise that such changes do not make any effect on the results in [GMY17]. In fact, condition $h(\Delta^*) \geq \mu(1)$ was only used to prove [[GMY17], Lemmas 2.4 and 4.2]. But, it is easy to show (see the proof of Lemma 3.2 below) that both lemmas there still hold as long as we replace the constant $2K_1$ there by $2K_1\big(1 \vee [1/\mu^{-1}(h(1))]\big)$ and this change does not affect any other results in [GMY17]. It is also easy to check that replacing $\Delta^{1/4}h(\Delta) \leq 1$ by $\Delta^{1/4}h(\Delta) \leq \hat{h}$ does not make any effect on the results in [GMY17].

The following lemma shows that the truncated functions defined by (2.6) preserve the Khasminskii-type condition (2.2) to a very nice degree.

Lemma 3.2. *Let Assumption 2.1 hold. Then, for every $\Delta \in (0,1]$, we have*

$$x^T f_\Delta(x,y) + \frac{1}{2}|g_\Delta(x,y)|^2 \leq \hat{K}(1 + |x|^2 + |y|^2) \tag{3.2}$$

for all $x, y \in \mathbb{R}^n$, where $\hat{K} = 2K_1\big(1 \vee [1/\mu^{-1}(h(1))]\big)$.

Proof. This lemma was essentially proved in [GMY17] but we here need only h to satisfy condition (3.1) instead of (2.5) and, in particular, we do not need condition $h(\Delta^*) \geq \mu(1)$ as we already pointed out in Remark 3.1.

Fix any $\Delta \in (0,1]$. For $x \in \mathbb{R}^n$ with $|x| \leq \mu^{-1}(h(\Delta))$ and any $y \in \mathbb{R}^n$, the assertion follows from (2.2) directly. For $x \in \mathbb{R}^n$ with $|x| > \mu^{-1}(h(\Delta))$ and any $y \in \mathbb{R}^n$, we have

$$\begin{aligned}
&x^T f_\Delta(x,y) + \frac{\bar{p}-1}{2}|g_\Delta(x,y)|^2 \\
&= \pi_\Delta(x)^T f(\pi_\Delta(x), \pi_\Delta(y)) + \frac{\bar{p}-1}{2}|g(\pi_\Delta(x), \pi_\Delta(y))|^2 \\
&+ (x - \pi_\Delta(x))^T f(\pi_\Delta(x), \pi_\Delta(y)) \\
&\leq K_1(1 + |\pi_\Delta(x)|^2 + |\pi_\Delta(y)|^2) \\
&+ \Big(\frac{|x|}{\mu^{-1}(h(\Delta))} - 1\Big)\pi_\Delta(x)^T f(\pi_\Delta(x), \pi_\Delta(y)),
\end{aligned} \tag{3.3}$$

where (2.2) has been used. But it also follows from (2.2) that

$$\pi_\Delta(x)^T f(\pi_\Delta(x), \pi_\Delta(y)) \leq K_1(1 + |\pi_\Delta(x)|^2 + |\pi_\Delta(y)|^2).$$

Substituting this into (3.3) yields

$$\begin{aligned}
&x^T f_\Delta(x,y) + \frac{1}{2}|g_\Delta(x,y)|^2 \\
&\leq \frac{K_1|x|}{\mu^{-1}(h(\Delta))}(1 + |\pi_\Delta(x)|^2 + |\pi_\Delta(y)|^2) \\
&\leq K_1\big(1 \vee [1/\mu^{-1}(h(1))]\big)(|x| + |x|^2 + |x||y|) \\
&\leq 2K_1\big(1 \vee [1/\mu^{-1}(h(1))]\big)(1 + |x|^2 + |y|^2).
\end{aligned} \tag{3.4}$$

Namely, we have showed that the required assertion (3.2) also holds for $x \in \mathbb{R}^n$ with $|x| > \mu^{-1}(h(\Delta))$ and any $y \in \mathbb{R}^n$. The proof is hence complete. $\square$

The following lemma shows that the truncated functions f_Δ and g_Δ preserve Assumption 2.4 perfectly.

Lemma 3.3. *Let Assumption 2.4 hold. Then, for every $\Delta \in (0, 1]$, we have*

$$\begin{aligned}&|f_\Delta(x, y) - f_\Delta(\bar{x}, \bar{y})|^2 \vee |g_\Delta(x, y) - g_\Delta(\bar{x}, \bar{y})|^2 \\ &\le K_4(|x - \bar{x}|^2 + |y - \bar{y}|^2)(1 + |x|^r + |\bar{x}|^r + |y|^r + |\bar{y}|^r)\end{aligned} \tag{3.5}$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$.

Proof. For any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$,

$$\begin{aligned}&|f_\Delta(x, y) - f_\Delta(\bar{x}, \bar{y})|^2 \vee |g_\Delta(x, y) - g_\Delta(\bar{x}, \bar{y})|^2 \\ &= |f(\pi_\Delta(x), \pi_\Delta(y)) - f(\pi_\Delta(\bar{x}), \pi_\Delta(\bar{y}))|^2 \vee |g(\pi_\Delta(x), \pi_\Delta(y)) - g(\pi_\Delta(\bar{x}), \pi_\Delta(\bar{y}))|^2 \\ &\le K_4(|\pi_\Delta(x) - \pi_\Delta(\bar{x})|^2 + |\pi_\Delta(y) - \pi_\Delta(\bar{y})|^2)(1 + |\pi_\Delta(x)|^r + |\pi_\Delta(\bar{x})|^r + |\pi_\Delta(y)|^r + |\pi_\Delta(\bar{y})|^r).\end{aligned}$$

This implies the assertion by noting that

$$|\pi_\Delta(x)| \le |x|, \quad |\pi_\Delta(x) - \pi_\Delta(\bar{x})|^2 \le |x - \bar{x}|^2,$$

etc. The proof is complete. $\square$

Recalling Remark 3.1, we can then cite two lemmas from [GMY17] on the continuous-time truncated EM solutions defined by (2.9) and (2.11) for the use of this paper. *From now on we will fix $T > 0$ arbitrarily and let C stand for generic positive real constants dependent on T, K_1, K_2, ξ etc. but independent of Δ and its values may change between occurrences.*

Lemma 3.4. *For any $\Delta \in (0, 1]$ and any $p \ge 2$, we have*

$$\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^p \le c_p \Delta^{p/2}(h(\Delta))^p, \quad \forall t \ge 0, \tag{3.6}$$

where c_p is a positive constant dependent only on p. Consequently

$$\lim_{\Delta \to 0} \mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^p = 0, \quad \forall t \ge 0. \tag{3.7}$$

Lemma 3.5. *Let Assumption 2.1 hold. Then*

$$\sup_{0<\Delta\le 1} \sup_{0\le t\le T} \mathbb{E}|x_\Delta(t)|^{\bar{p}} \le C. \tag{3.8}$$

Lemma 3.4 shows that $x_\Delta(t)$ and $\bar{x}_\Delta(t)$ are close to each other in the sense of L^p. We also observe that $\bar{x}_\Delta(t)$ is computable, but $x_\Delta(t)$ is not in general. It is therefore $\bar{x}_\Delta(t)$ that we use in practice. However, for our analysis, it is more convenient to work on both of them. We also emphasize that Lemma 3.4 holds for any $p \ge 2$ but Lemma 3.5 holds only for the specified $\bar{p}$.

3.2 Convergence rates. The following theorem is one of our main results in this paper.

Theorem 3.6. *Let Assumptions 2.1 - 2.4 hold and assume that $\bar{p} > r+2$. Then, for every $\Delta \in (0,1]$,*

$$\mathbb{E}|x(T) - x_\Delta(T)|^2 \le C\big(\Delta^{2\gamma} \vee \Delta(h(\Delta))^2 \vee (\mu^{-1}(h(\Delta)))^{-(\bar{p}-r-2)}\big) \tag{3.9}$$

and

$$\mathbb{E}|x(T) - \bar{x}_\Delta(T)|^2 \le C\big(\Delta^{2\gamma} \vee \Delta(h(\Delta))^2 \vee (\mu^{-1}(h(\Delta)))^{-(\bar{p}-r-2)}\big). \tag{3.10}$$

Proof. The proof is very technical so we divide it into three steps.

Step 1. We fix $\Delta \in (0,1]$ and the initial data ξ arbitrarily. Let i be any integer such that $i \ge \|\xi\|$. Define the stopping time

$$\theta_i = \inf\{t \ge 0 : |x(t)| \wedge |x_\Delta(t)| \ge i\},$$

where we set $\inf \emptyset = \infty$ (and $\emptyset$ denotes the empty set as usual). Set $e_\Delta(t) = x(t) - x_\Delta(t)$ for $t \in [-\tau, T]$ and we know $e_\Delta(t) = 0$ for $t \in [-\tau, 0]$. By the Itô formula as well as the elementary inequality $(a+b)^2 \le (1+\alpha)a^2 + (1+\alpha^{-1})b^2$ for any real numbers a and b, we derive that, for $0 \le t \le T$,

$$\begin{aligned}
&\mathbb{E}|e_\Delta(t\wedge\theta_i)|^2 \\
&= \mathbb{E}\int_0^{t\wedge\theta_i} \Big(2e_\Delta^T(s)[f(x(s),x(s-\tau)) - f_\Delta(\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau))] \\
&\quad + |g(x(s),x(s-\tau)) - g_\Delta(\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau))|^2\Big)ds \\
&= \mathbb{E}\int_0^{t\wedge\theta_i} \Big(2e_\Delta^T(s)\big([f(x(s),x(s-\tau)) - f(x_\Delta(s),x_\Delta(s-\tau))] \\
&\quad + [f(x_\Delta(s),x_\Delta(s-\tau)) - f_\Delta(\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau))]\big) \\
&\quad + \big|g(x(s),x(s-\tau)) - g(x_\Delta(s),x_\Delta(s-\tau)) \\
&\quad + g(x_\Delta(s),x_\Delta(s-\tau)) - g_\Delta(\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau))\big|^2\Big)ds \\
&\le \mathbb{E}\int_0^{t\wedge\theta_i} \Big(2e_\Delta^T(s)[f(x(s),x(s-\tau)) - f(x_\Delta(s),x_\Delta(s-\tau))] \\
&\quad + (1+\alpha)|g(x(s),x(s-\tau)) - g(x_\Delta(s),x_\Delta(s-\tau))|^2 \\
&\quad + 2e_\Delta^T(s)[f(x_\Delta(s),x_\Delta(s-\tau)) - f_\Delta(\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau))] \\
&\quad + (1+\alpha^{-1})|g(x_\Delta(s),x_\Delta(s-\tau)) - g_\Delta(\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau))|^2\Big)ds.
\end{aligned} \tag{3.11}$$

By Assumption 2.3, we get

$$\mathbb{E}|e_\Delta(t\wedge\theta_i)|^2 \le H_1 + H_2, \tag{3.12}$$

where

$$\begin{aligned}H_1 := \mathbb{E}\int_0^{t\wedge\theta_i} \Big(&|e_\Delta(s)|^2 + 2K_3(|e_\Delta(s)|^2 + |e_\Delta(s-\tau)|^2)\\ &- 2U(x(s), x_\Delta(s)) + 2U(x(s-\tau), x_\Delta(s-\tau))\Big)ds\end{aligned}$$

and

$$\begin{aligned}H_2 := \mathbb{E}\int_0^{t\wedge\theta_i} \Big(&|f(x_\Delta(s), x_\Delta(s-\tau)) - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2\\ &+ (1+\alpha^{-1})|g(x_\Delta(s), x_\Delta(s-\tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2\Big)ds.\end{aligned}$$

Step 2. Let us now estimate H_1 and H_2. Noting that $x(s) = x_\Delta(s)$ whence $U(x(s), x_\Delta(s)) = 0$ for $s \in [-\tau, 0]$, we see that

$$\int_0^{t\wedge\theta_i} U(x(s-\tau), x_\Delta(s-\tau))ds \le \int_{-\tau}^{t\wedge\theta_i} U(x(s), x_\Delta(s))ds = \int_0^{t\wedge\theta_i} U(x(s), x_\Delta(s))ds.$$

Consequently, we have

$$H_1 \le (1+4K_3)\mathbb{E}\int_0^{t\wedge\theta_i} |e_\Delta(s)|^2 ds \le (1+4K_3)\int_0^t \mathbb{E}|e_\Delta(s\wedge\theta_i)|^2 ds. \tag{3.13}$$

To estimate H_2, we observe that

$$H_2 \le H_{21} + H_{22}, \tag{3.14}$$

where

$$\begin{aligned}H_{21} := 2\int_0^T \Big(&\mathbb{E}|f(x_\Delta(s), x_\Delta(s-\tau)) - f_\Delta(x_\Delta(s), x_\Delta(s-\tau))|^2\\ &+ (1+\alpha^{-1})\mathbb{E}|g(x_\Delta(s), x_\Delta(s-\tau)) - g_\Delta(x_\Delta(s), x_\Delta(s-\tau))|^2\Big)ds\end{aligned}$$

and

$$\begin{aligned}H_{22} := 2\int_0^T \Big(&\mathbb{E}|f_\Delta(x_\Delta(s), x_\Delta(s-\tau)) - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2\\ &+ (1+\alpha^{-1})\mathbb{E}|g_\Delta(x_\Delta(s), x_\Delta(s-\tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2\Big)ds,\end{aligned}$$

where $t\wedge\theta_i$ has been replaced by T as $t\wedge\theta_i \le T$ and the order of integrations has also been exchanged. By Assumption 2.4 and the Hölder inequality as well as Lemma 3.5, we derive that

$$\begin{aligned}&\mathbb{E}|f(x_\Delta(s), x_\Delta(s-\tau)) - f_\Delta(x_\Delta(s), x_\Delta(s-\tau))|^2\\ =&\mathbb{E}|f(x_\Delta(s), x_\Delta(s-\tau)) - f(\pi_\Delta(x_\Delta(s)), \pi_\Delta(x_\Delta(s-\tau)))|^2\\ \le& K_4\mathbb{E}\big[(|x_\Delta(s) - \pi_\Delta(x_\Delta(s))|^2 + |x_\Delta(s-\tau) - \pi_\Delta(x_\Delta(s-\tau))|^2)\\ &\quad\times(1 + 2|x_\Delta(s)|^r + 2|x_\Delta(s-\tau)|^r)\big]\\ \le& C\Big(\mathbb{E}|x_\Delta(s) - \pi_\Delta(x_\Delta(s))|^{2\bar{p}/(\bar{p}-r)} + \mathbb{E}|x_\Delta(s-\tau) - \pi_\Delta(x_\Delta(s-\tau))|^{2\bar{p}/(\bar{p}-r)}\Big)^{(\bar{p}-r)/\bar{p}}\\ &\quad\times\big(1 + \mathbb{E}|x_\Delta(s)|^{\bar{p}} + \mathbb{E}|x_\Delta(s-\tau)|^{\bar{p}}\big)^{r/\bar{p}}\\ \le& C\Big(\mathbb{E}\Big[I_{\{|x_\Delta(s)|>\mu^{-1}(h(\Delta))\}}|x_\Delta(s))|^{2\bar{p}/(\bar{p}-r)}\Big]\\ &\quad+\mathbb{E}\Big[I_{\{|x_\Delta(s-\tau)|>\mu^{-1}(h(\Delta))\}}|x_\Delta(s-\tau))|^{2\bar{p}/(\bar{p}-r)}\Big]\Big)^{(\bar{p}-r)/\bar{p}}.\end{aligned}$$

But

$$
\begin{aligned}
&\mathbb{E}\Big[I_{\{|x_\Delta(s)|>\mu^{-1}(h(\Delta))\}}|x_\Delta(s))|^{2\bar p/(\bar p-r)}\Big]\\
\le&\big[\mathbb{P}\{|x_\Delta(s)|>\mu^{-1}(h(\Delta))\}\big]^{(\bar p-r-2)/(\bar p-r)}\big[\mathbb{E}|x_\Delta(s))|^{\bar p}\big]^{2/(\bar p-r)}\Big]\\
\le&C\big[\mathbb{P}\{|x_\Delta(s)|>\mu^{-1}(h(\Delta))\}\big]^{(\bar p-r-2)/(\bar p-r)}\\
\le&C\Big(\Big[\frac{\mathbb{E}|x_\Delta(s)|^{\bar p}}{(\mu^{-1}(h(\Delta)))^{\bar p}}\Big]^{(\bar p-r-2)/(\bar p-r)}\\
\le&C(\mu^{-1}(h(\Delta)))^{-\bar p(\bar p-r-2)/(\bar p-r)}.
\end{aligned}
$$

Similarly,

$$
\mathbb{E}\Big[I_{\{|x_\Delta(s-\tau)|>\mu^{-1}(h(\Delta))\}}|x_\Delta(s-\tau))|^{2\bar p/(\bar p-r)}\Big]\le C(\mu^{-1}(h(\Delta)))^{-\bar p(\bar p-r-2)/(\bar p-r)}.
$$

Consequently,

$$
\mathbb{E}|f(x_\Delta(s),x_\Delta(s-\tau))-f_\Delta(x_\Delta(s),x_\Delta(s-\tau))|^2\le C(\mu^{-1}(h(\Delta)))^{-(\bar p-r-2)}.
$$

Similarly, we can show

$$
\mathbb{E}|g(x_\Delta(s),x_\Delta(s-\tau))-g_\Delta(x_\Delta(s),x_\Delta(s-\tau))|^2\le C(\mu^{-1}(h(\Delta)))^{-(\bar p-r-2)}.
$$

We therefore have

$$
H_{21}\le C(\mu^{-1}(h(\Delta)))^{-(\bar p-r-2)}. \tag{3.15}
$$

Let us now estimate H_{22}. By Lemma 3.3 and the Hölder inequality as well as Lemma 3.4 and Assumption 2.2, we derive that

$$
\begin{aligned}
&\mathbb{E}|f_\Delta(x_\Delta(s),x_\Delta(s-\tau))-f_\Delta(\bar x_\Delta(s),\bar x_\Delta(s-\tau))|^2\\
\le&K_4\mathbb{E}\big[(|x_\Delta(s)-\bar x_\Delta(s)|^2+|x_\Delta(s-\tau)-\bar x_\Delta(s-\tau)|^2)\\
&\quad\times(1+|x_\Delta(s)|^r+|\bar x_\Delta(s)|^r+|x_\Delta(s-\tau)|^r+|\bar x_\Delta(s-\tau)|^r)\big]\\
\le&C\Big(\mathbb{E}|x_\Delta(s)-\bar x_\Delta(s)|^{2\bar p/(\bar p-r)}+\mathbb{E}|x_\Delta(s-\tau)-\bar x_\Delta(s-\tau)|^{2\bar p/(\bar p-r)}\Big)^{(\bar p-r)/\bar p}\\
&\quad\times\big(1+\mathbb{E}|x_\Delta(s)|^{\bar p}+\mathbb{E}|\bar x_\Delta(s)|^{\bar p}+\mathbb{E}|x_\Delta(s-\tau)|^{\bar p}+\mathbb{E}|\bar x_\Delta(s-\tau)|^{\bar p}\big)^{r/\bar p}\\
\le&C(\Delta^{2\gamma}\vee\Delta(h(\Delta))^2).
\end{aligned}
$$

Similarly, we have

$$
\mathbb{E}|g_\Delta(x_\Delta(s),x_\Delta(s-\tau))-g_\Delta(\bar x_\Delta(s),\bar x_\Delta(s-\tau))|^2\le C(\Delta^{2\gamma}\vee\Delta(h(\Delta))^2).
$$

Consequently

$$
H_{22}\le C(\Delta^{2\gamma}\vee\Delta(h(\Delta))^2). \tag{3.16}
$$

Step 3. Combining (3.12) - (3.16) together yields

$$
\mathbb{E}|e_\Delta(t\wedge\theta_i)|^2\le C\int_0^t\mathbb{E}|e_\Delta(s\wedge\theta_i)|^2ds+C\big(\Delta^{2\gamma}\vee\Delta(h(\Delta))^2\vee(\mu^{-1}(h(\Delta)))^{-(\bar p-r-2)}\big).
$$

The Gronwall inequality shows

$$\mathbb{E}|e_\Delta(T\wedge\theta_i)|^2 \le C\big(\Delta^{2\gamma}\vee\Delta(h(\Delta))^2\vee(\mu^{-1}(h(\Delta)))^{-(\bar p-r-2)}\big)$$

Letting $i\to\infty$ gives the first assertion (3.9). The second assertion (3.10) follows from the first one and Lemma 3.4. The proof is therefore complete. □

We observe from Assumption 2.4 that

$$\sup_{|x|\vee|y|\le u}\big(|f(x,y)|\vee|g(x,y)|\big)\le K_5u^{(2+r)/2},\quad u\ge 1, \tag{3.17}$$

for some $K_5>0$. We can therefore let $\mu(u)=K_5u^{(2+r)/2}$ and $h(\Delta)=\Delta^{-\epsilon}$ for some $\epsilon\in(0,1/4]$. This implies the following corollary immediately.

Corollary 3.7. *Let Assumptions 2.1 - 2.4 hold and assume that $\bar p>r+2$. Let $\mu(u)=K_5u^{(2+r)/2}$ and $h(\Delta)=\Delta^{-\epsilon}$ for some $\epsilon\in(0,1/4]$. Then, for every $\Delta\in(0,1]$,*

$$\mathbb{E}|x(T)-x_\Delta(T)|^2\le C\,\Delta^{2\gamma\wedge(1-2\epsilon)\wedge(2\epsilon(\bar p-r-2)/(2+r))} \tag{3.18}$$

and

$$\mathbb{E}|x(T)-\bar x_\Delta(T)|^2\le C\,\Delta^{2\gamma\wedge(1-2\epsilon)\wedge(2\epsilon(\bar p-r-2)/(2+r))}. \tag{3.19}$$

The following corollary is more useful sometimes.

Corollary 3.8. *Let all the conditions of Corollary 3.7 hold. In particular, let Assumption 2.1 hold for any $\bar p>2$ (K_1 depends on $\bar p$ of course). Let $\mu(u)=K_5u^{(2+r)/2}$ and $h(\Delta)=\Delta^{-\epsilon}$ for some $\epsilon\in(0,1/4]$. Then, for every $\Delta\in(0,1]$,*

$$\mathbb{E}|x(T)-x_\Delta(T)|^2\le C\,\Delta^{2\gamma\wedge(1-2\epsilon)} \tag{3.20}$$

and

$$\mathbb{E}|x(T)-\bar x_\Delta(T)|^2\le C\,\Delta^{2\gamma\wedge(1-2\epsilon)}. \tag{3.21}$$

Proof. Choose $\bar p>2$ sufficiently large for

$$2\epsilon(\bar p-r-2)/(2+r)>1-2\epsilon.$$

The assertions then follow from Corollary 3.7. □

It is reasonable to regard the initial data $\xi=\{\xi(t):-\tau\le t\le 0\}$ as the observation of the solution on $t\in[-\tau,0]$. Recalling that the Brownian motion is α-Hölder continuous for $\alpha\in(0,0.5)$ (see, e.g., [KR88]), we may assume that Assumption 2.2 holds for some $\gamma\in(0,0.5)$ close to 0.5. In this case, Corollary 3.8 shows the order of the convergence rate is close to 0.5. This is almost optimal if we recall the order of the classical EM method applied to stochastic differential equations (SDEs) is 0.5 under the global Lipschiz condition (see, e.g., [KP,M97]).

4 Comparison. Let us now compare our new Theorem 3.6 with the main result of [GMY17], namely Theorem 2.5 in order to highlight the significant contribution of our new result. Although the assumptions imposed in both theorems are almost the same, we observe the following key differences:

i). The key feature of Theorem 3.6 is that it does not require the restrictive condition (2.14).

ii). The assertions of Theorem 3.6 hold for any $\Delta \in (0, 1]$ while the assertions of Theorem 2.5. hold only for sufficiently small Δ which satisfies condition (2.14).

iii). Theorem 3.6 needs a slightly stronger condition on the parameters, namely $\bar{p} > r + 2$, while Theorem 2.5 needs $\bar{p} > r$ only.

iv). The assertions of Theorem 3.6 look slightly worse than those of Theorem 2.5 but could be the same when $\bar{p}$ is sufficiently large, for example, $2\epsilon(\bar{p}-r-2)/(2+r) \geq 1 - 2\epsilon$ in Corollary 3.7.

The key advantage of our new Theorem 3.6 lies in that it does not need condition (2.14). In Section 2.5, we have shown, via the example, that condition (2.14) could sometimes make Theorem 2.5 inapplicable and hence our new Theorem 3.6 without condition (2.14) is particularly useful in this situation.

5 Further Results. In Section 3, we showed that both truncated EM solutions $x_\Delta(T)$ and $\bar{x}_\Delta(T)$ converge to the true solution $x(T)$ in L^2 for any $T > 0$. This is sufficient for some applications e.g. when we need to approximate the European put or call option value at time T (see, e.g., [HM05]). However, we sometimes need to approximate quantities that are path-dependent, for example, the European barrier option value. In these situations, we will need a stronger convergence result like

$$\lim_{\Delta \to 0} \mathbb{E}\Big(\sup_{0 \leq t \leq T} |x_\Delta(t) - x(t)|^2 \Big) = 0.$$

We aim to establish such stronger results in this section. For this purpose, we need to replace Assumption 2.4 with the following slightly stronger one.

Assumption 5.1. *Assume that there is a pair of positive constants r and K_4 such that*

$$|f(x, y) - f(\bar{x}, \bar{y})|^2 \leq K_4(|x - \bar{x}|^2 + |y - \bar{y}|^2)(1 + |x|^r + |\bar{x}|^r + |y|^r + |\bar{y}|^r) \tag{5.1}$$

and

$$|g(x, y) - g(\bar{x}, \bar{y})|^2 \leq K_4(|x - \bar{x}|^2 + |y - \bar{y}|^2) \tag{5.2}$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$.

Let us make a useful remark.

Remark 5.2. As Assumption 5.1 is stronger than Assumption 2.4 so all the results before hold if Assumption 2.4 is replaced with Assumption 5.1. Moreover, it is easy to see that if Assumption 2.1 holds for some $\bar{p} > 0$, then it must hold for any $\bar{p} > 2$ as long as Assumption 5.1 holds as well. In fact, if Assumption 2.1 holds for some $\bar{p} > 0$, then together with (5.2), there holds for any $p > 2$

$$x^T f(x, y) + \frac{p-1}{2}|g(x, y)|^2 \leq \bar{K}_1(1 + |x|^2 + |y|^2) \tag{5.3}$$

for some $\bar{K}_1$. Recalling Corollary 3.8 and its proof, we can then conclude that the term

$$C\big(\Delta^{2\gamma} \vee \Delta(h(\Delta))^2 \vee (\mu^{-1}(h(\Delta)))^{-(\bar{p}-r-2)}\big)$$

throughout Section 3 can be replaced by

$$C\,\Delta^{2\gamma\wedge(1-2\epsilon)}$$

if Assumption 2.4 is replaced with Assumption 5.1 and we let $\mu(u) = K_5 u^{(2+r)/2}$ and $h(\Delta) = \Delta^{-\epsilon}$ for some $\epsilon \in (0, 1/4]$.

We can now state our stronger result under the stronger conditions.

Theorem 5.3. *Let Assumptions 2.1 - 2.3 and 5.1 hold and assume that $\bar{p} > r + 2$. Let $\mu(u) = K_5 u^{(2+r)/2}$ and $h(\Delta) = \Delta^{-\epsilon}$ for some $\epsilon \in (0, 1/4]$. Then, for every $\Delta \in (0, 1]$,*

$$\mathbb{E}\Big(\sup_{0\le t\le T} |x(t) - x_\Delta(t)|^2\Big) \le C\,\Delta^{2\gamma\wedge(1-2\epsilon)}. \tag{5.4}$$

Proof. We will use the same notation as in the proof of Theorem 3.6. The Itô formula shows that

$$|e_\Delta(t)|^2 \le H_3(t) + H_4(t), \quad 0 \le t \le T, \tag{5.5}$$

where

$$H_3(t) := \int_0^t \Big(2e_\Delta^T(s)[f(x(s), x(s-\tau)) - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))] \\ + |g(x(s), x(s-\tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2\Big) ds$$

and

$$H_4(t) := \int_0^t 2e_\Delta^T(s)\big[g(x(s), x(s-\tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))\big] dB(s).$$

In the same way as in the proof of Theorem 3.6, we can show that

$$\begin{aligned} H_3(t) \le{}& (1 + 4K_3)\int_0^t |e_\Delta(s)|^2 ds \\ &+ \int_0^t \Big(|f(x_\Delta(s), x_\Delta(s-\tau)) - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2 \\ &\quad + (1+\alpha^{-1})|g(x_\Delta(s), x_\Delta(s-\tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2\Big) ds. \end{aligned}$$

Recalling (3.14), we then get

$$\mathbb{E}\Big(\sup_{0\le t\le T} H_3(t)\Big) \le (1 + 4K_3)\int_0^T \mathbb{E}|e_\Delta(s)|^2 ds + H_{21} + H_{22}.$$

By Theorem 3.6 and Remark 5.2 as well as (3.15) and (3.16), we get

$$\mathbb{E}\Big(\sup_{0\le t\le T} H_3(t)\Big) \le C\,\Delta^{2\gamma\wedge(1-2\epsilon)}. \tag{5.6}$$

On the other hand, by the Burkholder-Davis-Gundy inequality (see, e.g., [DZ92]), we derive

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T} H_4(t)\Big)\\
\le&6\mathbb{E}\Big(\int_0^T |e_\Delta(s)|^2|g(x(s),x(s-\tau))-g_\Delta(\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau))|^2ds\Big)^{1/2}\\
\le&6\mathbb{E}\Big(\Big[\sup_{0\le t\le T}|e_\Delta(s)|^2\Big]\int_0^T |g(x(s),x(s-\tau))-g_\Delta(\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau))|^2ds\Big)^{1/2}\\
\le&0.5\mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(s)|^2\Big]+9\mathbb{E}\int_0^T |g(x(s),x(s-\tau))-g_\Delta(\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau))|^2ds.
\end{aligned}\tag{5.7}$$

But

$$\begin{aligned}
&\mathbb{E}\int_0^T |g(x(s),x(s-\tau))-g_\Delta(\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau))|^2ds\\
\le&2\mathbb{E}\int_0^T |g(x(s),x(s-\tau))-g(x_\Delta(s),x_\Delta(s-\tau))|^2ds\\
+&2\mathbb{E}\int_0^T |g(x_\Delta(s),x_\Delta(s-\tau))-g_\Delta(\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau))|^2ds.
\end{aligned}$$

Recalling the estimate on H_2 in the proof of Theorem 3.6 as well as Remark 5.2, we see that

$$\mathbb{E}\int_0^T |g(x_\Delta(s),x_\Delta(s-\tau))-g_\Delta(\bar{x}_\Delta(s),\bar{x}_\Delta(s-\tau))|^2ds\le C\,\Delta^{2\gamma\wedge(1-2\epsilon)}.$$

Moreover, by Assumption 5.1,

$$\begin{aligned}
&\mathbb{E}\int_0^T |g(x(s),x(s-\tau))-g(x_\Delta(s),x_\Delta(s-\tau))|^2ds\\
\le&\int_0^T \big(\mathbb{E}|x(s)-x_\Delta(s)|^2+\mathbb{E}|x(s-\tau)-x_\Delta(s-\tau)|^2\big)ds\\
\le&C\,\Delta^{2\gamma\wedge(1-2\epsilon)}.
\end{aligned}$$

It therefore follows from (5.7)that

$$\mathbb{E}\Big(\sup_{0\le t\le T} H_4(t)\Big)\le C\,\Delta^{2\gamma\wedge(1-2\epsilon)}+0.5\mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(s)|^2\Big].\tag{5.8}$$

Hence the required assertion (5.4) follows from (5.5) along with (5.6) and (5.8). The proof is complete. □

As pointed out in Section 3, $\bar{x}_\Delta(t)$ is computable, but $x_\Delta(t)$ is not in general. It would therefore be very useful to have a convergence result like (5.4) but $x_\Delta(t)$ there is replaced by $\bar{x}_\Delta(t)$. For this purpose, let us present a lemma.

Lemma 5.4. *Let $\Delta \in (0,1]$ and $\epsilon \in (0,1/4]$. Let ν be a sufficiently large integer for which*

$$\frac{2\nu}{2\nu-1}(T+1)^{1/\nu} \le 2 \quad \textit{and} \quad \frac{1}{\nu} < \epsilon. \tag{5.9}$$

We then have

$$\mathbb{E}\Big(\sup_{0\le t\le T} |x_\Delta(t) - \bar{x}_\Delta(t)|^2\Big) \le 2(2\nu+1)(h(\Delta))^2\Delta^{1-\epsilon}. \tag{5.10}$$

Proof. Let κ be the integer part of T/Δ, namely $\kappa = \lfloor T/\Delta \rfloor$. Recalling (2.7), we derive

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T} |x_\Delta(t) - \bar{x}_\Delta(t)|^2\Big)\\
&\le \mathbb{E}\Big(\max_{0\le k\le\kappa} \sup_{t_k\le t\le t_{k+1}} |f_\Delta(\bar{x}_\Delta(t_k))(t-t_k) + g_\Delta(\bar{x}_\Delta(t_k))(B(t)-B(t_k))|^2\Big)\\
&\le 2\mathbb{E}\Big(\max_{0\le k\le\kappa} \sup_{t_k\le t\le t_{k+1}} |f_\Delta(\bar{x}_\Delta(t_k))|^2(t-t_k)^2 + |g_\Delta(\bar{x}_\Delta(t_k))|^2|B(t)-B(t_k)|^2\Big)\\
&\le 2(h(\Delta))^2\Big[\Delta^2 + \mathbb{E}\Big(\max_{0\le k\le\kappa} \sup_{t_k\le t\le t_{k+1}} |B(t)-B(t_k)|^2\Big)\Big].
\end{aligned} \tag{5.11}$$

By the Hölder inequality and the Doob martingale inequality, we then derive that

$$\begin{aligned}
&\mathbb{E}\Big(\max_{0\le k\le\kappa} \sup_{t_k\le t\le t_{k+1}} |B(t)-B(t_k)|^2\Big)\\
&\le \Big[\mathbb{E}\Big(\max_{0\le k\le\kappa} \sup_{t_k\le t\le t_{k+1}} |B(t)-B(t_k)|^{2\nu}\Big)\Big]^{1/\nu}\\
&\le \Big[\sum_{k=0}^{\kappa} \mathbb{E}\Big(\sup_{t_k\le t\le t_{k+1}} |B(t)-B(t_k)|^{2\nu}\Big)\Big]^{1/\nu}\\
&\le \Big[\sum_{k=0}^{\kappa} \Big(\frac{2\nu}{2\nu-1}\Big)^{2\nu} \mathbb{E}|B(t_{k+1})-B(t_k)|^{2\nu}\Big]^{1/\nu}\\
&\le \Big[\sum_{k=0}^{\kappa} \Big(\frac{2\nu}{2\nu-1}\Big)^{2\nu} (2\nu-1)!!\Delta^{\nu}\Big]^{1/\nu}\\
&\le \Big[\Big(\frac{2\nu}{2\nu-1}\Big)^{2\nu} (T+1)(2\nu-1)!!\Delta^{\nu-1}\Big]^{1/\nu},
\end{aligned} \tag{5.12}$$

where $(2\nu-1)!! = (2\nu-1)\times(2\nu-3)\times\cdots\times 3\times 1$. But

$$[(2\nu-1)!!]^{1/\nu} \le \frac{1}{\nu}\sum_{i=1}^{\nu}(2i-1) = \nu.$$

Hence

$$\mathbb{E}\Big(\max_{0\le k\le\kappa} \sup_{t_k\le t\le t_{k+1}} |B(t)-B(t_k)|^2\Big) \le \frac{2\nu}{2\nu-1}(T+1)^{1/\nu}\nu\Delta^{(\nu-1)/\nu} \le 2\nu\Delta^{1-\epsilon}. \tag{5.13}$$

Substituting this into (5.11) yields the required assertion (5.9). The proof is complete. □

The following more useful theorem follows from Theorem 5.3 and Lemma 5.4 immediately.

Theorem 5.5. *Let Assumptions 2.1 - 2.3 and 5.1 hold and assume that $\bar{p} > r+2$. Let $\mu(u) = K_5 u^{(2+r)/2}$ and $h(\Delta) = \Delta^{-\epsilon}$ for some $\epsilon \in (0, 1/4]$. Then, for every $\Delta \in (0,1]$,*

$$\mathbb{E}\Big(\sup_{0\le t\le T} |x(t) - \bar{x}_\Delta(t)|^2\Big) \le C\,\Delta^{2\gamma\wedge(1-3\epsilon)}. \tag{5.14}$$

6 Examples. Let us discuss more examples in this section to illustrate our theory.

Example 6.1. Let us first return to the SDDE (2.17). In Section 2.5 we have verified Assumptions 2.1 - 2.4 and, in particular, we have $\bar{p} = 18.5$, $r = 4$ and $\gamma = 0.5$.

First of all, let $h(\Delta) = \Delta^{-1/5}$ as in Section 2.5. By Corollary 3.7, we can then conclude that the truncated EM solutions will approximate the true solution $x(t)$ of the SDDE (2.17) in the sense that

$$\mathbb{E}|x_\Delta(T) - x(T)|^2 \vee \mathbb{E}|\bar{x}_\Delta(T) - x(T)|^2 \le C\Delta^{3/5} \tag{6.1}$$

for all $\Delta \in (0,1]$. We emphasise that the order $\Delta^{3/5}$ is the same as that in (2.18) but (6.1) holds for any $\Delta \in (0,1]$ while (2.18) holds only if $\Delta \le 10^{-11}$ which may be too small in practice.

To improve the convergence order, we next let $h(\Delta) = \Delta^{-1/6}$ for $\Delta \in (0,1]$. Then condition (2.14) becomes

$$\Delta^{-1/6} \ge 10\Delta^{-1/8.25}, \text{ namely, } \Delta \le 10^{-22}.$$

By Theorem 2.5, we can only conclude that

$$\mathbb{E}|x_\Delta(T) - x(T)|^2 \vee \mathbb{E}|\bar{x}_\Delta(T) - x(T)|^2 \le C\Delta^{2/3} \tag{6.2}$$

for $\Delta \le 10^{-22}$. But it is almost impossible to use such a small stepsize in practice. On the other hand, by Corollary 3.7, we can conclude that

$$\mathbb{E}|x_\Delta(T) - x(T)|^2 \vee \mathbb{E}|\bar{x}_\Delta(T) - x(T)|^2 \le C\Delta^{2/3} \tag{6.3}$$

for all $\Delta \in (0,1]$. In other words, we do not only get the same convergence order but also allow the stepsize $\Delta \in (0,1]$.

Example 6.2. Consider the scalar SDDE

$$dx(t) = f(x(t), x(t-\tau))dt + g(x(t), x(t-\tau))dB(t), \quad t \ge 0, \tag{6.4}$$

with the initial data $\{x(\theta) : -\tau \le \theta \le 0\} = \xi \in C([-\tau, 0]; \mathbb{R})$ satisfying Assumption 2.2, where

$$f(x,y) = a_1 + a_2|y|^{4/3} - a_3x^3 \quad \text{and} \quad g(x,y) = a_4|x|^{4/3} + a_5y, \quad x, y \in \mathbb{R},$$

and $a_1, \cdots, a_5$ are all real numbers with $a_3 > 0$. Clearly, the coefficients f and g are locally Lipschitz continuous. Moreover, for any $\bar{p} > 2$, we have

$$xf(x,y) + \frac{\bar{p}-1}{2}|g(x,y)|^2 \le |a_1||x| + |a_2||x||y|^{4/3} - a_3|x|^4 + (\bar{p}-1)(a_4^2|x|^{8/3} + a_5^2|y|^2).$$

But, by the well-known Young inequality,

$$|x||y|^{4/3} = (|x|^3)^{1/3}(|y|^2)^{2/3} \le |x|^3 + |y|^2.$$

We therefore have

$$\begin{aligned} & xf(x,y) + \frac{\bar{p}-1}{2}|g(x,y)|^2 \\ & \le |a_1||x| + |a_2||x|^3 + a_4^2(\bar{p}-1)|x|^{8/3} - a_3|x|^4 + (|a_2| + a_5^2(\bar{p}-1))|y|^2 \\ & \le K_1(1+|y|^2), \end{aligned}$$

where $K_1 = (|a_2| + a_5^2(\bar{p}-1)) \vee \beta_1$ and

$$\beta_1 = \sup_{u\ge 0}\big[|a_1|u + |a_2|u^3 + a_4^2(\bar{p}-1))u^{8/3} - a_3u^4\big] < \infty.$$

That is, Assumption 2.1 is satisfied for any $\bar{p} > 2$.

To verify Assumption 2.3, we note that, for $x, \bar{x}, y, \bar{y} \in \mathbb{R}$,

$$\begin{aligned} (x-\bar{x})(f(x,y) - f(\bar{x},\bar{y})) &= (x-\bar{x})\big[-a_3(x^3-\bar{x}^3) + a_2(|y|^{4/3} - |\bar{y}|^{4/3})\big] \\ &\le -0.5a_3|x-\bar{x}|^2(x^2+\bar{x}^2) + 0.5a_2^2(x-\bar{x})^2 + 0.5a_2^2(|y|^{4/3} - |\bar{y}|^{4/3})^2 \end{aligned}$$

while

$$|g(x,y) - g(\bar{x},\bar{y})|^2 \le 2a_4^2(|x|^{4/3} - |\bar{x}|^{4/3})^2 + 2a_5^2|y-\bar{y}|^2.$$

But, by the mean-valued theorem,

$$\begin{aligned} &(|y|^{4/3} - |\bar{y}|^{4/3})^2 \le \frac{16}{9}|y-\bar{y}|^2(|y|^{1/3} + |\bar{y}|^{1/3})^2 \\ \le &|y-\bar{y}|^2(|y|^{2/3} + |\bar{y}|^{2/3}) \le |y-\bar{y}|^2\big(\beta_2 + (a_3/2a_2^2)(|y|^2 + |\bar{y}|^2)\big), \end{aligned}$$

where

$$\beta_2 = \sup_{u\ge 0}(2u^{2/3} - a_3u^2/a_2^2) < \infty.$$

Similarly,

$$(|x|^{4/3} - |\bar{x}|^{4/3})^2 \le |x-\bar{x}|^2\big(\beta_3 + (a_3/8a_4^2)(|x|^2 + |\bar{x}|^2)\big),$$

where

$$\beta_3 = \sup_{u\ge 0}(2u^{2/3} - a_3u^2/8a_4^2) < \infty.$$

Hence, for $\alpha = 1$,

$$\begin{aligned} &(x-\bar{x})(f(x,y) - f(\bar{x},\bar{y})) + \frac{1+\alpha}{2}|g(x,y) - g(\bar{x},\bar{y})|^2 \\ &\le K_3[|x-\bar{x}|^2 + |y-\bar{y}|^2] - U(x,\bar{x}) + U(y,\bar{y}), \end{aligned}$$

where $K_3 = 0.5a_2^2(1+\beta_2) + 2a_4^2\beta_3 + 2a_5^2$ and

$$U(x,\bar{x}) = 0.5a_3|x-\bar{x}|^2(x^2+\bar{x}^2).$$

We have hence verified Assumption 2.3. It is also straightforward to show that Assumption 2.4 is satisfied with $r=4$ (and some K_4 which is not important).

To apply Theorem 3.6, we still need to design functions μ and h satisfying (2.4) and (3.1). Note that

$$\sup_{|x|\le u}(|f(x)| \vee |g(x)|) \le \hat{a}u^3, \quad \forall u \ge 1,$$

where $\hat{a} = (|a_1|+|a_2|+a_3) \vee (|a_4|+|a_5|)$. We can hence have $\mu(u) = \hat{a}u^3$ and its inverse function $\mu^{-1}(u) = (u/\hat{a})^{1/3}$ for $u \ge 0$. For $\epsilon \in (0,1/4]$, we define $h(\Delta) = \Delta^{-\epsilon}$ for $\Delta > 0$. By Theorem 3.6, we can then conclude that the truncated EM solutions will converge to the true solution of the SDE (6.4) in the sense that

$$\mathbb{E}|x_\Delta(T) - x(T)|^2 \vee \mathbb{E}|\bar{x}_\Delta(T) - x(T)|^2 \le C\Delta^{2\gamma\wedge(1-2\epsilon)} \tag{6.5}$$

for all $\Delta \in (0,1]$. In particular, if γ is close to 0.5 (or bigger than half), this shows that the order of convergence is close to 0.5.

Example 6.3. Let us still consider the scalar SDDE (6.4) but change the diffusion coefficient into $g(x,y) = a_4x + a_5y$. We see clearly Assumptions 2.1 - 2.3 and 5.1 hold. We also let $\mu(u) = \hat{a}u^3$ for $u \ge 0$ and $h(\Delta) = \Delta^{-\epsilon}$ for $\epsilon \in (0,1/4]$. Then, by Theorem 5.5, we can conclude that for every $\Delta \in (0,1]$,

$$\mathbb{E}\Big(\sup_{0\le t\le T}|x(t) - \bar{x}_\Delta(t)|^2\Big) \le C\,\Delta^{2\gamma\wedge(1-3\epsilon)}. \tag{6.6}$$

7 Conclusion. In this paper, we reviewed one of the main results of [GMY17] and pointed out a restrictive condition imposed there via an example. We then successfully established the strong convergence theory without this restrictive condition. We compared our new result with the one in [GMY17] and highlighted our significant contribution in this paper. We also established a new strong convergence theory for the solutions over a finite time interval, and this was not discussed in [GMY17]. Examples were used to motivate our paper and to illustrate our new theory.

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