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The average order of the Möbius function for Beurling primes

Ammar Ali Neamah[1,2] and Titus W Hilberdink[1]

[1]Department of Mathematics, University of Reading, Whiteknights, PO Box 220, Reading RG6 6AX, UK; t.w.hilberdink@reading.ac.uk

[2] Faculty of Computer Science and Mathematics, University of Kufa, Najaf, Iraq; ammara.meamah@uokufa.edu.iq, a.a.n.al-rammahi@pgr.reading.ac.uk

Abstract

In this paper, we study the counting functions $\psi_{\mathcal{P}}(x)$, $N_{\mathcal{P}}(x)$ and $M_{\mathcal{P}}(x)$ of a generalized prime system $\mathcal{N}$. Here $M_{\mathcal{P}}(x)$ is the partial sum of the Möbius function over $\mathcal{N}$ not exceeding x. In particular, we study these when they are asymptotically well-behaved, in the sense that $\psi_{\mathcal{P}}(x) = x + O(x^{\alpha+\varepsilon})$, $N_{\mathcal{P}}(x) = \rho x + O(x^{\beta+\varepsilon})$ and $M_{\mathcal{P}}(x) = O(x^{\gamma+\varepsilon})$, for some $\rho > 0$ and $\alpha, \beta, \gamma < 1$. We show that the two largest of α, β, γ must be equal and at least $\frac{1}{2}$.

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1. Introduction

A *Beurling generalized prime system* $\mathcal{P}$ is an unbounded sequence of real numbers $p_1, p_2, p_3, \ldots$ satisfying

$$1 < p_1 \leq p_2 \leq \cdots \leq p_n \leq \cdots .$$

We call these numbers *generalized primes* (or g-primes), and from them we form the system $\mathcal{N}$ of *generalized integers* (or g-integers) associated to $\mathcal{P}$. These are the numbers of the form

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

where $\alpha_1, \ldots, \alpha_k \in \mathbb{N}_0$. In other words, $\mathcal{N}$ (viewed as a multi-set) is the semi-group generated by the (multi-set) $\mathcal{P}$ under multiplication. Such systems were first defined and investigated by Beurling [1] in 1937 and have been studied by many researchers since then (see for instance [2], [5] and the numerous references therein). Attached to these systems are the counting functions

$$\pi_{\mathcal{P}}(x) = \sum_{\substack{p \leq x \\ p \in \mathcal{P}}} 1, \quad N_{\mathcal{P}}(x) = \sum_{\substack{n \leq x \\ n \in \mathcal{N}}} 1, \quad \psi_{\mathcal{P}}(x) = \sum_{\substack{p^k \leq x \\ p \in \mathcal{P} \\ k \in \mathbb{N}}} \log p,$$

which generalize the usual counting functions. In each case, the sum is over all possible elements from the multi-set $\mathcal{P}$ or $\mathcal{N}$ with the given constraint. We are also interested in the generalized *Möbius* function defined to be $\mu_{\mathcal{P}}(1) = 1$, $\mu_{\mathcal{P}}(p_{i_1} \cdots p_{i_k}) = (-1)^k$ for distinct g-primes (i.e. $i_1, \ldots, i_k$ are distinct) and zero otherwise. Strictly speaking this need not be a function if two such products are numerically the same. In any case, we define the sum function

$$M_{\mathcal{P}}(x) = \sum_{\substack{n \leq x \\ n \in \mathcal{N}}} \mu_{\mathcal{P}}(n).$$

This generalizes the usual $M(x) = \sum_{n \le x} \mu(n)$. The associated *Beurling zeta function* is defined as usual by

$$\zeta_{\mathcal{P}}(s) = \prod_{p \in \mathcal{P}} \frac{1}{1 - \frac{1}{p^s}} = \sum_{n \in \mathcal{N}} \frac{1}{n^s}.$$

We are interested in systems for which one or more of $\psi_{\mathcal{P}}(x) - x$, $N_{\mathcal{P}}(x) - \rho x$, or $M_{\mathcal{P}}(x)$ is $O(x^\theta)$ for some $\theta < 1$ (and $\rho > 0$). More precisely, we define three numbers α, β, γ by the following:

$$\psi_{\mathcal{P}}(x) = x + O(x^{\alpha+\varepsilon}) \tag{1}$$

$$N_{\mathcal{P}}(x) = \rho x + O(x^{\beta+\varepsilon}) \tag{2}$$

$$M_{\mathcal{P}}(x) = O(x^{\gamma+\varepsilon}) \tag{3}$$

hold for all $\varepsilon > 0$ but no $\varepsilon < 0$. For example, for $\mathcal{N} = \mathbb{N}$, $\beta = 0$ while $\alpha = \gamma \ge \frac{1}{2}$ due to the Riemann zeros. At the outset we are only interested in those systems for which the abscissa of convergence of the Dirichlet series for $\zeta_{\mathcal{P}}$ is 1. Thus $\alpha, \beta, \gamma \in [0, 1]$ in any case.

For (1) and (2) to hold simultaneously for some $\alpha, \beta < 1$ is akin to having a kind of Riemann Hypothesis being true for such a system. In [11], it was shown that such a system does exist with $\alpha, \beta \le \frac{1}{2}$. On the other hand, in [6], it was shown that it is impossible to have both α and β less than $\frac{1}{2}$.

We note that (3) is related to an interesting problem in its own right: *how small can $M_{\mathcal{P}}(x)$ be made for a system with abscissa*[1] *equal to* 1*?* In other words, how much cancellation can occur in the sum for $M_{\mathcal{P}}(x)$? Of course, for $\mathcal{N} = \mathbb{N}$, $M(x) = \Omega(\sqrt{x})$ on account of the Riemann zeros, but without this knowledge it is not clear how to even prove $M(x) = \Omega(x^a)$ for some $a > 0$. This is similar to a question of Kahane and Saias [9] who ask how small $\sum_{n \le x} f(n)$ can be for f completely multiplicative.

It is also related to the more general question of the size of $M_{\mathcal{P}}(x)$ and how it relates to the other functions. For example, much work has been done to determine under what conditions one has $M_{\mathcal{P}}(x) = o(x)$ (see for example Chapter 14 of [5]). Zhang [10] was the first to note that PNT is not equivalent to $M_{\mathcal{P}}(x) = o(x)$. For the most general results giving $M_{\mathcal{P}}(x) = o(x)$, see the very recent papers [3] and [4].

Our main result is the following:

Theorem 1
Of the numbers α, β, γ, the two largest must be the same and at least $\frac{1}{2}$.

This result implies that for $M_{\mathcal{P}}(x) = O(x^\gamma)$ with $\gamma < \frac{1}{2}$ to hold, we need the system to have somewhat chaotic g-primes and g-integers; i.e. the errors in (1) and (2) have to be $\Omega(x^{\frac{1}{2}-\varepsilon})$ for every $\varepsilon > 0$. It may be conjectured that having $\gamma < \frac{1}{2}$ is actually impossible.

2. Some Relevant Results

In order to prove the main result we shall need some relevant notions as well as existing results about g-prime systems.

Let $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{b_n^s}$ be a generalized Dirichlet series where $b_n > 0$ is strictly increasing with finite abscissae of absolute convergence σ_a. Suppose f has a meromorphic continuation to $H_\alpha := \{s \in \mathbb{C} : \Re s > \alpha\}$. We say f has *finite order* in H_α if

$$f(\sigma + it) \ll |t|^\lambda \qquad (|t| \ge 1)$$

[1] The abscissa of convergence of $\zeta_{\mathcal{P}}(s)$. With abscissa σ_c, we trivially have $M_{\mathcal{P}}(x) \ll x^{\sigma_c+\varepsilon}$. Without the condition on the abscissa, $M_{\mathcal{P}}(x)$ can even be bounded: take $\mathcal{P} = \{2^{2^n} : n \in \mathbb{N}_0\}$. Then $M_{\mathcal{P}}(x) = 0, 1$ or -1.

for $\sigma > \alpha$. As such, we can define the *Lindelöf function* $\mu_f(\sigma)$ to be the infimum of such λ. It is well-known that μ_f is non-negative, decreasing and convex and for $\sigma > \sigma_a$, $\mu_f(\sigma) = 0$.

The following result about such Dirichlet series and "counting function"

$$A(x) := \sum_{b_n \le x} a_n$$

was essentially proved in [6], Proposition 3 (see also [8], Theorem 2.1). It was proven for the case where $a_n \ge 0$ such that $a_n \ll n^\varepsilon$ for all $\varepsilon > 0$. This latter condition however is not necessary. Also we shall require a particular case when a_n is also sometimes negative.

Theorem A
Let $f(s) = \sum_{n=1}^\infty \frac{a_n}{b_n^s}$ have abscissa of convergence $\sigma_c \le 1$. Suppose that for some $\alpha \in [0,1)$ and $c \in \mathbb{C}$, we have

$$A(x) = cx + O(x^{\alpha+\varepsilon}) \quad \text{for all } \varepsilon > 0. \tag{4}$$

Then $f(s)$ has an analytic continuation to $H_\alpha \setminus \{1\}$ with a simple pole at $s = 1$ with residue[2] c and f has finite order; indeed $\mu_f(\sigma) \le 1$ for $\sigma > \alpha$.

Conversely, suppose that for some $\alpha \in [0,1)$, $f(s)$ has an analytic continuation to H_α except for a simple pole at $s = 1$ with residue c. Further assume that $\mu_f(\sigma) = 0$ for $\sigma > \alpha$ and either
(i) $a_n \ge 0$ or
(ii)

$$\sum_{x-1<b_n\le x} |a_n| = O(x^{\alpha+\varepsilon}) \quad \text{for all } \varepsilon > 0. \tag{5}$$

Then (4) holds.

Proof. The proof of the first part is standard and follows on writing

$$f(s) = s\int_1^\infty \frac{A(x)}{x^{s+1}}\,dx = \frac{cs}{s-1} + s\int_1^\infty \frac{A(x)-cx}{x^{s+1}}\,dx,$$

and noting that the integral on the right converges absolutely to a holomorphic function on H_α.

For the converse, we follow the proof of Proposition 3 in [6] as much as possible. This leads to

$$A(x) - cx \ll \frac{x}{T^{1-\varepsilon}} + x^{\alpha+\varepsilon}T^\varepsilon + \frac{x^{1+\varepsilon}}{T} + \frac{x}{T}\sum_{\frac{x}{2}<b_n<2x} \frac{|a_n|}{|b_n - x|} \tag{6}$$

for every $T > 1$ and $\varepsilon > 0$ — see equation (3.7) of [6].

Now, as in [6], consider x such that

$$\left(x - \frac{1}{x^2}, x + \frac{1}{x^2}\right) \cap \{b_k : k \in \mathbb{N}\} = \emptyset. \tag{7}$$

For such x, $|b_n - x| \ge \frac{1}{x^2}$ for all n and the sum on the right in (6) is at most

$$x^2 \sum_{\frac{x}{2}<b_n<2x} |a_n|.$$

In case (i), this is $O(x^{3+\varepsilon})$, while in case (ii), it is $O(x^{3+\alpha+\varepsilon})$ by (5).

Taking $T = x^4$ in (6) shows that (4) holds whenever $x \to \infty$ satisfying (7). As shown in [8] (see (2.3)), for every x there exist $x_1 \in (x-1, x)$, $x_2 \in (x, x+1)$ such that x_1, x_2 satisfy (7). Thus (4) holds for x_1 and x_2.

[2] Of course, if $c = 0$, the pole is removable.

For case (i), positivity of a_n implies $A(x_1) \le A(x) \le A(x_2)$. Hence (4) follows for x.

For case (ii), we use (5). We have

$$|A(x) - A(x_1)| \le \sum_{x-1<b_n\le x} |a_n| \ll x^{\alpha+\varepsilon}$$

by (5). Hence (4) follows.

□

We also require the following result from [8] (Theorem 2.3).

Theorem B

Suppose (1) *and* (2) *hold for some* $\alpha, \beta < 1$. *Then for* $\sigma > \Theta := \max\{\alpha, \beta\}$ *and uniformly for* $\sigma \ge \Theta + \delta$ *(any* $\delta > 0$*),*

$$\frac{\zeta'_{\mathcal{P}}(\sigma+it)}{\zeta_{\mathcal{P}}(\sigma+it)} = O\Big((\log|t|)^{\frac{1-\sigma}{1-\Theta}+\varepsilon}\Big) \quad \text{and} \quad \zeta_{\mathcal{P}}(\sigma+it), \frac{1}{\zeta_{\mathcal{P}}(\sigma+it)} = O\Big(\exp\Big\{(\log|t|)^{\frac{1-\sigma}{1-\Theta}+\varepsilon}\Big\}\Big)$$

for all $\varepsilon > 0$. *In particular, for* $\sigma > \Theta$*, the Lindelöf functions for* $\zeta_{\mathcal{P}}$ *and* $\frac{1}{\zeta_{\mathcal{P}}}$ *are zero.*

Actually, the statement of Theorem 2.3 in [8] does not mention $\frac{1}{\zeta_{\mathcal{P}}}$ but the proof, which argues from $\log \zeta_{\mathcal{P}}$ clearly applies also to $-\log\zeta_{\mathcal{P}} = \log\frac{1}{\zeta_{\mathcal{P}}}$.

Also, we have the following two consequences as described at the end of section 2 in [8]:

(a) If $\alpha > \beta$, then $\zeta_{\mathcal{P}}$ has infinitely many zeros on, or arbitrarily close to, the line $\sigma = \alpha$.

(b) If $\alpha < \beta$, then $\zeta_{\mathcal{P}}$ and $\frac{1}{\zeta_{\mathcal{P}}}$ have infinite order in the strip $\{s \in \mathbb{C} : \alpha < \Re s < \beta\}$.

Proof of Theorem 1

Let $\Theta := \max\{\alpha, \beta\}$. We use the converse part of Theorem A with $f(s) = \frac{1}{\zeta_{\mathcal{P}}(s)}$. This function has an analytic continuation to H_Θ and, by Theorem B, has zero order here. Further, $A(x) = M_{\mathcal{P}}(x)$ and

$$\sum_{\substack{x-1<n\le x\\ n\in\mathcal{N}}} |\mu_{\mathcal{P}}(n)| \le N_{\mathcal{P}}(x) - N_{\mathcal{P}}(x-1) \ll x^{\beta+\varepsilon} \le x^{\Theta+\varepsilon}.$$

Thus (5), and hence (4), holds (with $c = 0$). That is, $M_{\mathcal{P}}(x) = O(x^{\Theta+\varepsilon})$; i.e. $\gamma \le \Theta$.

Now suppose $\alpha > \beta$. Then $\zeta_{\mathcal{P}}$ has infinitely many zeros on, or arbitrarily close to, the line $\sigma = \alpha$. Thus $\gamma \ge \alpha - \delta$ for any $\delta > 0$; i.e. $\gamma \ge \alpha$ and so $\gamma = \alpha$.

Now suppose $\alpha < \beta$. Then the Lindelöf functions for $\zeta_{\mathcal{P}}$ and $1/\zeta_{\mathcal{P}}$ are infinite for $\sigma < \beta$. Thus we cannot have $\gamma < \beta$ by the first part of Theorem A with $A(x) = M_{\mathcal{P}}(x)$; i.e. $\gamma = \beta$.

Thus if $\alpha \ne \beta$, then $\gamma = \Theta$. Hence the two largest numbers are always equal. Finally, since $\max\{\alpha,\beta\} \ge \frac{1}{2}$, we see that in all three cases the largest pair is always at least $\frac{1}{2}$.

□

2. Systems with different α, β, γ.

It is perhaps of interest to see if it really is possible that each of α, β or γ can be strictly less than the other two and whether it can be less than $\frac{1}{2}$.

(a) $\beta < \alpha = \gamma$. For $\mathcal{N} = \mathbb{N}$, we have $\beta = 0$ and, under the Riemann Hypothesis, $\alpha = \gamma = \frac{1}{2}$. Unconditionally, we only have $\alpha = \gamma = \Theta$ where $\Theta = \sup\{\Re\rho : \zeta(\rho) = 0\}$.

(b) $\alpha < \beta = \gamma$. In the final discussion of [6], a g-prime system was given with $\alpha = 0$. Namely, take $p_n = R^{-1}(n)$, where R is the strictly increasing function on $[1, \infty)$ defined by

$$R(x) = \sum_{k=1}^{\infty} \frac{(\log x)^k}{k!k\zeta(k+1)},$$

where $\zeta(\cdot)$ is the Riemann zeta-function. As such, one has $\psi_{\mathcal{P}}(x) = x + O(\log x \log\log x)$. By Theorem 1, $\beta = \gamma$, but what this common value is is not clear, except that it lies in $[\frac{1}{2}, 1]$.

(c) $\gamma < \alpha = \beta$. For this we can use the example $\mathcal{P} = \mathbb{P} \sqcup \mathbb{P}^{1/\beta}$ with $\beta \in (0,1)$. Using Dirichlet's hyperbola method, we have

$$N_{\mathcal{P}}(x) = \sum_{mn^{1/\beta} \leq x} 1 = \zeta\Big(\frac{1}{\beta}\Big)x + \zeta(\beta)x^{\beta} + O(x^{\frac{\beta}{1+\beta}})$$

(see [7] where this calculation was done). Furthermore, $\psi_{\mathcal{P}}(x) = \psi(x) + \psi(x^{\beta}) = x + x^{\beta} + O(x^{\frac{1}{2}+\varepsilon})$ on RH. Thus $\alpha = \beta$. But, with $M(x) = \sum_{n \leq x} \mu(n)$,

$$M_{\mathcal{P}}(x) = \sum_{mn^{1/\beta} \leq x} \mu(m)\mu(n) = \sum_{n \leq a^{\beta}} M\Big(\frac{x}{n^{1/\beta}}\Big) + \sum_{n \leq b} M\Big(\Big(\frac{x}{n}\Big)^{\beta}\Big) - M(a^{\beta})M(b)$$

for any $ab = x$. Putting $a = x^{\lambda}$ and using the bound $M(x) \ll x^{\frac{1}{2}+\varepsilon}$ gives

$$\begin{aligned} M_{\mathcal{P}}(x) &\ll \sum_{n \leq x^{\lambda\beta}} \Big(\frac{x}{n^{1/\beta}}\Big)^{\frac{1}{2}+\varepsilon} + \sum_{n \leq x^{1-\lambda}} \Big(\Big(\frac{x}{n}\Big)^{\beta}\Big)^{\frac{1}{2}+\varepsilon} + (x^{\lambda\beta})^{\frac{1}{2}+\varepsilon} x^{(1-\lambda)(\frac{1}{2}+\varepsilon)} \\ &\ll \Big(x^{\frac{\beta}{2}+\lambda(1-\frac{\beta}{2})} + x^{\frac{1}{2}+(1-\lambda)(\beta-\frac{1}{2})} + x^{\frac{\lambda}{2}+(1-\lambda)\frac{\beta}{2}}\Big)x^{\varepsilon}. \end{aligned}$$

Choosing $\lambda = \frac{\beta}{1+\beta}$ optimally shows that $M_{\mathcal{P}}(x) \ll x^{\frac{3\beta}{2(1+\beta)}+\varepsilon}$ for all $\varepsilon > 0$. Thus $\gamma \leq \frac{3\beta}{2(1+\beta)} < \beta$. Note that $\gamma \geq \frac{1}{2}$, since $\frac{1}{\zeta_{\mathcal{P}}(s)} = \frac{1}{\zeta(s)\zeta(s/\beta)}$ has poles on the $\frac{1}{2}$-line.

Open problems

1) From (a) and (b) above we have systems with $(\alpha, \beta, \gamma) = (a, 0, a)$ and $(0, b, b)$ for some $a, b \in [\frac{1}{2}, 1]$. Can we find, unconditionally, such systems with $a < 1$ and $b < 1$?

2) In (c) above we have a system, conditional on RH, with $(\alpha, \beta, \gamma) = (c, c, d)$ with $\frac{1}{2} \leq d < c < 1$. Can we find one unconditionally, with $d < 1$. Furthermore, can we find one with $d < \frac{1}{2}$?

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