Learning trajectories and fractions: primary teachers' meaning attributions

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A learning trajectory constitutes a hypothesis and a description of students' thinking related to learning a mathematical notion. The study reported here, employing a multiple case study approach, investigates the use of learning trajectories in the teaching of fractions by three 5th grade Greek primary school teachers. Particularly, the research problem pursued concerns the teachers' meaning making and use of the concept of learning trajectories introduced by a recent Mathematics Curriculum reform. The results of analyzing the teachers' answers to two semi-structured interviews and the transcripts of a non-participant observation of their lessons show that learning trajectories were understood as means of planning teaching and used as 'maps' of navigating classroom instruction, predominately successful when students' previous knowledge and thinking, and not just their difficulties, are taken into consideration.

Keywords: Learning trajectories, fractions, mathematics education, teaching mathematics.

Learning trajectories in mathematics education

Learning trajectories consist of three parts: the goal, the developmental path and a sequence of appropriate tasks that correspond to each level of thought and help the child proceed to the next level. There is also a relationship between developmental progress and the activities selected (Clements & Sarama, 2009). The "big ideas" of mathematics, i.e., sets of ideas and skills that are coherent and are central to mathematics, are the objectives (Kuntze et al., 2011). Developmental paths refer to the levels of thinking, each of which is more sophisticated than the previous one and leads to the achievement of the mathematical objective (Clements & Sarama, 2009). Notwithstanding the distinct levels, it is not to be assumed that the track is linear or specified. Various factors, such as differences between formal and informal learning, affect the development of a learning trajectory (IEP, 2014). Finally, the sequence of the tasks is a set of work corresponding to each level of thinking, supporting the learning of mathematical ideas and skills belonging to each stage and including solving, investigating, experimenting and communicating in the context of the student's activity.

Learning trajectories are promising tools that can help in curriculum development, mathematics teaching and assessment. A substantial use of learning trajectories can bring about several reversals in the expected developmental paths of students and in the expected research findings as it opens new routes in learning. Moreover, it seems to give new possibilities both for the design and the implementation of the curriculum as well as for teaching due to the detailed theoretical basis it can offer (Clements & Sarama, 2009). There are two studies worth mentioning that make use of the notion of learning trajectory. The first is part of a wider research program emphasizing the development of formative assessment tools for the trajectory of fractions (Confrey & Maloney, 2010). The results of the study reveal that a coherent trajectory leads to its redesigning, distinguishing between the levels of thought on the one hand and the parameters of the tasks on the other, clarifying the non-linearity of the procedure. The second study concerns the work of Steffe (2004), who has been extensively involved with students' learning and knowledge of fractional numbers. According to the findings of this study, a trajectory allows teachers to understand how students construct mathematical concepts in order to influence the evolution of notions and processes so that, through productive teaching, productive learning is supported.

The study aims to explore the understanding and the use of the notion of learning trajectories by three primary teachers teaching fractions in the 5th grade, where this mathematical notion holds a central position in the respective curriculum, concluding their 'fractional' thinking at the threshold to secondary education and to being introduced to the set of rational numbers in a mainly formal way.

Fractions in school mathematics: Students' difficulties

Understanding fractions is crucial for pupils' mathematical thinking because of its significance in becoming numerate citizens; also for its relation to learning related fundamental mathematical notions like proportion and percentage or developing powerful mathematical modes of thinking like algebra. In the current discussion in mathematics education the focus is on educating pupils as numerate future citizens, i.e., citizens who can operate successfully in the modern society by understanding and handling quantitative data that they encounter in everyday life. Being competent in dealing with the notion of rational numbers constitutes a basic prerequisite. During schooling, many hours are devoted to teaching fractions, but the difficulties and misconceptions of students identified as early as in the 1970's persist, fueling substantial research activity on children's understanding of fractions and on developing new teaching tools and approaches to the subject matter. The numerous studies on fractions tended to initially focus their interest on the way students learn but also on the difficulties they meet. Subsequently, research interest shifted to teaching fractions with emphasis on the conceptual or the procedural mode of the relevant knowledge pursued and on how students responded to these two modes of knowing and thinking. Finally, attention was drawn to the teachers' role and teaching practices (Van Dooren, Lehtinen & Verschaffel, 2015).

The main students' difficulties identified in the relevant studies arise from the different nature of fractions compared to natural numbers. Rational numbers indicate the relation between two quantities, whereas natural numbers refer to one quantity. Also, rational numbers are dense because of continuity and infiniteness, while natural numbers are distinct. Unlike natural numbers, where each number is represented in a unique way, rational numbers can be denoted by fractions, decimals and percentages or even by a set of equivalent fractions. Furthermore, operations may differ in the two sets of numbers. For example, the result of division and multiplication of fractions is not less or bigger than the original number respectively, when the divisor or the multiplier is less than 1, as happens with natural numbers.

Apart from students' difficulties arising from the close connection of rational numbers and thus of fractions with natural numbers, difficulties with fractions have been found to also be related to the abstract nature of the processes involved in operating with fractions as well as to the erroneous application of the related rules. Finally, teachers' inadequate knowledge of fractions and poor teaching practices as well as cultural issues have been identified to contribute to students' learning of this important mathematical notion.

The study

The reported study focuses on: (a) teachers' meaning making of the learning trajectories in general; (b) teachers' understanding and use of the notion of learning trajectories in teaching highly demanding mathematical learning contexts: fractions in the 5th grade. The method adopted is a 'multiple case study' of three female teachers working in an experimental primary school in an urban area of northern Greece.

The three teachers have had almost the same years of teaching experience but different scientific and professional profiles. Teacher A is a graduate of both a primary education and a mathematics department, with 17 years of teaching experience; teacher B, a graduate of a primary education department, holds a postgraduate diploma and a PhD and has 14 years of teaching experience. Finally, teacher C is a graduate of a two-year course with 16 years of teaching experience. Teachers A and C participated actively in the piloting of a new mathematics curriculum which promoted active learning and approaches like 'learning trajectories' that support this type of learning, while teacher B did not. The research tools employed were a semi-structured interview (one initial and one at the end of the study) and a series of non-participant observations of their mathematics instruction. The non-participant observation was used to provide information for the final interview and to offer a further source of data thus enhancing the credibility of the study.

The first semi-structured interview, consisting of two parts, served the first research question. The questions in the first part explored teachers' understandings of conceptual aspects of the construction of learning trajectory in mathematics and how its use influences the choice of the mathematical content. The questions in the second part were related to the teaching and learning benefits of exploiting learning trajectories in the mathematics classroom (in the context of the new mathematics curriculum).

For the second research question, a second semi-structured interview was conducted, which also consisted of two parts. The questions in the first part were common to all three teachers: on planning, implementing and reflecting on teaching fractions based on the notion of learning trajectory. In the second part of the interview, based on extracts from each teacher's instruction identified in the observations, the teachers were asked to comment on how they used the learning trajectory approach. For each participating teacher, two double teaching sessions on fractions were observed (non-participant observation). Field notes were kept, with the emphasis being on teachers' communication with students, on tasks used and generally on teachers' actions and discourse. The aim was to identify ways in which the trajectory of fractions was used and adapted to the students' needs. Where necessary, a brief discussion with the teachers was carried out to clarify and explain choices.

A combination of Grounded Theory and Content Analysis techniques was used for analyzing the data. More specifically, a three-stage analysis was followed for the interview data: (a) careful reading of the data and detection of relevant extracts (b) coding of the extracts and grouping them in sub-groups of similar meaning and (c) repetition of stage (b) within each of the emerging sub-groups for the formation of third level categories. For reliability reasons, the whole process was realized simultaneously by the three researchers. The non-participant observation data collected for the second research question were used for completing and refining categories emerging from the interview data analysis, but also for providing critical teaching incidents used in the final interview for the teachers to comment on.

Results

Due to the limited space, only the results of the 1st and 2nd semi-structured interviews will be presented. As far as the first research question is concerned, the meanings attached to the notion of a Learning Trajectory (LT) developed by the three teachers converge only to some respect to the relevant literature.

In particular, teacher A considers that the trajectories are organized into thematic areas (algebra, geometry, etc.). She recognizes a similarity between LT and mathematics as a discipline. This similarity concerns both the way mathematical knowledge is organized and the historical progress of human thinking in mathematics.

> The trajectory matches with mathematics in my view, how they are structured…in this way the approach of knowledge suits with the construction of knowledge and of human's thought because it seems that the notion of number has progressed developmentally in a similar way. And this helps a lot… (teacher A)

Teachers B and C seem to be more influenced by learning theories and interpret the organization of knowledge through LTs in a similar way.

> In the trajectory you are interested in student's understanding, you want to activate his thought rather than to help him practice his memory and to simply remember the terms... (teacher B)

Teacher B, recognizing constructivist elements in the idea of LTs, considers them as providing a specifically arranged infrastructure for the construction of the mathematical knowledge, where the pre-existing knowledge serves to shape the new.

> In the trajectory, I think that somehow a gradual construction of knowledge must be carried out…that is, starting with the natural numbers, we will go to the fractions... (teacher B)

Teacher C, on the other hand, indicates various and somewhat contradictory views on how knowledge is organized in learning paths through LT notion (e.g., constructivism as well as behaviorism).

> We start to break the notion into parts and after each one is completed, we say that we have completed the particular part... (teacher C)

It appears that the three teachers' understandings of LTs are not identical but are overall compatible to the relevant literature with respect to the way in which a LT organizes and represents the progress of knowledge. But what they certainly do not distinguish is that a LT is not a simple organization of cognitive objectives but an empirically formulated proposal on how a student learns a mathematical concept. In addition, organizing knowledge along a LT suggests a learning approach with specific characteristics derived from the constructivist perspective. These basic characteristics are conceptual understanding (teacher B) and engagement in active learning (teacher C).

> Now, I do not know what exactly they learn. But I see that they start to investigate, they ask questions, I like that. In traditional contexts they wouldn't do that because they do not work this way…. But now they pose questions, they ask, "Why do we do that in this way? (teacher B)

> We felt that our experience was positive, and we saw that children responded very well, they liked this approach. Through an informally provided feedback, we see that their high school mathematics teachers are very pleased with these students… (teacher C)

This line of thinking is compatible with the concept of LT, since the first use of the term was made by Simon (1995), who proposed it as a teaching tool for the "constructivist" teacher.

On the other hand, teacher A considers LT to be independent from learning and teaching issues, limiting its use to the level of organizing mathematical content, which can be then framed by a variety of teaching approaches chosen by the teacher. This is a position also adopted by teacher C. However, teacher B argues that LT use in teaching improved her own knowledge of mathematical concepts resulting in her understanding better students' thinking. Finally, all three teachers report that LTs, combined with various teaching practices, contributed both to the learning process (teacher B) and to the learning outcomes (all teachers) in a variety of ways, such as, by facilitating logical connections and strengthening exploratory learning.

> But if you can use the model or remember the wall of fractions, you can reach the equivalence much more easily... Or let's say the addition 1/4 + 1/2, if the student depicts it and puts 1/4 and 1/2 close, he ends up much more easily to rules of addition than if you dictate the rule to him. (teacher B)

Teacher A also reports that she has too identified improvement in students' attitudes towards the lesson, but this is not the case for teacher C, who believes that the improvement noticed can be attributed to various factors. This finding is in line with that of Clements and Sarama's (2009) about LTs and the reinforcement of mathematical knowledge.

As far as the second research question is concerned, that is, how teachers claim to use LTs, the analysis of the relevant data reveals that LTs provided a general framework for organizing teaching, within which each teacher worked in her own way. Thus, the intermediate stages of the trajectory of fractions appeared to have provided successive intermediate learning targets for the instruction. Thus, teacher A attempted to incorporate LTs from different areas of the curriculum to simultaneously meet different objectives. This choice is of interest since the relevant research refers to the contribution of interacting LTs to pupils' learning. For achieving these objectives, tasks were organized, which not only corresponded to the objectives but also incorporated socio-cultural features, such as the history of mathematics and issues of everyday life, according to teacher A. Thus, in terms of lesson planning, LTs were used by teachers as a 'curriculum' that organizes the learning objectives. However, it is noteworthy that during the implementation of the lesson plan, LTs tended to be adapted by the teachers accordingly to the students' emerging needs.

All three teachers attempted to obtain relevant information about students through informal assessment processes during the instruction. This was mainly done through students' answers to the tasks, with the emphasis being on the explanations and the justification provided. At the same time, feedback was provided through students' work (teacher A) and by the way the students used the manipulatives and the models of fractions while dealing with the tasks (teachers B and C).

> Also, the difference is that with LT I also use manipulatives. I do not just use the textbook or representations only, I use materials that students use, test them, change, see them differently... (teacher C)

Teacher A provided additional comments for each student individually about the progress and the difficulties in relation to the basic learning objectives included in the trajectory of fractions. Thus, she could have a "picture" of the whole class with regard to the points of the trajectory that students found more difficult. In one way or another, therefore, the teachers collected information about their students' learning. Likewise, the three participating teachers interpreted the information collected in their own way and proceeded to make decisions about their instruction. Their interpretations influenced their teaching choices. It seems, however, that LT supported a more focused interpretation taking into account students' past knowledge and thinking in order to guide the learner to the desired objective through a developmental path.

> Maybe the knowledge they are building using the trajectories is more solid…When you tell them about the multiplication or the division, if they have learnt only the procedures without understanding them, they cannot go on…but with the trajectories it is so easy for them to perceive 3/6, 4/8, 5/10 and they say "this is a half" without any stress. (teacher C)

The LT of fractions seems to have played an active role in teaching, whereas the developmental path that it offered made it easier for the teachers to link the pupils' previous knowledge to the new one. Thus, there were several occasions where, in order to understand some issues and to deal with the difficulties of fractions, teachers used students' knowledge coming from other trajectories (i.e., natural numbers). This approach is consistent with the literature claiming that teachers should adjust LTs on the basis of children's pre-existing knowledge and preferences. It also presents similarities with the use of learning trajectories by the Dutch TAL project, where the aims of a trajectory were linked to each other in ways that could be achieved through this association (Simon & Tzur, 2004).

When dealing with students' difficulties, however, the choices for the development of the instruction did not always include the active use of the trajectory of fractions. For example, an occasion in which the trajectory did not seem to have played an important role was the extended stay on a target, backing it up with extra tasks and supporting material (often digital).

> We had difficulties in finding ½ of 3/4. We had difficulties in these things, but again, the models and various digital applications in the internet showed the way…How one can use ½ of ¾ … and we also used manipulatives where the students, in groups, were experimenting…how I can find 1/3 of 3/6, let's say. (teacher A)

Another option was to use LT to simplify activities to suit the potential of students. This may indicate a case in which, as the relevant literature suggests, the teacher adapted the trajectory to students and chose activities appropriate to their level (Clements & Sarama, 2009). Thus, on the one hand, the objectives were not always at the ideal level, but on the other, students were given the opportunity to follow a course compatible with their level and needs.

In addition, teachers tended to allow each student to go at their own pace and go along with their own progress on a "micro-scale". This personal progress often began from the intuitive and ended up to the formal level or started from a simple and reached to a more complex stage (teacher A). This appears to be very close to the proposal of the new Mathematics Curriculum (2014) according to which the sense of number is cultivated progressively and at successive levels of abstraction and generalization. Furthermore, there were many occasions where different solutions to problems corresponding to different levels of understanding would appear (teacher B) indicating that a LT allows students to expose individual differences in learning (IEP, 2014). It therefore follows that, in general, the notion of LT has been used in the class by teachers in various ways as a general framework and as a useful tool, not just as a list of cognitive objectives and corresponding tasks.

> You can go ahead and see that something has not been understood…don't you come back to negotiate it again? You have to. So, you can come back and deal with the notion again (teacher B)

In general, however, LT's incorporation into teaching was proven to be more substantial when using the knowledge held by the students. In other words, a learning trajectory may be more useful and effective in teaching when the teacher takes into account the basic principles of constructivism, such as the developmental progress of learning and the importance of pre-existing knowledge. This goes back to its original purpose, when the term was first mentioned by Simon (1995), considering it as a tool for designing teaching according to constructivist principles.

Discussion and concluding remarks

The participating teachers interpret the concept of LT in ways that present similarities and deviations from the official literature. They tend to recognize that there is a certain sequence in teaching and to identify constructivist elements such as building on pre-existing knowledge and active engagement of the learner in the learning process. However, they view LT as a way of organizing mathematical content rather than as assumptions about the development of students' thinking when learning a mathematical notion, and so it seems that they perceive it yet as another local curriculum. As to the meaning attributions related to the application of LTs, overall, the contribution of LT to teaching is found either in organizing the content or in improving teachers' knowledge of mathematical notions and consequently in improving teaching. Thus, the participating teachers treated LTs just as means of organizing the mathematical content. At the same time, the use of the notion of LT did had a positive effect both on the learning process and the learning outcomes, as it encouraged exploratory learning procedures, exploited learning through connections between the notions and led to a solid and structured formulation of knowledge.

Learning trajectories appear to have provided the teachers with a general framework for organizing their teaching in ways that shared common features but also varied. These ways were not limited just to organizing the content. At the first phase, the use of LTs was related to lesson planning, including the development of individual objectives for a mathematical notion and the corresponding tasks. However, during the instruction and on several occasions, the teachers modified the originally planned instruction under the light of emerging issues and students' thought. Thus, in order to teach the new mathematical content, the pre-existing knowledge that was part of the trajectory of fractions or of a different trajectory was used. Also, LT was adapted to the capabilities and the level of students' thinking through simplification of objectives and varied problem solving processes. Furthermore, a personal route of the students from the intuitive to the formal and from the simple to the complex was realized, for example, in the comparison of fractions. In contrast to the above, the LT of fractions was not particularly effective in cases where difficulties were simply identified without further interpretation of students' thinking. Teachers tended to stay for a long time in a learning objective without actually taking into account students' background so as to combine it with the trajectory at hand.

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