Essential stability in large square economies

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Abstract

Exchange economies are defined by a mapping between an atomless space of agents and a space of characteristics where the commodity space is a separable Banach space. We characterize equilibrium stability of economies relaying on the continuity of the equilibrium correspondence. We provide a positive answer to an open question about the continuity of the Walras correspondence in infinite dimensional spaces. In addition, we do not assume neither differentiability nor a fixed set of agents for the different economies, like it is usually assumed in the stability literature.

Keywords: Essential Stability, Walras Correspondence, Infinitely Many Commodities, Large Economies, Nowhere Equivalence

1. Introduction

The existence of a competitive equilibrium is followed by questions regarding the characterization of the equilibrium set in order to analyze efficiency, uniqueness or regularity properties. These results, and specifically those of reg-

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ularity, are closely related to the finiteness of the equilibrium set. It is the finite property that allows to define a concept of locally stable equilibria. For this purpose, it is required to analyze how the set of equilibria responds to small perturbations in exogenous parameters that characterize agents and, therefore, economies. This relation between parameters and equilibrium sets has been captured in the literature through correspondences that associate economies with its equilibria. The approach generally consists of proving conditions over this *equilibrium correspondence* in order to conclude that it defines finite sets. This is the aim of the pioneering work of Debreu (1970) assuming differentiability conditions and using the Theorem of Sard (1942).

Furthermore, Kannai (1970), Hildenbrand (1970) and Hildenbrand and Mertens (1972) introduce the study on the continuity of the *equilibrium correspondence* for pure exchange economies. All these studies, also including Balasko (1975), understand parameters as exogenous characteristics that define the agents (i.e. consumption sets, tastes or endowments). In particular, it turns to be a crucial point the way in which a topology in the space of economies is defined.

In this study, we use a concept of stability for competitive economies related to the continuity property of the equilibrium correspondence, i.e., essential stability that was introduced in the fixed point theory by Fort (1950) and, accordingly to game theory by Wen-Tsun and Jia-He (1962). In particular, the translation from game theory to economies states that an equilibrium is *essentially stable* if it is possible to approximate it by equilibria of "similar" economies, i.e. economies that are close to the economy of reference under a metric in the space of economies that has to be precised. Generally speaking, defining the space of economies by a metric space requires to parameterize the family of economies of interest with respect to the dimensions of similarity. In our case, the dimensions are consumption sets, preference relations and endowments. It is possible to extend our analysis to other parameterizations of economies, e.g. externalities, tax structures or information, by requiring that the metric space of economies remains complete. Mas-Colell (1977a) raised the following question regarding the equilibrium set:

> Is there a dense set of economies having a finite set of equilibria? We shall see the answer is yes, but this is not by itself a very interesting property; what one wants (for, say, estimation or prediction purposes) is that those equilibria be "essential", i.e., that they do not disappear by performing an arbitrarily small perturbation of the economy.

This quotation emphasizes that for our purposes the most accurate definition should be regarding essentiality instead of regularity. We remark that every regular equilibrium is essential but the converse is not true. Furthermore, in order to characterize this concept, we need to study the relation between parameters and equilibria instead of the equilibrium set.

Recently, the continuity of the equilibrium correspondence in general equilibrium theory was stated by Dubey and Ruscitti (2015) and He et al. (2017). We extend their results taking into consideration infinite dimensional commodity spaces and by characterizing stability when the continuity property in the equilibrium correspondence can not be obtained directly. In fact, our results answer the question posited in Dubey and Ruscitti (2015) about the possibility of getting stability results in infinite dimensional economies. In addition we remark that we have not restricted the economies to have the same space of agents as it has typically been done in the literature.

2. The model

We characterize an economy by a relation between agents and characteristics. The characteristics of the agents and, also, the quantity of them may vary across economies. Therefore, the representation of an economy could be a map between the space of agents and the space of characteristics for which it is necessary to define at first, the commodity space. In turn, it induces a distribution over the space of characteristics.

In the following subsections we define a general spaces of agents, characteristics and characteristics types that are common to the space of economies we study.

2.1. Space of agents

The **space of economic agents** is an atomless measure space $(A, \mathcal{A}, \mu)$ which is also separable. That is, for any coalition $A' \in \mathcal{A}$ such that $\mu(A') > 0$ there is a $\mathcal{A}$-measurable coalition $A'' \subset A$ such that $0 < \mu(A'') < \mu(A')$ where $\mathcal{A}$ is separable with respect to a suitable distance.

2.2. The commodity space and the space of characteristics

The commodity space is defined by a separable Banach space $(L, \|\cdot\|)$ whose positive cone has a non-empty interior. Consequently, the price space is given by the positive cone of the topological dual of L, L^*_+. We endow this space with the weak-star topology w^*. Each consumption set X is a subset of L_+. Endowed with the weak topology w, (L, w) is also a complete topological vector space. By $\|\cdot\|$-topology we mean the topology induced by the norm $\|\cdot\|$. An analogous notation is given to the w-topology.

We consider a convex and w-compact subset Q of the space L which includes all consumption sets and the vectors 0 and u, where $u \in \text{int}L_+$ and $\|u\| = 1$. Clearly, Q is w-closed (whence $\|\cdot\|$-closed since Q is convex) and $\|\cdot\|$-bounded (Diestel (1984), p. 17). The norm on Q, $\|\cdot\|_Q$, is induced from $\|\cdot\|$. Notice that $(Q, \|\cdot\|_Q)$ is a Polish space since L is Polish (Fristedt and Gray (1996), Proposition 3, p. 350). The weak topology on Q, w_Q, is the relativization to Q of w. Even though (L, w) is not metrizable, it is (Q, w_Q) since Q is w-compact (Dunford and Schwartz (1958), Theorem 3. p. 434). Furthermore, (Q, w_Q) is separable (Aliprantis and Border (2006), Lemma 3.26, p. 85) and obviously complete. Consequently (Q, w_Q) is a locally compact Polish space. The positive cone of Q, denoted Q_+, is $Q_+ = L_+ \cap Q$. Since Q_+ is a closed subset of Q then it also a locally compact Polish space. The vector u belongs to the norm interior of Q_+ since it belongs to $\text{int}L_+ \cap Q$ and $\|u\|_Q = 1$. A typical element of Q, X, is a consumption set. We denote by F a $\|\cdot\|$-closed subset of Q such that $(F, \|\cdot\|_F)$ is the corresponding topological subspace, where $\|\cdot\|_F$ is the relativization of

$\|\cdot\|$ to F.[1] Taking into account the considerations at the beginning of this section, we have that $(F, \|\cdot\|_F)$ is a Polish space since $(L, \|\cdot\|)$ is also Polish. As we shall assume later, all initial endowments and Walrasian allocations belong to F.

Regarding the consideration of having a w-compact subset of the commodity space L, we note that similar assumptions are made in large economies even if the commodity space is finite dimensional. Indeed, Hildenbrand (1974) p. 85-86 states a condition on consumption sets which in turn implies that the family of such spaces is a compact set (Theorem 1, p. 96). The works of Khan and Yannelis (1991) and Noguchi (1997) assume that each consumption set is weak-compact. Bewley (1991) takes as commodity space the non-separable space l_∞ and assumes the existence of a common consumption set which is a weak*-compact subset of l_∞^+. More recently, Khan and Sagara (2016) also assumes the existence of a common consumption set which is weak-compact and metrizable. On the other hand, several papers on topologies on the space of preferences take as commodity spaces locally compact ones (see Back (1986), Chichilnisky (1980), Kannai (1970) Remark 1, Mas-Colell (1977b) among others). In this sense, the consideration of Q is consistent with that literature and we shall make use of some important results of it.

Consider the preference relation $(X, \succ)$ such that $\succ \subset X \times X$ is a transitive and irreflexive binary relation on X. Let $\mathcal{P}$ be the set preference relations. For each $(X, \succ) \in \mathcal{P}$ we associate the set $P := \{(x, y) \in X \times X \mid (x, y) \notin \succ\}$. In addition we shall also consider the endowment vector e belonging to X.

Let us consider now the set of all monotonic preference relations in $\mathcal{P}$ as follows $\mathcal{P}_{mo} := \{(X, \succ) \in \mathcal{P},$ such that for all x, y in $X,$ if $x \geq y$ and $x \neq y$ then $x \succ y\}$. Hence, the **space of characteristics** is given by $\mathcal{P}_{mo} \times F$

[1] See our Assumption BA (3) in section 4. This space plays the role of the norm compact subset of the commodity space in the definition of the economy number (4) p. 236 in Khan and Yannelis (1991). Note that this assumption is always satisfied for the finite dimensional case.

where a typical element is $((X, \succ), e) \in \mathcal{P}_{mo} \times F$.

2.3. Space of characteristic types

Fix a particular space of agents $(A, \mathcal{A}, \mu)$. Let $X : (A, \mathcal{A}, \mu) \to Q_+$ the correspondence that associates a consumption set to each agent in the space. Consequently, denote $\succ(a) \subset X(a) \times X(a)$ the preference relation of agent $a \in A$ provided the space of agents. Given the relation $((X(a), \succ(a))$ and $x, y \in X(a)$, we shall say that $x \succ (a)\ y$ if and only if $(x, y) \in \succ(a)$. Similarly, define a Bochner integrable, X-valued and measurable function $e : (A, \mathcal{A}, \mu) \to F$ that specifies the endowments for the agents in the given space. Thus, we shall say that $e \in L_1(\mu, F)$ which allows us to well define the aggregate endowment by $\int_A e d\mu$.

We formally define an economy by means of a mapping between the space of agents and the space of characteristics.

Definition 1. *An economy is a function $\mathcal{E} : (A, \mathcal{A}, \mu) \to \mathcal{P}_{mo} \times F$ which is measurable with respect to a given countably-generated sub-σ-algebra $\mathcal{G}$ of $\mathcal{A}$, and the Borel σ-algebra $\mathcal{B}(\mathcal{P}_{mo} \times F)$*

The sub-σ-algebra $\mathcal{G}$ gives place to the measure space $(A, \mathcal{G}, \mu)$ which defines the characteristic type space since $\mathcal{G}$ can be viewed as the σ-algebra induced by $\mathcal{E}$.

Thus, for a given $a \in A$ we have $\mathcal{E}(a) = ((X_{\mathcal{E}}(a), \succ_{\mathcal{E}}(a)), e_{\mathcal{E}}(a)) \in \mathcal{P}_{mo} \times F$. When it is clear which mapping $\mathcal{E}$ is considered, we shall represent an agent $a \in A$ in a shorter way, that is, by $((X(a), \succ(a)), e(a)) \in \mathcal{P} \times F$. Even more, for the sake of simplicity, sometimes we shall write directly $((X, \succ), e) \in \mathcal{P} \times F$.

The image of the map that defines the economy should have a measurable structure that we assume to be the σ-algebra of the Borelians on the space of characteristics. For the sake of a simple notation, we omit those precisions. The **characteristic type space** is $(A, \mathcal{G}, \mu)$ for a given economy $\mathcal{E}$ whose sub-σ-algebra is $\mathcal{G}$.

2.4. Walrasian equilibirum

A Bochner-integrable function $f : (A, \mathcal{A}, \mu) \longrightarrow Q_+$ is an *allocation* for the economy $\mathcal{E}$ if $f \in L_1(\mu, X_{\mathcal{E}})$. Further, f is said to be *attainable* for $\mathcal{E}$ if $\int_A f d\mu = \int_A e_{\mathcal{E}} d\mu$.

The demand for agent a at prices $p \in L^*_+$ in the economy $\mathcal{E}$ is given by $D_{\mathcal{E}(a)}(p)$, i.e., maximal elements for $\succ_{\mathcal{E}} (a)$ in

$$B_{\mathcal{E}(a)}(p) = \{x \in X_{\mathcal{E}}(a) : p(x - e_{\mathcal{E}}(a)) \leq 0\}.$$

Definition 2. *An allocation for $\mathcal{E}$, $\overline{f} : (A, \mathcal{A}, \mu) \to Q_+$, is walrasian if there is a price vector $\overline{p} \in L^*_+$ such that:*

(i) $\overline{f}(a) \in D_{\mathcal{E}(a)}(\overline{p})$ *for μ-almost all $a \in A$,*

(ii) $\int_A \overline{f} d\mu = \int_A e d\mu$.

Thus, a walrasian allocation jointly with a corresponding price, $(\overline{f}, \overline{p})$, is called a walrasian equilibrium.

3. Difficulties with a double infinity of agents and commodities

The aim of the present paper is to extend previous results on essential stability by allowing both infinitely many commodities and varying atomless space of agents. In order to compare what has been done previously, we can identify two kinds of stability results: those regarding continuity of walrasian and Cournot-Nash correspondences. In the first case, as far as we know, all commodity spaces are finite dimensional explicit or implicitly. Furthermore, sequences of economies typically have varying sets of finite agents. In the second case, the set of agents is assumed fixed (see Carbonell-Nicolau (2010) or Correa and Torres-Martínez (2014)). Notice that even if we only consider economies with finitely many agents and an infinite dimensional commodity space, essential stability is not ensured (See Dubey and Ruscitti (2015), p. 2). In addition, we add uncountable varying sets of agents.

In extending the analysis to the previously described contexts, we have the following drawbacks

- Essential stability relies on Fort's Theorem (Fort (1950)) which requires a metric set of prices. However, when considering infinite dimensional spaces, price simplex is endowed with the weak*-topogy which is not metrizable on the whole price space.

- Also due to Fort's Theorem, the set of economies is required to be Baire. This is the case if the set of characteristics is Polish. For this, it is sufficient to take a locally compact commodity space. However, Hausdorff topological vector spaces are locally compact if and only if they are finite dimensional (Aliprantis and Border (2006), Theorem 4.63, p. 150).

As for the first difficulty, if the price simplex is compact in the weak*-topology then it is metrizable. This solves the first problem. Regarding the second one, we consider a locally compact subset of a non-locally compact commodity space by taking a suitable weak-compact subset Q of the space L (see Section 2.2). Then, we shall have that the set of characteristics is contained in a compact set yet the commodity space is not locally compact. An alternative approach is to embed the space $(L, \|\cdot\|)$ into the Hilbert cube $(\mathcal{H}, d_{\mathcal{H}})$ which is a compact metrizable space. We profit from the fact that the topology of $\mathcal{H}$ (the $d_{\mathcal{H}}$-topology) induced on L is equivalent to the $\|\cdot\|$-topology. Besides, no additional assumption is needed in comparison with the first approach. We postpone the second approach to the Appendix B.

4. Space of Economies

We first consider economies without strongly convex preferences. In Section 6, we introduce and assume this condition that simplifies the assumption required.

Basic Assumptions (BA). For each economy $\mathcal{E}$ we have:

1. X is contained in Q_+ and contains 0 and u. It is a norm-closed and convex subset of L_+.
2. $(X, \succ)$ is weak-relatively open in $X \times X$.

3. F is contained in every X and each walrasian allocation f belongs to the space F.

Assumptions BA (1) and (2) are natural in these configurations. Assumption BA (1) implies that each X is a w-closed subset of Q_+ whence w-compact. Because of Assumption BA (2) each preference P is a $w \times w$-closed subset of $L_+ \times L_+$ and then a $w_Q \times w_Q$-closed subset of $Q_+ \times Q_+$.

BA (3) says that all individual endowments and Walras allocations belong to a common set. An analogous assumption is made in Hart et al. (1974). Assumption BA (3) becomes relevant in two aspects. First, because we are not assuming neither that the commodity space is Q_+, as most papers with infinitely many commodities do, nor that there is a common consumption set X as in Khan and Sagara (2016). Consequently, this assumption allows us to consider relevant sequences of economies through Skorokhod's Theorem which are essential in many proofs of the paper. Second, because every converging sequence in the metric space $(F, \|\cdot\|_F)$ is a converging sequence in the normed space $(L, \|\cdot\|)$. Hence, in the proof of Theorems 1 and 2 we can deal with the well known difficulty of joint continuity in infinite dimensional spaces.

We restrict the commodity space to the relevant subset Q and, within it, to F where we assume that all initial endowments and Walras allocations take values. Since every economy considers both aggregate initial endowments and aggregate feasible allocations, we show that every allocation with range in F is Bochner integrable

Proposition 1. *Every function $f : A \to F \subset L$ which is $(\mathcal{A}, \mathcal{B}(F))$-measurable is Bochner integrable.*

Proof. We start by claiming that f is $(\mathcal{A}, \mathcal{B}(F))$-measurable if and only if it is $(\mathcal{A}, \mathcal{B}(L))$-measurable. Indeed, suppose that f is $(\mathcal{A}, \mathcal{B}(F))$-measurable. Then, for any $B \in \mathcal{B}(L)$, $f^{-1}(B) = f^{-1}(B \cap F) \cup f^{-1}(B \cap (L \setminus F))$. $f^{-1}(B \cap F) \in \mathcal{A}$ since f is $(\mathcal{A}, \mathcal{B}(F))$-measurable and $f^{-1}(B \cap (L \setminus F)) = \emptyset$ since f takes values only in F. The converse is obvious.

Since F is separable, there exists a sequence of $(\mathcal{A}, \mathcal{B}(F))$-measurable simple functions $\{f_n\}_{n\in\mathbb{N}}$ from A into F which converges in norm to f a.e. (see Aliprantis and Border (2006) Theorem 4.38 1. p. 145). By previous claim, the simple functions are $(\mathcal{A}, \mathcal{B}(L))$-measurable and since f is $\|\cdot\|$-bounded it is Bochner integrable (Diestel (1977), Theorem 2, p. 45) □

Remark 1. *Let us consider the space $(F, \|\cdot\|_F)$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of $(\mathcal{A}, \mathcal{B}(F))$-measurable functions from A into F, and let $f : A \to F$ be a $(\mathcal{A}, \mathcal{B}(F))$-measurable function such that $f_n(a)$ $\|\cdot\|$-converges to $f(a)$ a.e. By the above remark, each f_n and f are Bochner integrable and since Q is $\|\cdot\|$-bounded one can use the Dominated Convergence Theorem in Dunford and Schwartz (1958), p. 328 to claim that the limit with respect to the norm $\|\cdot\|$ of $\int_A f_n(a)d\mu$ is equal to $\int_A f(a)d\mu$.*

The following result shows that we can even follow an alternative version of the Dominated Convergence Theorem when the weak topology w_Q is considered. This result is required to ensure the closed-graph of the equilibrium correspondence (See Appendix A.3).

Proposition 2. *Let $(A, \mathcal{A}, \mu)$ be a finite measure space. Let x and the sequence $\{x_n\}_{n\geq 1}$ be measurable functions from $(A, \mathcal{A}, \mu)$ into F such that the limit with respect to the w-topology of $x_n(a)$, or the $w - \lim_{n\to\infty} x_n(a)$, is equal to $x(a)$ a.e, then $w - lim_{n\to\infty} \int_A x_n(a)d\mu = \int_A x(a)d\mu$.*

Proof. Let $\{x_n\}_{n\geq 1}$ be a sequence from the measure space $(A, \mathcal{A}, \mu)$ into F such that $x_n(a)$ converges to $x(a)$ a.e with respect to w. Let $f \in (L^*, w^*)$ arbitrary. By weak-pointwise convergence of $\{x_n\}_{n\geq 1}$ one has that $\lim_{n\to\infty} f(x_n(a)) = f(x(a))$ a.e. Since f is w-continuous it is $\|\cdot\|$-bounded and thus the sequence $\{f \circ x_n\}_{n\geq 1}$ and $f \circ x$ are bounded and $(\mathcal{A}, \mathcal{B}(\mathbb{R}))$-measurable whence, strong measurable since $\mathbb{R}$ is separable. Consequently, for each $n \geq 1$ both $\{f \circ x_n\}_{n\in\mathbb{N}}$ and $f \circ x$ are integrable. Hence, by the Dominated Convergence Theorem $\lim_{n\to\infty} \int_A f(x_n(a))d\mu = \int_A f(x(a))d\mu$ (compare with Remark 1). On the other hand, by Proposition 1 every x_n and x are Bochner integrable, whence

$f\left(\int_A x_n(a)d\mu\right) = \int_A f(x_n(a))d\mu$ (Aliprantis and Border (2006) Lemma 11.45, p. 427). Consequently, we deduce that $w - \lim_n \int_A x_n(a)d\mu = \int_A x(a)d\mu$. $\square$

Let $\mathcal{C}^{w_Q}(Q \times Q)$ be the set of all $w_Q \times w_Q$-closed subsets of $Q \times Q$. We denote by τ_C the topology of closed convergence on $\mathcal{C}^{w_Q}(Q \times Q)$. Since every P belongs to $\mathcal{C}^{w_Q}(Q \times Q)$, we can define a mapping $g : \mathcal{P} \to \mathcal{C}^{w_Q}(Q \times Q)$ by $(X, \succ) \mapsto P$. It is easily verified that g is an injection. Indeed, let $(X, \succ) \neq (X', \succ')$ in $\mathcal{P}$ and let us assume that $P = P'$. If $X = X'$, then we have that $(X \times X) \setminus \succ = (X' \times X') \setminus \succ'$, whence $\succ = \succ'$ which contradicts $(X, \succ) \neq (X', \succ')$. If $X \neq X'$ one can assume without loss of generality that $X \setminus X' \neq \emptyset$. It follows from $(X \times X) \setminus \succ = (X' \times X') \setminus \succ'$ that for any $y \in X \setminus X'$ that $(y, y) \in \succ$ which contradicts irreflexivity. Consequently, we must have $P \neq P'$ whenever $(X, \succ) \neq (X', \succ')$.

We define the topology $\tau_C^{\mathcal{P}}$ on $\mathcal{P}$ by $\tau_C^{\mathcal{P}} = \{g^{-1}(U) : U \in \tau_C\}$. Thus $\tau_C^{\mathcal{P}}$ can be seen as the topology τ_C induced on $\mathcal{P}$.[2] We characterize the preferences of the space of characteristics in the following lemma which follows and adapts the arguments of Theorem 1 in Hildenbrand (1974).

Lemma 1. *Under Assumption BA (1)-(2) the following holds:*

1. $(\mathcal{P}, \tau_C^{\mathcal{P}})$ *is compact and metrizable (and hence, a Polish space)*
2. *A sequence of preferences* $\{(X^n, \succ^n)\}_{n \geq 1}$ *converges to* $(X, \succ)$ *in* $(\mathcal{P}, \tau_C^{\mathcal{P}})$ *if and only if* $Li(P_n) = P = Ls(P_n)$
3. *The set* $\{((X, \succ), x, y) \in \mathcal{P} \times Q \times Q : x, y \in X \text{ and } x \not\succ y\}$ *is closed for the product topology* $\tau_C^{\mathcal{P}} \times w_Q \times w_Q$. *Furthermore,* $\tau_C^{\mathcal{P}}$ *is the weakest topology on* $\mathcal{P}$ *for which the above set is closed.*

The proof is given in Appendix A.1. The following corollary is useful to achieve individual optimality in the proof of Theorem 1.

[2] See Hervés-Beloso et al. (1999) for another applications of the closed convergence topology for infinite dimensional space of characteristic.

Corollary 1. *Let $(X, \succ) \in \mathcal{P}$ such that $x, y \in X$ and $x \succ y$. There exists an $\tau_C^{\mathcal{P}}$-open neighborhood $U_{(X, \succ)}$, a w_Q-open neighborhood V_x and a w_Q-open neighborhood V_y, such that for all $(X', \succ') \in U_{(X, \succ)}$ and for all $(x', y') \in (X' \cap V_x) \times (X' \cap V_y)$ we have $x' \succ' y'$.*

Proof. Since the set $\{((X, \succ), x, y) \in \mathcal{P} \times Q \times Q : x, y \in X \, and \, x \not\succ y\}$ is closed for the product topology $\tau_C^{\mathcal{P}} \times w_Q \times w_Q$, then $\mathcal{P} \times Q \times Q \backslash \{((X, \succ), x, y) \in \mathcal{P} \times Q \times Q : x, y \in X \, and \, x \not\succ y\}$ is $\tau_C^{\mathcal{P}} \times w_Q \times w_Q$-open. □

The following lemma shows that $\mathcal{P}_{mo}$ is also Polish.

Lemma 2. *Under Assumptions BA (1)-(2), the subset $\mathcal{P}_{mo}$ is a Polish space.*

The proof is given in Appendix A.2. We state now an additional assumption which concerns "small" perturbations of consumption sets. We note that this assumption is not necessary when strongly convex preferences are concerned (see Section 6).

Assumption C. Let $\{X_n, \succ_n\}_{n \geq 1}$ be a sequence converging to $(X, \succ)$ with respect to $\tau_C^{\mathcal{P}}$ such that $X_n, X : (A, \mathcal{A}, \mu) \twoheadrightarrow Q$. For all $x \in L_1(\mu, X)$, there exists a sequence $\{x_n\}_{n \geq 1}$ in $L_1(\mu, X_n)$ which $\| \cdot \|$-converges pointwise to x.

From an economic point of view, it means that a small change in the consumption set has a relatively small impact in consumption bundles. Notice that from a mathematical point of view, since X is the closed limit of $\{X_n\}_{n \geq 1}$ for every $x \in X$ there exists a sequence $\{x_n\}_{n \geq 1}$ in $L_1(\mu, X_n)$ which w-converges pointwise to x. So, Assumption C imposes a stronger convergence. This assumption is automatically satisfied when the commodity space is the positive cone L_+ as it is usually assumed in the literature on infinite dimensional commodity spaces.

Together with the assumption in the following section, we provide examples of economies satisfying our configuration in Section 4.2.

4.1. Nowhere Equivalence

At this point we characterize the relationship between an agent space $(A, \mathcal{A}, \mu)$ and its characteristic type space $(A, \mathcal{G}, \mu)$, where $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{A}$ induced by the measurable mapping $\mathcal{E}$. Indeed, $\mathcal{G}$ is the σ-algebra generated by $\mathcal{E}^{-1}(\mathcal{B}(\mathcal{P}_{mo}) \otimes \mathcal{B}(F))$. Next, we introduce the nowhere equivalence condition of He et al. (2017) as follows.

Let $\mathcal{G}$ be a countably generated sub-σ-algebra of the σ-algebra $\mathcal{A}$. For any $A' \in \mathcal{A}$, such that $\mu(F) > 0$, the restricted probability space $(A', \mathcal{A}^{A'}, \mu^{A'})$ is defined by: $\mathcal{A}^{A'} = \{A' \cap \ A'' : A'' \in \mathcal{A}\}$ and $\mu^{A'}$ is the probability measure rescaled from the restriction of μ to $\mathcal{A}^{A'}$.

We shall say that $\mathcal{A}$ is nowhere equivalent to $\mathcal{G}$ if for every $A' \in \mathcal{A}$ with $\mu(A') > 0$, there exists a $\mathcal{A}$-measurable subset A'_0 of A' such that $\mu(A'_0 \triangle A'_1) > 0$ for any $A'_1 \in \mathcal{G}^{A'}$, where $A'_0 \triangle A'_1$ is the symmetric difference $(A'_0 \backslash A'_1) \cup (A'_1 \backslash A'_0)$. In what follows we require that the sub-σ-algebra and the algebra associated to an economy to be nowhere equivalent.

Assumption NE. For each economy $\mathcal{E} : (A, \mathcal{A}, \mu) \to \mathcal{P}_{mo} \times F$ we have that $\mathcal{A}$ is nowhere equivalent to the sub-σ-algebra $\mathcal{G}$.

From an economic viewpoint, the above definition means that for a non-trivial collection of agents A', if $(A, \mathcal{A}, \mu)$ and $(A, \mathcal{G}, \mu)$ represent the respective spaces of agents and characteristics types, then $\mathcal{A}^{A'}$ and $\mathcal{G}^{A'}$ are the sets or subcoalitions in A' and the characteristic-generated subcoalitions of A' respectively. Nowhere equivalence means that the σ-algebra $\mathcal{A}$ is strictly richer than its sub-σ-algebra $\mathcal{G}$ when they are restricted to the group of agents A'.

4.2. Examples

The space $C(K)$ of continuous functions on the compact metric space K with the sup norm is a separable Banach space whose positive $C(K)_+$ has a nonempty norm interior.

Consequently, our analysis covers $C(K)$. Although, for a given measure space $(M, \mathcal{M}, \nu)$, the spaces $L_\infty(M, \mathcal{M}, \nu)$ and l_∞ of essentially bounded measurable functions and of essentially bounded sequences respectively are not separable, our analysis cover these two spaces. On the one hand, due to the fact that the weakly compact subsets of $L_\infty(M, \mathcal{M}, \nu)$ are norm separable (Diestel and Uhl (1977), Theorem 13, p. 252) with nonempty norm interior. On the other hand, norm-bounded subsets of l_∞ are weak*-compact by Alaoglu's Theorem and by Bewley (1991), p. 226, these are complete and separable metric spaces. Some examples of these spaces are given below.

1. *Perfectly competitive economies (Rustichini and Yannelis (1991)).* Let us consider the space $L = C([0,1])$ of continuous functions on $[0,1]$ being $C_+([0,1])$ its positive cone. Let Q be a convex subset of $C_+([0,1])$ containing both 0 and u. We define the commodity space as the weak*-closure of Q, $\bar{Q}$, which is weakly*-compact (Dunford and Schwartz (1958), Theorem 14 (1), p. 269), metrizable and separable. Since Q is convex, it is closed with respect to the norm topology (Schaefer (1971), 3.1, p. 130). Thus, $\mathcal{P}_{mo} \times \bar{Q}$ is a Polish space. Let $(A, \mathcal{A}, \mu)$ be an atomless measure space . The economy is defined by a mapping from $(A, \mathcal{A}, \mu)$ into $\mathcal{P}_{mo} \times \bar{Q}$ which is measurable with respect to the σ-algebra generated by $\mathcal{E}^{-1}(\mathcal{P}_{mo} \times \bar{Q})$. For all $a \in A$, $X(a) = \bar{Q}$. Individual preferences are given by the utility function $u_a : \bar{Q} \to \mathbb{R}$ so that for an allocation $x : A \to \bar{Q}$, $u_a(x(a)) = \sqrt{\|x(a)\|}$. Individual endowments belong to $\bar{Q}$ and we take $F = \bar{Q}$.
   Let us assume that the space $(A, \mathcal{A} \setminus \mathcal{E}^{-1}(\mathcal{B}(\mathcal{P}_{mo}) \otimes \mathcal{B}(\bar{Q})), \mu)$ satisfies the "many more agents than commodities" condition of Rustichini and Yannelis (1991).[3] This economy satisfies Assumption BA while Assumption C holds trivially. As for Assumption NE it is satisfied because of Lemma 4 in He et al. (2017).
2. *Discrete time infinite horizon economies (Bewley (1991), Suzuki (2013)).*

[3] Assumption A1, p. 255. Take into account that because of their Theorem 4.1, p. 259, $(A, \mathcal{A}, \mu)$ may be considered as an agent space satisfying this assumption

Let $(A, \mathcal{A}, \mu)$ be an atomless agent space and let us consider the space l_∞. The set $Q = \{x \in \ l_\infty^+ : \|x\|_\infty \leq c\}$, $c > 3$, is the common consumption set, i.e., $X(a) = Q$ for all $a \in A$ (Bewley (1991)). Thus, $\mathcal{P}_{mo}$ is defined on $Q \times Q$. Individual endowments satisfy $(1, 1, ...) \leq e(a) \leq (c-2)(1, 1, ...)$ for all $a \in A$ and the utility functions are $u_a(x) = \sum_{t=1}^{\infty} 2^{-t} x(a)$ for all $x : A \to \bar{Q}$. Thus, an economy is a function $\mathcal{E}$ from $(A, \mathcal{A}, \mu)$ to $\mathcal{P}_{mo} \times Q$ such that a.e. $\mathcal{E}(a) = ((Q, u_a), e(a)) \in \mathcal{P}_{mo} \times Q$.

Let us observe that the vectors $(0, 0, ..., 0)$ and $u = (1, 1, ..., 1)$ belong to Q which is weak*-compact and metrizable. Preferences are monotone and if we take $F = Q$ then each endowment and each walrasian allocation belong to F. Consequently, one easily checks that all items in Assumption BA and C hold. Furthermore, $\mathcal{P}_{mo} \times Q$ is a Polish space. On the other hand, if we assume that $\mathcal{E}$ is $\mathcal{A}$-measurable, then the σ-algebra $\mathcal{G}$ generated by $\mathcal{E}^{-1}(\mathcal{B}(\mathcal{P}_{mo}) \otimes \mathcal{B}(Q))$ is a sub-σ-algebra of $\mathcal{A}$. $\mathcal{G}$ is countably generated since $\mathcal{P}_{mo} \times Q$ is second countable. Hence, if $(A, \mathcal{A}, \mu)$ is saturated [4] $\mathcal{A}$ is nowhere equivalent to $\mathcal{G}$ accordingly to He et al. (2017) Corollary 3 (ii), p. 792 and Assumption NE is satisfied.

3. *Standard representation (Hildenbrand (1974), Hart et al. (1974))*. Let us consider the space L_∞. Let $Q = \{x \in L_\infty^+ : \|x\|_\infty \leq c\}$, $c > 1$ and let $F = \{x \in L_\infty^+ : \|x\|_\infty \leq b\}$ for $b < c$. Q is w^*-compact and metrizable, hence second countable. Let us consider the agent space $(A, \mathcal{A}, \mu)$ given by $A = (\mathcal{P}_{mo} \times Q) \times [0, 1]$, $\mathcal{A} = \mathcal{B}(\mathcal{P}_{mo} \times Q) \otimes \mathcal{B}([0, 1])$ and $\mu = \delta \otimes \lambda$ where δ is a distribution on $P_{mo} \times Q$ and λ is the Lebesgue measure. Consumption sets are equal to the commodity space Q and preferences are representable

[4] He et al. (2017) p. 791: An atomless probability space $(A, \mathcal{A}, \mu)$ is said to have the saturation property for a probability distribution μ on the product of Polish spaces X and Y if for every random variable $f : A \to X$, which induces the distribution as the marginal distribution of μ over X, there is a random variable $g : A \to Y$ such that the induced distribution of the pair (f, g) on $(A, \mathcal{A}, \mu)$ is μ. A probability space $(A, \mathcal{A}, \mu)$ is said to be saturated if for any Polish spaces X and Y, $(A, \mathcal{A}, \mu)$ has the saturation property for every probability distribution μ on $X \times Y$.

by a norm continuous, strictly monotone, concave function $u_a : Q \to \mathbb{R}_+$ in such a way that for a concave, continuous, strictly monotone function $\nu_a : [0, \infty) \to \mathbb{R}_+$ and every commodity bundle $x : A \to Q$, we posit $u_a(x) = \int_A \nu_a(x(a)) d\mu(a)$.

Thus the economy $\mathcal{E} : (A, \mathcal{A}, \mu) \to \mathcal{P}_{mo} \times Q$ is the standard representation of δ (Hart et al. (1974)). It induces a sub-σ-algebra $\mathcal{G} = \mathcal{B}(\mathcal{P}_{mo} \times Q) \otimes \{[0, 1], \emptyset\}$ which is countably generated and for which $\mathcal{A}$ is nowhere equivalent. Furthermore $\mathcal{E}$ is $\mathcal{G}$-measurable.

4.3. Similarity between atomless economies

In this setup, two economies may have different agents in contrast to a fixed set of agents whose characteristics vary. Consequently, two economies may differ in size as the support of the distribution of agents' characteristics varies. In order to define the space of economies and a concept of convergence in it, we state some results over the space of characteristics.

Let $\mathbb{E}$ be the set all economies according to Definition 1 satisfying Assumption NE. Let $\mathcal{M}(\mathcal{P}_{mo} \times F)$ be the set of all probability distributions on the Borel σ-algebra $\mathcal{B}(\mathcal{P}_{mo} \times F) = \mathcal{B}(\mathcal{P}_{mo}) \otimes \mathcal{B}(F)$, where $\mathcal{B}(F)$ is the σ-algebra generated by the $\|\cdot\|_F$-open subsets of F. We endow the space $\mathcal{M}(\mathcal{P}_{mo} \times F)$ with the weak* topology.[5] Let $\mathcal{E} : (A, \mathcal{A}, \mu) \to \mathcal{P}_{mo} \times F$ and $\mathcal{E}' : (A', \mathcal{A}', \mu') \to \mathcal{P}_{mo} \times F$ be two elements of $\mathbb{E}$. Accordingly to He et al. (2017) we shall say that $\mathcal{E}$ and $\mathcal{E}'$ are similar if they are close in the sense of having similar distributions and total endowments.

We need to define a metric space of economies, since we require the use of Theorem 2 in Fort (1951) in our Theorem 3. Therefore, we need a metrizable topology over the set of distributions. Since $\mathcal{P}_{mo} \times F$ is separable, $\mathcal{M}(\mathcal{P}_{mo} \times F)$ is separable and the weak* topology is metrizable by the Prohorov metric ρ

[5] A sequence $\{\mu_n\}_{n \geq 1}$ in $\mathcal{M}(\mathcal{P}_{mo} \times F)$ converges to the measure μ in the weak* topology $\sigma(\mathcal{M}(\mathcal{P}_{mo} \times F), C_b(\mathcal{P}_{mo} \times F))$ if and only if $\int_{\mathcal{P}_{mo} \times F} f d\mu_n \to \int_{\mathcal{P}_{mo} \times F} f d\mu$ for all $f \in C_b(\mathcal{P}_{mo} \times F)$ which is the Banach lattice of all bounded continuous real functions on $\mathcal{P}_{mo} \times F$.

(Billingsley (1999), Theorem 5 Appendix III). Hence, for $\mathcal{E}$ and $\mathcal{E}'$, we posit the distance:

$$d_{\mathbb{E}}(\mathcal{E},\mathcal{E}') = \rho\left(\mu o(\mathcal{E})^{-1},\ \mu' o(\mathcal{E}')^{-1}\right) + \left\| \int_A e d\mu - \int_{A'} e' d\mu' \right\|.$$

Notice that $d_{\mathbb{E}}$ is a pseudo-metric and hence $(\mathbb{E}, d_{\mathbb{E}})$ is a pseudo-metric space. In contrast, Theorem 2 in Fort (1951) requires a metric space. Thus, we construct a metric space from this in a standard way: let us define the equivalence relation $\sim$ as $\mathcal{E} \sim \mathcal{E}'$ if and only if $d_{\mathbb{E}}(\mathcal{E},\mathcal{E}') = 0$. Consequently, if $[\mathcal{E}]$ and $[\mathcal{E}']$ are two equivalence classes containing $\mathcal{E}$ and $\mathcal{E}'$ respectively and if $\mathcal{E}'' \in [\mathcal{E}]$ and $\mathcal{E}''' \in [\mathcal{E}']$ then $d_{\mathbb{E}}(\mathcal{E}'',\mathcal{E}''') = d_{\mathbb{E}}(\mathcal{E},\mathcal{E}') = 0$. More generally, we have that $\hat{d}_{\mathbb{E}}([\mathcal{E}],[\mathcal{E}']) := d_{\mathbb{E}}(\mathcal{E},\mathcal{E}')$ is a metric in the quotient space $\mathbb{E}/\sim$ for any $\mathcal{E},\mathcal{E}' \in \mathbb{E}$, i.e, $(\mathbb{E}/\sim, \hat{d}_{\mathbb{E}})$ is a metric space.

In words, we can consider the distance $d_{\mathbb{E}}$ as a metric on $\mathbb{E}$ if we define the space of economies $\mathbb{E}$ as the equivalence classes of economies according to Definition 1. Hereafter it will be always the case.

5. Walras correspondences

The price simplex is given by $S = \{p \in L^*_+ : p(u) = 1\}$. By Jameson (1970), Theorem 3.8.6, S is weak*-compact and since L is separable, the topology induced on S by the weak*-topology is metrizable by a translation invariant metric on L^* (Dunford and Schwartz (1958), Theorem 1, p. 426). Furthermore, S is a norm-bounded subset of L^* accordingly to Alaoglu's Theorem (Dunford and Schwartz (1958), Corollary 3, p. 424).

Let x belongs to $L_1(\mu, L)$. Endowed with the norm $\|x\|_1 = \int_A \|x\| d\mu$, $(L_1(\mu, L), \|\cdot\|_1)$ is a Banach space (Diestel (1977), p. 50) wich is also locally convex (Schaefer (1971), p. 48). Furthermore, since every measure space concerning agents is assumed to be separable, $L_1(\mu, L)$ is a separable Banach space (Kolmogorov and Fomin (1975), p. 381). The topological dual of $L_1(\mu, L)$ is $L_\infty(\mu, L)$ and for the weak-topology w_{L_1} on $L_1(\mu, L)$ the space $(L_1(\mu, L), w_{L_1})$ is also a locally convex topological vector space (Schaefer (1971) p. 52). It is known that we can construct an invariant metric on $L_1(\mu, L)$ that generates a weaker topology than w_{L_1}. However, for every compact subset of $(L_1(\mu, L), w_{L_1})$ both topologies induce equivalent topologies (Dunford and Schwartz (1958), Theorem 3, p. 434).

Let $\mathcal{E} : (A, \mathcal{A}, \mu) \to \mathcal{P}_{mo} \times F$ be an economy where $\mathcal{E}(a) = ((X(a), \succ (a)), e(a)) \in \mathcal{P}_{mo} \times F$ μ-a.e. $a \in A$. Then, the attainable set for the economy $\mathcal{E}$ is $\mathbb{A}(\mathcal{E}) := \left\{x \in L_1(\mu, X) : \int_A x d\mu = \int_A e d\mu\right\}$. We enunciate the next proposition in order to show that for every economy $\mathcal{E} \in \mathbb{E}$ the walrasian correspondence is contained in a compact metric set.

Proposition 3. *Given Assumptions BA, for every $\mathcal{E} \in \mathbb{E}$ the set $\mathbb{A}(\mathcal{E})$ is a weakly compact metric subset of $L_1(\mu, L)$. Hence, it is a weakly compact metric subset of $L_1(\mu, Q)$*

Proof. Let $(A, \mathcal{A}, \mu)$ be the measure space of agents corresponding to the economy $\mathcal{E}$. First, by Theorem 2 in Diestel (1977), we know that $L_1(\mu, X_{\mathcal{E}})$ is weakly compact in $L_1(\mu, L)$ and by the above argument it is metrizable so that it is a compact metric subset of $L_1(\mu, L)$. Let $\{x_n\}_{n \geq 1}$ be a sequence in $\mathbb{A}(\mathcal{E}) \subset L_1(\mu, X_{\mathcal{E}})$ which converges weakly to x. Since $\int_A x_n d\mu = \int_A e d\mu$ for every $n \geq 1$ and the fact that for every $n \geq 1$ and a.e. $a \in A$, $x_n(a)$ belongs to $X_{\mathcal{E}}$ which is norm-bounded accordingly to Remark 1, it follows by Proposition 2 that $w - \lim_{n\to\infty} \int_A x_n d\mu = \int_A x d\mu = \int_A e d\mu$. Thus $x \in \mathbb{A}(\mathcal{E})$ and the proof is complete. $\square$

Thus, if we restrict the analysis to a set of finitely many economies, the attainable of all such economies is weakly compact as the following Corollary states.

Corollary 2. *If $|\mathbb{E}| < \infty$ and there is a common measure space of agents $(A, \mathcal{A}, \mu)$, then $\bigcup_{\mathcal{E} \in \mathbb{E}} \mathbb{A}(\mathcal{E})$ is a weakly compact and metrizable subset of $L_1(\mu, Q)$.*

We endow the set $\bigcup_{\mathcal{E} \in \mathbb{E}} \mathbb{A}(\mathcal{E})$ with the pseudo-metric $d_{\bigcup_{\mathcal{E} \in \mathbb{E}} \mathbb{A}(\mathcal{E})}$: for x, x' in $\mathbb{A}(\mathcal{E})$ and $\mathbb{A}(\mathcal{E}')$ respectively, $d_{\bigcup_{\mathcal{E} \in \mathbb{E}} \mathbb{A}(\mathcal{E})}(x,\ x') = \rho_Q(\mu \circ (x)^{-1},\ \mu' o (x')^{-1})$ where ρ_Q is the Prohorov metric on the space of probability measures $\mathcal{M}(Q)$ on $(Q,\ \mathcal{B}(Q))$. Since Q is separable, ρ_Q is a metrization of the topology of weak*-convergence on $\mathcal{M}(Q)$. Let us recall the discussion at the end of Section 4.3 and then let us consider $\bigcup_{\mathcal{E}\in\mathbb{E}} \mathbb{A}(\mathcal{E})$ as a set of equivalence classes so that the metric $d_{\bigcup_{\mathcal{E}\in\mathbb{E}} \mathbb{A}(\mathcal{E})}$ is well defined. Thus, the notion of convergence we shall consider on $\bigcup_{\mathcal{E}\in\mathbb{E}} \mathbb{A}(\mathcal{E})$ is that of convergence in distribution, i.e., a sequence of measurable attainable allocations $\{x_n\}_{n\geq 1}$ in $\bigcup_{\mathcal{E}\in\mathbb{E}} \mathbb{A}(\mathcal{E})$ converges to the measurable allocation x if the sequence of distributions $\{\mu_n \circ x_n^{-1}\}_{n\geq 1}$ converges weakly to the distribution $\mu \circ x^{-1}$. Let $\mathbb{A}$ be a compact subset of $\bigcup_{\mathcal{E}\in\mathbb{E}} \mathbb{A}(\mathcal{E})$ which contains all walrasian allocations. We endow $\mathbb{A}$ with the topology induced by $d_{\bigcup_{\mathcal{E}\in\mathbb{E}} \mathbb{A}(\mathcal{E})}$ on $\mathbb{A}$ and denoted by $d_{\mathbb{A}}$. Since $\mathbb{A}$ is assumed to be compact, our definition should induce some equilibrium selection of feasible allocations. This is also the case in Correa and Torres-Martínez (2014) or Carbonell-Nicolau (2010).

Definition 3. *The Walras allocation correspondence $WA : \mathbb{E} \twoheadrightarrow \mathbb{A}$ assigns to every economy $\mathcal{E} \in \mathbb{E}$ its corresponding walrasian allocation set $WA(\mathcal{E}) \subset \mathbb{A}$.*

Definition 4. *The walrasian equilibrium correspondence $WE : \mathbb{E} \twoheadrightarrow S \times \mathbb{A}$ associates each economy $\mathcal{E} \in \mathbb{E}$ to its corresponding equilibria $WE(\mathcal{E}) \subset S \times \mathbb{A}$.*

Related to the above definition there is that of Walras equilibrium distribution as stated in Hildenbrand (1974) p. 158. We adapt it to the context of our model. For $p \in S$ we define the set

$$E_p = \{(\tilde{\mathcal{E}},\ \tilde{x}) \in (\mathcal{P}_{mo} \times F) \times F : \tilde{x} \in D_{\tilde{\mathcal{E}}}(p)\}$$

Definition 5. *A Walras equilibrium distribution for a distribution θ of agents' characteristics in $\mathcal{P} \times F$ is a probability measure η on $\mathcal{P} \times F \times F$ equipped with its Borel σ-algebra such that:*

1. *The marginal distribution $\eta_{\mathcal{P}\times F}$ equals θ,*
2. *Mean demand equals mean supply, and*
3. *There exists $p \in S$ such that $\eta(E_p) = 1$.*

It is straightforward to verify that if (p,x) belongs to $WE(\mathcal{E})$, where $\mathcal{E} : (A, \mathcal{A}, \mu) \to \mathcal{P}_{mo} \times F$ is the corresponding economy and agent space respectively, then $\eta = \mu \circ (\mathcal{E}, x)^{-1}$ in $\mathcal{M}(\mathcal{P}_{mo} \times F \times F)$ is a Walras equilibrium distribution for $\mu \circ (\mathcal{E})^{-1}$. We are now ready for stating the next result.

Theorem 1. *Given $\mathbb{E}' \subset \mathbb{E}$ such that $WA(\mathcal{E}) \neq \emptyset$ for each $\mathcal{E} \in \mathbb{E}$, the correspondence WA has a $(d_{\mathbb{E}}, d_{\mathbb{A}})$-closed graph if Assumptions BA, NE and C hold.*

The proof is given in Appendix A.3. Let us note that since $\mathbb{A}$ is $d_{\mathbb{A}}$-compact the correspondence WA is $(d_{\mathbb{E}}, d_{\mathbb{A}})$-upper hemi-continuous (Hildenbrand (1974) p. 23). We now state an immediate consequence of the above theorem.

Corollary 3. *The correspondence WE is $(d_{\mathbb{E}}, w^* \times d_{\mathbb{A}})$-upper hemi-continuous*

Next, we show a special case of Theorem 1, namely, when there exist countably many economies and the set of agents is fixed.

Corollary 4. *If $|\mathbb{E}| = \aleph_0$ and there is a common measure space of agents $(A, \mathcal{A}, \mu)$, then the correspondence $WA : (\mathbb{E}, d_{\mathbb{E}}) \twoheadrightarrow \left(\bigcup_{\mathcal{E} \in \mathbb{E}} \mathbb{A}(\mathcal{E}), \| \cdot \|_1 \right)$ has a $\left(d_{\mathbb{E}}, d_{\bigcup_{\mathcal{E} \in \mathbb{E}} \mathbb{A}(\mathcal{E})} \right)$-closed graph under the conditions of Theorem 1. If $|\mathbb{E}| < \infty$ then WA is $\left(d_{\mathbb{E}}, d_{\bigcup_{\mathcal{E} \in \mathbb{E}} \mathbb{A}(\mathcal{E})} \right)$-upper hemi-continuous.*

Proof. The first part follows since $\|\cdot\|_1$-convergence implies $d_{\bigcup_{\mathcal{E} \in \mathbb{E}} \mathbb{A}(\mathcal{E})}$-convergence. Indeed, for the sequence $\{x_n\}_{n \geq 1}$ where each $x_n \in WA(\mathcal{E}_n)$ converging in $\| \cdot \|_1$ to x one has that $\lim_{n \to \infty} \|x_n - x\|_1 = \lim_{n \to \infty} \int_A \|x_n - x\| d\mu = 0$. Since $\| \int_A (x_n - x) d\mu \| \leq \int_A \|x_n - x\| d\mu$ (Diestel and Uhl (1977) Theorem 4. pg. 46) we get that $\lim_{n \to \infty} \int_A x_n d\mu = \int_A x d\mu$. Consequently by change of variable $\lim_{n \to \infty} \mu \circ x_n^{-1} = \lim_{n \to \infty} \int_{\mathcal{B}(\mathcal{P}_{mo}) \otimes \mathcal{B}(F)} d(\mu \circ x_n^{-1}) = \lim_{n \to \infty} \int_A x_n d\mu = \int_A x d\mu$, that is equal to $\int_{\mathcal{B}(\mathcal{P}_{mo}) \otimes \mathcal{B}(F)} d(\mu \circ x^{-1}) = \mu \circ x^{-1}$. Consequently, $\lim_{n \to \infty} \rho_Q(\mu \circ x_n^{-1}, \mu \circ x^{-1}) = 0$

For the second part, i.e. if $|\mathbb{E}| < \infty$ note that by Corollary 2 the set $\bigcup_{\mathcal{E} \in \mathbb{E}} \mathbb{A}(\mathcal{E})$ is weak-compact and metrizable, whence (Hildenbrand (1974) p. 23) upper hemi-continuous. □

6. Economies with strongly convex preferences

In this section we introduce the strongly convex preference condition. Together with the following modification of item (3) of Assumption BA it allows us to avoid Assumption C.

Assumption BA′ (3)

Each endowment e and each walrasian allocation f belong to a set F which is $\|\cdot\|$-closed and contains both 0 and u. In addition, there is $\alpha > 0$ such that F is a subset of $V(0,\alpha) \cap L_+ \subset X$, $\forall X \subset Q_+$ where $V(0,\alpha)$ denotes a neighborhood centered in zero of radius α.

As for strongly convexity of preferences we follow the definition 4.7 (b) of Debreu (1959). Other convexity definitions are given in Mas-Colell (1989) Definition 2.2.3 or Hildenbrand (1974) p. 88.

Assumption SCO (Strong convexity) Let $(X, \succ)$ be a preference relation and let x, and y be two vectors of X. If $y \succ x$ then $ty + (1-t)x \succ x$ for all $t \in (0,1)$.

Let $\mathcal{P}_{sco}$ be the set of all strongly convex preference relations. Then:

Lemma 3. *The subset $\mathcal{P}_{mo,sco} := \mathcal{P}_{mo} \cap \mathcal{P}_{sco}$ is a Polish space.*

The proof is provided in Appendix A.4.

In the following we shall consider the characteristic space given by $\mathcal{P}_{mo,sco} \times F$. As usual, $\mathcal{B}(\mathcal{P}_{mo,sco}) \otimes \mathcal{B}(F)$ is the Borel σ-algebra of $\mathcal{P}_{mo,sco} \times F$ and $\mathcal{M}(\mathcal{P}_{mo,sco} \times F)$ is the set of all probability distributions on $\mathcal{B}(\mathcal{P}_{mo,sco}) \otimes \mathcal{B}(F)$. Since $\mathcal{P}_{mo,sco}$ is Polish, we can follow the same argument of Section 4.3.

We conclude this section with the following result whose proof is in Appendix A.5.

Theorem 2. *The correspondence WA has a $(d_{\mathbb{E}}, d_{\mathbb{A}})$-closed graph if Assumptions BA(1)-(2), BA′ (3), NE and SCO hold.*

7. Stability results

We have all the prerequisites to characterize essential equilibria. We remark that not every economy in $\mathbb{E}$ may reach an equilibrium. Since the stability analysis require existence, we concentrate the attention on assumptions as those given in Khan and Yannelis (1991) or Noguchi (1997). This allows us to state the following.

Proposition 4. *Let* $\widehat{\mathbb{E}}$ *be any subset of the space* $\mathbb{E}$ *such that* $WE(\mathcal{E}) \neq \emptyset$ *for all* $\mathcal{E} \in \widehat{\mathbb{E}}$*, then we also have that* $WE(\mathcal{E}) \neq \emptyset$ *in the* $d_{\mathbb{E}}$*-closure of* $\widehat{\mathbb{E}}$*.*

Proof. Let $\mathcal{E}'$ be an element of the $d_{\mathbb{E}}$-closure of $\widehat{\mathbb{E}}$. Hence, there is a sequence $\{\mathcal{E}_n\}_{n\geq 1}$ where $\mathcal{E}_n \in \widehat{\mathbb{E}}$ for all $n \geq 1$ such that $\mathcal{E}' = \lim_{n\to\infty} \mathcal{E}_n$. Consequently, there exists a sequence $\{(p_n, x_n)\}_{n\geq 1}$ where each $(p_n, x_n) \in WE(\mathcal{E}_n) \subset S \times \mathbb{A}$. By taking a subsequence if necessary, it $\sigma^* \times d_{\mathbb{A}}$-converges to (p, x). By Corollary 3, $(p, x) \in WE(\mathcal{E}')$. □

In what follows, denote by $\overline{\mathbb{E}}$ a $d_{\mathbb{E}}$-closed subset of $\widehat{\mathbb{E}}$ defined in Proposition 4 above. We shall now study the stability of large economies with infinitely many commodities by analyzing how a Walras equilibrium for an economy $\mathcal{E}$ changes when their characteristics are perturbed. Formally, we need the following definition.

Definition 6. *Let* $\mathbb{E}' \subseteq \overline{\mathbb{E}}$ *and* $\mathcal{E} \in \mathbb{E}'$*. A walrasian equilibrium* (p, x) *of* $\mathcal{E}$ *is an essential equilibrium of* $\mathcal{E}$ *relative to* $\mathbb{E}'$ *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for every* $\mathcal{E}' \in V(\mathcal{E}, \delta) \cap \mathbb{E}'$ *it follows that* $WE(\mathcal{E}') \cap V((p, x), \varepsilon) \neq \emptyset$*.*

Thus, essential stability is equivalent to the lower hemi-continuity of the Walras equilibrium correspondence. We remark that the open ball $V(\mathcal{E}, \delta)$ is generated by $d_{\mathbb{E}}$ and the open ball $V((p, x), \varepsilon)$ is generated by the metric $d_{\mathbb{A}\times S}$ which exists since both S and $\mathbb{A}$ are metrizable. We would like to ensure the following properties:

(S1) The collection of essential economies $\mathbb{E}' \subset \overline{\mathbb{E}}$ is a dense residual subset of $\overline{\mathbb{E}}$.

(S2) If for $\mathcal{E} \in \overline{\mathbb{E}}$ we have that $WE(\mathcal{E})$ is singleton, then it is essential.

(S3) There exists a minimal essential subset of $WE(\mathcal{E})$ for $\mathcal{E} \in \overline{\mathbb{E}}$ and any of such sets is connected.

(S4) Given a essential and connected set $(\mathcal{E}) \subset WE(\mathcal{E})$, there exists an essential component of $WE(\mathcal{E})$ that contains $m(\mathcal{E})$.

(S5) Every essential subset of $WE(\mathcal{E})$ is stable.

In order to ensure these properties we invoke Theorem 2 in Fort (1950). Consequently, in our setting, the WE correspondence should be defined on a complete metric space. In the following result we verify that this is the case.

Proposition 5. *Under Assumption BA the space* $(\mathbb{E}, d_{\mathbb{E}})$ *is complete.*

The proof is in Appendix A.6.

Theorem 3. *Consider* $(\overline{\mathbb{E}}, d_{\mathbb{E}})$. *Then, (S1) is satisfied as well as for any* $\mathcal{E} \in \overline{\mathbb{E}}$ *we have that properties (S2)-(S5) hold.*

Proof. Given $WE(\mathcal{E}) \neq \emptyset$ for $\mathcal{E} \in \overline{\mathbb{E}}$ jointly with the fact that WE is compact-valued and upper hemi-continuous, we can apply Theorem 2 in Fort (1951) in order to achieve that there exists a dense residual subset $\mathbb{E}'$ of $\overline{\mathbb{E}}$ where WE is lower-hemicontinuous (see also (Carbonell-Nicolau, 2010, Lemma 5)). Thus, as every economy $\mathcal{E} \in \mathbb{E}'$ is a point of lower-hemicontinuity of WE, it follows from Yu (1999), Theorem 4.1, that $\mathcal{E}$ is essential with respect to $\mathbb{E}'$. With a similar argument, if $WE(\mathcal{E})$ is a singleton, the equilibrium correspondence is continuous at that point and essential relative to $\overline{\mathbb{E}}$ by Theorem 4.3. in Yu (1999), that is (S1). Properties (S2)-(S3) follows from applying Theorem 2.1 of Yu et al. (2005). Property (S4) follows from Theorem 4.1. of Yu et al. (2005) since by its definition of stability (Def. 8 (iii)) it is sufficient to show that minimal essential sets are stable. □

Appendix A. Proofs

Appendix A.1. Proof of Lemma 1

We would like to use Theorem 1 p. 96 of Hildenbrand (1974) since it works for locally compact spaces Q other than $\mathbb{R}^L$. For that, we have to ensure that $(\mathcal{C}^{w_Q}(Q \times Q), \tau_C)$ is compact metrizable. This follows from the application of Theorem 2 p. 19 of Hildenbrand (1974) to $(Q \times Q, w_Q \times w_Q)$ that is a locally compact Polish space.

1. Let $\{(X_n, \succ_n)\}_{n \geq 0}$ be a sequence in $\mathcal{P}$ such that it has a closed limit $(X, \succ)$. We shall prove that it belongs to $\mathcal{P}$. This is equivalent to $g(\mathcal{P})$ being closed in $\mathcal{C}^{w_Q}(Q \times Q)$. Indeed, let us consider the sequence $\{g\left((X_n, \succ_n)\right)\}_{n \in \mathbb{N}} = \{P_n\}_{n \in \mathbb{N}}$ in $\mathcal{C}^{w_Q}(Q \times Q)$ where $P_n = \{(x, y) \in X_n \times X_n : x \not\succ_n y\}$. We already noted that $(\mathcal{C}^{w_Q}(Q \times Q), \tau_C)$ is a compact metric space. Then, the sequence $\{P_n\}_{n \geq 1}$ converges to P if and only if $P = \mathrm{Li}(P_n) = \mathrm{Ls}(P_n)$ (Hildenbrand (1974), B.II. Theorem 2, p. 19). Let us define $X = \mathrm{proj}_{Q_+} P$ and $\succ = X \times X \setminus P$. We have to prove that $g((X, \succ)) = P$.
   Let us note that for $x \in X$ it follows that $(x, x) \in P$. Indeed, for $x \in X$ there exists $x' \in Q_+$ such that $(x, x') \in P$. Since $P = \mathrm{Li}(P_n) = \mathrm{Ls}(P_n)$, there exists a sequence $\{(x_n, x'_n)\}_{n \geq 1}$ belonging to P_n for each $n \geq 1$ and $\lim_{n \to \infty}(x_n, x'_n) = (x, x')$ for the topology $w_{Q_+} \times w_{Q_+}$ (Hildenbrand (1974), p. 15). Since $\succ_n$ is irreflexive for each $n \geq 1$ it follows that $(x_n, x_n) \in P_n$ and then $(x, x) \in P$.
   The argument above implies that X is the closed limit of the sequence $\{X_n\}_{n \geq 1}$ and it is nonempty since $0 \in X_n$ for all $n \geq 1$. Following the arguments of Hildenbrand (1974), p. 97, we note that X is convex and $\succ$ is irreflexive and transitive.
   Finally, we only need to show that $g((X, \succ)) = P$ which is direct since $g((X, \succ)) = \{(x, y) \in \mathrm{proj}_{Q_+} P \times \mathrm{proj}_{Q_+} P : (x, y) \in P\} = P$.
2. and 3. follows from mimicking the proof of Theorem 1(b) of Hildenbrand (1974). □

Appendix A.2. Proof of Lemma 2

First, we shall prove that $\mathcal{P}_{mo}$ with the metric of the closed convergence is a G_δ-set, i.e., a countable intersection of open sets in $\mathcal{P}$. We follow the approach given in Lemma of p. 98 by Hildenbrand (1974). Let d_{w_Q} be the metric for which (Q, w_Q) is metrizable. For every $m \in \mathbb{N}$ we define the set $\mathcal{P}_m = \big\{(X, \succ) \in \mathcal{P} : \exists\ x, y \in X, x \geq y,\ x \not\succ y \ and \ d_{w_Q}(x, y) \geq \frac{1}{m}\big\}$. Let $\{(X_n, \succ_n)\}_{n\geq 1}$ be a sequence in $\mathcal{P}_m$, then there exists a sequence $\{(x_n, y_n)\}_{n\geq 1}$ such that $x_n \geq y_n$, $x_n \not\succ_n y_n$ and $d_{w_Q}(x_n, y_n) \geq \frac{1}{m}$. Since both x_n and y_n belong to Q which is w-compact, there are subsequences also denoted by x_n and y_n which w-converge to x and y respectively. Let $P_n = \{(x', y') \in (X_n, X_n) : x' \not\succ_n y'\}$ from which we deduce that $(x_n, y_n) \in P_n$ for each $n \geq 1$. By Lemma 1, $(X, \succ) \in \mathcal{P}$ and $\text{Li}(P_n) = \text{Ls}(P_n) = P$. We want to prove that the closed limit $(X, \succ)$ belongs to $\mathcal{P}_m$. It is easily verified that both x and y belong to $\text{Ls}(X_n) = X$. Notice that $(x, y) \in \text{Ls}(P_n)$ so that $x \not\succ y$. Since Q_+ is w-closed, it follows that $x \geq y$. We claim that $d_{w_Q}(x, y) \geq \frac{1}{m}$. Otherwise, we would have that there exists n_0 such that for all $n > n_0$, $d_{w_Q}(x_n, y_n) < \frac{1}{m}$ which is a contradiction. Consequently, $(X, \succ) \in \mathcal{P}_m$ whence $\mathcal{P}_m$ is $\tau_C^{\mathcal{P}}$-closed. Note that $\mathcal{P}_{mo} = \bigcap_{m\in\mathbb{N}} (\mathcal{P} \setminus \mathcal{P}_m)$ and thus $\mathcal{P}_{mo}$ is a G_δ-set.

Second, by the classical Alexandroff lemma (see Aliprantis and Border (2006), Lemma 3.34 p.88), we conclude that $\mathcal{P}_{mo}$ is completely metrizable. In addition, by Corollary 3.5 p. 73 in Aliprantis and Border (2006) we know that $\mathcal{P}_{mo}$ as subset of a separable metric space $\mathcal{P}$ is separable. Thus, $\mathcal{P}_{mo}$ is a Polish space.

□

Appendix A.3. Proof of Theorem 1

Regarding the closed graph property, let $\{(A_n, \mathcal{A}_n, \mu_n)\}_{n\geq 1}$ be a sequence of agent spaces and let $\{(A_n, \mathcal{G}_n, \mu_n)\}_{n\geq 1}$ be a sequence of characteristic type spaces such that for $n \geq 1$, $\mathcal{G}_n$ is a countably generated sub-σ algebra of $\mathcal{A}_n$. Let $\{\mathcal{E}_n\}_{n\geq 1}$ be a sequence of $\mathcal{G}_n$-measurable mappings form $(A_n, \mathcal{A}_n, \mu_n)$ into $\mathcal{P}_{mo} \times F$ which converges weakly to $\mathcal{E} : (A, \mathcal{A}, \mu) \to \mathcal{P}_{mo} \times F$ in the sense that $\lim_{n\to\infty} \rho(\mu_n \circ \mathcal{E}_n,\ \mu o \mathcal{E}) = 0$ and $\lim_{n\to\infty} \left\| \int_{A_n} e_n d\mu_n - \int_A e d\mu \right\| = 0$. The economy $\mathcal{E}$ is

$\mathcal{G}$-measurable where $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{A}$ induced by $\mathcal{E}^{-1}(\mathcal{B}(\mathcal{P}_{mo})\otimes\mathcal{B}(F))$. Let $\{x_n\}_{n\geq 1}$ be a sequence such that $x_n \in WA(\mathcal{E}_n)$ and x_n is $\mathcal{A}_n$-measurable for all $n \geq 1$ and which $d_{\mathbb{A}}$-converges. It means that $\mu_n \circ x_n^{-1} \xrightarrow[n\to\infty]{\rho} \gamma$ for $\gamma \in \mathcal{M}(F)$. We want to prove that there exists a $\mathcal{A}$-measurable allocation $x \in \mathbb{A}$ such that $\mu \circ x^{-1} = \gamma$. We notice that a similar result is proved in Theorem 1 of He et al. (2017) but with a finite dimensional commodity space. In their proof, the authors make use of Lemma 2.1 (iii) in Keisler and Sun (2009) and the fact that the nowhere equivalence holds. Applied to our setting, that lemma works since the space $\mathcal{P}_{mo}\times F\times F$ is Polish. Consequently, we can follow the guidelines of He et al. (2017) to obtain a $\mathcal{A}$-measurable allocation $x \in \mathbb{A}$ such that $\mu \circ x^{-1} = \gamma$. In addition, we have that the sequence $\left\{\eta_n = \mu_n \circ (\mathcal{E}_n, x_n)^{-1}\right\}_{n\geq 1}$ converges to $\eta = \mu \circ (\mathcal{E}, x)^{-1}$ and the marginals $\eta_{\mathcal{P}_{mo}\times B}$ and η_B are $\mu \circ \mathcal{E}^{-1}$ and $\mu \circ x^{-1}$ respectively.

Now, we have to prove that $x \in WA(\mathcal{E})$. In doing so we shall make use of Skorokhod's Theorem (Billingsley (1999), Theorem 6.7, p. 70) to the sequence $\{\eta_n\}_{n\geq 1} \xrightarrow[n\to\infty]{} \eta$. So, there is a measure space $(\Omega, \mathcal{O}, \nu)$ and measurable mappings $\{(\hat{\mathcal{E}}_n,\ \hat{x}_n)\}_{n\geq 1}$ and $(\hat{\mathcal{E}},\ \hat{x})$ from $(\Omega, \mathcal{O}, \nu)$ into $(\mathcal{P}_{mo}\times F)\times F$ such that $\{(\hat{\mathcal{E}}_n(\omega),\ \hat{x}_n(\omega))\}_{n\geq 1}$ converges with respect to $\tau_C^{\mathcal{P}} \times \|\cdot\|_F \times \|\cdot\|_F$ a.e. in Ω to $(\hat{\mathcal{E}}(\omega),\ \hat{x}(\omega))$, and we have that $\{\eta_n = \nu \circ (\hat{\mathcal{E}}_n, \hat{x}_n)^{-1}\}_{n\geq 1}$and $\eta = \nu \circ (\hat{\mathcal{E}}, \hat{x})^{-1}$. Since $\eta_n = \nu \circ (\hat{\mathcal{E}}_n, \hat{x}_n)^{-1} = \mu_n \circ (\mathcal{E}_n, x_n)^{-1}$ and considering the fact that there exists a price sequence $\{p_n\}_{n\geq 1}$ where each $p_n \in S$ such that $(p_n, x_n) \in WE(\mathcal{E}_n)$ for all $n \geq 1$, we deduce that $(p_n,\ \hat{x}_n)$ is a Walras equilibrium for $\hat{\mathcal{E}}_n$ for all $n \geq 1$ whence $\hat{x}_n \in WA(\hat{\mathcal{E}}_n)$ for all $n \geq 1$. The argument is identical to the one in the last paragraph of this proof. Since $\{(\hat{\mathcal{E}}_n(\omega),\ \hat{x}_n(\omega))\}_{n\geq 1}$ converges a.e. to $(\hat{\mathcal{E}}(\omega),\ \hat{x}(\omega))$ with respect to $\tau_C^{\mathcal{P}}\times\|\cdot\|_F\times\|\cdot\|_F$, we claim that $\hat{x} \in WA(\hat{\mathcal{E}})$. Indeed, take into account that $\int_\Omega \hat{x}_n d\nu = \int_\Omega \hat{e}_n d\nu$ for all $n \geq 1$, $\lim_{n\to\infty} \hat{x}_n(\omega) = \hat{x}(\omega)$ a.e. and the fact that $\hat{x}_n(\omega)$ is norm-bounded a.e. ω in Ω. Thus, we can apply the Dominated Convergence Theorem (Dunford and Schwartz (1958), Th. 10 p. 328) to get $\int_\Omega \hat{x} d\nu = \lim_{n\to\infty} \int_\Omega \hat{x}_n d\nu$. Since $\lim_{n\to\infty} \hat{e}_n(\omega) = \hat{e}(\omega)$ a.e. (in $\|\cdot\|$) and it is norm-bounded, we deduce by the Dominated Convergence Theorem, again, that $\int_\Omega \hat{e} d\nu = \lim_{n\to\infty} \int_\Omega \hat{e}_n d\nu$. Hence, $\int_\Omega \hat{x} d\nu = \int_\Omega \hat{e} d\nu$. Furthermore, since the sequence $\{p_n\}_{n\geq 1}$ belongs to S there is a subsequence also denoted by $\{p_n\}_{n\geq 1}$ which converges to $p \in S$ in the weak*-topology. Hence, since $\{\hat{x}_n(\omega)\}_{n\geq 1}$ and $\{\hat{e}_n(\omega)\}_{n\geq 1}$ converge for $\|\cdot\|$ to $\hat{x}(\omega)$ and $\hat{e}(\omega)$ respectively a.e. $\omega \in \Omega$, $(p,\ \hat{x}(\omega)) \mapsto p(\hat{x}(\omega))$ and $(p,\ \hat{e}(\omega)) \mapsto p(\hat{e}(\omega))$ are jointly continuous a.e. $\omega \in \Omega$. Hence, $p_n(\hat{x}_n(\omega)) = p_n(\hat{e}_n(\omega))$ for all n implies $p(\hat{x}(\omega)) = p(\hat{e}(\omega))$ a.e. $\omega \in \Omega$.

Finally, we show that $\hat{x}(\omega) \in D_{\hat{\mathcal{E}}(\omega)}(p)$ a.e. $\omega \in \Omega$. Suppose not, then there exists $\xi \in L_1(\nu, \hat{X})$ such that $\xi(\omega) \succ_\omega \hat{x}(\omega)$ and $p(\xi(\omega)) < p(\hat{e}(\omega))$ for ω in a non-null subset of Ω. By Assumption C and Corollary 1 there exists a sequence $\{\xi_n\}_{n\geq 1}$ converging to ξ pointwise in norm such that $\xi_n \in L_1(\nu, \hat{X}_n)$ and $\xi_n(\omega) \succ_\omega \hat{x}_n(\omega)$ a.e. for n large enough. Because of equilibrium conditions in $\hat{\mathcal{E}}_n$ it follows that $p_n(\xi_n(\omega)) > p_n(\hat{e}_n(\omega))$ a.e. and, taking limits, we get $p(\xi(\omega)) \geq p(\hat{e}(\omega))$ which contradicts the above converse inequality.

Consequently, $\eta = \nu \circ (\hat{\mathcal{E}}, \hat{x})^{-1}$ is a Walras equilibrium distribution for $\hat{\mathcal{E}}$ which is equal to $\mu \circ (\mathcal{E}, x)^{-1}$ Therefore, $\nu \circ \hat{\mathcal{E}}^{-1} = \mu \circ \mathcal{E}^{-1}$ that is to say, both economies have the same distribution. Further, $\nu \circ \hat{e}^{-1} = \mu \circ e^{-1}$ where $\hat{e}$ is the endowment of the economy $\hat{\mathcal{E}}$. By Lemma 8 (f), p. 182, in Dunford and Schwartz (1958) we get $\int_\Omega \hat{e} d\nu = \int_A e d\mu$ which means that both economies $\mathcal{E}$ and $\hat{\mathcal{E}}$ have the same mean endowment.

Recall that $(\hat{x},\ p) \in WE(\hat{\mathcal{E}})$ implies $\int_\Omega \hat{x} d\nu = \int_\Omega \hat{e} d\nu$. In addition, note that the marginal $\eta_B = \nu \circ \hat{x}^{-1} = \mu \circ x^{-1}$ whence, again by Lemma 8 (f), p. 182, in Dunford and Schwartz (1958), we obtain $\int_\Omega \hat{x} d\nu = \int_A x d\mu$ from which we get $\int_A x d\mu = \int_A e d\mu$ since mean endowments are equal.

Let us note that $\nu \circ (\hat{\mathcal{E}}, \hat{x})^{-1}(E_p) = 1$ which implies that $\mu \circ (\mathcal{E}, x)^{-1}(E_p) = 1$. Thus $\mu(\{a \in A : (\mathcal{E}(a),\ x(a)) \in E_p\}) = 1$ which implies that, a.e. $a \in A$, $x(a) \in D_{\mathcal{E}(a)}(p)$. Hence $(p,\ x)$ is a walrasian equilibrium for $\mathcal{E}$. □

Appendix A.4. Proof of Lemma 3

First we shall prove that $\mathcal{P}_{sco}$ with the metric of the closed convergence is a G_δ-set in a Polish space $\mathcal{P}$ (see Lemma 1 and 2). Let d_{w_Q} be the metric for which (Q, w_Q) is metrizable. For every m and k in $\mathbb{N}$, $k \geq 2$, we define the set $\mathcal{P}_{mk} =$

$\{(X,\succ) \in \mathcal{P} : \textit{there exists } x, y \in X, \textit{ and } t \in \mathbb{R}, \textit{ such that } d_{w_Q}(x,y) \geq \frac{1}{m}, \frac{1}{k} \leq t \leq 1-\frac{1}{k}, y \succ x \textit{ and } ty+(1-t)x \not\succ x\}$. Let $\{(X_n,\succ_n)\}_{n\geq 1}$ be a sequence in $\mathcal{P}_{mk}$ which converges in the closed topology to $(X,\succ)$. Consequently, there exist sequences $\{(x_n,y_n)\}_{n\geq 1}$ and $\{t_n\}_{n\geq 1}$ such that $x_n, y_n \in X_n$, $d_{w_Q}(x_n,y_n) \geq \frac{1}{m}$, $\frac{1}{k} \leq t_n \leq 1-\frac{1}{k}$, $y_n \succ_n x_n$ and $t_n y_n + (1-t_n)x_n \not\succ_n x_n$ for all $n \geq 1$.

Since (x_n,y_n) belongs to $Q \times Q$ for $n \geq 1$ which is weak-compact there is a subsequence also denoted by $\{(x_n,y_n)\}_{n\geq 1}$ which $w_Q \times w_Q$-converges to (x,y). In the same way, the sequence $\{t_n\}_{n\geq 1}$ belongs to $[\frac{1}{k}, 1-\frac{1}{k}] \subset \mathbb{R}$ whence there exists a subsequence also denoted by $\{t_n\}_{n\geq 1}$ which converges to t in $[\frac{1}{k}, 1-\frac{1}{k}]$. By Lemma 1 $(X,\succ) \in \mathcal{P}$ and $\mathrm{Li}(P_n) = \mathrm{Ls}(P_n) = P$. We want to prove that the closed limit $(X,\succ)$ actually belongs to $\mathcal{P}_{mk}$. Let us note that $(x,y) \in \mathcal{P}$ and because of Corollary 1 it follows that $ty+(1-t)x \not\succ x$. Finally, because of continuity of the distance function d_{w_Q} it follows that $d_{w_Q}(x,y) \geq \frac{1}{m}$. Thus $\mathcal{P}_{mk}$ is a closed subset of $\mathcal{P}$. It is straightforward that $\mathcal{P}_{sco} = \bigcap_{m\geq 1} \bigcap_{k\geq 2} (\mathcal{P} \setminus \mathcal{P}_{mk})$, whence it is a G_δ-set which, in turn, implies that $\mathcal{P}_{mo,sco}$ is also a G_δ-set in $\mathcal{P}$. In order to conclude the proof, we follow the last part of the proof of Lemma 2 to conclude that $\mathcal{P}_{mo,sco}$ is a Polish space. □

Appendix A.5. Proof of Theorem 2

For the first part, we transcript the proof of Theorem 1 taking into account that we are considering the set $\mathcal{P}_{mo,sco}$ instead of $\mathcal{P}_{mo}$. Let us take the proof since the third paragraph. Hence, suppose that there exists $\xi \in L_1(\nu, \hat{X})$ such that $\xi(\omega) \succ (\omega)\hat{x}(\omega)$ and $p(\xi(\omega)) < p(\hat{e}(\omega))$ for ω in a non-null subset of Ω. Since both preferences (Assumption SCO) and the consumption set are convex it follows that for all $t \in (0,1)$, $t\xi(\omega) + (1-t)\hat{x}(\omega) \succ (\omega)\hat{x}(\omega)$. Furthermore $p(t\xi(\omega) + (1-t)\hat{x}(\omega)) < p(\hat{e}(\omega))$ for all $t \in (0,1)$. For t close enough to 0, one easily checks that $t\xi(\omega) + (1-t)\hat{x}(\omega)$ belongs to $V(0,\alpha) \cap L_+$. Consequently, by Assumption BA′ (3) $t\xi(\omega) + (1-t)\hat{x}(\omega)$ belongs to X_n for all $n \geq 1$ and by Corollary 1 $t\xi(\omega) + (1-t)\hat{x}(\omega) \succ_n (\omega)\, \hat{x}_n(\omega)$ a.e. for n large enough. Because of equilibrium conditions in $\hat{\mathcal{E}}_n$ it follows that $p_n(t\xi(\omega) + (1-t)\hat{x}(\omega)) > p_n(\hat{e}_n(\omega))$ a.e. Taking limits we get $p(t\xi(\omega) + (1-t)\hat{x}(\omega)) \geq p(\hat{e}(\omega))$ which contradicts the above converse inequality. The rest of the proof is identical with that of Theorem 2. □

Appendix A.6. PROOF OF PROPOSITION 5

The proof is equivalent for preferences $\mathcal{P}_{mo}$ or $\mathcal{P}_{mo,sco}$ since both spaces are Polish (see Lemmata 2 and 3). We use the notation $\mathcal{P}_{mo}$ for simplicity. Let $\{\mathcal{E}_n\}_{n\geq 1}$ be a Cauchy sequence on $\mathbb{E}$ where, by definition, each $\mathcal{E}_n$ is a $\mathcal{G}_n$-measurable function from $(A_n, \mathcal{A}_n, \mu_n)$ into $\mathcal{P}_{mo} \times F$, being $\mathcal{G}_n$ a sub-σ-algebra of a countably generated sub-σ-algebra $\mathcal{A}_n$ such that $\mathcal{G}_n$ is generated by $\mathcal{E}_n^{-1}(\mathcal{B}(\mathcal{P}_{mo}) \otimes \mathcal{B}(Q))$ and for which $\mathcal{A}_n$ is nowhere equivalent for all $n \geq 1$. We have that $\{\mu_n \circ (\mathcal{E}_n)^{-1}\}_{n\geq 1}$ and $\{\int_{A_n} e_n d\mu_n\}_{n\geq 1}$ are also Cauchy sequences on $\mathcal{M}(\mathcal{P} \times F)$ and F respectively. Since $(\mathcal{M}(\mathcal{P}_{mo} \times F),\ \rho)$ is complete there exists a measure $\delta \in \mathcal{M}(\mathcal{P}_{mo} \times F)$ such that $\lim_{n\to\infty} \rho(\mu_n \circ (\mathcal{E}_n)^{-1},\ \delta) = 0$ and since $(F,\ \|\cdot\|_F)$ is a complete normed space, there exists a vector z in F such that $\lim_{n\to\infty} \| \int_{A_n} e_n d\mu_n - z\|_F = 0$. Since $F \subset L_+$ is $\|\cdot\|$- closed we have that $z \in F \subset L_+$. Thus the sequence $\{\mathcal{E}_n\}_{n\geq 1}$ converges. It only remains to show that it does in $\mathbb{E}$, that is to say, that there exists a $\mathcal{G}$-measurable function $\mathcal{E}$ from $(A, \mathcal{A}, \mu)$ to $\mathcal{P}_{mo} \times F$ such that $(X(a), \succ(a), e(a)) \in \mathcal{P}_{mo} \times F$ for all $a \in A$, $\mu \circ \mathcal{E}^{-1} = \delta$, $\int_A e d\mu = z$, and $\mathcal{G}$ being a sub-σ-algebra of $\mathcal{A}$ that is countably generated for which $\mathcal{A}$ is nowhere equivalent.

Let us consider the following measure space $(A, \mathcal{A}, \mu)$ where $A = (\mathcal{P}_{mo} \times F) \times [0,\ 1]$, $\mathcal{A} = \mathcal{B}(\mathcal{P}_{mo} \times F) \otimes \mathcal{B}([0,\ 1])$ and $\mu = \delta \otimes \lambda$ where λ is the Lebesgue measure. Thus, $\mathcal{E} : (A, \mathcal{A}, \mu) \to \mathcal{P}_{mo} \times F$ is the standard representation of δ which induces the sub-σ-algebra $\mathcal{G} = \mathcal{B}(\mathcal{P}_{mo} \times F) \otimes \{[0,\ 1],\ \emptyset\}$ of $\mathcal{A}$.[6] It can be shown that $(A, \mathcal{A}, \mu)$ is atomless, $\mathcal{A}$ is nowhere equivalent to $\mathcal{G}$ which, in turn, is countably generated since $(\mathcal{P}_{mo} \times F) \times [0, 1]$ is second countable. Furthermore, $\mathcal{E}$ is $\mathcal{G}$-measurable and it is the distributional limit of the sequence $\{\mathcal{E}_n\}_{n\geq 1}$. Furthermore, since $(\mathcal{P}_{mo} \times F) \times [0, 1]$ is a Hausdorff space, $\mathcal{B}((\mathcal{P}_{mo} \times F) \times [0, 1])$ separates points and then it is separable (Dudley (1999), Theorem 5.3.1, p.

[6]For details on the standard representation, see Hildenbrand (1974) p. 156.

186). Since both $\mathcal{P}_{mo} \times F$ and $[0,1]$ are separable, $\mathcal{B}((\mathcal{P}_{mo} \times F) \times [0,1]) = \mathcal{B}(\mathcal{P}_{mo} \times F) \otimes \mathcal{B}([0,1])$.

For each agent $a \in A$, his/her initial endowment is given by $e(a) := \mathrm{Proj}_F \mathcal{E}(a)$. Hence, let us note that the marginal distribution $(\mu_n \circ (\mathcal{E}_n)^{-1})_F = \mu_n \circ (e_n)^{-1}$ converges weakly to the marginal distribution $(\mu \circ e^{-1})$, each e_n is integrable and e is $\mathcal{A}$-measurable and Bochner integrable. By Skorokhod's Theorem (Billingsley (1999), Theorem 6.7, p. 70), there exist a measure space $(\Omega, \mathfrak{A}, \nu)$ and measurable mappings $\widehat{\mathcal{E}}_n = ((\hat{X}_n, \hat{\succ}_n), \hat{e}_n)$ and $\widehat{\mathcal{E}} = ((\hat{X}, \hat{\succ}), \hat{e})$ from $(\Omega, \mathfrak{A}, \nu)$ into $\mathcal{P}_{mo} \times F$ such that (i) $\{\widehat{\mathcal{E}}_n(\omega)\}_{n \geq 1}$ converges a.e. $\omega \in \Omega$ to $\widehat{\mathcal{E}}$ with respect to the topology $\tau_C^{\mathcal{P}} \times \| \cdot \|_F$, (ii) $\nu o \widehat{\mathcal{E}}_n^{-1} = \mu_n \circ \mathcal{E}_n^{-1}$, and $\nu o \widehat{\mathcal{E}}^{-1} = \mu \circ \mathcal{E}^{-1}$. In consequence, since $\widehat{\mathcal{E}}_n$ and $\widehat{\mathcal{E}}$ are measurable, $\widehat{e}_n$ and $\widehat{e}$ as projection on F are measurable. Moreover, the convergence given by (i) implies that $\nu \circ \widehat{e}_n^{-1}$ converges to $\nu \circ \widehat{e}^{-1}$ and the distributions in (ii) imply that $\nu o \widehat{e_n}^{-1} = \mu_n \circ e_n^{-1}$ and $\nu o \widehat{e}^{-1} = \mu \circ e^{-1}$. Since the sequence $\{\widehat{e}_n(\omega)\}_{n \geq 1}$ is norm-bounded, we have by the Dominated Convergence Theorem (Dunford and Schwartz (1958), Th. 10 p. 328)

$$\lim_{n \to \infty} \int_\Omega \widehat{e}_n(\omega) d\nu = \int_\Omega \widehat{e}(\omega) d\nu,$$

Thus, using repeatedly a change of variables and Dominated Convergence:

$$\begin{aligned}
\lim_{n \to \infty} \int_{A_n} e_n d\mu_n &= \lim_{n \to \infty} \int_{\mathcal{B}(F)} d\left(\mu_n \circ e_n^{-1}\right) = \lim_{n \to \infty} \int_{\mathcal{B}(F)} d\left(\nu \circ \widehat{e}_n^{-1}\right) \\
&= \lim_{n \to \infty} \int_\Omega \widehat{e}_n d\nu = \int_\Omega \widehat{e} d\nu \\
&= \int_{\mathcal{B}(F)} d\left(\nu \circ \widehat{e}^{-1}\right) \\
&= \int_{\mathcal{B}(F)} d\left(\mu \circ e^{-1}\right) = \int_A e d\mu
\end{aligned}$$

□

Thus, $z = \int_A e(a) d\mu$ and the proof is complete.[7]

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[7] Actually, since $\int_\Omega \hat{e}(\omega) d\nu = \int_A e(a) d\mu$ and $\mu \circ \mathcal{E}^{-1} = \nu \circ \hat{\mathcal{E}}^{-1}$, $\mathcal{E}$ and $\hat{\mathcal{E}}$ are equivalents modulo $\sim$.

Appendix B. An alternative approach: compactification of the commodity space.

Appendix B.1. Basic Assumptions and compactification of the commodity space

Given that L is a Banach separable space, it is possible to define a compactification (see Corollary 3.41 in Aliprantis and Border (2006) p. 91). Moreover, L can be understood as a subset of the Hilbert cube $\mathcal{H}$.[8] Thus, the metric compact space $(\mathcal{H}, d_{\mathcal{H}})$ is the compactification of $(L, \|\cdot\|)$ where the former is a Polish space. The metric $d_{\mathcal{H}}$ induces a metric on L, $d_{\mathcal{H},L}$ which is equivalent to $\|\cdot\|$. Thus $\|\cdot\|$-open sets are $d_{\mathcal{H},L}$-open and viceversa. Let $(\mathcal{H} \times \mathcal{H}, d_{\mathcal{H}\times\mathcal{H}})$ be the metrizable product space which is a compactification of the product space $(L \times L, \|\cdot\| \times \|\cdot\|)$.[9]

Basic Assumptions (BA$''$)

For each economy $\mathcal{E}$ we have:

1. There exists a vector $u \in \text{int}L_+$ such that $\|u\| = 1$.
2. There is a set $E \subset L_+$ which is closed and convex, satisfies $E - E \subset E$ and $X \subset E$.
3. Every X is a convex and closed subset of L_+ containing both u and 0.
4. $(X, \succ)$ is relatively open in $X \times X$.
5. Aggregate endowments are strictly positive.
6. Individual endowments and walrasian allocations belong to F which is a closed subset L_+ such that $F \subset X$ for all X.

[8] More precisely, let $f : L \to \mathcal{H}$ be an embedding between L and $\mathcal{H}$. Then L is a topological subspace of $\mathcal{H}$ by identifying L with its image $f(L)$ which is a topological subspace of $\mathcal{H}$. Recall that $f : L \to f(L)$ is an homeomorphism.

[9] Let us take $(f, f) : L \times L \to f(L) \times f(L) \subset \mathcal{H} \times \mathcal{H}$. From previous footnote, it is clear that (f, f) is an homeomorphism between $L \times L$ and $f(L) \times f(L)$.

Remark 2. *Assumption BA$''$(1) says that there exists a reference commodity bundle u with such properties. This is a technical Assumption. BA$''$(2) restricts all individual consumption vectors to E. This condition will allow us to define an appropriate topological structure in the set of preferences. Condition BA$''$(3) implies that each consumption set is closed for the topology $d_{\mathcal{H}}$ induced on L while Assumption BA$''$(4) is a classical one. BA$''$(5) restricts all total endowments to be strictly positive while BA$''$(6) says that individual endowments belong to a common subset as well as those allocations that are walrasian. Notice that F is $\|\cdot\|$-closed so it is $d_{\mathcal{H}}$-closed and thus $d_{\mathcal{H}}$-bounded since $\mathcal{H}$ is $d_{\mathcal{H}}$-compact. Since $F \subset L_+$ it is $\|\cdot\|$-bounded.*

Remark 3. *Let us note that because of Assumption BA$''$(3) it follows that P is $d_{\mathcal{H}\times\mathcal{H}}$-closed in $\mathcal{H}\times\mathcal{H}$ and thus it is closed in $L \times L$ for the induced topology.*

Appendix B.2. Space of Economies

Let $\mathcal{C}(\mathcal{H}\times\mathcal{H})$ be the set of all closed subsets of $\mathcal{H}\times\mathcal{H}$. We denote by τ_C the topology of closed convergence on $\mathcal{C}(\mathcal{H}\times\mathcal{H})$. Since every P belongs to $\mathcal{C}(\mathcal{H}\times\mathcal{H})$, we can define a mapping $g : \mathcal{P} \to \mathcal{C}(\mathcal{H}\times\mathcal{H})$ by $(X, \succ) \mapsto P$. As in Section 4 one can observe that g is an injection. Then we define a topology $\tau_C^{\mathcal{P}}$ on $\mathcal{P}$ by $\tau_C^{\mathcal{P}} = \{g^{-1}(U) : U \in \tau_C\}$. The proof of our Lemma 1 is a direct adaptation for the present topology. In particular, take into account the subset E in BA$''$(2)

When $L = \mathbb{R}^\ell$ for $\ell > 0$, Hildenbrand (1974) uses the topology induced by the closed convergence on the space $\mathcal{C}(L \times L)$ rather than $\mathcal{C}(\mathcal{H}\times\mathcal{H})$. This is so because $\mathbb{R}^\ell$ is locally compact.

Note that above, though we are considering the space $\mathcal{H}$, the relevant topological results take place in L (or $L \times L$) with the relative topologies. Indeed, $X \subset L$ and $\succ \subset X \times X \subset L \times L$. $\mathcal{P}$ is also a subset of $L \times L$ and it is a closed subset of the latter for the topology $d_{\mathcal{H}\times\mathcal{H}, L\times L} = \|\cdot\| \times \|\cdot\|$.

We state now Assumption C$'$ which concerns "small" perturbations of consumption sets which takes into account the topological space $(\mathcal{P}, \tau_C^{\rho})$ and its connection with $(\mathcal{H}, d_{\mathcal{H}})$ rather than $(Q, \|\cdot\|_Q)$ as in Assumption C.

Assumption C'

Let $(X_n, \succ_n)$ be a sequence converging to $(X, \succ)$ with respect to $\tau_C^{\mathcal{P}}$ such that $X_n, X : (A, \mathcal{A}, \mu) \twoheadrightarrow E$. For all $x \in L_1(\mu, X)$, there exists a sequence (x_n) in $L_1(\mu, X_n)$ which $d_{\mathcal{H},L}$-converges pointwise to x.

The set S and the correspondences WA and WE are the same as those in Section 4. Then, the results of Sections 5 to 7 follow with this approach.

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