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Giffen goods: an illustration

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Abstract

Over time, there has been a growth in the literature exploring preferences that can allow for Giffen goods to appear; most of the times, they occur under unusual circumstances, such as corner solutions or non-strictly increasing utility functions. One of these cases, in particular, finds them by eliminating the substitution effect, and proposes a generalization that might yield the same result. I explore, mostly through numerical simulations, how in this generalization, which features "well-behaved" preferences and continuously differentiable indifference curves, Giffen goods appear through a progressive reduction and, in the limit, elimination of the substitution effect provided that the good in question is an inferior good, and not necessarily being accompanied by a strengthening of the income effect. In addition, I develop an alternative set of preferences, that share the same original case as their limit case, and illustrate how Giffen goods appear in a similar manner.

Keywords: Giffen goods, consumer preferences, substitution effect, income effect

Resumo

Ao longo do tempo, tem aumentado a literatura que explora preferências que permitam o aparecimento de bens de Giffen; a maior parte das vezes, estes ocorrem sob circunstâncias menos habituais, como soluções de canto ou em funções utilidade não estritamente crescentes. Um destes casos em particular, descobre-os por via da eliminação do efeito substituição, e propõe uma generalização que possa gerar o mesmo resultado. Eu exploro, sobretudo através de simulações numéricas, como é que nesta generalização, que apresenta preferências "bem-comportadas" e curvas de indiferença continuamente diferenciáveis, os bens de Giffen aparecem através de uma redução progressiva e, no limite, eliminação do efeito substituição, desde que o bem em questão seja um bem inferior, sem que tal seja necessariamente acompanhado por um aumento do efeito rendimento. Para além disso, desenvolvo um conjunto alternativo de preferências, que partilham o mesmo caso original como caso limite, e ilustro como os bens de Giffen surgem de modo semelhante.

Palavras-chave: Bens de Giffen, preferências do consumidor, efeito substituição, efeito rendimento

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1 Introduction

Giffen goods have long been talked about, starting off as a mere theoretical curiosity based upon reports on the behavior of demand for bread in 19th century Britain, attributed by Alfred Marshall to Sir Robert Giffen. As we can read in the 1890 edition of his *Principles of Economics*:

> (...) as Sir R. Giffen has pointed out, a rise in the price of bread makes so large a drain on the resources of the poorer labouring families and raises so much the marginal utility of money to them, that they are forced to curtail their consumption of meat and the more expensive farinaceous foods: and, bread being still the cheapest food which they can get and will take, they consume more, and not less of it. But such cases are rare; when they are met with, each must be treated on its own merits. (p. 81)

Since then, we have been calling *Giffen goods* to any goods which, under certain conditions, feature an increase in their demand quantity when responding to a price increase, and vice-versa, in defiance of the general case in which consumers respond to a price change with a change in the demand in the opposite direction.

Even though it might appear strange, in the quotation above Marshall provides a simple example that may help see that this can, indeed, happen: let us take a household consisting of, for simplicity, one student, who must buy his own lunch at university every weekday; if possible, he'd rather eat at the university canteen; his alternative for lunch is a sandwich at the bar. If any part of his lunch budget is still left over, he can buy extra sandwiches for his afternoon break. Let us say that this student has a weekly budget of $12; lunch at the canteen costs $3 per meal, and a sandwich costs $1.5. Then, to achieve a minimum of 5 meals per week, he'll have lunch at the canteen 3 times a week and still be able to buy 2 sandwiches for the other two lunches.

What if, now, the price of sandwiches rises to $2? This student can no longer afford to eat three times a week in the canteen, for the $3 that are still left on his budget are not enough for 2 sandwiches. Then, he will now choose to eat at the canteen only twice a week, and have $6 that he can spend on three sandwiches for the other three days of the week. The price of sandwiches increased, and in response so did the consumption of sandwiches, meaning that, in this case, they are a Giffen good.

With this simple example it becomes clear that Giffen goods are, at least on an individual level, a possibility; its transfer to the aggregate level relies mostly on how widespread such peculiar and specific preference structures are. This dissertation, then, using the final proposal by Sørensen (2007) as a starting point, seeks to review that it indeed provides for the existence of Giffen goods and to find a new structure of preferences that also generates Giffen goods.

Arising from the experience of previous authors, it is expected that there will be difficulties finding functional forms. Therefore, we will rely mostly on numerical examples, particularly when exploring demand curves. In section 2 we will go through Sørensen (2007)'s model; we develop its generalization as proposed by the author in section 3. In Appendix A, specifically, we will present and correct a minor mistake that was found in that paper, and which must necessarily be addressed in order to seamlessly discuss Example 2 in section 3.2. In section 4 we will explore an alternative set of preferences, which will be based upon the same Sørensen (2007)'s Example 1.

1.1 Related literature and motivation

Giffen goods have been studied mostly through two paths: empirically, so as to demonstrate their existence under real world conditions, and theoretically, so as to study preferences that might generate Giffen behavior for certain goods. This is important because, typically, Giffen goods are presented even in microeconomics textbooks using simple graphical analysis and immediately brushed off as something that, in practice, might not even exist or, at least, be so rare that it does not deserve particular attention. However, in both sides there have been developments that counter this usual attitude towards the "Giffen paradox".

The first path had been, until recently, largely unsuccessful; in their literature review regarding the search for utility functions generating Giffen goods, Heijman and van Mouche (2009) claim that most proposed examples of real life Giffen goods "have been discredited because the data were not correctly interpreted or analysed" (p. 1), including the famous and widely mentioned example of potatoes in the days of the Great Famine in Ireland. As far as we are aware, to this day only Jensen and Miller (2008) have provided statistical evidence that there is at least one case of Giffen goods, namely in the Chinese provinces of Hunan and Gansu, with rice and wheat, respectively. These results were heavily dependent on income levels, corroborating traditional assumptions about Giffen goods. In the case of Hunan, evidence was significant enough to suspect that similar results might be waiting to be found elsewhere in the world, provided that poverty is dominant and some basic goods have similar characteristics to those of rice.

In terms of the second path, there has been a series of slow but steady developments. Heijman and van Mouche (2009) have categorized this search for a utility function u into three parts: the weak Giffen problem: "u is continuous, strictly increasing and quasi-concave" (p. 2), meaning that preferences are well-behaved; the text book Giffen problem: "u is continuous and has the property that Gossen's Second Law is applicable" (p. 2), meaning that Giffen goods are generated through an internal solution; and, finally, the strong Giffen problem: "u simultaneously solves the weak and the text book Giffen problem" (p. 2), joining both well-behaved preferences and interior solutions potentially generating Giffen goods in a single utility function.

Spiegel (1994) found a continuous but not always increasing utility function that generated Giffen goods that featured an interior solution; he also attempted to find well-behaved utility functions that allowed for the occurrence of Giffen goods but, being unable to do so, he proposed that they were non-existent. This was disproved by Moffat (2002), who showed the existence of preferences compatible with the typical assumptions of consumer theory (continuity, monotonicity and convexity) and also displaying an interior solution, but was not able to provide a specific functional form for a utility function corresponding to such preferences. As we shall see, explicit functional forms are a frequent problem while studying Giffen goods.

Examples of well-behaved utility functions allowing for Giffen goods were found, then, by Landi (2015) and, more noticeably, by Sørensen (2007) who provided us an entire family of Giffen-compatible utility functions. Their findings rely on a kink in the indifference curves that generates a downwards-sloped kink curve, generating Giffen behavior for a given good depending on the exact specifications of the budget constraint. The work of Sørensen (2007) is of particular interest, given that his own class of utility functions can be described as a particular case of a wider class that is capable of allowing for Giffen goods to occur in the context of interior solutions of the consumer's problem. Nevertheless, the author finishes his work without exploring such a proposition.

Doi, Isawa, and Shimomura (2009), then, provide a utility function that, even though it is divided in branches, it is in such a way that there is no discontinuity in the slope of the indifference curves at the point of splicing, and therefore the problem of a marginal rate of substitution not existing is avoided; Gossen's Second Law is applicable and Giffen goods appear in interior solutions. This utility function has the particular feature of the existence of Giffen goods not being a result of poverty, defying conventional assumptions about them if they exist in the real world and providing a theoretical backing if they are ever discovered in the real world under conditions very different from those studied by Jensen and Miller (2008). Finally, Biederman (2015) provides a non-spliced, well-behaved utility function that allows for Giffen goods to appear in interior solutions to the consumer's problem. Nevertheless, this author is faced with an impossibility to derive an explicit demand function, needing to approach his own set of preferences either numerically or by analysing elasticities of substitution. He finds that Giffen goods appear if the elasticity of substitution is sufficiently high in absolute value.

One thing is to find utility functions that have the property of allowing for Giffen goods; another is to isolate characteristics of preferences that allow for that same property. So far, two have been put forward: the most frequently mentioned one is the (at least implicit) existence of at least one constraint other than the budget constraint: the traditional examples involving food, such as the one regarding the Irish Famine or the original one with the bread, are based upon a minimum in nutritional requirements: as one given food item becomes more expensive, but is still cheaper than the alternative, and under a very tight budget, the consumer can no longer afford to diversify between both types of food and, then, so as to be able to afford a minimum quantity of food, buys more and more only from the cheapest item (even though it is the one whose price increased); the statistical findings by Jensen and Miller (2008) seem to be based upon this reasoning. A similar type of reasoning is put forward for the case of public transport in Spiegel (1994). The other characteristic, which is based upon the Slustky equation and is the key to Sørensen (2007), consists of eliminating the substitution effect in an inferior good.

All the previous authors studied Giffen goods using typical two-good models. For the existence of Giffen goods in a model with more goods, Sørensen (2011) uses a modification of the model in Sørensen (2007) to generate Giffen goods, once again relying on kinks.

2 Sørensen's "modified Leontief" model

Sørensen (2007) has shown, by means of examples, that Giffen goods can be generated by a utility function of the type

$$U(x_1,x_2) = min\{u_1(x_1,x_2), u_2(x_1,x_2)\} \tag{1}$$

as long as u_1 and u_2 represent continuous, monotonic and (weakly) convex preferences over x_1,x_2. To explain this fact, we must look at the Slutsky equation and which is a landmark tool in the study of Giffen goods:

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - x_j \frac{\partial x_i}{\partial m} \tag{2}$$

which can be read as "total effect of a price change equals substitution effect plus income effect". If we set $i = j$, we know that the substitution effect term $\partial h_i/\partial p_i$ (in which h_i is the Hicksian demand function for good i) must be negative if preferences are monotonic; therefore, for good x_i to become a Giffen good, i.e., $\partial x_i/\partial p_i > 0$, we must have that it is an inferior good, i.e., $\partial x_i/\partial m < 0$. However, even if we have it, it is typical that the substitution effect dominates, and in that case we do not have a Giffen good. However, this can be changed by the presence of the minimum operator, which generates a kink in the indifference curves where $u_1 = u_2$ and, therefore, eliminates all substitution effect at least in part of the demand curves for each good; a good, then, becomes a Giffen good if it is an inferior good. More specifically, if the continuous locus of kinks, which the author calls kink curve, is downwards sloping, then, by expanding the budget constraint (generating a pure income effect) the consumer's optimal solution may (depending on the marginal rates of substitution and relative prices) consist in moving along the kink curve, reducing demand for one of the goods, and having therefore an inferior good; with the elimination of the substitution effect it becomes a Giffen good as well, at least locally.

Now, for the sake of simplicity, we shall focus on Sørensen (2007)'s Example 1, in which the author defines

$$u_1(x_1,x_2) = x_1 + B \tag{3}$$

$$u_2(x_1,x_2) = A(x_1 + x_2) \tag{4}$$

with $B > 0, A > 1$, which are necessary conditions for Giffen goods to appear in the limit "modified Leontief" case. In section 3.2, we shall move to Example 2 of the same paper; in both cases, it is good x_2 that has the potential to become a Giffen good. We will illustrate how it eventually occurs.

The consumer solves the problem

$$\max_{x_1,x_2} U(x_1,x_2) = min\{x_1+B, A(x_1+x_2)\}$$

s.t.

$$p_1x_1 + p_2x_2 \leq m$$

We see that this cannot be solved by a typical $MRS_{12} = p_1/p_2$ rule, as it it typical of a Leontief utility function. Then, it becomes easier to find the optimal solution to the problem by use of graphs.

We start off by determining the indifference curves. The utility function can be written as

$$U = min\{x_1+B, A(x_1+x_2)\} = \begin{cases} x_1+B & x_1+B \leq A(x_1+x_2) \\ A(x_1+x_2) & x_1+B > A(x_1+x_2) \end{cases}$$

Rearranging, we get that

$$U = \begin{cases} x_1+B & x_2 \geq \frac{x_1(1-A)+B}{A} \\ A(x_1+x_2) & x_2 < \frac{x_1(1-A)+B}{A} \end{cases}$$

Meaning that, at $x_2 = [x_1(1-A)+B]/A$, the utility function changes branches and indifference curves feature a kink; this locus of kinks is the kink curve. In this example, looking at the two-goods space, above the kink curve, utility curves are vertical straight lines; below it, they continue into straight lines of slope -1. We must notice that it has slope $(1-A)/A$ and it intercepts the vertical axis at B/A, and the horizontal axis at $B/(A-1) > B/A$. The prerequisites $B > 0, A > 1$ ensure that we have both a negative slope and a positive intercept; they will become relevant when we demonstrate that good 2 may indeed become a Giffen good. Figure 1 is an illustration, based in the original example in the paper: $A = 2, B = 10, m = 60, p_1 = 12$, of how the agent responds to change in p_2. The thicker lines represent the successive changes in the budget constraint, as p_2 takes values 8, 12 and 15; the thinner lines are indifference curves (please notice that one of the indifference curves, below the kink curve, coincides with the budget constraint for $p_2 = 12$, and above it with the vertical axis); finally, the dashed line is the kink curve.

Now, to find optimal points, we must take the budget constraint into account, which is always fulfilled in equality:

Case 1: $p_1/p_2 < 1$**:**

For any $p_1/p_2 < 1$, and independently from his income level m and the prince of good 1 p_1, the agent will always choose to spend all of his income in good 1. Therefore, in this situation, demand is given by $(x_1, x_2) = (m/p_1, 0)$.

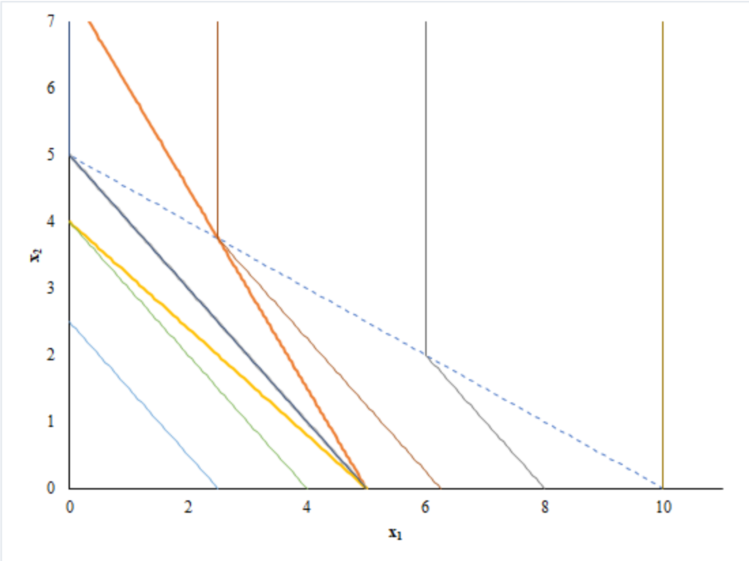

Figure 1: "Modified Leontief" optimal choices in the two-goods space.

Case 2: $p_1/p_2 > 1$**:**

For $p_1/p_2 \geq 1$, things get more complex and we must take into account both income and price levels and how they relate to the kink curve. For ease of analysis, we shall take m/p_1, fix it, move p_2, and analyse how this differs under different cases of m/p_1.

Case 2.1: $m/p_1 < B/A$**:**

Here we can distinguish between two distinct situations: if $m/p_2 \leq B/A$, the agent decides to spend all his income on good 2; demand is given by $(x_1, x_2) = (0, m/p_2)$. If, on the other hand, $m/p_2 > B/A$, there optimal bundle is on the kink curve; then, the agent's optimal choice is the intersection between the budget constraint and the kink curve:

$$\begin{cases} p_1x_1 + p_2x_2 = m \\ x_2 = \frac{x_1(1-A)+B}{A} \end{cases} \iff \begin{cases} x_1 = \frac{Am-p_2B}{A(p_1-p_2)+p_2} \\ x_2 = \frac{m}{p_2} - \frac{p_1}{p_2}\frac{Am-p_2B}{A(p_1-p_2)+p_2} \end{cases}$$

Case 2.2: $B/A \leq m/p_1 < B/(A-1)$**:**

In this situation, the optimal choice is always the point where the budget constraint intersects the kink curve and the optimal choice of the agent is given by

$$\begin{cases} x_1 = \frac{Am-p_2B}{A(p_1-p_2)+p_2} \\ x_2 = \frac{m}{p_2} - \frac{p_1}{p_2}\frac{Am-p_2B}{A(p_1-p_2)+p_2} \end{cases}$$

We should notice, then, that cases 2.1 and 2.2 can be summarized with m/p_2 as the break-point: since $p_1 > p_2$, we necessarily have $m/p_1 < m/p_2$. Then, $m/p_2 \leq B/A$ necessarily implies $m/p_1 < B/A$; therefore, this condition is necessary and sufficient, given the relative price condition, to have $(x_1, x_2) = (0, m/p_2)$ as the optimal choice. Otherwise, and provided that $m/p_1 < B/(A-1)$, $m/p_2 > B/A$ is necessary and sufficient for the demand functions to be given by the system of equations that describes the intersection between the budget constraint and the kink curve.

Case 2.3: $m/p_1 \geq B/(A-1)$**:**

For any $x_1 > B/(A-1)$ utility depends exclusively on the level of x_1, which translates into utility curves being given by straight vertical lines. The agent, then, chooses to consume good 1 only: his choice will be $(x_1, x_2) = (m/p_1, 0)$.

Case 3: $p_1/p_2 = 1$**:**

For $p_1/p_2 = 1$, in most cases we have the budget constraint coinciding, at least partially, with a single indifference curve in the section where the latter has slope -1. This implies that, in these cases, there will be a continuum of optimal solutions for the agent's problem:

Case 3.1: $m/p_1 < B/A$**:**

Any point in the budget constraint will be optimal. The optimal choice is given by $(x_1, x_2) = (m/p_1 - k, k)$ with $k \in [0, m/p_2]$.

Case 3.2: $B/A \leq m/p_1 < B/(A-1)$**:**

Any point in the budget constraint *at or below the kink curve* will be optimal. The optimal choice will be given by $(x_1, x_2) = (m/p_1 - k, k)$ with $k \in \left[0, \frac{m}{p_2} - \frac{p_1}{p_2}\frac{Am - p_2 B}{A(p_1 - p_2) + p_2}\right]$.

Case 3.3: $m/p_1 \geq B/(A-1)$**:**

For the same reasons as in case 2.3, the optimal bundle for the consumer is $(x_1, x_2) = (m/p_1, 0)$.

Then, the Marshallian demand functions for both goods are given by

$$x_1^M = \begin{cases} \frac{m}{p_1} & ,\left(\frac{p_1}{p_2} < 1\right) \vee \left(\frac{m}{p_1} \geq \frac{B}{A-1}\right) \\ 0 & ,\left(\frac{p_1}{p_2} > 1\right) \wedge \left(\frac{m}{p_2} \leq \frac{B}{A}\right) \\ \frac{Am - p_2 B}{A(p_1 - p_2) + p_2} & ,\left(\frac{p_1}{p_2} > 1\right) \wedge \left(\frac{m}{p_1} < \frac{B}{A-1}\right) \wedge \left(\frac{m}{p_2} > \frac{B}{A}\right) \\ \frac{m}{p_1} - k & ,\left(\frac{p_1}{p_2} = 1\right) \wedge \left(\frac{m}{p_1} < \frac{B}{A-1}\right), \forall k \in \left[0, min\left\{\frac{m}{p_2}, \frac{m}{p_2} - \frac{p_1}{p_2}\frac{Am - p_2 B}{A(p_1 - p_2) + p_2}\right\}\right] \end{cases}$$

$$x_2^M = \begin{cases} 0 & ,\left(\frac{p_1}{p_2} < 1\right) \vee \left(\frac{m}{p_1} \geq \frac{B}{A-1}\right) \\ \frac{m}{p_2} & ,\left(\frac{p_1}{p_2} > 1\right) \wedge \left(\frac{m}{p_2} \leq \frac{B}{A}\right) \\ \frac{m}{p_2} - \frac{p_1}{p_2}\frac{Am - p_2 B}{A(p_1 - p_2) + p_2} & ,\left(\frac{p_1}{p_2} > 1\right) \wedge \left(\frac{m}{p_1} < \frac{B}{A-1}\right) \wedge \left(\frac{m}{p_2} > \frac{B}{A}\right) \\ k & ,\left(\frac{p_1}{p_2} = 1\right) \wedge \left(\frac{m}{p_1} < \frac{B}{A-1}\right), \forall k \in \left[0, min\left\{\frac{m}{p_2}, \frac{m}{p_2} - \frac{p_1}{p_2}\frac{Am - p_2 B}{A(p_1 - p_2) + p_2}\right\}\right] \end{cases}$$

From here, we can take the derivatives of the demand for good 2 in order to the respective price so as to confirm that, in some of these cases, good 2 is, in fact, a Giffen good:

$$\frac{dx_2}{dp_2} = \begin{cases} 0 & ,\left(\frac{p_1}{p_2} < 1\right) \vee \left(\frac{m}{p_1} \geq \frac{B}{A-1}\right) \\ -\frac{m}{p_2^2} & ,\left(\frac{p_1}{p_2} > 1\right) \wedge \left(\frac{m}{p_2} \leq \frac{B}{A}\right) \\ a & ,\left(\frac{p_1}{p_2} > 1\right) \wedge \left(\frac{m}{p_1} < \frac{B}{A-1}\right) \wedge \left(\frac{m}{p_2} > \frac{B}{A}\right) \\ 0 & ,\left(\frac{p_1}{p_2} = 1\right) \wedge \left(\frac{m}{p_1} < \frac{B}{A-1}\right), \forall k \in \left[0, min\left\{\frac{m}{p_2}, \frac{m}{p_2} - \frac{p_1}{p_2}\frac{Am-p_2B}{A(p_1-p_2)+p_2}\right\}\right] \end{cases}$$

with

$$a = -\frac{m}{p_2^2} + \frac{p_1}{p_2^2}\frac{Am - p_2B}{A\left(p_1 - p_2\right) + p_2} + \frac{p_1}{p_2}\frac{B}{A\left(p_1 - p_2\right) + p_2} + \frac{p_1}{p_2}\frac{\left(1 - A\right)\left(Am - p_2B\right)}{\left[A\left(p_1 - p_2\right) + p_2\right]^2}$$

While the derivative is zero in the first and last branches of the system (meaning that the demand for good 2 does not vary with price), and negative in the second one (meaning that it varies negatively, which would be the usual case under standard preferences), it can be shown that, given the conditions for the third branch and remembering that $A > 1$ and $B > 0$, the derivative in the third branch is positive, i.e., $a > 0$. This proof will be provided in Appendix B.1. Then, in the third branch, good 2 is, indeed, a Giffen good.

3 The "modified CES" model

At the end of Sørensen (2007), the author notes that equation (1), which he calls "modified Leontief", can be approximated by an equation of the type

$$U(x_1,x_2) = \left[u_1^{\rho}(x_1,x_2) + u_2^{\rho}(x_1,x_2)\right]^{1/\rho} \tag{5}$$

with $\rho < 1$, when ρ converges to $-\infty$; we can call this type of function a "modified CES" utility function. Therefore, it is reasonable to expect that, for sufficiently low values of ρ, part of the Marshallian demand curve for one of the goods will be upwards sloped if u_1 and u_2 are such that they generate Giffen goods in the "modified Leontief" case. Given the characteristics of both CES functions and functions u_1 and u_2, U fulfills the prerequisites to solve the strong Giffen problem; now, one only needs to confirm that this utility function results in locally upwards sloped Marshallian demand functions to solve the strong Giffen problem.

For the sake of illustration, we will, once again, use the definitions for u_1 and u_2 provided by Example 1.

The consumer solves the problem

$$\max_{x_1,x_2} U(x_1,x_2) = \left[(x_1+B)^{\rho} + [A(x_1+x_2)]^{\rho}\right]^{1/\rho}$$

s.t.

$$p_1x_1 + p_2x_2 \leq m$$

together with non-negativity restrictions for both goods.

Except for corner solutions (in which one of the non-negativity restrictions is binding) the solution to the problem is given by

$$MRS_{12} \equiv \frac{1}{A^{\rho}}\left(\frac{x_1+B}{x_1+x_2}\right)^{\rho-1} + 1 = \frac{p_1}{p_2} \tag{6}$$

together with the budget constraint holding in equality. We could not derive explicit functional forms for the Marshallian demand functions for either good; however, one can resort to finding numerical solutions to the problem.

In original example, the author sets $A = 2, B = 10, m = 60, p_1 = 12$. Figure 2 there are demand curves for several levels of ρ and p_2. The "modified Leontief" case is depicted as $\rho \to -\infty$.

It should be noted that, for any levels of $p_2 > 12$, demand will be $x_2 = 0$. This does not change with changes in income levels.

Here we see that, for the parameters in question, there exists a critical $\rho^* \in (-20,-10)$ such that, if $\rho < \rho^*$, there is a portion of the demand curve that is upwards sloping (meaning that x_2 is indeed behaving as a Giffen good). Additional computations have shown that, in this case,

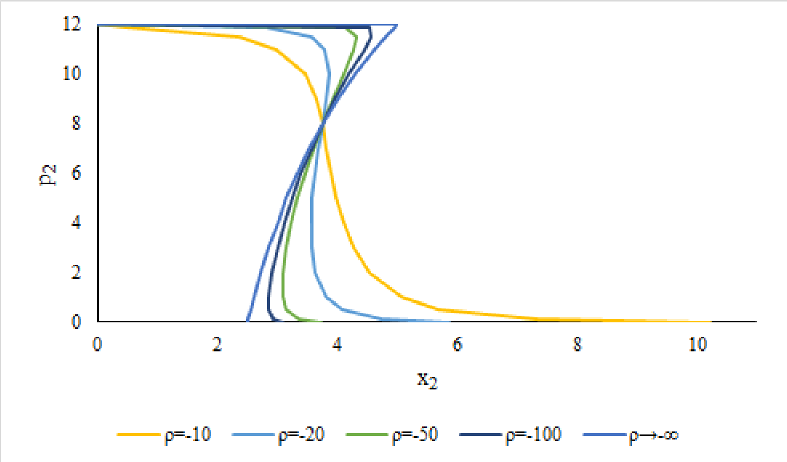

Figure 2: Marshallian demand curves for x_2 with $m = 60$

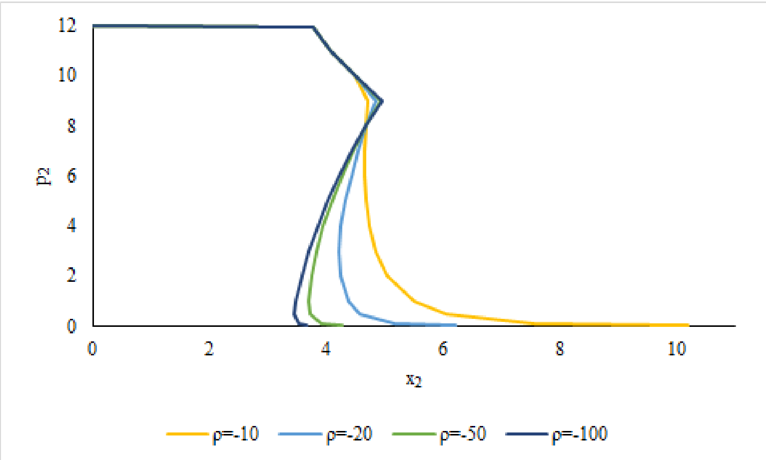

Figure 3: Marshallian demand curves for x_2 with $m = 45$

$\rho^* \approx -13.9$. Indeed, as ρ grows more negative, we see the shape of demand curve approach what it would be for the "modified Leontief" case, when $\rho \to -\infty$. Good 2 is confirmed to be a Giffen good.

Now, we should investigate some characteristics of these demand curves and verify whether they fulfill some predictions about Giffen goods: first, as it was previously stated and mathematically shown, they should be inferior goods; therefore, with increases in income demand should decrease (meaning that demand curves shift to the left when income increases, and vice-versa). It has also been widely assumed that not only should inferior goods be what we can call "poor goods", goods of bad quality that should be replaced with similar ones of better quality as long as the consumer is able to acquire them, but Giffen goods should, in the same spirit, be the "poorest of the poor goods". Therefore, as income increases, it should become "harder" for a good to be a Giffen good, as it becomes easier to be substituted, meaning that a lower level of ρ should be necessary for Giffen goods to appear. Figures 3 and 4 depict the demand curves for x_2 when income is changed to $m = 45$ and $m = 90$. For all three figures in this section, detailed

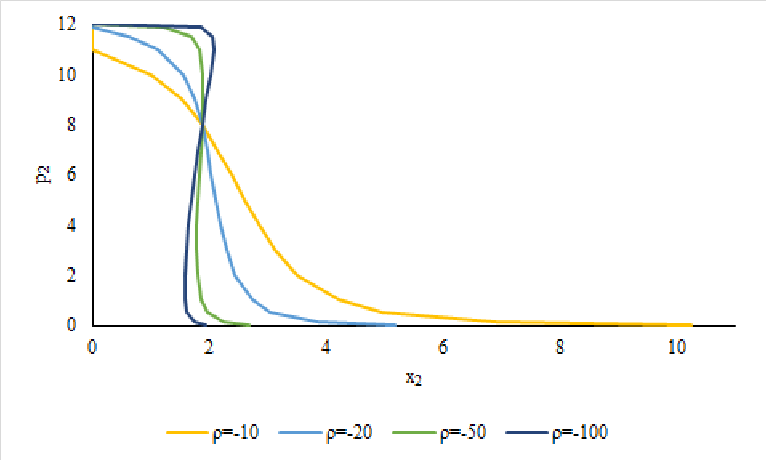

Figure 4: Marshallian demand curves for x_2 with $m = 90$

results are displayed in Appendix C.1.

We can, first, verify that, independently from the levels of ρ and p_2, demand for x_2 (our potential Giffen good) has moved in the opposite direction from changes in income, thus providing clear evidence that it is indeed an inferior good (except for some higher levels of p_2 when income has decreased to $m = 45$, which is due to corner solutions in the optimization problem). Moreover, we see that, as expected, changes in income level result in changes in the critical ρ^*, making it harder or easier for the good to exhibit Giffen behavior: when m is raised to 90, the critical ρ^* has moved to the interval $(-50, -20)$, meaning that levels of ρ (such as $\rho = -20$) no longer allow for x_2 to be a Giffen good as they did when we had $m = 60$; in a similar way, when income is reduced to $m = 45$, we will have that ρ^* belongs to the interval $(-10, -5)$ (the demand curves for $\rho = -5$ have been computed, even though they do not appear in the figures), meaning that levels of ρ (such as $\rho = -10$), that previously would not make x_2 a Giffen good, now do; more specifically, We have computed that $\rho^* \approx -9.22$. It is also worth noticing that, namely in the cases of income levels of $m = 60$ and $m = 90$ and higher levels of $-\rho$ (such as $\rho = -50$ and $\rho = -100$), the expenditure share in x_2 (depending on the price level) may not exceed .4 and can be even much lower; this appears to be in agreement with evidence from Biederman (2015) that “Giffen behavior is compatible with relatively low expenditure shares on the Giffen good” (p.27), following the comparable discoveries by Doi et al. (2009) that were previously mentioned.

3.1 Substitution and income effects in Sørensen (2007)’s Example 1

The theoretical advantage of a kink in the “modified Leontief” case, as it was previously explained, is that it eliminates the substitution effect and turns the income effect of an inferior good into the only effect triggered by a price change, resulting into said good becoming a Giffen good. Therefore, as it is approximated by the “modified CES” utility function, it would be most helpful to decompose changes in demand into both effects and verify how each behaves individually; we should expect, then, to see a significant decrease in the magnitude of the substitution

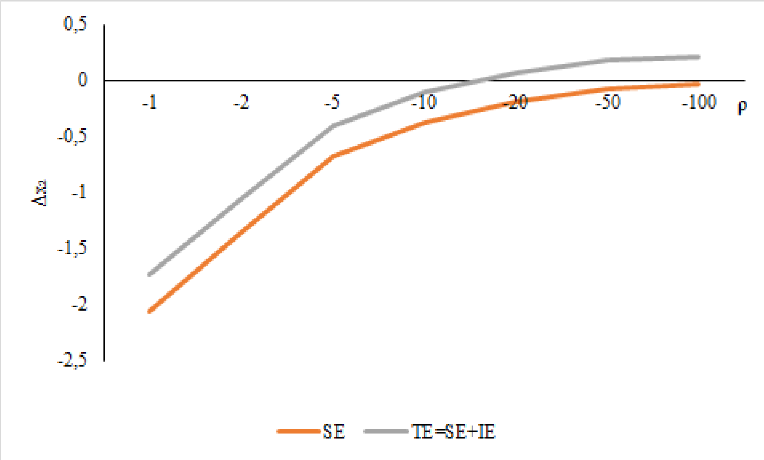

Figure 5: Slutsky decomposition of substitution and income effects in good x_2

effects as ρ grows more negative (since it is approaches zero as ρ tends to $-\infty$).

Figure 5 illustrates the Slutsky decomposition between substitution and income effects of a price change from $p_2 = 8$ to $p_2 = 9$ in the benchmark of preference parameters $A = 2$, $B = 10$ and income $m = 60$ and how it evolves with changes in ρ. The initial price level $p_2 = 8$ was chosen because, given the expression for the MRS (6), and under said parameters, it makes the initial levels of x_1 and x_2 become independent from ρ; under the Slutsky equation (2), even though it applies infinitesimally, this ensures comparability as the income effect term directly depends on the initial level of the good in question. Please notice that the axis for ρ is inverted and not in scale:

As we can see, there is a major decline in absolute value in the substitution effect as ρ grows negative, eventually converging to zero. The income effect (which is positive, confirming once more that we are in presence of an inferior good) also suffers a decline that is too small to become evident in the graph, but it does not converge to zero. This corroborates our expectations.

I have also calculated these effects for the same price change $\Delta p_2 = 1$ but for initial price levels $p_2 = 1$ and $p_2 = 4$. Detailed results will be displayed in Appendix C.3. The behavior of the substitution effect is essentially the same, even in terms of levels in the case of $p_2 = 4$. The behavior in terms of income effect is slightly more complex: a lower initial price level for good 2 leads to a smaller income effect for most levels of ρ; for lower values of ρ (in absolute value), however, it varies in great measure, to the point that for initial level $p_2 = 1$ and $\rho \geq -2$, there is evidence of a negative income effect in response to the increase in prices, which would turn x_2 into a normal good (and not inferior anymore). This provides further evidence that the ocurrence of inferior goods (and, therefore, Giffen goods) is closely related not only to characteristics in preferences but also income levels, as we can consider (everything else being equal) agents facing initial $p_2 = 1$ to be richer than agents facing initial price level $p_2 > 1$.

How can we explain the fact, however, that good 2 can change from being a normal to an inferior good under some circumstances? Biederman (2015) establishes that, in a two-goods case, a given good i is inferior if and only if $MRS_{ij} \equiv \frac{\partial U/\partial x_i}{\partial U/\partial x_j}$ is decreasing in x_j, meaning that in this specific case, for good 2 to be an inferior good, we need that

$$\frac{\partial MRS_{21}}{\partial x_1} < 0 \tag{7}$$

After rearranging and simplifying this expression, and under the previously established assumptions that $\rho < 1, A > 1, B > 0$, it simplifies to $x_2 < B$. In our numerical example, we had that $B = 10$, meaning that x_2 is inferior if and only if it is lower than 10; in our estimations, the situations in which we find it to be a normal good coincide with situations in which $x_2 > 10$, which is consistent with such a condition. Given the Slutsky equation once again, we have that, under the structure of preferences established by equations (3)-(5), it is necessary that, for good 2 to be a Giffen good, it fulfills the condition $x_2 < B$.

It should be important to verify whether or not these characteristics can be compared to what happens with other utility functions generate Giffen goods. Not straying away from Sørensen (2007)'s work, we can explore its Example 2 and modify it so as to fit the "modified CES" case. However, before we proceed, we must take note that there is an error in Sørensen (2007) regarding which conditions make the demand for either good feature Giffen behavior. This will be shown in Appendix A.

3.2 A modification of Sørensen (2007)'s Example 2

First of all, we will now drop the definitions of c_1 and c_2 as marginal rates of substitution at a specific point (as they were considered in the proof in Appendix A), and let them be simple parameters in their own right. Now we will have, for the same structure expressed in equation (3), the following branches u_1 and u_2:

$$u_1(x_1, x_2) = \left(x_1 x_2^{c_1}\right)^{1/(1+c_1)} \tag{8}$$

$$u_2(x_1, x_2) = \left(x_1 x_2^{c_2}\right)^{1/(2+2c_2)} \tag{9}$$

with $c_1 > c_2 > 0$ and which individually correspond to Cobb-Douglas utility function. The objective of this modification is simply a re-branding of variables, so as to make good 2 become the one to potentially behave in a Giffen manner. The marginal rates of substitution MRS_{12}^k for each of these functions u_k are, respectively,

$$MRS_{12}^1 = \frac{x_2}{c_1 x_1} \tag{10}$$

$$MRS_{12}^2 = \frac{x_2}{c_2 x_1} \tag{11}$$

I have attempted to apply condition (7) to the "modified CES" case involving equations (8) and (9); however, no meaningful results were achieved from it. For this reason, we suggest the following reasoning, based on the "modified Leontief" case, which relies on the kink curve to find a case where Giffen goods are expected to appear:

In the original "modified Leontief" case, it is crucial for the appearance of Giffen goods that the kink curve (where $u_1 = u_2$) is downwards sloping. In this case, we have that it will be given by the expression

$$x_2 = \left[x_1^{1/(2+2c_2)-1/(1+c_1)}\right]^{1/[c_1/(1+c_1)-c_2/(2+2c_2)]} \tag{12}$$

By taking the first derivative of x_2 with respect to x_1, and taking into account that $c_1 > c_2 > 0$, we learn that, for this derivative to be negative (and therefore for the possibility of Giffen goods to appear), we need that $c_1 < 1 + 2c_2$. This will be proved in Appendix B.2. Therefore, we should choose parameters that fulfill these conditions, for example, $(c_1, c_2) = (1.5, 0.5)$.We also need a reasonable price and income vector $(\hat{p}_1, \hat{p}_2, \hat{m})$ such that we can expect to, in the "modified Leontief" case at least, be in presence of a Giffen good.

Then, it should be noticed that the point $\hat{x} = (x_1, x_2) = (1, 1)$ is, independently from the values of c_1, c_2, part of the kink curve. According the aforementioned conditions for the existence of a Giffen good, we have $\partial u_1(\hat{x})/\partial x_1 < \partial u_2(\hat{x})/\partial x_1$ being fulfilled in this point if the same condition $c_1 < 1 + 2c_2$ applies (once again, conditional on $c_1 > c_2$), meaning that in the limit, as $\rho \to -\infty$, good 2 will be Giffen. Therefore, it is advisable to focus our attention to this point, which offers hope of locally generating Giffen behavior for good 2.

The "modified CES" problem the agent will solve is:

$$\max_{x_1,x_2} U(x_1, x_2) = \left[\left(x_1 x_2^{c_1}\right)^{\rho/(1+c_1)} + \left(x_1 x_2^{c_2}\right)^{\rho/(2+2c_2)}\right]^{1/\rho}$$

s.t.

$$p_1 x_1 + p_2 x_2 \leq m$$

To ensure that we can achieve such behavior at least in the vicinity of point $\hat{x}$, we need a baseline budget constraint that fulfills $\hat{m} = \hat{p}_1 + \hat{p}_2$. Regarding the relative price, for this point to be a corner solution (making the kink relevant) between u_1 and u_2 we should have $\hat{p}_1/\hat{p}_2$ between the marginal rates of substitution (10) and (11) measured at the point in question. We see that, in this point $\hat{x}$, $MRS_{12}^k = c_k$. Therefore, in accordance with the values of c_1, c_2 that were previously proposed, we should have the relative price within the interval $(0.5, 1.5)$. To achieve this point, we can simply settle on a baseline relative price of 1, and for example set $(\hat{p}_1, \hat{p}_2, \hat{m}) = (10, 10, 20)$. The problem's solution is achieved, then, by the usual condition:

$$MRS_{12} = \frac{x_2}{x_1} \times \frac{\frac{1}{1+c_1}\left(x_1 x_2^{c_1}\right)^{\rho/(1+c_1)} + \frac{1}{2+2c_2}\left(x_1 x_2^{c_2}\right)^{\rho/(2+2c_2)}}{\frac{c_1}{1+c_1}\left(x_1 x_2^{c_1}\right)^{\rho/(1+c_1)} + \frac{c_2}{2+2c_2}\left(x_1 x_2^{c_2}\right)^{\rho/(2+2c_2)}} = \frac{p_1}{p_2} \tag{13}$$

together with the budget constraint set in equality. Just like in Example 1, these conditions

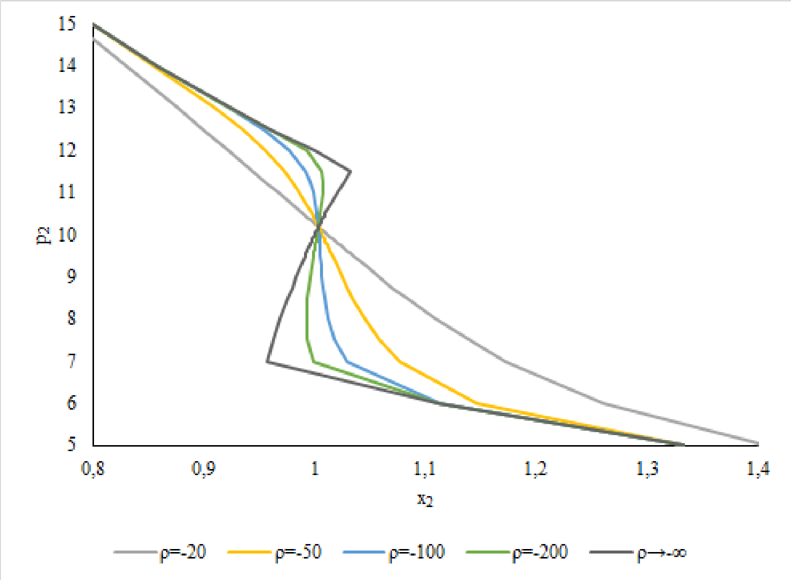

Figure 6: Marshallian demand curves for x_2 with $m = 20$ and $p_1 = 10$

do not seem to allow for derivation of explicit functional forms for the demand curve.

Using the same numerical methods as when we were analysing the previous example, the approximated demand curves were obtained (please take notice that the graph does not have its origin at point $(0,0)$ and is not in scale) and can be seen in Figure 6. Details are available in Appendix C.1.

It becomes clear that, under these specifications and for any level of ρ lower than an undetermined $\rho^* \in (-200, -100)$ there is clear evidence of good 2 behaving as a Giffen good around $p_2 = 10$ and very close to $x_2 = 1$, as we had predicted; after further computations, it could be seen that, in this case, we have $\rho^* \approx -118.1$. It must also be noticed that, as ρ grows more and more negative, it approaches the curve from the "modified Leontief" case (which is the one labeled as $\rho \to -\infty$). Even in relative terms, these positive changes in changes in demand when the price level increases seem to be very small (.1 units in 1 between each local maximum and minimum in the "modified Leontief" case, compared with, for example, more than 1 unit in 4 in the "modified CES" case with $\rho = -20$ and $m = 60$ in Example 1, as it can be seen in Figure 2). Other than that, the shape of these functions is similar to those in Example 1; the main difference is in that Example 1 admits corner solutions (as it is built upon linear preferences), and so the demand for good 2 is allowed to become zero as price grows, unlike this case in which it just approaches zero as p_2 grows, which is derived from the fact that Example 2's preferences are derived from standard Cobb-Douglas utility functions.

3.2.1 Substitution and income effects in Sørensen (2007)'s Example 2

The main goal of this exercise with Example 2 was to compare the behavior of substitution and income effects; as it was stated before, the advantage of the "modified CES" structure was that it can generate, in parts of the demand curve, a significant reduction of the substitution effect, until it reaches zero in the limit "modified Leontief" case. Using, once again, the Slutsky decomposition between substitution and income effects, and keeping p_1 and m fixed at 10 and 20, respectively, we have computed (using the same numerical methods) these effects for changes in price of good 2 $\Delta p_2 = .5$ for several levels of ρ starting at the levels of p_2 of 7.5, 9.5 and 11. These values were chosen because the three of them, at some levels of ρ, indeed generate Giffen behavior in x_2. Figure 7 shows the evolution of both effects in the change of price from $p_2 = 9.5$ to $p_2 = 10$ as ρ grows more and more negative, and it should be noticed that, once again, the axis for ρ is inverted and not in scale. Details for these price changes are displayed in Appendix C.3.

What happens when using 7.5 or 11 as starting points for p_2 in this price change is very similar to Figure 7, with the biggest difference being that the line that shows total effect crosses the horizontal axis at a higher level of ρ than in those cases (meaning, to the left of the points when it crosses the same axis under those specifications), reflecting that the closer to $p_2 = 10$ we are the easier it is for good 2 to become a Giffen good. Like in Example 1, and as expected due to the model's specifications, as $\rho \to -\infty$ we see the substitution effect gradually disappearing until it is completely eliminated in the limit. The most noticeable difference is in the income effect term, which leads to believe that its behavior depends very much on the specifications for u_1 and u_2. More specifically, while in Example 1 the income effect was almost always in direction opposite from the price movement in the domain of ρ that was considered, meaning that x_2 could be an inferior good (which is, once again, a necessary condition for the appearance of Giffen goods as per the Slutsky equation) without needing for ρ to be very low, now we have the transition point from being an ordinary into an inferior good set at a much lower level of ρ, shortly "before" said good turns into a Giffen good; this happened for the three price changes that were considered.

We can look at this fact by taking into account the structure of preferences, and the demand curves in Figure 6 can provide some intuition; for low levels of $-\rho$, we have a highly smoothed mixture between two typical Cobb-Douglas utility functions, resulting in demand curves resembling a typical Cobb-Douglas demand curve: see, for example, the demand curve for $\rho = -20$. However, as $-\rho$ grows and the change from u_1 to u_2 becomes sharper, preferences start to behave as two different Cobb-Douglas utility functions each taking up its own part of the (x_1, x_2) domain and with an increasingly small transition area. In terms of demand functions, as ρ grows more negative, we have the Marshallian demand curve starts off by imitating the individual demand for one of the individual utility functions u_i, eventually transitions to imitating the demand curve for u_j, and potentially generates locally upwards sloping demand curves in that transition area, which is the origin of the Giffen behavior for the good in question.

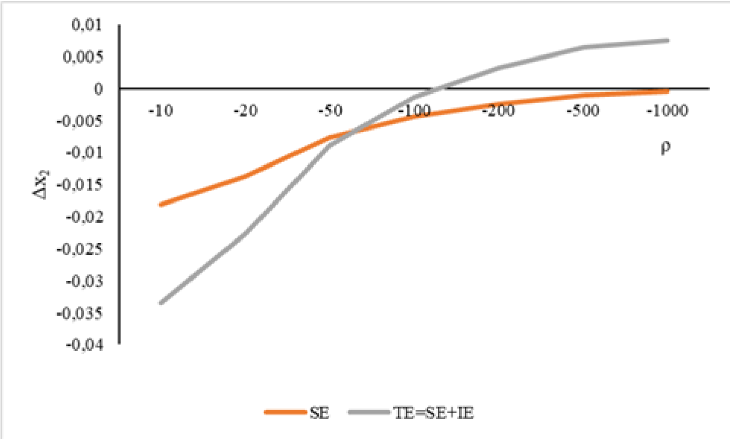

Figure 7: Slutsky decomposition between substitution and income effects in x_2 in Example 2

4 An alternative model

The main point of the previous chapter was to show that the "modified CES" utility function proposed by Sørensen (2007) does allow for the appearance of Giffen goods. Now, we propose a new structure of preferences for two goods that, in the limit, converges to Sørensen (2007)'s Example 1 in its original "modified Leontief" form and, therefore, must allow for the appearance of Giffen goods if a given parameter, which will be disclosed soon, converges to $+\infty$.

However, one clarifying point must be made: we could not isolate one single functional form for an utility function that describes this family of preferences; instead, we have described it using the equation $x_2 = f(x_1, U)$ that describes indifference curves in the space of goods (x_1, x_2) given utility level U. The equation is the following:

$$x_2 = k(x_1 - U + B + 1)^{-n} + \frac{U}{A} - x_1 \tag{14}$$

with $A > 1, B > 0, k > 0, n > 0$ and in which A and B are the same parameters as in Example 1. This equation is valid in the domain $(x_1, x_2) \in \mathbb{R}^2_+ \bigcap x_1 > U - B - 1$. Upon further verification, we have come to the conclusion that these indifference curves, being based on an hyperbolic structure, resemble those proposed by Moffatt (2002).

4.1 Basic characteristics

To check that these preferences are "well-behaved", we need to verify both monotonicity and convexity.

Monotonicity:

To show that utility is monotonic, we can take differences on equation (14):

$$dx_2 = dx_1 \frac{\partial x_2}{\partial x_1} + dU \frac{\partial x_2}{\partial U}$$

$$dx_2 = dU\left[kn\left(x_1 - U + B + 1\right)^{-n-1} + \frac{1}{A}\right] - dx_1\left[kn\left(x_1 - U + B + 1\right)^{-n-1} + 1\right]$$

Keeping x_2 constant (meaning $dx_2 = 0$), we have that

$$dU\left[kn\left(x_1 - U + B + 1\right)^{-n-1} + \frac{1}{A}\right] = dx_1\left[kn\left(x_1 - U + B + 1\right)^{-n-1} + 1\right]$$

knowing that $kn\left(x_1 - U + B + 1\right)^{-n-1} > 0$, we necessarily have that, if dU must share the same sign as dx_1: utility levels increase when consumption of good 1 increases and vice-versa.

Keeping x_1 constant, that is, having $dx_1 = 0$, we have instead

$$dx_2 = dU\left[kn\left(x_1 - U + B + 1\right)^{-n-1} + \frac{1}{A}\right]$$

and it becomes clear that utility levels always vary in the same direction as the quantity of good 2.

As U is increasing in both x_1 and x_2, preferences are monotonic.

Convexity:

In a two-goods space (x_1, x_2), if preferences are monotonic, it is a necessary and sufficient condition for preferences to be strictly convex that the slope of the indifference curves declines in absolute value along x_1, meaning that, as they are negatively sloped,

$$\frac{d^2 x_2}{d\left(x_1\right)^2}|_{U=\bar{U}} > 0$$

or, taking the second derivative,

$$kn\left(n+1\right)\left(x_1 - \overline{U} + B + 1\right)^{-n-2} > 0$$

which always happens. These preferences are both monotonic and convex, and therefore they are "well-behaved".

4.2 Potential to generate Giffen goods

As $n \to +\infty$, these indifference curves will converge to those generated by the utility function

$$U\left(x_1, x_2\right) = min\left\{x_1 + B, A\left(x_1 + x_2\right)\right\} \tag{15}$$

and good 2 can, then, easily become a Giffen good.

To demonstrate this, we can see that the marginal rate of substitution can be written as

$$MRS_{12} \equiv -\frac{dx_2}{dx_1}|_{U=\bar{U}} = kn\left(x_1 - \overline{U} + B + 1\right)^{-n-1} + 1 \tag{16}$$

and therefore we can write

$$\underset{n\to+\infty}{Lim}\, MRS_{12} = k \underset{n\to+\infty}{Lim}\, n\left(x_1 - \overline{U} + B + 1\right)^{-n-1} + 1 = \begin{cases} +\infty, & x_1 < \overline{U} - B \\ 1, & x_1 > \overline{U} - B \end{cases} \tag{17}$$

How are these indifference curves shaped, then? Keeping U constant, at $x_1 = U - B$, we have a vertical line which, for higher values of x_1, becomes a straight line with slope -1. So far this is consistent with the behavior of the “modified Leontief” curves. Given that in terms of shape they are equal to the “modified Leontief” ones, now we need to find also one more thing in common; in a very narrowing perspective, if for the same U the kink occurs at the same point (x_1, x_2) in both curves, they correspond to the same preferences; more widely, taking into account that, except for situations corresponding to the Expected Utility theory, utility is merely ordinal, if both types of preferences generate the same kink curve then they correspond to the same preferences.

For the “modified Leontief” case, the kink curve is, looking at equation (15), given by the equations

$$\begin{cases} U = x_1 + B \\ U = A\left(x_1 + x_2\right) \end{cases} \tag{18}$$

and for the new preferences, the indifference curves are, from (14) (with $n \to +\infty$) and from the previous paragraph,

$$\begin{cases} x_2 = \frac{U}{A} - x_1 & , x_1 > U - B \\ x_2 \in \left[\frac{U}{A} - x_1, +\infty\right) & , x_1 = U - B \end{cases} \tag{19}$$

And the kink curve, for a given utility level U, is given by

$$\begin{cases} x_2 = \frac{U}{A} - x_1 \\ x_1 = U - B \end{cases} \tag{20}$$

We can see that systems (18) and (20) are the same given the same parameters A and B (meaning that the kink curve is the same), and even for the same utility level U. This means that, as n grows, the preferences implied by the indifference curves given by (14) converge to the ones first proposed by Sørensen (2007) in his Example 1 and given by equation (15) and, by continuity, are capable of generating Giffen goods (in this case, it should be good 2) if n is sufficiently high. That happens because the growth in this parameter makes part of the indifference curves evolve and approximate a kink, progressively reducing and, in the limit, eliminating the substitution effect and enacting the same principle that makes Sørensen’s preferences work for Giffen goods; therefore, at some point it would be practically enough to have these new preferences making good 2 an inferior good for it to be Giffen, as per Slutsky’s equation (2). We can verify this using a reinterpretation of condition (7): keeping x_2 constant but increasing x_1, and remembering that $MRS_{21} = 1/MRS_{12}$, the marginal rate of substitution at the new bundle should now be higher. Therefore, we should have

$$\frac{dMRS_{12}}{dx_1} > 0$$

(we are writing total derivatives so as to take into account that utility level U is a function of (x_1, x_2)). This develops into

$$kn(n+1)\left(\frac{\partial U}{\partial x_1} - 1\right)(x_1 - U + B + 1)^{-n-2} > 0$$

in which $\partial U/\partial x_1$ is marginal utility of good 1. From our calculations to demonstrate that these preferences are monotonic we have that, keeping x_2 constant, we have that

$$\frac{dU}{dx_1} = \frac{kn(x_1 - U + B + 1)^{-n-1} + 1}{kn(x_1 - U + B + 1)^{-n-1} + \frac{1}{A}}$$

meaning that, since $A > 1$, $\partial U/\partial x_1 > 1$. This makes the development of equation (7) hold and makes good 2 an inferior good, allowing for it to display Giffen behavior provided that the substitution effect is small enough.

Now, we should verify that it effectively occurs. Figure 8 features some indifference curves and budget constraints yielding optimal points for $U = 11, 12, 13, 14, 14.5$, under the original parameters $A = 2, B = 10$ and $n = 10$, together with $m = 60, p_1 = 12$ which were approximated numerically, once again, using MS Excel's Solver tool.

However, it should be noted that, unlike the previous numerical estimations, which were computed by maximizing the utility level U subject to the budget constraint by choosing x_1 and x_2, and paying attention to the relation between the marginal rate of substitution and the relative price so as to distinguish between corner and interior solutions, now these estimations were performed by forcing the marginal rate of substitution to be equal to the relative price while fulfilling the budget constraint in equality; given p_1, m and U, we choose p_2 and x_1, with x_2 being found by the equation for the indifference curve. This means that this procedure is only valid for interior solutions. Figure 8, which shows the Marshallian demand curves for the same parameters A and B and under the same budget $m = 60$ and $p_1 = 12$ for some levels of n, was obtained by using this last method calculating each curve for several utility levels $\overline{U} \in (10, 15)$. This is shown in more detail in Appendix C.2.

As p_2 decreases, p_1/p_2 increases and the budget constraint expands. We see that the shape of the indifference curves is such that, for the first four pairs of budget constraints/indifference curves, x_2 is also decreasing, confirming that in this case good 2 is, in fact, a Giffen good. At some point, p_2 becomes so low that, in the end, x_2 increases. For the same example, in Figure 9 we find, among some other levels of n, an approximation of the corresponding Marshallian demand curve, among points that can be calculated. As $p_2 \to 0$, the demand for good 2 con-

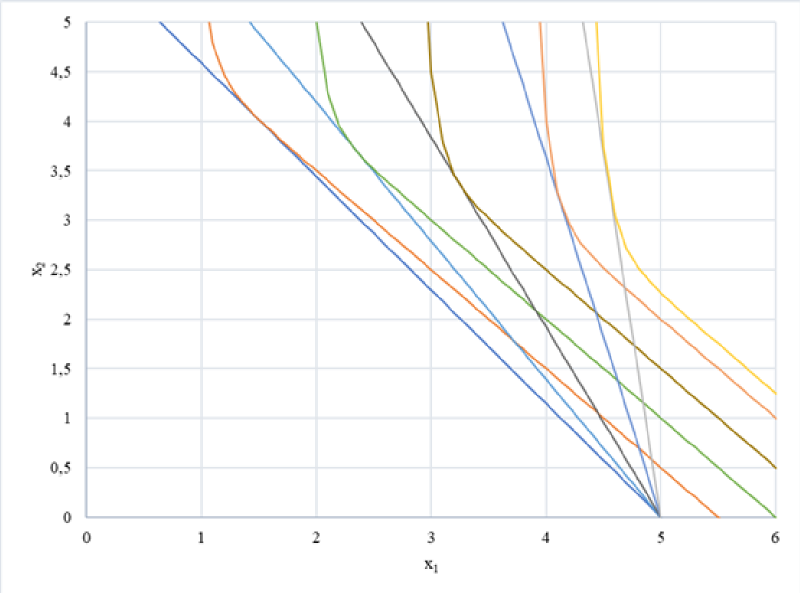

Figure 8: Optimal choices under $m = 60, p_1 = 12, n = 10$ in the two-goods space

verges to infinity, and for values of p_2 significantly higher than 12 demand is zero. Then, it should be noticed that the shape of the demand curve for $n = 10$ is similar to those in Figure 2 (for the "modified CES" version of Sørensen (2007)'s Example 1), resembling mostly (and by coincidence) the curve corresponding to $\rho = -50$. This fact is not surprising, since such an approximation, since such a resemblance should be expected by construcion, as these indifference curves and the ones from the "modified CES" case are built so that, in the limit, they converge to the "modified Leontief". It should also be noticed that the fact that the Marshallian demand curves in Figures 2 and 9 differ from each other, namely in the fact that Figure 9 does not show them intersecting at the same point $(x_2, p_2) = (3.75, 8)$ confirms that the new preferences are not the same as the "modified CES" ones.

In any case, one should notice how, as n increases, the shape of the demand curves converges to that of the "modified Leontief" one, and under $n = 3$ we can already see that a portion of the curve is upwards sloped, confirming that good 2 is, in a portion of the domain for p_2, a Giffen good. If we can find, for these preferences, and given those values of A, B and n, an explicit utility function, it solves the strong Giffen problem as defined by Heijman and van Mouche (2009).

However, it would be most useful if it would be possible to find, at least in the case of interior solutions (in whose context good 2 becomes a Giffen good), a functional form for the Marshallian demand function. It turns out that such a feat is possible, by an indirect approach, through Roy's identity.

First, we find the expenditure function, which requires Hicksian demand functions for goods 1 and 2. In the case of interior solutions, they would usually be found through the system

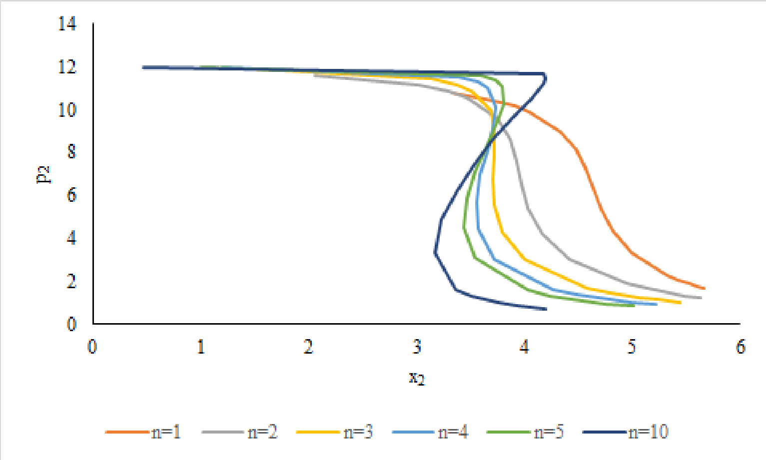

Figure 9: Marshallian demand curves for x_2 under $m = 60, p_1 = 12$

$$\begin{cases} MRS_{12} = \frac{p_1}{p_2} \\ U = u(x_1, x_2) \end{cases}$$

This can still be applicable: on one hand, we have MRS_{12} defined in equation (16). At the same time, we do not have a functional form for the utility function; however, since Hicksian demand functions and, therefore, expenditure functions, take utility level U as given, the corresponding equation in this system can be replaced by the equation for the indifference curves, which is equation (14).

The Hicksian demands in interior solutions are, then, found by simultaneously solving the following equations:

$$MRS_{12} = kn(x_1 - U + B + 1)^{-n-1} + 1 = \frac{p_1}{p_2}$$

$$x_2 = k(x_1 - U + B + 1)^{-n} + \frac{U}{A} - x_1$$

subject to the previously mentioned condition of $x_1 > U - B - 1$. The Hicksian demand for x_1 can be found by rearranging the first equation:

$$kn(x_1 - U + B + 1)^{-n-1} + 1 = \frac{p_1}{p_2}$$

$$x_1^H(p_1, p_2, U) = \left(\frac{knp_2}{p_1 - p_2}\right)^{1/(n+1)} + U - B - 1$$

Please notice that, as long as $p_1 > p_2$, the previously mentioned condition of $x_1 > U - B - 1$ is fulfilled. Inserting this result into the second equation, we find the Hicksian demand for x_2:

$$x_2 = k\left[\left(\frac{knp_2}{p_1-p_2}\right)^{1/(n+1)} + U - B - 1 - U + B + 1\right]^{-n} + \frac{U}{A} - \left(\frac{knp_2}{p_1-p_2}\right)^{1/(n+1)} - U + B + 1$$

$$x_2^H(p_1,p_2,U) = k\left(\frac{p_1-p_2}{knp_2}\right)^{n/(n+1)} - \left(\frac{knp_2}{p_1-p_2}\right)^{1/(n+1)} + U\left(\frac{1-A}{A}\right) + B + 1$$

Then, we can find the expenditure function:

$$E(p_1,p_2,U) = p_1x_1^H + p_2x_2^H$$

$$= (p_1-p_2)\left[\left(\frac{knp_2}{p_1-p_2}\right)^{1/(n+1)} - B - 1\right] + p_2k\left(\frac{p_1-p_2}{knp_2}\right)^{n/(n+1)} + U\left[p_1 + p_2\left(\frac{1-A}{A}\right)\right]$$

By inverting this equation with respect to the utility level U, we can find the indirect utility function:

$$V(p_1,p_2,m) = \frac{m - (p_1-p_2)\left[\left(\frac{knp_2}{p_1-p_2}\right)^{1/(n+1)} - B - 1\right] - p_2k\left(\frac{p_1-p_2}{knp_2}\right)^{n/(n+1)}}{p_1 + p_2\left(\frac{1-A}{A}\right)}$$

Applying Roy's identity, we can find the Marshallian demand functions. Since it is good 2 we are most interested in, as it is the one we expect to possibly be a Giffen good, we should find its Marshallian demand function:

$$x_2^M = -\frac{\partial V/\partial p_2}{\partial V/\partial m}$$

We can start by computing the denominator:

$$\frac{\partial V}{\partial m} = \frac{1}{p_1 + p_2\left(\frac{1-A}{A}\right)}$$

And then the numerator, taking derivatives and simplifying:

$$\frac{\partial V}{\partial p_2} = \frac{-kp_1\left(\frac{p_1-p_2}{knp_2}\right)^{n/(n+1)} + \left(\frac{A-1}{A}\right)m + \frac{p_1}{A}\left[\left(\frac{knp_2}{p_1-p_2}\right)^{1/(n+1)} - B - 1\right]}{\left[p_1 + p_2\left(\frac{1-A}{A}\right)\right]^2}$$

Now, we can join both so as to get the Marshallian demand function for good 2:

$$x_2^M(p_1,p_2,m) = \frac{kp_1\left(\frac{p_1-p_2}{knp_2}\right)^{n/(n+1)} - \left(\frac{A-1}{A}\right)m - \frac{p_1}{A}\left[\left(\frac{knp_2}{p_1-p_2}\right)^{1/(n+1)} - B - 1\right]}{p_1 + p_2\left(\frac{1-A}{A}\right)}$$

which will be valid as long as $p_1 > p_2$. This result is consistent with the values calculated for Figures 8 and 9 and which can be found in Appendix C.2 (which comply with the price condition), confirming their validity. For $p_1 < p_2$, it is easy to see in Figure 8 that it necessarily yields a corner solution with $(x_1,x_2) = (m/p_1, 0)$. Conditions for corner solutions with $p_1 > p_2$ are yet to be found.

From this equation, it can be easily concluded that, as long as $p_1 > p_2$, interior solutions necessarily have good 2 behaving as an inferior good, as it must happen for it to be a Giffen good. At the same time, under certain conditions, it is perfectly possible that $dx_2^M/dp_2 > 0$, meaning that good 2 would be, then, a Giffen good.

4.2.1 Income and substitution effects

Finally, we should remember that what originates the appearance of Giffen goods is the disappearance of the substitution effect when an inferior good is present in the "modified CES" case, as it was established in sections 2.3 and 2.5.1; this was originally suspected because the "modified Leontief" utility function, which displays Giffen behavior due to its kink totally eliminating any substitution effect, is its limit when ρ grows to $-\infty$. Then, since the new preferences established in equation (28) also converge, when $n \to +\infty$, to the "modified Leontief" ones, we should also guess that, as n grows, the substitution effect gradually disappears as well.

To find evidence of this fact, we should, like in the same sections, decompose price changes into substitution and income effect; however, while in sections 2.3 and 2.5.1 we used the Slutsky decomposition, now we shall use the Hicks decomposition. This is due to the fact that the procedure required to approximate demand for the new preferences, which was explained above, requires specific utility levels, and taking into consideration that the Hicks decomposition between substitution and income effects relies on the conservation of the initial level of utility when reacting to the price change (and not purchasing power of the initial bundle, as in Slutsky).

Then, given that the same procedure relies on specific utility levels rather than budget constraints, it becomes critical to decide how to choose a given price change. We suggest the following reasoning:

We can see in Figures 2-4 that the "modified CES" case for Sørensen (2007)'s Example 1 has the feature that, around $p_2 = 8$, the level of x_2 is, by a numerical coincidence, constant as ρ changes, and equal to 3.75 when $m = 60$; that extends into the original "modified Leontief" case for the same example.

Then, if we are building our preferences so as to converge to the same situation and using the same basic parameters, then it should be initially suspected that, around $p_2 = 8$, there should be

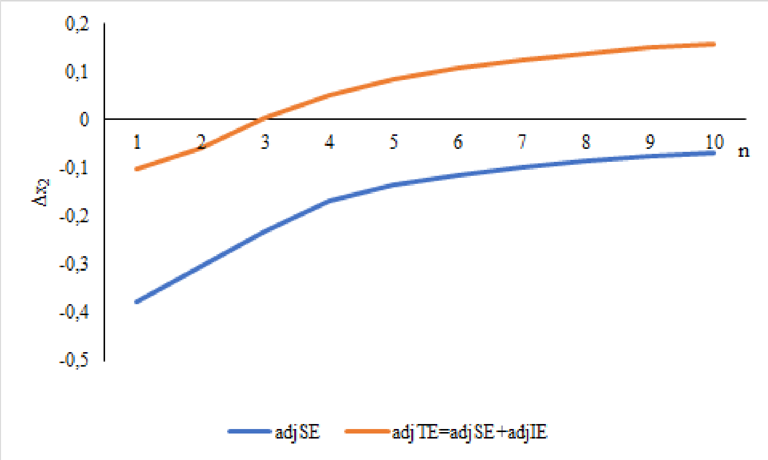

Figure 10: "Adjusted" income and substitution effects with price changes in the vicinity of $p_2 = 8$

at least a convergence of x_2 to the same level of 3.75 as n grows; Figure 9 seems to corroborate this guess. In the "modified CES" model, one of the reasons for us to focus on price changes around $p_2 = 8$ was exactly this feature, which ensured a better degree of comparability. Then, we should also focus our analysis around that point.

Since we now need changes in utility levels instead so as to compute the impact of the change in prices, and since, for the building of Figure 9, we relied on successive changes in utility level subject to the budget constraint rather than changes in p_2, we decided on using as our reference points the changes in utility level that corresponded to price changes between p_2^* and p_2', with $p_2^* < p_2'$, such that $p_2^* < 8 < p_2'$, and to show their evolution as parameter n increases. However, for each distinct n, the absolute price change $\Delta p_2 = p_2' - p_2^*$ was different, which threatened comparability. Therefore, after obtaining both values for substitution and income effects for each given n, we have divided these values by the corresponding Δp_2 so as to facilitate comparability. Therefore, the results presented in Figure 10 are not true substitution and income effects, but what we have called "adjusted" effects. More details, including the price changes and "adjusted" and "non-adjusted" Hicksian substitution and income effects, can be found in Appendix C.3.

Just like it was expected, as n increases, the substitution effect shows signs of converging to zero, eventually approaching it when n approaches infinity, which is the "modified Leontief" model. This convergence is evident even before adjusting the effects for the respective price changes. Moreover, the income effect is always positive, confirming that we are in the presence of an inferior good. However, its evolution is unclear: without adjusting for the price change, it is constant; however, the adjusted income effect features small decreases and increases. In any case, the behavior of the substitution effect provides evidence that it was through its reduction and eventual disappearance that good 2 became a Giffen good.

5 Concluding Remarks

Throughout this dissertation, we explored two distinct preference structures that, as they approach the "modified Leontief" ones proposed by Sørensen (2007), have the capability of allowing for a Giffen good. The first one, the "modified CES" structure, was originally proposed in the same work but remained unexplored; the original motivation to develop it further consisted of starting to close a gap in the literature that was left open by the original paper. Further developments await to be explored, namely in the form of explicit demand functions (which I could not find, not even indirectly through Roy's identity), and in the determination of the critical ρ^* that marks the threshold between there being or not a Giffen good. The second one was in the form of indifference curves, and does not feature a general explicit utility function; it can be, however, postulated that for some individual values of n, single utility functions may be derived in the future. However, for the general case, I have found part of the demand curve for good 2, and the same can be done for good 1. All of this provides some hints so as to overcome the problem faced by Biederman (2015), of finding Marshallian demand curves for Giffen goods that appear in interior solutions in well-behaved, non-spliced functions.

The main point with the "modified Leontief" model, as it was said before, was that it eliminated the substitution effect on an inferior good, making it necessarily a Giffen good due to the Slutsky equation. Then, it is not surprising that, in these two separate and distinct generalizations of the model, Giffen goods appear together with a significant reduction in the substitution effect, with the inferiority of the same goods, while existing, featuring varying behavior. This being known, it is not unusual to hear the Slutsky equation being interpreted, in the context of Giffen goods, as us having a very strong income effect − such that it dominates over the substitution effect, defying what is a very typical macroeconomic assumption − in the presence of an inferior good. If nothing else, at least these types of Giffen goods might become easier to understand if they were, instead, described as featuring a smaller substitution effect component when reacting to a price change in the presence of an inferior good.

Graphically speaking, this translates in a very sudden (but not necessarily discontinuous) change in the marginal rate of substitution along the same indifference curve, with the "region" of the two-goods space where these changes happen in indifference curves corresponding to successive utility levels resembling a downwards sloped "cloud", if not a line in the case of a discontinuity (the "modified Leontief"'s kink curve). This is in line with Biederman (2015)'s result that Giffen goods require a sufficiently high MRS elasticity (in absolute value), prompting that a large change in the relative price can be addressed in the case of interior solutions, and considering the substitution effect only, with a smaller change in the optimal bundle.

It should be remarked, however, that for these results to hold at the aggregate level, that is, to potentially resemble real-world situations, similar preferences needed to be held, regarding the same goods, by a significant amount of consumers in the market. Given the usual assumptions regarding inferiority and poverty (the only type of circumstances where Giffen goods were empirically found), this should not occur frequently in wealthier societies at the individual level, and even less at the aggregate one.

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Appendix A - An error in Sørensen (2007)

From Sørensen (2007):

> (...) By the implicit function theorem applied to $u_1(x) = u_2(x)$, if $\partial u_1(\hat{x})/\partial x_2 \neq \partial u_2(\hat{x})/\partial x_2$, the locus of kinks extends locally through $\hat{x}$ with slope
>
> $$\frac{dx_2}{dx_1} = -\frac{\partial u_1(\hat{x})/\partial x_1 - \partial u_2(\hat{x})/\partial x_1}{\partial u_1(\hat{x})/\partial x_2 - \partial u_2(\hat{x})/\partial x_2} \tag{21}$$
>
> By the assumptions, the demand $x(p,m)$ is on the kink curve when (p,m) is near $(\hat{p},\hat{m})$. If $\partial u_1(\hat{x})/\partial x_2 > \partial u_2(\hat{x})/\partial x_2$, then $c_1 > c_2$ implies $0 > dx_2/dx_1 > -c_2$. The kink curve is flatter than the indifference curves, (...) so good 2 is Giffen. If $\partial u_2(\hat{x})/\partial x_1 > \partial u_1(\hat{x})/\partial x_1$, likewise $0 > dx_1/dx_2 > -1/c_1$, and good 1 is Giffen. (p. 369)

Below, We show that the conditions for either good 1 or good 2 to be Giffen in the "modified Leontief" case are switched. This does not invalidate the conclusions of the paper. The basic premises of the following work build on the same logic as the paper.

Conditions for a good to be Giffen

Sørensen (2007) defines the marginal rate of substitution at the point $\hat{x}$, c_k, as

$$c_k = \frac{\partial u_k(\hat{x})/\partial x_1}{\partial u_k(\hat{x})/\partial x_2} \tag{22}$$

and establishes that $c_1 > c_2 > 0$.

Then, to allow for a good to be Giffen, it is mandatory that the kink curve is downwards sloped, meaning that $dx_2/dx_1 < 0$. This means that, as the price of the Giffen good i increases and the budget constraint contracts, and assuming that it still crosses the kink curve at a point near $\hat{x}$, it will cross it at a point with a higher level of good i than initially.

Now, exactly which good is the Giffen good depends on the shape of the kink curve relatively to the indifference curves. If the indifference curves are flatter than the kink curve, meaning that $0 < c_2 < c_1 < -dx_2/dx_1$, then good 1 is the Giffen good. Similarly, if the kink curve is flatter than the indifference curves, meaning that $0 < -dx_2/dx_1 < c_2 < c_1$, then good 2 is the Giffen good. So far, this is in agreement with Sørensen (2007).

For good 2 to be Giffen we need

$$0 < -dx_2/dx_1 < c_2 \tag{23}$$

or

$$0 < \frac{\partial u_1(\hat{x})/\partial x_1 - \partial u_2(\hat{x})/\partial x_1}{\partial u_1(\hat{x})/\partial x_2 - \partial u_2(\hat{x})/\partial x_2} < c_2 \tag{24}$$

If $c_1 > c_2$ it follows that $c_1 = c_2 + \alpha$, with $\alpha > 0$. Then:

$$(c_2 + \alpha) \times \partial u_1(\hat{x})/\partial x_2 = \partial u_1(\hat{x})/\partial x_1 \tag{25}$$

$$c_2 \times \partial u_2(\hat{x})/\partial x_2 = \partial u_2(\hat{x})/\partial x_1 \tag{26}$$

Inserting into (24):

$$0 < \frac{(c_2 + \alpha) \times \partial u_1(\hat{x})/\partial x_2 - c_2 \times \partial u_2(\hat{x})/\partial x_2}{\partial u_1(\hat{x})/\partial x_2 - \partial u_2(\hat{x})/\partial x_2} < c_2 \tag{27}$$

Rearranging:

$$0 < c_2 + \alpha \frac{\partial u_1(\hat{x})/\partial x_2}{\partial u_1(\hat{x})/\partial x_2 - \partial u_2(\hat{x})/\partial x_2} < c_2 \tag{28}$$

From here it follows that, for this condition to verify and good 2 to be Giffen, we need that $\partial u_1(\hat{x})/\partial x_2 < \partial u_2(\hat{x})/\partial x_2$, which also requires $\partial u_1(\hat{x})/\partial x_1 < \partial u_2(\hat{x})/\partial x_1$ for the kink curve to be negatively sloped. Sørensen identified this condition as being that made the other good, good 1, be a Giffen good. Actually, if we have $c_1 > c_2$ and this last condition, the former will necessarily follow. A similar reasoning will make good 1 be a Giffen good:

For good 1 to be Giffen we need that

$$-dx_2/dx_1 > c_1 \tag{29}$$

or

$$\frac{\partial u_1(\hat{x})/\partial x_1 - \partial u_2(\hat{x})/\partial x_1}{\partial u_1(\hat{x})/\partial x_2 - \partial u_2(\hat{x})/\partial x_2} > c_1 \tag{30}$$

If $c_1 > c_2$ it follows that $c_1 - \alpha = c_2$, with $\alpha > 0$. Then:

$$c_1 \times \partial u_1(\hat{x})/\partial x_2 = \partial u_1(\hat{x})/\partial x_1 \tag{31}$$

$$(c_1 - \alpha) \times \partial u_2(\hat{x})/\partial x_2 = \partial u_2(\hat{x})/\partial x_1 \tag{32}$$

Inserting into (30):

$$\frac{c_1 \times \partial u_1(\hat{x})/\partial x_2 - (c_1 - \alpha) \times \partial u_2(\hat{x})/\partial x_2}{\partial u_1(\hat{x})/\partial x_2 - \partial u_2(\hat{x})/\partial x_2} > c_1 \tag{33}$$

Rearranging:

$$c_1 + \alpha \frac{\partial u_2(\hat{x})/\partial x_2}{\partial u_1(\hat{x})/\partial x_2 - \partial u_2(\hat{x})/\partial x_2} > c_1 \tag{34}$$

Which makes it clear that, for good 1 to be Giffen, we need $\partial u_1(\hat{x})/\partial x_2 > \partial u_2(\hat{x})/\partial x_2$, which is the condition the author established for good 2 to be Giffen.

Conclusions regarding these conditions

The conditions that necessary and sufficient for each good to be a Giffen good are, after imposing $c_1 > c_2$:

1. For good 1, $\partial u_1(\hat{x})/\partial x_2 > \partial u_2(\hat{x})/\partial x_2$;
2. For good 2, $\partial u_1(\hat{x})/\partial x_1 < \partial u_2(\hat{x})/\partial x_1$.

These conditions are in agreement with what happens in both examples proposed by Sørensen (2007). In Example 1, we have

$$u_1 = x_1 + B$$

$$u_2 = A(x_1 + x_2)$$

with $A > 1, B > 0$. If we take marginal utilities we see that, for any point $\hat{x}$,

$$\partial u_1(\hat{x})/\partial x_1 = 1 < A = \partial u_2(\hat{x})/\partial x_1$$

which accordingly with our conclusions, should make good 2 a Giffen good, and not good 1 as it would be expected from Sørensen's results. In fact, in this example, it is in fact good 2 that is Giffen.

Also, in Example 2, we have

$$u_1 = \left(x_1^{c_1} x_2\right)^{1/(1+c_1)}$$

$$u_2 = \left(x_1^{c_2} x_2\right)^{1/(2+2c_2)}$$

in which the author states that, for $\hat{x} = (1,1)$, if $1 + 2c_2 > c_1$ (leading to $\partial u_1(\hat{x})/\partial x_2 > \partial u_2(\hat{x})/\partial x_2$) then good 2 is Giffen. However, numerical simulations using $c_1 = 1.5$, $c_2 = .5$, $p_1/p_2 = 1$ and $\hat{m} = p_1 + p_2$ as the baseline case led to the conclusion that it is in fact good 1 that is Giffen, which is in agreement with our conclusions. Actually, it was through these simulations that we first became aware of the possibility of a problem with Sørensen (2007).

Appendix B - Proofs

Appendix B.1

We need to show that

$$a = -\frac{m}{p_2^2} + \frac{p_1}{p_2^2}\frac{Am - p_2B}{A(p_1 - p_2) + p_2} + \frac{p_1}{p_2}\frac{B}{A(p_1 - p_2) + p_2} + \frac{p_1}{p_2}\frac{(1-A)(Am - p_2B)}{[A(p_1 - p_2) + p_2]^2} > 0$$

given that $A > 1, B > 0, p_1 > p_2, m/p_1 < B/(A-1), m/p_2 > B/A$.
Multiplying both sides by $p_2[A(p_1 - p_2) + p_2]$:

$$-\frac{m}{p_2}[A(p_1 - p_2) + p_2] + \frac{p_1}{p_2}(Am - p_2B) + p_1B + p_1\frac{(1-A)(Am - p_2B)}{A(p_1 - p_2) + p_2} > 0$$

Rearranging, both p_1B and Amp_1/p_2 cancel out:

$$Am - m + p_1\frac{(1-A)(Am - p_2B)}{A(p_1 - p_2) + p_2} > 0$$

Rearranging:

$$m(A-1) > p_1\frac{(A-1)(Am - p_2B)}{A(p_1 - p_2) + p_2}$$

Since $A > 1$, we can divide both sides by $A - 1$ and keep the inequality to the same side:

$$m > p_1\frac{(Am - p_2B)}{A(p_1 - p_2) + p_2}$$

Given that $p_1 > p_2$ and that A is positive, both sides can be, again, multiplied by $A(p_1 - p_2) + p_2$ while keeping the inequality unchanged:

$$mA(p_1 - p_2) + mp_2 > p_1(Am - p_2B)$$

Rearranging, cancelling Amp_1 out and dividing both sides by $-p_2$:

$$mA - m < p_1B$$

This can be rewritten as

$$\frac{m}{p_1} < \frac{B}{A-1}$$

which is one of the assumptions behind the corresponding branch of the demand function. $a > 0$ has been confirmed.

Appendix B.2

Given that $c_1 > c_2$, we need to prove that the kink curve, which is given by the function

$$x_2 = \left[x_1^{1/(2+2c_2)-1/(1+c_1)}\right]^{1/[c_1/(1+c_1)-c_2/(2+2c_2)]}$$

is downwards sloped, i.e, $dx_2/dx_1 < 0$.

$$\frac{dx_2}{dx_1} = \frac{1}{c_1/(1+c_1)-c_2/(2+2c_2)}\left[x_1^{1/(2+2c_2)-1/(1+c_1)}\right]^{1/[c_1/(1+c_1)-c_2/(2+2c_2)]-1}$$

$$\times\left(\frac{1}{2+2c_2}-\frac{1}{1+c_1}\right)x_1^{1/(2+2c_2)-1/(1+c_1)-1} < 0$$

Since x_1 and x_2 are positive, this crumbles down to

$$\frac{1}{c_1/(1+c_1)-c_2/(2+2c_2)}\left(\frac{1}{2+2c_2}-\frac{1}{1+c_1}\right) < 0$$

Meaning that one of the factors must be positive and the other negative:

$$\begin{cases}\frac{1}{c_1/(1+c_1)-c_2/(2+2c_2)} > 0\\ \frac{1}{2+2c_2}-\frac{1}{1+c_1} < 0\end{cases} \vee \begin{cases}\frac{1}{c_1/(1+c_1)-c_2/(2+2c_2)} < 0\\ \frac{1}{2+2c_2}-\frac{1}{1+c_1} > 0\end{cases}$$

Rearranging:

$$\begin{cases}\frac{c_1}{1+c_1} > \frac{c_2}{2+2c_2}\\ \frac{1}{2+2c_2} < \frac{1}{1+c_1}\end{cases} \vee \begin{cases}\frac{c_1}{1+c_1} < \frac{c_2}{2+2c_2}\\ \frac{1}{2+2c_2} > \frac{1}{1+c_1}\end{cases}$$

Inverting both inequations in both systems and simplifying:

$$\begin{cases}\frac{2}{c_2}+1 > \frac{1}{c_1}\\ c_1 < 1+2c_2\end{cases} \vee \begin{cases}\frac{2}{c_2}+1 < \frac{1}{c_1}\\ c_1 > 1+2c_2\end{cases}$$

Knowing that $c_1 > c_2$, we verify that $2/c_2+1 > 1/c_1$ always happens. Therefore, we are left with the second inequation from the left-hand side system. $c_1 < 1+2c_2$ is the condition that makes the kink curve be downwards sloped.

Appendix C - Tables

Appendix C.1 - "Modified CES" demand for x_2

Table 1: Values of x_2 in the "Modified CES" (Example 1) exercise with $m = 60$

p_2	$\rho = -0.5$	$\rho = -1$	$\rho = -2$	$\rho = -5$	$\rho = -10$	$\rho = -20$	$\rho = -50$	$\rho = -100$	$\rho \to -\infty$
0.01	1246.6554	355.3166	94.9367	22.4236	10.2166	5.8658	3.7377	3.1019	2.5010
0.1	245.7585	104.8390	40.7657	13.6427	7.3452	4.7466	3.3619	2.9291	2.5105
0.5	74.4714	41.7117	21.1745	9.2173	5.6916	4.0677	3.1452	2.8468	2.5532
1	43.3231	27.1361	15.4974	7.6585	5.0804	3.8234	3.0893	2.8509	2.6087
2	24.4333	17.0645	11.0165	6.2873	4.5364	3.6371	3.0923	2.9120	2.7273
3	16.9982	12.6598	8.8303	5.5635	4.2657	3.5751	3.1476	3.0029	2.8571
4	12.7906	10	7.4154	5.0754	4.0985	3.5655	3.2302	3.1162	3
5	9.9271	8.1013	6.3525	4.6999	3.9846	3.5865	3.3332	3.2462	3.1579
6	7.6997	6.5688	5.4595	4.3792	3.8989	3.6283	3.4544	3.3946	3.3333
7	5.7375	5.1801	4.6257	4.0751	3.8264	3.6848	3.5933	3.5617	3.5294
8	3.75	3.75	3.75	3.75	3.75	3.75	3.75	3.75	3.75
9	1.3495	2.0204	2.6868	3.3471	3.6449	3.8143	3.9236	3.9606	4
10	0	0	1.1106	2.7422	3.4554	3.8543	4.1090	4.1967	4.2857
11	0	0	0	1.5060	2.9909	3.7855	4.2792	4.4466	4.6154
11.5	0	0	0	0	2.3663	3.5806	4.3122	4.5561	4.8
11.9	0	0	0	0	0.4938	2.8185	4.1257	4.5461	4.9587
12	0	0	0	0	0	0	0	0	5
12	-	-	-	-	-	-	-	-	0

Table 2: Values of x_2 in the "Modified CES" (Example 1) exercise with $m = 45$

p_2	$\rho = -5$	$\rho = -10$	$\rho = -20$	$\rho = -50$	$\rho = -100$
0.01	21.3877	10.1985	6.2105	4.2599	3.6771
0.1	13.3375	7.5676	5.1868	3.9182	3.5216
0.5	9.2844	6.0609	4.5762	3.7327	3.4599
1	7.8651	5.5108	4.3684	3.6992	3.4791
2	6.6354	5.0486	4.2336	3.7398	3.5745
3	6.0071	4.8391	4.2176	3.8328	3.7026

4	5.6031	4.7310	4.2549	3.9555	3.8534
5	5.3114	4.6787	4.3265	4.1025	4.0255
6	5.0818	4.6618	4.4248	4.2726	4.2201
7	4.8831	4.6683	4.5460	4.4670	4.4396
8	4.6875	4.6875	4.6875	4.6875	4.6875
9	4.4559	4.7041	4.8452	4.9363	4.9679
10	4.1031	4.5	4.5	4.5	4.5
11	3.3262	4.0909	4.0909	4.0909	4.0909
11.5	2.2907	3.9130	3.9130	3.9130	3.9130
11.9	0	3.7815	3.7815	3.7815	3.7815
12	0	0	0	0	0

Table 3: Values of x_2 in the "Modified CES" (Example 1) exercise with $m = 90$

p_2	$\rho = -5$	$\rho = -10$	$\rho = -20$	$\rho = -50$	$\rho = -100$
0.01	24.4954	10.2527	5.1763	2.6934	1.9516
0.1	14.2533	6.9002	3.8661	2.2494	1.7440
0.5	9.0831	4.9530	3.0508	1.9700	1.6205
1	7.2453	4.2074	2.7334	1.8698	1.5860
2	5.5912	3.5119	2.4441	1.7969	1.5804
3	4.6762	3.1188	2.2901	1.7771	1.6035
4	4.0202	2.8342	2.1867	1.7795	1.6406
5	3.4768	2.5964	2.1065	1.7947	1.6877
6	2.9740	2.3740	2.0354	1.8180	1.7430
7	2.4593	2.1427	1.9625	1.8461	1.8058
8	1.875	1.875	1.875	1.875	1.875
9	1.1295	1.5265	1.7524	1.8981	1.9486
10	0.0205	1.0011	1.5497	1.8999	2.0204
11	0	0	1.1222	1.8275	2.0666
11.5	0	0	0.6179	1.6870	2.0435
11.9	0	0	0	1.2368	1.8639
12	0	0	0	0	0

Table 4: Values of x_2 in the "Modified CES" (Example 2) exercise with $m = 60$

p_2	$\rho = -20$	$\rho = -50$	$\rho = -100$	$\rho = -200$	$\rho = -500$	$\rho = -1000$	$\rho \to -\infty$
0.01	666.6667	666.6667	666.6667	666.6667	666.6667	666.6667	666.6667

0.1	66.6667	66.6667	66.6667	66.6667	66.6667	66.6667	66.6667
0.2	33.3333	33.3333	33.3333	33.3333	33.3333	33.3333	33.3333
0.5	13.3333	13.3333	13.3333	13.3333	13.3333	13.3333	13.3333
1	6.6667	6.6667	6.6667	6.6667	6.6667	6.6667	6.6667
2	3.3334	3.3333	3.3333	3.3333	3.3333	3.3333	3.3333
3	2.2246	2.2222	2.2222	2.2222	2.2222	2.2222	2.2222
4	1.6862	1.6667	1.6667	1.6667	1.6667	1.6667	1.6667
5	1.4064	1.3333	1.3333	1.3333	1.3333	1.3333	1.3333
6	1.2602	1.1467	1.1136	1.1111	1.1111	1.1111	1.1111
7	1.1721	1.0764	1.029	0.9991	0.9769	0.968	0.9571
7.5	1.1386	1.0584	1.0182	0.9934	0.9761	0.9697	0.963
8	1.1093	1.0449	1.0124	0.9926	0.9791	0.9743	0.9693
8.5	1.0827	1.0341	1.009	0.9938	0.9835	0.9799	0.9761
8.75	1.0702	1.0292	1.0079	0.9948	0.986	0.9829	0.9797
9	1.058	1.0246	1.0069	0.9961	0.9887	0.9861	0.9834
9.25	1.0462	1.0202	1.0062	0.9975	0.9916	0.9895	0.9873
9.5	1.0346	1.0158	1.0055	0.999	0.9946	0.993	0.9913
9.55	1.0323	1.0149	1.0053	0.9993	0.9952	0.9937	0.9922
9.6	1.03	1.0141	1.0052	0.9996	0.9958	0.9944	0.993
9.65	1.0277	1.0132	1.0051	0.9999	0.9964	0.9952	0.9938
9.7	1.0254	1.0123	1.0049	1.0003	0.9971	0.9959	0.9947
9.75	1.0232	1.0114	1.0048	1.0006	0.9977	0.9966	0.9956
9.8	1.0209	1.0105	1.0047	1.0009	0.9983	0.9974	0.9964
9.85	1.0186	1.0096	1.0045	1.0013	0.999	0.9982	0.9973
9.9	1.0164	1.0087	1.0044	1.0016	0.9996	0.9989	0.9982
9.95	1.0141	1.0078	1.0042	1.0019	1.0003	0.9997	0.9991
10	1.0119	1.0069	1.0041	1.0022	1.001	1.0005	1,0000
10.05	1.0096	1.006	1.0039	1.0026	1.0016	1.0013	1.0009
10.1	1.0074	1.0051	1.0038	1.0029	1.0023	1.0021	1.0018
10.15	1.0051	1.0042	1.0036	1.0032	1.003	1.0029	1.0028
10.2	1.0029	1.0032	1.0034	1.0035	1.0036	1.0037	1.0037
10.25	1.0006	1.0023	1.0032	1.0039	1.0043	1.0045	1.0047
10.3	0.9984	1.0013	1.003	1.0042	1.005	1.0053	1.0056
10.35	0.9962	1.0003	1.0028	1.0045	1.0057	1.0061	1.0066
10.4	0.9939	0.9993	1.0026	1.0048	1.0064	1.007	1.0076
10.45	0.9917	0.9983	1.0024	1.0051	1.0071	1.0078	1.0086
10.5	0.9895	0.9973	1.0021	1.0054	1.0078	1.0086	1.0096
10.75	0.9783	0.992	1.0006	1.0067	1.0112	1.0129	1.0148
11	0.9671	0.9862	0.9985	1.0075	1.0145	1.0173	1.0203

11.25	0.9559	0.9798	0.9956	1.0076	1.0174	1.0215	1.0261
11.5	0.9447	0.9727	0.9915	1.0062	1.0191	1.0248	1.0324
12	0.922	0.9558	0.9777	0.993	0.9999	1	1
12.5	0.899	0.9348	0.9535	0.9597	0.96	0.96	0.96
13	0.8759	0.91	0.9217	0.9231	0.9231	0.9231	0.9231
14	0.8295	0.8541	0.8571	0.8571	0.8571	0.8571	0.8571
15	0.7841	0.7993	0.8	0.8	0.8	0.8	0.8

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Appendix C.2 - Demand for x_2 under the new preferences

Table 5: Values of x_2 and p_2 for given levels of U and $m = 60$ from $n = 1$ to 3

	$n=1$		$n=2$		$n=3$	
U	x_2	p_2	x_2	p_2	x_2	p_2
10.2	3.7193	10.4219	3.03	11.1587	3.1412	11.4447
10.35	3.9074	10.1525	3.302	10.8519	3.3808	11.1537
10.5	4.0476	9.8891	3.4706	10.5579	3.5116	10.8715
11	4.3325	9.0248	3.75	9.6	3.6868	9.9325
11.5	4.4815	8.1521	3.8646	8.6246	3.7226	8.9521
12	4.5729	7.2529	3.9242	7.6058	3.7163	7.9063
12.5	4.6434	6.3194	3.9721	6.5341	3.7047	6.7844
13	4.7183	5.3491	4.0393	5.4084	3.7158	5.5834
13.5	4.824	4.3449	4.1633	4.2377	3.788	4.3118
14	5	3.3167	4.4128	3.0467	4	3
14.5	5.3259	2.2865	4.9588	1.8875	4.5824	1.7228
14.6	5.4232	2.0835	5.1395	1.6673	4.7989	1.4838
14.7	5.5369	1.8826	5.3605	1.4534	5.0774	1.2543
14.75	5.601	1.7831	5.4898	1.3492	5.2473	1.1439
14.8	5.6707	1.6843	5.6344	1.2471	5.4429	1.0367

Table 6: Values of x_2 and p_2 for given levels of U and $m = 60$ from $n = 4$ to 6

	$n=4$		$n=5$		$n=6$	
U	x_2	p_2	x_2	p_2	x_2	p_2
10.2	3.3869	11.5602	3.6064	11.6165	3.7807	11.6486
10.35	3.5667	11.2898	3.7358	11.3616	3.8727	11.4046
10.5	3.6549	11.0234	3.7913	11.1075	3.9044	11.1595
11	3.7383	10.1195	3.808	10.2337	3.8733	10.3093
11.5	3.7121	9.1572	3.7377	9.2915	3.7716	9.3847
12	3.654	8.115	3.6428	8.2607	3.6502	8.3662
12.5	3.5939	6.9802	3.5473	7.127	3.529	7.2382
13	3.5556	5.7463	3.4714	5.8801	3.4251	5.9877
13.5	3.5739	4.4175	3.4457	4.5193	3.3653	4.6091
14	3.7268	3.0224	3.543	3.0676	3.4156	3.1186

14.5	4.2718	1.6473	4.0257	1.6134	3.8314	1.6012
14.6	4.4975	1.3905	4.2463	1.3406	4.04	1.3143
14.7	4.8019	1.1456	4.5571	1.0811	4.3457	1.0414
14.75	4.9948	1.0286	4.7612	0.9577	4.5531	0.9119
14.8	5.2234	0.9158	5.0097	0.8394	4.812	0.7881

Table 7: Values of x_2 and p_2 for given levels of U and $m = 60$ from $n = 7$ to 10

	$n = 7$		$n = 8$		$n = 9$		$n = 10$	
U	x_2	p_2	x_2	p_2	x_2	p_2	x_2	p_2
10.2	3.9173	11.6689	4.0254	11.6829	4.1126	11.693	4.184	11.7006
10.35	3.9816	11.4329	4.0691	11.4528	4.1404	11.4675	4.1994	11.4787
10.5	3.996	11.1946	4.0705	11.2196	4.1318	11.2383	4.1829	11.2527
11	3.9301	10.3626	3.9783	10.402	4.0195	10.4322	4.0547	10.4561
11.5	3.8053	9.4525	3.8363	9.5039	3.864	9.5441	3.8885	9.5764
12	3.6644	8.4454	3.6805	8.5068	3.6966	8.5557	3.712	8.5955
12.5	3.524	7.3246	3.5255	7.3933	3.5302	7.449	3.5364	7.495
13	3.3989	6.0748	3.3841	6.1461	3.3758	6.2054	3.3717	6.2553
13.5	3.3126	4.6863	3.277	4.7524	3.2522	4.8093	3.2346	4.8584
14	3.3247	3.169	3.2578	3.2162	3.2076	3.2596	3.169	3.2991
14.5	3.6772	1.601	3.5535	1.6076	3.4531	1.618	3.3708	1.6306
14.6	3.8707	1.3017	3.731	1.2974	3.615	1.2985	3.5179	1.3028
14.7	4.1648	1.0166	4.0101	1.0013	3.8776	0.9924	3.7636	0.9879
14.75	4.3702	0.8814	4.2104	0.8608	4.0707	0.847	3.9485	0.838
14.8	4.6328	0.7524	4.4719	0.7269	4.3279	0.7085	4.1992	0.6951

Appendix C.3 - Decomposition between substitution and income effects

Table 8: Slutsky decomposition of the effect on x_2 of a price change from $p_2 = 1$ to 2 in the "modified CES" model (Example 1) under $m = 60$

$p_2 = 1, p_2' = 2$							
ρ	x_2	x_2^s	x_2'	m^s	SE	IE	TE
-1	27.1361	18.2624	17.0645	87.1361	-8.8737	-1.1979	-10.0716
-2	15.4974	11.1149	11.0165	75.4974	-4.3825	-0.0985	-4.4809
-5	7.6585	6.1096	6.2873	67.6585	-1.5489	0.1777	-1.3712
-10	5.0804	4.3629	4.5364	65.0804	-0.7175	0.1735	-0.544
-20	3.8234	3.485	3.6371	63.8234	-0.3383	0.152	-0.1863
-50	3.0893	2.9588	3.0923	63.0893	-0.1305	0.1335	0.003
-100	2.8509	2.7835	2.912	62.8509	-0.0674	0.1285	0.0611

Table 9: Slutsky decomposition of the effect on x_2 of a price change from $p_2 = 4$ to 5 in the "modified CES" model (Example 1) under $m = 60$

$p_2 = 4, p_2' = 5$							
ρ	x_2	x_2^s	x_2'	m^s	SE	IE	TE
-1	10	7.9552	8.1013	70	-2.0448	0.1461	-1.8987
-2	7.4154	6.1444	6.3525	67.4154	-1.271	0.2081	-1.063
-5	5.0754	4.493	4.6999	65.0754	-0.5825	0.2069	-0.3756
-10	4.0985	3.7949	3.9846	64.0985	-0.3035	0.1896	-0.1139
-20	3.5655	3.4106	3.5865	63.5655	-0.1549	0.1759	0.021
-50	3.2302	3.1676	3.3332	63.2302	-0.0626	0.1657	0.103
-100	3.1162	3.0843	3.2462	63.1162	-0.0319	0.1619	0.13

Table 10: Slutsky decomposition of the effect on x_2 of a price change from $p_2 = 8$ to 9 in the "modified CES" model (Example 1) under $m = 60$

$p_2 = 8, p_2' = 9$							
ρ	x_2	x_2^s	x_2'	m^s	SE	IE	TE
-1	3.75	1.6879	2.0204	63.75	-2.0621	0.3325	-1.7296
-2	3.75	2.382	2.6868	63.75	-1.368	0.3047	-1.0632

-5	3.75	3.0699	3.3471	63.75	-0.6801	0.2772	-0.4029
-10	3.75	3.3801	3.6449	63.75	-0.3699	0.2648	-0.1051
-20	3.75	3.5566	3.8143	63.75	-0.1934	0.2577	0.0643
-50	3.75	3.6704	3.9236	63.75	-0.0796	0.2532	0.1736
-100	3.75	3.7098	3.9606	63.75	-0.0402	0.2508	0.2106

Table 11: Slutsky decomposition of the effect on x_2 of a price change from $p_2 = 7.5$ to 8 in the "modified CES" model (Example 2) under $m = 20$

$p_2 = 7.5, p_2' = 8$							
ρ	x_2	x_2^s	x_2'	m^s	SE	IE	TE
-10	1.21029	1.18338	1.16473	20.60514	-0.0269	-0.01865	-0.04556
-20	1.13863	1.11898	1.10927	20.5693	-0.01966	-0.00971	-0.02936
-50	1.0584	1.04671	1.04494	20.5292	-0.01169	-0.00177	-0.01346
-100	1.01822	1.01075	1.01242	20.50913	-0.00747	0.00166	-0.00581
-200	0.99337	0.98885	0.99259	20.49669	-0.00453	0.00374	-0.00079
-500	0.97605	0.97395	0.97912	20.48803	-0.0021	0.00517	0.00307
-1000	0.96971	0.96858	0.97432	20.48486	-0.00113	0.00574	0.0046

Table 12: Slutsky decomposition of the effect on x_2 of a price change from $p_2 = 9.5$ to 10 in the "modified CES" model (Example 2) under $m = 20$

$p_2 = 9.5, p_2' = 10$							
ρ	x_2	x_2^s	x_2'	m^s	SE	IE	TE
-10	1.04902	1.03087	1.0155	20.52451	-0.01815	-0.01537	-0.03353
-20	1.03459	1.02097	1.01186	20.51729	-0.01361	-0.00912	-0.02273
-50	1.01581	1.00814	1.00693	20.5079	-0.00766	-0.00121	-0.00887
-100	1.00548	1.00107	1.00409	20.50274	-0.00441	0.00302	-0.00139
-200	0.99899	0.99661	1.00224	20.4995	-0.00238	0.00563	0.00325
-500	0.99455	0.99355	1.00095	20.49728	-0.001	0.0074	0.0064
-1000	0.99297	0.99246	1.00049	20.49646	-0.00051	0.00803	0.00752

Table 13: Slutsky decomposition of the effect on x_2 of a price change from $p_2 = 11$ to 11.5 in the "modified CES" model (Example 2) under $m = 20$

$p_2 = 11, p_2' = 11.5$							
ρ	x_2	x_2^s	x_2'	m^s	SE	IE	TE
-10	0.95388	0.93937	0.92534	20.47694	-0.0145	-0.01403	-0.02853
-20	0.96714	0.95523	0.94468	20.48357	-0.01191	-0.01055	-0.02246
-50	0.98621	0.97796	0.97268	20.49311	-0.00825	-0.00528	-0.01353
-100	0.99855	0.99281	0.99151	20.49927	-0.00574	-0.0013	-0.00704
-200	1.00752	1.00383	1.0062	20.50376	-0.00369	0.00236	-0.00133
-500	1.01453	1.0127	1.01906	20.50727	-0.00183	0.00636	0.00453
-1000	1.01727	1.01627	1.02484	20.50863	-0.001	0.00858	0.00757

Table 14: Hicksian decomposition of the effect on x_2 of a price change from p_2 to p_2', in the example with the new preferences

n	p_2	p_2'	x_2	x_2^s	x_2'	SE	IE	TE	adj SE	adj IE	adj TE
1	7.2529	8.1521	4.5729	4.2315	4.4815	-0.3414	0.25	-0.0914	-0.3797	0.278	-0.1017
2	7.6058	8.6246	3.9242	3.6146	3.8646	-0.3096	0.25	-0.0596	-0.3039	0.2454	-0.0585
3	7.9063	8.9521	3.7163	3.4726	3.7226	-0.2437	0.25	0.0063	-0.233	0.2391	0.0061
4	6.9802	8.115	3.5939	3.404	3.654	-0.1899	0.25	0.0601	-0.1673	0.2203	0.053
5	7.127	8.2607	3.5473	3.3928	3.6428	-0.1546	0.25	0.0954	-0.1363	0.2205	0.0842
6	7.2382	8.3662	3.529	3.4002	3.6502	-0.1288	0.25	0.1212	-0.1142	0.2216	0.1074
7	7.3246	8.4454	3.524	3.4144	3.6644	-0.1096	0.25	0.1404	-0.0978	0.2231	0.1252
8	7.3933	8.5068	3.5255	3.4305	3.6805	-0.095	0.25	0.155	-0.0853	0.2245	0.1392
9	7.449	8.5557	3.5302	3.4466	3.6966	-0.0836	0.25	0.1664	-0.0755	0.2259	0.1504
10	7.495	8.5955	3.5364	3.462	3.712	-0.0744	0.25	0.1756	-0.0676	0.2272	0.1596