ORDER OF CONVERGENCE OF THE FINITE ELEMENT METHOD FOR THE $p(x)-$LAPLACIAN

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ABSTRACT. In this work, we study the rate of convergence of the finite element method for the $p(x)-$Laplacian $(1 < p_1 \leq p(x) \leq p_2 \leq 2)$ in a bounded convex domain in $\mathbb{R}^2$.

1. INTRODUCTION

Let Ω be a bounded convex domain in $\mathbb{R}^n$ and $p : \Omega \to (1, +\infty)$ be a measurable function. In this work, we first consider the Dirichlet problem for the $p(x)-$Lapalacian

$$(1.1) \qquad \begin{cases} -\Delta_{p(x)}u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_{p(x)}u = \text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)-$Laplacian and $|\cdot|^2 = \langle \cdot, \cdot \rangle_{\mathbb{R}^n}$. The assumptions over p, f and g will be specified later.

Note that the $p(x)-$Laplacian extends the classical Laplacian $(p(x) \equiv 2)$ and the $p-$Laplacian $(p(x) \equiv p$ with $1 < p < +\infty)$. This operator has been recently used in image processing and in the modeling of electrorheological fluids, see [3, 7, 23].

For the applications in image processing, in [7] the authors introduce a model that involves the $p(x)-$Laplacian, for some function $p : \Omega \to [p_1, 2]$, with $p_1 > 1$. Recently, in [3] the authors propose a variant of this model. More precisely, they consider the functional

$$\int_\Omega |\nabla u|^{p(x)} + \frac{\lambda}{2} \int_\Omega |u - \xi|^2 \, dx,$$

with $p : \Omega \to [1, 2]$ a function such that $p(x) = P_M(|\nabla G_\delta * \xi|(x))$, where ξ is the observe image (the real image with a white noise), $G_\delta(x)$ is an approximation of the identity, $M >> 1$ and P_M is a function that satisfies $P_M(0) = 2$ and $P_M(x) = 1$ for all $|x| > M$.

A function $u \in W_g^{1,p(\cdot)}(\Omega) \coloneqq \{v \in W^{1,p(\cdot)}(\Omega) \colon v = g \text{ on } \partial\Omega\}$ is a weak solution of (1.1) if

$$(1.2) \qquad \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx = \int_\Omega f v \, dx$$

for all $v \in W_0^{1,p(\cdot)}(\Omega)$.

Motivated by the applications to image processing problems, in [9] the authors study the convergence of the discontinuous Galerkin finite element method and the continuous Galerkin finite element method (FEM) to approximate weak solutions of the equations of the type (1.1). On the other hand, motivated by the application to electrorheological fluids, in [6, 22] the authors prove weak convergence of an implicit finite element discretization for a parabolic equation involving the $p(x)-$Laplacian.

In [10], for the case $n = 2$, we prove the H^2 regularity of the solution of (1.2) when Ω is a bounded domain with convex boundary and under certain assumptions for p, f and g (see Section 2 for details).

In the present work, we study the rate of convergence of the continuous Galerking FEM in the case where $n = 2$ and $p \colon \Omega \to [p_1, p_2]$ with $1 < p_1 \le p_2 \le 2$. To this end, we follow the ideas of [1, 18, 20], where the authors study the case $p(x) \equiv p \ (1 < p < +\infty)$ and $n = 2$. More precisely, let $h > 0$, Ω^h be a polygonal subset of Ω and $\mathcal{T}^h$ be a nondegenerate triangulation of Ω^h, where each triangle $\kappa \in \mathcal{T}^h$ has maximum diameter bounded by h. Let S^h denote the space of C^0 piecewise linear functions. Our finite element approximation of (1.1) is:

Find $u^h \in S_g^h$ such that

$$(1.3) \qquad \int_{\Omega^h} |\nabla u^h|^{p(x)-2} \nabla u^h \nabla v \, dx = \int_{\Omega^h} f v \, dx \quad \forall v \in S_0^h$$

where

$$S_0^h \coloneqq \{v \in S^h \colon v = 0 \text{ on } \partial\Omega^h\}, \quad S_g^h \coloneqq \{v \in S^h \colon v = g^h \text{ on } \partial\Omega^h\},$$

and $g^h \in S^h$ is chosen to approximate the Dirichlet boundary data.

In [9, Theorem 7.2], the authors prove that if $p(x)$ is a log-Hölder continuous function (see Section 2 for the definition) then the sequence of solutions of (1.3) converges to the solution of (1.2). In the present work, we study the rate of convergence of this method. In general, all the error bounds depend on the global regularity of the second derivatives of the solution. For example, in the case $p(x) \equiv p \in (1, 2]$ and $n = 2$, there exists a constant $C = C(\|u\|_{W^{2,p}(\Omega)})$ such that

$$\|u - u^h\|_{W^{1,p}(\Omega_h)} \le Ch^{p/2},$$

where $u \in W^{2,p}(\Omega)$ is the weak solution of (1.1) and u^h is the solution of (1.3), see [1]. It is shown in [1] that if $1 < p < 2$, $u \in W^{3,1}(\Omega) \cap C^{2,1+2/p}(\overline{\Omega})$ and

$$(1.4) \qquad \int_\Omega |\nabla u|^{p-2} |D^2 u|^2 \, dx < \infty$$

then

$$\|u - u_h\|_{W^{1,p}(\Omega)} \leq Ch.$$

In [18], it is shown that if $f \in L^\infty(\Omega)$, $\partial\Omega \in C^2$ and $g = 0$ then (1.4) holds for $1 < p < 2$.

However, the regularity $u \in W^{3,1}(\Omega) \cap C^{2,1+2/p}(\overline{\Omega})$ is only shown for p−harmonic functions with $1 < p \leq 2$, that is the case $f \equiv 0$, see [19]. Indeed it does not seem that such higher regularity is in general achievable for the p−Laplacian ($1 < p \leq 2$) even with very smooth data. In [18], this regularity condition is weakened to: (1.4) and $u \in W^{1+2/p,p}(\Omega)$. It is still not clear whether or not this weakened regularity is in general achievable for smooth data. Recently, in [15], it is shown that the regularity condition (1.4) is indeed sufficient for the optimal error bounds. Therefore, if $f \in L^\infty(\Omega)$, $\partial\Omega \in C^2$ and $g = 0$ they have optimal error bounds.

The main results of the present paper are the following theorems.

Theorem 1.1. *Let Ω be a bounded convex domain in $\mathbb{R}^2$ and $p\colon \Omega \to [p_1, p_2]$ be a Lipschitz continuous function with $1 < p_1 \leq p_2 \leq 2$, $f \in L^{q(\cdot)}(\Omega)$ with $q(x) \geq q_1 > 2$, $g \in H^2(\Omega)$, u and u^h be the unique solutions of (1.2) and (1.3) respectively. If $h \leq 1$, then*

$$\|u - u^h\|_{W^{1,p(\cdot)}(\Omega_h)} \leq Ch^{p_1/2},$$

where C is a constant that depends on $p(x)$, $\|f\|_{L^{q(\cdot)}(\Omega)}$ and $\|g\|_{H^2(\Omega)}$.

We want to mention that, in the previous theorem, we take $n = 2$ since the H^2 regularity is only known in this case.

For sufficiently regular solutions, we obtain optimal order of convergence.

Theorem 1.2. *Let Ω be a bounded convex domain in $\mathbb{R}^2$ and $p\colon \Omega \to [p_1, p_2]$ be a Lipschitz continuous function with $1 < p_1 \leq p_2 \leq 2$, u and u^h be the unique solutions of (1.2) and (1.3) respectively. If*

$$(1.5) \qquad \int_\Omega |\nabla u|^{p(x)-2} H[u]^2 \, dx < +\infty$$

where $H[u] = |u_{x_1 x_1}| + |u_{x_1 x_2}| + |u_{x_2 x_2}|$ and

$$(1.6) \qquad u \in C^{2,\alpha^+}(\tau) \text{ for each } \tau \in \mathcal{T}^h$$

with $\alpha^+ = {}^{(2-p^+)}/{}_{p^+}$ and $p^+ = \max\limits_{x \in \tau} p(x)$, then

$$\|u - u^h\|_{1,p,\Omega^h} \leq Ch.$$

Observe that the assumption (1.5) implies that the solution u of (1.2) belongs to $W^{2,p_1}(\Omega)$. Then $u \in C(\overline{\Omega})$ due to $n = 2$, see Remark 2.9. Therefore the interpolant of u is well defined, see Subsection 2.3.

On the other hand, the regularity assumption (1.6) on u is local and depends only on p^+. We also note that the assumption (1.5) holds for example if $u \in W^{3,1}(\Omega)$, see Remark 3.4.

It is still not clear whether or not the regularity assumption of Theorem 1.2 is in general achievable for smooth data. However we show that if Ω is a ball, $p(x)$ and $f(x)$ are radially symmetric functions, g is constant and

$$(1.7) \qquad p \in C^{1,\beta}(\tau), f \in C^{\beta}(\tau) \text{ with } \beta \geq \alpha^+ \quad \forall \tau \in \mathcal{T}^h$$

then the assumptions of Theorem 1.2 are satisfied. So in this case we have optimal order of convergence. Observe that these regularity assumptions on the data are local and depend only on p^+.

So in this case, in order to have optimal order, by (1.7), we only need $p \in C^1$ and $f \in C^0$, in regions where the maximum of p is 2, and we need, for example, $p \in C^{1,1}$ and $f \in C^1$ only in regions where the function $p(x)$ is near 1.

Organization of the paper. In Section 2 we collect some preliminary facts concerning variable Sobolev spaces, the weak solution of (1.1), finite element spaces and the Decomposition–Coordination method; in Section 3 we prove Theorem 1.1, Theorem 1.2 and we study the radially symmetric case. In Section 4 we show a family of numerical examples where we study the behaviour of the error when we use the Decomposition–Coordination method to approximate the solution (1.3). Finally we give an example where we have optimal order of convergence although assumptions of Theorem 1.2 are not satisfied.

2. Preliminaries

We begin with a review of the basic results that will be needed in subsequent sections. The known results are generally stated without proofs, but we provide references where the proofs can be found. Also, we introduce some of our notational conventions.

2.1. General Properties of Variable Sobolev Spaces.
We first introduce the space $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ and state some of their properties.

Let Ω be a bounded open set of $\mathbb{R}^n$ and $p\colon \Omega \to [1, +\infty]$ be a measurable function, called a variable exponent on Ω. Denote

$$p_1 := \operatorname*{ess\,inf}_{x\in\Omega} p(x) \text{ and } p_2 := \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$

We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u\colon \Omega \to \mathbb{R}$ for which the modular

$$\varrho_{p(\cdot),\Omega}(u) := \int_{\Omega} \varphi(|u(x)|, p(x)) \, dx$$

is finite, where $\varphi\colon [0, +\infty) \times [1, +\infty] \to [0, +\infty]$

$$\varphi(t,p) = \begin{cases} t^p & \text{if } p \neq \infty, \\ \infty \chi_{(1,\infty)}(t) & \text{if } p = \infty, \end{cases}$$

with the notation $\infty \cdot 0 = 0$.

We define the Luxemburg norm on this space by

$$\|u\|_{p(\cdot),\Omega} := \inf\{k > 0\colon \varrho_{p(\cdot),\Omega}(u/k) \leq 1\}.$$

This norm makes $L^{p(\cdot)}(\Omega)$ a Banach space.

We will write it simply $\varrho_{p(\cdot)}(u)$ and $\|u\|_{p(\cdot)}$ when no confusion can arise.

Lemma 2.1. *For any $p, \delta : \Omega \to \mathbb{R}_{\geq 0}$ be measurable functions with $1 < p_1 \leq p(x) \leq p_2 \leq 2$, then for any $\xi, \eta \in \mathbb{R}^2$, $|\xi| + |\eta| \neq 0$, $x \in \Omega$ we have*

$$(2.1) \qquad ||\xi|^{p(x)-2}\xi - |\eta|^{p(x)-2}\eta| \leq C_1|\xi - \eta|^{1-\delta(x)}(|\xi| + |\eta|)^{p(x)-2+\delta(x)},$$

and

$$(2.2) \quad (|\xi|^{p(x)-2}\xi - |\eta|^{p(x)-2}\eta)(\xi - \eta) \geq C_2|\xi - \eta|^{2+\delta(x)}(|\xi| + |\eta|)^{p(x)-2-\delta(x)}$$

where $C_1 = 2^{2-p_1}$ and $C_2 = (p_1 - 1)2^{2-p_2}$.

Proof. First, we will prove the second inequality. By [5, Lemma 2.4], we have that

$$(|\xi|^{p(x)-2}\xi - |\eta|^{p(x)-2}\eta)(\xi - \eta) \geq (p(x) - 1)2^{2-p(x)}|\xi - \eta|^2(|\xi| + |\eta|)^{p(x)-2}.$$

Therefore (2.2) holds due to

$$|\xi - \eta|^{-\delta(x)}(|\xi| + |\eta|)^{\delta(x)} \geq 1.$$

Finally, we follow ideas of [5, Lemma 3.5] to show that (2.1) holds.

We can assume that $|\eta| \geq |\xi| > 0$. Let

$$\eta' = \frac{\eta}{|\eta|}, \quad \xi' = \frac{\xi}{|\xi|}, \quad k = \frac{|\eta|}{|\xi|} \geq 1 \quad \text{and} \quad \gamma = \eta'\xi'.$$

Observe that $|\gamma| \leq 1$. Then,

$$||\xi|^{p(x)-2}\xi - |\eta|^{p(x)-2}\eta| \leq 2^{2-p(x)}|\xi - \eta|(|\xi| + |\eta|)^{p(x)-2},$$

is equivalent with

$$|k^{p(x)-1}\eta' - \xi'| \leq 2^{2-p(x)}(k + 1)^{p(x)-2}|k\eta' - \xi'|,$$

that is equivalent to proving

$$(2.3) \quad k^{2(p(x)-1)} + 1 - 2k^{p(x)-1}\gamma \leq 2^{4-2p(x)}(k + 1)^{2(p(x)-2)}(k^2 + 1 - 2k\gamma).$$

Let $f \colon [1, +\infty) \times [-1, 1] \to (0, +\infty)$

$$f(k, \gamma) = \frac{k^{2(p(x)-1)} + 1 - 2k^{p(x)-1}\gamma}{(k + 1)^{2(p(x)-2)}(k^2 + 1 - 2k\gamma)}.$$

Then

$$\frac{\partial f}{\partial \gamma}(k, \gamma) = \frac{2k(k^{p(x)-2} - 1)(k^{p(x)} - 1)}{(k + 1)^{2(p(x)-2)}(k^2 + 1 - 2k\gamma)^2} \leq 0$$

due to $p(x) \leq 2$. Hence

$$f(k, \gamma) \leq f(k, -1) = \frac{(k^{p(x)-1} + 1)^2}{(k + 1)^{2(p(x)-1)}}.$$

We therefore examine

$$g(k) = \sqrt{f(k, -1)} = \frac{k^{p(x)-1} + 1}{(k + 1)^{p(x)-1}}.$$

Observe that $g(1) = 2^{2-p(x)}$ and

$$\frac{\partial g}{\partial k}(k) = (p(x) - 1)\frac{k^{p(x)-2} - 1}{(k + 1)^{p(x)}} < 0 \quad \forall k > 1.$$

Hence
$$f(k,\gamma) \leq f(k,-1) = g(k)^2 \leq g(1)^2 = 2^{4-2p(x)}$$
for all $(k,\gamma) \in [1,+\infty) \times [-1,1]$. Therefore (2.3) holds. $\square$

For the proofs of the following theorems, we refer the reader to [13].

Lemma 2.2. *Let* $p : \Omega \to [1,+\infty]$ *be a measurable function with* $p_1 < \infty$. *If* $\varrho_{p(\cdot)}(u) > 0$ *or* $p_2 < \infty$ *then*

$$\min\{\varrho_{p(\cdot)}(u)^{1/p_1}, \varrho_{p(\cdot)}(u)^{1/p_2}\} \leq \|u\|_{p(\cdot)} \leq \max\{\varrho_{p(\cdot)}(u)^{1/p_1}, \varrho_{p(\cdot)}(u)^{1/p_2}\}$$

for all $u \in L^{p(\cdot)}(\Omega)$.

Theorem 2.3 (Hölder's inequality)**.** *Let* $p,q,s : \Omega \to [1,+\infty]$ *be measurable functions such that*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = \frac{1}{s(x)} \quad in\ \Omega.$$

Then

$$\|fg\|_{s(\cdot)} \leq 2\|f\|_{p(\cdot)}\|g\|_{q(\cdot)},$$

for all $f \in L^{p(\cdot)}(\Omega)$ *and* $g \in L^{q(\cdot)}(\Omega)$

Let $W^{1,p(\cdot)}(\Omega)$ denote the space of measurable functions u such that, u and the distributional derivative ∇u are in $L^{p(\cdot)}(\Omega)$. The norm

$$\|u\|_{1,p(\cdot),\Omega} := \|u\|_{p(\cdot),\Omega} + \|\nabla u\|_{p(\cdot),\Omega}$$

makes $W^{1,p(\cdot)}(\Omega)$ a Banach space.

We note

$$|u|_{1,p(\cdot),\Omega} := \|\nabla u\|_{p(\cdot),\Omega}$$

and we just write $\|u\|_{1,p(\cdot)}$ instead of $\|u\|_{1,p(\cdot),\Omega}$ and $|u|_{1,p(\cdot)}$ instead of $|u|_{1,p(\cdot),\Omega}$ when no confusion arises.

Theorem 2.4. *Let* $p,p' : \Omega \to [1,+\infty]$ *be measurable functions such that*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \quad in\ \Omega.$$

Then $L^{p'(\cdot)}(\Omega)$ *is the dual of* $L^{p(\cdot)}(\Omega)$. *Moreover, if* $p_1 > 1$, $L^{p(\cdot)}(\Omega)$ *and* $W^{1,p(\cdot)}(\Omega)$ *are reflexive.*

We define the space $W_0^{1,p(\cdot)}(\Omega)$ as the closure of the $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Then we have the following version of Poincaré inequity (see Theorem 3.10 in [17]).

Lemma 2.5 (Poincaré inequity)**.** *If* $p : \Omega \to [1,+\infty)$ *is continuous in* $\overline{\Omega}$, *there exists a constant* C *such that*

$$\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)}$$

for all $u \in W_0^{1,p(\cdot)}(\Omega)$.

In order to have better properties of these spaces, we need more hypotheses on the regularity of $p(x)$.

We say that p is log-*Hölder continuous* in Ω if there exists a constant C_{log} such that

$$|p(x) - p(y)| \leq \frac{C_{log}}{\log \left(e + \frac{1}{|x-y|} \right)} \quad \forall\, x, y \in \Omega.$$

It was proved in [12, Theorem 3.7], that if one assumes that p is log–Hölder continuous then $C^\infty(\bar{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$, see also [11, 13, 14, 17, 24].

Proposition 2.6. *Let $p : \Omega \to [1, \infty)$ be a bounded* log*-Hölder continuous function. Let $\beta > 0$, $D \subset \Omega$ and $h = diam(D)$. Then there exist constants C independent of h such that*

$$(2.4) \qquad h^{\beta(p(x)-p(y))} \leq C \quad \forall x, y \in D.$$

Moreover, if $p(x)$ is continuous in $\overline{D}$ then the inequality (2.4) holds for all $x, y \in \overline{D}$.

2.2. **The weak solution of** (1.1).

Lemma 2.7. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, $p : \Omega \to [p_1, p_2]$ be a bounded* log*-Hölder continuous function, $f \in L^{q(\cdot)}(\Omega)$ with $q(x) \geq q_1 > (p_1^*)'$, $g \in W^{1,p(\cdot)}(\Omega)$, and u be the weak solution of (1.1). Then*

$$\|\nabla u\|_{p(\cdot)} \leq C$$

where C is a constant depending on $\|f\|_{q(\cdot)}, \|g\|_{1,p(\cdot)}, p_1, p_2$ and Ω.

Proof. Let

$$J(v) := \int_\Omega \frac{1}{p(x)} |\nabla v|^{p(x)} \, dx.$$

By the convexity of J and using (1.2) we have that,

$$J(u) \leq J(g) - \int_\Omega |\nabla u|^{p(x)-2} \nabla u (\nabla g - \nabla u) \, dx$$
$$= J(g) - \left(\int_\Omega f(g - u) \, dx \right).$$

Using Hölder inequality, that $q'(x) \leq q_1' < p_1^*$, $W^{1,p_1}(\Omega) \hookrightarrow L^{q_1'}(\Omega)$ continuously and Poincare inequality, we have that

$$J(u) \leq \left(J(g) + 2\|f\|_{q(\cdot)} \|u - g\|_{q'(\cdot)} \right)$$
$$\leq \left(J(g) + C\|f\|_{q(\cdot)} \|u - g\|_{q_1'} \right)$$
$$\leq \left(J(g) + C\|f\|_{q(\cdot)} \|\nabla u - \nabla g\|_{p_1} \right)$$
$$\leq C \left(1 + \|f\|_{q(\cdot)} \|\nabla u - \nabla g\|_{p(\cdot)} \right)$$

where the constanat $C = C(\|g\|_{1,p(\cdot)}, \|f\|_{q(\cdot)}, \Omega, p_1)$.

Thus, we have that

$$\int_\Omega |\nabla u|^{p(x)} \, dx \leq C \left(1 + \|\nabla u\|_{p(\cdot)} \right)$$

where $C = C(\|g\|_{1,p(\cdot)}, \|f\|_{q(\cdot)}, \Omega, p_1, p_2)$. Using the properties of the $L^{p(\cdot)}(\Omega)-$ norms this means that

$$\|\nabla u\|_{p(\cdot)}^m \leq C\left(1 + \|\nabla u\|_{p(\cdot)}\right)$$

for some $m > 1$. Therefore $\|\nabla u\|_{p(\cdot)}$ is bounded by a constant C that depends on $\|g\|_{1,p(\cdot)}, \|f\|_{q(\cdot)}, \Omega, p_1$, and p_2. $\qquad\square$

The following results can be found in [10].

Theorem 2.8. *Let Ω be a bounded domain in $\mathbb{R}^2$ with convex boundary, $p \in \mathrm{Lip}(\overline{\Omega})$ with $1 < p_1 \leq p(x) \leq 2$, $f \in L^{q(x)}(\Omega)$ with $q(x) \geq q_1 > 2$, and $g \in H^2(\Omega)$. Then the weak solution of (1.1) belongs to $H^2(\Omega)$.*

Remark 2.9. If Ω is a bounded domain with Lipschitz boundary in $\mathbb{R}^2$, we have that $W^{2,p_1}(\Omega)$ is continuously imbedded in $C(\overline{\Omega})$, see [21, Theorem 5.7.8]. Therefore, if the weak solution u of (1.1) belongs to $W^{2,p_1}(\Omega)$ then $u \in C(\overline{\Omega})$.

Remark 2.10. The proof of Theorem 2.8 follows using that there exists $\{u_n\}_{n\in\mathbb{N}} \subset H^2(\Omega)$ such that

$$\|u_n\|_{2,2} \leq C = C(p(\cdot), \|f\|_{q(\cdot)}, \|g\|_{2,2}) \quad \forall n \in \mathbb{N},$$

and

$$u_n \rightharpoonup u \quad \text{weakly in } H^2(\Omega)$$

where u is the weak solution of (1.1). Therefore,

$$\|u\|_{2,2} \leq C = C(p(\cdot), \|f\|_{q(\cdot)}, \|g\|_{2,2}).$$

See the proofs of Theorem 1.1 and Theorem 1.2 in [10].

2.3. **Finite Element Spaces.** Let Ω be a bounded convex domain in $\mathbb{R}^2$ with Lipschitz boundary. Let Ω^h be a polygonal approximation to Ω defined by $\overline{\Omega^h} = \bigcup_{\kappa \in \mathcal{T}^h} \kappa$ where $\mathcal{T}^h$ is a partition of Ω^h into a finite number of disjoint open triangles κ, each of maximum diameter bounded above by h. We assume that $\mathcal{T}^h$ is nondegenerate, that is

$$\max_{\kappa \in \mathcal{T}^h} \frac{h_\kappa}{\rho_\kappa} \leq \gamma_0$$

where $h_\kappa = \mathrm{diam}(\kappa)$ and $\rho_\kappa = \sup\{\mathrm{diam}\, S : S \subset \kappa \text{ is a ball}\}$. In addition, for any two different triangles, their closures are either disjoint, or have a common vertex, or a common side. We also assume that $\Omega^h \subset \Omega$, and if a vertex belongs to $\partial\Omega^h$ then it also belongs to $\partial\Omega$.

Let

$$S^h := \{v \in C(\overline{\Omega^h}) : v|_\kappa \text{ is linear } \forall \kappa \in T^h\},$$

and $\pi_h : C(\overline{\Omega^h}) \to S^h$ denote the interpolation operator such that for any $v \in C(\overline{\Omega^h})$, $\pi_h v$ satisfies

$$\pi_h v(P) = v(P)$$

for all vertex P associated to $\mathcal{T}^h$.

The finite element approximation of (1.2) is: Find $u^h \in S_g^h$ such that

$$(2.5) \qquad \int_{\Omega^h} |\nabla u^h|^{p(x)-2} \nabla u^h \nabla v \, dx = \int_{\Omega^h} f v \, dx \quad \forall v \in S_0^h$$

where
$$S_g^h := \{v \in S^h : v = g^h \text{ on } \partial\Omega^h\},$$
and $g^h = \pi_h u$ with u the solution of (1.2).

Observe that $\pi_h u$ is well defined due to $u \in C(\overline{\Omega})$, see Remark 2.9.

Lemma 2.11. *Let $p : \Omega \to [p_1, p_2]$ be a bounded* log-*Hölder continuous function, $f \in L^{q(\cdot)}(\Omega)$ with $q(x) \geq q_1 > (p_1^*)'$, $g^h \in W^{1,p(\cdot)}(\Omega^h)$, and u^h be the solution of (2.5). Then*

(2.6)
$$\|\nabla u^h\|_{p(\cdot),\Omega^h} \leq C$$

where C is a constant depending on $\|f\|_{q(\cdot),\Omega}$ and $\|g^h\|_{1,p(\cdot),\Omega^h}$.

Proof. The proof follows as in Lemma 2.7, changing u by u^h and g by g^h. □

Let $m \in \{0, 1, 2\}$ and $q \in [1, +\infty]$. We define the following seminorm on $W^{m,q}(\Omega)$

$$|u|_{m,q,\Omega} = \begin{cases} \|u\|_{L^q(\Omega)} & \text{if } m = 0, \\ \|\nabla u\|_{L^q(\Omega)} & \text{if } m = 1, \\ \|D^2 u\|_{L^q(\Omega)} & \text{if } m = 2. \end{cases}$$

The following interpolation theorem can by found in [8].

Theorem 2.12. *For $m = 0, 1$ and for all $q \in [1, \infty]$ we have that,*
$$|v - \pi_h v|_{m,q,\Omega^h} \leq Ch^{2-m}|v|_{2,q,\Omega}$$
for all $v \in W^{2,q}(\Omega)$.

2.4. **Decomposition–Coordination method.** Let V, H be topological vectors spaces, $B \in \mathcal{L}(V, H)$ and $F : H \to \overline{\mathbb{R}}$, $G : V \to \overline{\mathbb{R}}$ be convex proper, lower semicontinuous functionals. To approximate the solution of variational problems of the following kind

(2.7)
$$\min_{v \in V} F(Bv) + G(v)$$

we use the following algorithm:

Given $r > 0$ and
$$\{\eta_0, \lambda_1\} \in H \times H;$$
then, $\{\eta_{n-1}, \lambda_n\}$ known, we define $\{u_n, \eta_n, \lambda_{n+1}\} \in V \times H \times H$ by
$$G(v) - G(u_n) + \langle \lambda_n, B(v - u_n) \rangle_H + r\langle Bu_n - \eta_{n-1}, B(v - u_n) \rangle_H \geq 0$$
for all $v \in V$;
$$F(\eta) - F(\eta_n) - \langle \lambda_n, \eta - \eta_n \rangle_H + r\langle \eta_n - Bu_n, \eta - \eta_n \rangle_H \geq 0$$
for all $\eta \in H$;
$$\lambda_{n+1} = \lambda_n + \rho_n(Bu_n - \eta_n)$$
where $\rho_n > 0$.

The following theorem can be found in [16].

Theorem 2.13. *Assume that V and H are finite dimensional and that (2.7) has a solution u. If*

- *B is an injection;*
- *G is convex, proper and lower semicontinous functional;*

- $F = F_0 + F_1$ with F_1 convex, proper and lower semicontinous functional over H and F_0 strictly convex and C^1 over H;
- $0 < \rho_n = \rho < \dfrac{1 + \sqrt{5}}{2}$;

then

$$
\begin{aligned}
u_n &\to u && \text{strongly in } V, \\
\eta_n &\to Bu && \text{strongly in } H, \\
\lambda_{n+1} - \lambda_n &\to 0 && \text{strongly in } H,
\end{aligned}
$$

and λ_n is bounded in H.

For more details about the Decomposition–Coordination method, we refer the reader to [16] and references therein.

3. Proofs of Theorem 1.1 and Theorem 1.2

In the remainder of this work we use the notation $0^0 = 1$.

Let $1 < p_1 \leq p(x) \leq p_2 < \infty$ and $\sigma(x) \geq 0$, we define for any $v \in W^{1,p(\cdot)}(\Omega^h)$

$$
\|v\|_{(p(\cdot),\sigma(\cdot))} := \|\xi(|\nabla u|, |\nabla v|, \cdot)\|_{\sigma(\cdot),\Omega^h},
$$

and

$$
|v|_{(p(\cdot),\sigma(\cdot))} := \int_{\Omega^h} \xi(|\nabla u|, |\nabla v|, x)^{\sigma(x)}\, dx,
$$

where u is the solution of (2.5), and $\xi \colon [0, +\infty) \times [0, +\infty) \times \Omega \to \mathbb{R}$

$$
\xi(a, b, x) = \begin{cases} (a+b)^{\frac{p(x)-\sigma(x)}{\sigma(x)}} a & \text{if } a+b > 0, \\ 0 & \text{if } a+b = 0. \end{cases}
$$

Observe that when σ is constant we have $\|v\|_{(p(\cdot),\sigma)}^{\sigma} = |v|_{(p(\cdot),\sigma)}$.

Before proving Theorem 1.1, we need some technical lemmas.

Lemma 3.1. *Let $p, \sigma \colon \Omega \to (1, +\infty)$ be measurable functions such that*

$$
1 < p_1 \leq p(x) \leq \sigma(x) \leq \sigma_2 < +\infty.
$$

Then

(3.1)
$$
\|v\|_{(p(\cdot),\sigma(\cdot))} \leq \||\nabla v|^{p(\cdot)/\sigma(\cdot)}\|_{\sigma(\cdot),\Omega^h}.
$$

Moreover, if there exits a constant M such that

(3.2)
$$
\varrho_{p(\cdot),\Omega^h}(|\nabla u| + |\nabla v|) \leq M
$$

then

(3.3)
$$
\|\nabla v\|_{p(\cdot),\Omega^h} \leq C \max\{M^{1/\alpha_1}, M^{1/\alpha_2}\}\|v\|_{(p(\cdot),\sigma(\cdot))}
$$

where

$$
\alpha_1 = \operatorname*{ess\,inf}_{x \in \Omega^h} \frac{\sigma(x)p(x)}{\sigma(x) - p(x)} \quad \text{and} \quad \alpha_2 = \operatorname*{ess\,sup}_{x \in \Omega^h} \frac{\sigma(x)p(x)}{\sigma(x) - p(x)}.
$$

Proof. If $\sigma(x) \equiv p(x)$ a.e. then both inequalities are trivial.

Then, we will assume that $\operatorname{ess\,sup}\{\sigma(x) - p(x)\colon x \in \Omega^h\} > 0$. Therefore, the inequality (3.1) holds due to $|\nabla u| + |\nabla v| \ge |\nabla v|$.

To prove inequality (3.3), we will assume that $|\nabla u| + |\nabla v| > 0$ in a set of positive measure; the other case is trivial.

Let $w\colon \Omega^h \to \mathbb{R}$, $w(x) := (|\nabla u(x)| + |\nabla v(x)|)^{p(x) - \sigma(x)}$. Then, by Hölder's inequality, we have

$$
\begin{aligned}
\|\nabla v\|_{p(\cdot),\Omega^h} &= \|w^{-1/\sigma(\cdot)} w^{1/\sigma(\cdot)} |\nabla v|\|_{p(\cdot),\Omega^h} \\
&\le C \|w^{-1/\sigma(\cdot)}\|_{\alpha(\cdot),\Omega^h} \|w^{1/\sigma(\cdot)} |\nabla v|\|_{\sigma(\cdot),\Omega^h} \\
&= C \|w^{-1/\sigma(\cdot)}\|_{\alpha(\cdot),\Omega^h} \|v\|_{(p(\cdot),\sigma(\cdot))},
\end{aligned}
$$
(3.4)

where $\alpha(x) := \frac{\sigma(x)p(x)}{\sigma(x) - p(x)}$. Observe that $\alpha(x) = \infty$ if only if $\sigma(x) = p(x)$.

On the other hand, by the definition of $\varrho_{\alpha(\cdot),\Omega^h}$ the fact that $\varrho_{\infty,A}(1) = 0$ for any $A \subset \Omega$ and (3.2), we get

$$
\begin{aligned}
\varrho_{\alpha(\cdot),\Omega^h}\left(w^{\frac{-1}{\sigma}}\right) &= \varrho_{\alpha(\cdot),\Omega^h}\left(w^{\frac{-1}{\sigma}} \chi_{\{p \ne \sigma\}}\right) + \varrho_{\alpha(\cdot),\Omega^h}\left(w^{\frac{-1}{\sigma}} \chi_{\{p = \sigma\}}\right) \\
&= \varrho_{p(\cdot),\Omega^h}\left((|\nabla u| + |\nabla v|)\chi_{\{p \ne \sigma\}}\right) + \varrho_{\alpha(\cdot),\Omega^h}\left(\chi_{\{\alpha = \infty\}}\right) \\
&= \varrho_{p(\cdot),\Omega^h}\left((|\nabla u| + |\nabla v|)\chi_{\{p \ne \sigma\}}\right) \\
&\le \varrho_{p(\cdot),\Omega^h}(|\nabla u| + |\nabla v|) \\
&\le M.
\end{aligned}
$$

Finally, let $\alpha_1 = \operatorname*{ess\,inf}_{x \in \Omega^h} \alpha(x)$ and $\alpha_2 = \operatorname*{ess\,sup}_{x \in \Omega^h} \alpha(x)$. Observe that $\alpha_1 < \infty$ due to $\operatorname{ess\,sup}\{\sigma(x) - p(x)\colon x \in \Omega^h\} > 0$. Therefore, by Lemma 2.2, we have that

$$
\|w^{\frac{-1}{\sigma}}\|_{\alpha(\cdot),\Omega^h} \le \max\{M^{1/\alpha_1}, M^{1/\alpha_2}\}.
$$

Combining this inequality with (3.4) we obtain (3.3). $\qquad\square$

Remark 3.2. Let u and u^h be the unique solutions of (1.2) and (2.5), respectively. Then

$$
\begin{aligned}
J_\Omega(u) &\le J_\Omega(v) \quad \forall v \in W_g^{1,p(\cdot)}(\Omega), \\
J_{\Omega^h}(u^h) &\le J_{\Omega_h}(v) \quad \forall v \in S_g^h,
\end{aligned}
$$

where

$$
J_\Lambda(v) := \int_\Lambda \frac{1}{p(x)} |\nabla v|^{p(x)} \, dx - \int_\Lambda f v \, dx
$$

with $\Lambda = \Omega$ or $\Lambda = \Omega^h$.

Observe that J_Λ is Gâxteaux differentiable with

$$
J_\Lambda'(u)(v) = \int_\Lambda |\nabla u|^{p(x) - 2} \nabla u \nabla v \, dx - \int_\Lambda f v \, dx.
$$

for any $v \in W^{1,p(\cdot)}(\Lambda)$.

Lemma 3.3. *Let* $p\colon \Omega \to [p_1, p_2]$ *be a log–Hölder continuous function with* $1 < p_1 \le p_2 \le 2$. *Let* u *and* u^h *be the solutions of* (1.2) *and* (2.5), *respectively. Then, for any* $\delta_1, \delta_2\colon \Omega \to [0, +\infty)$ *measurable functions such that* $0 \le \delta_1(x) \le \delta^+ < 2$, *we have*

$$|u - u^h|_{(p(\cdot), 2+\delta_2(\cdot))} \le C|u - v|_{(p(\cdot), 2-\delta_1(\cdot))}$$

for all $v \in S_g^h$.

Proof. We first observe that for all $v \in S_g^h$

$$(3.5) \qquad J_{\Omega^h}(v) - J_{\Omega^h}(u) = A(v) + J'_{\Omega^h}(u)(v - u),$$

where

$$A(v) = \int_0^1 \int_{\Omega^h} \left(|\nabla(u + sw)|^{p(x)-2}\nabla(u + sw) - |\nabla u|^{p(x)-2}\nabla u \right) \nabla w \, dx \, ds,$$

with $w = v - u$.

Observe that, for all v_1, v_2. and $s \in [0, 1]$ we have

$$(3.6) \qquad \frac{s}{2}(|\nabla v_1| + |\nabla v_2|) \le |\nabla(v_1 + sv_2)| + |\nabla v_1| \le 2(|\nabla v_1| + |\nabla v_2|).$$

By (2.1) and (3.6), for $q_1(x) = 1 - \delta_1(x)$ and $q_2(x) = p(x) - 2 + \delta_1(x)$ we have

$$
\begin{aligned}
(3.7) \quad |A(v)| \le &\, C \int_0^1 \int_{\Omega^h} (|\nabla(u+sw)| + |\nabla u|)^{q_2(x)} |\nabla w|^{1+q_1(x)} s^{q_1(x)} dx ds \\
\le &\, C \int_{\Omega^h} (|\nabla w| + |\nabla u|)^{q_2(x)} |\nabla w|^{1+q_1(x)} \left(\int_0^1 s^{q_1(x)} ds \right) dx \\
\le &\, \frac{C}{2 - \delta^+} \int_{\Omega^h} (|\nabla w| + |\nabla u|)^{p(x)-2+\delta_1(x)} |\nabla w|^{2-\delta_1(x)} dx \\
= &\, C|w|_{(p(\cdot), 2-\delta_1(\cdot))} \\
= &\, C|u - v|_{(p(\cdot), 2-\delta_1(\cdot))}.
\end{aligned}
$$

On the other hand, by (2.2) and (3.6), for $q_3(x) = 1 + \delta_2(x)$ and $q_4(x) = p(x) - 2 - \delta_2(x)$ we have

$$
\begin{aligned}
(3.8) \quad |A(v)| \ge &\, C \int_0^1 \int_{\Omega^h} (|\nabla(u + sw)| + |\nabla u|)^{q_4(x)} |\nabla w|^{1+q_3(x)} s^{q_3(x)} dx ds \\
\ge &\, C \int_{\Omega^h} (|\nabla w| + |\nabla u|)^{q_4(x)} |\nabla w|^{1+q_3(x)} \left(\int_0^1 s^{p(x)-1} ds \right) dx \\
\ge &\, \frac{C}{p_2} \int_{\Omega^h} (|\nabla w| + |\nabla u|)^{p(x)-2-\delta_2(x)} |\nabla w|^{2+\delta_2(x)} dx \\
= &\, C|w|_{(p(\cdot), 2+\delta_2(\cdot))} \\
= &\, C|u - v|_{(p(\cdot), 2+\delta_2(\cdot))}
\end{aligned}
$$

for all $v \in S_g^h$.

Using (3.5), we have that

$$A(u^h) + J'_{\Omega^h}(u)(u^h - u) \le A(v) + J'_{\Omega^h}(u)(v - u) \quad \forall v \in S_g^h$$

due to u^h is a minimizer of J_{Ω^h}. Then,

$$A(u^h) \leq A(v) + J'_{\Omega^h}(u)(v - u^h) \quad \forall v \in S_g^h.$$

Therefore, by (3.7) and (3.8), we have

$$|u - u^h|_{(p(\cdot), 2+\delta_2(\cdot))} \leq C|u - v|_{(p(\cdot), 2-\delta_1(\cdot))} + |J'_{\Omega^h}(u)(v - u)| \quad \forall v \in S_g^h.$$

Finally, for any $v \in S_g^h$, since Ω^h is Lipschitz, $\Omega^h \subset \Omega$ and $\varphi = v - u^h \in S_0^h$, we can extend φ to be zeros in $\Omega \setminus \Omega^h$, by a function $\hat{\varphi} \in W_0^{1,p(\cdot)}(\Omega)$. Then

$$J'_{\Omega^h}(u)(\varphi) = J'_\Omega(u)(\hat{\varphi}) = 0$$

due to u is a minimizer of J_Ω. Therefore $J'_{\Omega^h}(u)(v - u^h) = 0$ for all $v \in S_g^h$. This completes the proof. $\qquad \square$

Now we are able to prove Theorem 1.1.

Proof of Theorem 1.1. We begin by noting that, by Lemma 2.7 and Lemma 2.11 we can apply Lemma 3.1. Using (3.3) with $\sigma = 2$, we get

$$|u - u^h|^2_{1,p(\cdot),\Omega_h} \leq C\|u - u^h\|^2_{(p(\cdot),2)} = C|u - u^h|_{(p(\cdot),2)}.$$

Then, taking $\delta_1(x) = 2 - p(x)$ and $\delta_2(x) \equiv 0$ in Lemma 3.3, we have that

$$|u - u^h|^2_{1,p(\cdot),\Omega_h} \leq C|u - v|_{(p(\cdot),p(\cdot))} = C\rho_{p(\cdot),\Omega_h}(|\nabla u - \nabla v|) \quad \forall v \in S_g^h.$$

By Lemma 2.2, we have that

$$(3.9) \quad |u - u^h|_{1,p(\cdot),\Omega_h} \leq C \max \left\{ |u - v|^{p_1/2}_{1,p(\cdot),\Omega_h}, |u - v|^{p_2/2}_{1,p(\cdot),\Omega_h} \right\} \quad \forall v \in S_g^h.$$

On the other hand, by Poincaré inequality and triangle inequality,

$$\|u - u^h\|_{1,p(\cdot),\Omega^h} \leq \|u - \pi_h u\|_{1,p(\cdot),\Omega^h} + \|u^h - \pi_h u\|_{1,p(\cdot),\Omega^h}$$

$$(3.10) \qquad \leq C \left(\|u - \pi_h u\|_{1,p(\cdot),\Omega^h} + |u^h - \pi_h u|_{1,p(\cdot),\Omega^h} \right)$$

$$\leq C \left(\|u - \pi_h u\|_{1,p(\cdot),\Omega^h} + |u^h - u|_{1,p(\cdot),\Omega^h} \right).$$

Using Theorem 2.12 for $m = 0, 1$ and $q = p_2$, Theorem 2.8, Remark 2.10 and that $p_2 \leq 2$, , we have that

$$(3.11) \qquad |u - \pi_h u|_{m,p(\cdot),\Omega^h} \leq C|u - \pi_h u|_{m,p_2,\Omega^h} \leq Ch^{2-m}|u|_{2,p_2,\Omega}.$$

Taking $v = \pi_h u$ in (3.9) and, using (3.10) and (3.11), we get

$$\|u - u^h\|_{1,p(\cdot),\Omega^h} \leq C \left(h|u|_{2,p_2,\Omega} + \max \left\{ |u|^{p_1/2}_{2,p_2,\Omega}, |u|^{p_2/2}_{2,p_2,\Omega} \right\} h^{p_1/2} \right),$$

for all $h \leq 1$. Hence

$$\|u - u^h\|_{1,p(\cdot),\Omega^h} \leq C \left(|u|_{2,p_2,\Omega} + |u|^{p_1/2}_{2,p_2,\Omega} + |u|^{p_2/2}_{2,p_2,\Omega} \right) h^{p_1/2} \quad \forall h \leq 1.$$

Moreover

$$\|u - u^h\|_{1,p(\cdot),\Omega^h} \leq C \left(|u|_{2,2,\Omega} + |u|^{p_1/2}_{2,2,\Omega} + |u|^{p_2/2}_{2,2,\Omega} \right) h^{p_1/2} \quad \forall h \leq 1$$

duet to $p_2 \leq 2$.

Finally, using Remark 2.10, we obtain the desired result. $\qquad \square$

Lastly, we prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 3.1 with $\sigma = 2$ and taking $\delta_1(x) = \delta_2(x) \equiv 0$ in Lemma 3.3, we obtain

$$
\begin{aligned}
|u - u^h|^2_{1,p(\cdot),\Omega^h} &\le C|u - u^h|_{(p(\cdot),2)} \\
&\le C|u - \pi_h u|_{(p(\cdot),2)} \\
&= C\sum_{\tau \in \mathcal{T}_h} \int_\tau (|\nabla u| + |\nabla(u - \pi_h u)|)^{p(x)-2} |\nabla(u - \pi_h u)|^2 \, dx \\
&=: I.
\end{aligned}
$$

On the other hand, by interpolation inequality, we have

$$(3.12) \quad |\nabla(u - \pi_h u)(x)| \le Ch\|H[u]\|_{L^\infty(\tau)} \le ChH[u](x) + Ch^{1+\alpha^+} \quad \forall x \in \tau,$$

due to $u \in C^{2,\alpha^+}(\tau)$.

For any fixed x, we have that $q(t) = (a+t)^{p(x)-2}t^2$ with $a \ge 0$ is increasing and $q(|t_1 + t_2|) \le 2(q(|t_1|) + q(|t_2|))$. Then, taking $a = |\nabla u(x)|$, by (3.12) we get

$$q(|\nabla(u - \pi_h u)(x)|) \le 2\left(q(ChH[u](x)) + q(Ch^{1+\alpha^+}) \right).$$

Hence, since $p(x) \le 2$, we have

$$
\begin{aligned}
I &\le C\sum_{\tau \in \mathcal{T}_h} h^2 \int_\tau (|\nabla u| + ChH[u])^{p(x)-2} H[u]^2 \, dx \\
&\quad + \sum_{\tau \in \mathcal{T}_h} \int_\tau \left(|\nabla u| + Ch^{1+\alpha^+} \right)^{p(x)-2} h^{2(1+\alpha^+)} \, dx \\
&\le Ch^2 \int_{\Omega^h} |\nabla u|^{p(x)-2} H[u]^2 \, dx + C\sum_{\tau \in \mathcal{T}_h} \int_\tau h^{p(x)(1+\alpha^+)} \, dx \\
&\le Ch^2 \int_{\Omega^h} |\nabla u|^{p(x)-2} H[u]^2 \, dx + Ch^2
\end{aligned}
$$

where in the last inequality we are using Proposition 2.6. This completes the proof. □

Remark 3.4. Since,

$$\int_\Omega |\nabla u|^{p(x)-2} H[u]^2 \, dx \le \int_\Omega |\nabla u|^{p_2-2} H[u]^2 \, dx + \int_\Omega |\nabla u|^{p_1-2} H[u]^2 \, dx$$

we have, by Lemma 3.1 in [1], that (1.5) holds if $u \in W^{3,1}(\Omega)$.

Remark 3.5. We can see that (1.6) can be interpreted as follows: in order to have optimal rate of convergence we only need C^2 regularity of the solution, in regions where the maximum of $p(x)$ is 2, and we need, for example, $C^{2,1}$ regularity of the solution, only in regions where the function $p(x)$ is near 1.

The next example is a generalization of [20, Example 3.1].

Example 1. We consider the radially symmetric version of the problem. Let $\Omega = B_1(0)$, $p(x) = P(r)$ be a log-Hölder continuous function, $f(x) = F(r) \in L^q(\Omega)$ for some $q \ge 2$, where $r = |x|$ and g be constant. We assume that

$$(3.13) \qquad P(r) \ne 2 \quad \text{if} \quad \frac{1}{r}\int_0^r tF(t)\,dt = 0,$$

and for each $\tau \in \mathcal{T}^h$

(3.14) $$p \in C^{1,\beta}(\tau), f \in C^\beta(\tau) \text{ with } \beta \geq \alpha^+.$$

We will see that (1.5) and (1.6) of Theorem 1.2 hold.

First observe that $f \in L^{q(\cdot)}(\Omega)$ if only if $t^{1/q(\cdot)}F \in L^{q(\cdot)}(0,1)$.

If we consider

$$Z(r) = -\frac{1}{r}\int_0^r tF(t)\,dt.$$

then, by Hölder's inequality, we have that

$$\int_0^1 |Z(r)|^{P'(r)}\,dr \leq \int_0^1 \left(\left\| \frac{t^{1/q'(\cdot)}}{r} \right\|_{L^{q'(\cdot)}(0,r)} \|t^{1/q(\cdot)}F(t)\|_{L^{q(\cdot)}(0,1)} \right)^{P'(r)}\,dr < \infty$$

due to $q' \leq 2$ and $t^{1/q(\cdot)}F \in L^{q(\cdot)}(0,1)$. Then $Z(r) \in L^{P'(\cdot)}(0,1)$ and we can define $U \in W^{1,p(\cdot)}(0,1)$ as the solution of

$$\left(\left| \frac{dU}{dr} \right|^{P-2} \frac{dU}{dr} \right)(r) = Z(r).$$

Therfore the solution of (1.2) is

$$u(x) = U(r) = -\int_r^1 Z(t)|Z(t)|^{\frac{2-P(t)}{P(t)-1}}\,dt + g$$

If we derive Z, using that $|Z| = \left| \frac{dU}{dr} \right|^{P-1}$, we have that

$$\frac{d^2U}{dr^2} = \frac{1}{P-1}\frac{dZ}{dr}|Z|^{\frac{2-P}{P-1}} - \frac{1}{(P-1)^2}|Z|^{\frac{2-P}{P-1}}\frac{dP}{dr}\log(|Z|)Z.$$

Observe that $\dfrac{d^2U}{dr^2}$ is well define since (3.13) implies

(3.15) $$Z(r) \neq 0 \text{ if } P(r) = 2.$$

On the other hand, for each $\tau \in \mathcal{T}^h$ there exist a cloused set $A_\tau \subset \{(r,\theta) \colon r \geq 0 \text{ and } \theta \in [0,2\pi)\}$ such that $\tau = \phi(A_\tau)$ where $\phi(r,\theta) = (r\cos(\theta), r\sin(\theta))$. Then, by (3.14), we have that

(3.16) $$P \in C^{1,\beta}(A_\tau) \text{ and } F \in C^\beta(A_\tau)$$

for any $\tau \in \mathcal{T}^h$.

Then

(3.17) $$Z \in C^{1,\beta}(A_\tau) \quad \forall \tau \in \mathcal{T}^h$$

and therefore

(3.18) $$|Z|^{\frac{2-P}{P-1}} \in C^{\frac{2-P^+}{P^+-1}}(A_\tau)$$

where $P^+ = \max_{A_\tau} P$ for all $\tau \in \mathcal{T}^h$.

On the other hand, since $\log(t)t$ is a Hölder continuous for any exponent, we have that

(3.19) $$|Z|^{\frac{2-P}{P-1}}\log(|Z|)Z \in C^{\frac{2-P^+}{P^+-1}}(A_\tau) \quad \forall \tau \in \mathcal{T}^h,$$

and then, by (3.16)–(3.19), for each $\tau \in \mathcal{T}^h$ we have that

$$\frac{d^2U}{dr^2} \in C^\gamma(A_\tau) \text{ where } \gamma = \min\left\{\beta, \frac{2-P^+}{P^+ - 1}\right\}.$$

Finally, since $Z(0) = 0$ and by (3.15), we have that $\dfrac{dU}{dr}(0) = \dfrac{d^2U}{dr^2}(0) = 0$ so $u \in C^{2,\gamma}(\tau)$ for all $\tau \in \mathcal{T}^h$ and (1.6) holds.

If we define $\hat{H}[u]^2 = (u_{x_1x_1})^2 + 2(u_{x_1x_2})^2 + (u_{x_2x_2})^2$ we have

$$H[u] \leq 3\hat{H}[u],$$

$$(3.20) \qquad \hat{H}[u]^2 |\nabla u|^{p-2} = \left(\frac{d^2U}{dr^2}\right)^2 \left|\frac{dU}{dr}\right|^{P-2} + \frac{1}{r^2}\left|\frac{dU}{dr}\right|^P.$$

First, since $P, Z \in C^1$ and by (3.15), we have that

$$(3.21) \qquad \left(\frac{d^2U}{dr^2}\right)^2 \left|\frac{dU}{dr}\right|^{P-2} = G(Z, P) \in L^\infty(0,1),$$

where

$$G(Z, P) = \frac{1}{(P-1)^2}\left(\frac{dZ}{dr}\right)^2 |Z|^{\frac{2-P}{P-1}} - \frac{2}{(P-1)^3}|Z|^{\frac{2-P}{P-1}}\frac{dP}{dr}\log(|Z|)Z$$

$$+ \frac{2}{(P-1)^4}\left(\frac{dP}{dr}\right)^2 |Z|^{\frac{2-P}{P-1}}\log^2(|Z|)Z^2.$$

On the other hand using, that $Z(0) = 0$ and $Z \in C^1$, we have that

$$(3.22) \qquad \frac{1}{r^2}\left|\frac{dU}{dr}\right|^p = \frac{1}{r}|Z|^{\frac{P}{P-1}} \in L^\infty(0,1).$$

Therefore, by (3.20)–(3.22)

$$\int_\Omega \hat{H}[u]^2 |\nabla u|^{p-2} \, dx = 2\pi \int_0^1 \left(\frac{d^2U}{dr^2}\right)^2 \left|\frac{dU}{dr}\right|^{P-2} r + \frac{1}{r}\left|\frac{dU}{dr}\right|^P \, dr < \infty,$$

so (1.5) holds.

4. Numerical examples

In this section, for each $h \geq 0$ we approximate the solution u^h of (2.5) by the sequence u_n^h driven by the algorithm described in Subsection 2.4. For simplicity we will denote $u_n^h = u_n$.

Let $V = S_g^h$,

$$H = \left\{\eta : \mathbb{R}^2 \to \mathbb{R}^2 : \eta|_\kappa = \text{ constant }\right\},$$

$$F(\eta) = \int_\Omega \frac{|\eta|^{p(x)}}{p(x)} \, dx, \quad G(v) = -\int_\Omega fv \, dx,$$

and $B : V \to H$ defined by $B(v) = \nabla v$. Then

$$J_{\Omega^h}(v) = F(B(v)) + G(v).$$

V and H are endowed with the L^2-norm and the $L^2 \times L^2-$norm, respectively.

If we take $\rho_n = r = 1$ then the algorithm is:

Given
$$\{\eta_0, \lambda_1\} \in H \times H,$$
then, $\{\eta_{n-1}, \lambda_n\}$ known, we define $\{u_n, \eta_n, \lambda_{n+1}\} \in V \times H \times H$ by

$$(4.1) \qquad \int_\Omega \nabla u_n \nabla v \, dx = \int_\Omega fv \, dx + \int_\Omega (\eta_{n-1} - \lambda_n) \nabla v \, dx, \qquad \forall v \in V,$$

$$(4.2) \qquad \int_\Omega (|\eta_n|^{p(x)-2} \eta_n + \eta_n) \eta \, dx = \int_\Omega (\lambda_n + \nabla u_n) \eta \, dx \qquad \forall \eta \in H,$$

$$\lambda_{n+1} = \lambda_n + (\nabla u_n - \eta_n).$$

Remark 4.1. Since V, H, F, G, B, ρ_n and r satisfy the assumptions of Theorem 2.13, the conclusions of Theorem 2.13 are satisfied, that is $u_n \to u^h$ and $\nabla u_n \to \nabla u^h$.

Observe that (4.1) can be replace by,
$$MU_n = F_n,$$
where
$$M_{ij} = \int_\Omega \nabla \varphi_i \nabla \varphi_j \, dx,$$
$$F_{n,j} = \int_\Omega \varphi_j f \, dx + \int_\Omega (\eta_{n-1} - \lambda_n) \nabla \varphi_j \, dx,$$
and $\{\varphi_j\}_{j \le N}$ is a basis of V with $N = \dim(V)$. Thus
$$u_n = \sum_{j=1}^N u_{n,j} \varphi_j.$$

On the other hand, we define $\eta_{n,\kappa} = \eta_n|_\kappa$, in the same way we define $\lambda_{n,\kappa}$ and $\nabla_\kappa u_n$. We can see from (4.2) that $\eta_{n,\kappa}$ satisfies

$$\left(\frac{1}{|\kappa|} \int_\kappa |\eta_{n,\kappa}|^{p(x)-2} \, dx + 1 \right) \eta_{n,\kappa} = \lambda_{n,\kappa} + \nabla_\kappa u_n.$$

Let $\bar{p}_\kappa = p(\bar{x}_\kappa)$, where $\bar{x}_\kappa$ is the varicenter of κ. Then using a quadrature rule for the first term, we can approximate $\eta_{n,\kappa}$ by the equation,
$$(|\eta_{n,\kappa}|^{\bar{p}_\kappa - 2} + 1) \eta_{n,\kappa} = \lambda_{n,\kappa} + \nabla_\kappa u_n,$$
thus $|\eta_{n,\kappa}|$ solves
$$|\eta_{n,\kappa}|^{\bar{p}_\kappa - 1} + |\eta_{n,\kappa}| = |\lambda_{n,\kappa} + \nabla_\kappa u_n|,$$
and therefore
$$\eta_{n,\kappa} = \frac{\lambda_{n,\kappa} + \nabla_\kappa u_n}{|\eta_{n,\kappa}|^{\bar{p}_\kappa - 2} + 1}.$$

Summarizing, each iteration of the algorithm can be reduce to the following:

Find $\{u_n, \eta_n, \lambda_{n+1}\} \in V \times H \times H$ such that
$$u_n = \sum_{j=1}^N U_{n,j} \varphi_j,$$

where U_n solves,

$$(4.3) \qquad\qquad\qquad MU_n = F_n;$$

$$\eta_{n,\kappa} = \frac{\lambda_{n,\kappa} + \nabla_\kappa u_n}{b^{\bar{p}_\kappa - 2} + 1}$$

where $b \in \mathbb{R}_{\geq 0}$ solves

$$(4.4) \qquad\qquad\qquad b^{\bar{p}_\kappa - 1} + b = |\lambda_{n,\kappa} + \nabla_\kappa u_n|,$$

and

$$\lambda^{n+1} = \lambda_n + (\nabla u_n - \eta_n).$$

Observe that each step of the algorithm consists in solving the linear equation (4.3) and then the one dimensional nonlinear equation (4.4).

We now apply the algorithm to a family of examples. For each h, we use a stooping time criterion and we approximate u^h by u_n^h, and finally we compute $\|u_n^h - u\|_{W^{1,p(\cdot)}(\Omega)}$.

In the following examples, we have considered a rectangular domain $\Omega = [-1\ 1] \times [-1\ 1]$ and a uniform mesh, with linear finite elements in all triangles. We denote by N the number of degrees of freedom in the finite element approximation.

Example 1. In this example we consider the case $f = 0$, and the following function $p(x)$,

$$p(x) = \begin{cases} 1 + \left(\dfrac{b}{2}(x_1 + x_2) + 1 + b \right)^{-1} & \text{if } b \neq 0, \\ 2 & \text{if } b = 0. \end{cases}$$

It is easy to see that the solution of (1.1) is

$$u(x) = \begin{cases} \dfrac{\sqrt{2}e^{b+1}}{b} \left(e^{\frac{b}{2}(x_1 + x_2)} - 1 \right) & \text{if } b \neq 0, \\ \dfrac{\sqrt{2}e}{2}(x_1 + x_2) & \text{if } b = 0. \end{cases}$$

The experimental results for different values of b and N are shown in the following table, where $e = u - u_n^h$.

b \ $N^{1/2}$	20	40	60	80	100	120	140
0.1	0.0200	0.0100	0.0067	0.0050	0.0040	0.0033	0.0029
0.5	0.1707	0.0848	0.0567	0.0427	0.0342	0.0286	0.0245
1	0.6704	0.3341	0.2244	0.1692	0.1357	0.1135	0.0973
2	5.5457	2.7592	1.8683	1.3750	1.1055	0.9250	0.7940
2.5	5.5457	2.7592	1.8683	1.3750	1.1055	2.3770	2.0434
3	14.2471	7.2017	4.8641	3.6136	2.8534	6.6850	5.8923

TABLE 1. $\|e\|_{1,p(\cdot)}$ respect to $N^{1/2}$ and b

Figure 1 exhibits a plot, for different values of b, of $\log(\|e\|_{1,p(\cdot)})$ respect to $N^{1/2}$.
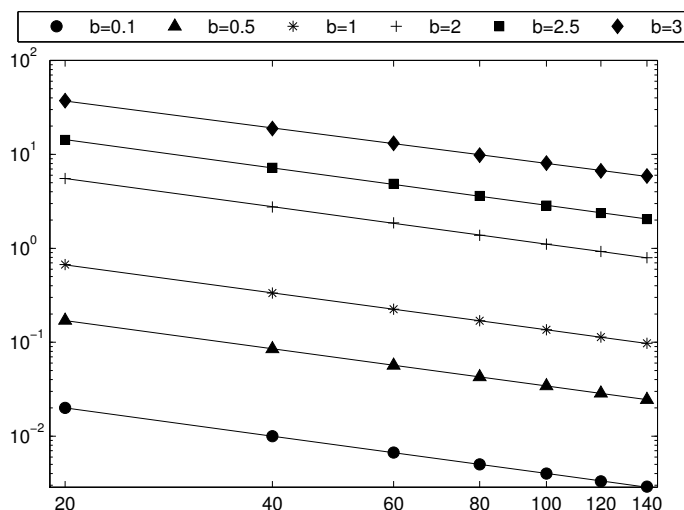

FIGURE 1. $\|e\|_{1,p(\cdot)}$ respect to $N^{1/2}$ in loglog scale

Fitting these values by the model $\|e\|_{1,p(\cdot)} \sim CN^{-\alpha/2}$ using least square approximation gives us the results of Table 2.

b	p_1	α	C
0.1	1.83	0.9984	0.1992
0.5	1.5	0.9961	1.6842
1	1.33	0.9900	6.52289
2	1.2	0.9998	55.3856
2.5	1.16	1.0007	143.9890
3	1.14	0.9495	329.2832

TABLE 2. Numerical order

Observe that the numerical rate of convergence is still of order one.

We also observe that p_1 is close to one when $b \gg 1$, for example $p_1 = 1.14$ if $b = 3$. Table 2 shows that the constant C increases when p_1 is near to one. In fact, the bound of the $\|u\|_{H^2(\Omega)}$ and the constants C in Lemma 3.3 depend on $1/(p_1-1)$. See [10] and Lemma 2.1.

Finally, we give an example where the regularity assumption of Theorem 1.2 are not satisfied. This example is a generalization of an example used for the case $p(x) \equiv p$. See [1, (5.9)].

Example 2. Let $p \in C^1$ be a radially symmetric function, $\rho \in (0,1)$, $C = 1/(1+\rho)$ and

$$f(x) = f(|x|) = |x|^{-1+\rho(p(|x|)-1)}(1 + \rho(p(|x|) - 1) + \rho|x|\log(|x|)p'(|x|))$$

then $u(x) = C(1 - |x|^{1+\rho})$ is a solution of

$$-\Delta_{p(x)}u = f.$$

Observe that $u \in W^{2,s}(\Omega)$ if only if $s < \frac{2}{1-\rho}$. Thus, in the limit case $(\rho \to 0)$, we only have H^2 regularity.

For our example, we take $p(x) = 2 + (p_1 - 2)|x|$ with $1 < p_1 \leq 2$. Note that $p_1 \leq p(x) \leq 2$.

The experimental results for $p_1 = 1.5$ and different values of ρ and N are shown in the following tables, where $\|e\|_{1,2} = \|u - u_n^h\|_{1,2}$ is given with respect to $N^{1/2}$ and ρ.

$p_1 = 1.5$		
$N^{1/2}$ \ ρ	0.1	0.5
10	0.2956	0.1773
20	0.1550	0.0883
30	0.1057	0.0588
40	0.0803	0.0441
50	0.06491	0.0353
60	0.0545	0.0294
70	0.0470	0.0252
80	0.0413	0.0220
90	0.0369	0.0196
100	0.0333	0.0176

Fitting these values by the model $\|e\|_{1,2} \sim CN^{-\alpha/2}$ using least square approximation gives us the results of Table 3.

ρ	p_1	α
0.1	1.5	0.9498
0.5	1.5	1.0027

TABLE 3. Numerical order

On other hand, in all these cases, we can apply Theorem 1.1 since all the solutions belong to $H^2(\Omega)$. Then we can conclude order 0.75 when $p_1 = 1.5$ in the $W^{1,p(\cdot)}(\Omega)$ norm. However, in our numerical examples, we can observe a better order in a better norm since, in all the cases, the order of the H^1-error is near to one. Hence, the conclusions of Theorem 1.1 are pessimistic for these cases. For this reason, we believe that the error bound obtained in Theorem 1.1 can be improved. That means that we expect that the regularity assumptions in Theorem 1.2 can be weakened.

5. Some Comments

During the refereeing process of this article, Breit et al published the work [4] where they show, for the general dimension case that if $p \in C^\alpha(\overline{\Omega})$ (without the restriction $p(x) \leq 2$), $g = 0$ and

$$(5.1) \qquad\qquad F(\cdot, \nabla u) \in H^1(\Omega)$$

then

$$\|F(\cdot, \nabla u) - F(\cdot, \nabla u)\|_{L^2(\Omega)} \leq Ch^\alpha$$

where C depends on $\|F(\cdot, \nabla u)\|_{H^1(\Omega)}$. Here $F(x, \xi) = |\xi|^{\frac{p(x)-2}{2}}\xi$.

In the case $\alpha < 1$ one cannot expect that $F(\cdot, \nabla u) \in H^1(\Omega)$ even locally, see [4, Remark 4.5]. In fact, as far as we know the best regularity proved for the solution is $H^2(\Omega)$ for the two dimensional case. Therefore, Theorem (1.1) is the best order that we could prove without assuming extra assumptions over the solution. In fact, this is the reason why we assume along the paper that we are in the two dimensional case.

Theorem 4.4 in [4] and Theorem 1.2 have extra assumptions over u. For the case $\alpha = 1$ both theorems prove order one, in the first case for the quasi-norm $\|F(\cdot, \nabla u) - F(\cdot, \nabla u)\|_{L^2(\Omega)}$ and in the other for the error in the $W^{1,p(\cdot)}(\Omega)$ norm.

Observe that, by Lemma 2.1 replacing $p(x)$ by $\frac{p(x)}{2} + 1$ and taking $\delta = 0$ we have that

$$||\xi|^{\frac{p(x)-2}{2}}\xi - |\eta|^{\frac{p(x)-2}{2}}\eta|^2 \sim |\xi - \eta|^2(|\xi| + |\eta|)^{p(x)-2}.$$

Then, as in the case $p(x) =$constant (see [2, Remark 2.2]), we have that

$$(5.2) \qquad\qquad \|F(\cdot, \nabla u) - F(\cdot, \nabla u)\|_{L^2(\Omega)}^2 \sim |u - u^h|_{(p(\cdot), 2)}.$$

Therefore, since in the proof of Theorem 1.2 we arrive at

$$|u - u^h|_{1,p(\cdot),\Omega^h}^2 \leq C|u - u^h|_{(p(\cdot),2)} \leq Ch^2,$$

we also obtain, assuming (1.5) and (1.6), order one for the quasinorm defined in [4].

On the other hand, it is possible to see that if $p(x)$ is Lipschitz then

$$\|\nabla F(\cdot, \nabla u)\|_{L^2(\Omega)}^2 =$$

$$\int_\Omega |\nabla u|^{p(x)-2} \sum_{i,j} \left(\frac{(p(x) - 2)}{2} u_{x_j} \frac{\nabla u \nabla u_{x_i}}{|\nabla u|^2} + \log(|\nabla u|)u_{x_j}p_{x_i} + u_{x_i x_j} \right)^2 dx.$$

Therefore,

$$\|\nabla F(\cdot, \nabla u)\|_{L^2(\Omega)}^2 \leq C \int_\Omega (|\nabla u|^{p(x)-2}|D^2 u|^2 + (\log(|\nabla u|))^2 |\nabla u|^{p(x)}|\nabla p|^2)\, dx.$$

Thus if u satisfies (1.5) and

$$(5.3) \qquad \int_\Omega (\log(|\nabla u|))^2 |\nabla u|^{p(x)}|\nabla p|^2\, dx < \infty$$

then $F(\cdot, \nabla u) \in H^1(\Omega)$.

In particular, in Example 2, we have that

$$\int_\Omega |\nabla u|^{p(x)-2}|D^2 u|^2\, dx = (\rho^2 + 1)\int_\Omega |x|^{\rho p(x)-1}\, dx$$

and

$$\|\nabla F(\cdot, \nabla u)\|_{L^2(\Omega)}^2 = \int_\Omega |x|^{\rho p(\cdot)-1}(1 + C(1 + C\log(|x|)|x|)^2\, dx$$

and since $\log(|x|)|x|$ is bounded, in this case

$$\|\nabla F(\cdot, \nabla u)\|_{L^2(\Omega)}^2 \sim \int_\Omega |\nabla u|^{p(x)-2}|D^2 u|^2\, dx.$$

Then, in this example our assumption (1.5) and their assumption (5.1) are equivalent. Observe that this example is not consider in [4] since here $g \neq 0$.

Finally we want to emphasize that we only use that we are in the two dimensional case to ensure the H^2 regularity of the solution in Theorem 1.1. Therefore, as in Theorem 1.2 we are assuming the regularity of the solution, the result it is also valid for any dimension.

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