Partial isometries in semi-Hilbertian spaces

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Abstract

In this work the concepts of isometries, unitaries and partial isometries operators on a Hilbert space are generalized when a semi-inner product is considered. These new concepts are described by mean of oblique projections.

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This paper is devoted to the study of operators on a Hilbert space $\mathcal{H}$ with an additional semi inner product defined by a positive semidefinite operator A namely $\langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle$ for every $\xi, \eta \in \mathcal{H}$.

This additional structure induces an adjoint operation. However, this operation is defined for not every bounded linear operator on $\mathcal{H}$, unless A is invertible. Operators T which are selfadjoint with respect to $\langle \ . \ , \ . \ \rangle_A$ in the sense that $\langle T\xi, \eta \rangle_A = \langle \xi, T\eta \rangle_A$ for every $\xi, \eta \in \mathcal{H}$ (or, equivalently, $AT = T^*A$) are called "symmetrizable" (with respect to A) and they have been studied since longtime. We refer the reader to the pioneer work by M. G. Krein [17], and also to papers by A. C. Zaanen [28], W. T. Reid [21], J. Dieudonné [9], P. Lax [18], and to the books of Zaanen [29] and Istrătescu [15].

Our goal is to describe classes of operators which are Hermitian, isometric, unitary or partially isometric with respect to $\langle \ . \ , \ . \ \rangle_A$. These classes admit treatments similar to their classical counterparts in many aspects of spectral theory.

It should be noticed that the extension is not trivial because not every operator admits an A- adjoint and, in case it admits one, it may have many others. In this extension, a key role

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is played by R. G. Douglas range inclusion theorem [10]. Briefly, Douglas theorem says that the operator equation $BX = C$ has a bounded linear solution X if and only if $R(C) \subseteq R(B)$; moreover, among its many solutions it has only one which satisfies $R(X) \subseteq \overline{R(B^*)}$. With this result we are able to choose for those operators T which admit an adjoint with respect to $\langle\ .\ ,\ .\ \rangle_A$, one, denoted by $T^\sharp$, which has similar, but not identical, properties as the classical T^*.

Many of the descriptions are easier if the range $R(A)$ is closed. Moreover, for describing the class of A-partial isometries which is the main goal of the paper, we need an additional hypothesis, known as compatibility in the recent literature. A closed subspace $\mathcal{S}$ and a positive (semidefinite)operator A are called compatible if there exists a (bounded linear) projection Q onto $\mathcal{S}$ which is A- selfadjoint, i.e., $AQ = Q^*A$. This condition says that a certain angle, namely that between $\mathcal{S}$ and $(A\mathcal{S})^\perp$, is not zero. As we shall see, it appears quite naturally in many computations. We shall describe elsewhere the relationship between the system generated by $\mathcal{H}$, A and the adjoint operation $^\sharp$ as studied here, with the Hilbert space $R(A^{1/2})$ with the inner product $\left\langle A^{1/2}\xi, A^{1/2}\eta\right\rangle' = \langle P\xi, P\eta\rangle$, where $\xi, \eta \in \mathcal{H}$ and $P = P_{\overline{R(A)}}$. These Hilbert spaces are relevant in the de Branges theory (see the book by T. Ando [1] for many references on this subject). Some papers by Z. Sebestyén and collaborators [22], [23], [24] and by P. Colujari and A. Gheondea [4] suggest a strong relationship between our results and some results of them on operators between Hilbert spaces like $R(A^{1/2})$ and $R(A)$.

The contents of the paper are the following. In section 1 we set up notation and terminology. We recall Douglas' theorem, we summarize the more relevant facts of the theory of compatibility, and we start the description of A-adjoints. It should be mentioned that a result of S. Hassi, Z. Sebestyén and H. S. V. De Snoo [14] allows to calculate the seminorm $\|T\|_A$ when T admits an A-adjoint. In section 2 we distinguish an adjoint operator for the semi-inner product in matter, which will be useful in this work. We study diverse properties of it. In addition, we give a decomposition of $L(\mathcal{H})$ in terms of this operator. Section 3 is devoted to the study of A-isometries. Such operators are, in particular, A-contractions which have been studied in several papers (see the papers L. Suciu [25]-[27] and the references therein). We relate the A-isometries with A-selfadjoint projections and obtain similar results to the ones existing for classical isometries. Furthermore, we establish a relationship between A-isometries and partial isometries by mean of a Douglas-type equation. Once the A-isometries are defined, the generalization of the concept of unitary operator emerges naturally. We call such operators A-unitary operators and study some of their properties. In section 4 we introduce the concept of A-partial isometries. Several properties are proved which result in a generalization of well known assertions of partial isometries. However, this concept is more difficult to handle. In fact, not only the nullspace of the operator but also that of A, both affect the A-partial isometry. We need to put an additional condition, namely that of compatibility, in order to get results which are extensions of those which are valid for classical partial isometries. In section 5 we describe the classes obtained from those above if we let A vary through the set $Gl(\mathcal{H})^+$ of all positive invertible operators.

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of A-partial isometries.

1 Preliminaries

Throughout $\mathcal{H}$ denotes a Hilbert space with inner product $\langle\ ,\ \rangle$. By $L(\mathcal{H})$ we denote the algebra of all linear bounded operators on $\mathcal{H}$. Given two closed subspaces of $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{T}$, $L(\mathcal{S},\mathcal{T}) = \{T \in L(\mathcal{H}) :\ T(\mathcal{S}^\perp) = \{0\} \text{ and } T(\mathcal{S}) \subseteq \mathcal{T}\}$. $L(\mathcal{H})^+$ is the the cone of positive (semidefinite) operators, i.e., $L(\mathcal{H})^+ = \{A \in L(\mathcal{H}) :\ \langle A\xi, \xi\rangle \geq 0\ \forall \xi \in \mathcal{H}\}$. $Gl(\mathcal{H})$ is the group of invertible operators of $L(\mathcal{H})$ and $Gl(\mathcal{H})^+ = Gl(\mathcal{H}) \cap L(\mathcal{H})^+$. For every $T \in L(\mathcal{H})$ its range is denoted by $R(T)$, its nullspace by $N(T)$ and its adjoint by T^*. Given a closed subspace $\mathcal{S}$ of $\mathcal{H}$, $P_{\mathcal{S}}$ denotes the orthogonal projection onto $\mathcal{S}$.

It is well known that for every bounded linear densely defined operator T there exists a unique extension in $L(\mathcal{H})$. During these notes such extension will be denoted $\overline{T}$. Moreover, it holds $\|\overline{T}\| = \|T\|$ and $\overline{T^*} = \overline{T}^* = T^*$. The following theorem due to R. G. Douglas will be used frequently (see [10] or [13] for its proof).

Theorem *Let $A, B \in L(\mathcal{H})$. The following conditions are equivalent:*

(1) $R(B) \subseteq R(A)$.

(2) There exists a positive number λ such that $BB^* \leq \lambda AA^*$.

(3) There exists $C \in L(\mathcal{H})$ such that $AC = B$.

If one of these conditions holds then there exists a unique operator $D \in L(\mathcal{H})$ such that $AD = B$, $R(D) \subseteq \overline{R(A^*)}$ and $N(D) = N(B)$. Moreover, $\|D\|^2 = \inf\{\lambda > 0 :\ BB^* \leq \lambda AA^*\}$. We shall call D the **reduced solution** *of $AX = B$.*

The reduced solution of the equation $AX = B$ can be obtained by mean of the Moore-Penrose inverse of A. Recall that given $A \in L(\mathcal{H})$ the Moore-Penrose inverse of A, denoted by $A^\dagger$, is defined as the unique linear extension of $\tilde{A}^{-1}$ to $\mathcal{D}(A^\dagger) := R(A) + R(A)^\perp$ with $N(A^\dagger) = R(A)^\perp$, where $\tilde{A}$ is the isomorphism $A|_{N(A)^\perp} : N(A)^\perp \longrightarrow R(A)$. Moreover, $A^\dagger$ is the unique solution of the four "Moore-Penrose equations":

$$AXA = A, \quad XAX = X, \quad XA = P_{N(A)^\perp}, \quad AX = P_{\overline{R(A)}}|_{\mathcal{D}(A^\dagger)}.$$

It is easy to prove that $A^\dagger$ has closed graph and it is bounded if and only if $R(A)$ is closed. As a consequence, given $A, B \in L(\mathcal{H})$ such that $R(B) \subseteq R(A)$ then $A^\dagger B \in L(\mathcal{H})$ even if $A^\dagger$ is not bounded. Proofs of these facts can be found in many places, e. g. the books [20], [3] and [12]. Moreover, $A^\dagger B$ is the reduced solution of the equation $AX = B$. In fact, $AA^\dagger B = P_{\overline{R(A)}}|_{\mathcal{D}(A^\dagger)}B = B$. Furthermore, as $A^\dagger B \in L(\mathcal{H})$ and $R(A^\dagger B) \subseteq \overline{R(A)}$, then $A^\dagger B$ is the reduced solution of the equation $AX = B$.

Any $A \in L(\mathcal{H})^+$ defines a positive semidefinite sesquilinear form:

$$\langle\ ,\ \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C},\ \langle \xi, \eta\rangle_A = \langle A\xi, \eta\rangle.$$

By $\| \, . \, \|_A$ we denote the seminorm induced by $\langle \ , \ \rangle_A$, i.e., $\|\xi\|_A = \langle \xi, \xi \rangle_A^{1/2}$. Observe that $\|\xi\|_A = 0$ if and only if $\xi \in N(A)$. Then $\| \, . \, \|_A$ is a norm if and only if A is an injective operator. Moreover, $\langle \ , \ \rangle_A$ induces a seminorm on a certain subset of $L(\mathcal{H})$, namely, on the subset of all $T \in L(\mathcal{H})$ for which there exists a constant $c > 0$ such that $\|T\xi\|_A \leq c\|\xi\|_A$ for every $\xi \in \mathcal{H}$. For these operators it holds

$$\|T\|_A = \sup_{\substack{\xi \in \overline{R(A)} \\ \xi \neq 0}} \frac{\|T\xi\|_A}{\|\xi\|_A} < \infty.$$

If $\mathcal{S} \subseteq \mathcal{H}$ then we denote by $S^{\perp_A} = \{\xi \in \mathcal{H} : \ \langle \xi, \eta \rangle_A = 0 \text{ for every } \eta \in \mathcal{S}\}$. It is easy to check that $S^{\perp_A} = (A\mathcal{S})^\perp = A^{-1}(\mathcal{S}^\perp)$. Moreover, since $A(A^{-1}(\mathcal{S})) = \mathcal{S} \cap R(A)$, then $(\mathcal{S}^{\perp_A})^{\perp_A} = (\mathcal{S}^\perp \cap R(A))^\perp$.

Definition 1.1 *Given $T \in L(\mathcal{H})$, an operator $W \in L(\mathcal{H})$ is called an* A**-adjoint** *of T if $\langle T\xi, \eta \rangle_A = \langle \xi, W\eta \rangle_A$ for every $\xi, \eta \in \mathcal{H}$, i.e., if $AW = T^*A$; T is called* A**-selfadjoint** *if $AT = T^*A$.*

Not every $T \in L(\mathcal{H})$ admits an A-adjoint. In fact, $T \in L(\mathcal{H})$ has an A-adjoint operator if and only if there exists $W \in L(\mathcal{H})$ such that $AW = T^*A$, if and only if the equation $AX = T^*A$ has solution; then, by Douglas theorem, T admits an A-adjoint if and only if $R(T^*A) \subseteq R(A)$.

From now on, $L_A(\mathcal{H})$ denotes the set of all $T \in L(\mathcal{H})$ which admit an A-adjoint, it is,

$$L_A(\mathcal{H}) = \{T \in L(\mathcal{H}) : \ R(T^*A) \subseteq R(A)\}.$$

$L_A(\mathcal{H})$ is a subalgebra of $L(\mathcal{H})$ which is neither closed nor dense in $L(\mathcal{H})$. In fact, let $A \in L(\mathcal{H})^+$ with non closed range and let $\eta \in \overline{R(A)} \setminus R(A)$. Hence, there exists $(\eta_n)_{n\in\mathbb{N}} \subseteq R(A)$ such that $\eta_n \underset{n\to\infty}{\longrightarrow} \eta$. Fixed $\xi \in R(A)$, for each $n \in \mathbb{N}$ define $T_n : \mathcal{H} = \langle \xi \rangle \oplus \langle \xi \rangle^\perp \to \mathcal{H}$ as $T_n\xi = \eta_n$ and $T_n \langle \xi \rangle^\perp = \{0\}$. Thus, $T_n \underset{n\to\infty}{\longrightarrow} T$ where T is defined as $T\xi = \eta$ and $T \langle \xi \rangle^\perp = \{0\}$. Obviously, for every $n \in \mathbb{N}$ $T_n^* \in L_A(\mathcal{H})$ and $T^* \notin L_A(\mathcal{H})$. Therefore, $L_A(\mathcal{H})$ is not a closed subalgebra of $L(\mathcal{H})$. On the other hand, it is easy to check that $\overline{L_A(\mathcal{H})} \subseteq \{T \in L(\mathcal{H}) : \ R(T^*A) \subseteq \overline{R(A)}\}$. Now, $\{T \in L(\mathcal{H}) : \ R(T^*A) \subseteq \overline{R(A)}\} \neq L(\mathcal{H})$. Indeed, if A is not injective then there exists $\eta \in \mathcal{H} \setminus \overline{R(A)}$. Let $\xi \in R(A)$ and define $S : \mathcal{H} = \langle \xi \rangle \oplus \langle \xi \rangle^\perp \to \mathcal{H}$ as $S\xi = \eta$ and $S \langle \xi \rangle^\perp = \{0\}$. Hence, $S \in L(\mathcal{H})$ and $T = S^* \notin \{T \in L(\mathcal{H}) : \ R(T^*A) \subseteq \overline{R(A)}\}$. Therefore, $L_A(\mathcal{H})$ is not dense in $L(\mathcal{H})$.

By Douglas theorem it results that $L_{A^{1/2}}(\mathcal{H}) = \{T \in L(\mathcal{H}) : \text{ there exists a constant } c > 0 \text{ such that } \|T\xi\|_A \leq c\|\xi\|_A \text{ for every } \xi \in \mathcal{H}\}$. In [9] it is proved that if T is A-selfadjoint then $T \in L_{A^{1/2}}(\mathcal{H})$ and it admits an unique $A^{1/2}$-adjoint operator which is selfadjoint. In Lemma 2.1 we give the expression of such $A^{1/2}$-adjoint. Hassi et al. [14], 5.1 proved the following general result: if $AB = C^*A$, where $B, C \in L(\mathcal{H})$ and $A \in L(\mathcal{H})^+$ then there exists a unique $S \in L(\mathcal{H})$ such that $N(A) \subseteq N(S)$, $A^{1/2}B = SA^{1/2}$ and $C^*A^{1/2} = A^{1/2}S$, namely, $S = (A^{1/2})^\dagger C^* A^{1/2}$. In particular, it implies that the inclusion $L_A(\mathcal{H}) \subseteq L_{A^{1/2}}(\mathcal{H})$ holds. It is easy to check that if A has closed range then the equality is obtained. Hence, if $T \in L_{A^{1/2}}(\mathcal{H})$ (in particular, if $T \in L_A(\mathcal{H})$) then $(A^{1/2})^\dagger T^* A^{1/2} \in L(\mathcal{H})$ and $A^{1/2}T(A^{1/2})^\dagger$ is bounded. This fact will be used during this work without mentioning. We refer the reader to the paper by B. A. Barnes [2], who describes completely the situation for an injective positive operator A. Summarizing, let $A \in L(\mathcal{H})^+$, $T \in L(\mathcal{H})$ and consider the following conditions:

(1) There exists a positive constant c such that $\|T\xi\|_A \leq c\|\xi\|_A$ for every $\xi \in \mathcal{H}$.
(2) There exists a positive constant c such that $\|T\xi\|_A \leq c\|\xi\|_A$ for every $\xi \in \overline{R(A)}$.
(3) $T^*R(A) \subseteq R(A)$.
(4) $T^*R(A^{1/2}) \subseteq R(A^{1/2})$.

A precise description of the relationships among them is provided by the implications: $1 \to 2$, $1 \leftrightarrow 4$ and $3 \to 4$. Moreover, if A is injective then $2 \to 1$ and if A has closed range then $1 \leftrightarrow 3 \leftrightarrow 4$.

If $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ is closed subspace of $\mathcal{H}$, $\mathcal{P}(A, \mathcal{S})$ denotes the set of A-selfadjoint projections with fixed range $\mathcal{S}$, i.e., $\mathcal{P}(A, \mathcal{S}) = \{Q \in L(\mathcal{H}) : Q^2 = Q,\ R(Q) = \mathcal{S}$ and $AQ = Q^*A\}$. If $\mathcal{P}(A, \mathcal{S})$ is not empty it is said that the pair $(A, \mathcal{S})$ is **compatible**. It is easy to see that a projection Q belongs to $\mathcal{P}(A, \mathcal{S})$ if and only if $R(Q) = \mathcal{S}$ and $N(Q) \subseteq \mathcal{S}^{\perp_A}$. Then

$$\text{the pair } (A, \mathcal{S}) \text{ is compatible if and only if } \mathcal{S} + \mathcal{S}^{\perp_A} = \mathcal{H}.$$

If the pair $(A, \mathcal{S})$ is compatible then the unique element in $\mathcal{P}(A, \mathcal{S})$ with nullspace $(A\mathcal{S})^\perp \ominus \mathcal{N}$, where $\mathcal{N} = N(A) \cap \mathcal{S}$, is denoted by $P_{A,\mathcal{S}}$. The theory of compatibility was introduced in [6]; a survey on properties and applications can be found in [7].

Theorem 1.2 *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace such that $(A, \mathcal{S})$ is compatible. Let $\mathcal{N} = N(A) \cap \mathcal{S}$ and Q the reduced solution of the equation $(P_\mathcal{S} A P_\mathcal{S})X = P_\mathcal{S} A$. Then*

(1) $Q = P_{A,\mathcal{S}\ominus\mathcal{N}}$,
(2) $P_{A,\mathcal{S}} = P_{A,\mathcal{S}\ominus\mathcal{N}} + P_\mathcal{N}$,
(3) $\mathcal{P}(A, \mathcal{S})$ is an affine manifold that can be parametrized as $\mathcal{P}(A, \mathcal{S}) = P_{A,\mathcal{S}} + L(\mathcal{S}^\perp, \mathcal{N})$. In particular, if $\mathcal{N} = \{0\}$ then $\mathcal{P}(A, \mathcal{S}) = \{P_{A,\mathcal{S}}\}$.

Remark 1.3 *Let $A \in L(\mathcal{H})^+$. Then the pair $(A, \overline{R(A)})$ is compatible and $P_{A,\overline{R(A)}} = P_{\overline{R(A)}}$. In fact, $AP_{\overline{R(A)}} = P_{\overline{R(A)}}A$, so that $P_{\overline{R(A)}} \in \mathcal{P}(A, \overline{R(A)})$ and the pair $(A, \overline{R(A)})$ is compatible. On the other hand, $N(P_{A,\overline{R(A)}}) = (A(\overline{R(A)}))^\perp = N(A) = N(P_{\overline{R(A)}})$. Thus, $P_{A,\overline{R(A)}} = P_{\overline{R(A)}}$.*

2 The A-adjoint operator $T^\sharp$

If $T \in L(\mathcal{H})$ admits an A-adjoint operator, i.e., if $R(T^*A) \subseteq R(A)$, then there exists a distinguished A-adjoint operator of T, namely, the reduced solution of equation $AX = T^*A$, i.e., $A^\dagger T^* A$. We denote this operator by $T^\sharp$. Therefore, $T^\sharp = A^\dagger T^* A$ and

$$AT^\sharp = T^*A, \;\; R(T^\sharp) \subseteq \overline{R(A)} \;\text{ and }\; N(T^\sharp) = N(T^*A).$$

Observe that if T is A-selfadjoint it does not mean, in general, that $T = T^\sharp$. For example, if $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_2(\mathbb{C})^+$ and $T = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$ then T is A-selfadjoint, but $T^\sharp = A^\dagger T^* A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq T$. In fact, $T = T^\sharp$ if and only if T is A-selfadjoint and $R(T) \subseteq \overline{R(A)}$. In this section we study some properties of $T^\sharp$. From now on, to simplify notation, we write P instead of $P_{\overline{R(A)}}$.

Lemma 2.1 *Let $A \in L(\mathcal{H})^+$ and $T \in L_A(\mathcal{H})$. Then the following statements hold:*

(1) $(A^t)^\sharp = A^t$ *for every* $t > 0$.
(2) If $AT = TA$ *then* $T^\sharp = PT^*$.
(3) If $AT = T^*A$ *then* $(A^{1/2})^\dagger T^* A^{1/2}$ *is selfadjoint.*
(4) If $W \in L_A(\mathcal{H})$ *then* $TW \in L_A(\mathcal{H})$ *and* $(TW)^\sharp = W^\sharp T^\sharp$.
(5) $T^\sharp \in L_A(\mathcal{H})$, $(T^\sharp)^\sharp = PTP$ *and* $((T^\sharp)^\sharp)^\sharp = T^\sharp$.
(6) $T^\sharp T$ *and* $TT^\sharp$ *are A-selfadjoint.*

Proof

(1) For every $t > 0$ it holds $(A^t)^\sharp = A^\dagger A^t A = A^\dagger A A^t = PA^t = A^t$.
(2) If $AT = TA$ then $T^\sharp = A^\dagger T^* A = A^\dagger A T^* = PT^*$.
(3) Since $L_A(\mathcal{H}) \subseteq L_{A^{1/2}}(\mathcal{H})$, then the equation $A^{1/2}X = T^*A^{1/2}$ has solution. Let see that its reduced solution, $(A^{1/2})^\dagger T^* A^{1/2}$, is selfadjoint. As $AT = T^*A$, then $A^{1/2}T = (A^{1/2})^\dagger AT = (A^{1/2})^\dagger T^* A$. Hence, we get that$A^{1/2}T(A^{1/2})^\dagger = (A^{1/2})^\dagger T^* A (A^{1/2})^\dagger = (A^{1/2})^\dagger T^* A^{1/2}|_{\mathcal{D}((A^{1/2})^\dagger)}$. Therefore, it holds that $((A^{1/2})^\dagger T^* A^{1/2})^* = \overline{A^{1/2}T(A^{1/2})^\dagger} = \overline{(A^{1/2})^\dagger T^* A^{1/2}|_{\mathcal{D}((A^{1/2})^\dagger)}} = (A^{1/2})^\dagger T^* A^{1/2}$. So, $(A^{1/2})^\dagger T^* A^{1/2}$ is selfadjoint.
(4) Since $R(W^\sharp T^\sharp) \subseteq \overline{R(A)}$, it is sufficient to show that $W^\sharp T^\sharp$ is solution of the equation $AX = (TW)^*A$. Now, $AW^\sharp T^\sharp = W^*AT^\sharp = W^*T^*A = (TW)^*A$. Then, by the uniqueness of the reduced solution, the assertion follows.
(5) Since $R((T^\sharp)^*A) = R(AT) \subseteq R(A)$ then $T^\sharp \in L_A(\mathcal{H})$. Furthermore, as $A(PTP) = ATP = (T^\sharp)^*A$ and $R(PTP) \subseteq \overline{R(A)}$. Then both, $(T^\sharp)^\sharp$ and PTP, are reduced solutions of $AX = (T^\sharp)^*A$. So, $(T^\sharp)^\sharp = PTP$. In particular, $((T^\sharp)^\sharp)^\sharp = PT^\sharp P = T^\sharp P = T^\sharp$.
(6) In fact, as $AT^\sharp = T^*A$, then $AT^\sharp T = T^*AT = T^*(T^\sharp)^*A = (T^\sharp T)^*A$ and $ATT^\sharp = (T^\sharp)^*AT^\sharp = (T^\sharp)^*T^*A = (TT^\sharp)^*A$. So, $T^\sharp T$ and $TT^\sharp$ are A-selfadjoint.

It is well known that every $T \in L(\mathcal{H})$ can be written as the sum of a selfadjoint operator and an antiselfadjoint operator. More precisely $T = \frac{T+T^*}{2} + \frac{T-T^*}{2}$. We generalize this result considering an A-selfadjoint operator and an A-antiselfadjoint operator when A has closed range. For this, the matrix representation of operators induced by a given $A \in L(\mathcal{H})^+$ will be useful. Note that for every $A \in L(\mathcal{H})^+$, the Hilbert space $\mathcal{H}$ can be decomposed as $\mathcal{H} = \overline{R(A)} \oplus N(A)$. Under this decomposition,

$$A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.1}$$

Observe that A has closed range if and only if $a \in Gl(R(A))$. In what follows $L_A^s(\mathcal{H})$ denotes the set of the A-selfadjoint operators of $L(\mathcal{H})$, i.e., $L_A^s(\mathcal{H}) = \{T \in L(\mathcal{H}) : AT = T^*A\}$ and $L_A^{as}(\mathcal{H})$ the set of A- antiselfadjoint operators, it is, $L_A^{as}(\mathcal{H}) = \{T \in L(\mathcal{H}) : iT \in L_A^s(\mathcal{H})\} = \{T \in L(\mathcal{H}) : AT = -T^*A\}$.

Proposition 2.2 *Let $A \in L(\mathcal{H})^+$. Then*

(1) $L_A(\mathcal{H}) = L_A^s(\mathcal{H}) \oplus L_A^{as}(\mathcal{H}) \subseteq \left\{ T \in L(\mathcal{H}) : T = \begin{pmatrix} t_{11} & 0 \\ t_{21} & t_{22} \end{pmatrix} \right\}$.

(2) A has closed range if and only if $L_A(\mathcal{H}) = \left\{ T \in L(\mathcal{H}) : T = \begin{pmatrix} t_{11} & 0 \\ t_{21} & t_{22} \end{pmatrix} \right\}$.

Proof

1. Let $T \in L_A(\mathcal{H})$. Then, $T = \frac{T + T^\sharp}{2} + \frac{T - T^\sharp}{2}$. It is easy to prove that $\frac{T + T^\sharp}{2} \in L_A^s(\mathcal{H})$ and $\frac{T - T^\sharp}{2} \in L_A^{as}(\mathcal{H})$, and so $T \in L_A^s(\mathcal{H}) + L_A^{as}(\mathcal{H})$. Conversely, if $T = T_1 + T_2$, where $T_1 \in L_A^s(\mathcal{H})$ and $T_2 \in L_A^{as}(\mathcal{H})$ then $T_1 - T_2$ is a solution of $AX = T^*A$. Therefore T admits A-adjoint, i.e., $T \in L_A(\mathcal{H})$.

On the other hand, let $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \in L_A(\mathcal{H})$. Hence, the range of $T^*A = \begin{pmatrix} t_{11}^*a & 0 \\ t_{12}^*a & 0 \end{pmatrix}$ is included in $R(A)$. Therefore $t_{12}^*a = 0$. So $R(t_{12}) \subseteq \overline{R(a)} \subseteq R(t_{12})^\perp$ and therefore $t_{12} = 0$.

2. Let $A \in L(\mathcal{H})^+$ with closed range and let $T = \begin{pmatrix} t_{11} & 0 \\ t_{21} & t_{22} \end{pmatrix}$. Then, $T^*A = \begin{pmatrix} t_{11}^*a & 0 \\ 0 & 0 \end{pmatrix}$ and so $R(T^*A) = R(t_{11}^*a) \subseteq \overline{R(A)} = R(A)$. Hence, $T \in L_A(\mathcal{H})$. The other inclusion follows by item 1.

Conversely, if $R(A)$ is not closed then there exists $\eta \in \overline{R(A)} - R(A)$. Hence, a bounded linear operator $t_{11} : \overline{R(A)} \to \overline{R(A)}$ can be defined such that $t_{11}^*a\xi = \eta$ for some $\xi \in \overline{R(A)}$. Consider

$T = \begin{pmatrix} t_{11} & 0 \\ 0 & 0 \end{pmatrix}$. Then $\eta \in R(T^*A)$ but $\eta \notin R(A)$, i.e., $R(T^*A) \nsubseteq R(A)$ or which is the same, $T \notin L_A(\mathcal{H})$.

Corollary 2.3 *The following conditions are equivalent:*

(1) $A \in L(\mathcal{H})^+$ *has closed range.*
(2) $L(\mathcal{H}) = L_A^s(\mathcal{H}) \oplus L_A^{as}(\mathcal{H}) \oplus L(N(A), \overline{R(A)})$.

Proof It is immediate from Proposition 2.2.

3 A-isometries

Recall that $T \in L(\mathcal{H})$ is an isometry if $\|T\xi\| = \|\xi\|$ for every $\xi \in \mathcal{H}$. Obviously, every isometry is injective. Moreover, T is an isometry if and only if $T^*T = I$.

Given $A \in L(\mathcal{H})^+$ and $T \in L(\mathcal{H})$ we denote $N_A(T) = \{\xi \in \mathcal{H} : \|T\xi\|_A = 0\} = N(A^{1/2}T)$. Note that if $T \in L_A(\mathcal{H})$ then $N(A) \subseteq N_A(T)$. In addition, if $R(A)$ is closed then the converse holds.

Definition 3.1 *Let* $A \in L(\mathcal{H})^+$. $T \in L(\mathcal{H})$ *is called an* A-**isometry** *if* $\|T\xi\|_A = \|\xi\|_A$ *for every* $\xi \in \mathcal{H}$.

Note that $T \in L(\mathcal{H})$ is an A-isometry if and only if $T^*AT = A$. In what follows $\mathcal{I}_A(\mathcal{H})$ denotes the set of all A-isometries of $L(\mathcal{H})$.

Examples 1 *The following examples of A-isometries can be easily checked.*

(1) For every $A \in L(\mathcal{H})^+$, *there exist A-isometries. In fact, the identity operator and P are A-isometries. Moreover, if T is an A-isometry then* $T + L(\mathcal{H}, N(A))$ *is a set of A-isometries.*
(2) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_2^+(\mathbb{C})$ *then* $T = \begin{pmatrix} 1 & 1-n \\ 0 & n \end{pmatrix}$ *is an A-isometry for every* $n \in \mathbb{N}$. *Hence for every $M > 0$ there exists an A-isometry T such that* $\|T\| > M$, *i.e.,* $\mathcal{I}_A$ *is not bounded for the operator norm.*

Now, we summarize some elementary properties of A-isometries.

Proposition 3.2 *Let* $A \in L(\mathcal{H})^+$ *and* $T \in \mathcal{I}_A(\mathcal{H})$. *Hence,*

(1) $N_A(T) = N(A)$.
(2) $\|T\|_A = 1$.

(3) If $W \in \mathcal{I}_A(\mathcal{H})$ then $TW \in \mathcal{I}_A(\mathcal{H})$.
(4) $T \in L_{A^{1/2}}(\mathcal{H})$. Moreover, if A has closed range then $T \in L_A(\mathcal{H})$.

Proof We include only the proof of item 4. because the others are trivial. Since $T^*AT = A$, then, by Douglas theorem, $R(T^*A^{1/2}) \subseteq R(A^{1/2})$, i.e., $T \in L_{A^{1/2}}(\mathcal{H})$. On the other hand, by item 1, it holds $N(A) = N_A(T) = N(A^{1/2}T)$. So, if $R(A)$ is closed then $R(T^*A) = R(T^*A^{1/2}) \subseteq R(A)$. Therefore, $T \in L_A(\mathcal{H})$.

Observe that item 1 of the above proposition generalizes the fact that every isometry is injective.

The following result relates A-isometries with Douglas-type equations and their reduced solutions. Moreover, it establishes a relationship between A-isometries and partial isometries.

Proposition 3.3 *Let $A \in L(\mathcal{H})^+$. Then T is an A-isometry if and only if there exists a partial isometry $V \in L(\mathcal{H})$ with $R(V) = \overline{R(A)}$ such that $A^{1/2}V = T^*A^{1/2}$.*

Proof Let T be an A-isometry. Then, $(T^*A^{1/2})(T^*A^{1/2})^* = T^*AT = A$ and so, by Douglas theorem, there exists $V \in L(\mathcal{H})$ such that $A^{1/2}V = T^*A^{1/2}$ and $R(V) \subseteq \overline{R(A^{1/2})}$. We claim that V is a partial isometry. In fact, as $T^*AT = A^{1/2}VV^*A^{1/2} = A$ then $VV^*A^{1/2}$ is a solution of the equation $A^{1/2}X = A$. Now, since $R(VV^*A^{1/2}) \subseteq \overline{R(A^{1/2})}$ and by the uniqueness of the reduced solution, it holds $VV^*A^{1/2} = A^{1/2}$ or, which is the same, $A^{1/2}VV^* = A^{1/2}$. Then, applying Douglas theorem again, $VV^* = P_{\overline{R(A^{1/2})}}$ and so V is a A-partial isometry with $R(V) = \overline{R(A^{1/2})}$. The converse is straightforward.

Corollary 3.4 *Let $A \in L(\mathcal{H})^+$ be an injective operator. Then T is an A-isometry if and only if there exists a co-isometry $V \in L(\mathcal{H})$ (i.e., $VV^* = I$) such that $A^{1/2}V = T^*A^{1/2}$.*

As it happens with isometries and orthogonal projections, A-isometries can be characterized in terms of A-selfadjoint projections.

Remark 3.5 *Observe that two projections Q_1, Q_2 on $\mathcal{H}$ such that $N(Q_1) \subseteq N(Q_2)$ and $R(Q_1) \subseteq R(Q_2)$ are equal: every $\xi \in \mathcal{H}$ can be written as $\xi = \rho + \nu$ with $\rho \in R(Q_1), \nu \in N(Q_1)$; then $Q_1\xi = \rho$ and $Q_2\xi = \rho + Q_2\nu = \rho$, because $\nu \in N(Q_1) \subseteq N(Q_2)$.*

Proposition 3.6 *Let $A \in L(\mathcal{H})^+$ and $T \in L_A(\mathcal{H})$. Then,*

(1) T is an A-isometry if and only if $T^\sharp T = P$.
(2) If T is an A-isometry then the pair $(A, \overline{R(TA)})$ is compatible and $TT^\sharp = P_{A,\overline{R(TA)}}$.

Proof 1. Let T be an A-isometry. Then $T^*AT = A$ and so $T^\sharp T = A^\dagger T^*AT = A^\dagger A = P$. Conversely, if $T^\sharp T = A^\dagger T^*AT = P$ then $T^*AT = AA^\dagger T^*AT = AP = A$. Therefore, T is an A-isometry.

2. Let T be an A-isometry. Hence, by the above item and since $R(T^\sharp) \subseteq \overline{R(A)}$, $(TT^\sharp)^2 = TT^\sharp TT^\sharp = TPT^\sharp = TT^\sharp$. It is clear that $TT^\sharp$ is A-selfadjoint. We shall prove that $R(TT^\sharp) = \overline{R(TA)}$. Let $\eta \in R(TA)$; then there exists $\xi \in \mathcal{H}$ such that $TA\xi = \eta$. So, $TT^\sharp\eta = TT^\sharp TA\xi = TPA\xi = TA\xi = \eta$. Then, $\eta \in R(TT^\sharp)$, so $R(TA) \subseteq R(TT^\sharp)$ and therefore $\overline{R(TA)} \subseteq R(TT^\sharp)$. On the other hand, $R(TT^\sharp) = TR(T^\sharp) \subseteq T\overline{R(A)} \subseteq \overline{R(TA)}$. Hence, $R(TT^\sharp) = \overline{R(TA)}$ and $TT^\sharp \in \mathcal{P}(A, \overline{R(TA)})$.

It only remains to show that $TT^\sharp = P_{A,\overline{R(TA)}}$. By Remark 3.5, it is sufficient to prove that $N(P_{A,\overline{R(TA)}}) \subseteq N(TT^\sharp)$. First, note that $N(A^2T^\sharp) \subseteq N(T^\sharp)$. In fact, if $\xi \in N(A^2T^\sharp)$ then $T^\sharp\xi \in N(A)$. On the other hand, $T^\sharp\xi \in \overline{R(A)}$ then $T^\sharp\xi = 0$ and so $\xi \in N(T^\sharp)$. Now, $N(P_{A,\overline{R(TA)}}) = (A\overline{R(TA)})^\perp \ominus (N(A) \cap \overline{R(TA)}) \subseteq (A\overline{R(TA)})^\perp \subseteq R(ATA)^\perp = N(AT^*A) = N(A^2T^\sharp) \subseteq N(T^\sharp) \subseteq N(TT^\sharp)$. Therefore $TT^\sharp = P_{A,\overline{R(TA)}}$.

Corollary 3.7 *Let $A \in L(\mathcal{H})^+$ with closed range and $T \in L(\mathcal{H})$. Then:*

(1) T is an A-isometry if and only if $T \in L_A(\mathcal{H})$ and $T^\sharp T = P_{R(A)}$.
(2) If T is an A-isometry then $TT^\sharp = P_{A,\overline{R(TA)}}$.

Remark 3.8 *Let $B \in Gl(\mathcal{H})^+$. Then $X^*X = B$ if and only if $X = VB^{1/2}$, where $V \in L(\mathcal{H})$ is an isometry, i.e., $V^*V = I$. In fact, if $X^*X = B$ then $B^{-1/2}X^*XB^{-1/2} = I$. So $V = XB^{-1/2}$ is an isometry. Then $X = VB^{1/2}$. The converse is trivial.*

Proposition 3.9 *Let $A \in L(\mathcal{H})^+$ with closed range and $T \in L(\mathcal{H})$. Then the following conditions are equivalent:*

(1) T is an A-isometry.
(2) $T = \begin{pmatrix} a^{-1/2}va^{1/2} & 0 \\ t_{21} & t_{22} \end{pmatrix}$, where v is an isometry in $R(A)$.

Proof

$1 \Rightarrow 2$. Let $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$ be an A-isometry. Then $T^*AT = A$, i.e.,

$$\begin{pmatrix} t_{11}^*at_{11} & t_{11}^*at_{12} \\ t_{12}^*at_{11} & t_{12}^*at_{12} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore $t_{11}^*at_{11} = a$ and $t_{12}^*at_{12} = 0$. Hence, by Remark 3.8, $a^{1/2}t_{11} = va^{1/2}$, with $v \in L(R(A))$ an isometry. So, $t_{11} = a^{-1/2}va^{1/2}$. On the other hand, if $t_{12}^*at_{12} = 0$ then $N(A) = N(t_{12}^*at_{12}) = N(a^{1/2}t_{12}) = N(t_{12})$ and so, $t_{12} = 0$.

$2 \Rightarrow 1$. If $T = \begin{pmatrix} a^{-1/2}va^{1/2} & 0 \\ t_{21} & t_{22} \end{pmatrix}$, where v is an isometry, then it is straightforward that

$T^*AT = A$.

3.1 A-unitaries

Definition 3.10 *Let $A \in L(\mathcal{H})^+$ and $U \in L_A(\mathcal{H})$. Then, U is called A-unitary if U and $U^\sharp$ are A-isometries.*

For example, if $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{M}_2^+(\mathbb{C})$ then $T = \begin{pmatrix} 1 & 1-n \\ 0 & n \end{pmatrix}$ is an A-unitary operator for every $n \in \mathbb{N}$.

In the next proposition we sum up some simple properties of A-unitaries operators.

Proposition 3.11 *Let $U, V \in L(\mathcal{H})$ be A-unitary operators. Then,*

(1) $U^\sharp U = (U^\sharp)^\sharp U^\sharp = P$.
(2) $\|U\|_A = 1$.
(3) $U^\sharp$ *is A-unitary.*
(4) UV *is A-unitary.*
(5) $\|UTV^\sharp\|_A = \|T\|_A$ *for every* $T \in L_{A^{1/2}}(\mathcal{H})$.

Proof We only include the proof of item 5. since the others can be easily checked.

$$
\begin{aligned}
\|UTV^\sharp\|_A^2 &= \sup_{\substack{\xi \in \overline{R(A)} \\ \xi \neq 0}} \frac{\|UTV^\sharp \xi\|_A^2}{\|\xi\|_A^2} = \sup_{\substack{\xi \in \overline{R(A)} \\ \xi \neq 0}} \frac{\left\langle UTV^\sharp \xi, UTV^\sharp \xi \right\rangle_A}{\|\xi\|_A^2} = \sup_{\substack{\xi \in \overline{R(A)} \\ \xi \neq 0}} \frac{\left\langle TV^\sharp \xi, U^\sharp UTV^\sharp \xi \right\rangle_A}{\|\xi\|_A^2} \\
&= \sup_{\substack{\xi \in \overline{R(A)} \\ \xi \neq 0}} \frac{\left\langle ATV^\sharp \xi, PTV^\sharp \xi \right\rangle}{\|\xi\|_A^2} = \sup_{\substack{\xi \in \overline{R(A)} \\ \xi \neq 0}} \frac{\left\langle TV^\sharp \xi, TV^\sharp \xi \right\rangle_A}{\|\xi\|_A^2} = \sup_{\substack{\xi \in \overline{R(A)} \\ \xi \neq 0}} \frac{\|TV^\sharp \xi\|_A^2}{\|\xi\|_A^2} \\
&= \|TV^\sharp\|_A^2.
\end{aligned}
$$

Let $\omega = V^\sharp \xi$. Then $\omega \in \overline{R(A)}$ and $(V^\sharp)^\sharp \omega = (V^\sharp)^\sharp V^\sharp \xi = P\xi = \xi$. Therefore,

$$
\|UTV^\sharp\|_A^2 = \|TV^\sharp\|_A^2 = \sup_{\substack{\xi \in \overline{R(A)} \\ \xi \neq 0}} \frac{\|TV^\sharp \xi\|_A^2}{\|\xi\|_A^2} = \sup_{\substack{\omega \in \overline{R(A)} \\ \omega \neq 0}} \frac{\|T\omega\|_A^2}{\|(V^\sharp)^\sharp \omega\|_A^2} = \sup_{\substack{\omega \in \overline{R(A)} \\ \omega \neq 0}} \frac{\|T\omega\|_A^2}{\|\omega\|_A^2} = \|T\|_A^2.
$$

The A-unitary operators can be characterized by means of partial isometries as follows:

Proposition 3.12 *Let $A \in L(\mathcal{H})^+$ and $U \in L_A(\mathcal{H})$. Then, U is A-unitary if and only if there exists a partial isometry $V \in L(\mathcal{H})$ with $R(V) = R(V^*) = \overline{R(A)}$ such that $A^{1/2}V = U^*A^{1/2}$ and $A^{1/2}V^* = (U^\sharp)^*A^{1/2}$.*

Proof Let U be an A-unitary operator. By Proposition 3.3, there exist partial isometries V, W such that $R(V) = R(W) = \overline{R(A)}$, $A^{1/2}V = U^*A^{1/2}$ and $A^{1/2}W = (U^\sharp)^*A^{1/2}$. We claim that $W = V^*$. In fact, as $A^{1/2}(VA^{1/2}) = U^*A = AU^\sharp = A^{1/2}(A^{1/2}U^\sharp)$ then, by the uniqueness of the reduced solution, $VA^{1/2} = A^{1/2}U^\sharp$ or, which is the same, $A^{1/2}V^* = (U^\sharp)^*A^{1/2}$. Now, since $N(V) = N(U^*A^{1/2})$, we get $R(V^*) \subseteq \overline{R(A^{1/2})}$. Thus, applying again the uniqueness of the reduced solution, $W = V^*$. The converse is an immediate consequence of Proposition 3.3.

Corollary 3.13 *Let $A \in L(\mathcal{H})^+$ be an injective operator and $U \in L_A(\mathcal{H})$. Then, U is A-unitary if and only if there exists a unitary operator $V \in L(\mathcal{H})$ such that $A^{1/2}V = U^*A^{1/2}$ and $A^{1/2}V^* = (U^\sharp)^*A^{1/2}$.*

For the next proposition we consider the decomposition $\mathcal{H} = \overline{R(A)} \oplus N(A)$ and the representation (2.1) of A.

Proposition 3.14 *Let $A \in L(\mathcal{H})^+$ with closed range and $U \in L(\mathcal{H})$. Then the following conditions are equivalent:*

(1) U is an A-unitary operator.

(2) $U = \begin{pmatrix} a^{-1/2}va^{1/2} & 0 \\ t_{21} & t_{22} \end{pmatrix}$, where v is a unitary operator on $R(A)$.

Proof

$1 \Rightarrow 2$. Let $U \in L(\mathcal{H})$ be an A-unitary operator. In particular, U is an A-isometry. Then by Proposition 3.9, $U = \begin{pmatrix} a^{-1/2}va^{1/2} & 0 \\ u_{21} & u_{22} \end{pmatrix}$ where v is an isometry in $R(A)$. Let see that v is an unitary operator. Since $U^\sharp = \begin{pmatrix} a^{-1/2}v^*a^{1/2} & 0 \\ 0 & 0 \end{pmatrix}$ is also an A-isometry, it holds that $(U^\sharp)^*AU^\sharp = \begin{pmatrix} a^{1/2}vv^*a^{1/2} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = A$. Then $vv^* = I$ and so v is an unitary operator.

$2 \Rightarrow 1$. If $U = \begin{pmatrix} a^{-1/2}va^{1/2} & 0 \\ u_{21} & u_{22} \end{pmatrix}$, where v is an unitary operator, it is easy to check that U and $U^\sharp$ are A-isometries.

Corollary 3.15 *If $A \in M_n^+(\mathbb{C})$ then every A-isometry is an A-unitary operator.*

Proof It is immediate from Propositions 3.9, 3.14 and from the fact that in a finite dimensional space every isometry is a unitary operator.

In general, for infinite dimensional Hilbert spaces, the concepts of A-isometries and A-unitary operators do not coincide. The following example shows this assertion.

Example 3.16 *Let $\mathcal{H} = \ell^2$ be the space of all square-summable sequences. Consider $S : \ell^2 \to \ell^2$ defined by $S(\xi_1, \xi_2, \xi_3, \ldots) = (0, \xi_1, \xi_2, \ldots)$. This operator is usually called shift operator and it is an isometry, i.e., $S^*S = I$. Now, let $A = P_{R(S)}$ and $T = SA$. First, note that T is an A-isometry. In fact, $T^*AT = (AS^*)A(SA) = AS^*SA = A$. On the other hand, $T^\sharp = AS^*$ is not an A-isometry. Indeed, $(T^\sharp)^*AT^\sharp = SAS^*$. Now, suppose that $(T^\sharp)^*AT^\sharp = A$ or which is the same $SAS^* = A$. Since $S^*S = I$ it holds $AS^* = S^*A$, i.e., $SA = AS$. Since $A = P_{R(S)}$, $SA = S$ and then $A = S^*S = I$ which is a contradiction. Therefore $T^\sharp$ is not an A-isometry and so T is not an A-unitary operator.*

4 A-partial isometries

An operator $T \in L(\mathcal{H})$ is said to be a partial isometry if $\|T\xi\| = \|\xi\|$ for every $\xi \in N(T)^\perp$. It follows easily from the definition that T is a partial isometry if and only if one of the next conditions hold: T^* is a partial isometry, $T^*T = P_{\overline{R(T^*)}}$, $TT^* = P_{\overline{R(T)}}$, $TT^*T = T$ or $T^*TT^* = T^*$. Moreover, T is an injective partial isometry if and only if T is an isometry.

Definition 4.1 *Let $A \in L(\mathcal{H})^+$, $T \in L(\mathcal{H})$ is an* A**-partial isometry** *if $\|T\xi\|_A = \|\xi\|_A$ for every $\xi \in N_A(T)^{\perp_A}$.*

Remarks 4.2

(1) If $A = I$ then an A-partial isometry is a partial isometry.
(2) Every A-isometry is an A-partial isometry.
(3) If $T \in L(\mathcal{H})$ is an A-partial isometry then $N(A) \subseteq N_A(T)$.
(4) If A has closed range and $T \in L(\mathcal{H})$ is an A-partial isometry then $T \in L_A(\mathcal{H})$.

Lemma 4.3 *Let $T \in L_A(\mathcal{H})$. Then $\overline{R(T^\sharp T)} + N(A) = N_A(T)^{\perp_A}$.*

Proof First, observe that $\overline{R(T^\sharp T)} + N(A)$ is a closed subspace. Indeed, as $\overline{R(T^\sharp T)} \subseteq \overline{R(T^\sharp)} \subseteq \overline{R(A)}$, then $\overline{R(T^\sharp T)} + N(A)$ is closed. On the other hand, note that $R(T^\sharp T)^{\perp_A} = N_A(T)$. Indeed, $R(T^\sharp T)^{\perp_A} = (AR(T^\sharp T))^\perp = R(T^*AT)^\perp = N(A^{1/2}T) = N_A(T)$. Then, it holds $N_A(T)^{\perp_A} = (R(T^\sharp T)^{\perp_A})^{\perp_A} = (R(T^\sharp T)^\perp \cap R(A))^\perp = \overline{R(T^\sharp T)} + N(A)$, where the last equality holds by [16], Theorem 4.8, p. 221

Proposition 4.4 *Let $A \in L(\mathcal{H})^+$ and $T \in L_A(\mathcal{H})$. Then, T is an A-partial isometry if and only if $\|T\xi\|_A = \|\xi\|_A$ for every $\xi \in \overline{R(T^\sharp T)}$.*

Proof Let T be an A-partial isometry. By Lemma 4.3, $N_A(T)^{\perp_A} = \overline{R(T^\sharp T)} + N(A)$ then the assertion follows by the definition of A-partial isometry.

Conversely, since $T \in L_A(\mathcal{H})$ then $R(T^*A) \subseteq R(A)$, and so $N(A) \subseteq N(A^{1/2}T)$. Hence, $\|T\xi\|_A = \|\xi\|_A = 0$ for every $\xi \in N(A)$. Thus, $\|T\xi\|_A = \|\xi\|_A$ for every $\xi \in \overline{R(T^\sharp T)} + N(A) = N_A(T)^{\perp_A}$. Therefore, T is an A-partial isometry.

In the next proposition we shall modify the characterizations recalled above in order to hold for A-partial isometries. Observe that we need compatibility assumption. We do not know if it strictly necessary.

Proposition 4.5 *Let $A \in L(\mathcal{H})^+$ and $T \in L_A(\mathcal{H})$. If the pair $(A, \overline{R(T^\sharp T)})$ is compatible then the following statements are equivalent:*

(1) T is an A-partial isometry.
(2) $T^*AT = AP_{A,\overline{R(T^\sharp T)}}$.
(3) $T^\sharp T = P_{A,\overline{R(T^\sharp T)}}$.

Proof Abbreviate $\mathcal{T} = \overline{R(T^\sharp T)}$ and $Q = P_{A,\mathcal{T}}$.

$1 \Rightarrow 2$. Let $T \in L(\mathcal{H})$ be an A-partial isometry, i.e., $\|T\xi\|_A = \|\xi\|_A$ for every $\xi \in \mathcal{T}$ or, which is the same, $\left\langle T^\sharp T\xi, \xi \right\rangle_A = \langle \xi, \xi\rangle_A$ for every $\xi \in \mathcal{T}$. Thus, since the pair $(A, \mathcal{T})$ is compatible, $\left\langle T^\sharp TQ\xi, Q\xi \right\rangle_A = \langle Q\xi, Q\xi\rangle_A$ for every $\xi \in \mathcal{H}$. Therefore, $\left\langle AT^\sharp TQ\xi, \xi \right\rangle = \langle AQ\xi, \xi\rangle$ for every $\xi \in \mathcal{H}$. Hence $AT^\sharp TQ = AQ$. On the other hand, $(AT^\sharp TQ)^* = Q^*(T^\sharp T)^*A = Q^*AT^\sharp T = AQT^\sharp T = AT^\sharp T$. Then $T^*AT = AT^\sharp T = AQ$.

$2 \Rightarrow 1$. Let $\xi \in \mathcal{H}$, then

$$\begin{aligned}
\|TQ\xi\|_A^2 &= \langle TQ\xi, TQ\xi\rangle_A = \left\langle T^\sharp TQ\xi, Q\xi \right\rangle_A \\
&= \left\langle AT^\sharp TQ\xi, Q\xi \right\rangle = \langle T^*ATQ\xi, Q\xi\rangle \\
&= \langle AQ\xi, Q\xi\rangle = \|Q\xi\|_A^2.
\end{aligned}$$

Therefore $\|T\xi\|_A^2 = \|\xi\|_A^2$ for every $\xi \in \mathcal{T}$ and so T is an A-partial isometry.

$2 \Leftrightarrow 3$. If $T^*AT = AQ$, i.e., $AT^\sharp T = AQ$, then $T^\sharp T$ and Q are solutions of the equation $AX = T^*AT$. Moreover, $R(T^\sharp T) \subseteq \overline{R(A)}$ and $R(Q) \subseteq \overline{R(A)}$. Then, by the uniqueness of the reduced solution, $T^\sharp T = Q$. Conversely, if $T^\sharp T = Q$ then $AQ = AT^\sharp T = T^*AT$.

We will continue exploring the way in which compatibility properties enter when studying A-partial isometries. As $\overline{R(T^\sharp T)} + N(A) = N_A(T)^{\perp_A}$ is a closed subspace, if $R(A)$ is closed then the pair $(A, \overline{R(T^\sharp T)})$ is compatible ([6], Theorem 6.2). Moreover, since $\mathcal{N} = N(A) \cap \overline{R(T^\sharp T)} =$

$\{0\}$, then, by Theorem 1.2, $\mathcal{P}(A, \overline{R(T^\sharp T)}) = \{P_{A,\overline{R(T^\sharp T)}}\}$. As a consequence, if $R(A)$ is closed then Proposition 4.5 can be rephrased as follows:

Proposition 4.6 *Let $A \in L(\mathcal{H})^+$ with closed range and $T \in L(\mathcal{H})$. The following statements are equivalent:*

(1) T is an A-partial isometry.
(2) $T \in L_A(\mathcal{H})$ and $T^*AT = AP_{A,\overline{R(T^\sharp T)}}$.
(3) $T \in L_A(\mathcal{H})$ and $T^\sharp T = P_{A,\overline{R(T^\sharp T)}}$.

Corollary 4.7 *If $\mathcal{S} \subseteq \mathcal{H}$ is a closed subspace such that the pair $(A, \mathcal{S})$ is compatible then the elements of $\mathcal{P}(A, \mathcal{S})$ are A-partial isometries.*

Proof Let $Q \in \mathcal{P}(A, \mathcal{S})$. Then, $Q^\sharp$ exists and it is easy to prove that $Q^\sharp Q$ is A-selfadjoint. Furthermore, $Q^\sharp Q$ is a projection. In fact, $(Q^\sharp Q)^2 = A^\dagger Q^* A A^\dagger Q^* A = A^\dagger Q^* P|_{\mathcal{D}(A^\dagger)} Q^* A = A^\dagger Q^* Q^* A = A^\dagger Q^* A Q = Q^\sharp Q$. Moreover, $R(Q^\sharp Q) \subseteq \overline{R(A)}$ and so, by Theorem 1.2, it holds $Q^\sharp Q = P_{A,R(Q^\sharp Q)}$. Therefore, by Proposition 4.5, Q is an A-partial isometry.

Corollary 4.8 *Let $\mathcal{S}$ a closed subspace of $\overline{R(A)}$ and $A \in L(\mathcal{H})^+$ such that the pair $(A, \mathcal{S})$ is compatible. Then there exists an A-partial isometry T such that $R(T^\sharp T) = R(T^\sharp) = \mathcal{S}$.*

Proof Since the pair $(A, \mathcal{S})$ is compatible, by Corollary 4.7 it holds that $P_{A,\mathcal{S}}$ is an A-partial isometry. Moreover, as $\mathcal{S} \subseteq \overline{R(A)}$ then $P^\sharp_{A,\mathcal{S}} = P_{A,\mathcal{S}}$ and so $R(P^\sharp_{A,\mathcal{S}} P_{A,\mathcal{S}}) = R(P^\sharp_{A,\mathcal{S}}) = \mathcal{S}$.

Examples 2 *We present some examples of A-partial isometries.*

(1) If T is an A-partial isometry then $T + L(\mathcal{H}, N(A))$ is a set of A-partial isometries.
(2) If $A \in M_n^+(\mathbb{C})$ has rank 1 then every non zero A-partial isometry is an A-isometry. In fact, if T is an A-partial isometry then $T^\sharp T = P_{A,R(T^\sharp T)}$. Furthermore, as $R(T^\sharp T) \subseteq R(A)$ and $dim(R(A)) = 1$, $R(T^\sharp T) = R(A)$. Therefore $T^\sharp T = P_{A,R(A)} = P_{R(A)}$. Then T is an A-isometry.
(3) Let $A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix} \in M_3^+(\mathbb{C})$. Then $T = \begin{pmatrix} n & 0 & 2-n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is an A-partial isometry for every $n \in \mathbb{N}$, but this is not an A-isometry. In addition, for every $M > 0$ there exists an A-partial isometry T such that $\|T\| > M$.

Proposition 4.9 *Let $A \in L(\mathcal{H})^+$ and $T \in L_A(\mathcal{H})$ such that $T^\sharp$ is non-zero. Hence, if T is an A-partial isometry and the pair $(A, \overline{R(T^\sharp T)})$ is compatible then $\|T\|_A = 1$.*

Proof As the pair $(A, \overline{R(T^\sharp T)})$ is compatible and $R(T^\sharp T)^{\perp_A} = N_A(T)$, then $\mathcal{H} = \overline{R(T^\sharp T)} + \overline{R(T^\sharp T)}^{\perp_A} = \overline{R(T^\sharp T)} + N_A(T)$. Therefore, as T is an A-partial isometry, $\|T\xi\|_A \leq \|\xi\|_A$ for every $\xi \in \mathcal{H}$. So, $\|T\|_A \leq 1$. Now, as $T^\sharp$ is non-zero, it is easy to prove that $T^\sharp T$ is non-zero. Therefore, $\overline{R(A)} \cap R(T^\sharp T) = R(T^\sharp T) \neq \{0\}$. Thus, there exists $0 \neq \xi \in R(T^\sharp T)$ such that $\|T\xi\|_A = \|\xi\|_A$, and so $\|T\|_A = 1$.

Note that in the above Proposition, if $T^\sharp = 0$ the assertion fails since $\|T\|_A = \|T^\sharp\|_A = 0$. The next proposition provides a characterization of A-isometries in terms of A-partial isometries.

Proposition 4.10 *Let $A \in L(\mathcal{H})^+$ with closed range and $T \in L(\mathcal{H})$. Then, T is an A-isometry if and only if T is an A-partial isometry and $N_A(T) \subseteq N(A)$.*

Proof If T is an A-isometry the implication is trivial. For the converse, if T is an A-partial isometry and $N_A(T) \subseteq N(A)$ then $N_A(T) = N(A)$ or, which is the same, $R(T^*AT) = R(A)$. Therefore $R(T^\sharp T) = A^\dagger R(T^*AT) = A^\dagger R(A) = R(A)$ and so T is an A-isometry.

The theorem below procures a matrix representation of the A-partial isometries.

Theorem 4.11 *Let $A \in L(\mathcal{H})^+$ with closed range and $T \in L(\mathcal{H})$. Then, under the matrix representation (2.1) of A, the following statements are equivalent:*

(1) T is an A-partial isometry.

(2) $T = \begin{pmatrix} a^{-1/2}v(aP_{a,\mathcal{S}})^{1/2} & 0 \\ t_{21} & t_{22} \end{pmatrix}$, where $v \in L(R(A))$ is an isometry and $\mathcal{S}$ is a closed subspace of $R(A)$.

Proof

$1 \Rightarrow 2$. First, let note that $P_{a,\mathcal{S}}$ exists and $aP_{a,\mathcal{S}}$ is a positive operator. In fact, since A has closed range then $a \in Gl(R(A))$ and so $(a, \mathcal{T})$ is compatible for every $\mathcal{T} \in R(A)$. In particular, $(a, \mathcal{S})$ is compatible and so $P_{a,\mathcal{S}}$ exists. On the other hand, $\langle aP_{a,\mathcal{S}}\xi, \xi\rangle = \langle P_{a,\mathcal{S}}\xi, P_{a,\mathcal{S}}\xi\rangle_a = \langle aP_{a,\mathcal{S}}\xi, P_{a,\mathcal{S}}\xi\rangle \geq 0$, for every $\xi \in R(A)$. Therefore the matrix representation of T is well defined. Let $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$ be an A-partial isometry. Then, as $N(A) \subseteq N(A^{1/2}T)$, it is straightforward that $N(A) \subseteq N(a^{1/2}t_{12}) \subseteq N(A)$. Now, since $a \in Gl(R(A))^+$, we get that $N(t_{12}) = N(A)$ and so $t_{12} = 0$. Moreover, $T^\sharp T = A^\dagger T^*AT = \begin{pmatrix} a^{-1}t_{11}^*at_{11} & 0 \\ 0 & 0 \end{pmatrix}$ is an A-selfadjoint projection. Hence, $t_{11}^*at_{11} = aP_{a,\overline{R(T^\sharp T)}}$. Then, by Remark 3.8, $t_{11} = a^{-1/2}v(aP_{a,\overline{R(T^\sharp T)}})^{1/2}$ where $v \in L(R(A))$ is an isometry.

$2 \Rightarrow 1$. If $T = \begin{pmatrix} a^{-1/2}v(aP_{a,\mathcal{S}})^{1/2} & 0 \\ t_{21} & t_{22} \end{pmatrix}$, for some isometry $v \in L(R(A))$ and $\mathcal{S}$ a closed subspace of $R(A)$ then, an easy computation shows that $R(T^*A) \subseteq R(A)$, i.e., $T^\sharp$ exists and $T^\sharp T = P_{A,\overline{R(T^\sharp T)}}$. Therefore T is an A-partial isometry.

Remark 4.12 *We have not been able to omit the compatibility and closed range hypotheses which appear in most statements of this section.*

5 Collecting some results

As we mentioned in the introduction, one aim of this paper is to extend known results for Hermitian operators, isometries or partial isometries to more general classes as A-selfadjoint operators, A-isometries or A-partial isometries. This section is devoted to characterize these classes, when A varies in $Gl(\mathcal{H})^+$. Before we introduce some notations. We denote $L^s(\mathcal{H}) = \{T \in L(\mathcal{H}) : T \text{ is a selfadjoint operator}\}$, $\mathcal{I}(\mathcal{H}) = \{T \in L(\mathcal{H}) : T \text{ is an isometry}\}$, $\mathcal{U}(\mathcal{H}) = \{U \in L(\mathcal{H}) : U \text{ is an unitary operator}\}$ and $\mathcal{J}(\mathcal{H}) = \{T \in L(\mathcal{H}) : T \text{ is a partial isometry}\}$. For a fixed $A \in L(\mathcal{H})^+$, we denote $L^r_A(\mathcal{H}) = \{T \in L^s_A(\mathcal{H}) : T^\sharp = T\}$, $\mathcal{I}_A(\mathcal{H}) = \{T \in L(\mathcal{H}) : T \text{ is an } A\text{-isometry}\}$, $\mathcal{U}_A(\mathcal{H}) = \{U \in L(\mathcal{H}) : U \text{ is an } A\text{-unitary operator}\}$ and $\mathcal{J}_A(\mathcal{H}) = \{T \in L(\mathcal{H}) : T \text{ is an } A\text{-partial isometry}\}$.

If $A \in Gl(\mathcal{H})^+$ then it is easy to prove that:

$$
\begin{aligned}
&\text{(i) } T \in L^r_A(\mathcal{H}) \text{ if and only if } T = A^{-1/2}RA^{1/2} \text{ for some } R \in L^s(\mathcal{H}).\\
&\text{(ii) } T \in \mathcal{I}_A(\mathcal{H}) \text{ if and only if } T = A^{-1/2}VA^{1/2} \text{ for some } V \in \mathcal{I}(\mathcal{H}).\\
&\text{(iii) } T \in \mathcal{U}_A(\mathcal{H}) \text{ if and only if } T = A^{-1/2}UA^{1/2} \text{ for some } U \in \mathcal{U}(\mathcal{H}).\\
&\text{(iv) } T \in \mathcal{J}_A(\mathcal{H}) \text{ if and only if } T = A^{-1/2}VA^{1/2} \text{ for some } V \in \mathcal{J}(\mathcal{H}).
\end{aligned}
$$

The next proposition contains a description of all operators which are similar to one which is Hermitian (resp. isometric, resp. unitary, resp. partially isometric) in terms of the A notions when A varies in $Gl(\mathcal{H})^+$.

Proposition 5.1 . *The following identities hold:*

(1) $\bigcup\limits_{A \in Gl(\mathcal{H})^+} L^r_A(\mathcal{H}) = \bigcup\limits_{G \in Gl(\mathcal{H})} G^{-1}L^s(\mathcal{H})G.$

(2) $\bigcup\limits_{A \in Gl(\mathcal{H})^+} \mathcal{I}_A(\mathcal{H}) = \bigcup\limits_{G \in Gl(\mathcal{H})} G^{-1}\mathcal{I}(\mathcal{H})G.$

(3) $\bigcup\limits_{A \in Gl(\mathcal{H})^+} \mathcal{U}_A(\mathcal{H}) = \bigcup\limits_{G \in Gl(\mathcal{H})} G^{-1}\mathcal{U}(\mathcal{H})G.$

(4) $\bigcup\limits_{A \in Gl(\mathcal{H})^+} \mathcal{J}_A(\mathcal{H}) = \bigcup\limits_{G \in Gl(\mathcal{H})} G^{-1}\mathcal{J}(\mathcal{H})G.$

Proof We only include the proof of item 1., since the others can be proved following the same lines. By item (i) above, it is clear that $\bigcup\limits_{A\in Gl(\mathcal{H})^+} L_A^r(\mathcal{H}) \subseteq \bigcup\limits_{G\in Gl(\mathcal{H})} G^{-1}L^s(\mathcal{H})G$. On the other hand, let $T = G^{-1}RG$ with $G \in Gl(\mathcal{H})$ and $R \in L^s(\mathcal{H})$. If $G = U|G|$ is the polar decomposition of G, with $|G| = (G^*G)^{1/2}$ and U the partial isometry, then it is easy to check that $G^{-1} = |G|^{-1}U^*$. Therefore $T = |G|^{-1}(U^*RU)|G|$. Then, taking $A = (G^*G)^{-1}$, we get the other inclusion.

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