Reduced solutions of Douglas equations and angles between subspaces

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AMS Classification: Primary 46C05 47A05 47A30
Keywords: Generalized inverses, angles, projections.

Abstract

In this paper we study some particular solutions of Douglas type equations by means of generalized inverses and angles. We apply this result to characterize positive solutions and some special projections which are symmetrizable for a semidefinited positive bounded linear operator.

Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Generalized inverses of bounded linear operators $T : \mathcal{H} \to \mathcal{K}$ can be parametrized by means of the sets $\mathcal{Q}^{\mathcal{H}}$ and $\mathcal{Q}^{\mathcal{K}}$ of linear projections (not necessarily bounded) in $\mathcal{H}$ and $\mathcal{K}$, respectively. On the other side, a projection can be studied by means of the angle (inclination) between its nullspace and its image. This paper explores in some detail the relationships between generalized inverses, projections and angles. Before stating the main results, we describe some previous facts. The main tool in the paper is the use of a particular type of solutions of operator equations like $BX = C$ where B, C are bounded linear operators between suitable Hilbert spaces. We refer to this sort of equations as **Douglas equations**, because R. G. Douglas [8] proved the following theorem which appears once and again in the literature.

Theorem. *Let $B \in L(\mathcal{H}, \mathcal{K})$ and $C \in L(\mathcal{G}, \mathcal{K})$. The following conditions are equivalent:*

1. *There exists $D \in L(\mathcal{G}, \mathcal{H})$ such that $BD = C$.*
2. *$R(C) \subseteq R(B)$.*
3. *There exists a positive number λ such that $CC^* \leq \lambda BB^*$.*

If one of these conditions holds then there exists a unique solution $X_{N(B)^\perp} \in L(\mathcal{G}, \mathcal{H})$ of the equation $BX = C$ such that $R(X_{N(B)^\perp}) \subseteq N(B)^\perp$. This solution will be called the **Douglas reduced solution**. *It also satisfies $\|X_{N(B)^\perp}\| = \inf\{\|D\| : D \in L(\mathcal{G}, \mathcal{H})\ BD{=}C\}$.*

[*]The authors were partially supported by grant PIP 5272 CONICET

We refer the reader to the papers by Douglas [8], Fillmore and Williams [10] and Nashed [12] for proofs and applications of the previous theorem. In [1], the orthogonal complement of $N(B)$ is replaced by an arbitrary closed complement $\mathcal{M}$ of $N(B)$. Then there exists a unique solution $X_{\mathcal{M}}$ of $BX = C$ such that $R(X_{\mathcal{M}}) \subseteq \mathcal{M}$; it is called the **reduced solution for** $\mathcal{M}$ and it has some analogous properties as those of Douglas reduced solution. In [1] we parametrize reduced solutions by means generalized inverses under the hypothesis $R(B)$ closed. Here, we omit this hypothesis. This forces us to deal with unbounded generalized inverses.

The main result of the paper states that, among all solutions of the equation $BX = C$, reduced solutions are those Y which can be factorized as $Y = B'C$ for some generalized inverse of B such that $R(B'B)$ is closed; this set coincides with the set of those solutions Y such that $R(Y)$ has positive Dixmier angle with $N(B)$ (see definition below). These results are applied to the problem of characterizing positive solutions of a Douglas type equations. This problem, originally studied by Z. Sebestyén [15], has been recently solved by A. Dajić and J. J. Koliha [5] for closed range operators. Our methods here extend the result to non closed range operators.

Finally we apply the previous results to get a partition of a certain set of projections. More precisely, given a closed subspace, $\mathcal{S}$, of a Hilbert space $\mathcal{H}$ and a positive semidefinite operator $A \in L(\mathcal{H})$ we consider the set $\mathcal{P}(A, \mathcal{S}) = \{Q \in L(\mathcal{H}) : Q^2 = Q,\ R(Q) = \mathcal{S},\ AQ = Q^*A\}$. This set is an affine submanifold of $L(\mathcal{H})$, which may be void. If $\mathcal{P}(A, \mathcal{S})$ is not void and we consider the matrix representation of operators of $L(\mathcal{H})$ induced by the decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$ then the elements of $\mathcal{P}(A, \mathcal{S})$ admit the matrix representation

$$Q = \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix},$$

where $y \in L(\mathcal{S}^\perp, \mathcal{S})$ is a solution of a certain Douglas type equation. Therefore we study the elements of $\mathcal{P}(A, \mathcal{S})$ for which the operator y is a reduced solution.

1 Preliminaries

Throughout $\mathcal{H}, \mathcal{K}$ and $\mathcal{G}$ denote separable complex Hilbert spaces with inner product $\langle\ ,\ \rangle$. By $L(\mathcal{H}, \mathcal{K})$ we denote the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. The algebra $L(\mathcal{H}, \mathcal{H})$ is abbreviated by $L(\mathcal{H})$. By $L(\mathcal{H})^+$ we denote the cone of positive (semidefinite) operators of $L(\mathcal{H})$, i.e., $L(\mathcal{H})^+ := \{A \in L(\mathcal{H}) :\ \langle A\xi, \xi\rangle \geq 0\ \forall \xi \in \mathcal{H}\}$. For every $T \in L(\mathcal{H}, \mathcal{K})$ its range is denoted by $R(T)$, its nullspace by $N(T)$ and its adjoint by T^*. In the sequel we denote by $\mathcal{S} \dot{+} \mathcal{T}$ the direct sum of the subspaces $\mathcal{S}$ and $\mathcal{T}$. In particular, if $\mathcal{S} \subseteq \mathcal{T}^\perp$ we denote $\mathcal{S} \oplus \mathcal{T}$. If $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ we denote $\mathcal{Q}_{\mathcal{S}} := \{Q \in L(\mathcal{H}) :\ Q^2 = Q \text{ and } R(Q) = \mathcal{S}\}$ and $P_{\mathcal{S}}$ the orthogonal projection onto $\mathcal{S}$. Furthermore, if $\mathcal{T}$ is an algebraic complement of $\mathcal{S}$, i.e., $\mathcal{S} \dot{+} \mathcal{T} = \mathcal{H}$, then $Q_{\mathcal{T}//\mathcal{S}}$ denotes the unique projection with $R(Q) = \mathcal{T}$ and $N(Q) = \mathcal{S}$. It is well known that $Q_{\mathcal{T}//\mathcal{S}}$ is bounded if and only if $\mathcal{T}$ is closed.

The next theorem introduces the notion of reduced solution which is the starting point of this article. The proof is similar to Douglas original proof [1].

Theorem 1.1. *Let $B \in L(\mathcal{H}, \mathcal{K})$ and $C \in L(\mathcal{G}, \mathcal{K})$ be such that the equation $BX = C$ has a solution and let $\mathcal{M}$ be a topological complement of $N(B)$. Then there exists a unique solution $X_{\mathcal{M}} \in L(\mathcal{G}, \mathcal{H})$ of the equation $BX = C$ such that $R(X_{\mathcal{M}}) \subseteq \mathcal{M}$. The operator $X_{\mathcal{M}}$ will be called the* **reduced solution for** $\mathcal{M}$ *of the equation $BX = C$.*

Corollary 1.2. *If $X_{\mathcal{M}}$ is a reduced solution for $\mathcal{M}$ of the equation $BX = C$ then $N(X_{\mathcal{M}}) = N(C)$.*

Proof. Since $BX_{\mathcal{M}} = C$ then $N(X_{\mathcal{M}}) \subseteq N(C)$. Now, if $\xi \in N(C)$ then $0 = C\xi = BX_{\mathcal{M}}\xi$. So $X_{\mathcal{M}}\xi \in R(X_{\mathcal{M}}) \cap N(B) \subseteq \mathcal{M} \cap N(B) = \{0\}$. Therefore $N(X_{\mathcal{M}}) = N(C)$. □

Remark 1.3. If the equation $BX = C$ has a solution and $\mathcal{M}$ is an algebraic complement of $N(B)$ then there exists a unique linear operator $X_{\mathcal{M}}$ such that $BX_{\mathcal{M}} = C$ and $R(X_{\mathcal{M}}) \subseteq \mathcal{M}$. In this case, as the next examples show, the boundedness of the solution $X_{\mathcal{M}}$ is not guaranteed.

1. Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and $\mathcal{M}$ a non closed subspace of $\mathcal{H}$ such that $\mathcal{M} \dot{+} \mathcal{S} = \mathcal{H}$. If $\xi \in \mathcal{M}$ then $P_{span\{\xi\}}$ is a solution of the equation $P_{\mathcal{S}^\perp}X = P_{\mathcal{S}^\perp}P_{span\{\xi\}}$ such that $R(P_{span\{\xi\}}) \subseteq \mathcal{M}$, and $P_{span\{\xi\}}$ is bounded.

2. Given the equation $BX = B$ and $\mathcal{M}$ a non closed subspace such that $\mathcal{M} \dot{+} N(B) = \mathcal{H}$, then $Q_{\mathcal{M}//N(B)}$ is a solution such that $R(Q_{\mathcal{M}//N(B)}) \subseteq \mathcal{M}$, but, since $\mathcal{M}$ is not closed, $Q_{\mathcal{M}//N(B)}$ is not bounded.

Our goal is to study reduced solutions. For this, the notion of generalized inverse will play a fundamental role. Note that, given $T \in L(\mathcal{H},\mathcal{K})$ and an algebraic complement of $N(T)$ then the operator $T_{\mathcal{M}} = T|_{\mathcal{M}} : \mathcal{M} \to R(T)$ is injective. Hence, there exists

$$T_{\mathcal{M}}^{-1} = (T|_{\mathcal{M}})^{-1} : R(T) \to \mathcal{M}.$$

Definition 1.4. *Given $T \in L(\mathcal{H},\mathcal{K})$, let T' be a linear operator $\mathcal{K} \to \mathcal{H}$ whose domain $\mathcal{D}(T')$ contains the range of T and let $\mathcal{M}$ be an algebraic complement of $N(T)$.*

1. *T' is an* **inner inverse** *of T if $T'|_{R(T)} = T_{\mathcal{M}}^{-1}$.*

2. *T' is a* **generalized inverse** *of T if T' is an inner inverse of T and $\mathcal{D}(T') = R(T) \dot{+} N(T')$.*

3. *T' is the* **Moore-Penrose inverse of** *T, if T' is the generalized inverse of T with $\mathcal{M} = N(T)^\perp$ and $N(T') = R(T)^\perp$. In the sequel, the Moore-Penrose inverse of T will be denoted by $T^\dagger$.*

These kinds of inverses have been extensively studied. We refer the reader to [9], [13] and [14] for the proof of the following characterizations.

Proposition 1.5. *Let $T \in L(\mathcal{H},\mathcal{K})$ and $T' : \mathcal{D}(T') \subseteq \mathcal{K} \to \mathcal{H}$ a linear operator with $R(T) \subseteq \mathcal{D}(T')$.*

1. *The next assertions are equivalent:*

   (a) *T' is an inner inverse of T;*

   (b) *$T'T = Q_{\mathcal{M}//N(T)}$ for some algebraic complement $\mathcal{M}$ of $N(T)$;*

   (c) *$TT' : \mathcal{D}(T') \to \mathcal{H}$ verifies $(TT')^2 = TT'$ and $R(TT') = R(T)$;*

   (d) *$TT'T = T$.*

2. *The next assertions are equivalent:*

   (a) *T' is a generalized inverse of T;*

   (b) *$TT'T = T$ and $T'TT' = T'$;*

   (c) *T' is an inner inverse with $R(T') = \mathcal{M}$.*

3. *The next assertions are equivalent:*

(a) T' *is the Moore-Penrose inverse of* T;

(b) $T' : R(T) \oplus R(T)^{\perp} \to \mathcal{H}$ *verifies* $TT'T = T$, $T'TT' = T'$, $TT' = P_{\overline{R(T)}}|_{\mathcal{D}(T')}$ *and* $T'T = P_{N(T)^{\perp}}$.

The inner inverses corresponding to a topological complement of the nullspace instead of an algebraic complement are particularly interesting because in such case $T'T = Q_{\mathcal{M}//N(T)}$ is bounded. Given $T \in L(\mathcal{H}, \mathcal{K})$, we shall denote by

$$\mathcal{I}(T) = \{T' : T'T = Q_{\mathcal{M}//N(T)} \in L(\mathcal{H})\}$$

and by

$$\mathcal{I}_g(T) = \{T' \in \mathcal{I}(T) : \ T' \text{ is a generalized inverse of } T\}.$$

Note that $T^{\dagger} \in \mathcal{I}(T)$. Moreover, if $T' \in \mathcal{I}(T)$ and $C \in L(\mathcal{G}, \mathcal{K})$ verifies that $R(C) \subseteq R(T)$ then $T'C \in L(\mathcal{G}, \mathcal{H})$. In fact, by Douglas theorem, there exists $D \in L(\mathcal{G}, \mathcal{H})$ such that $TD = C$. Now, if $T' \in \mathcal{I}(T)$ then $T'C = T'TD = Q_{\mathcal{M}//N(T)}D \in L(\mathcal{G}, \mathcal{H})$.

Another notion which is relevant in this paper is that of angle between subspaces. Recall that the **Friedrichs angle** between two closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{H}$ is the angle $\theta(\mathcal{S}, \mathcal{T}) \in [0, \frac{\pi}{2}]$ whose cosine is defined by

$$c(\mathcal{S}, \mathcal{T}) = \sup\{|\langle \xi, \eta \rangle| : \xi \in \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^{\perp}, \eta \in \mathcal{T} \cap (\mathcal{S} \cap \mathcal{T})^{\perp} \text{ and } \|\xi\| \leq 1, \ \|\eta\| \leq 1\}.$$

Furthermore, the **Dixmier angle** between $\mathcal{S}$ and $\mathcal{T}$ is the angle $\theta_0(\mathcal{S}, \mathcal{T}) \in [0, \frac{\pi}{2}]$ whose cosine is defined by

$$c_0(\mathcal{S}, \mathcal{T}) = \sup\{|\langle \xi, \eta \rangle| : \xi \in \mathcal{S}, \eta \in \mathcal{T} \text{ and } \|\xi\| \leq 1, \ \|\eta\| \leq 1\}.$$

Clearly, if $\mathcal{S} \cap \mathcal{T} = \{0\}$ then both angles coincide. We present some well known results about angles between subspaces that we shall use frequently during these notes. See [11] and [6] for their proofs.

Proposition 1.6. *Let* $\mathcal{S}, \mathcal{T}$ *be two closed subspaces of* $\mathcal{H}$.

1. $\mathcal{S} + \mathcal{T}$ *is a closed subspace if and only if* $c(\mathcal{S}, \mathcal{T}) < 1$.
2. $c(\mathcal{S}, \mathcal{T}) = 0$ *if and only if* $\mathcal{S} = \mathcal{S} \cap \mathcal{T} \oplus \mathcal{S} \cap \mathcal{T}^{\perp}$.
3. $\mathcal{S} \dot{+} \mathcal{T}$ *is a closed subspace if and only if* $c_0(\mathcal{S}, \mathcal{T}) < 1$.

2 Reduced solutions of Douglas equations

Given a solution D of the equation $BX = C$ then, in many cases, it is useful to express D as a product $\tilde{B}C$ for a certain $\tilde{B}$. In such cases, the operator $\tilde{B}$ acts as a sort of inverse of B; in fact, $\tilde{B}BD = D$. In the next proposition we provide equivalent conditions to the existence of such factorization of the solution.

Proposition 2.1. *Let* $D \in L(\mathcal{G}, \mathcal{H})$ *be a solution of the equation* $BX = C$. *The following conditions are equivalent:*

1. $D = \tilde{B}C$ *for some linear operator* $\tilde{B} : \mathcal{D}(\tilde{B}) \to \mathcal{H}$ *with* $R(C) \subseteq \mathcal{D}(\tilde{B})$;
2. $R(D) \cap N(B) = \{0\}$;
3. $N(D) = N(C)$.

Proof. 1. $\to$ 2. Let $\xi \in R(D) \cap N(B)$. Then, $\xi = D\eta$ for some $\eta \in \mathcal{G}$ and so $C\eta = BD\eta = 0$. Hence, $\xi = D\eta = \tilde{B}C\eta = 0$.

2. $\to$ 3. Suppose that $R(D) \cap N(B) = \{0\}$ and let $\xi \in N(C)$. Then $BD\xi = C\xi = 0$, i.e., $D\xi \in R(D) \cap N(B) = \{0\}$. So $\xi \in N(D)$ and thus $N(C) \subseteq N(D)$. It is trivial that $N(D) \subseteq N(C)$. Therefore, $N(C) = N(D)$.

3. $\to$ 1. If $N(C) = N(D)$ then the operator $\tilde{B} : R(C) \to \mathcal{H}$ defined by $\tilde{B}(C\xi) = D\xi$ is well defined. It is clear that $\tilde{B}$ is linear and that $\tilde{B}C = D$. □

Corollary 2.2. *Every reduced solution $X_{\mathcal{M}}$ of the equation $BX = C$ can be written as $X_{\mathcal{M}} = \tilde{B}C$, for some linear operator $\tilde{B}$ with $R(C) \subseteq \mathcal{D}(\tilde{B})$.*

Proof. Since $R(X_{\mathcal{M}}) \cap N(B) \subseteq \mathcal{M} \cap N(B) = \{0\}$ then, by Proposition 2.1, the assertion follows. □

In the next result we describe the operator $\tilde{B}$ of Corollary 2.2. It is well known that the Douglas reduced solution of $BX = C$ is given by $B^{\dagger}C$ (see [13]). We prove that a similar factorization holds for reduced solutions when the Moore-Penrose inverse of B is replaced by a generalized inverse in $\mathcal{I}_g(B)$. Moreover, we characterize the reduced solutions by means of angles. As a consequence, we shall prove that reduced solutions are exactly the solutions which can be written as $\tilde{B}C$ for some linear operator $\tilde{B}$ with $R(C) \subseteq \mathcal{D}(\tilde{B})$ if some additional hypotheses regarding the dimensions of the spaces involved are included.

Theorem 2.3. *Let $B \in L(\mathcal{H}, \mathcal{K})$ and $C \in L(\mathcal{G}, \mathcal{K})$ be such that $R(C) \subseteq R(B)$. Hence, if $Y \in L(\mathcal{G}, \mathcal{H})$ is a solution of the equation $BX = C$ then the following conditions are equivalent:*

1. *Y is a reduced solution of $BX = C$;*
2. *$Y = Q_{\mathcal{M}//N(B)} X_{N(B)^{\perp}}$, where $\mathcal{M}$ is a topological complement of $N(B)$;*
3. *$Y = B'C$, where $B' \in \mathcal{I}_g(B)$;*
4. *$Y = B'C$, where $B' \in \mathcal{I}(B)$;*
5. *$c_0(\overline{R(Y)}, N(B)) < 1$.*

Proof. 1. $\to$ 2. Let Y be the reduced solution for $\mathcal{M}$ of the equation $BX = C$. Observe that since $\mathcal{M}$ is a closed subspace then $Q_{\mathcal{M}//N(B)}$ is bounded and so $Q_{\mathcal{M}//N(B)} X_{N(B)^{\perp}} \in L(\mathcal{G}, \mathcal{H})$. Furthermore, $B(Q_{\mathcal{M}//N(B)} X_{N(B)^{\perp}}) = BX_{N(B)^{\perp}} = C$ and $R(Q_{\mathcal{M}//N(B)} X_{N(B)^{\perp}}) \subseteq \mathcal{M}$. Thus, by the uniqueness of the reduced solution, we get $Y = Q_{\mathcal{M}//N(B)} X_{N(B)^{\perp}}$.

2. $\to$ 3. Observe that the operator $B' : R(B) + R(B)^{\perp} \to \mathcal{H}$ defined by $B' = Q_{\mathcal{M}//N(B)} B^{\dagger}$ belongs to $\mathcal{I}_g(B)$. Furthermore, $B'B = Q_{\mathcal{M}//N(B)}$. Hence, $Y = Q_{\mathcal{M}//N(B)} X_{N(B)^{\perp}} = B'BX_{N(B)^{\perp}} = B'C$

3. $\to$ 4. It is trivial, because if $\mathcal{I}_g(B) \subseteq \mathcal{I}(B)$.

4. $\to$ 5. Let $Y = B'C$ for some inner $B' \in \mathcal{I}(B)$. Then, $\overline{R(Y)} = \overline{R(B'C)} \subseteq \overline{R(B'B)} = \mathcal{M}$ for some closed subspace $\mathcal{M}$ such that $\mathcal{M} \dot{+} N(B) = \mathcal{H}$. Thus, $\overline{R(Y)} \subseteq \mathcal{M}$ and so $\overline{R(Y)} \cap N(B) = \{0\}$. Moreover, since $\mathcal{M} \dot{+} N(B) = \mathcal{H}$ is closed, we have that $c_0(\mathcal{M}, N(B)) < 1$. Hence, as $\overline{R(Y)} \subseteq \mathcal{M}$, $c_0(\overline{R(Y)}, N(B)) \leq c_0(\mathcal{M}, N(B)) < 1$.

5. $\to$ 1. If $c_0(\overline{R(Y)}, N(B)) < 1$ then, by Proposition 1.6, it holds that $\overline{R(Y)} \dot{+} N(B)$ is closed. Hence, there exists $\mathcal{M} = (\overline{R(Y)} \dot{+} N(B))^{\perp} \dot{+} \overline{R(Y)}$, which is closed since $(\overline{R(Y)} \dot{+} N(B))^{\perp} \subseteq R(Y)^{\perp}$, such that $R(Y) \subseteq \mathcal{M}$ and $\mathcal{M} \dot{+} N(B) = \mathcal{H}$. Therefore, Y is the reduced solution for $\mathcal{M}$. □

Corollary 2.4. *Let $B \in L(\mathcal{H}, \mathcal{K})$, $C \in L(\mathcal{G}, \mathcal{K})$ and $Y \in L(\mathcal{G}, \mathcal{H})$ be such that $BY = C$. If $\mathcal{H}$ has finite dimension then the following conditions are equivalent:*

1. Y is a reduced solution of the equation $BX = C$;

2. $Y = \tilde{B}C$ for some linear operator $\tilde{B}$ with $R(C) \subseteq \mathcal{D}(\tilde{B})$.

Proof. 1. $\rightarrow$ 2. It follows by Theorem 2.3.

2. $\rightarrow$ 1. If $Y = \tilde{B}C$ for some linear operator $\tilde{B}$ with $R(C) \subseteq \mathcal{D}(\tilde{B})$ then, by Proposition 2.1, $R(Y) \cap N(B) = \{0\}$. Furthermore, as $R(Y) \dotplus N(B)$ has finite dimension, then $R(Y) \dotplus N(B)$ is a closed subspace and, by Proposition 1.6 we get that $c_0(R(Y), N(B)) < 1$. Therefore, by Theorem 2.3, Y is a reduced solution of the equation $BX = C$. □

As the next example shows, Corollary 2.4 fails, in general, in the infinite dimensional case.

Example 2.5. Let $D \in L(\mathcal{H})$ be a non closed range operator and let $\xi \in \overline{R(D)} \setminus R(D)$. Define $B = P_{\text{span}\{\xi\}^\perp}$. Clearly, D is a solution of the equation $BX = BD$. Moreover, as $N(B) = \text{span}\{\xi\}$, then $R(D) \cap N(B) = \{0\}$ and so, by Proposition 2.1, $D = \tilde{B}BD$ for some linear operator $\tilde{B} : \mathcal{D}(\tilde{B}) \rightarrow \mathcal{H}$ with $R(D) \subseteq \mathcal{D}(\tilde{B})$. However, D is not a reduced solution of the equation $BX = BD$. Indeed, as $\overline{R(D)} \cap N(B) = N(B) \neq \{0\}$ then $c_0(\overline{R(D)}, N(B)) = 1$ and, by Theorem 2.3, D is not a reduced solution.

Remark 2.6. By Douglas theorem, the Douglas reduced solution has minimal norm. A similar property holds for every reduced solution. Let $X_\mathcal{M}$ be a reduced solution for $\mathcal{M}$ of the equation $BX = C$. If we consider the generalized inverse, $B' : R(B) \dotplus R(B)^\perp \subseteq \mathcal{K} \rightarrow \mathcal{M} \subseteq \mathcal{H}$ of B such that $B'|_{R(B)} = B_\mathcal{M}^{-1}$ and $N(B') = R(B)^\perp$, then $X_\mathcal{M} = B'C$. Now, let $A = Q^*Q + (I - Q^*)(I - Q)$ where $Q = Q_{\mathcal{M}//N(B)}$. Thus, $A \in L(\mathcal{H})^+$ is invertible and the functional $\langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle$ for every $\xi, \eta \in \mathcal{H}$ defines an inner product. Hence, it can be checked that $\mathcal{M}$ is the orthogonal complement of $N(B)$ with respect to $\langle\ ,\ \rangle_A$. Thus, if we consider $\mathcal{H}$ with the inner product $\langle\ ,\ \rangle_A$ then the generalized inverse B', defined above, is the Moore-Penrose inverse of B. Then, $X_\mathcal{M} = B'C$ is the Douglas reduced solution of the equation $BX = C$ and so $|||X_\mathcal{M}||| = \inf\{|||D||| : \ BD = C\}$ where $|||D||| = \sup\limits_{0 \neq \xi \in \mathcal{G}} \frac{\|A^{1/2}D\xi\|}{\|\xi\|}$. Note that $\langle\ ,\ \rangle_A$ and $\langle\ ,\ \rangle$ are equivalent because A is invertible.

2.1 Positive reduced solutions

This subsection is devoted to study positive reduced solutions of Douglas type equations. The next result due to Z. Sebestyén [15] provides an equivalent condition for the existence of a positive solution.

Theorem (Sebestyén). *Consider operators $B, C \in L(\mathcal{H}, \mathcal{K})$ such that the equation $BX = C$ has a solution. Hence, the equation admits a positive solution if and only if $CC^* \leq \lambda BC^*$ for some constant $\lambda \geq 0$.*

Even though Sebestyén'result characterize the Douglas equations which admits a positive solution, it does not provide an expression of them. A formula of positive solutions is given by A. Dajić and J. J. Koliha in [5], but under some extra hypotheses. More precisely, they proved the following result.

Theorem 2.7. *Consider operators $B, C \in L(\mathcal{H}, \mathcal{K})$ such that $R(B)$ and $R(BC^*)$ are closed. If the equation $BX = C$ has a positive solution then the general positive solution is given by*

$$X = C^*(BC^*)'C + (I - B'B)S(I - B'B)^*, \quad S \in L(\mathcal{H})^+,$$

where $(BC^*)'$ and B' are arbitrary inner inverses of BC^* and B, respectively. Moreover, $X_0 = C^*(BC^*)'C$ is a particular positive solution of $BX = C$, independent of the choice of the inner inverse $(BC^*)'$.

Here, we shall prove that the formula given by A. Dajić and J. J. Koliha still holds if the conditions $R(B)$ and $R(BC^*)$ closed are replaced by certain angle condition. In Remark 2.10, we show that our hypotheses are weaker than Dajić and Koliha's ones. Before that we present the following technical lemma.

Lemma 2.8. *Let $B \in L(\mathcal{H}, \mathcal{K})$. The general selfadjoint solution of the homogeneous equation $BX = 0$ is given by $Y = (I - B'B)Z(I - B'B)^*$, where $Z \in L(\mathcal{H})$ is selfadjoint and $B' \in \mathcal{I}(B)$.*

Proof. Let Y be a solution of $BX = 0$. Then, $Y = (I - B'B)Y$ for every $B' \in \mathcal{I}(B)$. If in addition $Y = Y^*$ then $Y = Y(I - B'B)^*$ and thus $Y = (I - B'B)Y(I - B'B)^*$.

Conversely, if $Z \in L(\mathcal{H})$ is selfadjoint then $Y = (I - B'B)Z(I - B'B)^*$ is a selfadjoint solution of $BX = 0$ for every $B' \in \mathcal{I}(B)$. □

Theorem 2.9. *Consider operators $B, C \in L(\mathcal{H}, \mathcal{K})$ such that the equation $BX = C$ has a positive solution. If $R(C) \subseteq R(BC^*)$ and $c_0(\overline{R(C^*)}, N(B)) < 1$ then $X_0 = C^*(BC^*)^\dagger C$ is a positive reduced solution of the equation $BX = C$. Furthermore, the general positive solution is given by*

$$Y = C^*(BC^*)^\dagger C + (I - B'B)S(I - B'B)^*, \tag{1}$$

where $S \in L(\mathcal{H})^+$ and $B' \in \mathcal{I}(B)$.

Proof. Note that $X_0 = C^*(BC^*)^\dagger C \in L(\mathcal{H})$ because $R(C) \subseteq R(BC^*)$. Moreover, since the equation $BX = C$ admits a positive solution then, by Sebestyén'result, $BC^* \in L(\mathcal{K})^+$ and so $(BC^*)^\dagger$ is positive[1]. Therefore $X_0 \in L(\mathcal{H})^+$. Now, $BX_0 = B(C^*(BC^*)^\dagger C) = P_{\overline{R(BC^*)}}|_{\mathcal{D}((BC^*)^\dagger)} C = C$. On the other hand, $\overline{R(X_0)} = \overline{R(C^*)}$. Indeed, since $BX_0 = C$ then $N(X_0) \subseteq N(C)$ which implies that $\overline{R(C^*)} \subseteq \overline{R(X_0)}$. The other inclusion follows from the definition of X_0. Then $c_0(\overline{R(X_0)}, N(B)) = c_0(\overline{R(C^*)}, N(B)) < 1$ and so, by Theorem 2.3, we assert that X_0 is a positive reduced solution.

The last part of the proof is devoted to obtain formula (1). Let Y be a positive solution of $BX = C$. Then $Y - X_0$ is a selfadjoint solution of $BX = 0$. Now, consider $\hat{B} \in \mathcal{I}(B)$ such that $\hat{B}C = X_0$; the existence of such $\hat{B}$ is guaranteed by Theorem 2.3. Therefore, by Lemma 2.8, we have

$$Y - X_0 = (I - \hat{B}B)Z(I - \hat{B}B)^*, \tag{2}$$

for some $Z \in L(\mathcal{H})$ selfadjoint. Then

$$\begin{aligned} (I - \hat{B}B)Y(I - \hat{B}B)^* &= (I - \hat{B}B)X_0(I - \hat{B}B)^* + (I - \hat{B}B)Z(I - \hat{B}B)^* \\ &= (I - \hat{B}B)Z(I - \hat{B}B)^*. \end{aligned}$$

Note that since Y is positive then $S = (I - \hat{B}B)Z(I - \hat{B}B)^* \in L(\mathcal{H})^+$. Now, let $B' \in \mathcal{I}(B)$. Then $(I - B'B)(I - \hat{B}B) = (I - \hat{B}B)$ and so, from (2), we get $Y = X_0 + (I - B'B)S(I - B'B)^*$.

Conversely, if $Y = X_0 + (I - B'B)S(I - B'B)^*$ with $S \in L(\mathcal{H})^+$ then Y is a positive solution of $BX = C$. □

Remark 2.10. Let us note that the hypotheses of Theorem 2.9 are weaker than A. Dajić and J. J. Koliha's hypotheses. In fact, suppose that A. Dajić and J. J. Koliha's hypotheses hold. Thus, since the equation $BX = C$ admits a positive solution then $CC^* \leq \lambda BC^*$ for some positive constant λ. Hence, by Douglas theorem, $R(C) \subseteq R((BC^*)^{1/2})$. Thus, if $R(BC^*)$ is closed then $R(C) \subseteq R((BC^*)^{1/2}) = R(BC^*)$ and so $\overline{R(C)} \subseteq R(BC^*) = R(CB^*) \subseteq R(C)$, i.e., $R(C)$ is closed. Furthermore, from $R(C) \subseteq R(BC^*)$ we get that $N(BC^*) \subseteq N(C^*)$ which implies $R(C^*) \cap N(B) =$

[1]Here, the positivity of $T^\dagger$ means $\left\langle T^\dagger \xi, \xi \right\rangle \geq 0$ for every $\xi \in \mathcal{D}(T^\dagger)$.

$\{0\}$. Finally, if $X_0 = C^*(BC^*)^\dagger C$ then $R(C^*) = R(X_0B^*) \subseteq R(X_0) \subseteq R(C^*)$, i.e., $R(C^*) = R(X_0)$. Now, $\overline{R(C^*)} + N(B) = R(X_0) + N(B) = B^{-1}(R(C))$ is closed and so, by Proposition 1.6, $c_0(\overline{R(C^*)}, N(B)) < 1$.

On the other hand, let $C \in L(\mathcal{H})^+$ with non closed range and $B = P_{\overline{R(C)}}$. Hence the equation $BX = C$ admits a positive solution, $R(C) \subseteq R(BC^*)$ and $c_0(\overline{R(C^*)}, N(B)) = 0 < 1$, but $R(BC^*) = R(C)$ is not closed.

3 Reduced projections

In this section we will study some special projections $Q \in \mathcal{Q}_\mathcal{S}$ which are orthogonal respect to the semi-inner product induced by $A \in L(\mathcal{H})^+$, namely,

$$\langle\ ,\ \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C},\ \langle \xi, \eta \rangle_A := \langle A\xi, \eta \rangle .$$

This functional is an inner product if and only if A is injective. Moreover, if A is invertible then $\langle\ ,\ \rangle_A$ and $\langle\ ,\ \rangle$ are equivalent in an obvious sense.

Given $T \in L(\mathcal{H})$, we say that T is **A-selfadjoint** (or **symmetrizable for** A) if $\langle T\xi, \eta \rangle_A = \langle \xi, T\eta \rangle_A$ for every $\xi, \eta \in \mathcal{H}$ or, which is equivalent, if $AT = T^*A$. Given a closed subspace $\mathcal{S}$, we denote by $\mathcal{P}(A, \mathcal{S})$ the set of A-selfadjoint projections with fixed range $\mathcal{S}$:

$$\mathcal{P}(A, \mathcal{S}) = \{Q \in \mathcal{Q}_\mathcal{S} :\ AQ = Q^*A\}.$$

If $\mathcal{P}(A, \mathcal{S})$ is not empty, then the pair $(A, \mathcal{S})$ is called **compatible**. The compatibility of a pair $(A, \mathcal{S})$ can be read in terms of range inclusions. For this, let us consider the matrix representation of operators in $L(\mathcal{H})$ induced by the decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$. Thus, every $Q \in \mathcal{Q}_\mathcal{S}$ is represented by

$$Q = \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix}, \tag{3}$$

where $y \in L(\mathcal{S}^\perp, \mathcal{S})$; and every $A \in L(\mathcal{H})^+$ is represented by

$$A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}, \tag{4}$$

where $a \in L(\mathcal{S})^+$ and $c \in L(\mathcal{S}^\perp)^+$. Therefore, the pair $(A, \mathcal{S})$ is compatible if and only if $R(b) \subseteq R(a)$ (see [2]). Moreover, $Q \in \mathcal{P}(A, \mathcal{S})$ if and only if $ay = b$. The main goal of this section is to characterize the elements $Q \in \mathcal{P}(A, \mathcal{S})$ such that

$$Q = \begin{pmatrix} 1 & x_\mathcal{M} \\ 0 & 0 \end{pmatrix}, \tag{5}$$

where $x_\mathcal{M}$ is a reduced solution of the equation $ax = b$. This sort of projections will be called **reduced projections**. Observe that, by Douglas theorem, if the pair $(A, \mathcal{S})$ is compatible then there always exists a reduced projection in $\mathcal{P}(A, \mathcal{S})$, namely,

$$P_{A,\mathcal{S}} = \begin{pmatrix} 1 & x_{N(a)^\perp} \\ 0 & 0 \end{pmatrix}.$$

Different properties of the element $P_{A,\mathcal{S}}$ have been studied by Corach et al. in [4] and [3]. Here, we describe the set of reduced projections by means of the results obtained in Section 2. For this, we start providing an equivalent condition for a projection $Q \in \mathcal{Q}_\mathcal{S}$ to be A-selfadjoint.

In what follows, given a subspace $\mathcal{S}$ of $\mathcal{H}$ we denote

$$S^{\perp_A} = \{\xi \in \mathcal{H} : \langle \xi, \eta \rangle_A = 0 \ \forall \, \eta \in \mathcal{S}\}.$$

The following identities hold: $\mathcal{S}^{\perp_A} = (A\mathcal{S})^\perp = A^{-1}(\mathcal{S}^\perp)$.

Proposition 3.1. *Let $(A, \mathcal{S})$ be a compatible pair and $Q \in \mathcal{Q}_\mathcal{S}$. The following conditions are equivalent:*

1. *$Q \in \mathcal{P}(A, \mathcal{S})$;*
2. *$N(Q) = \mathcal{S}^{\perp_A} \cap \mathcal{M}$ for some topological complement,$\mathcal{M}$, of $\mathcal{S} \cap N(A)$.*

Proof. 1→ 2. Let $Q \in \mathcal{P}(A, \mathcal{S})$ and $\mathcal{N} = \mathcal{S} \cap \mathcal{S}^{\perp_A} = \mathcal{S} \cap N(A)$. First, let see that $\mathcal{S}^{\perp_A} = \mathcal{N} \dot{+} N(Q)$. Observe that since $Q \in \mathcal{Q}_\mathcal{S}$, then $\mathcal{N} \cap N(Q) = (\mathcal{S} \cap N(Q)) \cap N(A) = \{0\}$. Furthermore, as $Q \in \mathcal{P}(A, \mathcal{S})$, then $N(Q) \subseteq \mathcal{S}^{\perp_A}$ and so $\mathcal{N} \dot{+} N(Q) \subseteq \mathcal{S}^{\perp_A}$. On the other hand, if $\xi = \xi_1 + \xi_2 \in \mathcal{S}^{\perp_A}$, where $\xi_1 \in \mathcal{S}$ and $\xi_2 \in N(Q)$ then $\xi - \xi_2 = \xi_1 \in \mathcal{S}^{\perp_A} \cap \mathcal{S} = \mathcal{N}$. Therefore $\mathcal{S}^{\perp_A} = \mathcal{N} \dot{+} N(Q)$. Now, define $\mathcal{M} = N(Q) \dot{+} (\mathcal{S}^{\perp_A})^\perp$ which is closed because $N(Q) \subseteq \mathcal{S}^{\perp_A}$. We claim that $\mathcal{M} \dot{+} \mathcal{N} = \mathcal{H}$. Indeed, $\mathcal{M} + \mathcal{N} = N(Q) \dot{+} (\mathcal{S}^{\perp_A})^\perp + \mathcal{N} = N(Q) \dot{+} \mathcal{N} \dot{+} (\mathcal{S}^{\perp_A})^\perp = \mathcal{S}^{\perp_A} \dot{+} (\mathcal{S}^{\perp_A})^\perp = \mathcal{H}$. Moreover, let $\eta = \eta_1 + \eta_2 \in \mathcal{M} \cap \mathcal{N}$, where $\eta_1 \in N(Q)$ and $\eta_2 \in (\mathcal{S}^{\perp_A})^\perp$. Now, $\eta - \eta_1 = \eta_2 \in \mathcal{S}^{\perp_A} \cap (\mathcal{S}^{\perp_A})^\perp = \{0\}$. Then $\eta = \eta_1 \in N(Q) \cap \mathcal{S} = \{0\}$. Hence, $\mathcal{M} \dot{+} \mathcal{N} = \mathcal{H}$. It only remains to show that $N(Q) = \mathcal{S}^{\perp_A} \cap \mathcal{M}$. It is straightforward that $N(Q) \subseteq \mathcal{S}^{\perp_A} \cap \mathcal{M}$. On the contrary, let $\mu = \mu_1 + \mu_2 \in \mathcal{S}^{\perp_A} \cap \mathcal{M}$, where $\mu_1 \in N(Q)$ and $\mu_2 \in (\mathcal{S}^{\perp_A})^\perp$. Then $\mu - \mu_1 = \mu_2 \in \mathcal{S}^{\perp_A} \cap (\mathcal{S}^{\perp_A})^\perp = \{0\}$. Therefore $\mu = \mu_1 \in N(Q)$ and so $N(Q) = \mathcal{S}^{\perp_A} \cap \mathcal{M}$.

2→ 1. As $N(Q) \subseteq \mathcal{S}^{\perp_A}$, if $\xi = \xi_1 + \xi_2$ and $\eta = \eta_1 + \eta_2$, where $\xi_1, \eta_1 \in \mathcal{S}$ and $\xi_2, \eta_2 \in N(Q)$ then $\langle Q\xi, \eta \rangle_A = \langle \xi_1, \eta_1 \rangle_A = \langle \xi, Q\eta \rangle_A$. Therefore Q is A-selfadjoint. □

Lemma 3.2. *Let $A \in L(\mathcal{H})^+$ with the matrix representation (4) and let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. If the pair $(A, \mathcal{S})$ is compatible then $N(a) = \mathcal{S} \cap N(A)$.*

Proof. Let $\xi \in N(a) \subseteq \mathcal{S}$. Then, as the pair $(A, \mathcal{S})$ is compatible and a is positive it holds $N(a) \subseteq N(b^*)$ and therefore $b^*\xi = 0$. Hence, $A\xi = a\xi + b^*\xi = 0$, i.e., $\xi \in \mathcal{S} \cap N(A)$. The other inclusion is clear. □

Lemma 3.3. *Let $Q_1, Q_2 \in \mathcal{Q}$. If $N(Q_1) \subseteq N(Q_2)$ and $R(Q_1) \subseteq R(Q_2)$ then $Q_1 = Q_2$.*

Proof. The assertion is consequence of the following equivalence: (i) $N(Q_1) \subseteq N(Q_2) \Leftrightarrow Q_2Q_1 = Q_2$ and (ii) $R(Q_1) \subseteq R(Q_2) \Leftrightarrow Q_2Q_1 = Q_1$. □

In the next theorem we characterize the reduced projections. Observe that in Proposition 3.1 we describe the nullspace of the elements of $\mathcal{P}(A, \mathcal{S})$ by means of a complement of $\mathcal{S} \cap N(A)$. Now, we shall give an additional condition on such complement in order to $Q \in \mathcal{P}(A, \mathcal{S})$ be a reduced projection.

Theorem 3.4. *Let $(A, \mathcal{S})$ be a compatible pair and $Q \in \mathcal{P}(A, \mathcal{S})$. Then the following conditions are equivalent*

1. *Q is a reduced projection;*
2. *$N(Q) = \mathcal{S}^{\perp_A} \cap \mathcal{M}$, where $\mathcal{M}$ is a topological complement of $\mathcal{S} \cap N(A)$ such that $c(\mathcal{M}, \mathcal{S}) = 0$;*
3. *$c_0(\overline{R(QP_{\mathcal{S}^\perp})}, \mathcal{S} \cap N(A)) < 1$.*

Proof. Let $Q = \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix} \in \mathcal{P}(A, \mathcal{S})$ and, for simplicity, let $\mathcal{N} = \mathcal{S} \cap N(A)$.
1. $\rightarrow$ 2. If y a reduced solution of the equation $ax = b$ then $R(y) \subseteq \mathcal{W}$ for some closed subspace $\mathcal{W}$ such that $\mathcal{W} \dot{+} N(a) = \mathcal{S}$. Define $\mathcal{M} = \mathcal{W} + \mathcal{S}^{\perp}$. Then, $\mathcal{M} + \mathcal{N} = \mathcal{M} + N(a) = \mathcal{W} + \mathcal{S}^{\perp} + N(a) = \mathcal{S} + \mathcal{S}^{\perp} = \mathcal{H}$. Furthermore, $\mathcal{M} \cap \mathcal{N} = \{0\}$. Indeed, if $\xi = \xi_1 + \xi_2 \in \mathcal{M} \cap \mathcal{N}$ with $\xi_1 \in \mathcal{S}^{\perp}$ and $\xi_2 \in \mathcal{W}$ then $\xi - \xi_2 = \xi_1 \in \mathcal{S} \cap \mathcal{S}^{\perp} = \{0\}$. Thus, $\xi_1 = 0$ and so $\xi = \xi_2 \in \mathcal{W} \cap N(a) = \{0\}$. Hence, $\xi = 0$ and so $\mathcal{M} \dot{+} \mathcal{N} = \mathcal{H}$. It is straightforward that $\mathcal{S} \dot{+} (\mathcal{S}^{\perp_A} \cap \mathcal{M}) = \mathcal{H}$. By Proposition 3.1 and Lemma 3.3, in order to prove that $Q = Q_{\mathcal{S}//\mathcal{S}^{\perp_A} \cap \mathcal{M}}$ we must see that $N(Q) \subseteq \mathcal{S}^{\perp_A} \cap \mathcal{M}$. Now, as $Q \in \mathcal{P}(A, \mathcal{S})$, then $N(Q) \subseteq \mathcal{S}^{\perp_A}$. Let us see that $N(Q) \subseteq \mathcal{M}$. For this, let $\xi = \xi_1 + \xi_2 \in N(Q)$ with $\xi_1 \in \mathcal{S}$ and $\xi_2 \in \mathcal{S}^{\perp}$. Then $Q\xi = \xi_1 + y\xi_2 = 0$. Hence, $\xi_1 = -y\xi_2 \in \mathcal{W}$. So, $\xi = \xi_1 + \xi_2 \in \mathcal{W} + \mathcal{S}^{\perp} = \mathcal{M}$ and then $N(Q) \subseteq \mathcal{M}$. It only remains to show that $c(\mathcal{M}, \mathcal{S}) = 0$ or, which is equivalent, that $\mathcal{M} = (\mathcal{M} \cap \mathcal{S}) \oplus (\mathcal{M} \cap \mathcal{S}^{\perp})$. Clearly, $(\mathcal{M} \cap \mathcal{S}) \oplus (\mathcal{M} \cap \mathcal{S}^{\perp}) \subseteq \mathcal{M}$. Now, let $\eta = \eta_1 + \eta_2 \in \mathcal{M}$, where $\eta_1 \in \mathcal{W}$ and $\eta_2 \in \mathcal{S}^{\perp}$. Then $\eta - \eta_1 = \eta_2 \in \mathcal{M} \cap \mathcal{S}^{\perp}$ and $\eta - \eta_2 = \eta_1 \in \mathcal{M} \cap \mathcal{S}$.
2. $\rightarrow$ 1. It is sufficient to show that there exists a closed subspace $\mathcal{W}$ such that $\mathcal{W} \dot{+} N(a) = \mathcal{S}$ and $R(y) \subseteq \mathcal{W}$. Define $\mathcal{W} = \mathcal{M} \cap \mathcal{S}$. Then $\mathcal{W} + N(a) \subseteq \mathcal{S}$. Now, let $\xi = \xi_1 + \xi_2 \in \mathcal{S}$, where $\xi_1 \in \mathcal{M}$ and $\xi_2 \in \mathcal{N} = N(a)$. Then $\xi - \xi_2 = \xi_1 \in \mathcal{M} \cap \mathcal{S} = \mathcal{W}$ and so $\xi = \xi_1 + \xi_2 \in \mathcal{W} + N(a)$. Furthermore, $\mathcal{W} \cap N(a) = \mathcal{W} \cap \mathcal{N} = \mathcal{M} \cap \mathcal{N} = \{0\}$. Thus $\mathcal{W} \dot{+} N(a) = \mathcal{S}$. Now, given $\xi \in \mathcal{S}^{\perp}$, it holds $-y\xi + \xi \in N(Q) \subseteq \mathcal{M}$. Since $c(\mathcal{M}, \mathcal{S}) = 0$ then, by Proposition 1.6, $\mathcal{M} = (\mathcal{M} \cap \mathcal{S}) \oplus (\mathcal{M} \cap \mathcal{S}^{\perp})$ and so $-y\xi + \xi = \kappa_1 + \kappa_2$, where $\kappa_1 \in \mathcal{M} \cap \mathcal{S}$ and $\kappa_2 \in \mathcal{M} \cap \mathcal{S}^{\perp}$. Therefore $-y\xi = \kappa_1$ and $\xi = \kappa_2$ and so $R(y) \subseteq \mathcal{M} \cap \mathcal{S} = \mathcal{W}$.
1 $\leftrightarrow$ 3 It is consequence of Theorem 2.3. □

Remark 3.5. If the pair $(A, \mathcal{S})$ is compatible and $Q \in \mathcal{P}(A, \mathcal{S})$ then Q is the reduced projection $P_{A,\mathcal{S}}$ if and only if any of the following conditions hold:

i. $N(Q) = \mathcal{S}^{\perp_A} \cap (\mathcal{S} \cap N(A))^{\perp}$;

ii. $c_0(\overline{R(QP_{\mathcal{S}^{\perp}})}, \mathcal{S} \cap N(A)) = 0$.

The compatibility of a pair is related to the existence of solutions of certain Douglas type equations. For example, in [3], Proposition 4.2, it is proved that the pair $(A, \mathcal{S})$ is compatible if and only if the equation $A^{1/2}P_{\mathcal{S}}X = P_{\overline{A^{1/2}(\mathcal{S})}}A^{1/2}$ has a solution. As a consequence, in the next result we show that if the pair $(A, \mathcal{S})$ is compatible then the equation $A^{1/2}X = P_{\overline{A^{1/2}(\mathcal{S})}}A^{1/2}$ has a solution. In that case, the reduced solutions are A-selfadjoint projections. Moreover, if $\mathcal{S} \cap N(A) = \{0\}$ then $P_{A,\mathcal{S}}$ is a reduced solution of such equation.

Proposition 3.6. *Let $A \in L(\mathcal{H})^+$ and $(A, \mathcal{S})$ be a compatible pair. Then the following conditions hold:*

1. *The equation $A^{1/2}X = P_{\overline{A^{1/2}\mathcal{S}}}A^{1/2}$ has a solution;*

2. *If $X_{\mathcal{M}}$ is a reduced solution of the equation $A^{1/2}X = P_{\overline{A^{1/2}\mathcal{S}}}A^{1/2}$ then $X_{\mathcal{M}}$ is an A-selfadjoint projection with $N(X_{\mathcal{M}}) = \mathcal{S}^{\perp_A}$;*

3. *If $\mathcal{S} \cap N(A) = \{0\}$ then there exists a closed subspace $\mathcal{M}$ of $\mathcal{H}$ such that $\mathcal{S} \subseteq \mathcal{M}$ and $\mathcal{M} \dot{+} N(A) = \mathcal{H}$. In that case, $P_{A,\mathcal{S}}$ is the reduced solution for $\mathcal{M}$ of the equation $A^{1/2}X = P_{\overline{A^{1/2}\mathcal{S}}}A^{1/2}$. In other words, $P_{A,\mathcal{S}} = (A^{1/2})'P_{\overline{A^{1/2}\mathcal{S}}}A^{1/2}$ where $(A^{1/2})'$ is the generalized inverse for $\mathcal{M}$ of $A^{1/2}$.*

Proof. 1. If the pair $(A, \mathcal{S})$ is compatible then the equation $A^{1/2}P_{\mathcal{S}}X = P_{\overline{A^{1/2}\mathcal{S}}}A^{1/2}$ has a solution (see [3], Proposition 4.2). Hence, $R(P_{\overline{A^{1/2}\mathcal{S}}}A^{1/2}) \subseteq R(A^{1/2}P_{\mathcal{S}}) \subseteq R(A^{1/2})$. Thus, $A^{1/2}X = P_{\overline{A^{1/2}\mathcal{S}}}A^{1/2}$ has a solution.
2. Let D be a reduced solution for $\mathcal{M}$ of the equation $A^{1/2}X = P_{\overline{A^{1/2}\mathcal{S}}}A^{1/2}$. Hence, $A^{1/2}D^2 = A^{1/2}DD = P_{\overline{A^{1/2}\mathcal{S}}}A^{1/2}D = P^2_{\overline{A^{1/2}\mathcal{S}}}A^{1/2} = P_{\overline{A^{1/2}\mathcal{S}}}A^{1/2}$. Hence both, D and D^2 are reduced solution for $\mathcal{M}$ of the equation $A^{1/2}X = P_{\overline{A^{1/2}\mathcal{S}}}A^{1/2}$ and so, $D = D^2$. Furthermore, $AD = A^{1/2}P_{\overline{A^{1/2}\mathcal{S}}}A^{1/2}$ is selfadjoint. Thus, D is an A-selfadjoint projection. Finally, since $N(D) = N(P_{\overline{A^{1/2}\mathcal{S}}}A^{1/2})$, it is sufficient to show that $N(P_{\overline{A^{1/2}\mathcal{S}}}A^{1/2}) = \mathcal{S}^{\perp_A} = (A\mathcal{S})^{\perp}$. Now, since for every $\eta \in \mathcal{S}$ and $\xi \in \mathcal{H}$ it holds $\langle \xi, A\eta \rangle = \left\langle P_{\overline{A^{1/2}\mathcal{S}}}A^{1/2}\xi, A^{1/2}\eta \right\rangle$, then the assertion follows.
3. Let $\mathcal{M} = \mathcal{S} + (\mathcal{S} \dot{+} N(A))^{\perp}$. Since $(\mathcal{S} \dot{+} N(A))^{\perp} \subseteq \mathcal{S}^{\perp}$, then $\mathcal{M}$ is closed. Moreover, since $(A, \mathcal{S})$ is compatible then $\mathcal{S} + N(A)$ is a closed subspace of $\mathcal{H}$ (see [2], Theorem 6.2) and so $\mathcal{M} \dot{+} N(A) = \mathcal{H}$. Now, if $X_{\mathcal{M}}$ denotes the reduced solution for $\mathcal{M}$ of the equation $A^{1/2}X = P_{\overline{A^{1/2}\mathcal{S}}}A^{1/2}$ then, by item 2., $X_{\mathcal{M}}$ is an A-selfadjoint projection. Furthermore, $\mathcal{S} \subseteq R(X_{\mathcal{M}})$. In fact, if $\eta \in \mathcal{S}$ then $P_{\overline{A^{1/2}\mathcal{S}}}A^{1/2}\eta = A^{1/2}\eta = A^{1/2}X_{\mathcal{M}}\eta$. Thus, $\eta - X_{\mathcal{M}}\eta \in \mathcal{M} \cap N(A^{1/2}) = \{0\}$. Then, $\eta = X_{\mathcal{M}}\eta \in R(X_{\mathcal{M}})$. Since $\mathcal{S} \cap N(A) = \{0\}$, by item 2. it follows that $N(X_{\mathcal{M}}) = N(P_{A,\mathcal{S}})$. Therefore, $X_{\mathcal{M}}$ is an A-selfadjoint projection with $R(P_{A,\mathcal{S}}) = \mathcal{S} \subseteq R(\mathcal{M})$ and $N(X_{\mathcal{M}}) = N(P_{A,\mathcal{S}})$. So, by Lemma 3.3, $X_{\mathcal{M}} = P_{A,\mathcal{S}}$. □

Acknowledgment

The authors thank Professor A. Maestripieri for her useful comments which lead to the equivalence between angles and reduced solutions.

References

[1] M. L. Arias, G. Corach, M. C. Gonzalez, Generalized inverses and Douglas equations, Proc. Amer. Math. Soc. 136 (2008), 3177-3183.

[2] G. Corach, A. Maestripieri, D. Stojanoff, Oblique projections and Schur complements, Acta Sci. Math. 67 (2001), 439-459.

[3] G. Corach, A. Maestripieri, D. Stojanoff, Oblique projections and abstract splines, Journal of Approximation Theory, 117 (2002), 189-206.

[4] G. Corach, A. Maestripieri and D. Stojanoff, A classification of projectors, Topological algebras, their applications and related topics, Banach Center Publications 67, Polish Acad. Sci., Warsaw (2005), 145–160.

[5] A. Dajic, J. Koliha, Positive solutions to the equations $AX = C$ and $XB = D$ for Hilbert space operators, J. Math. Anal. Appl. 333 (2007), 567-576.

[6] F. Deutsch, The angles between subspaces in Hilbert space, “Approximation Theory, Wavelets and applications”, (S. P. Singh, ed.), Kluwer, Netherlands 1995, 107-130.

[7] F. Deutsch, Best Approximation in Inner Product Spaces, Springer, New York, 2001.

[8] R. G. Douglas, On majorization, factorization and range inclusion of operators in Hilbert space, Proc. Amer. Math. Soc. 17 (1966), 413-416.

[9] H. W. Engl, M. Z. Nashed, New extremal characterizations of generalized inverses of linear operators, J. Math. Anal. Appl. 82 (1981), 566-586.

[10] P. A. Fillmore, J. P. Williams, On operator ranges, Adv. Math. 7 (1971) 254-281.

[11] T. Kato, Perturbation theory of linear operators (Second edition), Springer, New York, 1966.

[12] M. Z. Nashed, Generalized Inverses and Applications, Academic Press, New York, 1976.

[13] M. Z. Nashed, On generalized inverses and operator ranges, Funct. Anal. and Approx. Birkhäuser, Basel (1981), 85-96.

[14] M. Z. Nashed, Inner, outer, and generalized inverses in Banach and Hilbert spaces, Numer. Funct. Anal. Optim. 9 (1987), 261-325.

[15] Z. Sebestyén, Restrictions of positive operators, Acta Sci. Math. (Szeged) 46 (1983), 299-301.

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