

INFINITELY MANY SOLUTIONS FOR NON-LOCAL PROBLEMS WITH BROKEN SYMMETRY

ROSSELLA BARTOLO, PABLO L. DE NÁPOLI, AND ADDOLORATA SALVATORE

ABSTRACT. The aim of this paper is investigating the existence of solutions of the non-local elliptic problem

$$(P_h^s) \quad \begin{cases} (-\Delta)^s u = |u|^{p-2}u + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $s \in (0, 1)$, $n > 2s$, Ω is an open bounded domain of \mathbb{R}^n with Lipschitz boundary $\partial\Omega$, $(-\Delta)^s$ is the non-local Laplacian operator, $2 < p < 2_s^*$ and $h \in L^2(\Omega)$. This problem requires the study of the eigenvalue problem related to the fractional Laplace operator, with or without potential.

1. INTRODUCTION

In the last decade, starting with the works [9, 10, 30], PDE models involving operators whose paradigm is the so called fractional Laplacian have been widely studied. This kind of operators acts by a global integration with respect to a very singular kernel and not by pointwise differentiation. Needless to say, the interest in these operators comes from very different fields: for instance, in Probability the fractional Laplace operators of the form $(-\Delta)^s$, $s \in (0, 1)$, are infinitesimal generators of a stable Lévy process (cf., e.g., [35]); on the other hand the famous Signorini problem gives a motivation from Mechanics and in [11, 21] one can find further applications in Fluid Mechanics (cf. also [17] for a list of other applications).

In this work we deal with the existence of multiple solutions of inhomogeneous superlinear boundary value problems involving the fractional Laplacian operator. Namely, we deal with the non-local elliptic problem

$$(P_h^s) \quad \begin{cases} (-\Delta)^s u = |u|^{p-2}u + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $s \in (0, 1)$, $n > 2s$, Ω is an open bounded domain of \mathbb{R}^n with Lipschitz boundary $\partial\Omega$, $(-\Delta)^s$ is the non-local Laplacian operator, i.e., for a suitable $C(n, s) > 0$

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(see for instance [17]),

$$(-\Delta)^s u(x) := C(n, s) \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy \quad \text{on } \mathbb{R}^n,$$

$2 < p < 2_s^*$, with $2_s^* = \frac{2n}{n-2s}$ fractional critical exponent, and $h \in L^2(\Omega)$. By a solution of (P_h^s) we mean a function in the fractional Sobolev space $H^s(\Omega)$ (cf. Section 2) which vanishes almost everywhere outside Ω and satisfies the first equation of (P_h^s) in the weak sense.

If $h \equiv 0$, i.e. the problem is symmetric, the existence of infinitely many solutions has been proved in [24, Theorem 1.1] by equivariant variational methods, by a proof similar to that performed in [27, Theorem 9.38] for the local case.

We recall that the local case corresponds to $s = 1$ and the problem corresponding to (P_h^s) is

$$(P_h) \quad \begin{cases} -\Delta u = |u|^{p-2}u + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

therefore, due to the presence of the term h , the problem is not symmetric. Problem (P_h) has been firstly studied using some methods devised by [2, 3, 26, 31], commonly referred to as the ‘‘perturbation from symmetry’’ ones. Later on, an improvement of these results was obtained in [34]. More recently, problem (P_h) has been studied in [6, 13] also for non-zero boundary condition.

As far as we know, in the non-local setting the only contribution related to our kind of problem is contained in [28] (cf. also [25, Chapter 11 - Part II]). Due to some non trivial technical difficulties, we limit here ourselves to the case $u = 0$ on $\partial\Omega$, i.e., to the case in which the non-homogeneity originates only in the equation.

More precisely we can state our main result, the non-local counterpart of [34, Theorem 1] with $u = 0$ on $\mathbb{R}^n \setminus \Omega$.

Theorem 1.1. *Let $s \in (0, 1)$ with $n > 2s$ and Ω be an open bounded domain of \mathbb{R}^n with Lipschitz boundary $\partial\Omega$. Then, problem (P_h^s) has infinitely many solutions for any $h \in L^2(\Omega)$ provided that*

$$2 < p < \frac{2n - 2s}{n - 2s}. \quad (1.1)$$

Remark 1.2. For the sake of simplicity, in Theorem 1.1 we deal with the model nonlinearity $g(s) = |s|^{p-2}s$, nevertheless our techniques work for any continuous $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following widely used set of assumptions:

- there exist $\mu > 2$ and $\rho \geq 0$ such that

$$(x, s) \in \Omega \times \mathbb{R}, |s| \geq \rho \Rightarrow 0 < \mu G(x, s) \leq sg(x, s)$$

with $G(x, s) = \int_0^s g(x, t) dt$;

- there exist $p \in]2, 2_s^*[$, $c > 0$ such that

$$|g(x, s)| \leq c(|s|^{p-1} + 1) \quad \text{for all } (x, s) \in \Omega \times \mathbb{R};$$

- $g(x, -s) = -g(x, s)$ for all $(x, s) \in \Omega \times \mathbb{R}$.

Remark 1.3. Also in the local case, without additional assumptions (such as radial symmetry, cf., e.g., [12] and [32]), it is still an open problem whether there exist infinitely many solutions for p up to $2^* = \frac{2n}{n-2}$: in fact, the result in [34, Theorem 1] holds for $p \in \left(2, \frac{2n-2}{n-2}\right)$, hence in Theorem 1.1 we have a full extension of this result to the non-local case.

The paper is organized as follows: in Section 2 we recall some basic facts about the fractional Laplacian and fractional Sobolev spaces. In Section 3 we obtain bounds for eigenvalue problems with the fractional Laplacian. In Section 4 we recall Bolle's method. Finally, in Section 5 we prove Theorem 1.1.

2. PRELIMINARIES ABOUT THE FRACTIONAL LAPLACIAN

It is important to point out that different notions of fractional Laplacian in domains $\Omega \subset \mathbb{R}^n$, with different interpretations of the Dirichlet boundary conditions, have been defined in the literature. Let us highlight the main differences between them.

In this paper we consider equations involving the standard fractional Laplacian on \mathbb{R}^n and require that the solutions vanish outside Ω . In this case the associated *Dirichlet form* is

$$\mathcal{E}_{\mathbb{R}^n}(u, v) = C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y)) \cdot (v(x) - v(y))}{|x - y|^{n+2s}} dx dy. \quad (2.1)$$

In probabilistic terms, the associated stochastic process is the standard symmetric α -stable Lévy process (with $\alpha = 2s$), killed upon living Ω . This operator has been also used for instance in [23], even in a more general form, called there the fractional p -Laplacian.

A second option is to consider the so called *regional fractional Laplacian*, which corresponds to the Dirichlet form

$$\mathcal{E}_{\Omega}(u, v) = C(n, s) \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y)) \cdot (v(x) - v(y))}{|x - y|^{n+2s}} dx dy.$$

The associated stochastic process is the censored stable process, for which jumps outside Ω are completely forbidden (cf., e.g., [4]).

Another approach is to consider the fractional powers of the Laplacian in Ω with Dirichlet conditions, defined from the spectral decomposition. The corresponding operator is called *the spectral fractional Laplacian* and coincides with the one obtained from the Caffarelli-Silvestre extension on a cylinder based in Ω (see [7]). The associated stochastic process is the subordinate killed Brownian motion studied in [33].

Next we recall some basic definitions. Let us take $s \in (0, 1)$ and a measurable function $u : \mathbb{R}^n \rightarrow \mathbb{R}$. Recalling that the Gagliardo seminorm is defined by

$$[u]_{s,2} = (\mathcal{E}_{\mathbb{R}^n}(u, u))^{1/2} = \left(C(n, s) \int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{\frac{1}{2}},$$

Then, the fractional Sobolev space is defined as

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : [u]_{s,2} < +\infty\}$$

and it is equipped with the norm

$$\|u\|_{s,2} = (|u|_2^2 + [u]_{s,2}^2)^{\frac{1}{2}}, \quad (2.2)$$

where $|\cdot|_2$ denotes the norm on $L^2(\mathbb{R}^n)$.

A fundamental tool is the following fractional Sobolev inequality (cf., e.g., [17, Theorem 6.5]).

Theorem 2.1 (Fractional Sobolev inequality). *Let $s \in (0, 1)$ be such that $n > 2s$. Then there exists a positive constant $K = K(n, s)$ such that, for any measurable and compactly supported function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, it results that*

$$\|f\|_{L^{2_s^*}(\mathbb{R}^n)}^2 \leq K \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x - y|^{n+2s}} dx dy.$$

Consequently, the space $H^s(\mathbb{R}^n)$ is continuously embedded in $L^p(\mathbb{R}^n)$ for any $p \in [2, 2_s^*]$.

Our problem is set in the space

$$H_0^s(\Omega) = \{u \in H^s(\mathbb{R}^n) : u(x) = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\} \quad (2.3)$$

which is a closed subspace of $H^s(\mathbb{R}^n)$. Since we are working in a bounded Lipschitz domain Ω , $C_0^\infty(\Omega)$ is dense in $H_0^s(\Omega)$, i.e., we could alternatively define $H_0^s(\Omega)$ as the completion of $C_0^\infty(\Omega)$ (cf., e.g., [8, Lemma 2.3]).

Moreover, since Ω is bounded, the following *Poincaré inequality* holds

$$\|u\|_{L^2} \leq C [u]_{s,2} \text{ for all } u \in C_0^\infty(\Omega)$$

with $C = C(n, s, p, \Omega)$ (cf. [8, Lemma 2.4]).

As a consequence we can use in $H_0^s(\Omega)$ the equivalent norm

$$\|\cdot\| = [\cdot]_{s,2}.$$

Moreover, $H_0^s(\Omega)$ then becomes a Hilbert space, with the associated inner product given by the Dirichlet form (2.1).

We remark that the following embeddings hold:

$$H_0^s(\Omega) \hookrightarrow L^\mu(\Omega) \text{ continuously for } \mu \in [1, 2_s^*]$$

(as a consequence of Theorem 2.1) and

$$H_0^s(\Omega) \hookrightarrow\hookrightarrow L^\mu(\Omega) \text{ compactly for } \mu \in [1, 2_s^*[\quad (2.4)$$

(cf. [17, Theorem 7.1]).

3. ESTIMATES FOR EIGENVALUES OF THE FRACTIONAL LAPLACIAN WITH WEIGHTS

In this section we prove some estimates for eigenvalues of equations involving the fractional Laplacian in domains; we need such estimates for the proof of our main result, though we believe they have an interest of their own. Indeed, eigenvalues of the fractional Laplacian in domains have recently received some attention in literature. For instance, Z.Q. Chen and R. Song used in [14] probabilistic methods to obtain two-sided estimates for the eigenvalues of the fractional Laplacian in domains, while B. Dyda, A. Kuznetsov and M. Kwanicki explicitly computed in [16] the eigenvalues in a ball. Moreover, we refer to [20] and references therein for some Weyl-type laws for p -fractional eigenvalue problems.

We obtain new estimates for eigenvalues of equations involving the fractional Laplacian with weights by adapting the technique of P. Li and S.T. Yau in [22]. Indeed the following theorem generalizes [22, Theorem 2].

Theorem 3.1. *Let $s \in (0, 1)$, $n > 2s$, Ω be an open bounded domain of \mathbb{R}^n with Lipschitz boundary $\partial\Omega$ and $q(x)$ a positive function in $L^r(\Omega)$, with $r > \frac{n}{2s}$. Moreover, let us denote by μ_k the k^{th} eigenvalue for the problem*

$$\begin{cases} (-\Delta)^s \phi(x) = \mu q(x)\phi(x) & \text{in } \Omega, \\ \phi = 0 & \text{on } \mathbb{R}^n \setminus \Omega. \end{cases} \quad (3.1)$$

Then there exists a positive constant $\bar{C} = \bar{C}(n, s)$ such that for all $k \in \mathbb{N}$

$$\mu_k^{n/2s} \int_{\Omega} q^{\frac{n}{2s}}(x) \, dx \geq \bar{C}k.$$

Proof. The arguments in [1, Subsection 1.3], joined with the compact embeddings in (2.4), guarantee the existence of a sequence $(\mu_k)_k$ of eigenvalues for problem (3.1). Let $(\varphi_k)_k$ be an orthogonal basis of eigenfunctions of $(-\Delta)^s$ in $L^2(\Omega, q(x)dx)$ with corresponding eigenvalues $(\mu_k)_k$. For every $k \in \mathbb{N}$, the first equation of (3.1) in the weak sense is

$$C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\varphi_k(x) - \varphi_k(y))(\psi(x) - \psi(y))}{|x - y|^{n+2s}} \, dx \, dy = \mu_k \int_{\Omega} q(x) \varphi_k(x) \psi(x) \, dx \quad (3.2)$$

for all test function $\psi \in H_0^s(\Omega)$. Let us consider (cf., e.g., [19, Propositions 6.4 and 6.5]) the *heat kernel*

$$H(x, y, t) = \sum_{i=1}^{+\infty} e^{-\mu_i t} \varphi_i(x) \varphi_i(y),$$

which is positive by the maximum principle in [30, Proposition 2.17], and the function

$$h(t) = \sum_{i=1}^{+\infty} e^{-2\mu_i t}.$$

Namely, fixing $(y, t) \in \Omega \times \mathbb{R}$, $H(\cdot, y, t)$ is a weak solution of

$$\frac{\partial H}{\partial t}(x, y, t) = -\frac{(-\Delta)_x^s}{q(x)} H(x, y, t),$$

i.e.,

$$\begin{aligned} & C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(H(x, y, t) - H(z, y, t))(\psi(x) - \psi(z))}{|x - z|^{n+2s}} \, dx \, dz \\ &= \int_{\Omega} q(x) \frac{\partial H}{\partial t}(x, y, t) \psi(x) \, dx, \end{aligned}$$

for all $\psi \in H_0^s(\Omega)$: in fact,

$$\begin{aligned} & \int_{\Omega} q(x) \frac{\partial H}{\partial t}(x, y, t) \psi(x) \, dx = - \int_{\Omega} q(x) \sum_{i=1}^{+\infty} \mu_i e^{-\mu_i t} \varphi_i(x) \varphi_i(y) \psi(x) \, dx \\ &= - \sum_{i=1}^{+\infty} \mu_i e^{-\mu_i t} \varphi_i(y) \int_{\Omega} q(x) \varphi_i(x) \psi(x) \, dx \\ &= -C(n, s) \sum_{i=1}^{+\infty} e^{-\mu_i t} \varphi_i(y) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\varphi_i(x) - \varphi_i(z))(\psi(x) - \psi(z))}{|x - z|^{n+2s}} \, dx \, dz \\ &= -C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(H(x, y, t) - H(z, y, t))(\psi(x) - \psi(z))}{|x - z|^{n+2s}} \, dx \, dz. \end{aligned}$$

We have that

$$\begin{aligned}
& \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} H^2(x, y, t) q(x) q(y) \, dx \, dy \\
& \int_{\Omega} \int_{\Omega} \left(\sum_{i=1}^{+\infty} e^{-\mu_i t} \varphi_i(x) \varphi_i(y) \right) \left(\sum_{j=1}^{+\infty} e^{-\mu_j t} \varphi_j(x) \varphi_j(y) \right) q(x) q(y) \, dx \, dy \\
& = \sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} e^{-\mu_i t} e^{-\mu_j t} \left(\int_{\Omega} \int_{\Omega} \varphi_i(x) \varphi_i(y) \varphi_j(x) \varphi_j(y) q(x) q(y) \, dx \, dy \right) \\
& = \sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} e^{-\mu_i t} e^{-\mu_j t} \delta_{ij} = h(t).
\end{aligned} \tag{3.3}$$

Therefore

$$\begin{aligned}
h'(t) &= \int_{\Omega} \int_{\Omega} 2H(x, y, t) \frac{\partial H}{\partial t}(x, y, t) q(x) q(y) \, dx \, dy \\
&= \int_{\Omega} \int_{\Omega} 2H(x, y, t) \left[- \sum_{i=1}^{+\infty} \mu_i e^{-\mu_i t} \varphi_i(x) \varphi_i(y) \right] q(x) q(y) \, dx \, dy \\
&= -2 \sum_{i=1}^{+\infty} \mu_i e^{-\mu_i t} \int_{\Omega} q(x) \varphi_i(x) \left[\int_{\Omega} \varphi_i(y) q(y) H(x, y, t) \, dy \right] \, dx.
\end{aligned}$$

Using $\psi = H(x, \cdot, t)$ as test function in (3.2), we obtain that

$$\begin{aligned}
h'(t) &= -2C(n, s) \cdot \\
& \sum_{i=1}^{+\infty} e^{-\mu_i t} \int_{\Omega} q(x) \varphi_i(x) \left[\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\varphi_i(y) - \varphi_i(z))(H(x, y, t) - H(x, z, t))}{|y - z|^{n+2s}} \, dy \, dz \right] \, dx \\
&= -2C(n, s) \cdot \\
& \int_{\Omega} q(x) \sum_{i=1}^{+\infty} e^{-\mu_i t} \varphi_i(x) \left[\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\varphi_i(y) - \varphi_i(z))(H(x, y, t) - H(x, z, t))}{|y - z|^{n+2s}} \, dy \, dz \right] \, dx \\
&= -2C(n, s) \cdot \\
& \int_{\Omega} q(x) \left[\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(H(x, y, t) - H(x, z, t))(H(x, y, t) - H(x, z, t))}{|y - z|^{n+2s}} \, dy \, dz \right] \, dx \\
&= -2C(n, s) \int_{\Omega} q(x) \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|H(x, y, t) - H(x, z, t)|^2}{|y - z|^{n+2s}} \, dy \, dz \right) \, dx.
\end{aligned}$$

Using the fractional Sobolev inequality in Theorem 2.1, it follows that

$$- h'(t) \geq \tilde{C}(n, s) \int_{\Omega} q(x) \left[\int_{\Omega} H^{\frac{2n}{n-2s}}(x, y, t) \, dy \right]^{\frac{n-2s}{n}} \, dx. \tag{3.4}$$

On the other hand, by (3.3) and using the Hölder inequality we have that

$$\begin{aligned}
h(t) &= \int_{\Omega} q(x) \left(\int_{\Omega} H^2(x, y, t) q(y) \, dy \right) \, dx \\
&\leq \int_{\Omega} q(x) \left(\int_{\Omega} H^{\alpha p}(x, y, t) \, dy \right)^{1/p} \left(\int_{\Omega} H^{\beta p'}(x, y, t) q^{p'}(y) \, dy \right)^{1/p'} \, dx
\end{aligned} \tag{3.5}$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad \alpha + \beta = 2.$$

It must be $\alpha p = 2_s^*$ and $\beta p' = 1$. Hence,

$$p = \frac{2n}{n-2s} - 1 = \frac{n+2s}{n-2s} \Rightarrow p' = \frac{n+2s}{4s}.$$

By (3.5) it follows that

$$h(t) \leq \int_{\Omega} q(x) \left(\int_{\Omega} H^{\frac{2n}{n-2s}}(x, y, t) dy \right)^{\frac{n-2s}{n+2s}} \left(\int_{\Omega} H(x, y, t) q^{\frac{n+2s}{4s}}(y) dy \right)^{\frac{4s}{n+2s}} dx.$$

By using again the Hölder inequality with exponents $m = \frac{n+2s}{n}$ and $m' = \frac{n+2s}{2s}$, we get that

$$h(t) \leq \left(\int_{\Omega} q(x) \left(\int_{\Omega} H^{\frac{2n}{n-2s}}(x, y, t) dy \right)^{\frac{n-2s}{n}} dx \right)^{\frac{n}{n+2s}} \left(\int_{\Omega} q(x) Q^2(x, t) dx \right)^{\frac{2s}{n+2s}} \quad (3.6)$$

where

$$Q(x, t) = \int_{\Omega} H(x, y, t) q^{\frac{n+2s}{4s}}(y) dy.$$

We claim that function Q satisfies the *heat equation*

$$\frac{\partial Q}{\partial t}(x, t) = -\frac{1}{q(x)} (-\Delta)_x^s Q(x, t)$$

in weak sense, i.e., for any test function $\psi \in H_0^s(\Omega)$:

$$\int_{\Omega} \frac{\partial Q}{\partial t}(x, t) \psi(x) q(x) dx = C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(Q(x) - Q(y))(\psi(x) - \psi(y))}{|x - y|^{n+2s}} dx dy$$

and is such that

$$Q(x, 0) = q^{\frac{n-2s}{4s}}(x),$$

since H is the fundamental solution of the heat equation. Therefore

$$\begin{aligned} \frac{\partial Q}{\partial t}(x, t) &= \int_{\Omega} \frac{\partial H}{\partial t}(x, y, t) q^{\frac{n+2s}{4s}}(y) dy \\ &= - \int_{\Omega} \frac{(-\Delta)_x^s}{q(x)} H(x, y, t) q^{\frac{n+2s}{4s}}(y) dy = - \frac{(-\Delta)_x^s}{q(x)} \int_{\Omega} H(x, y, t) q^{\frac{n+2s}{4s}}(y) dy \end{aligned}$$

and the claim follows.

Hence, we have that

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} Q^2(x, t) q(x) dx &= 2 \int_{\Omega} Q(x, t) \frac{\partial Q}{\partial t}(x, t) q(x) dx \\ &= -2C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|Q(x, t) - Q(z, t)|^2}{|x - z|^{n+2s}} dx dz \leq 0. \end{aligned}$$

This implies that

$$\int_{\Omega} Q^2(x, t) q(x) dx \leq \int_{\Omega} Q^2(x, 0) q(x) dx = \int_{\Omega} q^{\frac{n}{2s}}(x) dx.$$

Therefore by (3.6) we get that

$$h(t) \leq \left[\int_{\Omega} q(x) \left(\int_{\Omega} H^{\frac{2n}{n-2s}}(x, y, t) \, dy \right)^{\frac{n-2s}{n}} \, dx \right]^{\frac{n}{n+2s}} \left[\int_{\Omega} q^{\frac{n}{2s}}(x) \, dx \right]^{\frac{2s}{n+2s}}$$

and plainly

$$h^{\frac{n+2s}{n}}(t) \left(\int_{\Omega} q^{\frac{n}{2s}}(x) \, dx \right)^{-2s/n} \leq \int_{\Omega} q(x) \left(\int_{\Omega} H^{\frac{2n}{n-2s}}(x, y, t) \, dy \right)^{\frac{n-2s}{n}} \, dx.$$

By (3.4) we get that

$$h^{\frac{n+2s}{n}}(t) \left(\int_{\Omega} q^{\frac{n}{2s}}(x) \, dx \right)^{-2s/n} \leq -\frac{1}{\tilde{C}(n, s)} h'(t),$$

that is

$$h'(t) \leq -\tilde{C}(n, s) h^{\frac{n+2s}{n}}(t) \left(\int_{\Omega} q^{\frac{n}{2s}}(x) \, dx \right)^{-2s/n}.$$

Dividing by $h^{\frac{n+2s}{n}}(t)$ it results that

$$h^{-\frac{n+2s}{n}}(t) h'(t) \leq -\tilde{C}(n, s) \left(\int_{\Omega} q^{\frac{n}{2s}}(x) \, dx \right)^{-2s/n},$$

that is

$$\frac{n}{2s} \frac{d}{dt} \left(h^{-\frac{2s}{n}}(t) \right) \geq \tilde{C}(n, s) \left(\int_{\Omega} q^{\frac{n}{2s}}(x) \, dx \right)^{-2s/n}.$$

Now $h^{-\frac{2s}{n}}(t) \rightarrow 0$ as $t \rightarrow 0$, since $h(t) \rightarrow +\infty$. Hence, integrating in $(0, t)$, we get that

$$\frac{n}{2s} h(t)^{-\frac{2s}{n}} \geq \tilde{C}(n, s) t \left(\int_{\Omega} q^{\frac{n}{2s}}(x) \, dx \right)^{-2s/n}.$$

Finally,

$$h(t) \leq \left(\frac{2s}{n} \tilde{C}(n, s) \right)^{-\frac{n}{2s}} \left(\int_{\Omega} q^{\frac{n}{2s}}(x) \, dx \right) t^{-n/2s}.$$

By the definition of h ,

$$\sum_{i=1}^{+\infty} e^{-2\mu_i t} \leq \left(\frac{2s}{n} \tilde{C}(n, s) \right)^{-\frac{n}{2s}} \left(\int_{\Omega} q^{\frac{n}{2s}}(x) \, dx \right) t^{-n/2s}.$$

Setting $t = \frac{n}{4\mu_k}$, we get

$$\left(\frac{2s}{n} \tilde{C}(n, s) \right)^{-\frac{n}{2s}} \left(\int_{\Omega} q^{\frac{n}{2s}}(x) \, dx \right) \left(\frac{n}{4\mu_k} \right)^{-n/2s} \geq \sum_{i=1}^{+\infty} e^{-\frac{n\mu_i}{2\mu_k}} \geq k e^{-n/2}$$

and the proof is complete. \square

Remark 3.2. We conjecture that Theorem 3.1 holds for a larger class of non-local operators (like those considered in [18] for $p = 2$): in fact, only some structural properties are needed along the computations.

Now, given a potential $V \in L^{\frac{n}{2s}}(\Omega)$, let us consider the eigenvalue problem:

$$\begin{cases} (-\Delta)^s u + V(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega \end{cases}$$

and denote by $\mathcal{N}_-^s((-\Delta)^s + V)$ the number of its non-positive eigenvalues and by $V_-(x) := \min\{V(x), 0\}$. Then, by using Theorem 3.1, we can state the analogous of [22, Corollary 2] for the non-local case.

Corollary 3.3. *There exists a constant $\underline{C} = \underline{C}(n, s)$ such that*

$$\mathcal{N}_-^s((-\Delta)^s + V) \leq \underline{C} |V_-|_{\frac{n}{2s}}. \quad (3.7)$$

Proof. The proof is essentially the same of [22, Corollary 2]: indeed, (i) and (ii) of their proof work also in our case, therefore it is enough to consider only non-positive eigenvalues and to work with a strictly negative potential, step (iii) is not needed since we work on bounded domains of \mathbb{R}^n and moreover the number $\mathcal{N}_-^s((-\Delta)^s + V)$ is equal to the number of eigenvalues less than 1 for (3.1) with $q(x) = -V(x)$ (cf. formula (24) in [22]). Then, denoting by μ_k the greatest eigenvalue less than or equal to 1, by Theorem 3.1 it follows that

$$|V_-|_{\frac{n}{2s}} \geq \mu_k^{n/2s} |V_-|_{\frac{n}{2s}} \geq \bar{C} k \geq \bar{C} \mathcal{N}_-^s((-\Delta)^s + V).$$

□

4. BOLLE'S PERTURBATION METHOD

In order to apply the method devised by Bolle in [5] for dealing with problems with broken symmetry, let us recall the main theorem stated in [6] and generalized in [15] by considering C^1 instead of C^2 functionals. The key point is to handle a continuous path of functionals $(I_\theta)_{\theta \in [0,1]}$ starting at a symmetric functional I_0 (which corresponds in our case to $h = 0$) and ending at the non-even functional I_1 associated to the problem.

Let X be a Banach space equipped with the norm $\|\cdot\|$, $(X', \|\cdot\|_{X'})$ its dual and $I : [0, 1] \times X \rightarrow \mathbb{R}$ be a C^1 functional; let us set $I_\theta = I(\theta, \cdot) : X \rightarrow \mathbb{R}$ and $I'_\theta(\cdot) = \frac{\partial I}{\partial v}(\theta, \cdot)$ for all $\theta \in [0, 1]$.

Assume that

(A₁) I satisfies a variant of the Palais-Smale condition: each sequence $((\theta_n, v_n))_n$ in $[0, 1] \times X$ such that

$$(I(\theta_n, v_n))_n \text{ is bounded} \quad \text{and} \quad \lim_{n \rightarrow +\infty} \|I'_{\theta_n}(v_n)\| = 0$$

converges up to subsequences;

(A₂) for all $b > 0$ there exists $C_b > 0$ such that, if $(\theta, v) \in [0, 1] \times X$, then

$$|I_\theta(v)| \leq b \implies \left| \frac{\partial I}{\partial \theta}(\theta, v) \right| \leq C_b (\|I'_\theta(v)\|_{X'} + 1)(\|v\| + 1);$$

(A₃) there exist two continuous maps $\eta_1, \eta_2 : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$, with $\eta_1(\theta, \cdot) \leq \eta_2(\theta, \cdot)$ for all $\theta \in [0, 1]$, which are Lipschitz continuous with respect to the second variable and such that, if $(\theta, v) \in [0, 1] \times X$, then

$$I'_\theta(v) = 0 \implies \eta_1(\theta, I_\theta(v)) \leq \frac{\partial I}{\partial \theta}(\theta, v) \leq \eta_2(\theta, I_\theta(v)); \quad (4.1)$$

(A₄) I_0 is even and for each finite dimensional subspace V of X it results

$$\lim_{\substack{v \in V \\ \|v\| \rightarrow +\infty}} \sup_{\theta \in [0,1]} I_\theta(v) = -\infty .$$

For $i \in \{1, 2\}$, let $\psi_i : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be the solution of the problem

$$\begin{cases} \frac{\partial \psi_i}{\partial \theta}(\theta, s) = \eta_i(\theta, \psi_i(\theta, s)) \\ \psi_i(0, s) = s. \end{cases}$$

Note that for $i \in \{1, 2\}$ $\psi_i(\theta, \cdot)$ is continuous, non-decreasing on \mathbb{R} and $\psi_1(\theta, \cdot) \leq \psi_2(\theta, \cdot)$. Set

$$\bar{\eta}_1(s) = \max_{\theta \in [0,1]} |\eta_1(\theta, s)|, \quad \bar{\eta}_2(s) = \max_{\theta \in [0,1]} |\eta_2(\theta, s)| .$$

Now let us consider a sequence $(e_k)_k \subset X$ of linearly independent unitary vectors and set $E_k = \text{span}\{e_1, \dots, e_k\}$; moreover let us define

$$\Gamma = \{\gamma \in C(X, X) : \gamma \text{ is odd and } \exists R > 0 \text{ s.t. } \gamma(v) = v \text{ if } \|v\| \geq R\}$$

and

$$c_k = \inf_{\gamma \in \Gamma} \sup_{v \in E_k} I_0(\gamma(v)) \quad (4.2)$$

($c_k \leq c_{k+1}$ for all $k \geq 1$).

In this framework, the following abstract result can be proved, cf. [5, Theorem 3] and [6, Theorem 2.2].

Theorem 4.1. *Assume that the C^2 path of functionals $I : [0, 1] \times X \rightarrow \mathbb{R}$ satisfies (A₁) – (A₄) above. Then there exists $\bar{K} > 0$ such that for all $k \in \mathbb{N}$*

- (a) *either I_1 has a critical level \tilde{c}_k with $\psi_2(1, c_k) < \psi_1(1, c_{k+1}) \leq \tilde{c}_k$,*
- (b) *or $c_{k+1} - c_k \leq \bar{K} (\bar{\eta}_1(c_{k+1}) + \bar{\eta}_2(c_k) + 1)$.*

5. PROOF OF THEOREM 1.1

Let us consider, as in Theorem 1.1, $s \in (0, 1)$, $n > 2s$, Ω open bounded domain of \mathbb{R}^n with Lipschitz boundary and $h \in L^2(\Omega)$.

The weak solutions of (P_h^s) are the critical points of the functional

$$J_1(v) = \frac{1}{2} \|v\|^2 - \frac{1}{p} |v|_p^p - \int_{\Omega} hv \, dx$$

defined on the space $H_0^s(\Omega)$ introduced in (2.3).

Thus, according to the Bolle's perturbation method, we define the path of functionals $J : [0, 1] \times H_0^s(\Omega) \rightarrow \mathbb{R}$ as follows:

$$J(\theta, v) = \frac{1}{2} \|v\|^2 - \frac{1}{p} |v|_p^p - \theta \int_{\Omega} hv \, dx.$$

For simplicity we set $J_\theta = J(\theta, \cdot)$, hence the weak solutions of (P_h^s) are the critical points of $J(1, \cdot) = J_1$.

Next we check that, with this choice of the path, the assumptions in Theorem 4.1 are satisfied.

For what concerns the Palais–Smale type condition in assumption (A₁), we refer to [24, Lemmas 3.2 and 3.3]. Moreover, since for all $(\theta, v) \in [0, 1] \times H_0^s(\Omega)$ it results

$$\frac{\partial J}{\partial \theta}(\theta, v) = - \int_{\Omega} hv \, dx, \text{ also (A}_2\text{) holds.}$$

We prove that J satisfies assumption (A_3) : in fact, the following lemma holds.

Lemma 5.1. *If $(\theta, v) \in [0, 1] \times H_0^s(\Omega)$ is such that $J'_\theta(v) = 0$, then the inequality (4.1) holds with $\eta_1, \eta_2 : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ defined as*

$$-\eta_1(\theta, s) = \eta_2(\theta, s) = D (s^2 + 1)^{\frac{1}{2p}}$$

for a suitable constant $D > 0$.

Proof. If $v \in H_0^s(\Omega)$ is a critical point of J_θ , it results that

$$\|v\|^2 - |v|_p^p - \theta \int_{\Omega} hv \, dx = 0.$$

Hence

$$J_\theta(v) = \left(\frac{1}{2} - \frac{1}{p}\right) |v|_p^p + \left(\frac{1}{2} - 1\right) \theta \int_{\Omega} hv \, dx,$$

by which it follows that, for a suitable $D_1 > 0$,

$$|v|_p^p \leq D_1 (J_\theta(v) + 1). \quad (5.1)$$

By (5.1) we get for some $D_2 > 0$,

$$\left| \frac{\partial J}{\partial \theta}(\theta, v) \right| \leq D_2 (J_\theta^2(v) + 1)^{\frac{1}{2p}},$$

thus the lemma is proved. \square

Finally, let us remark that the functional J_0 is even on $H_0^s(\Omega)$, hence by the following lemma, also property (A_4) holds.

Lemma 5.2. *Let V be a finite dimensional subspace of $H_0^s(\Omega)$. Then*

$$\lim_{\substack{v \in V \\ \|v\| \rightarrow +\infty}} \sup_{\theta \in [0, 1]} J(\theta, v) = -\infty.$$

Proof. For all $(\theta, v) \in [0, 1] \times H_0^s(\Omega)$ it results that

$$J_\theta(v) \leq \frac{1}{2} \|v\|^2 - \frac{1}{p} |v|_p^p + |h|_2 |v|_2,$$

then the thesis follows, since $p > 2$ and on finite dimensional spaces all norms are equivalent. \square

Now we are ready to prove our main result.

Proof of Theorem 1.1. As the path of functionals J satisfies $(A_1) - (A_4)$, we can apply Theorem 4.1. Let us denote by $(\lambda_k)_k$ the sequence of the eigenvalues of $(-\Delta)^s$ on $H_0^s(\Omega)$ and by $(e_k)_k$ the corresponding sequence of eigenfunctions. According to [29, Section 3], the main features of the spectrum of the fractional Laplace operator are very closed to that of classical one. Then, we set

$$E_1 = \mathbb{R}e_1, \quad E_{k+1} = E_k \oplus \mathbb{R}e_{k+1} \quad \text{for } k \geq 1$$

and define a corresponding sequence of mini-max levels as

$$c_k = \inf_{\gamma \in \Gamma} \sup_{v \in E_k} J_0(\gamma(v))$$

where Γ is as in Section 4 with $X = H_0^s(\Omega)$. Next we have to estimate the growth of c_k 's. We claim that condition (b) in Theorem 4.1 can not hold for all k large enough. In fact, taking η_1, η_2 as in Lemma 5.1, condition (b) in Theorem 4.1 becomes

$$c_{k+1} - c_k \leq \underline{K} \left((c_k)^{\frac{1}{p}} + (c_{k+1})^{\frac{1}{p}} + 1 \right), \quad (5.2)$$

for a suitable $\underline{K} > 0$; hence, if (5.2) holds for all k large enough, then by [2, Lemma 5.3] it follows that there exist $L > 0$ and $\bar{k} \in \mathbb{N}$ such that

$$c_k \leq L k^{\frac{p}{p-1}} \quad \text{for all } k \geq \bar{k}. \quad (5.3)$$

On the other hand, [34, Theorem B] implies that for all $k \in \mathbb{N}$ there exists $v_k \in H_0^s(\Omega)$ such that

$$J_0'(v_k) = 0 \quad \text{and} \quad J(v_k) \leq c_k, \quad (5.4)$$

with large Morse index greater or equal than k , i.e., the operator

$$J_0''(v_k) = (-\Delta)^s - (p-1)|v_k|^{p-2}$$

has at least k non-positive eigenvalues. Therefore, by Corollary 3.3 with $V(x) = -(p-1)|v_k|^{p-2}$ we infer that for a suitable $C'_{n,s} > 0$

$$k \leq \mathcal{N}_-(J_0''(v_k)) \leq C'_{n,s} |v_k|_{(p-2)^{\frac{2s}{s}}}. \quad (5.4)$$

In particular, (5.4) implies $J_0'(v_k)[v_k] = 0$, then

$$\|v_k\|^2 = |v_k|_p^p;$$

hence for a suitable $C''_{n,s} > 0$ it follows that

$$c_k \geq C''_{n,s} k^{\frac{2sp}{n(p-2)}} \quad \forall k \geq k_0,$$

and by (5.3) this yields to a contradiction under assumption (1.1). Therefore condition (a) in Theorem 4.1 holds for infinitely many $k \in \mathbb{N}$. □

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ROSSELLA BARTOLO
DIPARTIMENTO DI MECCANICA, MATEMATICA E MANAGEMENT
POLITECNICO DI BARI
VIA E. ORABONA 4, 70125 BARI
ITALY
E-mail address: `rossella.bartolo@poliba.it`

PABLO L. DE NÁPOLI
IMAS (UBA-CONICET) AND DEPARTAMENTO DE MATEMÁTICA,
FACULTAD DE CIENCIAS EXACTAS Y NATURALES, UNIVERSIDAD DE BUENOS AIRES,
CIUDAD UNIVERSITARIA, 1428 BUENOS AIRES
ARGENTINA
E-mail address: `pdenapo@dm.uba.ar`

ADDOLORATA SALVATORE
DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DEGLI STUDI DI BARI “ALDO MORO”
VIA ORABONA 4, 70125 BARI
ITALY
E-mail address: `addolorata.salvatore@uniba.it`