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INFLUENCERS AND COMMUNITIES IN SOCIAL NETWORKS

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Influencers and communities in social networks (Job market paper)

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Abstract

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Keywords: social media, network, community, opinion mining, natural language processing

1 Introduction

Financial and social networks are often analyzed through *vector autoregression* model, for instance, in Zhu et al. (2017). Consider a network that produces a time series $Y_t \in \mathbb{R}^N$,

*The order of the authors is alphabetical.

$t = 1, \dots, T$ and dependencies between its elements are modeled through the equation

$$Y_t = \Theta Y_{t-1} + W_t, \tag{1.1}$$

where W_t are innovations that satisfy $\mathsf{E}[W_t | \mathcal{F}_{t-1}] = 0$, $\mathcal{F}_t = \sigma\{Y_{t-1}, Y_{t-2}, \dots\}$, so that the interactions between the nodes are described by an autoregression operator $\Theta \in \mathbb{R}^{N \times N}$. In terms of the network connections we say that a node i is connected tothe node j if

$$\Theta_{ij} \neq 0,$$

so that the adjacency matrix of such network is represented by nonzero coefficients and the sparsity of Θ represents the number of edges. For large-scale time series one encounters the curse of dimension, as estimating the matrix-parameter Θ with N^2 elements requires significantly large number of observations T.

Several attempts to reduce the dimensionality have been made in the past literature. Assuming that the elements of a time series form a connected network, Zhu et al. (2017) introduces a Network Autoregression model (NAR) with $\Theta_{ij} = \beta A_{ij} / \sum_{k=1}^{N} A_{ik}$, provided that the adjacency matrix $A \in \mathbb{R}^{N \times N}$ is known. Here, the regression operator, defined up to a single parameter β, which is called the *network effect*, can be estimated through simple least squares. Zhu et al. (2019) also extend this model for conditional quantiles. Furthermore, Zhu and Pan (2018) argue that a single network parameter may not be satisfactory as it treats all nodes of the network homogeneously. In particular, the NAR model implies that each node is affected by its neighbours in the same extent, while in reality we may have financial institutions that are affected less or more than the others (see Mihoci et al. (2019)). Then they propose to detect communities in the network based on the given adjacency matrix and suggest that the nodes in each community share a separate network effect parameter. A somewhat opposite direction is taken by Gudmundsson and Brownlees (2018): their BlockBuster algorithm determines the communities through the estimated autoregressive model, which, however, does not solve the dimensionality problem. Apart from this line of work, sparse regularisations have been extensively used, see Fan et al. (2009); Han et al. (2015); Melnyk and Banerjee (2016).

To sum up we want to address the following problems, which one encounters dealing with vector autoregression:

- the VAR parameter dimension is particularly large, one requires even larger time intervals for consistent estimation. Even if one can afford such a dataset, in the long run autoregressive parametric models tend to be violated, see e.g. Čížek et al. (2009). Naturally, we want to impose some structural assumptions on the operator Θ, so that it can be estimated by means of moderate sample sizes.

- The NAR model assumes that the adjacency matrix is known. In particular, this is justified for social networks with a natural friendship/follower-followee relationship, provided that it is stable. For a network of financial institutions, there is no explicitly defined adjacency matrix and one has to heuristically evaluate it using additional information (identical shareholders, trading volumes, etc.) or through analyzing correlations and lagged cross-correlations between returns or risk profiles, see Diebold and Yılmaz (2014) and Chen et al. (2019b). However, there is no rigorous reason to believe that the operator in (1.1) depends explicitly on such adjacency matrix, see also Cha et al. (2010).

Motivated by two important facts that one can observe in social networks we propose SONIC — *SOcial Network analysis with Influencers and Communities*. Based on well-known user experience on platforms like facebook, twitter, etc., one can assume that there are some alpha users that are followed significantly more than others. Take, for example, celebrities, sportsmen, politicians, or instagram divas. In a network view these users are nodes that have much more influence than the rest of the nodes: these nodes are *influencers*. In the framework of autoregression, a node j is an influencer if there is a significant amount of other nodes i such that $\Theta_{ij} \neq 0$. Assuming that the number of influencers is limited, we fix only a few columns of matrix Θ to be important. This allows us to take into account only the connections to the influencers, significantly reducing the number of parameters to be estimated. A similar idea is used in Chen et al. (2018), with a group-LASSO regularisation imposed, so that they find a solution with few active columns. Notice, however, that relying on the sparsity alone still requires $T > N$, see e.g. Fan et al. (2009); Chernozhukov et al. (2018).

It is also well-known that social networks have smaller communities, with the nodes exhibiting higher connection density or similar behaviour inside communities. Zhu and Pan (2018) makes one step to extend the NAR model from Zhu et al. (2017) into a more realistic set-up by allowing separate parameters for each community, instead of a single network effect parameter. In our notation the conditional mean of the response of the node i satisfies

$$\mathsf{E}[Y_{it}|\, \mathcal{F}_{t-1}] = \Theta_{i1} Y_{it-1} + \cdots + \Theta_{iN} Y_{Nt-1}.$$

Therefore, the behaviour of the node i is characterized by the coefficients $\Theta_{i1}, \ldots, \Theta_{iN}$, i.e. the nodes it depends on and to what extent. We assume that the nodes are separated into few clusters such that the nodes from the same cluster have the same dependencies. This brings a bigger picture into the view: instead of saying that two nodes from the same cluster are more likely to be connected, we say that they are connected to the same influencers.

Our main focus is application to the sentiment extracted from a microblogging platform dedicated to stock trading, StockTwits (available at `https://stocktwits.com`.) For each user one can extract average sentiment score over the messages he posts during the day. Analyzing the resulting high-dimensional time series, on one hand, we are able to identify influencers — the users whose opinion is overwhelmingly important, and on the other hand, we determine the community structure. One serious problem arises here: the presence of missing observations because on some days some users do not leave any messages. We treat this as follows: assume there is an underlying opinion process that follows autoregressive equation (1.1). During each day the user may express his opinion by posting one or a few messages. In such case we can observe his opinion, otherwise the default value 0 is assigned to the observation. This results in a popular model for missing observations that involves masked Bernoulli random variables. We return to it in detail in Section 3.3.

The rest of the paper is organized as follows. Section 2 introduces the reader to StockTwits 3.2 platform, describes in detail the available dataset and the process of sentiment weights extraction. In Section 3 we first introduce our SONIC model, then describe the estimation procedure and provide a consistency result. In Section 4 we provide simulation results that partially the theoretical properties of our estimator. Next, in Section 5 we present and discuss the results of application of our model to some datasets extracted from the StockTwits. Section 6 is dedicated to the proofs, as well as Sections A, B in the appendix.

2 StockTwits

Social media are an ideal platform where users can easily communicate with each other, exchange information and share opinions. The increasing popularity of social media is a clear evidence of such demand for exchanging opinions and information among granular users in a cyber world. Among social media platforms, we are particularly interested in StockTwits for a number of reasons. Firstly, it becomes predominantly popular and stands for a leading social network for investors and traders. Secondly, it is similar to Twitter but dedicated to financial discussion. One of the features that lead to its popularity is a well-designed reference between the message content and the referring stock symbols. Conversations are organized around 'cashtags' (e.g. '$AAPL' for APPLE; '$BTC.X' for BITCOIN) that allow to narrow down streams on specific assets. Thirdly and most importantly, users can also express their sentiment/opinions by labeling their messages as 'Bearish' (negative) or 'Bullish' (positive) *via* a toggle button. These are so-called *self-report sentiment*. Indeed, the user-generated messages and self reported sentiment attract the researchers for sentiment analysis. The available labeled data constitutes an advance on textual analysis that typically relies on the available training dataset. We use this convention and StockTwits Application Programming Interface (API) to download all messages containing the preferred cashtags. StockTwits API also provides for each message its unique user identifier, the time it was posted within one-second precision, and the sentiment associated by the user ('Bullish', 'Bearish' or unclassified).

Among over thousand tickers/symbols, we particularly pick up two selective symbols, $AAPL for APPLE; $BTC.X for BITCOIN, which represents the most popular security and cryptocurrency, respectively. Due to the fact that they attract investors/users with very distinct risk preference, we conjecture that the resulting opinion network and its dynamics may exhibit diverse structures. In Table 1 we summarize the messages' statistics with respect to AAPL and BTC. Even though we exclusively consider these two symbols, the message volume and number of users associated with these two symbols are tremendous. A glimpse on Table 1 shows different profiles between two symbols. Firstly, the users interested in BTC tend to disclose their sentiment, evident by 44% of labeled messages, while in AAPL only 28% of messages are labeled. It may lead to a better training accuracy in the case of BTC messages relative to the training model based on AAPL. Secondly, there is a clear imbalance between the numbers of positive and negative messages, showing that online investors are optimistic on average, as previously found by Kim and Kim (2014) and Avery et al. (2016). It seems that the imbalance is more evident in the case of AAPL. Judging by the reported average message volume per day, there is no doubt that AAPL is able to attract more attention of potential investors than BTC.

2.1 Quantifying message content

Conversion of text data into a quantitative sentiment variable can be done by two techniques, namely dictionary-based and machine learning-based analysis. Although a machine learning technique has many advantages compared to a dictionary-based approach, the latter offers better transparency, explication and less computational burden. Loughran and McDonald (2016) recommend that alternative complex methods should be

Symbols	AAPL	BTC
message volume	449,761	644,597
number of distinct users	26,521	25,492
number of bullish messages	133,316	196,555
number of bearish messages	48,186	90,677
percentage of bullish messages	20.6%	30.4%
percentage of bearish messages	7.4%	14.0%
percentage of labeled messages	28.0%	44.4%
size of positive training dataset	99,985	147,759
size of negative training dataset	36,100	67,752
message volume per day	730	305
number of positive terms in lexicon	4,000	3,775
number of negative terms in lexicon	4,000	3,759
sample period	2017-05-22	2013-03-21
	2019-01-27	2018-12-27

Table 1: Summary statistics of social media messages

considered only when they add substantive value beyond simpler and more transparent approaches such as bag-of word. We therefore opt for the lexicon approach in the task of sentiment quantification.

A dictionary, or lexicon, is a list of words labeled as positive, negative, or neutral. Given such a list, the classic *bag-of-words* approach consists of counting the number of positive and negative words in a document in order to assign it a sentiment value or a tone. For example, a simple dictionary containing only the words 'good' and 'bad' with positive and negative labels, respectively, would classify the sentence 'Bitcoin is a good investment' as positive with a tone +1. As the literature suggests (examples, please), the simplicity of the dictionary-based approach guarantees transparency and replicability provided, on the cons side, it comes with limitations associated with natural language analysis. First, referring in Deng et al. (2017) to the 'context of a discourse', one needs to be aware of the content domain, to which language interpretation is sensitive. For example, Loughran and McDonald (2011) point out that words like 'tax' or 'cost' are classified as negative by Harvard General Inquirer lexicon, whereas they should be considered neutral in financial context. Another example is about quantifying sentiment toward cryptocurrency, playing the role of non-standard assets and embracing new technologies as part of their characteristics. Chen et al. (2019a) point out that in many domain-specific terms, such as 'blockchain', 'ICO', 'hackers', 'wallet', 'shitcoin' and 'binance', 'hodl', are not covered in the existing financial and psychological dictionaries. They construct a novel cryptocurrency lexicon in response to the need of adopting a specific approach to measure sentiment about cryptocurrencies. The second limitation is the one of the language domain, which Deng et al. (2017) define as the 'lexical and syntactical choices of language'. One example would be the difference between newspapers where a formal and standardized tone is mostly used, and social media, where slang and emojis prevail (Loughran and McDonald, 2016). As shown by Chen et al. (2019a), online investors also use new 'emojis' such as 🚀 (positive) and 💩 (negative) when talking about cryptocurrencies. Obviously, these are not in the traditional dictionary.

To balance the complexity and transparency and also take into account the domain-specific terms in social media while applying lexicon approach, in the sentiment quantification for the messages of AAPL we employ the social media lexicon developed by Renault (2017) while in the quantification of BTC messages we advocate the lexicon tailored for cryptocurrency asset by Chen et al. (2019a). Renault (2017) demonstrates that his constructed lexicon significantly outperforms the benchmark dictionaries (Loughran and McDonald, 2016) used in the literature while remaining competitive with more complex machine learning algorithms. On the basis of 125,000 bullish and another 125,000 bearish messages published on StockTwits, using the lexicon for social media achieves 90% of classified messages, and 75.24% of correct classifications. With a collection of 1,533,975 messages from 38,812 distinct users, posted between March 2013 and December 2018, and related to 465 cryptocurrencies listed in StockTwits[1], Chen et al. (2019a) documents that implementing the crypto lexicon is able to classify 83% of messages, with 86% of them being correctly classified.[2]

The natural language processing (NLP) is a prerequisite for implementing textual analysis. Following Sprenger et al. (2014) and Renault (2017) we convert unstructured text into clean and manageable textual content as the grounding base throughout the textual analysis. First, all messages are lowercased. To account for lengthening of words, which has been shown to be a critical feature of sentiment expression on microblogs (Brody and Diakopoulos, 2011), but avoid noise in the lexicon, sequences of repeated letters are shrink to a maximum length of 3. Tickers ('$BTC.X', '$LTC.X'...), dollar or euro values, hyperlinks, numbers and mentions of users are respectively replaced by the words 'cashtag', 'moneytag', 'linktag', 'numbertag' and 'usertag'. The prefix "negtag_" is added to any word consecutive to 'not', 'no', 'none', 'neither', 'never' or 'nobody'. Finally, the three stopwords 'the', 'a', 'an' and all punctuation except the characters '?' and '!' are removed. Exclamation and interrogation marks are kept as it has been previously shown that they are often part of significant bigrams that improve lexicon accuracy (Renault, 2017).

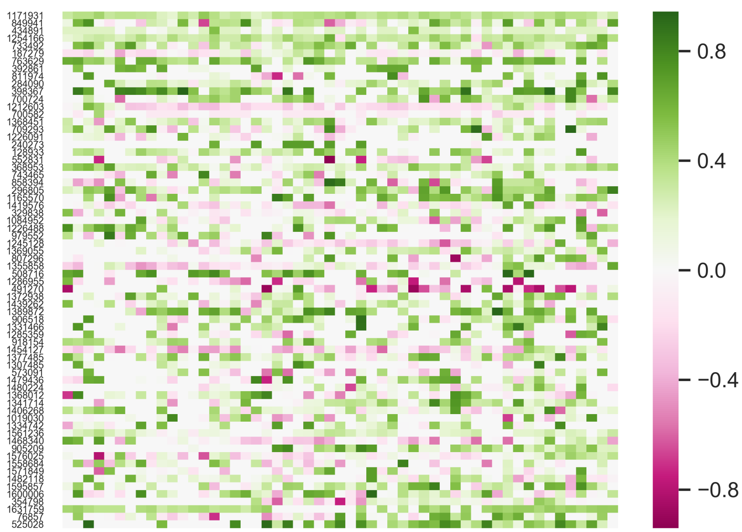
The next step is to undertake the lexicon approach in order to extract the semantic expression, sentiment or opinions. For any individual message in Table 1 we filter the terms being collected in the designated lexicon, and equally weight the filtered terms as the message sentiment score, which also means that the sentiment score of a sentence is estimated as the average over the weights of the lexicon terms it contains. Recall, that weights of the terms lexicon are in the range of -1 and $+1$. The sentiment score is automatically in the same range.

To visualize the quantified sentiment from individuals over time, we select the most active users and display their daily sentiment from 2018-11-01 to 2018-12-27. The heatmap shown in Figure 2.1 is a 2-dimensional matrix with y-axis for user's ID and x-axis for message posting date, the cell of heatmap is the quantified sentiment whose magnitude is represented as the color coded in the adjunct color bar. The evolution and dynamics of sentiment among users can be read in such heatmap presentation. From either Figure 2.1a (AAPL) or Figure 2.1b (BTC), one observes the similar color codes among a subset of users at particular date or period, indicating a contemporaneous common opinion/sentiment and an intertemporal opinion flow among users. Worth noting that

[1] This list can be found at https://api.stocktwits.com/symbol-sync/symbols.csv

[2] The percentage of of correct classification is defined as the proportion of correct classifications among all classified messages, while the percentage of classified messages is denoted as the proportion of classified messages among all messages. See more detain in Renault (2017) and Chen et al. (2019a)

(a) AAPL users



(b) BTC users

Figure 2.1: Social media users' sentiment over time
y-axis is the user's id, while x-axis is time stamp from 2018-11-01 — 2018-12-27.

some heterogeneity may exist as some users possess optimistic opinions and others are persistently pessimistic.

3 The SONIC approach

3.1 Notation

Let us first introduce some basic notation. Denote by $[N]$ the set of integers from 1 to N, i.e. $[N] = \{1, \dots, N\}$. For a vector $\mathbf{a} \in \mathbb{R}^d$ denote a square matrix $\mathrm{diag}(\mathbf{a}) \in \mathbb{R}^{d\times d}$ that has the values $a_1, \dots, a_n$ on the diagonal and zeros elsewhere. For a square matrix $A \in \mathbb{R}^{d\times d}$ we denote $\mathrm{Diag}(A) \in \mathbb{R}^{d\times d}$ as a diagonal matrix of the same size that coincides with A on the diagonal, i.e. $\mathrm{Diag}(A) = \mathrm{diag}(A_{11}, \dots, A_{dd})$. For the off-diagonal part we use the notation $\mathrm{Off}(A) = A - \mathrm{Diag}(A)$.

For a real vector $\mathbf{x} \in \mathbb{R}^d$ and $q \geq 1$ or $q = \infty$ denote the ℓ_q-norm $\|\mathbf{x}\|_q = (|x_1|^q + \dots + |x_d|^q)^{1/q}$; for $q = 2$ we ignore the index, i.e. $\|\mathbf{x}\| = \|\mathbf{x}\|_2$; we also denote a pseudo-norm $\|\mathbf{x}\|_0 = \sum_i \mathbf{1}(x_i \neq 0)$. For $A \in \mathbb{R}^{d_1\times d_2}$, $\sigma_1(A) \geq \sigma_2(A) \geq \dots \geq \sigma_{\min(d_1,d_2)}(A)$ denote the non-trivial singular values of A. We will also refer to $\sigma_{\min}(A)$ as the least nontrivial eigenvalue, i.e. $\sigma_{\min}(A) = \sigma_{\min(d_1,d_2)}(A)$. Furthermore, we write $|||A|||_{\mathsf{op}} = \max_j \sigma_j(A)$ for the spectral norm and $|||A|||_{\mathsf{F}} = \mathrm{Tr}^{1/2}(A^\top A) = \left(\sum_{j=1}^{\min(p,q)} \sigma_j(A)^2\right)^{1/2}$ for the Frobenius norm. Additionally, we introduce element-wise norms $\|A\|_{p,q}$ for $p, q \geq 1$ (including ∞) denotes ℓ_q norm of a vector composed of ℓ_p norms of rows of A, i.e. $\|A\|_{p,q} = \left(\sum_i \left(\sum_j |A_{ij}|^p\right)^{q/p}\right)^{1/q}$. Notice that $\|A\|_{2,2} = |||A|||_{\mathsf{F}}$.

3.2 Clusters of nodes and influencers

In our set-up the behaviour of each node $i \in [N]$ is characterized by the coefficients $\Theta_{i1}, \dots, \Theta_{iN}$, and when we group the nodes using their characteristics the notion of community is merged with the notion of cluster. We assume that the nodes are separated into clusters, such that these coefficients remain the same for the nodes within each cluster. Let us first give a precise definition of a clustering.

Definition 3.1. *A K-*clustering *of the set of the nodes $[N]$ is called a sequence $\mathcal{C} = (C_1, \dots, C_K)$ of K subsets of $[N]$, such that*

- *any two subsets are disjoint $C_i \cap C_j = \emptyset$ for $i \neq j$;*
- *the union of subsets C_j gives all nodes,*

$$C_1 \cup \dots \cup C_K = \{1, \dots, N\}.$$

Two clusterings $\mathcal{C}$ and $\mathcal{C}'$ are equivalent if the corresponding clusters are equal up to a relabelling, i.e. there is a permutation π on $\{1, \dots, K\}$, such that i.e. $C_j = C'_{\pi(j)}$ for every $j = 1, \dots, K$.

Furthermore, define a distance between two clusterings as

$$d(\mathcal{C}, \mathcal{C}') = \min_\pi \sum_{j=1}^{K} |C_j \setminus C'_{\pi(j)}|.$$

Remark 3.1. The distance between clusterings is in fact the minimal amount of node transfers from one cluster to another, that is required to make the clusterings equivalent. To see this, notice that each clustering can be defined as a sequence $(l_1, \dots, l_N)$ of N

labels taking values in $\{1,\dots,K\}$, so that each cluster defines as $C_j = \{i : \; l_i = j\}$. Then, if the clustering $\mathcal{C}'$ corresponds to the labels $l'_1,\dots,l'_N$, it is not hard to see, that the distance between them equals to

$$d(\mathcal{C},\mathcal{C}') = \min_{\pi} \sum_{i=1}^{N} \mathbf{1}(l_i \neq \pi(l'_i)).$$

We specify our model by putting structural assumptions which are motivated by both the communities and presence of the influencers.

Definition 3.2. *We say that* $\Theta \in \mathsf{SONIC}(s,K)$ *(Social Network with Influencers and Communities) if*

- *each user is influenced by at most* s *influencers, i.e.*

$$\max_i \sum_{j=1}^{N} \mathbf{1}(\Theta_{ij} \neq 0) \leq s;$$

- *there is a* K*-clustering* $\mathcal{C} = (C_1,\dots,C_K)$ *such that*

$$\Theta_{ij} = \Theta_{i'j}, \qquad j = 1,\dots,N$$

  whenever i, i' *are from the same cluster* C_l, $l = 1,\dots,K$.

We will also say that Θ *has clustering* $\mathcal{C}$.

Once $\Theta \in \mathsf{SONIC}(s,K)$ has clustering $\mathcal{C} = (C_1,\dots,C_K)$, the following factor representation takes place

$$\Theta = Z_{\mathcal{C}} V^{\top}, \tag{3.1}$$

where $Z_{\mathcal{C}}, V$ are $N \times K$ matrices such that

- $Z_{\mathcal{C}} = [\mathbf{z}_{C_1},\dots,\mathbf{z}_{C_K}]$ is a normalized index matrix of clustering $\mathcal{C}$, where for any $C \subset [N]$ we denote

$$\mathbf{z}_C = \frac{1}{\sqrt{|C|}}(\mathbf{1}(1 \in C),\dots,\mathbf{1}(N \in C)) \in \mathbb{R}^N$$

  — a normalized index vector for the cluster C;

- $V = [\mathbf{v}_1,\dots,\mathbf{v}_K]$ has sparse columns,

$$\|\mathbf{v}_j\|_0 \leq s.$$

A schematic picture of what we expect is shown in Figure 3.1. Here, the nodes from the same clusters depend on the same influencers (the grey nodes may be in any of the clusters), which also coincides with the idea of Rohe et al. (2016), who look for the right-hand side singular vectors of the Lagrangian in a directed network, grouping the nodes who tend to be affected by the same group of nodes.

The equation (3.1) is akin to bilinear factor models, which appear in Econometric models with factor loadings, see e.g. Moon and Weidner (2018) and the references therein.

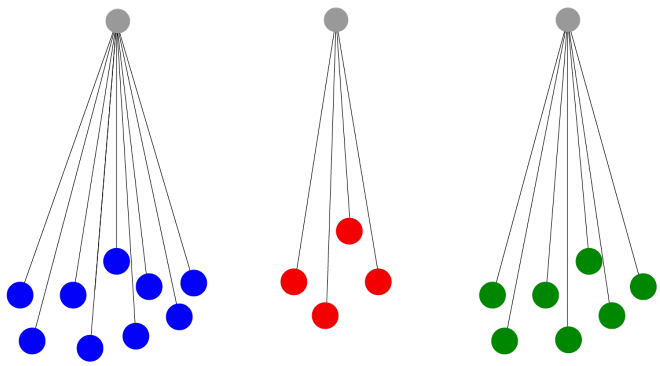


Figure 3.1: Example of a network with influencers.

It is also a popular machine learning technique for low rank approximation, see a thorough review in Udell et al. (2016). Chen and Schienle (2019) use sparse factors for a closely related model.

3.3 Model with missing observations

A network of size N represents a multivariate time series $Y_t = (Y_{1t}, \dots, Y_{Nt})^\top \in \mathbb{R}^N$, where Y_{it} is the response of a node $i = 1, \dots, N$ at a time $t = 1, \dots, T$, that follows the autoregressive equation

$$Y_t = \Theta^* Y_{t-1} + W_t, \tag{3.2}$$

with $\mathsf{E}[W_t | \mathcal{F}_{t-1}] = 0$ for $\mathcal{F}_{t-1} = \sigma(W_{t-1}, W_{t-2}, \dots)$. Once $|||\Theta^*|||_{\mathsf{op}} < 1$ the process exists as a converging series

$$Y_t = \sum_{k \geq 0} (\Theta^*)^k W_{t-k}, \tag{3.3}$$

and if the covariance of the innovations is $S = \operatorname{Var}(W_t)$, then the covariance of the process reads as

$$\Sigma = \operatorname{Var}(Y_t) = \sum_{k \geq 0} (\Theta^*)^k S \{(\Theta^*)^k\}^\top.$$

For simplicity we consider *subgaussian* vectors W_t, as it allows to have deviation bounds for covariance estimation with exponential probabilities. Recall the following definition, that appears, e.g., in Vershynin (2018).

Definition 3.3. *A random vector $W \in \mathbb{R}^d$ is called L-subgaussian if for every $\mathbf{u} \in \mathbb{R}^d$ it holds*

$$\|\mathbf{u}^\top W\|_{\psi_2} \leq L \|\mathbf{u}^\top X\|_{L_2},$$

where for a random variable $X \in \mathbb{R}$ we denote

$$\|X\|_{\psi_2} = \inf\left\{C > 0 : \ \mathsf{E}\exp\left\{\left(\frac{|X|}{C}\right)^2\right\} \leq 2\right\},$$
$$\|X\|_{L_2} = \mathsf{E}^{1/2}|X|^2.$$

Additionally, we adopt the framework of Lounici (2014) for vectors with missing observations, assuming that each variable Y_{it} is either observed or not independently and with some probability. Formally speaking, instead of having a realisation of the whole vector Y_t we only have the access to the vectors of form

$$Z_t = (\delta_{1t}Y_{1t}, \dots, \delta_{Nt}Y_{Nt})^\top, \qquad t = 1, \dots, T, \tag{3.4}$$

where $\delta_{it} \sim \mathrm{Be}(p_i)$ are independent Bernoulli random variables for every $i = 1, \dots, N$ and $t = 1, \dots, T$ and some $p_i \in (0, 1]$. This means that each variable Y_{it} is only observed with probability p_i independently from the other variables, with $\delta_{it} = 1$ corresponding to observed Y_{it} and $\delta_{it} = 0$ to missing Y_{it}, so instead we simply receive zero. Obviously, the case $p_i = 1$ for every $i = 1, \dots, N$ corresponds to the process without missing observations, therefore the new problem serves as a generalisation and the results for the missing observations model can be applied in the regular case as well.

Remark 3.2. In terms of the StockTwits sentiment we interpret the process Y_t as an unobserved underlying *opinion process*. During each day the users decide whether to express their opinion or not by posting a message on their page, which results in a masked process Z_t. Since some users are more active than the others, we need to account for different probabilities p_i.

Suppose that the probabilities p_i are given (otherwise they can easily be estimated) and set $\mathbf{p} = (p_1, \dots, p_N)^\top$. Due to Lounici (2014), set the observed empirical covariance $\Sigma^* = \frac{1}{T}\sum_{t=1}^T Z_t Z_t^\top$ and consider the following covariance estimator,

$$\hat{\Sigma} = \mathrm{diag}\{\mathbf{p}\}^{-1}\mathrm{Diag}(\Sigma^*) + \mathrm{diag}\{\mathbf{p}\}^{-1}\mathrm{Off}(\Sigma^*)\mathrm{diag}\{\mathbf{p}\}^{-1}.$$

It is straightforward to calculate that this is an unbiased estimator, i.e.

$$\mathsf{E}\hat{\Sigma} = \Sigma.$$

The state-of-the art bound for the error of such covariance estimator is due to Klochkov and Zhivotovskiy (2018), Theorem 4.2. In the case of independent vectors Y_t and equal probabilities of observations $p_1 = \cdots = p_N = p$ they show that for any $u \geq 1$ with probability at least $1 - e^{-u}$ it holds

$$\|\hat{\Sigma} - \Sigma\|_{\mathsf{op}} \leq C\|\Sigma\|_{\mathsf{op}} \left(\sqrt{\frac{\tilde{\mathbf{r}}(\Sigma)\log\tilde{\mathbf{r}}(\Sigma)}{Tp^2}} \bigvee \sqrt{\frac{u}{Tp^2}} \bigvee \frac{\tilde{\mathbf{r}}(\Sigma)(\log\tilde{\mathbf{r}}(\Sigma) + u)\log T}{Tp^2} \right),$$

where $\tilde{\mathbf{r}}(\Sigma) = \frac{\mathrm{Tr}(\Sigma)}{\|\Sigma\|_{\mathsf{op}}}$ denotes the *effective rank* of the covariance Σ. In a similar way the effective rank appears as well in the classic covariance estimation problem (i.e., $p = 1$), see, e.g., Koltchinskii and Lounici (2017) who even provide a matching lower bound. Notice that the effective rank takes values between 1 and the rank of Σ, however, if no specific restriction on the spectrum of Σ is given, the effective rank can grow as large as the full dimension N. This means that the bound above can only guarantee the error of order $\sqrt{\frac{N}{Tp^2}}$, not taking into account the logarithms. On the other hand, one often only needs to bound the error within specific low-dimensional subspaces. The following theorem provides such deviation bound for the autoregressive process (3.2), and in its turn accounts for possibly distinct probabilities p_i.

Theorem 3.4. *Assume the vectors W_t are independent L-subgaussian and also*

$$\|\Theta^*\|_{\mathsf{op}} \le \gamma < 1, \qquad p_i \ge p_{\min} > 0.$$

Let $P, Q \in \mathbb{R}^{N \times N}$ be two arbitrary orthogonal projectors of rank M_1, M_2, respectively. Then, for any $u \ge 1$it holds with probability at least $1 - e^{-u}$,

$$\|P(\hat{\Sigma} - \Sigma)Q\|_{\mathsf{op}} \le C\|S\|_{\mathsf{op}} \left(\sqrt{\frac{M_1 \vee M_2(\log N + u)}{T p_{\min}^2}} \bigvee \frac{\sqrt{M_1 M_2}(\log N + u)\log T}{T p_{\min}^2} \right),$$

where $C = C(\gamma, L)$ only depends on L and γ.

See proof of this result in Section A.

Additionally, we are interested in estimating lag-1 cross-covariance under the same scenario. Namely, based on the sample $Z_1, \dots, Z_T$ and given the probabilities $p_1, \dots, p_N$ we wish to estimate the matrix $A = \mathsf{E} Y_t Y_{t+1}^\top$. Since $\mathsf{E}[Y_{t+1} | \mathcal{F}_t] = \Theta^* Y_t$ for the linear process (A.1), the corresponding cross-covariance reads as

$$A = \Sigma(\Theta^*)^\top.$$

Consider the following estimator

$$\hat{A} = \mathrm{diag}\{\mathbf{p}\}^{-1} A^* \mathrm{diag}\{\mathbf{p}\}^{-1},$$

where A^* is the observed empirical cross-covariance

$$A^* = \frac{1}{T-1} \sum_{t=1}^{T-1} Z_t Z_{t+1}^\top.$$

For this estimator we provide an upper-bound, again with a restriction to some low-dimensional subspaces.

Theorem 3.5. *Let P, Q be two projectors of rank M_1 and M_2, respectively. Assume the independent vectors W_t are L-subgaussian and also*

$$\|\Theta^*\|_{\mathsf{op}} \le \gamma < 1, \qquad p_i \ge p_{\min} > 0.$$

Then, for any $u \ge 1$ it holds with probability at least $1 - e^{-u}$

$$\|P(\hat{A} - A)Q\|_{\mathsf{op}} \le C\|S\|_{\mathsf{op}} \left(\sqrt{\frac{(M_1 \vee M_2)(\log N + u)}{T p_{\min}^2}} \bigvee \frac{\sqrt{M_1 M_2}(\log N + u)\log T}{T p_{\min}^2} \right),$$

where $C = C(\gamma, L)$ only depends on γ and L.

The proof is postponed to Section A.

3.4 Alternating minimization algorithm

In order to estimate the matrix $\Theta = Z_{\mathcal{C}} V^\top$ we need to estimate both $\mathcal{C}$ and V simultaneously. Suppose, we have some clustering $\mathcal{C}$ at hand and we want to estimate the corresponding V. The mean squared loss from the fully observed sample would look like

$$R^*_{\mathcal{C}}(V) = \frac{1}{2(T-1)} \sum_{t=1}^{T-1} \|Y_{t+1} - Z_{\mathcal{C}} V^\top Y_t\|^2$$
$$= \frac{1}{2}\mathrm{Tr}(V^\top \tilde{\Sigma} V) - \mathrm{Tr}(V^\top \tilde{A} Z_{\mathcal{C}}) + \frac{1}{2(T-1)} \sum_{t=1}^{T-1} \|Y_{t+1}\|^2,$$

where we used the fact that $Z_{\mathcal{C}}^\top Z_{\mathcal{C}} = I_K$ and the trace of a matrix product is invariant with respect to transition $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Here, we also denote

$$\tilde{\Sigma} = \frac{1}{T-1} \sum_{t=1}^{T-1} Y_t Y_t^\top, \qquad \tilde{A} = \frac{1}{T-1} \sum_{t=1}^{T-1} Y_t Y_{t+1}^\top,$$

to be empirical covariance and empirical lag-1 covariance built on a sample $Y_1, \ldots, Y_T$, which we do not fully observe. Instead, since we only have access to the missing observation estimators $\hat{\Sigma}$ and $\hat{A}$, consider the loss function (notice that the star has disappeared)

$$R_{\mathcal{C}}(V) = \frac{1}{2}\mathrm{Tr}(V^\top \hat{\Sigma} V) - \mathrm{Tr}(V^\top \hat{A} Z_{\mathcal{C}}).$$

As we are searching for a sparse matrix V, we additionally put a LASSO regularization, so we end up with the following program,

$$\hat{V}_{\mathcal{C},\lambda} = \arg\min R_{\mathcal{C},\lambda}(V), \qquad R_{\mathcal{C},\lambda}(V) = R_{\mathcal{C}}(V) + \lambda \|V\|_{1,1}$$
$$= \frac{1}{2}\mathrm{Tr}(V^\top \hat{\Sigma} V) - \mathrm{Tr}(V^\top \hat{A} Z_{\mathcal{C}}) + \lambda \|V\|_{1,1},$$

where $\|V\|_{1,1} = \sum_{ij} |V_{ij}|$, and $\lambda > 0$ depends on the dimension N and number of observations T. Concerning this minimization problem we have the following observations:

- the problem reduces to a simple quadratic programming and therefore can be efficiently solved;
- since $\|V\|_{1,1} = \sum_{j=1}^K \|\mathbf{v}_j\|_1$ we can rewrite

$$R_{\lambda,\mathcal{C}}(V) = \frac{1}{2}\mathrm{Tr}\left(V^\top \hat{\Sigma} V\right) - \mathrm{Tr}\left(V^\top \hat{A} Z\right) + \lambda \|V\|_{1,1}$$
$$= \sum_{j=1}^K \frac{1}{2}\mathbf{v}_j^\top \hat{\Sigma} \mathbf{v}_j - \mathbf{v}_j^\top \hat{A} \mathbf{z}_j + \lambda \|\mathbf{v}_j\|_1,$$

  therefore we need to solve K independent problems of size N, which reduces computational complexity and may also be implemented in parallel.

Ideally, we want to solve the following problem (note that the number of clusters K and the tuning parameter λ are fixed here)

$$F_\lambda(\mathcal{C}) \to \min_{\mathcal{C}}, \qquad F_\lambda(\mathcal{C}) = \min_V R_{\lambda,\mathcal{C}}(V).$$

We can employ a simple greedy procedure. In the beginning we initialize $\mathcal{C}^{(0)} = (l_1, \ldots, l_N)$ randomly, each label takes values $1, \ldots, K$. Then, at a step t we try to change one label of a node that reduces the risk the most. This means that we try all the clusterings in the nearest vicinity of a current solution $\mathcal{C}^{(t)}$, i.e.

$$\mathcal{C}^{(t+1)} = \arg \min_{d(\mathcal{C},\mathcal{C}^{(t)})\le 1} F_\lambda(\mathcal{C}).$$

At each such step we would need to calculate $F_\lambda(\mathcal{C})$ for $\mathcal{O}\{N(K-1)\}$ different candidates.

Remark 3.3. In general, it is impossible to optimize arbitrary function $f(\mathcal{C})$ with respect to a clustering. For instance, there it is known that K-means is general NP-hard, however different solutions are widely used in practice, see Shindler et al. (2011) and Likas et al. (2003).

To speed up the trials of the greedy procedure we utilize an alternating minimization strategy. Suppose, at the beginning we initialize the clustering by $\mathcal{C}^{(0)}$ and compute the LASSO solution $V^{(0)} = V_{\mathcal{C}^{(0)},t}$. When we want to update the clustering, we fix the matrix $V = V^{(t)}$ and solve the problem

$$R_{\mathcal{C},\lambda}(V) = \frac{1}{2}\mathrm{Tr}(V^\top \hat{\Sigma} V) - \mathrm{Tr}(V^\top \hat{A} Z_{\mathcal{C}}) + \lambda \|V\|_{1,1} \to \min_{\mathcal{C}},$$

where only the term $-\mathrm{Tr}(V^\top \hat{A} Z_{\mathcal{C}})$ depends on $\mathcal{C}$. Minimizing by conducting a few steps of the greedy procedure we obtain the next clustering update $\mathcal{C}^{(t+1)}$. Then, we again update the V-factor by setting $V^{(t+1)} = V_{\mathcal{C}^{(t+1)},\lambda}$. We continue so until the clustering does not change or the number of iterations exceeds a certain limit. The pseudo code in Algorithm 1 summarizes this procedure.

Result: a pair $(\hat{\mathcal{C}}, \hat{V})$
initialize $\mathcal{C}^{(0)} = (l_1^{(0)}, \dots, l_N^{(0)})$ randomly;
$t \leftarrow 0$;
while $t < max\_iter$ **do**
  update $\hat{V}^{(t)} \leftarrow \arg\min R_{\mathcal{C}^{(t)},\lambda}(V)$;
  for $i = 1, \dots, N$ **do**
    for $l = 1, \dots, N$ **do**
      consider candidate $\mathcal{C}' = (l_1^{(t)}, \dots, l_{i-1}^{(t)}, l, l_{i+1}^{(t)}, \dots, l_N^{(t)})$;
      $r_{il} \leftarrow -\mathrm{Tr}(V^{(t)} \hat{A} Z_{\mathcal{C}'})$;
    end
  end
  $(i^*, l^*) = \arg\min r_{il}$;
  update $\mathcal{C}^{(t+1)} \leftarrow (l_1^{(t)}, \dots, l_{i^*-1}^{(t)}, l^*, l_{i^*+1}^{(t)}, \dots, l_N^{(t)})$;
  if $\mathcal{C}^{(t+1)} = \mathcal{C}^{(t)}$ **then**
    return $(\mathcal{C}^{(t)}, V^{(t)})$;
  else
    $t \leftarrow t + 1$;
  end
end

Algorithm 1: Alternating greedy clustering procedure.

3.5 Local consistency result

In this section we show the existence of a locally optimal solution in the neighbourhood of the true parameter with high probability. We call a clustering solution $\hat{\mathcal{C}}$ *locally optimal*, if the functional $F_\lambda(\cdot)$ has the minimum value at the point $\hat{\mathcal{C}}$ among its nearest neighbours $d(\mathcal{C}, \hat{\mathcal{C}}) \le 1$. In particular, Algorithm 1 obviously stops at such a solution.

Conditions

Here we describe the conditions that we need for the consistency result. The first condition concludes the requirements of Theorems 3.4 and 3.5.

Assumption 1. *There is some $\Theta^* \in \mathbb{R}^{N\times N}$ such that $\|\!|\Theta^*|\!\|_{\mathsf{op}} \le \gamma$ for some $\gamma < 1$ and the time series Y_t follows (3.3). The innovations W_t are independent with $\mathsf{E}W_t = 0$ and $Var(W_t) = S$. Moreover, each W_t is L-subgaussian.*

Furthermore, we impose structural assumptions on the true parameter Θ^* described in Section 3.2.

Assumption 2. *The true VAR operator admits decomposition with K-clustering $\mathcal{C}^*$*

$$\Theta^* = Z_{\mathcal{C}^*} V^*,$$

and meets the following conditions:

1. $\|\!|\Theta^*|\!\|_{\mathsf{op}} = \|\!|V^*|\!\|_{\mathsf{op}} \le \gamma < 1$;
2. *cluster separation*
$$\sigma_{\min}([V^*]^\top \Sigma V^*) \ge a_0; \tag{3.5}$$
3. *sparsity: for every $j = 1, \dots, K$ the active set $\Lambda_j = \operatorname{supp}(\mathbf{v}_j^*)$ satisfies*
$$|\Lambda_j| \le s;$$
4. *significant active coefficients:*
$$|v_{ij}^*| \ge \tau_0 s^{-1/2}, \qquad i \in \Lambda_j, \qquad j = 1, \dots, K\,. \tag{3.6}$$
Here each $\|\mathbf{v}_j^*\| \le 1$ has (at most) s nonzero values, hence the normalization;
5. *significant cluster sizes:*
$$\frac{\min_j |C_j^*|}{\max_j |C_j^*|} \ge \alpha, \qquad 0 < \alpha \le 1.$$

Notice that the condition (3.5) requires that the clusters appropriately separated, i.e. each $\mathbf{v}_j^*$ is far enough from a linear combination of the rest. Another assumption is concerned with the population covariance Σ.

Assumption 3. *The covariance of Y_t reads as*

$$\Sigma = \sum_{k=0}^{\infty} (\Theta^*)^k S [(\Theta^*)^k]^\top,$$

where $S = Var(W_t)$. We impose the following assumptions on this matrix.

1. *bounded operator norm*
$$\|\Sigma\|_{\mathsf{op}} \le \sigma_{\max};$$
2. *restricted least eigenvalue*
$$\sigma_{\min}(\Sigma_{\Lambda_j,\Lambda_j}) \ge \sigma_{\min}, \qquad j = 1, \ldots, K\,.$$
3. *bounded* $(1,1)$*-norm*
$$\|\Sigma^{-1}_{\Lambda_j,\Lambda_j}\|_{1,1} \le M, \qquad j = 1, \ldots, K. \tag{3.7}$$

Remark 3.4. Note, that we do not assume that the smallest eigenvalue of Σ is bounded away from zero, but only those corresponding to the small subsets of indices are. For sake of simplicity we additionally assume that the ratio

$$\frac{\sigma_{\max}}{\sigma_{\min}} \le \kappa,$$

is bounded by some constant $\kappa \ge 1$.

Note also, that the bias term of the LASSO estimator usually reads as $\hat{\Sigma}^{-1}_{\Lambda_j,\Lambda_j}\mathbf{g}$ with some $\|\mathbf{g}\|_\infty \le 1$, see Lemma B.3. We need (3.7) to control the sup-norm of this bias.

Finally, we present the assumption that allows to control exact recovery of sparsity patterns for the LASSO estimator.

Assumption 4. *For every $j = 1, \ldots, K$ it holds*

$$\|\Sigma_{\Lambda_j^c,\Lambda_j}\Sigma^{-1}_{\Lambda_j,\Lambda_j}\|_{1,\infty} \le \frac{1}{4},$$

Remark 3.5. The inequality $\|\Sigma_{\Lambda_j^c,\Lambda_j}\Sigma^{-1}_{\Lambda_j,\Lambda_j}\|_{1,\infty} < 1$ allows to derive exact recovery of the sparsity pattern at the LASSO procedure-step described above. In Section B we show a straightforward extension of results from Tropp (2006) to the case with the presence of missing observations.

Theorem 3.6. *Suppose, Assumptions 1-4 hold. There are constants $c, C > 0$ that depend on L, γ such that the following holds. Suppose,*

$$\sqrt{\frac{sn^* \log N}{Tp^2_{\min}}} \bigvee \sqrt{\frac{s \log N \log^2 T}{Tp^2_{\min}}} \le c, \tag{3.8}$$

where $n^* = \max_{j \le K} |C^*_j|$ and, additionally, $N \ge (C\alpha^2 \vee \kappa)K$. Then, with probability at least $1 - 1/N$ for any λ satisfying

$$C\sigma_{\max}\sqrt{\frac{\log N}{Tp^2_{\min}}} \le \lambda \le c\left\{\kappa^{-4}(a_0^2/\sigma_{\max})K^{-2}s^{-1} \bigwedge \sigma_{\min}\tau_0 s^{-1}\right\},$$

and, additionally, $\lambda \geq C\alpha^2 K/N$, there is a locally optimal solution $\hat{\mathcal{C}}$ satisfying

$$\|\!|Z_{\hat{\mathcal{C}}}\hat{V}_{\hat{\mathcal{C}},\lambda}^{\top} - \Theta^*|\!\|_{\mathsf{F}} \leq \left\{3\sigma_{\min}^{-1}\sqrt{Ks} + \frac{C\gamma}{a_0}\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2 K\sqrt{s}\right\}\lambda\,.$$

Remark 3.6. It also follows from the proof that under the assumptions of the theorem, the sparsity pattern of each vector is recovered precisely, i.e. we correctly identify the influencers for every cluster.

Let us take a closer look at the condition (3.8). Under the cluster size restriction from Assumption 2 we have that all clusters have the size of order N/K, since

$$\alpha\frac{N}{K} \leq |C_j^*| \leq \alpha^{-1}\frac{N}{K}, \qquad j = 1, \dots, K.$$

This means that, say if we ignore the missing observations, we only need

$$\frac{(sN/K)\log N}{T} \leq c(\alpha)$$

to hold, to be able to estimate the parameter. This means that once K is large enough the estimator works with the corresponding error. Notice that the ℓ_1-regularisation alone requires the number of the observations must be at least the number of edges times $\log N$, see Fan et al. (2009). In our setting the number of connections is up to Ns, so the condition reads as

$$\sqrt{\frac{sN\log N}{T}} \leq 1,$$

therefore our SONIC model is an improvement in this regards.

According to the model, say if $N/K \geq \sqrt{T}$, the best available choice of tuning parameter is

$$\lambda^* = C\sigma_{\max}\sqrt{\frac{\log N}{Tp_{\min}^2}},$$

in which case the error of the estimator reads as

$$\|\!|\hat{\Theta}_{\lambda^*} - \Theta^*|\!\|_{\mathsf{F}} \lesssim K\sqrt{\frac{s\log N}{Tp_{\min}^2}},$$

which suggests some kind of tradeoff between small and large K.

4 Simulation study

Take $N = T = 100$ and $s = 1$, while K will be changing in a range 2..30. We are particularly interesting in capturing this effect that larger amount of clusters allows better estimation. For every $K = 2, \dots, 30$ we contruct the following matrix Θ^*,

- pick clusters C_j^* having approximately the same size $\frac{N}{K} \pm 1$;
- for every $j = 1, \dots, K$ set

$$\mathbf{v}_j^* = 0.5\mathbf{e}_j = (0, \dots, 0.5, \dots, 0)^{\top},$$

with a single nonzero value at the place j, so that $s = 1$.

- by construction we have,

$$|||\Theta^*|||_{\mathsf{op}} = |||V^*|||_{\mathsf{op}} = 0.5, \qquad |||\Theta^*|||_{\mathsf{F}} = |||V^*|||_{\mathsf{F}} = 0.5\sqrt{K}.$$

Furthermore we generate i.i.d. $W_{-19}, W_{-18}, \ldots, W_T \sim \mathrm{N}(0, I)$ and set

$$Y_t = \sum_{k=0}^{20} (\Theta^*)^k W_{t-k}, \qquad t = 1, \ldots T,$$

where due to $0.5^{-20} \approx 10^{-6}$ the terms for $k > 20$ can be neglected. On Figure 4.1a we show the relative error $\mathsf{E}|||\hat{\Theta} - \Theta^*|||_{\mathsf{F}}/|||\Theta^*|||_{\mathsf{F}}$ along regularization paths for different choices of K. Picking the best λ we show the relative error against the number of clusters on Figure 4.1b. We also show the clustering error $\mathsf{E}d(\hat{\mathcal{C}}, \mathcal{C}^*)$ on Figure 4.1c depending on K. All expectations are estimated based on 20 simulations.

We conclude that the simulations confirm the following theoretical property of our estimator: the smaller the size of largest cluster, the better, while the total size of the network can be even as large as the number of observations.

5 Application to StockTwits sentiment

Here we present the results of experiment with two datasets described in Section 2. The first one contains daily average sentiment weights constructed from the messages containing the cashtag '\$AAPL' (Apple) and the second one from those containing the cashtag '\$BTC.X' (Bitcoin.)

The missing observation model presented in Section 3.3 relies on persistent observation frequency with the same probability p_i over a time period under consideration. Moreover, since in Theorems 3.4 and 3.5 the amount of observations scales with the factor $p_{\min}^2$, we need to avoid the users whose p_i is too little. Based on these remarks we suggest the following preprocessing steps:

1. pick users with estimated probability $\hat{p}_i \geq 0.5$;

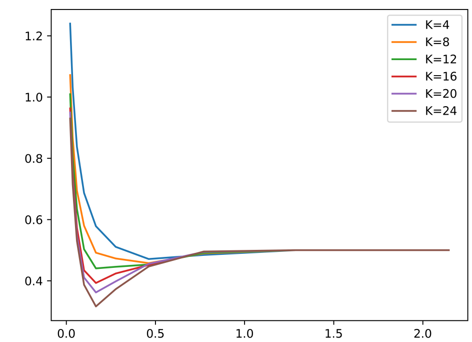
2. for every user left after step 1, pick the longest historical interval over which the user exhibits persistent probability of observation. One can look at a moving average estimation and ensure that for any window it remains within appropriate confidence interval;

3. take only users for whom the historical interval from step 2 is at least 50 days.

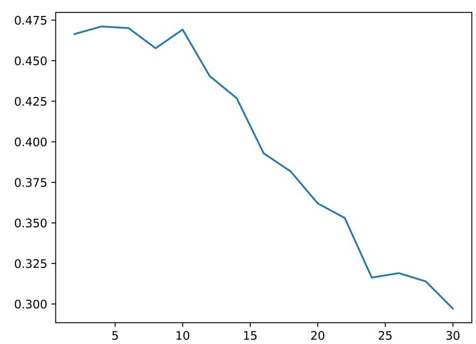
For AAPL dataset we are left with 46 users and 72 days, while for BTC we have 68 users and 52 days. The two datasets are visualized using heatmap in Figure 2.1.

We apply our SONIC model to AAPL dataset with $\lambda = 0.05$ and $K = 6$. A heatmap visualisation for estimated matrix $\hat{\Theta}$ is presented in Figure 5.1a. From here we can identify that the most important users have identification number 47688, 619769, 850976 and 1438287[3]. For the BTC dataset we use $\lambda = 0.05$ and $K = 5$, the results presented in Figure 5.1b. The influencers are 1171931 and 1254166.

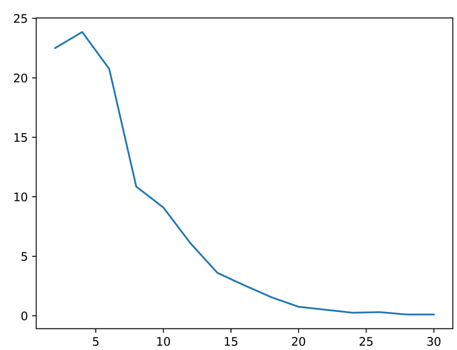
[3] To access a page via identification number type, for example, `https://stocktwits.com/123456` in a browser address line.



(a) Expected relative loss $\mathsf{E}\frac{|||\hat{\Theta}-\Theta^*|||_{\mathsf{F}}}{|||\Theta^*|||_{\mathsf{F}}}$ for different λ and $K = 4, 8, 12, 16, 20, 24$.

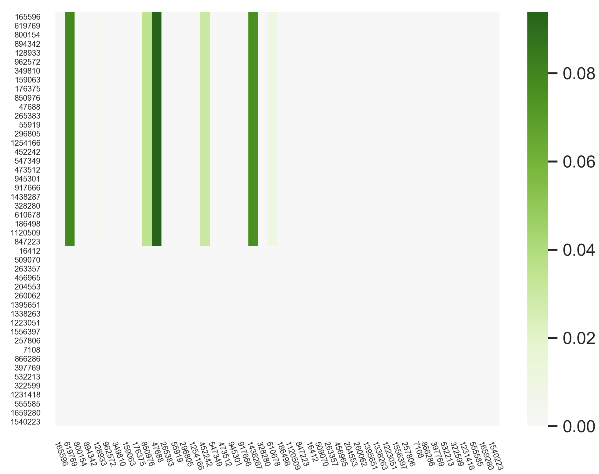


(b) Expected relative loss $\mathsf{E}\frac{|||\hat{\Theta}-\Theta^*|||_{\mathsf{F}}}{|||\Theta^*|||_{\mathsf{F}}}$ for the best λ and $K = 2, \ldots, 30$.

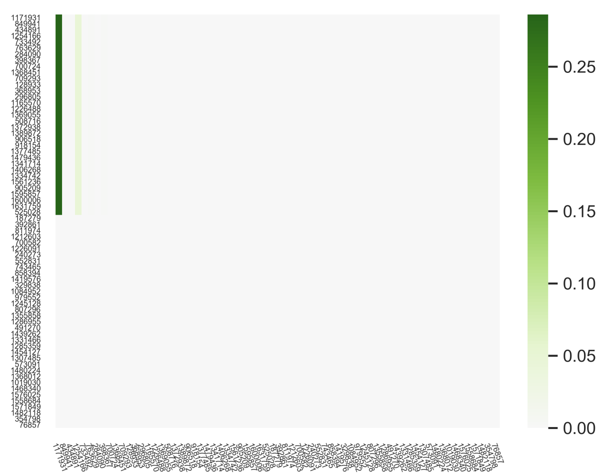


(c) Expected clustering error $\mathsf{E}d(\hat{\mathcal{C}}, \mathcal{C}^*)$ for the best λ and $K = 2, \ldots, 30$.

Figure 4.1: Simulation results for $N = T = 100$ and $s = 1$.



(a) AAPL dataset with $N = 46$, $T = 72$ and $K = 2$.



(b) BTC dataset with $N = 68$, $T = 52$ and $K = 2$.

Figure 5.1: Estimated $\hat{\Theta}$ for AAPL and BTC datasets. The axes correspond to user id's and are rearranged with respect to the estimated clusterings.

Remark 5.1. Choosing the tuning parameter λ and the number of clusters K remains beyond the scope of this paper. For this experiment we picked both numbers graphically: for λ based on the number of active columns with relatively small values, while for K we picked the smallest one for which there is no clusters that are much smaller than the others, as well as no clusters that are split into two or more. Development of a statistically-backed selection is left for further research.

Let us point out some observations based on the results of this experiment. The first one is that for the APPLE dataset we end up with users who have lots of followers, while from the BITCOIN dataset we have found two accounts that have a moderate amount of followers and as it seems to belong to companies that provide analytical tools for traders. We suggest that it highlights the difference between two assets of different nature — a classical one and a cryptocurrency. Secondly, in both cases, we have two communities, one with the coefficients $\Theta_{ij} = 0$, which simply corresponds to noise. For instance, this may represent users who only react to price changes. On the contrary, the second cluster may represent the users that follow the news.

6 Proof of main result

This section is devoted to the proof of Theorem 3.6. We start with some preliminary lemmas and then proceed with the proof that consists of several steps. Following the ideas in Gribonval et al. (2015), the proof is based on explicit representation of the loss function.

We exploit the following simplified notation. Denote, $\mathbf{z}_j^* = \mathbf{z}_{C_j^*}$ to be the columns of $Z^* = Z_{\mathcal{C}^*}$ and we also denote $n_j^* = |C_j^*|$ for every $j = 1, \dots, K$. When the clustering $\mathcal{C} = (C_1, \dots, C_K)$ is clear from the context we will also write Z for $Z_{\mathcal{C}}$, $\mathbf{z}_j$ for $\mathbf{z}_{C_j}$, and $n_j = |C_j|$ for every $j = 1, \dots, K$.

6.1 Preliminary lemmas

Lemma 6.1. *Suppose that C_j is such that $\|\mathbf{z}_{C_j} - \mathbf{z}_j^*\| \le 0.3$. Then,*

$$\frac{1}{1.1}|C_j^*| \le |C_j| \le 1.1|C_j^*|.$$

Proof. Suppose, $n_j = |C_j| > n_j^* = |C_j^*|$, then

$$r^2 = \|\mathbf{z}_j - \mathbf{z}_j^*\|^2 = 2 - \frac{2}{\sqrt{n_j n_j^*}}|C_j \cap C_j^*| \ge 2 - 2\sqrt{\frac{n_j^*}{n_j}},$$

since $|C_j \cap C_j^*| \le n_j^*$. Thus, $\sqrt{n_j} - \sqrt{n_j^*} \le (r^2/2)\sqrt{n_j}$, which due to $r \le 0.3$ implies by rearranging and taking square $n_j \le 1.1 n_j^*$.

If $n_j < n_j^*$ we have,

$$r^2 \ge \|\mathbf{z}_j - \mathbf{z}_j^*\|^2 = 2 - \frac{2|C_j \cap C_j'|}{\sqrt{n_j n_j^*}} \ge 2 - 2\sqrt{\frac{n_j}{n_j^*}},$$

and the fact that $r \le 0.3$ implies $n_j^* \le 1.1 n_j$.

□

Lemma 6.2. *Let* $\|\mathbf{z}_{C_1} - \mathbf{z}_{C_2}\| \le 0.3$. *Then,*

$$\|\mathbf{z}_{C_1} - \mathbf{z}_{C_2}\|_1 \le 1.55\sqrt{N_1}\|\mathbf{z}_{C_1} - \mathbf{z}_{C_2}\|^2 .$$

Proof. Let $N_j = |C_j|$ and $a = |C_1 \cap C_2|$, $b = |C_1 \setminus C_2|$, $c = |C_2 \setminus C_1|$, so that $N_1 = a + b$, $N_2 = a + c$, and $|C_1 \triangle C_2| = b + c$. We have,

$$\|\mathbf{z}_{C_1} - \mathbf{z}_{C_2}\|^2 = \left(\frac{1}{\sqrt{N_1}} - \frac{1}{\sqrt{N_2}}\right)^2 a + \frac{b}{N_1} + \frac{c}{N_2} \ge \frac{b}{N_1} + \frac{c}{N_2} .$$

On the other hand,

$$\begin{aligned} \|\mathbf{z}_{C_1} - \mathbf{z}_{C_2}\|_1 &= \left|\frac{1}{\sqrt{N_1}} - \frac{1}{\sqrt{N_2}}\right| a + \frac{b}{\sqrt{N_1}} + \frac{c}{\sqrt{N_2}} \\ &\le \left|\frac{1}{\sqrt{N_1}} - \frac{1}{\sqrt{N_2}}\right| a + \sqrt{N_1 \vee N_2}\|\mathbf{z}_{C_1} - \mathbf{z}_{C_2}\|^2 . \end{aligned}$$

Since $|N_1 - N_2| \le b + c$ we obviously have,

$$\begin{aligned} \left|\frac{1}{\sqrt{N_1}} - \frac{1}{\sqrt{N_2}}\right| a &= \frac{|N_1 - N_2| a}{\sqrt{(a+b)(a+c)}(\sqrt{a+b} + \sqrt{a+c})} \\ &\le \frac{(b+c)a}{\sqrt{N_1 \vee N_2}\sqrt{a}(2\sqrt{a})} \\ &\le \sqrt{N_1 \wedge N_2}\|\mathbf{z}_{C_1} - \mathbf{z}_{C_2}\|^2/2, \end{aligned}$$

and it is left to apply Lemma 6.1. □

Lemma 6.3. *Suppose,* $\frac{\min_j n_j^*}{\max_j n_j^*} \ge \alpha$ *for some* $\alpha \in (0, 1]$ *and let* $\|\mathbf{z}_j - \mathbf{z}_j^*\| \le r$. *Suppose,* $r \le 0.3$. *Then,*

$$\|[Z^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*)\|_1 \le 3.05\alpha^{-1/2}r^2.$$

Proof. 1) We first consider the case $|C_j| = n_j^*$. It holds then

$$[\mathbf{z}_j^*]^\top(\mathbf{z}_j^* - \mathbf{z}_j) = \frac{1}{n_j^*}(n_j^* - |C_j \cap C_j^*|) = \frac{1}{n_j^*}|C_j^* \setminus C_j|.$$

Moreover, for every $k \ne j$ it holds

$$|[\mathbf{z}_k^*]^\top(\mathbf{z}_j^* - \mathbf{z}_j)| = |[\mathbf{z}_k^*]^\top \mathbf{z}_j| = \frac{1}{\sqrt{n_k^* n_j^*}}|C_k^* \cap C_j| \le \frac{\alpha^{-1/2}}{n_j^*}|C_k^* \cap C_j|.$$

Summing up, we get

$$\begin{aligned}\|[Z^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*)\|_1 &\le \frac{\alpha^{-1/2}}{n_j^*}\left(|C_j^* \setminus C_j| + \sum_{k\ne j}|C_k^* \cap C_j|\right)\\ &\le \frac{\alpha^{-1/2}}{n_j^*}\left(|C_j^* \setminus C_j| + |C_j \setminus C_j^*|\right)\\ &= \frac{\alpha^{-1/2}}{n_j^*}|C_j \triangle C_j^*|.\end{aligned}$$

It is left to notice that in the case $|C_j| = |C_j^*| = n_j^*$ we have exactly $\|\mathbf{z}_j - \mathbf{z}_j^*\|^2 = \frac{1}{n_j^*}|C_j \triangle C_j^*|$.

2) Suppose, $n_j = |C_j| > n_j^*$. Obviously, we can decompose $C_j = C_j' \cup B$ such that $|C_j'| = n_j^*$ and $B \cap C_j^* = \emptyset$. Setting $\mathbf{z}_j' = \mathbf{z}_{C_j'}$ we get by the above derivations that $\|[Z^*]^\top(\mathbf{z}_j' - \mathbf{z}_j^*)\|_1 \le \alpha^{-1/2}\|\mathbf{z}_j' - \mathbf{z}_j^*\|^2$. Since $C_j' \cap C_j^* = C_j \cap C_j^*$ we can compare the distances

$$\|\mathbf{z}_j - \mathbf{z}_j^*\|^2 = 2 - \frac{2}{\sqrt{n_j n_j^*}}|C_j \cap C_j^*| > 2 - \frac{2}{n_j^*}|C_j \cap C_j^*| = \|\mathbf{z}_j' - \mathbf{z}_j^*\|^2.$$

Taking the remainder $\mathbf{b} = \mathbf{z}_j - \mathbf{z}_j'$ we have,

$$b_i = \begin{cases} n_j^{-1/2} - (n_j^*)^{-1/2}, & i \in C_j',\\ n_j^{-1/2}, & i \in B,\\ 0 & \text{otherwise.}\end{cases}$$

Setting $d = n_j - n_j^* = |B|$ it is easy to obtain $|n_j^{-1/2} - (n_j^*)^{-1/2}| \le \frac{d}{n_j}\frac{1}{\sqrt{n_j^*}}$. Thus, we get

$$\begin{aligned}\sum_{k=1}^{K}|[\mathbf{z}_k^*]^\top\mathbf{b}| &\le \sum_{i=1}^{k}\frac{1}{\sqrt{n_k^*}}\left(\frac{d}{n_j}\frac{1}{\sqrt{n_j^*}}|C_j' \cap C_k^*| + |B \cap C_k^*|\frac{1}{\sqrt{n_j}}\right)\\ &\le \frac{\alpha^{-1/2}d}{n_j^* n_j}|C_j'| + \frac{\alpha^{-1/2}}{\sqrt{n_j^* n_j}}d\\ &< \frac{2\alpha^{-1/2}d}{\sqrt{n_j n_j^*}}.\end{aligned}$$

We show that the latter is at most $2.05\alpha^{-1/2}r^2$. Indeed, it is not hard to show that from $n_j \le 1.1 n_j^*$ (see Lemma 6.1) it follows

$$\frac{n_j - n_j^*}{\sqrt{n_j n_j^*}} \le 2.05\left(1 - \frac{n_j^*}{\sqrt{n_j n_j^*}}\right) \le 2.05 \times \frac{r^2}{2},$$

thus $\|[Z^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*)\|_1 \le 3.05\alpha^{-1/2}r^2$ and the result follows.

3) The case $n_j < n_j^*$ can be resolved similarly to the previous one. Since $|C_j^* \setminus C_j| \geq n_j^* - n_j$ we can pick a subset $B \subset C_j^* \setminus C_j$ of size $d = n_j^* - n_j$ and set $C_j' = B \cup C_j$ with $|C_j'| = n_j^*$; set also $\mathbf{z}_j' = \mathbf{z}_{C_j'}$. Then, we have

$$\|\mathbf{z}_j' - \mathbf{z}_j^*\|^2 = 2 - 2\frac{|C_j' \cap C_j^*|}{n_j^*} \leq 2 - \frac{2|C_j \cap C_j'|}{\sqrt{n_j n_j^*}} = \|\mathbf{z}_j - \mathbf{z}_j^*\|^2,$$

and it is not hard to derive that $\|\mathbf{z}_j' - \mathbf{z}_j^*\|^2 \leq \|\mathbf{z}_j - \mathbf{z}_j^*\|^2$. Thus, by the first part of this proof it holds $\|[Z^*]^\top(\mathbf{z}_j' - \mathbf{z}_j^*)\|_1 \leq \alpha^{-1/2} r^2$. Setting $\mathbf{b} = \mathbf{z}_j' - \mathbf{z}_j$ we have,

$$b_i = \begin{cases} (n_j^*)^{-1/2} - {n_j}^{-1/2}, & i \in C_j, \\ n_j^{*-1/2}, & i \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Since $|{n_j}^{-1/2} - (n_j^*)^{-1/2}| \leq \frac{d}{n_j^*}\frac{1}{\sqrt{n_j}}$ we obtain,

$$\begin{aligned}\sum_{k=1}^{K} |[\mathbf{z}_k^*]^\top \mathbf{b}| &\leq \sum_{i=1}^{k} \frac{1}{\sqrt{n_k^*}} \left(\frac{d}{n_j^*}\frac{1}{\sqrt{n_j}} |C_j \cap C_k^*| + |B \cap C_k^*| \frac{1}{\sqrt{n_j^*}} \right) \\ &\leq \frac{\alpha^{-1/2} d}{(n_j^*)^{3/2} n_j^{1/2}} |C_j| + \frac{\alpha^{-1/2}}{n_j^*} d \\ &< \frac{2\alpha^{-1/2} d}{n_j^*}.\end{aligned}$$

It is left to notice that

$$r^2 \geq 2 - \frac{2n_j}{\sqrt{n_j n_j^*}} = \frac{2(\sqrt{n_j^*} - \sqrt{n_j})}{\sqrt{n_j}} = \frac{2(n_j^* - n_j)}{n_j^* + \sqrt{n_j n_j^*}} \geq \frac{2d}{2n_j^*},$$

therefore $\|[Z^*]^\top \mathbf{b}\|_1 \leq 2\alpha^{-1/2} r^2$, thus $\|[Z^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*)\|_1 \leq 3\alpha^{-1/2} r^2$. □

Lemma 6.4. *Let $r = \|Z_{\mathcal{C}} - Z^*\|_{\mathsf{F}}$ and suppose that $r \leq 0.3$. Then $\|P_{\mathcal{C}} - P_{\mathcal{C}^*}\|_{\mathsf{F}}^2 \geq 2r^2(1 - 10\alpha^{-1} r^2)$.*

Proof. Denote $\mathbf{z}_j = \mathbf{z}_{C_j}$ and $r_j = \|\mathbf{z}_j - \mathbf{z}_j^*\|$. It holds,

$$\|P_{\mathcal{C}} - P_{\mathcal{C}^*}\|_{\mathsf{F}}^2 = 2K - 2\mathrm{Tr}(P_{\mathcal{C}} P_{\mathcal{C}^*}) = 2K - \sum_{j,k} (\mathbf{z}_j^\top \mathbf{z}_k^*)^2.$$

Notice, that $2\mathbf{z}_j^\top \mathbf{z}_j^* = 2 - \|\mathbf{z}_j\|^2 - \|\mathbf{z}_j^*\|^2 + 2\mathbf{z}_j^\top \mathbf{z}_j^* = 2 - \|\mathbf{z}_j - \mathbf{z}_j^*\|^2$, i.e. $\mathbf{z}_j^\top \mathbf{z}_j^* = 1 - r_j^2/2$. In particular, $1 - (\mathbf{z}_j^\top \mathbf{z}_j^*)^2 = r_j^2 - r_j^4/4$, whereas $([\mathbf{z}_j^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*))^2 = r_j^4/4$. Since we additionally have $[\mathbf{z}_k^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*) = [\mathbf{z}_k^*]^\top \mathbf{z}_j$ for $k \neq j$, it holds

$$\begin{aligned} 2K - 2\sum_{j,k}(\mathbf{z}_j^\top \mathbf{z}_k^*)^2 &= 2\sum_j r_j^2 - r_j^4/4 - 2\sum_j \sum_{k\neq j} \left([\mathbf{z}_k^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*)\right)^2 \\ &= 2r^2 - 2\sum_{j,k}\left([\mathbf{z}_k^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*)\right)^2 \\ &= 2r^2 - 2\sum_j \|[Z^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*)\|^2 \end{aligned}$$

By Lemma 6.3 we have for every $j = 1, \dots, K$

$$\|[Z^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*)\| \leq \|[Z^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*)\|_1 \leq 3.05\alpha^{-1/2} r_j^2,$$

therefore

$$\sum_j \|[Z^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*)\|^2 \leq 10\alpha^{-1}\sum_j r_j^4 \leq 10\alpha^{-1} r^4,$$

thus inequality follows. □

Lemma 6.5. *Let C, C' be such that $|C \triangle C'| = 1$. Then $\|\mathbf{z}_C - \mathbf{z}_{C'}\|^2 \leq \frac{2}{|C|\vee|C'|}$.*

Proof. Suppose, $|C'| > |C|$ then $C' = C \cup \{a\}$ and denoting $n = |C|$ we have

$$\|\mathbf{z}_C - \mathbf{z}_{C'}\|^2 = n\left(\sqrt{\frac{1}{n+1}} - \sqrt{\frac{1}{n}}\right)^2 + \frac{1}{n+1} = \frac{(\sqrt{n+1} - \sqrt{n})^2 + 1}{n+1} \leq \frac{2}{n+1}.$$

□

6.2 Proof of Theorem 3.6

The proof consists of several steps, each represented by a separate lemma.

Lemma 6.6. *Suppose, Assumption 1 holds and let $N \geq 2$. There is a constant $C = C(\gamma, L)$, so that if*

$$\frac{s \log N \log^2 T}{T p_{\min}^2} \leq \frac{1}{3},$$

then with probability at least $1 - 1/N$ and for with $\Delta_1 = C\sigma_{\max}\sqrt{\frac{\log N}{T p_{\min}^2}}$ the following inequalities take place for every $j = 1, \dots, K$

- $$\|\hat{A} - A\|_{\infty,\infty} \leq \Delta_1, \qquad \|\Sigma_{\Lambda_j,\Lambda_j}^{-1}(\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot})\|_{\infty,\infty} \leq \sigma_{\min}^{-1}\Delta_1; \tag{6.1}$$

- $$\|(\hat{A} - A)\mathbf{z}_j^*\|_\infty \leq \Delta_1, \qquad \|\Sigma_{\Lambda_j,\Lambda_j}^{-1}(\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot})\mathbf{z}_j^*\|_\infty \leq \sigma_{\min}^{-1}\Delta_1; \tag{6.2}$$

- $$\|\hat{\Sigma} - \Sigma\|_{\infty,\infty} \leq \Delta_1, \qquad \|(\hat{\Sigma}_{\Lambda_j,\cdot} - \Sigma_{\Lambda_j,\cdot})\mathbf{v}_j^*\|_\infty \leq \Delta_1; \tag{6.3}$$

- $$\|\Sigma_{\Lambda_j,\Lambda_j}^{-1}(\hat{\Sigma}_{\Lambda_j,\cdot} - \Sigma_{\Lambda_j,\cdot})\mathbf{v}_j^*\|_\infty \leq \sigma_{\min}^{-1}\Delta_1; \tag{6.4}$$

-
$$|||\hat{\Sigma}_{\Lambda_j,\Lambda_j} - \Sigma_{\Lambda_j,\Lambda_j}|||_{\mathsf{op}} \le \sqrt{s}\Delta_1. \tag{6.5}$$

Proof. By Theorem 3.5 for any pair $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$ with $\|\mathbf{a}\| \le 1$, $\|\mathbf{b}\| \le 1$ it holds probability $\ge 1 - N^{-m}$,

$$|\mathbf{a}^\top(\hat{A} - A)\mathbf{b}| \le C\sigma_{\max}\left(\sqrt{\frac{(m+1)\log N}{Tp_{\min}^2}} \bigvee \frac{(m+1)\log N \log T}{Tp_{\min}^2}\right).$$

Suppose for a moment that m is such that

$$\sqrt{\frac{(m+1)s\log N}{Tp_{\min}^2}}\log T \le 1, \tag{6.6}$$

so that we can neglect the second term. Set,

$$A_0 = \{(\mathbf{e}_i, \mathbf{e}_{i'}) : \ i, i' \le N\}, \qquad B_0 = \{(\mathbf{e}_i, \mathbf{z}_l^*) : \ i \le N, l \le K\},$$

as well as for every $j = 1, \dots, K$

$$A_j = \{(\sigma_{\min}\Sigma_{\Lambda_j,\Lambda_j}^{-1}\mathbf{e}_i, \mathbf{e}_{i'}) : \ i \in \Lambda_j, i' \le N\},$$
$$B_j = \{(\sigma_{\min}\Sigma_{\Lambda_j,\Lambda_j}^{-1}\mathbf{e}_i, \mathbf{z}_l^*) : \ i \in \Lambda_j, l \le K\}.$$

Obviously we have $|A_0| \le N^2, |B_0| \le NK$ and $|A_j| \le sN, |B_j| \le sK$ for $j = 1, \dots, N$, so since $s, K \le N$ together they have not more than $4N^3$ pairs of vectors $(\mathbf{a}, \mathbf{b})$, each having norm bounded by one. Taking a union bound, we have that the inequalities (6.1) and (6.2) hold with probability at least $1 - 4N^{3-m}$. By analogy, we can show that (6.3) and (6.4) hold with probability at least $1 - 4N^{3-m}$.

As for the last inequality, for every $j = 1, \dots, K$ pick $P_j = \sum_{i\in\Lambda_j}\mathbf{e}_i\mathbf{e}_i^\top$, i.e. projectors onto the subspace of vectors supported on Λ_j. Then by Theorem 3.4 it holds with probability at least $1 - KN^{-m}$ for every $j = 1, \dots, K$ (taking into account (6.6))

$$|||\hat{\Sigma}_{\Lambda_j,\Lambda_j} - \Sigma_{\Lambda_j,\Lambda_j}|||_{\mathsf{op}} = |||P_j(\hat{\Sigma} - \Sigma)P_j|||_{\mathsf{op}} \le C\sigma_{\max}\sqrt{\frac{s(m+1)\log N}{Tp_{\min}^2}}.$$

The total probability will be at least $1 - 8N^{3-m} - KN^{-m}$, which is at least $1 - 1/N$ whenever $m \ge 7$ and $N \ge 2$.

□

In the following we apply the technique from Gribonval et al. (2015). Suppose, that the LASSO solution $\hat{\mathbf{v}}_j$ for a given clustering $\mathcal{C}$ is not only supported exactly on Λ_j, but the signs are matching those of the true $\mathbf{v}_j^*$. Then, $\|\hat{\mathbf{v}}_j\|_1 = \bar{\mathbf{s}}_j^\top(\hat{\mathbf{v}}_j)_{\Lambda_j}$. Therefore, we can write

$$(\hat{\mathbf{v}}_j)_{\Lambda_j} = \arg\min_{\mathbf{v}\in\mathbb{R}^{\Lambda_j}} \frac{1}{2}\mathbf{v}^\top\hat{\Sigma}_{\Lambda_j,\Lambda_j}\mathbf{v} - \mathbf{v}^\top\hat{A}_{\Lambda_j,\cdot}\mathbf{z}_j + \lambda\bar{\mathbf{s}}_j^\top\mathbf{v}$$
$$= \hat{\Sigma}_{\Lambda_j,\Lambda_j}^{-1}(\hat{A}_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\bar{\mathbf{s}}_j),$$

and plugging this solution into the risk function we get that $F_\lambda(\mathcal{C}) = \Phi_\lambda(\mathcal{C})$, where the latter is defined explicitly

$$\Phi_\lambda(\mathcal{C}) = -\frac{1}{2}\sum_{j=1}^{K}(\hat{A}_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\bar{\mathbf{s}}_j)^\top \hat{\Sigma}^{-1}_{\Lambda_j,\Lambda_j}(\hat{A}_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\bar{\mathbf{s}}_j).$$

The next lemma shows that such representation takes place in the local vicinity of the true clustering $\mathcal{C}^*$.

Lemma 6.7. *Suppose, the inequalities (6.1)–(6.5) take place. Assume,*

$$s\Delta_1 \le 1/16, \qquad 12\Delta_1 \le \lambda \le \frac{\sigma_{\min}}{4}\tau_0 s^{-1}. \tag{6.7}$$

Then, for any $\mathcal{C} = (C_1, \dots, C_K)$ satisfying

$$\max_j \|\mathbf{z}_{C_j} - \mathbf{z}_{C_j^*}\| \le 0.3 \wedge 0.22\sqrt{\left(2\sigma_{\max}\alpha^{-1/2} + \sqrt{n^*}\Delta_1\right)^{-1}\lambda} \tag{6.8}$$

it holds

$$|||\hat{V}_{\lambda,\mathcal{C}} - V^*|||_{\mathsf{F}} \le 3\sigma_{\min}^{-1}\sqrt{Ks}\lambda,$$

and the equality $F_\lambda(\mathcal{C}) = \Phi_\lambda(\mathcal{C})$ takes place.

Proof. Taking into account $Z^\top Z = I_K$, it holds

$$\begin{aligned} R_{\lambda,\mathcal{C}}(V) =& \frac{1}{2}\mathrm{Tr}\left(V^\top\hat{\Sigma}V\right) - \mathrm{Tr}\left(V^\top\hat{A}Z\right) + \lambda\|V\|_{1,1} \\ =& \sum_{j=1}^{K}\frac{1}{2}\mathbf{v}_j^\top\hat{\Sigma}\mathbf{v}_j - \mathbf{v}_j^\top\hat{A}\mathbf{z}_j + \lambda\|\mathbf{v}_j\|_1, \end{aligned}$$

so that the optimization problem separates into K independent subproblems. Solving each of the problems

$$\frac{1}{2}\mathbf{v}_j^\top\hat{\Sigma}\mathbf{v}_j - \mathbf{v}_j^\top\hat{A}\mathbf{z}_j + \lambda\|\mathbf{v}_j\|_1 \to \min_{\mathbf{v}_j}$$

corresponds to Corollary B.3 with $\hat{D} = \hat{\Sigma}$ and $\hat{\mathbf{c}} = \hat{A}\mathbf{z}_j$, whereas the "true" version of the problem corresponds to $\bar{D} = \Sigma$ and $\bar{\mathbf{c}} = A\mathbf{z}_j^* = \Sigma(\Theta^*)^\top\mathbf{z}_j^* = \Sigma\mathbf{v}_j^*$. We need to control the differences between $\hat{\mathbf{c}}$ and $\bar{\mathbf{c}}$, and between $\hat{D}$ and $\bar{D}$. It holds,

$$\|\hat{A}\mathbf{z}_j - A\mathbf{z}_j^*\|_\infty \le \|A(\mathbf{z}_j - \mathbf{z}_j^*)\|_\infty + \|(\hat{A} - A)\mathbf{z}_j^*\|_\infty + \|(\hat{A} - A)(\mathbf{z}_j - \mathbf{z}_j^*)\|_\infty .$$

Since $A = \Sigma V^*[Z^*]^\top$, we bound the first term using Lemma 6.3

$$\|A(\mathbf{z}_j - \mathbf{z}_j^*)\|_\infty \le \|\Sigma V^*\|_{\infty,\infty}\|[Z^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*)\|_1 \le 3.05\alpha^{-1/2}\|\Sigma V^*\|_{\infty,\infty}r_j^2.$$

The second term is bounded by Δ_1, whereas the fourth term satisfies

$$\|(\hat{A} - A)(\mathbf{z}_j - \mathbf{z}_j^*)\|_\infty \le \|\hat{A} - A\|_{\infty,\infty}\|\mathbf{z}_j - \mathbf{z}_j^*\|_1 \le 1.55\Delta_1\sqrt{n^*}r_j^2,$$

where we also used Lemma 6.2. Summing up we get,

$$\|\hat{\mathbf{c}} - \mathbf{c}\|_\infty \le 1.55(2\sigma_{\max}\alpha^{-1/2} + \sqrt{n_j^*}\Delta_1)r_j^2 + \Delta_1 .$$

Similarly, we bound $\|\Sigma_{\Lambda_j,\Lambda_j}(\hat{\mathbf{c}}_{\Lambda_j} - \bar{\mathbf{c}}_{\Lambda_j})\|_\infty$ as follows

$$\begin{aligned}\|\Sigma^{-1}_{\Lambda_j,\Lambda_j}(\hat{A}_{\Lambda_j,\cdot}\mathbf{z}_j - A_{\Lambda_j,\cdot}\mathbf{z}_j^*)\|_\infty \le& \|\Sigma^{-1}_{\Lambda_j,\Lambda_j}A(\mathbf{z}_j - \mathbf{z}_j^*)\|_\infty + \|\Sigma^{-1}_{\Lambda_j,\Lambda_j}(\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot})\mathbf{z}_j^*\|_\infty \\ &+ \|\Sigma^{-1}_{\Lambda_j,\Lambda_j}(\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot})(\mathbf{z}_j - \mathbf{z}_j^*)\|_\infty \\ \le& \|\Sigma^{-1}_{\Lambda_j,\Lambda_j}A(\mathbf{z}_j - \mathbf{z}_j^*)\|_\infty + 1.55\sigma_{\min}^{-1}\Delta_1\sqrt{n^*}r_j^2 + \sigma_{\min}^{-1}\Delta_1 \\ \le& 1.55\sigma_{\min}^{-1}(2\sigma_{\max}\alpha^{-1/2} + \sqrt{n_j^*}\Delta_1)r_j^2 + \sigma_{\min}^{-1}\Delta_1\end{aligned}$$

To sum up, Corollary B.3 is applied with

$$\begin{aligned}\delta_c =& 1.55(2\sigma_{\max}\alpha^{-1/2} + \sqrt{n^*}\Delta_1)r_j^2 + \Delta_1, \\ \delta_c' =& 1.55\sigma_{\min}^{-1}(2\sigma_{\max}\alpha^{-1/2} + \sqrt{n^*}\Delta_1)r_j^2 + \sigma_{\min}^{-1}\Delta_1 \\ \delta_D =& \Delta_1, \qquad \delta_D' = \Delta_1, \qquad \delta_D'' = \sigma_{\min}^{-1}\Delta_1.\end{aligned}$$

It requires the conditions,

$$3\{1.55(2\sigma_{\max}\alpha^{-1/2} + \sqrt{n^*}\Delta_1)r_j^2 + 2\Delta_1\} \le \lambda, \qquad s\Delta_1 \le \frac{1}{16},$$

and due to the fact that $\|D^{-1}_{\Lambda_j,\Lambda_j}\|_{1,\infty} \le \sqrt{s}|||D^{-1}_{\Lambda_j,\Lambda_j}|||_{\mathsf{op}}$ and Assumption 3.6,

$$2\sigma_{\min}^{-1}(1.55(2\sigma_{\max}\alpha^{-1/2} + \sqrt{n^*}\Delta_1)r_j^2 + 2\Delta_1 + \sqrt{s}\lambda) < \tau_0 s^{-1/2},$$

which are not hard to derive from the given inequalities. All this that $\hat{\mathbf{v}}_j$ is supported on Λ_j and the solution satisfies

$$(\hat{\mathbf{v}}_j)_{\Lambda_j} = \hat{\Sigma}^{-1}_{\Lambda_j,\Lambda_j}\left(\hat{A}_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\mathbf{s}_j^*\right),$$

and the corresponding minimum is equal to

$$\frac{1}{2}\hat{\mathbf{v}}_j^\top\hat{\Sigma}\hat{\mathbf{v}}_j^\top - \hat{\mathbf{v}}_j^\top\hat{A}\mathbf{z}_j + \lambda(\hat{\mathbf{v}}_j)_{\Lambda_j}^\top\mathbf{s}_j^* = -\frac{1}{2}\left(\hat{A}_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\mathbf{s}_j^*\right)^\top\hat{\Sigma}^{-1}_{\Lambda_j,\Lambda_j}\left(\hat{A}_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\mathbf{s}_j^*\right).$$

Summing up we get the corresponding expression for $F_\lambda(\mathcal{C})$. Moreover, we have

$$\begin{aligned}\|\hat{\mathbf{v}}_j - \mathbf{v}_j^*\| \le& 2\sqrt{s}\left\{2\Delta_1 + 1.55(2\sigma_{\max}\alpha^{-1} + \sqrt{n^*}\Delta_1)r_j^2 + \lambda\right\} \\ \le& 2\sigma_{\min}^{-1}\sqrt{s}\left(\frac{\lambda}{6} + \frac{1.55\lambda}{20} + \lambda\right) \\ \le& 3\sigma_{\min}^{-1}\sqrt{s}\lambda,\end{aligned}$$

and together it provides a bound on $|||\hat{V}_{\lambda,\mathcal{C}} - V^*|||_{\mathsf{F}}$. □

Consider the function,

$$\bar{\Phi}_\lambda(\mathcal{C}) = -\frac{1}{2}\sum_{j=1}^k\left(A_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\mathbf{s}_j^*\right)^\top\Sigma^{-1}_{\Lambda_j,\Lambda_j}\left(A_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\mathbf{s}_j^*\right).$$

The following lemma shows how this function grows with $\mathcal{C}$ receding from the true clustering $\mathcal{C}^*$.

Lemma 6.8. *Suppose, $\mathcal{C}$ is some clustering such that $r = \|\|Z_{\mathcal{C}} - Z^*\|\|_{\mathsf{F}} \le 0.3$. Then,*

$$\bar{\Phi}_\lambda(\mathcal{C}) - \bar{\Phi}_\lambda(\mathcal{C}^*) \ge \frac{a_0}{2} r^2(1 - 10\alpha^{-1}r^2) - \lambda\sqrt{Ks}\|\|V^*\|\|_{\mathsf{F}} r.$$

Proof. Denoting $\bar{\Phi}_0(\mathcal{C}) = -\frac{1}{2}\sum_{j=1}^k \mathbf{z}_j^\top \hat{A}_{\Lambda_j,\cdot}^\top \hat{\Sigma}_{\Lambda_j,\Lambda_j}^{-1} \hat{A}_{\Lambda_j,\cdot}\mathbf{z}_j$ (which indeed corresponds to $\lambda = 0$), we have the decomposition

$$\bar{\Phi}_\lambda(\mathcal{C}) - \bar{\Phi}_\lambda(\mathcal{C}^*) = \bar{\Phi}_0(\mathcal{C}) - \bar{\Phi}_0(\mathcal{C}^*) - \lambda\sum_{j=1}^K [\mathbf{s}_j^*]^\top \Sigma_{\Lambda_j,\Lambda_j}^{-1} A_{\Lambda_j,\cdot}(\mathbf{z}_j - \mathbf{z}_j^*).$$

Let us first deal with the term $\bar{\Phi}_0(\mathcal{C}) - \bar{\Phi}_0(\mathcal{C}^*)$. Note that since $[\mathbf{v}_j^*]_{\Lambda_j} = \Sigma_{\Lambda_j,\Lambda_j}^{-1} A_{\Lambda_j,\cdot}\mathbf{z}_j^*$, we have

$$\bar{\Phi}_0(\mathcal{C}^*) = -\frac{1}{2}\sum_{j=1}^K [\mathbf{v}_j^*]^\top \Sigma \mathbf{v}_j^* = -\frac{1}{2}\mathrm{Tr}([V^*]^\top \Sigma V^*) = -\frac{1}{2}\mathrm{Tr}(\Theta^*\Sigma[\Theta^*]^\top).$$

whereas

$$\bar{\Phi}_0(\mathcal{C}) = \min_{V=[\mathbf{v}_1,\ldots,\mathbf{v}_k]} \frac{1}{2}\mathrm{Tr}(V^\top \Sigma V) - \mathrm{Tr}(V^\top A Z_{\mathcal{C}})$$

where the minimum is taken s.t. the restrictions $\mathrm{supp}(\mathbf{v}_j) \subset \Lambda_j$. Dropping the restrictions we get,

$$\begin{aligned}
\bar{\Phi}_0(\mathcal{C}) - \bar{\Phi}_0(\mathcal{C}^*) &\ge \min_V \frac{1}{2}\mathrm{Tr}(V^\top \Sigma V) - \mathrm{Tr}(V^\top A Z_{\mathcal{C}}) + \frac{1}{2}\mathrm{Tr}(\Theta^*\Sigma[\Theta^*]^\top) \\
&= \min_V \frac{1}{2}\|\|Z_{\mathcal{C}} V^\top \Sigma^{1/2}\|\|_{\mathsf{F}}^2 - \mathrm{Tr}(Z_{\mathcal{C}} V^\top \Sigma[\Theta^*]^\top) + \|\|\Theta^*\Sigma^{1/2}\|\|_{\mathsf{F}}^2 \\
&= \min_V \frac{1}{2}\|\|(Z_{\mathcal{C}} V^\top - \Theta^*)\Sigma^{1/2}\|\|_{\mathsf{F}}^2.
\end{aligned}$$

It is not hard to calculate that the minimum is attained for $V = [\Theta^*]^\top Z_{\mathcal{C}}$ and therefore

$$\bar{\Phi}_0(\mathcal{C}) - \bar{\Phi}_0(\mathcal{C}^*) \ge \frac{1}{2}\|\|(Z_{\mathcal{C}} Z_{\mathcal{C}}^\top - I)\Theta^*\Sigma^{1/2}\|\|_{\mathsf{F}}^2 \ge \frac{a_0}{2}\|\|(Z_{\mathcal{C}} Z_{\mathcal{C}}^\top - I)Z^*\|\|_{\mathsf{F}}^2,$$

where the latter follows using $\Theta^* = Z^*[V^*]^\top$ and from the fact that $\lambda_{\min}([V^*]^\top \Sigma V^*) \ge \sigma_0$. Moreover,

$$\begin{aligned}
\|\|(Z_{\mathcal{C}} Z_{\mathcal{C}}^\top - I)Z^*\|\|_{\mathsf{F}}^2 &= \mathrm{Tr}((P_{\mathcal{C}} - I)P_{\mathcal{C}^*}(P_{\mathcal{C}} - I)) = \mathrm{Tr}(P_{\mathcal{C}^*}) - \mathrm{Tr}(P_{\mathcal{C}} P_{\mathcal{C}^*}) \\
&= \frac{1}{2}\|\|P_{\mathcal{C}} - P_{\mathcal{C}^*}\|\|_{\mathsf{F}}^2,
\end{aligned}$$

where we used the fact that $\mathrm{Tr}(P_{\mathcal{C}}) = \mathrm{Tr}(P_{\mathcal{C}^*}) = K$. It is left to recall the result of Lemma 6.4, so that we get

$$\bar{\Phi}_0(\mathcal{C}) - \bar{\Phi}_0(\mathcal{C}^*) \ge \frac{a_0 r^2}{2}(1 - 10\alpha^{-1}r^2).$$

As for the linear term, it holds

$$\left(\sum_{j=1}^{K}[\mathbf{s}_j^*]^\top \Sigma_{\Lambda_j,\Lambda_j}^{-1} A_{\Lambda_j,\cdot}(\mathbf{z}_j - \mathbf{z}_j^*)\right)^2 \le \left(\sum_{j=1}^{K}\|[\mathbf{s}_j^*]^\top \Sigma_{\Lambda_j,\Lambda_j}^{-1} A_{\Lambda_j,\cdot}\|^2\right) r^2$$

Since $A = \Sigma[\Theta^*]^\top$, we have $A_{\Lambda_j,\cdot}^\top \Sigma_{\Lambda_j,\Lambda_j}^{-1}\mathbf{s}_j^* = \Theta^*\Sigma_{\cdot,\Lambda_j}\Sigma_{\Lambda_j,\Lambda_j}^{-1}\mathbf{s}_j^*$. Denote, $\mathbf{x} = \Sigma_{\cdot,\Lambda_j}\Sigma_{\Lambda_j,\Lambda_j}^{-1}\mathbf{s}_j^*$, then we have $\mathbf{x}_{\Lambda_j} = \mathbf{s}_j$ and $\|\mathbf{x}_{\Lambda_j}\|_\infty = 1$. Moreover, by the ERC property

$$\|\mathbf{x}_{\Lambda_j^c}\|_\infty = \|\Sigma_{\Lambda_j^c,\Lambda_j}\Sigma_{\Lambda_j,\Lambda_j}^{-1}\mathbf{s}_j\|_\infty \le \|\Sigma_{\Lambda_j^c,\Lambda_j}\Sigma_{\Lambda_j,\Lambda_j}^{-1}\|_{1,\infty} \le 1/2.$$

We have

$$\|A_{\Lambda_j,\cdot}^\top \Sigma_{\Lambda_j,\Lambda_j}^{-1}\mathbf{s}_j^*\|^2 = \|\sum \mathbf{z}_j^*[\mathbf{v}_j^*]^\top \mathbf{x}\|^2 = \sum_{k=1}^{K}|[\mathbf{v}_k^*]^\top \mathbf{x}|^2,$$

where, since $\mathbf{v}_k^*$ is supported on Λ_k of size at most s,

$$|[\mathbf{v}_k^*]^\top \mathbf{x}| \le \|\mathbf{v}_k^*\|_1\|\mathbf{x}\|_\infty \le \sqrt{s}\|\mathbf{v}_k^*\|.$$

Summing up we get $\|A_{\Lambda_j,\cdot}^\top \Sigma_{\Lambda_j,\Lambda_j}^{-1}\mathbf{s}_j^*\|^2 \le s\|\!|V^*|\!\|_{\mathsf{F}}^2$, so that

$$\left|\sum_{j=1}^{K}[\mathbf{s}_j^*]^\top \Sigma_{\Lambda_j,\Lambda_j}^{-1} A_{\Lambda_j,\cdot}(\mathbf{z}_j - \mathbf{z}_j^*)\right| \le \sqrt{Ks}\|\!|V^*|\!\|_{\mathsf{F}} r.$$

The lemma now follows from the two terms put together. □

The next step is to bound the difference $\Phi_\lambda(\mathcal{C}) - \bar{\Phi}_\lambda(\mathcal{C})$ uniformly in the neighbourhood of $\mathcal{C}^*$.

Lemma 6.9. *Suppose that the inequalities (6.1)–(6.5) hold and let*

$$\Delta_1 \le \sigma_{\min}/(2\sqrt{s}) \vee \frac{\lambda}{12}, \qquad \sigma_{\max}/\sigma_{\min} \le n^*, \qquad \lambda \le \sigma_{\min}s^{-1}$$

Let some $r \le 0.3$ *satisfies* $\sqrt{sn^*}\Delta_1 r^2 \le \sigma_{\max}$. *Then,*

$$\sup_{\|\!|Z - Z^*|\!\|_{\mathsf{F}} \le r} |\Phi_\lambda(\mathcal{C}) - \bar{\Phi}_\lambda(\mathcal{C}) - \Phi_\lambda(\mathcal{C}^*) + \bar{\Phi}_\lambda(\mathcal{C}^*)|$$
$$\le 4\left(\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2\sqrt{s}\|\!|V^*|\!\|_{\mathsf{F}} + \frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{K}\right)\Delta_1 r + 15\frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{sn^*}\Delta_1 r^2.$$

Proof. Denote,

$$\tilde{\Phi}_\lambda(\mathcal{C}) = -\frac{1}{2}\sum_{j=1}^{K}\left(A_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\mathbf{s}_j^*\right)^\top \hat{\Sigma}_{\Lambda_j,\Lambda_j}^{-1}\left(A_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\mathbf{s}_j^*\right),$$

so that we have

$$
\begin{aligned}
&|\tilde{\Phi}_\lambda(\mathcal{C}) - \bar{\Phi}_\lambda(\mathcal{C}) - \tilde{\Phi}_\lambda(\mathcal{C}^*) + \bar{\Phi}_\lambda(\mathcal{C}^*)| \\
&\qquad \le \frac{1}{2}\sum_{j=1}^{K}\left|\left(A_{\Lambda_j,\cdot}(\mathbf{z}_j + \mathbf{z}_j^*) - 2\lambda \mathbf{s}_j^*\right)^\top (\hat{\Sigma}^{-1}_{\Lambda_j,\Lambda_j} - \Sigma^{-1}_{\Lambda_j,\Lambda_j}) A_{\Lambda_j,\cdot}(\mathbf{z}_j - \mathbf{z}_j^*)\right|
\end{aligned}
$$

First of all, due to (6.5) it holds,

$$
|||\hat{\Sigma}^{-1}_{\Lambda_j,\Lambda_j} - \Sigma^{-1}_{\Lambda_j,\Lambda_j}|||_{\mathsf{op}} \le \frac{\sigma_{\min}^{-2}\sqrt{s}\Delta_1}{1 - \sigma_{\min}^{-1}\sqrt{s}\Delta_1} \le 2\sigma_{\min}^{-2}\sqrt{s}\Delta_1.
$$

Since $A = \Sigma[\Theta^*]^\top$, we have

$$
\begin{aligned}
\|A_{\Lambda_j,\cdot}(\mathbf{z}_j - \mathbf{z}_j^*)\| &\le \sigma_{\max} r_j \\
\|A_{\Lambda_j,\cdot}(\mathbf{z}_j + \mathbf{z}_j^*) - 2\lambda \mathbf{s}_j^*\| &\le \sigma_{\max}(2\|\mathbf{v}_j^*\| + r_j) + 2\lambda\sqrt{s}.
\end{aligned}
$$

Then by Cauchy-Schwartz,

$$
\begin{aligned}
|\tilde{\Phi}_\lambda(\mathcal{C}) - \bar{\Phi}_\lambda(\mathcal{C}) - \tilde{\Phi}_\lambda(\mathcal{C}^*) + \bar{\Phi}_\lambda(\mathcal{C}^*)| &\le \sigma_{\min}^{-2}\sqrt{s}\Delta_1\left(\sum_{j=1}^{K}\sigma_{\max} r_j\left\{\sigma_{\max}(2\|\mathbf{v}_j\| + r_j) + 2\lambda\sqrt{s}\right\}\right) \\
&\le 2\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2\sqrt{s}|||V^*|||_{\mathsf{F}}\Delta_1 r + 2\frac{\sigma_{\max}}{\sigma_{\min}^2}\lambda s\sqrt{K}\Delta_1 r \\
&\quad + \left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2\sqrt{s}\Delta_1 r^2.
\end{aligned}
$$

Going further,

$$
\Phi_\lambda(\mathcal{C}) - \tilde{\Phi}_\lambda(\mathcal{C}) = -\frac{1}{2}\sum_{j=1}^{K}\left((A_{\Lambda_j,\cdot} + \hat{A}_{\Lambda_j,\cdot})\mathbf{z}_j - 2\lambda \mathbf{s}_j^*\right)^\top \hat{\Sigma}^{-1}_{\Lambda_j,\Lambda_j}(\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot})\mathbf{z}_j,
$$

which implies that

$$
\begin{aligned}
&|\Phi_\lambda(\mathcal{C}) - \tilde{\Phi}_\lambda(\mathcal{C}) - \Phi_\lambda(\mathcal{C}^*) + \tilde{\Phi}_\lambda(\mathcal{C}^*)| \\
&\quad \le \frac{1}{2}\sum_{j=1}^{K}\left|\left((A_{\Lambda_j,\cdot} + \hat{A}_{\Lambda_j,\cdot})(\mathbf{z}_j - \mathbf{z}_j^*)\right)^\top \hat{\Sigma}^{-1}_{\Lambda_j,\Lambda_j}(\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot})\mathbf{z}_j\right| \\
&\quad\ \ \frac{1}{2}\sum_{j=1}^{K}\left|\left((A_{\Lambda_j,\cdot} + \hat{A}_{\Lambda_j,\cdot})\mathbf{z}_j^* - 2\lambda \mathbf{s}_j^*\right)^\top \hat{\Sigma}^{-1}_{\Lambda_j,\Lambda_j}(\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot})(\mathbf{z}_j - \mathbf{z}_j^*)\right|
\end{aligned}
\tag{6.9}
$$

First notice, that due to Lemma 6.2 and (6.1) it holds,

$$
\begin{aligned}
\|(\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot})(\mathbf{z}_j - \mathbf{z}_j^*)\| &\le \sqrt{s}\|\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot}\|_{\infty,\infty}\|\mathbf{z}_j - \mathbf{z}_j^*\|_1 \\
&\le 1.55\sqrt{sn^*}\Delta_1 r_j^2.
\end{aligned}
$$

Therefore, it follows

$$\|(\hat{A}_{\Lambda_j,\cdot} + A_{\Lambda_j,\cdot})(\mathbf{z}_j - \mathbf{z}_j^*)\| \le 2\sigma_{\max} r_j + 1.55\sqrt{sn^*}\Delta_1 r_j^2.$$

Moreover, using (6.2) we get

$$\begin{aligned}\|(\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot})\mathbf{z}_j\| &\le \Delta_1 + 1.55\sqrt{sn^*}\Delta_1 r_j^2\\ \|(\hat{A}_{\Lambda_j,\cdot} + A_{\Lambda_j,\cdot})\mathbf{z}_j^* - 2\lambda\mathbf{s}_j^*\| &\le 2\sigma_{\max}\|\mathbf{v}_j\| + \Delta_1 + 2\lambda\sqrt{s}.\end{aligned}$$

and we also have $|||\hat{\Sigma}^{-1}_{\Lambda_j,\Lambda_j}|||_{\mathsf{op}} \le 2\sigma_{\min}^{-1}$ due to the condition $\sigma_{\min}^{-1}\sqrt{s}\Delta_1 \le 1/2$. Thus we get that the first sum of (6.9) is bounded by

$$\begin{aligned}&\sigma_{\min}^{-1}\sum_{j=1}^{K}\left(2\sigma_{\max} r_j + 1.55\sqrt{sn^*}\Delta_1 r_j^2\right)\left(\Delta_1 + 1.55\sqrt{sn^*}\Delta_1 r_j^2\right)\\ &\qquad\le 2\frac{\sigma_{\max}}{\sigma_{\min}}\Delta_1\sqrt{K}r + 1.55\sigma_{\min}^{-1}\sqrt{sn^*}\Delta_1^2 r^2 + 3.1\frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{sn^*}\Delta_1 r^3 + 2.5\sigma_{\min}^{-1} sn^*\Delta_1^2 r^4,\end{aligned}$$

while the second sum is bounded by

$$\begin{aligned}\sigma_{\min}^{-1}\sum_{j=1}^{K}&\left(2\sigma_{\max}\|\mathbf{v}_j^*\| + \Delta_1 + 2\lambda\sqrt{s}\right)\left(1.55\sqrt{sn^*}\Delta_1 r_j^2\right)\\ &\le \frac{1.55}{\sigma_{\min}}\left(\sigma_{\max}\sqrt{sn^*} + \sqrt{sn^*}\Delta_1 + 2\lambda s\sqrt{n^*}\right)\Delta_1 r^2\\ &\le \frac{5}{\sigma_{\min}}\left(\sigma_{\max}\sqrt{sn^*} + \lambda s\sqrt{n^*}\right)\Delta_1 r^2\end{aligned}$$

where we used the fact that $\max_j\|\mathbf{v}_j^*\| \le |||V^*|||_{\mathsf{op}} = |||\Theta^*|||_{\mathsf{op}} < 1$ together with the condition $\Delta_1 \le \sigma_{\max}$. Combining all the bounds we get

$$\begin{aligned}&|\Phi_\lambda(\mathcal{C}) - \bar{\Phi}_\lambda(\mathcal{C}) - \Phi_\lambda(\mathcal{C}^*) + \bar{\Phi}_\lambda(\mathcal{C}^*)|\\ &\quad\le 2\left(\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2\sqrt{s}|||V^*|||_{\mathsf{F}} + 2\frac{\sigma_{\max}}{\sigma_{\min}^2}\lambda s\sqrt{K} + 2\frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{K}\right)\Delta_1 r\\ &\qquad+\left(5\frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{sn^*} + 5\sigma_{\min}^{-1}\lambda s\sqrt{n^*} + 1.55\sigma_{\min}^{-1}\sqrt{sn^*}\Delta_1 + \left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2\sqrt{s}\right)\Delta_1 r^2\\ &\qquad+3.1\frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{sn^*}\Delta_1 r^3\\ &\qquad+2.5\sigma_{\min}^{-1} sn^*\Delta_1^2 r^4,\end{aligned}$$

where by $r \le 0.3$ and $\sqrt{sn^*}\Delta_1 \le \sigma_{\max}$ we can neglect the third and the fourth power, respectively, and thus the required bound follows. □

Lemma 6.10. *There are numerical constant $c, C > 0$ such that the following holds. Suppose, the inequalities take place:*

$$\sqrt{\frac{sn^*\log N}{Tp_{\min}^2}} \le c\frac{a_0\sigma_{\min}}{\sigma_{\max}^2}, \qquad n^* \ge \sigma_{\max}/\sigma_{\min}. \tag{6.10}$$

Let $C\sigma_{\max}\sqrt{\frac{\log N}{Tp_{\min}^2}} \le \lambda \le c\sigma_{\min}\tau_0 s^{-1}$*, and set*

$$\bar{r} = 0.3 \wedge 0.18\sqrt{\alpha} \wedge 0.22\sqrt{\left(2\sigma_{\max}\alpha^{-1/2} + \sqrt{n^*}\Delta_1\right)^{-1}\lambda}.$$

Then under the inequalities (6.1)–(6.5) *the clustering*

$$\hat{\mathcal{C}} = \arg\min_{\|\!|Z_{\mathcal{C}} - Z^*\|\!|_{\mathsf{F}} \le r_{\max}} F_\lambda(\mathcal{C})$$

satisfies

$$\|\!|Z_{\hat{\mathcal{C}}} - Z^*\|\!|_{\mathsf{F}} \le \frac{C}{a_0}\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2 \lambda K\sqrt{s}\,.$$

Proof. It is not hard to see that for $\Delta_1 = \sqrt{\frac{\log N}{Tp_{\min}^2}}$ the inequalities required by Lemmas 6.7–6.9 are satisfied for $r \le \bar{r}$ due to (6.10) and conditions on λ and $\bar{r}$. Since obviously $\hat{\mathcal{C}}$ satisfies $F_\lambda(\hat{\mathcal{C}}) \le F_\lambda(\mathcal{C}^*)$, we have for $r = \|\!|Z_{\hat{\mathcal{C}}} - Z_{\mathcal{C}^*}\|\!|_{\mathsf{F}} \le r_{\max}$

$$\begin{aligned}
F_\lambda(\hat{\mathcal{C}}) - F_\lambda(\mathcal{C}^*) \ge& \bar{\Phi}_\lambda(\mathcal{C}) - \bar{\Phi}_\lambda(\mathcal{C}) - |F_\lambda(\mathcal{C}) - \bar{\Phi}_\lambda(\mathcal{C}) - F_\lambda(\mathcal{C}^*) + \bar{\Phi}_\lambda(\mathcal{C}^*)| \\
\ge& \frac{a_0 r^2}{2}\left(1 - 10\alpha^{-1}r^2\right) - \lambda\sqrt{Ks}\|\!|V^*\|\!|_{\mathsf{F}} r \\
&- 4\left(\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2 \sqrt{s}\|\!|V^*\|\!|_{\mathsf{F}} + \frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{K}\right)\Delta_1 r - 15\frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{sn^*}\Delta_1 r^2 \\
=& \frac{a_0 r^2}{2}\left(1 - 10\alpha^{-1}r^2 - \frac{30}{a_0}\frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{sn^*}\Delta_1\right) \\
&- \lambda\sqrt{Ks}\|\!|V^*\|\!|_{\mathsf{F}} r - 4\left(\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2 \sqrt{s}\|\!|V^*\|\!|_{\mathsf{F}} + \frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{K}\right)\Delta_1 r\,.
\end{aligned}$$

Since $\bar{r} \le 0.2\sqrt{\alpha}$ implies $10\alpha^{-1}r^2 \le \frac{1}{3}$, it holds by (6.10)

$$1 - 10\alpha^{-1}r^2 - \frac{30}{a_0}\frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{sn^*}\Delta_1 \ge \frac{1}{2}.$$

Therefore, after dividing by r, we get that such optimal clustering must satisfy

$$\frac{a_0}{4} r \le \lambda\sqrt{Ks}\|\!|V^*\|\!|_{\mathsf{F}} + 4\left(\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2 \sqrt{s}\|\!|V^*\|\!|_{\mathsf{F}} + \frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{K}\right)\Delta_1.$$

Recalling that $\|\!|V^*\|\!|_{\mathsf{F}} \le \sqrt{K}$, $\Delta_1 = C\sigma_{\max}\sqrt{\frac{\log N}{Tp_{\min}^2}}$ and $\Delta_2 = C\sqrt{\frac{s\log N}{Tp_{\min}^2}}$ yields the result. □

Now we are ready to finalize the proof of Theorem 3.6. Firstly, we need to show that the clustering $\hat{\mathcal{C}}$ from the lemma above is locally optimal. By Lemma 6.5, any neighbouring to it clustering $\mathcal{C}'$ satisfies $\|\!|Z_{\mathcal{C}'} - Z_{\hat{\mathcal{C}}}\|\!|_{\mathsf{F}} \le \frac{2}{\sqrt{\alpha N/K}}$. Therefore,

$$\|\!|Z_{\mathcal{C}'} - Z_{\mathcal{C}^*}\|\!|_{\mathsf{F}} \le \frac{C}{a_0}\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2 \lambda K\sqrt{s} + 2\alpha^{-1/2}\sqrt{\frac{K}{N}}\,,$$

and it is enough to check that this value is at most $\bar{r}$. We check that each of the terms is at most $\bar{r}/2$. For the first one it is enough to have,

$$\frac{C}{a_0}\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2 \alpha^{-1/2}\lambda K\sqrt{s} \le 0.09,$$
$$\frac{C^2}{a_0^2}\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^4 \lambda\left(2\sigma_{\max}\alpha^{-1/2} + \sqrt{n^*}\Delta_1\right) K^2 s \le 0.012,$$

and both are satisfied due to the upper bound $\lambda \le c\kappa^{-4}(a_0^2/\sigma_{\max})K^{-2}s^{-1}$ and the requirement $\sqrt{\frac{sn^* \log N}{Tp_{\min}^2}} \le c$. For the second term we need

$$\alpha^{-1}\frac{K}{N} \le 0.008\alpha, \qquad \alpha^{-1}\left(2\sigma_{\max}\alpha^{-1/2} + \sqrt{n^*}\Delta_1\right)\frac{K}{N} \le \lambda,$$

both are satisfied once $N \ge C\alpha^2 K$ and $\lambda \ge C\sigma_{\max}\alpha^{-3/2}\frac{K}{N}$.

Moreover, by Lemma 6.7 we have for $\hat{\Theta} = Z_{\hat{\mathcal{C}}}\hat{V}_{\hat{\mathcal{C}},\lambda}$

$$\begin{aligned}\|\hat{\Theta} - \Theta^*\|_{\mathsf{F}} &\le \|Z_{\hat{\mathcal{C}}}(\hat{V}_{\hat{\mathcal{C}},\lambda} - V^*)^\top\|_{\mathsf{F}} + \|(Z_{\hat{\mathcal{C}}} - Z^*)V^*\|_{\mathsf{F}} \\ &\le 3\sigma_{\min}^{-1}\sqrt{Ks}\lambda + \frac{C}{a_0}\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2 \gamma K\sqrt{s}\lambda,\end{aligned}$$

which finishes the proof.

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A Proof of Theorems 3.4 and 3.5

Recall that we have a time series,

$$Y_t = \sum_{k \geq 0} \Theta^k W_{t-k}, \qquad t \in \mathbb{Z}, \tag{A.1}$$

where $W_t \in \mathbb{R}^N$, $t \in \mathbb{Z}$ are independent vectors with $\mathsf{E} W_t = 0$ and $\mathrm{Var}(W_t) = S$. We also have that $\|\Theta\|_{\mathsf{op}} \le \gamma$ for some $\gamma < 1$ and the covariance $\Sigma = \mathrm{Var}(Y_t)$ reads as

$$\Sigma = \sum_{k \ge 0} \Theta^k S [\Theta^k]^\top.$$

We have the observations

$$Z_t = (\delta_{1t} Y_{1t}, \ldots, \delta_{Nt} Y_{Nt})^\top, \qquad t = 1, \ldots, T, \tag{A.2}$$

where $\delta_{it} \sim \mathrm{Be}(p_i)$ are independent Bernoulli random variables for every $i = 1, \ldots, N$ and $t = 1, \ldots, T$ and some $p_i \in (0, 1]$.

The proofs of both statements are based on the following version of Bernstein matrix inequality, which does not require bounded summands. Recall, that for a random variable $X \in \mathbb{R}$ the value

$$\|X\|_{\psi_j} = \inf\{C > 0 : \mathsf{E} \exp\left(\left|\frac{X}{C}\right|^j\right) \le 2\}$$

denotes a ψ_j-norm. For $j = 1$ the norm is referred to as *subexponential* and for $j = 2$ as *subgaussian*.

Theorem A.1 (Klochkov and Zhivotovskiy (2018), Proposition 4.1)**.** *Suppose, the matrices A_t for $t = 1, \ldots, T$ are independent and let $M = \max_t \big\| \|A_t\|_{\mathsf{op}} \big\|_{\psi_1}$ is finite. Then, $S_T = \sum_{t=1}^T A_t$ satisfies for any $u \ge 1$*

$$\mathsf{P}\left(\|S_T - \mathsf{E} S_T\|_{\mathsf{op}} > C \left(\sqrt{\sigma^2 (\log N + u)} + M \log T (\log N + u) \right) \right) \le e^{-u},$$

where $\sigma^2 = \| \sum_{t=1}^T \mathsf{E} A_t^\top A_t \|_{\mathsf{op}} \vee \| \sum_{t=1}^T \mathsf{E} A_t A_t^\top \|_{\mathsf{op}}$ and C is an absolute constant.

Let $\boldsymbol{\delta}_t = (\delta_{t1}, \ldots, \delta_{tN})^\top$ denotes the vector with Bernoilli variables from above corresponding to the time point t. In what follows we consider the following matrices,

$$A_{t,t'}^{k,j} = \mathrm{diag}\{\boldsymbol{\delta}_t\} \Theta^k W_{t-k} W_{t'-j}^\top [\Theta^j]^\top \mathrm{diag}\{\boldsymbol{\delta}_{t'}\},$$

so that since $Z_t = \sum_{k \ge 0} \mathrm{diag}\{\boldsymbol{\delta}_t\} \Theta^k W_{t-k}$, we have

$$Z_t Z_t^\top = \sum_{k,j \ge 0} \mathrm{diag}\{\boldsymbol{\delta}_t\} \Theta^k W_{t-k} W_{t-j}^\top [\Theta^j]^\top \mathrm{diag}\{\boldsymbol{\delta}_t\} = \sum_{k,j \ge 0} A_{t,t}^{k,j}.$$

Therefore, the decomposition takes place

$$\Sigma^* = \sum_{k,j \ge 0} S_{k,j}, \qquad S_{k,j} = \frac{1}{T} \sum_{t=1}^T A_{t,t}^{k,j}, \tag{A.3}$$

and we shall analyze the sum for every pair of $k, j \ge 0$ separately. We first introduce two technical lemmas. In what follows we assume w.l.o.g. that $\|S\|_{\mathsf{op}} = 1$, since if we scale it, all the covariances and estimators scale correspondingly.

Lemma A.2. *Under the assumptions of Theorem 3.4 it holds,*

$$\|\|\|P\mathrm{diag}\{\mathbf{p}\}^{-1}\mathrm{Diag}(A^{k,j}_{t,t'})Q\|\|_{\mathsf{op}}\|_{\psi_1} \le Cp_{\min}^{-1}\sqrt{M_1M_2}\gamma^{k+j},$$
$$\|\|\|P\mathrm{diag}\{\mathbf{p}\}^{-1}\mathrm{Off}(A^{k,j}_{t,t'})\mathrm{diag}\{\mathbf{p}\}^{-1}Q\|\|_{\mathsf{op}}\|_{\psi_1} \le Cp_{\min}^{-2}\sqrt{M_1M_2}\gamma^{k+j},$$

with some $C = C(L) > 0$.

Proof. Denote for simplicity $\mathbf{x} = \Theta^k W_{t-k}$, $\mathbf{y} = \Theta^j W_{t'-j}$, as well as $\mathbf{x}^\delta = \mathrm{diag}\{\boldsymbol{\delta}_t\}\mathbf{x}$, $\mathbf{y}^\delta = \mathrm{diag}\{\boldsymbol{\delta}_t\}\mathbf{y}$, such that $A^{k,j}_{t,t'} = \mathbf{x}^\delta[\mathbf{y}^\delta]^\top$. Since W_t are subgaussian and $\|\|\Theta^k S\Theta^k\|\|_{\mathsf{op}} \le \gamma^{2k}$, we have for any $\mathbf{u} \in \mathbb{R}^N$

$$\log \mathsf{E}\exp(\mathbf{u}^\top\mathbf{x}) \le C'\gamma^{2k}\|\mathbf{u}\|^2, \tag{A.4}$$

and since δ_t takes values in $[0,1]^N$, same takes place for $\mathbf{x}^{\boldsymbol{\delta}}$. By Theorem 2.1 in Hsu et al. (2012) it holds for any matrix A and vector $\mathbf{u} \in \mathbb{R}^N$,

$$\|\|\|A\mathbf{x}^\delta\|\|\|_{\psi_2} \le C''\gamma^k\|\|A\|\|_{\mathsf{F}}, \qquad \|\mathbf{u}^\top\mathbf{x}^\delta\|_{\psi_2} \le C''\gamma^k\|\mathbf{u}\|, \tag{A.5}$$

and, similarly,

$$\|\|\|A\mathbf{y}^\delta\|\|\|_{\psi_2} \le C''\gamma^j\|\|A\|\|_{\mathsf{F}}, \qquad \|\mathbf{u}^\top\mathbf{y}^\delta\|_{\psi_2} \le C''\gamma^j\|\mathbf{u}\|.$$

We first deal with the diagonal term. Let $P = \sum_{i=1}^{M_1}\mathbf{u}_j\mathbf{u}_j^\top$ be its eigen-decomposition with $\|\mathbf{u}_j\| = 1$, then

$$\begin{aligned}\|\|\|P\mathrm{diag}(\mathbf{x}^\delta)\|\|_{\mathsf{op}}\|^2_{\psi_2} =& \|\|\|\mathrm{diag}(\mathbf{x}^\delta)P\mathrm{diag}(\mathbf{x}^\delta)\|\|_{\mathsf{op}}\|_{\psi_1} \le \sum_{j=1}^{M_1}\|\|\|\mathrm{diag}(\mathbf{x}_\delta)\mathbf{u}_j\mathbf{u}_j^\top\mathrm{diag}(\mathbf{x}^\delta)\|\|_{\mathsf{op}}\|_{\psi_1}\\ =& \sum_{j=1}^{M_1}\|\|\|\mathrm{diag}(\mathbf{u}_j)\mathbf{x}^\delta\|\|\|^2_{\psi_2},\end{aligned}$$

where each term in the latter is bounded by γ^{2k} due the fact that $\|\|\mathrm{diag}(\mathbf{u}_j)\|\|_{\mathsf{F}} = 1$. Summing up and taking square root we arrive at $\left\|\|\|P\mathrm{diag}(\mathbf{x}^\delta)\|\|_{\mathsf{op}}\right\|_{\psi_2} \le \sqrt{C''M_1}\gamma^k$. Taking into account similar bound for $Q\mathrm{diag}(\mathbf{y}^\delta)$, we have by Hölder inequality

$$\begin{aligned}\|\|\|P\mathrm{diag}\{\delta\}^{-1}\mathrm{diag}(\mathbf{x}^\delta)\mathrm{diag}(\mathbf{y}^\delta)Q\|\|_{\mathsf{op}}\|_{\psi_1} \le& p_{\min}^{-1}\left\|\|\|P\mathrm{diag}(\mathbf{x}^\delta)\|\|_{\mathsf{op}}\right\|_{\psi_2}\|\|\|Q\mathrm{diag}(\mathbf{y}^\delta)\|\|_{\mathsf{op}}\|_{\psi_2}\\ \le& C''\sqrt{M_1M_2}\gamma^{k+j},\end{aligned}$$

which yields the bound for the diagonal. As for the off-diagonal, consider first the whole matrix,

$$\|\|\|P\mathbf{x}^\delta[\mathbf{y}^\delta]^\top Q\|\|_{\mathsf{op}}\|_{\psi_1} \le \|\|\|P\mathbf{x}^\delta\|\|\|_{\psi_2}\|\|\|Q\mathbf{y}^\delta\|\|\|_{\psi_2} \le (C'')^2\sqrt{M_1M_2}\gamma^{j+k},$$

and since $\mathrm{Off}(A^{j,k}_{t,t'}) = A^{j,k}_{t,t'} - \mathrm{Diag}(A^{j,k}_{t,t'})$, the bound follows from the triangular inequality. □

The following technical lemma will help us to upper-bound σ^2 in Theorem A.1.

Lemma A.3. *Let $\delta_1, \dots, \delta_N$ consists of independent Bernoilli components with probabilities of success $p_1, \dots, p_N$ and set $p_{\min} = \min_{i \le N} p_i$. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$ be two arbitrary* *vectors. It holds,*

$$\begin{aligned} \mathsf{E}\left(\sum_i \frac{\delta_i}{p_i} a_i b_i\right)^2 &\le p_{\min}^{-1}\|\mathbf{a}\|^2\|\mathbf{b}\|^2, \\ \mathsf{E}\left(\sum_{i\ne j} \frac{\delta_i\delta_j}{p_i p_j} a_i b_j\right)^2 &\le 32 p_{\min}^{-2}\|\mathbf{a}\|^2\|\mathbf{b}\|^2 + 4\left(\sum_i a_i\right)^2\left(\sum_i b_i\right)^2. \end{aligned}$$

Additionally, if $\delta_1', \dots, \delta_N'$ *are independent copies of* $\delta_1, \dots, \delta_N$*, it holds*

$$\mathsf{E}\left(\sum_{i,j} \frac{\delta_i\delta_j'}{p_i p_j} a_i b_j\right)^2 \le 4 p_{\min}^{-2}\|\mathbf{a}\|^2\|\mathbf{b}\|^2 + 4\left(\sum_i a_i\right)^2\left(\sum_i b_i\right)^2.$$

Proof. It holds,

$$\begin{aligned} \mathsf{E}\left(\sum_i \frac{\delta_i}{p_i} a_i b_i\right)^2 &= \sum_{i,j} \mathsf{E}\frac{\delta_i\delta_j}{p_i p_j} a_i b_i a_j b_j = \sum_{i,j}\{1 + \mathbf{1}(i=j)(p_i^{-1} - 1)\} a_i b_i a_j b_j \\ &\le \left(\sum_i a_i b_i\right)^2 + (p_{\min}^{-1} - 1)\sum_i a_i^2 b_i^2 \\ &\le \|\mathbf{a}\|^2\|\mathbf{b}\|^2 + (p_{\min}^{-1} - 1)\|\mathbf{a}\|^2\|\mathbf{b}\|^2. \end{aligned}$$

To show the second inequality we use decoupling (Theorem 6.1.1 in **?**) and the trivial inequality $(x+y)^2 \le 2x^2 + 2y^2$,

$$\begin{aligned} \mathsf{E}\left(\sum_{i\ne j} \frac{\delta_i\delta_j}{p_i p_j} a_i b_j\right)^2 &\le 2\left(\sum_{i\ne j} a_i b_j\right)^2 + 2\mathsf{E}\left(\sum_{i\ne j} \frac{(\delta_i - p_i)(\delta_j - p_j)}{p_i p_j} a_i b_j\right)^2 \\ &\le 2\left(\sum_{i\ne j} a_i b_j\right)^2 + 32\mathsf{E}\left(\sum_{i\ne j} \frac{(\delta_i - p_i)(\delta_j' - p_j)}{p_i p_j} a_i b_j\right)^2. \end{aligned} \tag{A.6}$$

Denote for simplicity $\overline{\delta}_i = \delta_i - p_i$ and $\overline{\delta}_i' = \delta_i' - p_i$. Since the latter are centered we have,

$$\mathsf{E}\left(\sum_{i\ne j} \frac{\overline{\delta}_i\overline{\delta}_j'}{p_i p_j} a_i b_j\right)^2 = \sum_{\substack{i\ne j\\ k\ne l}} \frac{\mathsf{E}\overline{\delta}_i\overline{\delta}_k}{p_i p_k}\frac{\mathsf{E}\overline{\delta}_j'\overline{\delta}_l'}{p_j p_j} a_i a_k b_j b_l \tag{A.7}$$

note that the expectation $\mathsf{E}\overline{\delta}_i\overline{\delta}_k$ is only non-vanishing when $i = k$, in which case it holds $\mathsf{E}\overline{\delta}_i^2 = p_i - p_i^2$. Taking into account similar property of $\mathsf{E}\overline{\delta}_j'\overline{\delta}_l'$ we have that the sum above is equal to

$$\sum_{i\ne j} \frac{(p_i - p_i^2)(p_j - p_j^2)}{p_i^2 p_j^2} a_i^2 b_j^2 \le (p_{\min}^{-1} - 1)^2 \sum_{i,j} a_i^2 b_j^2 \le (p_{\min}^{-1} - 1)^2\|\mathbf{a}\|^2\|\mathbf{b}\|^2.$$

It is left to notice that

$$\left(\sum_{i\neq j} a_i b_j\right)^2 \le 2\left(\sum_{i,j} a_i b_j\right)^2 + 2\left(\sum_{i} a_i b_j\right)^2 \le 2\left(\sum_{i} a_i\right)^2\left(\sum_{i} b_i\right)^2 + 2\|\mathbf{a}\|^2\|\mathbf{b}\|^2,$$

which recalling (A.6) and noting that $32(p_{\min}^{-1}-1)^2+4 \le 32p_{\min}^{-2}$ for $p_{\min} \in [0,1]$, completes the proof.

Similarly to (A.7) we can show the third inequality. □

Now we apply Bernstein matrix inequality to the sum S_{kj} defined in (A.3), dealing separately with diagonal and off-diagonal parts. After that we present the proof of Theorem 3.4.

Lemma A.4. *Under the assumptions of Theorem 3.4 for any $u \ge 1$ it holds with probability at least $1 - e^{-u}$*

$$\begin{aligned}&\|\!|P\mathrm{diag}\{\mathbf{p}\}^{-1}(\mathrm{Diag}(S_{k,j}) - \mathsf{E}\mathrm{Diag}(S_{k,j}))Q\|\!|_{\mathsf{op}}\\ &\qquad \le C\gamma^{k+j}\left(\sqrt{\frac{M_1 \vee M_2(\log N + u)}{Tp_{\min}}} \bigvee \frac{\sqrt{M_1M_2}(\log N + u)}{Tp_{\min}}\right)\end{aligned}$$

where $C = C(K)$ only depends on K.

Proof. Note that,

$$P\mathrm{diag}\{\mathbf{p}\}^{-1}\mathrm{Diag}(S_{kj})Q = T^{-1}\sum_{t=1}^{T} A_t, \qquad A_t = P\mathrm{diag}\{\mathbf{p}\}^{-1}\mathrm{Diag}(A_{t,t}^{k,j})Q.$$

By Lemma A.2 we have $\|\|\!|A_t\|\!|_{\mathsf{op}}\|_{\psi_1} \le Cp_{\min}^{-1}\sqrt{M_1M_2}\gamma^{k+j}$. Moreover, using decomposition $Q = \sum_{j=1}^{M_2}\mathbf{u}_j\mathbf{u}_j$, we have

$$\begin{aligned}\|\!|\mathsf{E}A_tA_t^\top\|\!|_{\mathsf{op}} &\le \|\!|\mathsf{E}\mathrm{diag}\{\mathbf{p}\}^{-1}\mathrm{Diag}(A_{t,t}^{k,j})Q\mathrm{Diag}(A_{t,t}^{k,j})\mathrm{diag}\{\mathbf{p}\}^{-1}\|\!|_{\mathsf{op}}\\ &\le \sum_{j=1}^{M_2}\|\!|\mathsf{E}\mathrm{diag}\{\mathbf{p}\}^{-1}\mathrm{Diag}(A_{t,t}^{k,j})\mathbf{u}_j\mathbf{u}_j^\top\mathrm{Diag}(A_{t,t}^{k,j})\mathrm{diag}\{\mathbf{p}\}^{-1}\|\!|_{\mathsf{op}}\\ &\le \sum_{j=1}^{M_2}\sup_{\|\boldsymbol{\gamma}\|=1}\mathsf{E}(\boldsymbol{\gamma}^\top\mathrm{diag}\{\mathbf{p}\}^{-1}\mathrm{Diag}(A_{t,t}^{k,j})\mathbf{u}_j)^2\end{aligned}$$

By definition, $\mathrm{Diag}(A_{t,t}^{k,j}) = \mathrm{diag}\{\delta_{ti}x_iy_i\}_{i=1}^N$ for $\mathbf{x} = \Theta^kW_{t-k}$, $\mathbf{y} = \Theta^jW_{t-j}$. Let E_δ denotes the expectation w.r.t. the Bernoulli variables and conditioned on everything else. Setting $\mathbf{a} = (x_1\gamma_1,\dots,x_N\gamma_N)^\top$) and $\mathbf{b} = (y_1u_1,\dots,y_Nu_N)^\top$, we have by the first inequality of Lemma A.3,

$$\begin{aligned}\mathsf{E}(\boldsymbol{\gamma}^\top\mathrm{diag}\{\mathbf{p}\}^{-1}\mathrm{Diag}(A_{t,t}^{k,j})\mathbf{u}_j)^2 &= \mathsf{E}\mathsf{E}_\delta\left(\sum_i \gamma_i x_i\frac{\delta_{ti}}{p_i}y_iu_i\right)^2\\ &\le p_{\min}^{-1}\mathsf{E}\|\mathbf{a}\|^2\|\mathbf{b}\|^2\\ &\le p_{\min}^{-1}\mathsf{E}^{1/2}\|\mathbf{a}\|^4\mathsf{E}^{1/4}\|\mathbf{b}\|^4.\end{aligned}$$

Observe that,

$$\|\mathbf{a}\|^2 = \sum_i \gamma_i^2 x_i^2 = \mathbf{x}^\top \mathrm{diag}\{\boldsymbol{\gamma}\}^2 \mathbf{x},$$

so since $\mathrm{Tr}(\mathrm{diag}\{\boldsymbol{\gamma}\}^2) = 1$ and due to (A.4) and by Theorem 2.1 Hsu et al. (2012) it holds $\mathsf{E}^{1/2}\|\mathbf{a}\|^4 \le \|\|\mathbf{a}\|^2\|_{\psi_1} \le C'\gamma^{2k}$. Similarly, it holds $\mathsf{E}^{1/2}\|\mathbf{a}\|^4 \le C'\gamma^{2j}$, which together implies

$$\|\|\mathsf{E}A_t A_t^\top\|\|_{\mathsf{op}} \vee \|\|\mathsf{E}A_t^\top A_t^\top\|\|_{\mathsf{op}} \le C'' M_2 \vee M_1 \gamma^{2k+2j}.$$

Now notice that A_t is not necessary an independent sequence, as A_t depends directly on $(W_{t-k}, W_{t-j}, \boldsymbol{\delta}_t)$, which might intersect with e.g. $t' = t + |j-k|$. However, if we take a set $I \subset [1,T]$ such that any two $t, t' \in I$ satisfy $|t'-t| \ne |j-k|$ then obviously the sequence $(A_t)_{t\in I}$ is independent. We separate the whole interval $[1,T]$ into two such independent sets,

$$\begin{aligned} I_1 =& \{t \in [1,T] : \lceil t/|j-k| \rceil \text{ is odd }\}, \\ I_2 =& \{t \in [1,T] : \lceil t/|j-k| \rceil \text{ is even }\} \\ =& [1,T] \setminus I_1. \end{aligned} \tag{A.8}$$

Indeed, if for $t, t' \in I_1$ then $\lceil t/|j-k| \rceil$ and $\lceil t'/|j-k| \rceil$ are either equal or differ in at least two, so that in the first case we have $|t-t'| < |j-k|$ and in the second $|t-t'| > |j-k|$. Since both intervals have, very roughly, at most T elements, it holds by Theorem A.1 with probability at least $1 - e^{-u}$ for both j,

$$\begin{aligned} &\|\| \sum_{t \in I_j} A_t - \mathsf{E}A_t \|\|_{\mathsf{op}} \\ &\quad \le C\gamma^{j+k} \left(\sqrt{p_{\min}^{-1}(M_1 \vee M_2) T (\log N + u)} \vee p_{\min}^{-1} \sqrt{M_1 M_2} (\log N + u) \log T \right), \end{aligned}$$

so summing up the two and dividing by T we get the result. □

Lemma A.5. *Under the assumptions of Theorem 3.4 for any $u \ge 1$ it holds with probability at least $1 - e^{-u}$*

$$\begin{aligned} &\|\| P \mathrm{diag}\{\mathbf{p}\}^{-1} (\mathrm{Off}(S_{k,j}) - \mathsf{E}\mathrm{Off}(S_{k,j})) \mathrm{diag}\{\mathbf{p}\}^{-1} Q \|\|_{\mathsf{op}} \\ &\qquad \le C\gamma^{k+j} \left(\sqrt{\frac{M_1 \vee M_2 (\log N + u)}{T p_{\min}^2}} \bigvee \frac{\sqrt{M_1 M_2} (\log N + u) \log T}{T p_{\min}^2} \right) \end{aligned}$$

where $C = C(K)$ only depends on K.

Proof. It holds,

$$P \mathrm{diag}\{\mathbf{p}\}^{-1} \mathrm{Off}(S_{kj}) \mathrm{diag}\{\mathbf{p}\}^{-1} Q = T^{-1} \sum_{t=1}^T B_t, \quad B_t = P \mathrm{diag}\{\mathbf{p}\}^{-1} \mathrm{Off}(A_{t,t}^{k,j}) \mathrm{diag}\{\mathbf{p}\}^{-1} Q.$$

By Lemma A.2 we have $\|\|\|B_t\|\|_{\mathsf{op}}\|_{\psi_1} \le C p_{\min}^{-2} \sqrt{M_1 M_2} \gamma^{k+j}$. Using decomposition $Q =$

$\sum_{j=1}^{M_2} \mathbf{u}_j \mathbf{u}_j$ with $\|\mathbf{u}_j\| = 1$ we get that

$$
\begin{aligned}
\|\mathsf{E} B_t B_t^\top\|_{\mathsf{op}} \leq & \|\mathsf{E}\mathrm{diag}\{\mathbf{p}\}^{-1}\mathrm{Off}(A_{t,t}^{k,j})\mathrm{diag}\{\mathbf{p}\}^{-1} Q \mathrm{diag}\{\mathbf{p}\}^{-1}\mathrm{Off}(A_{t,t}^{k,j})\mathrm{diag}\{\mathbf{p}\}^{-1}\|_{\mathsf{op}} \\
\leq & \sum_{j=1}^{M_2} \|\mathsf{E}\mathrm{diag}\{\mathbf{p}\}^{-1}\mathrm{Off}(A_{t,t}^{k,j})\mathrm{diag}\{\mathbf{p}\}^{-1}\mathbf{u}_j \mathbf{u}_j^\top \mathrm{diag}\{\mathbf{p}\}^{-1}\mathrm{Off}(A_{t,t}^{k,j})\mathrm{diag}\{\mathbf{p}\}^{-1}\|_{\mathsf{op}} \\
\leq & \sum_{j=1}^{M_2} \sup_{\|\boldsymbol{\gamma}\|=1} \mathsf{E}(\boldsymbol{\gamma}^\top \mathrm{diag}\{\mathbf{p}\}^{-1}\mathrm{Off}(A_{t,t}^{k,j})\mathrm{diag}\{\mathbf{p}\}^{-1}\mathbf{u}_j)^2
\end{aligned}
$$

Again, using the notation $\mathbf{x} = \Theta^k W_{t-k}$, $\mathbf{y} = \Theta^j W_{t-j}$ and $\mathbf{a} = \mathrm{diag}\{\boldsymbol{\gamma}\}\mathbf{x}$, $\mathbf{b} = \mathrm{diag}\{\mathbf{u}\}\mathbf{y}$, we have $\mathrm{Off}(A_{t,t}^{j,k}) = \mathrm{Off}(\mathbf{x}\mathbf{y}^\top)$, therefore by Lemma A.3

$$
\begin{aligned}
\mathsf{E}(\boldsymbol{\gamma}^\top \mathrm{diag}\{\mathbf{p}\}^{-1}\mathrm{Off}(A_{t,t}^{k,j})\mathrm{diag}\{\mathbf{p}\}^{-1}\mathbf{u}_j)^2 = & \mathsf{E}\mathsf{E}_\delta \left(\sum_{i \neq j} \gamma_i \frac{\delta_{it}}{p_i} x_i y_j \frac{\delta_{jt}}{\delta_j} u_j \right)^2 \\
= & \mathsf{E}\mathsf{E}_\delta \left(\sum_{i \neq j} \frac{\delta_{it}}{p_i} \frac{\delta_{jt}}{\delta_j} a_i b_j \right)^2 \\
\leq & 32 p_{\min}^{-2} \mathsf{E}\|\mathbf{a}\|^2 \|\mathbf{b}\|^2 + 4\mathsf{E} \left(\sum_i a_i \right)^2 \left(\sum_i b_i \right)^2 .
\end{aligned}
$$

From the proof of Lemma A.5 we know that $\mathsf{E}\|\mathbf{a}\|^2\|\mathbf{b}\|^2 \leq C'\gamma^{2k+2j}$. Moreover, we have $\sum_i a_i = \boldsymbol{\gamma}^\top \mathbf{x}$ and $\sum_i b_i = \mathbf{u}^\top \mathbf{y}$. Thus, by (A.5) it holds $\mathsf{E}^{1/4}\|\boldsymbol{\gamma}^\top \mathbf{x}\|^4 \leq \|\boldsymbol{\gamma}^\top \mathbf{x}\|_{\psi_2} \leq C'\gamma^j$ and, similarly, $\mathsf{E}^{1/4}\|\mathbf{u}^\top \mathbf{y}\|^4 \leq C'\gamma^k$. Putting those bounds together and applying Cauchy-Schwarz inequality, we have

$$
\|\mathsf{E} B_t B_t^\top\|_{\mathsf{op}} \leq C'' p_{\min}^{-2} M_2 \gamma^{2k+2j}.
$$

By analogy, we have

$$
\|\mathsf{E} B_t B_t^\top\|_{\mathsf{op}} \vee \|\mathsf{E} B_t^\top B_t\|_{\mathsf{op}} \leq C'' p_{\min}^{-2} M_1 \vee M_2 \gamma^{2k+2j}.
$$

Applying the same sample splitting (A.8) we obtain the bound

$$
\|\sum_t A_t - \mathsf{E} A_t\|_{\mathsf{op}} \leq C\gamma^{j+k} \left(\sqrt{p_{\min}^{-2}(M_1 \vee M_2) T(\log N + u)} \vee p_{\min}^{-2}\sqrt{M_1 M_2}(\log N + u) \right),
$$

which divided by T provides the result. □

Proof of Theorem 3.4. Set,

$$
S_{k,j}^{\boldsymbol{\delta}} = \mathrm{diag}\{\mathbf{p}\}^{-1}\mathrm{Diag}(S_{k,j}) - \mathrm{diag}\{\boldsymbol{\delta}\}^{-1}\mathrm{Off}(S_{k,j})\mathrm{diag}\{\boldsymbol{\delta}\}^{-1},
$$

so that by the union of bounds in Lemmas A.5, A.4 for any $u \geq 1$

$$
\|P(S_{k,j}^{\boldsymbol{\delta}} - \mathsf{E}S_{k,j}^{\boldsymbol{\delta}})Q\|_{\mathsf{op}} > C\gamma^{k+j} \left(\sqrt{\frac{M_1 \vee M_2(\log N + u)}{T p_{\min}^2}} \bigvee \frac{\sqrt{M_1 M_2}(\log N + u)}{T p_{\min}^2} \right)
$$

holds with probability at least $1-e^{-u}$. Take a union of those bounds for every k, j with $u = u_{k,j} = k + j + 1 + u'$. The total probability of complementary event is at most

$$\sum_{k,j\geq 0} e^{-k-j-1-u} = e^{-1-u}\left(\sum_{k\geq 0} e^{-k}\right)^2 = e^{-u}/(e-1) < e^{-u}.$$

On such event it holds

$$\begin{aligned}
\|P(\hat{\Sigma} - \mathsf{E}\Sigma)Q\|_{\mathsf{op}} \leq & \sum_{k,j\geq 0} \|P(S^{\boldsymbol{\delta}}_{k,j} - \mathsf{E}S^{\boldsymbol{\delta}}_{k,j})Q\|_{\mathsf{op}} \\
\leq & C\sum_{k,j\geq 0}\gamma^{k+j}\left(\sqrt{\frac{M_1\vee M_2(\log N + u_{k,j})}{Tp^2_{\min}}}\bigvee\frac{\sqrt{M_1M_2}(\log N + u_{k,j})}{Tp^2_{\min}}\right) \\
\leq & C'\left[\sum_{k,j\geq 0}\gamma^{k+j}\right]\left(\sqrt{\frac{(M_1\vee M_2)\log N}{Tp^2_{\min}}}\bigvee\frac{\sqrt{M_1M_2}\log N}{Tp^2_{\min}}\right) \\
& + C\left[\sum_{k,j}(k+j)\gamma^{k+j}\right]\left(\sqrt{\frac{(M_1\vee M_2)u}{Tp^2_{\min}}}\bigvee\frac{\sqrt{M_1M_2}u}{Tp^2_{\min}}\right),
\end{aligned}$$

which completes the proof due to the equalities

$$\begin{aligned}
\sum_{k,j\geq 0}\gamma^{k+j} =& \left(\sum_{k\geq 0}\gamma^k\right)^2 = \frac{1}{(1-\gamma)^2} \\
\sum_{k,j\geq 0}(k+j)\gamma^{k+j} =& 2\sum_{k,j\geq 0}k\gamma^{k+j} = \frac{2}{(1-\gamma)}\sum_{k\geq 0}k\gamma^k = \frac{2}{(1-\gamma)^3}.
\end{aligned}$$

□

Proof of Theorem 3.5. Recall the definition,

$$A^{k,j}_{t,t'} = \operatorname{diag}\{\boldsymbol{\delta}_t\}\Theta^k W_{t-k}W^\top_{t'-j}[\Theta^j]^\top\operatorname{diag}\{\boldsymbol{\delta}_{t'}\}.$$

Then, it holds

$$Z_tZ^\top_{t+1} = \sum_{k,j\geq 0}\operatorname{diag}\{\boldsymbol{\delta}_t\}\Theta^k W_{t-k}W^\top_{t+1-j}[\Theta^j]^\top\operatorname{diag}\{\boldsymbol{\delta}_{t+1}\} = \sum_{k,j\geq 0}A^{k,j}_{t,t+1},$$

and the decomposition takes place,

$$A^* = \sum_{k,j\geq 0}S_{k,j}, \qquad S_{k,j} = \frac{1}{T-1}\sum_{t=1}^{T-1}A^{k,j}_{t,t+1}.$$

We first apply Bernstein matrix for each $S_{k,j}$ separately. Observe that

$$P\operatorname{diag}\{\mathbf{p}\}^{-1}S_{k,j}\operatorname{diag}\{\mathbf{p}\}^{-1}Q = \frac{1}{T-1}\sum_{t=1}^{T-1}B_t, \qquad B_t = P\operatorname{diag}\{\mathbf{p}\}^{-1}A^{k,j}_{t,t+1}\operatorname{diag}\{\mathbf{p}\}^{-1}Q.$$

By Lemma A.2 each term satisfies,

$$\max_t \| \| B_t \|_{\mathsf{op}} \|_{\psi_1} \le C\sqrt{M_1 M_2}\gamma^{k+j}.$$

Furthermore, let $Q = \sum_{j=1}^{M_2} \mathbf{u}_j \mathbf{u}_j^\top$ with unit vectors $\mathbf{u}_j$. Also, denoting $\mathbf{x} = \Theta^k W_{t-k}$ and $\mathbf{y} = \Theta^k W_{t+1-k}$ it holds $A_{t,t+1}^{k,j} = \operatorname{diag}\{\boldsymbol{\delta}_t\}\mathbf{x}\mathbf{y}^\top \operatorname{diag}\{\boldsymbol{\delta}_{t+1}\}$. Then, we have for any unit $\boldsymbol{\gamma} \in \mathbb{R}^N$ and using Lemma A.3,

$$\begin{aligned}
&\mathsf{E}(\boldsymbol{\gamma}^\top \operatorname{diag}\{\mathbf{p}\}^{-1} A_{t,t+1}^{k,j} \operatorname{diag}\{\mathbf{p}\}^{-1}\mathbf{u}_j)^2 \\
&\quad = \mathsf{E}\mathsf{E}_\delta \left(\sum_{i,j} \gamma_i x_i \frac{\delta_{ti}}{p_i} \frac{\delta_{t+1,j}}{p_j} y_j u_j \right)^2 \\
&\quad \le p_{\min}^{-2} \mathsf{E}\|\operatorname{diag}\{\boldsymbol{\gamma}\}\mathbf{x}\|^2 \|\operatorname{diag}\{\mathbf{u}\}\mathbf{y}\|^2 + \mathsf{E}(\boldsymbol{\gamma}^\top \mathbf{x})(\mathbf{u}^\top \mathbf{y})^2,
\end{aligned}$$

which due to the subgaussianity of $\mathbf{x}$ and $\mathbf{y}$ yields,

$$\begin{aligned}
\mathsf{E}\|\operatorname{diag}\{\boldsymbol{\gamma}\}\mathbf{x}\|^2 \|\operatorname{diag}\{\mathbf{u}\}\mathbf{y}\|^2 &\le \mathsf{E}^{1/2}\|\operatorname{diag}\{\boldsymbol{\gamma}\}\mathbf{x}\|^4 \mathsf{E}^{1/2}\|\operatorname{diag}\{\mathbf{u}\}\mathbf{y}\|^4 \\
&\le C' \gamma^{2k+2j} \\
\mathsf{E}(\boldsymbol{\gamma}^\top \mathbf{x})(\mathbf{u}^\top \mathbf{y})^2 &\le \mathsf{E}^{1/2}(\boldsymbol{\gamma}^\top \mathbf{x})^4 \mathsf{E}^{1/2}(\mathbf{u}^\top \mathbf{y})^4 \\
&\le C' \gamma^{2k+2j}.
\end{aligned}$$

Therefore, we get that

$$\| \mathsf{E} B_t B_t^\top \|_{\mathsf{op}} = \sup_{\|\boldsymbol{\gamma}\|=1} \sum_{j=1}^{M_2} \mathsf{E}\left(\boldsymbol{\gamma}^\top \operatorname{diag}\{\mathbf{p}\}^{-1} A_{t,t+1}^{k,j} \operatorname{diag}\{\mathbf{p}\}^{-1} \mathbf{u}_j\right)^2 \le C'' p_{\min}^{-2} M_2 \gamma^{2k+2j}.$$

Taking similar derivations we can arrive at

$$\sigma^2 = \| \mathsf{E} B_t B_t^\top \|_{\mathsf{op}} \vee \| \mathsf{E} B_t^\top B_t \|_{\mathsf{op}} \le C'' p_{\min}^{-2} (M_1 \vee M_2) \gamma^{2k+2j}.$$

Now we separate the indices $t = 1, \dots, T$ into four subsets, such that each corresponds to a set of independent matrices B_t. Since each B_t is generated by $W_{t-k}, W_{t+1-j}, \boldsymbol{\delta}_t$, and $\boldsymbol{\delta}_{t+1}$, we simply need to ensure that none of the pair of indices t, t' from the same subset satisfies $|t - t'| = |k - j + 1|$ nor $|t - t'| = 1$. This can be satisfied by the following separation. First, we separate the indices into two subsets with odd and even indices, respectively, so that none of the subsets contains two indices with $|t - t'| = 1$. Then, both of the subsets need to be separated into two others according to the scheme (A.8), so that the assertion $|t - t'| = |k - j + 1|$ is avoided within each subset. Therefore, applying Bernstein inequality, Theorem A.1, to each sum separately and then summing up, we get that for any $u \ge 1$ with probability at least $1 - e^{-u}$,

$$\begin{aligned}
&\| P \operatorname{diag}\{\boldsymbol{\delta}\}^{-1} (S_{k,j} - \mathsf{E} S_{k,j}) \operatorname{diag}\{\boldsymbol{\delta}\}^{-1} Q \|_{\mathsf{op}} \\
&\qquad \le C \left(\sqrt{p_{\min}^{-2} (M_1 \vee M_2) T (\log N + u)} \bigvee \sqrt{M_1 M_2} (\log N + u) \log T \right).
\end{aligned}$$

Similarly to the proof of Theorem 3.4 we take the union of those bounds for every i, j with $u = j + k + u'$ and then the result follows. □

B LASSO and missing observations

Suppose, we observe a signal $\mathbf{y} \in \mathbb{R}^n$ of the form

$$\mathbf{y} = \Phi \mathbf{b}^* + \boldsymbol{\varepsilon},$$

where $\Phi = [\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_p] \in \mathbb{R}^{n \times p}$ is a dictionary of words $\boldsymbol{\phi}_j \in \mathbb{R}^n$ and $\mathbf{b}^*$ is some sparse parameter with support $\Lambda \subset \{1, \dots, p\}$. We want to recover exact sparse representation by solving quadratic program

$$\frac{1}{2}\|\mathbf{y} - \Phi \mathbf{b}\|^2 + \gamma \|\mathbf{b}\|_1 \to \min_{\mathbf{b} \in \mathbb{R}^p} . \tag{B.1}$$

Denote by $\mathbb{R}^\Lambda$ the set of vectors with elements indexed by Λ, for $\mathbf{b} \in \mathbb{R}^n$ let $\mathbf{x}_\Lambda \in \mathbb{R}^\Lambda$ be the result of taking only elements indexed by Λ. With some abuse of notation we will also associate each vector $\mathbf{x}_\Lambda \in \mathbb{R}^\Lambda$ with a vector $\mathbf{x}$ from $\mathbb{R}^n$ that has same coefficients on Λ and zeros elsewhere. Let us also $\Phi_\Lambda = [\boldsymbol{\phi}_j]_{j \in \Lambda}$ be a subdictionary composed of words indexed by Λ and P_Λ is the projector onto the corresponding subspace.

The following sufficient conditions for the global minimizer of (B.1) to be supported on Λ are due to Tropp (2006), who uses the notion of *exact recovery coefficient*,

$$\mathrm{ERC}_\Phi(\Lambda) = 1 - \max_{j \notin \Lambda} \|\Phi_\Lambda^+ \boldsymbol{\phi}_j\|_1,$$

The results are summarized in the next theorem.

Theorem B.1 (Tropp (2006))**.** *Let $\tilde{\mathbf{b}}$ be a solution to (B.1). Suppose, that $\|\Phi^\top \boldsymbol{\varepsilon}\|_\infty \leq \gamma \mathrm{ERC}(\Lambda)$. Then,*

- *the support of $\tilde{\mathbf{b}}$ is contained in Λ;*
- *the distance between $\tilde{\mathbf{b}}$ and optimal (non-penalized) parameter satisfies,*

$$\begin{aligned} \|\tilde{\mathbf{b}} - \mathbf{b}^*\|_\infty &\leq \|\Phi_\Lambda^+ \boldsymbol{\varepsilon}\|_\infty + \gamma \|(\Phi_\Lambda \Phi_\Lambda^\top)^{-1}\|_{1,\infty}, \\ \|\Phi_\Lambda(\tilde{\mathbf{b}} - \mathbf{b}^*) - P_\Lambda \boldsymbol{\varepsilon}\|_2 &\leq \gamma \|(\Phi_\Lambda^+)^\top\|_{2,\infty}; \end{aligned}$$

In what follows we want to extend this result for the possibility of using missing observations model. Observe that the program (B.1) is equivalent to

$$\frac{1}{2}\mathbf{b}^\top [\Phi^\top \Phi] \mathbf{b} - \mathbf{b}^\top [\Phi^\top \mathbf{y}] + \gamma \|\mathbf{b}\|_1 \to \min_{\mathbf{b} \in \mathbb{R}^p},$$

so that for the minimization procedure only knowing $D = \Phi^\top \Phi$ and $\mathbf{c} = \Phi^\top \mathbf{y}$ is required. Suppose, that instead we have only access to some estimators $\hat{D} \geq 0$ and $\hat{\mathbf{c}}$ that are close enough to the original matrix and vector, which may come e.g. from missing observations model. Then, we can solve instead the following problem,

$$\frac{1}{2}\mathbf{b}^\top \hat{D} \mathbf{b} - \mathbf{b}^\top \hat{\mathbf{c}} + \gamma \|\mathbf{b}\|_1 \to \min_{\mathbf{b} \in \mathbb{R}^p} . \tag{B.2}$$

In what follows we provide a slight extension of Tropp's result towards missing observations, the proof mainly follows the same steps.

Further, for a matrix D and two sets of indices A, B we denote the submatrix on those indices as $D_{A,B}$ and for a vector $\mathbf{c}$, the corresponding subvector is $\mathbf{c}_A$.

Lemma B.2. *Suppose, that*

$$\|\hat{D}_{\Lambda^c,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}\hat{\mathbf{c}}_\Lambda - \hat{\mathbf{c}}_{\Lambda^c}\|_\infty \leq \gamma(1 - \|\hat{D}_{\Lambda^c,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}\|_{1,\infty}).$$

Then, the solution $\tilde{\mathbf{b}}$ *to* (B.2) *is supported on* Λ.

Proof. Let $\tilde{\mathbf{b}}$ be the solution to (B.2) with the restriction $\operatorname{supp}(\mathbf{b}) \subset \Lambda$. Since $\hat{D} \geq 0$ this is a convex problem and therefore the solution is unique and satisfy,

$$\hat{D}_{\Lambda,\Lambda}\tilde{\mathbf{b}} - \hat{\mathbf{c}}_\Lambda + \gamma\mathbf{g} = 0, \qquad \mathbf{g} \in \partial\|\tilde{\mathbf{b}}\|_1,$$

where $\partial f(\mathbf{b})$ denotes subdifferential of a convex function f at a point $\mathbf{b}$, in the case of ℓ_1 norm we have $\|\mathbf{g}\|_\infty \leq 1$. Thus,

$$\tilde{\mathbf{b}} = \hat{D}^{-1}_{\Lambda,\Lambda}\hat{\mathbf{c}}_\Lambda - \gamma\hat{D}^{-1}_{\Lambda,\Lambda}\mathbf{g}. \tag{B.3}$$

Next, we want to check that $\tilde{\mathbf{b}}$ is a global minimizer. To do so, let us compare the objective function at a point $\overline{\mathbf{b}} = \tilde{\mathbf{b}} + \delta\mathbf{e}_j$ for arbitrary index $j \notin \Lambda$. Since $\|\overline{\mathbf{b}}\|_1 = \|\tilde{\mathbf{b}}\|_1 + |\delta|$, we have

$$\begin{aligned} L(\tilde{\mathbf{b}}) - L(\overline{\mathbf{b}}) &= \frac{1}{2}\tilde{\mathbf{b}}^\top\hat{D}\tilde{\mathbf{b}} - \frac{1}{2}\overline{\mathbf{b}}^\top\hat{D}\overline{\mathbf{b}} - \hat{\mathbf{c}}^\top(\tilde{\mathbf{b}} - \overline{\mathbf{b}}) - \gamma|\delta| \\ &= \frac{\delta^2}{2}\mathbf{e}_j^\top\hat{D}\mathbf{e}_j + |\delta|\gamma - \delta\mathbf{e}_j^\top\hat{D}\tilde{\mathbf{b}} + \delta\widehat{c}_j \\ &> |\delta|\gamma - \delta\mathbf{e}_j^\top\hat{D}\tilde{\mathbf{b}} + \delta\widehat{c}_j, \end{aligned}$$

where the latter comes from the fact that $\hat{D}$ is positively definite. Applying the equality (B.3) yields,

$$\mathbf{e}_j^\top\hat{D}\tilde{\mathbf{b}} = \hat{D}_{j,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}\hat{\mathbf{c}}_\Lambda - \gamma\hat{D}_{j,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}\mathbf{g},$$

therefore, taking into account $\|\mathbf{g}\|_\infty \leq 1$ we have,

$$L(\tilde{\mathbf{b}}) - L(\overline{\mathbf{b}}) > |\delta|\left[\gamma(1 - \|\hat{D}_{\Lambda^c,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}\|_{1,\infty}) - \left|\hat{D}_{j,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}\hat{\mathbf{c}}_\Lambda - \widehat{c}_j\right|\right],$$

where the right-hand side is nonnegative by the condition of the lemma. Since $j \notin \Lambda$ is arbitrary, $\tilde{\mathbf{b}}$ is a global solution as well.

□

Remark B.1. It is not hard to see that in the exact case $\hat{D} = \Phi^\top\Phi$ and $\hat{\mathbf{c}} = \Phi^\top\mathbf{y}$ the condition of the lemma above turns into the condition $\|\Phi^\top_{\Lambda^c}P_\Lambda\boldsymbol{\varepsilon}\|_\infty \leq \gamma\mathrm{ERC}(\Lambda)$ of Theorem B.1.

Since we are particularly interested in an application to time series, the features matrix Φ should in fact be random, thus stating a ERC-like condition onto it might result in additional unnecessary technical difficulties. Instead, let us assume that there is some other matrix $\bar{D}$, potentially the expectation of $\Phi^\top\Phi$, such that it is close enough to $\hat{D}$ (with some probability, but we are stating all the results deterministically in this section), and the value that controls the exact recovery looks like

$$\mathrm{ERC}(\Lambda; \bar{D}) = 1 - \|\bar{D}_{\Lambda^c,\Lambda}\bar{D}_{\Lambda,\Lambda}^{-1}\|_{1,\infty}.$$

Additionally, we set $\bar{\mathbf{c}} = \bar{D}\mathbf{b}^* = \bar{D}_{\cdot,\Lambda}\mathbf{b}^*_\Lambda$ — the vector that $\hat{\mathbf{c}}$ is intended to approximate. Note that in this case we have $\bar{D}_{\Lambda^c,\Lambda}\bar{D}_{\Lambda,\Lambda}^{-1}\bar{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_{\Lambda^c} = \bar{D}_{\Lambda^c,\Lambda}\mathbf{b}^*_\Lambda - \bar{\mathbf{c}}_{\Lambda^c} = 0$, thus the conditions of Lemma B.2 hold for $\bar{D}, \bar{\mathbf{c}}$ once $\mathrm{ERC}(\Lambda; \bar{D})$ and γ are nonnegative. In what follows we control the values appearing in the lemma for $\hat{D}$ and $\hat{\mathbf{c}}$ through the differences between $\bar{\mathbf{c}}$, $\bar{D}$ and $\hat{\mathbf{c}}$, $\hat{D}$, respectively, thus allowing the exact recovery of the sparsity pattern. Lemma 6.7

Corollary B.3. *Let $\bar{D}$ and $\bar{\mathbf{c}}$ be such that $\bar{\mathbf{c}} = \bar{D}\mathbf{b}^*$. Assume that*

$$\|\hat{\mathbf{c}} - \bar{\mathbf{c}}\|_\infty \le \delta_c, \qquad \|\bar{D}_{\Lambda,\Lambda}^{-1}(\hat{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_\Lambda)\|_\infty \le \delta_c', \qquad \|\bar{D}_{\Lambda,\Lambda}^{-1}(\hat{D}_{\Lambda,\cdot} - \bar{D}_{\Lambda,\cdot})\|_{\infty,\infty} \le \delta_D,$$
$$\|(\hat{D}_{\cdot,\Lambda} - \bar{D}_{\cdot,\Lambda})\mathbf{b}^*_\Lambda\|_\infty \le \delta_D', \qquad \|\bar{D}_{\Lambda,\Lambda}^{-1}(\bar{D}_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})\mathbf{b}^*_\Lambda\|_\infty \le \delta_D''.$$

Suppose, $\mathrm{ERC}(\Lambda) \ge 3/4$ *and*

$$3\delta_c + 3\delta_D' \le \gamma, \qquad s\delta_D \le \frac{1}{16},$$

where $|\Lambda| = s$. Then, the solution to (B.2) is supported on Λ and satisfies

$$\tilde{\mathbf{b}}_\Lambda = \hat{D}_{\Lambda,\Lambda}^{-1}\hat{\mathbf{c}}_\Lambda - \gamma\hat{D}_{\Lambda,\Lambda}^{-1}\mathbf{g}, \tag{B.4}$$

with some $\mathbf{g} \in \mathbb{R}^s$ satisfying $\|\mathbf{g}_\Lambda\|_\infty \le 1$ and the max-norm error satisfies

$$\|\tilde{\mathbf{b}} - \mathbf{b}^*\|_\infty \le 2(\delta_D'' + \delta_c' + \gamma\|\bar{D}_{\Lambda,\Lambda}^{-1}\|_{1,\infty}),$$

while the ℓ_2-norm error satisfies

$$\|\tilde{\mathbf{b}} - \mathbf{b}^*\| \le 2\sqrt{s}(\delta_D'' + \delta_c' + \gamma\sigma_{\min}^{-1}).$$

If additionally $2(\delta_D'' + \delta_c' + \gamma\|\bar{D}_{\Lambda,\Lambda}^{-1}\|_{1,\infty}) \le \min_{j\in\Lambda}|\mathbf{b}^*_j|$, then we have the exact recovery, so that the following equality takes place

$$\tilde{\mathbf{b}}_\Lambda = \hat{D}_{\Lambda,\Lambda}^{-1}\hat{\mathbf{c}}_\lambda - \gamma\hat{D}_{\Lambda,\Lambda}^{-1}\mathbf{s}_\Lambda,$$

where $\mathbf{s} = sign(\mathbf{b}^*)$.

Proof. First observe that $D_{\Lambda^c,\Lambda}D_{\Lambda,\Lambda}^{-1}\mathbf{c}_\Lambda - \mathbf{c}_{\Lambda^c} = \Phi_{\Lambda^c}^\top(\Phi_\Lambda^+\mathbf{y} - \mathbf{y}) = \Phi_{\Lambda^c}^\top(P_\Lambda - I)\boldsymbol{\varepsilon}$. By Lemma B.4 we have,

$$\|\hat{D}_{\Lambda^c,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\|_{1,\infty} \le \|\bar{D}_{\Lambda^c,\Lambda}\bar{D}_{\Lambda,\Lambda}^{-1}\|_{1,\infty} + 4s\delta_D \le 1/2,$$

while since $\bar{\mathbf{c}}_{\Lambda^c} = \bar{D}_{\Lambda^c,\Lambda}\mathbf{b}^*_\Lambda = \bar{D}_{\Lambda^c,\Lambda}\bar{D}_{\Lambda,\Lambda}^{-1}\bar{\mathbf{c}}_\Lambda$,

$$\begin{aligned}
\|\hat{D}_{\Lambda^c,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\hat{\mathbf{c}}_\Lambda - \hat{\mathbf{c}}_{\Lambda^c}\|_\infty &\le \|\hat{D}_{\Lambda^c,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\hat{\mathbf{c}}_\Lambda - \bar{D}_{\Lambda^c,\Lambda}\bar{D}_{\Lambda,\Lambda}^{-1}\bar{\mathbf{c}}_\Lambda\|_\infty + \|\hat{\mathbf{c}}_{\Lambda^c} - \bar{\mathbf{c}}_{\Lambda^c}\|_\infty \\
&\le \|\hat{D}_{\Lambda^c,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}(\hat{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_\Lambda)\|_\infty + \|\hat{D}_{\Lambda^c,\Lambda}(\hat{D}_{\Lambda,\Lambda}^{-1} - \bar{D}_{\Lambda,\Lambda}^{-1})\bar{\mathbf{c}}_\Lambda\|_\infty \\
&\quad + \|(\hat{D}_{\Lambda^c,\Lambda} - \bar{D}_{\Lambda^c,\Lambda})\bar{D}_{\Lambda,\Lambda}^{-1}\bar{\mathbf{c}}_\Lambda\|_\infty + \delta_c \\
&\le \|\hat{D}_{\Lambda^c,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}(\hat{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_\Lambda)\|_\infty + \|\hat{D}_{\Lambda^c,\Lambda}(\hat{D}_{\Lambda,\Lambda}^{-1} - \bar{D}_{\Lambda,\Lambda}^{-1})\bar{\mathbf{c}}_\Lambda\|_\infty + \delta_D' + \delta_c.
\end{aligned}$$

Here, $\|\hat{D}_{\Lambda^c,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}(\hat{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_\Lambda)\|_\infty \le \delta_c/2$ due to $\|\hat{D}_{\Lambda^c,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}\|_{1,\infty} \le 1/2$. Moreover, we have

$$\begin{aligned}\|\hat{D}_{\Lambda^c,\Lambda}(\hat{D}^{-1}_{\Lambda,\Lambda} - \bar{D}^{-1}_{\Lambda,\Lambda})\bar{\mathbf{c}}_\Lambda\|_\infty &= \|\hat{D}_{\Lambda^c,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}(\bar{D}_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})\bar{D}^{-1}_{\Lambda,\Lambda}\bar{\mathbf{c}}_\Lambda\|_\infty \\ &\le \|\hat{D}_{\Lambda^c,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}\|_{1,\infty}\|(\bar{D}_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})\bar{D}^{-1}_{\Lambda,\Lambda}\bar{\mathbf{c}}_\Lambda\|_\infty \\ &\le \delta'_D/2.\end{aligned}$$

Using the condition on γ, we get that

$$\|\hat{D}_{\Lambda^c,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}\hat{\mathbf{c}}_\Lambda - \hat{\mathbf{c}}_{\Lambda^c}\|_\infty \le \frac{3}{2}(\delta'_D + \delta_c) \le \frac{\gamma}{2} \le \gamma(1 - \|\hat{D}_{\Lambda^c,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}\|_{1,\infty}),$$

so that the conditions of Lemma B.2 are satisfied and (B.4) takes place. This allows us to write

$$\begin{aligned}\tilde{\mathbf{b}}_\Lambda - \mathbf{b}^*_\Lambda &= \hat{D}^{-1}_{\Lambda,\Lambda}\hat{\mathbf{c}}_\Lambda - \bar{D}^{-1}_{\Lambda,\Lambda}\bar{\mathbf{c}}_\Lambda - \gamma\hat{D}^{-1}_{\Lambda,\Lambda}\mathbf{g}, \\ &= \hat{D}^{-1}_{\Lambda,\Lambda}(\bar{D}_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})\bar{D}^{-1}_{\Lambda,\Lambda}\bar{\mathbf{c}}_\Lambda + \hat{D}^{-1}_{\Lambda,\Lambda}(\hat{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_\Lambda) - \gamma\hat{D}^{-1}_{\Lambda,\Lambda}\mathbf{g} \\ &= \hat{D}^{-1}_{\Lambda,\Lambda}(\bar{D}_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})\mathbf{b}^*_\Lambda + \hat{D}^{-1}_{\Lambda,\Lambda}(\hat{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_\Lambda) - \gamma\hat{D}^{-1}_{\Lambda,\Lambda}\mathbf{g} \\ &= \hat{D}^{-1}_{\Lambda,\Lambda}\bar{D}_{\Lambda,\Lambda}\left(\bar{D}^{-1}_{\Lambda,\Lambda}(\bar{D}_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})\mathbf{b}^*_\Lambda + \bar{D}^{-1}_{\Lambda,\Lambda}(\hat{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_\Lambda) - \gamma\bar{D}^{-1}_{\Lambda,\Lambda}\mathbf{g}\right)\end{aligned}$$

By Lemma B.4 we have $\|\hat{D}^{-1}_{\Lambda,\Lambda}\bar{D}_{\Lambda,\Lambda}\|_{\infty\mapsto\infty} \le 2$ so that

$$\|\tilde{\mathbf{b}}_\Lambda - \mathbf{b}^*_\Lambda\|_\infty \le 2\|\bar{D}^{-1}_{\Lambda,\Lambda}(\bar{D}_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})\mathbf{b}^*_\Lambda\|_\infty + 2\|\bar{D}^{-1}_{\Lambda,\Lambda}(\hat{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_\Lambda)\|_\infty + 2\gamma\|\bar{D}^{-1}_{\Lambda,\Lambda}\|_{1,\infty}\,.$$

and since we also have $\|\hat{D}^{-1}_{\Lambda,\Lambda}\bar{D}_{\Lambda,\Lambda}\|_{\mathsf{op}} \le 2$ and $\|\mathbf{g}\| \le \sqrt{s}$, it holds

$$\|\tilde{\mathbf{b}}_\Lambda - \mathbf{b}^*_\Lambda\| \le 2\sqrt{s}\left(\|\bar{D}^{-1}_{\Lambda,\Lambda}(\bar{D}_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})\mathbf{b}^*_\Lambda\|_\infty + \|\bar{D}^{-1}_{\Lambda,\Lambda}(\hat{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_\Lambda)\|_\infty + \gamma\|\bar{D}^{-1}_{\Lambda,\Lambda}\|_{\mathsf{op}}\right).$$

□

Before we proceed with the proof of this corollary, we present a technical lemma that collects some trivial inequalities.

Lemma B.4. *Set* $\delta_c = \|\hat{\mathbf{c}} - \bar{\mathbf{c}}\|_\infty$, $\delta_D = \|(\hat{D}_{\Lambda^c,\Lambda} - \bar{D}_{\Lambda^c,\Lambda})\bar{D}^{-1}_{\Lambda,\Lambda}\|_{\infty,\infty}$. *Suppose,* $\|\bar{D}_{\Lambda^c\Lambda}\bar{D}^{-1}_{\Lambda\Lambda}\|_{1,\infty} \le 1$ *and* $s\delta_D \le 1/2$. *It holds,*

- *for any* $q \ge 1$

$$\|D_{\Lambda,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}\|_{q\to q} \le 2, \qquad \|\hat{D}^{-1}_{\Lambda,\Lambda}D_{\Lambda,\Lambda}\|_{q\to q} \le 2\,;$$

- $$\|\hat{D}_{\Lambda^c,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda} - D_{\Lambda^c,\Lambda}D^{-1}_{\Lambda,\Lambda}\|_{1,\infty} \le 4s\delta_D.$$

Proof. First, we have

$$\begin{aligned}\|D_{\Lambda,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}\|_{q\to q} &= \|I + (D_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})\hat{D}^{-1}_{\Lambda,\Lambda}\|_{q\to q} \\ &\le 1 + \|(D_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})D^{-1}_{\Lambda,\Lambda}\|_{q\to q}\|D_{\Lambda,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}\|_{q\to q} \\ &\le 1 + s\delta_D\|D_{\Lambda,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}\|_{q\to q},\end{aligned}$$

which solving the inequality and since $s\delta_D \leq 1/2$ turns into

$$\|D_{\Lambda,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}\|_{q\to q} \leq \frac{1}{1-s\delta_D} \leq 2.$$

Similarly, $\|\hat{D}^{-1}_{\Lambda,\Lambda}D_{\Lambda,\Lambda}\|_{q\to q} \leq 2$.

Furthermore,

$$\begin{aligned}\|(\hat{D}_{\Lambda^c,\Lambda} - D_{\Lambda^c,\Lambda})\hat{D}^{-1}_{\Lambda,\Lambda}\|_{1,\infty} &\leq \|(\hat{D}_{\Lambda^c,\Lambda} - D_{\Lambda^c,\Lambda})D^{-1}_{\Lambda,\Lambda}\|_{1,\infty}\|D_{\Lambda,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}\|_{1\to 1} \\ &\leq 2s\delta_D.\end{aligned}$$

and

$$\begin{aligned}\|D_{\Lambda^c,\Lambda}(D^{-1}_{\Lambda,\Lambda} - \hat{D}^{-1}_{\Lambda,\Lambda})\|_{1,\infty} \leq& \|D_{\Lambda,\Lambda^c}D^{-1}_{\Lambda,\Lambda}\|_{1,\infty}\|\hat{D}^{-1}_{\Lambda,\Lambda}(\hat{D}_{\Lambda,\Lambda} - D_{\Lambda,\Lambda})\|_{1\to 1} \\ \leq& \|D_{\Lambda,\Lambda^c}D^{-1}_{\Lambda,\Lambda}\|_{1,\infty}\|\hat{D}^{-1}_{\Lambda,\Lambda}D_{\Lambda,\Lambda}\|_{1\to 1}\|D^{-1}_{\Lambda,\Lambda}(\hat{D} - D)\|_{1\to 1} \\ \leq& 2\|D_{\Lambda,\Lambda^c}D^{-1}_{\Lambda,\Lambda}\|_{1,\infty}s\delta_D,\end{aligned}$$

which together give us the second inequality. □