Highest weight vectors for classical reductive groups

Adam David John Dent

Submitted in accordance with the requirements for the degree of

Doctor of Philosophy

The University of Leeds

School of Mathematics

September 2018

The candidate confirms that the work submitted is his own, except where work which has formed part of jointly authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

Most of the work in Chapter 2 of the thesis has been accepted for for publication in Algebras and Representation Theory as [6]:

I was responsible for the explanatory text, the definitions and preliminaries, and also the lemmas and theorems on the divisibility of the basis elements, including Subsection 2.2.1, which was not included in the paper submitted for publication, and also the examples, particularly 2.6; the contribution of Rudolf Tange, the other author, was the majority of the remarks, particularly at the end of each subsection, other examples including 2.5, and Theorem 2.23 and its corollaries including Theorem 2.30.

Acknowledgements

First, above all others and always, I shall acknowledge my wife Magdalena, without whom nothing would be possible, and without whose patience, wisdom and generosity, especially over the last year, this thesis would not have been finished. I am also indebted to my supervisor Rudolf Tange for all his own patience as well as his academic guidance, my advisors Robert Marsh and Bill Crawley-Boevey for their general guidance and support, and to EPSRC for funding my stipend for this project. My mental health during the last four years also owes a great deal to my colleagues and friends at the University of Leeds School of Mathematics, particularly fellow PhD students, including Josh C, Josh K, Cesare, Richard, Jakob, Anja, and others. The care I received from Mariusz, Anna and Marysia during the weeks I spent writing up in their house was invaluable, and the thesis certainly also owes to them. Lastly but not leastly, I could not have carried out or written up any of the research below had it not been for Penny and Chris Dent, as well as my brother Jim, and all the rest of my family hanc marginis exiguitas non caperet.

Abstract

A result by Tange from 2015 [26] gave bases for the spaces of highest weight vectors for the action of $\mathrm{GL}_r \times \mathrm{GL}_s$ on $k\left[\mathrm{Mat}_{rs}^m\right]$ over a field of characteristic zero, and in arbitrary characteristic for certain weights; here, we generalise this to give bases for the spaces of highest weight vectors in $k\left[\mathrm{Mat}_{rs}^m\right]$ of any given weight in arbitrary characteristic. The motivation for this is to apply the technique of transmutation to describe the highest weight vectors for the conjugation action of GL_n on $k\left[\mathrm{Mat}_n\right]$. Then, we use similar methods but in characteristic zero to describe finite spanning sets for the spaces of highest weight vectors for a certain polynomial action of GL_r on $k\left[\mathrm{Mat}_r^l\right]$ (derived from the GL_r-action on Mat_r given by $g \cdot A = gAg^T$), and apply this to the conjugation action of the symplectic group Sp_n on $k\left[\mathfrak{sp}_n\right]$.

Contents

Introduction

Definition. Let $\mathbb{A}^n$ denote n-dimensional affine space and k a field. An *affine algebraic set* is a subset $\{x \in \mathbb{A}^n | \, f(x) = 0 \forall f \in S\}$ for some set S of polynomials in $k[x_1, \ldots, x_n]$. An *affine variety* is an affine algebraic set that is *irreducible*, i.e. that is not equal to the union of two proper subsets that are both also affine algebraic sets. Throughout this thesis, affine varieties will be referred to simply as *varieties.*

Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be varieties; a map $\varphi : X \to Y$ is called a *morphism* of varieties if each $\varphi_i : X \to \mathbb{A}^1$, where $\varphi(x) = (\varphi_1, \ldots, \varphi_m)$, can be expressed as a polynomial on the co-ordinates of X. An *algebraic group* is a variety with a group structure defined on all its points such that the multiplication and inverse functions of the group are morphisms of varieties.

For G and H algebraic groups over an algebraically closed field, a function $f : G \to H$ is a *homomorphism* of algebraic groups if it is both a morphism of varieties and a group homomorphism.

In Chapter 1 of this thesis, I will discuss the background theory behind the research in the latter two chapters; the first chapter will not contain any original results. This chapter is mostly a summary of the research I did in the first year of my PhD. In that time, I had frequent meetings with my supervisor Dr Rudolf Tange, who gave me problems to try to solve by myself in between our meetings, and helped me fill in gaps in my solutions as well as explaining further interesting concepts in the meetings themselves. The problems I worked on, which were mainly related to decompositions of modules of Lie algebras as direct sums of irreducible modules, followed on from each other and, together with the books I was reading, eventually led me to classical Schur-Weyl duality, which is described in Section 1.6.

Chapter 2 is adapted from the paper [6], to appear in Algebras and Representation Theory, which was jointly authored with Dr Tange in 2016 and revised in 2017. Generally the results and examples arose from simultaneous collaboration, or followed from the other results; in terms of authorship, to the paper I contributed a lot of explanatory text, the definitions and preliminaries (especially almost all of the first section), and also the lemmas and theorems on the divisibility of the basis elements (especially the results before Theorem 2.23, which inspired the later results), and also most of the examples; the contribution of Dr Tange was the majority of the remarks, particularly at the end of each subsection, other examples including 2.5, and Theorem 2.23 and its corollaries including Theorem 2.30; in the thesis I have also included more of my own material that did not appear in the paper, in particular Subsection 2.2.1, Example 2.6, and in addition I have expanded some of the explanations and included certain definitions and Lemmas from the sources, for example my interpretation of Lemma 2.16.

The first goal in Chapter 2 is to give bases of the vector spaces $k[\mathrm{Mat}^m_{rs}]^{U_r\times U_s}_{(\mu,\lambda)}$. In

[26] this was done under the assumption that k is of characteristic 0. The method there was to reduce the problem via a few simple isomorphisms to certain results from the representation theory of the symmetric group which were originally due to Donin in [8]. Although this method is rather straightforward, it is hard to generalise to arbitrary characteristic. Now we solve the problem in arbitrary characteristic using results on bideterminants from the work of Kouwenhoven [21] which is based on work of Clausen [3], [4]. We introduce "twisted bideterminants" to construct an explicit "good" filtration and, in particular, give bases for the spaces of highest weight vectors in $k[\mathrm{Mat}_{rs}^m]$, see Theorem 2.23 and its two corollaries in Section 2.2. It turns out that these bases can also be obtained by dividing the basis elements from [26, Thm. 4] by certain integers in the obvious $\mathbb{Z}$-form and then reducing mod p. As an application we give in Section 2.3 explicit finite homogeneous spanning sets of the $k[\mathrm{Mat}_n]^{\mathrm{GL}_n}$-modules of highest weight vectors in the coordinate ring $k[\mathrm{Mat}_n]$ under the conjugation action of GL_n, see Theorem 2.30. Although this problem is difficult to tackle directly, [26] contains a method in arbitrary characteristic called "transmutation" to reduce this problem to giving spanning sets for the vector spaces $k[\mathrm{Mat}_{rs}^m]_{(\mu,\lambda)}^{U_r\times U_s}$, see Theorem 2.27 below. So the problem is reduced to the problem solved in Section 2.2.

The goal for Chapter 3 is to apply the knowledge and techniques developed for [26] and Chapter 2 for the action of the general linear group, to the related case of the symplectic group Sp_n (where n is now an even integer) acting via conjugation on its own Lie algebra $\mathfrak{sp}_n$ and the algebra $k[\mathfrak{sp}_n]$ of polynomial functions on $\mathfrak{sp}_n$; we consider the problem of giving finite homogeneous spanning sets for the $k[\mathfrak{sp}_n]^{\mathrm{Sp}_n}$-modules of highest weight vectors for such action. In particular, we prove in Subsection 3.1.3 that we can apply transmutation to move to an action of GL_r on the space of l-tuples of $r \times r$-matrices. This time we assume k is of characteristic 0 when looking for the highest weight vectors. Finite spanning sets for the spaces of highest weight vectors on this transmuted variety are described at the end of Subsection 3.2.2. Then these can be pulled back, i.e. we can apply to them the comorphism of a particular morphism of varieties, in order to give finite spanning sets for the spaces of highest weight vectors for the conjugation action of the symplectic group, see 3.14.

Some notation. Let V be a vector space over a field k. By $\mathrm{GL}(V)$, we mean the general linear group on V, that is, the group of linear maps $V \to V$; if $V \cong k^n$, then $\mathrm{GL}(V) \cong \mathrm{GL}_n(k)$, the space of invertible $n \times n$-matrices with entries in k, which will hereafter be denoted simply by GL_n. We denote the Lie algebra of $\mathrm{GL}(V)$ by $\mathfrak{gl}(V)$. The Lie algebra of GL_n, that is, the tangent to GL_n at its identity I_n (the $n \times n$-matrix with ones everywhere on the leading diagonal and zeros everywhere else), is the space $\mathrm{Mat}_n(k)$, consisting of all $n \times n$-matrices with entries in k; we shall also omit the (k) from this notation. A Lie algebra $\mathfrak{g}$ is a vector space together with an alternating bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x, y) \mapsto [x, y]$ satisfying the Jacobi identity, $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$. The adjoint action or adjoint representation of a Lie algebra $\mathfrak{g}$ on itself is the map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ given by $x \mapsto \mathrm{ad}x$, where for an element $x \in \mathfrak{g}$, $\mathrm{ad}x : \mathfrak{g} \to \mathfrak{g}$ denotes the map $y \mapsto [x, y]$. An abelian Lie algebra $\mathfrak{g}$ is one where $[x, y] = 0$ for all $x, y \in \mathfrak{g}$ or equivalently where $\mathrm{ad}x$ is the zero map for all $x \in \mathfrak{g}$.

Chapter 1

Background theory

1.1 Preliminaries

Definitions 1.1. Let $\mathfrak{g}$ be a Lie algebra, and G an algebraic group, over an algebraically closed field k.

1. A *symmetric bilinear form* on $\mathfrak{g}$ is a function $\beta : \mathfrak{g}^2 \to k$ satisfying

   (a) $\beta\left(ax_1 + x_2, y\right) = a\beta\left(x_1, y\right) + \beta\left(x_2, y\right),$

   (b) $\beta\left(x_1, y\right) = \beta\left(y, x_1\right),$

   for all $a \in k$, $x_1, x_2, y \in \mathfrak{g}$.

2. [16, p22] The *radical* of a symmetric bilinear form β is the subspace $S = \{x \in \mathfrak{g} \,|\beta\left(x, y\right) = 0\, \forall y \in \mathfrak{g}\}$. A symmetric bilinear form is called *nondegenerate* if its radical is $\{0\}$.

3. [16, p21] Let κ be the symmetric bilinear form on $\mathfrak{g}$ given by $\kappa(x, y) = \mathrm{tr}(\mathrm{ad}x.\mathrm{ad}y)$. We call κ the *Killing form*.

4. A k-vector space V is a *module* over $\mathfrak{g}$ if there is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$ or respectively a module over G if there is an algebraic group homomorphism $G \to \mathrm{GL}(V)$. This homomorphism, or equivalently the module itself, is called a *representation* of $\mathfrak{g}$ (resp. G). A representation is called *irreducible* if it has no nonzero proper submodules.

5. For $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$ and V a $\mathfrak{g}$-module, let $V^{\mathfrak{h}} := \{v \in V \,|\, h.v = 0 \;\forall h \in \mathfrak{h}\}$; similarly, for $H \leq G$ and V a G-module, denote $V^H := \{v \in V \,|\, h.v = v \;\forall h \in H\}$.

6. $\mathfrak{g}$ is called *simple* if it is non-abelian and has no non-zero proper (left- or right-) ideals; it is called *semisimple* if it is a (finite) direct product of simple Lie algebras.

Proposition 1.2. *A Lie algebra $\mathfrak{g}$ over a field of characteristic 0 is semisimple if and only if it satisfies one of the following equivalent conditions;*

- *The Killing form of $\mathfrak{g}$ is nondegenerate;*
- *Every representation of $\mathfrak{g}$ is fully reducible, i.e. is a sum of irreducible representations.*

From now on, let $\mathfrak{g}$ denote a finite-dimensional semisimple Lie algebra over an algebraically closed field of characteristic zero. A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called *toral* if it contains no non-zero nilpotent elements; a toral subalgebra is *maximal* if it is not contained in any other toral subalgebra. From now on let $\mathfrak{h}$ be a fixed maximal toral subalgebra of $\mathfrak{g}$, and $\mathfrak{h}^*$ its dual vector space.

Let $\Phi \subseteq \mathfrak{h}^*$ denote the root system of $\mathfrak{g}$. Recall that a *base* of the root system is a subset $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ of Φ that is a basis of $\mathrm{GL}\,(\mathbb{E})$ (where $\mathbb{E}$ is the Euclidean space of which Φ is a subset) such that for any $\alpha \in \Phi$, we can write

$\alpha = \sum_{\alpha_i \in \Delta} d_i \alpha_i$ with the integers d_i either all non-negative (in which case the root α is called a *positive* root) or all non-positive (in which case the root α is called a *negative* root); roots in the base are called *simple* roots. Let $\mathscr{W}$ denote the *Weyl group* of Φ; that is, the subgroup of $\mathrm{GL}(\mathbb{E})$ generated by reflections σ_α (for $\alpha \in \Phi$). The Weyl group is isomorphic to a subset of the symmetric group on Φ.

Using a base Δ, we can define a partial ordering on $\mathfrak{h}^*$ in the following way [16, 10.1]: for $\lambda, \mu \in \mathfrak{h}^*$, set $\lambda \prec \mu$ if and only if $\mu - \lambda$ is a sum of positive roots (and since every positive root is a sum of simple roots by the above definitions, this is equivalent to requiring that $\mu - \lambda$ be a sum of simple roots) or $\lambda = \mu$. We can also write $\alpha \succ 0$ for positive roots and $\alpha \prec 0$ for negative roots.

Let $\alpha \in \Phi \subseteq \mathfrak{h}^*$ be a root, then denote by h_α the image of $\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)}$ (where $(-,-)$ is the inner product in $\mathbb{E}$) under the inverse of the linear map $\theta : \mathfrak{h} \to \mathfrak{h}^*$ given by $\theta : h \mapsto \kappa(h, -)$ (where κ is the Killing form on $\mathfrak{g}$), which we know is invertible because the Killing form is nondegenerate which means θ is at least injective, and because $\mathfrak{h}$ and $\mathfrak{h}^*$ have the same dimension (see [16, Sect. 9] for more on this).

Definition 1.3. Now denote by Λ the subset of $\mathfrak{h}^*$ containing all integral linear maps λ, i.e. those for which all $\lambda(h_{\alpha_i})$ $(\alpha_i \in \Delta)$ and hence all $\lambda(h_\alpha)$ $(\alpha \in \Phi)$ are integral. Call elements of Λ *(integral) weights*. If all the $\lambda(h_{\alpha_i})$ are non-negative integers then λ is called *dominant*, and the subset of Λ containing all such weights is denoted Λ^+.

Any finite-dimensional $\mathfrak{g}$-module V can be decomposed [16, Sect. 7] as a direct sum $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, with the subspaces $V_\lambda := \{v \in V \,|\, h.v = \lambda(h)\, v \;\forall h \in \mathfrak{h}\}$ being called *weight spaces* whenever non-zero, in which case we say that λ is a weight of $\mathfrak{h}$ on V or simply a weight of V. (The set of weights of a particular finite-dimensional $\mathfrak{g}$-module V is a finite subset of the infinite set Λ of all weights.) Denote $\mathfrak{n}_+ := \sum_{\alpha \succ 0} \mathfrak{g}_\alpha$ (in case $\mathfrak{g} = \mathfrak{sl}_n$, then $\mathfrak{n}_+$ is the set of strictly upper triangular matrices). A *highest weight vector* in V is any non-zero weight vector killed by $\mathfrak{g}_\alpha$ for all $\alpha \in \Delta$.

We can also state some of the above definitions for modules over the algebraic group $G = \mathrm{GL}_n$, in almost the same way: let $H \subseteq G$ be the subgroup consisting of the diagonal invertible matrices, then *weights* are homomorphisms $H \to k$, the collection of which is denoted $X(H)$. $X(H)$ is isomorphic to the set $\mathbb{Z}^n$ of n-tuples of integers, (the isomorphism is given by $e_i \mapsto \varepsilon_i$ where $\{e_i\}$ is the standard basis of $\mathbb{Z}^n$ and $\varepsilon_i : h \mapsto h_i$ for h_i the entry in the i-th row (and column) of $h \in H$) and a weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ is called *dominant* here if $\lambda_1 \geq \ldots \geq \lambda_n$. The weights of a specific G-module V are those for which the space $V_\lambda := \{v \in V \,|h.v = \lambda(h)\, v\, \forall h \in H\}$ is non-zero, in which case such space is once again called a *weight space*. The *roots* are the non-zero weights of H on $\mathfrak{gl}_n$, which are $(\varepsilon_i - \varepsilon_j) : h \mapsto h_i h_j^{-1}$, $i \neq j$, and we can define again a "root ordering" on $X(H)$. Let us denote by $U^+ \subset \mathrm{GL}_n$ the set of upper unitriangular matrices; that is, upper triangular matrices with 1 everywhere on the diagonal. We can now state the following two results.

Theorem 1.4. *[16, 7.2] Let V be an irreducible module over the general linear group GL_n, or a finite-dimensional irreducible module over any semisimple Lie algebra. Then V has a highest weight μ (in the ordering of weights described above); this weight is dominant, and its weight space V_μ is one-dimensional and equal to V^{U^+} in the group case or $V^{\mathfrak{n}_+}$ in the Lie algebra case.*

Moreover, every dominant weight is the highest weight of some irreducible module, and two irreducible modules are isomorphic if and only if they have the same highest weight.

This means that we can describe unique irreducible modules by denoting them $V(\lambda)$, where λ is the highest weight. I will also make use later on of the following piece of notation from [16, 21.1]: for a module V, $\Pi(V)$ denotes the set of all weights of V, and for an irreducible $V(\lambda)$, we abbreviate the notation from $\Pi(V(\lambda))$ to $\Pi(\lambda)$.

Lemma 1.5. *Consider a finite-dimensional $\mathfrak{g}$-module V and an irreducible $\mathfrak{g}$-module $V(\mu)$ (with highest weight μ). Then we have*

$$\mathrm{Hom}_{\mathfrak{g}}\left(V(\mu), V\right) \cong V_{\mu}^{\mathfrak{n}+}.$$

There is an analogue of this for the general linear group: let $G = \mathrm{GL}_n$ and consider a G-module V and an irreducible G-module $V(\mu)$ of V (with highest weight μ). Then we have

$$\mathrm{Hom}_{G}\left(V(\mu), V\right) \cong V_{\mu}^{U+}.$$

In both of these cases, the isomorphism is given by $f \mapsto f(v_\mu)$, where $v_\mu \in V(\mu)$ denotes a highest weight vector of weight μ.

Now let $Z(\lambda)$ be the module $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(B)} D_\lambda$, ($\mathfrak{U}(A)$ denoting the universal enveloping algebra of a Lie algebra A,) where D_λ is the one-dimensional vector space with $\{v_\lambda\}$ as its basis, and $B = B(\Delta)$ is the *Borel subagebra* $\mathfrak{h} + \bigsqcup_{\alpha \succ 0} \mathfrak{g}_\alpha$, with an action of B on D_λ defined by $\left(h + \sum_{\alpha \succ 0} x_\alpha\right).v_\lambda = hv_\lambda = \lambda(h)v_\lambda$ making D_λ a B-module and therefore also a $\mathfrak{U}(B)$-module, hence the definition of $Z(\lambda)$ makes sense and indeed we see that $Z(\lambda)$ is a $\mathfrak{U}(\mathfrak{g})$-module (these $Z(\lambda)$, λ a weight, are called *Verma modules*). Now, we can also define $V(\lambda)$ to be the irreducible $\mathfrak{g}$-module $Z(\lambda)/Y(\lambda)$, where $Y(\lambda)$ is the unique maximal submodule of $Z(\lambda)$.

Theorem 1.6. *If $\lambda \in \Lambda^+$ then the irreducible $\mathfrak{g}$-module $V = V(\lambda)$ is finite-dimensional and its set of weights, denoted $\Pi(\lambda)$, is permuted by $\mathscr{W}$, with $\dim V_\mu = \dim V_{\sigma(\mu)}$ for $\sigma \in \mathscr{W}$.*

1.2 Characters and multiplicity formulas

Denote by $\mathbb{Z}[\Lambda]$ the *group ring* of Λ over $\mathbb{Z}$, the free $\mathbb{Z}$-module with basis elements $e(\lambda)$ in one-to-one correspondence with the elements λ of Λ, where as well as standard addition, we can define multiplication by $e(\lambda)e(\mu) = e(\lambda+\mu)$. $\mathscr{W}$ acts naturally on $\mathbb{Z}[\Lambda]$ by permuting the basis elements, so that $\sigma(e(\lambda)) = e(\sigma(\lambda))$. We can also think of $\mathbb{Z}[\Lambda]$ as a space of functions on $\mathfrak{h}^*$ with $e(\lambda)$ corresponding to ε_λ (where ε denotes the *characteristic function*, $\varepsilon_\lambda(\lambda) = 1$ and $\varepsilon_\lambda(\mu) = 0$ for any $\mu \neq \lambda$); the multiplication in this case is *convolution*, $(f * g)(\lambda) := \sum_{\substack{\mu,\nu\in\Lambda,\\ \mu+\nu=\lambda}} f(\mu).g(\nu)$. It can be checked that convolution is both associative and commutative and closed on the set $\{f : \mathfrak{h}^* \to k \mid f(\lambda) \neq 0 \Rightarrow \lambda \prec \mu \text{ for some } \mu \in M\}$ for a finite $M \subseteq \mathfrak{h}^*$. Note that $\varepsilon_\lambda * \varepsilon_\mu = \varepsilon_{\lambda+\mu}$, and $\varepsilon_{\sigma(\lambda)} = \sigma^{-1}\varepsilon_\lambda$ for $\sigma \in \mathscr{W}$.

Define the *formal character* of a finite-dimensional module V, denoted ch_V (or ch_λ in case V is the irreducible $V(\lambda)$, $\lambda \in \Lambda^+$), to be the element $\sum_{\mu\in\Pi(\lambda)} m(\mu)e(\mu)$ of $\mathbb{Z}[\Lambda]$, where $m(\mu)$ is the *multiplicity* of μ in V, defined so that $m(\mu) := \dim V_\mu$ (which equals 0 in case μ is not a weight of V). Let $\rho = \frac{1}{2}\sum_{\alpha\succ 0}\alpha$, then we can state the Weyl character formula as in [16, 24.3].

Theorem 1.7 (Weyl's formula)**.** *Let* $\lambda \in \Lambda^+$*, then*

$$\left(\sum_{\sigma\in\mathscr{W}} \operatorname{sgn}(\sigma')\, \varepsilon_{\sigma(\rho)}\right) * ch_\lambda = \sum_{\sigma\in\mathscr{W}} \operatorname{sgn}(\sigma')\, \varepsilon_{\sigma(\lambda+\rho)}.$$

Using this formula, we can also now define the function ch_λ for any $\lambda \in \Lambda$ (i.e. not necessarily dominant) by $ch_{\sigma(\lambda+\rho)-\rho} = \operatorname{sgn}(\sigma) ch_\lambda$ for $\sigma \in \mathscr{W}$, that is, $ch_\lambda = 0$ if $\lambda + \rho$ is fixed by a reflection, and otherwise $\lambda + \rho$ will be conjugate to some strictly dominant weight μ (that is, the integers $\mu(h_{\alpha_i})$ are strictly positive), in which case $\mu - \rho$ is dominant, and then $ch_{\mu-\rho}$ can be defined as it was for dominant weights before the theorem, and in this case the two definitions indeed coincide. Weyl's formula also gives us $\omega(\rho) * ch_\lambda = \omega(\lambda+\rho)$, where $\omega : \Lambda \to \mathbb{Z}[\Lambda]$ is given by

$$\omega(\mu) := \sum_{\sigma\in\mathscr{W}} \operatorname{sgn}(\sigma) \varepsilon_{\sigma(\mu)}$$

.

Let $p \in \mathbb{Z}[\Lambda]$ denote the *Konstant function*, defined so that

$$p(\mu) := \#\left\{ \{ r_\alpha |\, \alpha \succ 0\} \,\middle|\, -\mu = \sum_{\alpha\succ 0} r_\alpha \alpha \right\}.$$

If we define the functions $f_\alpha : \mathfrak{h}^* \to k$ (indexed by positive roots) by $f_\alpha(-r\alpha) = 1$ for $r \in \mathbb{Z}^+$ and $f_\alpha(\mu) = 0$ otherwise, then we have $p = \prod_{\alpha\succ 0} f_\alpha$. One way we can look at $p(\nu)$ ($\nu \in \mathfrak{h}^*$) is as the number of sets of non-negative integers $\{ r_\alpha |\, \alpha \succ 0\}$ for which $-\nu = \sum_{\alpha\succ 0} r_\alpha \alpha$. We can now also state Konstant's formula, from [16, 24.2].

Theorem 1.8 (Konstant's formula)**.** *Let* $\lambda \in \Lambda^+$*. Then the multiplicities of* $V(\lambda)$ *are given by*

$$m_\lambda(\mu) = \sum_{\sigma\in\mathscr{W}} \operatorname{sgn}(\sigma')\, p(\mu + \rho - \sigma(\lambda+\rho)).$$

Steinberg's formula [16, 24.2] combines those of Konstant and Weyl to give a method for decomposing a tensor product of two irreducible modules as a direct sum of irreducible modules.

Theorem 1.9 (Steinberg's formula)**.** *Let $\lambda', \lambda'' \in \Lambda^+$. Then the number of times $V(\lambda)$ ($\lambda \in \Lambda^+$) occurs in the direct sum decomposition of $V(\lambda') \otimes V(\lambda'')$ is*

$$\sum_{\sigma \in \mathscr{W}} \sum_{\tau \in \mathscr{W}} \operatorname{sgn}(\sigma\tau) p\left(\lambda + 2\rho - \sigma(\lambda' + \rho) - \tau(\lambda'' + \rho)\right).$$

The following is my solution to Exercise 9 from [16, Sect. 24], for which I followed the hints given there by the author. Starting by assuming the above formulas, we attempt to obtain an alternative formula for decomposing a tensor product of two irreducible modules as a direct sum of irreducible modules, assuming that we have explicit knowledge of the weights of one of the modules.

First, fix a pair $\lambda', \lambda'' \in \Lambda^+$, then for each $\lambda \in \Lambda^+$ let $n(\lambda)$ denote the number of times $V(\lambda)$ appears in the decomposition of the tensor product $V(\lambda') \otimes V(\lambda'')$, so that we can write the formal character of the tensor product as

$$ch_{\lambda'} * ch_{\lambda''} = \sum_{\lambda \in \Lambda^+} n(\lambda) ch_\lambda$$

Now multiply both sides of the above by $\omega(\rho)$ and apply Weyl's formula for λ and λ'' to obtain:

$$ch_{\lambda'} * \omega\left(\lambda'' + \rho\right) = \sum_{\lambda \in \Lambda^+} n(\lambda) \omega(\lambda + \rho).$$

We can replace $ch_{\lambda'}$ with the formula $\sum\limits_{\lambda \in \Lambda} m_{\lambda'}(\lambda) \varepsilon_\lambda$ and then the left-hand side of the above becomes:

$$\sum_{\lambda \in \Lambda} m_{\lambda'}(\lambda) \varepsilon_\lambda * \sum_{\sigma \in \mathscr{W}} \operatorname{sgn}(\sigma) \varepsilon_{\sigma(\lambda'' + \rho)}$$

then using the fact that $\mathscr{W}$ permutes the weight spaces of $V(\lambda')$ and that the leftmost sum is $\mathscr{W}$-invariant, we can distribute the product over the rightmost sum, bringing the summation over $\mathscr{W}$ to the outside of the expression and replacing λ by $\sigma(\lambda)$, to give

$$\sum_{\sigma\in\mathscr{W}} \operatorname{sgn}(\sigma) \sum_{\lambda\in\Pi(\lambda')} m_{\lambda'}(\lambda) \left(\varepsilon_{\sigma(\lambda)} * \varepsilon_{\sigma(\lambda''+\rho)}\right)$$

and then because of the distributivity of this action, as well as the property of the characteristic function that $\varepsilon_\mu * \varepsilon_\nu = \varepsilon_{\mu+\nu}$, we can again rewrite this, as

$$\sum_{\sigma\in\mathscr{W}} \operatorname{sgn}(\sigma) \sum_{\lambda\in\Pi(\lambda')} m_{\lambda'}(\lambda)\varepsilon_{\sigma(\lambda+\lambda''+\rho)}.$$

So, we are left with

$$\sum_{\sigma\in\mathscr{W}} \operatorname{sgn}(\sigma) \sum_{\lambda\in\Pi(\lambda')} m_{\lambda'}(\lambda)\varepsilon_{\sigma(\lambda+\lambda''+\rho)} = \sum_{\sigma\in\mathscr{W}} \operatorname{sgn}(\sigma) \sum_{\lambda\in\Lambda^+} n(\lambda)\varepsilon_{\sigma(\lambda+\rho)}.$$

Then applying Weyl's formula, we have

$$\sum_{\sigma\in\mathscr{W}} \operatorname{sgn}(\sigma) \sum_{\lambda\in\Pi(\lambda')} m_{\lambda'}(\lambda)\varepsilon_{\sigma(\lambda+\lambda''+\rho)} = \sum_{\lambda\in\Pi(\lambda')} m_{\lambda'}(\lambda)\omega(\rho) * ch_{\lambda+\lambda''};$$

but as we obtained $\sum_{\sigma\in\mathscr{W}} \operatorname{sgn}(\sigma) \sum_{\lambda\in\Lambda^+} n(\lambda)\varepsilon_{\sigma(\lambda+\rho)}$ by multiplying $ch_{\lambda'} * ch_{\lambda''}$ by $\omega(\rho)$ in the first place, this gives us a new formula:

$$ch_{\lambda'} * ch_{\lambda''} = \sum_{\lambda\in\Pi(\lambda')} m_{\lambda'}(\lambda) ch_{\lambda+\lambda''}.$$

This is known as Brauer's formula, or sometimes the Brauer-Klimyk formula.

Remark 1.10. Brauer's formula remains valid in characteristic p if we replace irreducible modules by induced modules (see Definition 2.11).

1.2.1 Example: deriving the Clebsch-Gordan formula

We apply Brauer's formula to the Lie algebra $\mathfrak{sl}_2$, consisting of 2×2 zero-trace matrices, which gives a very explicit formula for the direct sum decomposition of the tensor product of two finite-dimensional irreducible $\mathfrak{sl}_2$-modules.

$\mathfrak{sl}_2$ has a basis $\{E, F, H\}$, where

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that for these three matrices as defined, we have $[E, F] = H$, $[H, E] = 2E$, and $[H, F] = -2F$. The maximal toral subalgebra is simply $\mathfrak{h} = \langle H \rangle$, and so the roots are the eigenvalues of H, which are just 2 and -2 (we can consider weights of H on $\mathfrak{sl}_2$ as integers, because of the bijection taking any integer r to the linear map $x_r : H \mapsto r$ in $\mathfrak{h}^*$). Thus we have only one positive root $\alpha = 2$; this gives $\rho = \frac{1}{2}\alpha = 1$ and implies that the Weyl group is of order 2, $\mathscr{W} \cong \{1, -1\}$. We also have that for a finite-dimensional irreducible $\mathfrak{sl}_2$-module $V(n)$, $\Pi(n) = \{n, n-2, \ldots, -n\}$ and $V(n) = V_n \oplus V_{n-2} \oplus \ldots \oplus V_{-n}$ with $\dim V_\pi = 1$ for each $\pi \in \Pi(n)$ (hence, $\dim V(n) = n + 1$) so, we have that for every weight π, $m_n(\pi) = 1$.

So, for $\mathfrak{sl}_2$, Brauer's formula becomes:

$$V(m) \otimes V(n) \cong \bigoplus_{\pi \in \Pi(n)} V(\pi + m),$$

which we can state even more explicitly, as

$$V(m) \otimes V(n) \cong V(m+n) \oplus V(m + (n-2)) \oplus \ldots \oplus V(m + (-n)),$$

which is also known (e.g. in [16, 22.Ex. 7], [11, 1.2]) as the Clebsch-Gordan formula.

1.3 Symmetric and exterior powers of the natural module over $\mathfrak{sl}_n$

Throughout this section and the next, unless stated otherwise, V will necessarily denote the n-dimensional vector space k^n over a field k of characteristic 0, as well as the so-called *natural* module over $\mathfrak{sl}_n(k)$, which one defines by endowing the aforementioned vector space with the action $x.v = xv$, viewing $v \in V$ as an $n \times 1$ matrix and multiplying it on the left by the $n \times n$ matrix $x \in \mathfrak{sl}_n$.

1.3.1 Definitions

The *tensor algebra* $T(V)$ of the vector space V is the direct sum of the tensor spaces $V^{\otimes r}$ for $r = 1, 2, \ldots$ where the tensor space $V^{\otimes r} := \underbrace{V \otimes \ldots \otimes V}_{r}$ is the n^r-dimensional space spanned by tensors of the form $v_{i_1} \otimes \ldots \otimes v_{i_r}$, $1 \leq i_j \leq n$ for a basis $\{v_1, \ldots, v_n\}$ of V. The tensor algebra is indeed an algebra, with tensor multiplication acting as the algebra multiplication. In fact, because by this multiplication we have that the subspace $V^{\otimes r} V^{\otimes s}$, spanned by the products xy, $x \in V^{\otimes r}$, $y \in V^{\otimes s}$, is contained in $V^{\otimes (r+s)}$, this algebra is a *graded* algebra.

Consider the two-sided ideal of the tensor algebra, which we shall denote by I, generated by differences of products $v \otimes w - w \otimes v$ for $v, w \in V$. The quotient algebra $V^{\otimes}/I$ is called the *symmetric algebra* and is usually denoted $S(V)$. I is in fact a graded ideal of $V^{\otimes}$, and because of this, the quotient algebra $S(V)$ is again a graded algebra. The r-th graded piece of the symmetric algebra, denoted S^rV, is called the r-th *symmetric power* of the vector space V. Consider the standard basis $\{e_1, \ldots, e_n\}$ of V, then the products $e_{i_1} \ldots e_{i_r}$ with $i_1 \leq \ldots \leq i_r$ in S^rV, which can also be written as $e_1^{t_1} \ldots e_n^{t_n}$ with $\sum_i t_i = r$, form a basis of the r-th symmetric power, where if $\pi : T(V) \to S(V)$ is the canonical map, then the product of two elements of $S(V)$ is defined as $\pi(v)\pi(w) = \pi(v \otimes w)$ $(v, w \in T(V))$.

Now consider the ideal J of $V^{\otimes}$ generated by tensors of the form $v \otimes v$ for $v \in V$. Again, this is a graded ideal, and the graded quotient algebra $V^{\otimes}/J$ is called the *exterior algebra*, denoted $\wedge(V)$. Similarly to the symmetric power, the r-th *exterior power* of V for some r is simply the r-th graded piece of $\wedge(V)$ and is denoted $\wedge^r V$; a basis for this space is made of the *wedge products* $e_{i_1} \wedge \ldots \wedge e_{i_r}$ of the basis elements of V, with $i_1 < \cdots < i_r$. It should be noted here that clearly if $r > n$ then we will have $\wedge^r V = \{0\}$.

Weights of $T(V)$ as a module

We can consider $T(V)$ and subspaces and quotients of it as $\mathfrak{sl}_n$-modules themselves, defining the action by applying to the action on V a "derivation" rule

$$X.(v \otimes w) = (X.v) \otimes w + v \otimes (X.w),$$

for X in $\mathfrak{sl}_n$. This means that we can define weight spaces, weights and weight vectors for the tensor algebra and quotients of it. Let us now state a couple of lemmas that will help to determine the weights of the symmetric and exterior powers of V, and the dimensions of the weight spaces, in the following subsections.

Lemma 1.11. *Let v and w in $T(V)$ have weights λ and μ, respectively. Then the weight of $v \otimes w$ is $\lambda + \mu$. This property is also valid for $S(V)$ and $\wedge(V)$.*

Proof. Consider the action of elements of the Lie algebra on tensors in the module as defined above. Then if λ and μ are the of v and w respectively, we have

$$X.(v \otimes w) = (\lambda(X)v) \otimes w + v \otimes (\mu(X)w)$$

and then since $\lambda(X)$ and $\mu(X)$ are scalars, by the bilinearity of the tensor product this becomes

$$X.(v \otimes w) = \lambda(X)(v \otimes w) + \mu(X)(v \otimes w)$$

and finally, distributivity of scalar multiplication and the fact that λ and μ are homomorphisms give us

$$X.(v \otimes w) = (\lambda + \mu)(X)(v \otimes w)$$

as required. This also holds in the symmetric and exterior algebras because the canonical maps $\pi : T(V) \to S(V)$, $\rho : T(V) \to \wedge(V)$ are $\mathfrak{sl}_n$-module homomorphisms. $\square$

By the above lemma, we have that the weight of $e_1^{t_1} \dots e_n^{t_n}$ in $S^r V$ is $t_1\varepsilon_1 + \cdots + t_n\varepsilon_n$, where ε_i are the weights of the standard basis elements e_i of V.

Lemma 1.12. *Let $\mathfrak{h}$ be the maximal toral subalgebra of $\mathfrak{sl}_n$, that is, the space of diagonal $n \times n$ zero-trace matrices. Then the weights $\varepsilon_i : \mathfrak{h} \to k$, $i = 1, \dots, n$ on $\mathfrak{h}$ are given by*

$$\varepsilon_i : \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \mapsto d_i.$$

Proof. Defining a map ε_i as above, we have for any $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$ in $\mathfrak{h}$ that

$$D.e_i = De_i = d_i e_i = \varepsilon_i(D) e_i.$$

□

1.3.2 The symmetric powers

Weight spaces

By the results of the previous subsubsection, the weights of $\mathfrak{h}$ on $S^r V$ are $t_1\varepsilon_1 + \cdots + t_n\varepsilon_n$, with $t_1 + \cdots + t_n = r$. But what are the weight spaces? By definition, the weight space of a given weight $t_1\varepsilon_1 + \cdots + t_n\varepsilon_n$ is the span of its weight vectors (i.e. vectors of that weight) in the basis of $S^r V$.

Lemma 1.13. *Each weight of $\mathfrak{h}$ has a unique expression of the form $t_1\varepsilon_1 + \cdots + t_n\varepsilon_n$ with $t_i \in \mathbb{Z}_{\geq 0}$ and $t_1 + \cdots + t_n = r$.*

Proof. Let $(t_1, \ldots, t_n)$ and $(t'_1, \ldots, t'_n)$ be two n-tuples of positive integers such that $\sum\limits_i t_i = \sum\limits_i t'_i = r$ and $\sum\limits_i t_i\varepsilon_i = \sum\limits_i t'_i\varepsilon_i$. Then

$$0 = \sum_i t_i\varepsilon_i - \sum_i t'_i\varepsilon_i = \sum_i \left(t_i - t'_i\right)\varepsilon_i.$$

$\sum \varepsilon_i = 0$ is the defining relation for the ε_i, so any other linear relation is a scalar multiple of this, which means that all the coefficients $t_i - t'_i$ in the sum above are equal; but then

$$0 = r - r = \sum_i t_i - \sum_i t'_i = \sum_i \left(t_i - t'_i\right) = n\left(t_1 - t'_1\right),$$

implying $t_1 - t'_1 = 0$ and thus $t_i - t'_i = 0$ for all i. Hence, if two elements in the basis of $S^r V$ have the same weight, they must be equal. □

By this lemma, we have that each weight space S^rV_λ of S^rV is one-dimensional, i.e. the weight space for a weight $\lambda = t_1\varepsilon_1 + \cdots + t_n\varepsilon_n$ is the span of the single vector $e_1^{t_1} \ldots e_n^{t_n}$.

Irreducibility

Theorem 1.14. *The whole space S^rV is irreducible as an $\mathfrak{sl}_n$-module in the sense that it has no nonzero proper submodules.*

Proof. Let U be a nonzero submodule of S^rV. Then U contains at least one of the weight spaces in S^rV, including the basis element $e_1^{t_1} \ldots e_n^{t_n}$ corresponding to that weight. By assumption, U is a submodule and so must be closed under multiplication by elements of the underlying Lie algebra, but if we multiply $e_1^{t_1} \ldots e_i^{t_i} \ldots e_j^{t_j} \ldots e_n^{t_n}$ on the left by the matrix E_{ij} with a single 1 as the j-th entry of the i-th row and 0 elsewhere, then it can be checked using the derivation rule given above and the fact that the multiplication in the quotient algebra $S(V)$ is commutative, that we get $t_i e_1^{t_1} \ldots e_i^{t_i+1} \ldots e_j^{t_j-1} \ldots e_n^{t_n}$; but as long as $i \neq j$, the matrix E_{ij} will be in $\mathfrak{sl}_n$, which means that $E_{ij}.e_1^{t_1} \ldots e_n^{t_n}$ must also be in U. Of course, this is a scalar multiple of a basis element of S^rV, and in this way we can manipulate the powers of the e_i to show that any combination thereof must be in U; in other words, $U = S^rV$. Hence, S^rV is irreducible. □

Highest weight and highest weight vectors

We wish to find the highest weight of the irreducible module S^rV, and its highest weight vectors. First, we must choose which of the roots $\varepsilon_i - \varepsilon_j$ $(i \neq j)$ of $\mathfrak{sl}_n$ to define as positive: let us call the positive roots all those where $i < j$.

Theorem 1.15. *The highest weight of S^rV is $r\varepsilon_1$.*

Proof. Consider a weight $\mu = t_1\varepsilon_1 + \cdots + t_n\varepsilon_n$ in S^rV. Then the difference between this and $r\varepsilon_1$ is

$$r\varepsilon_1 - \mu = (r - t_1)\,\varepsilon_1 - t_2\varepsilon_2 - \cdots - t_n\varepsilon_n = t_2\,(\varepsilon_1 - \varepsilon_2) + \cdots + t_n\,(\varepsilon_1 - \varepsilon_n)$$

(because $t_1 + \cdots + t_n = r$, so $r - t_1 = t_2 + \cdots + t_n$). Each root $\varepsilon_1 - \varepsilon_j$ is positive since $j > 1$ in every case here, and the t_j are all positive integers, so $r\varepsilon_1 - \mu$ is a sum of positive roots for any weight μ, thus $r\varepsilon_1$ must be the highest weight of S^rV. □

So, a highest weight vector in S^rV is any scalar multiple of e_1^r.

Remark 1.16. In general, to check that something is a highest weight vector it only needs to be checked that it is a weight vector that is fixed by the upper unitriangular matrices under the action of SL_n or that it is killed by the strictly upper triangular matrices under the action of $\mathfrak{sl}_n$. What we checked in the above proof is a stronger property (although it is equivalent for irreducible modules): we showed that not only is the weight $r\varepsilon_1$ maximal in the set of weights (which would be enough to make it a highest weight vector), but that it is in fact greater than every other weight.

1.3.3 The exterior powers

Now let us consider the r-th exterior power of V. Basis elements here are $e_{i_1} \wedge \ldots \wedge e_{i_r}$ (with $i_1 < \cdots < i_r$, and this is well-defined because we are assuming that $\wedge^r V$ is nonzero and therefore that $r \leq n$). Each of these has weight $\varepsilon_{i_1} + \cdots + \varepsilon_{i_r}$, and these are the weights of $\wedge^r V$. Now to find the weight spaces: again by Lemma 1.13, all these weights (the weights of all the basis elements) are distinct so the weight spaces are all one-dimensional.

Theorem 1.17. *The r-th exterior power $\wedge^r V$ of the natural module V is an irreducible $\mathfrak{sl}_n$-module.*

Proof. As in Theorem 1.14, let U be a nonzero submodule in $\wedge^r V$. Then as before, U must contain at least one basis element, and in exactly the same way, we can exchange e_j for e_i any number of times (for any j and i) to obtain any combination of the e_is, so that the entire basis of $\wedge^r V$ is contained in U. Hence $\wedge^r V$ is irreducible. □

If we choose the same positive roots as we did for the symmetric power, then clearly the highest weight of the exterior power will be $\varepsilon_1 + \cdots + \varepsilon_r$. Vectors in $\wedge^r V$ with this weight are the scalar multiples of $e_1 \wedge \ldots \wedge e_r$.

1.4 Decomposition of $V \otimes V^*$ and $V \otimes V$ (as $\mathfrak{sl}_n$-modules) as direct sums of irreducibles

1.4.1 $\mathfrak{gl}_n \cong V \otimes V^*$ as an $\mathfrak{sl}_n$-module

Since $\mathfrak{gl}_n$ is a $\mathfrak{gl}_n$-module under the adjoint action, it is also an $\mathfrak{sl}_n$-module (by restriction). This module is isomorphic to $V \otimes V^*$, the tensor product of the natural module with its dual vector space. The isomorphism is given by $v \otimes f \mapsto (x \mapsto f(x)\,v)$ (Here f acts on x so x must be in V and $f(x) \in k$; then $f(x)\,v \in V$ so $(x \mapsto f(x)\,v) \in \mathrm{End}V \cong \mathfrak{gl}_n$). In order to visualise the inverse of this isomorphism, it is necessary to define bases for the two spaces. For example, choosing the standard bases $\{e_1, \ldots, e_n\}$, $\{e_1^*, \ldots, e_n^*\}$ and $\{E_{11}, \ldots, E_{nn}\}$, one can easily see that $E_{ij} \mapsto e_i \otimes e_j^*$ does the job.

We have that the weights of $\mathfrak{gl}_n$ as an $\mathfrak{sl}_n$-module are all the roots $\varepsilon_i - \varepsilon_j$, $i \neq j$, along with the zero weight. The zero weight space is the maximal toral subalgebra $\mathfrak{h}$, the algebra of diagonal matrices with trace zero. $\mathfrak{sl}_n \cong \mathfrak{h} \oplus \sum_{i \neq j} (\mathfrak{gl}_n)_{\varepsilon_i - \varepsilon_j}$ is a Lie subalgebra of $\mathfrak{gl}_n$ and an $\mathfrak{sl}_n$-submodule of $\mathfrak{gl}_n$ under the adjoint action (indeed, $[X, Y] \in \mathfrak{sl}_n$ for all $X, Y \in \mathfrak{gl}_n$). $\mathfrak{sl}_n$ is a simple Lie algebra and hence irreducible as a module over itself under the adjoint action (in fact we can think of this as a definition of a simple Lie algebra).

Now, $\mathfrak{sl}_n$ has dimension $n^2 - 1$ over k and $\mathfrak{gl}_n$ has dimension n^2, so the direct complement of $\mathfrak{sl}_n$ in $\mathfrak{gl}_n$ must be a one-dimensional submodule (and obviously this will also be irreducible, so then we will be done with our decomposition). There is only one (up to module isomorphism) one-dimensional representation of the Lie algebra $\mathfrak{sl}_n$, which is the representation corresponding to the module consisting of scalar multiples of the identity matrix $I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ in $\mathfrak{gl}_n$. Hence, we have the direct sum decomposition

$$V \otimes V^* \cong \mathfrak{gl}_n \cong \mathfrak{sl}_n \oplus \langle I_n \rangle \,,$$

where $\mathfrak{sl}_n$ and $\langle I_n \rangle$ are both irreducible submodules of $V \otimes V^*$.

1.4.2 $V \otimes V$

The weights of the natural module V are $\varepsilon_1, \ldots, \varepsilon_n$, where as above ε_i is the weight of the standard basis element e_i of V. This means that weights of $V \otimes V$ are $\varepsilon_i + \varepsilon_j$ (see Lemma 1.11). Clearly, the basis elements $e_i \otimes e_j$ and $e_j \otimes e_i$ of $V \otimes V$ both have weight $\varepsilon_i + \varepsilon_j = \varepsilon_j + \varepsilon_i$. By Lemma 1.13, we see that no other basis element in $V \otimes V$ (i.e. none involving an e_t in the tensor product where $i \neq t \neq j$)

will have the same weight $\varepsilon_i + \varepsilon_j$. Thus, each weight space $(V \otimes V)_{\varepsilon_i+\varepsilon_j}$, $i \neq j$ is two-dimensional, while the weight spaces $(V \otimes V)_{\varepsilon_i+\varepsilon_i}$ must have dimension one.

We can consider a symmetry group $\{1, \sigma\}$ of order two acting on the tensor space $V \otimes V$ and preserving the weight spaces, with id the identity and σ sending any $e_i \otimes e_j$ to $e_j \otimes e_i$. Now, let us define two representations 1 and sgn, by $1(\mathrm{id}) = 1(\sigma) = \mathrm{sgn}(\mathrm{id}) = 1 \in k$ and $\mathrm{sgn}(\sigma) = -1 \in k$. Then the stabilisers of these representations,

$$\{v \in V | s(v) = 1(s)\, v, \forall s \in \{\mathrm{id}, \sigma\}\} = \langle e_i \otimes e_i, e_i \otimes e_j + e_j \otimes e_i \rangle$$

and

$$\{v \in V | s(v) = \mathrm{sgn}(s)\, v, \forall s \in \{\mathrm{id}, \sigma\}\} = \langle e_i \otimes e_j - e_j \otimes e_i | i \neq j \rangle$$

are isomorphic to the symmetric square S^2V and exterior square $\wedge^2 V$ of V respectively (both of which have been shown in Section 3 to be irreducible) and thus we have

$$V \otimes V \cong S^2V \oplus \wedge^2 V.$$

1.5 Representations of the symmetric group

In this section and the next, unless stated otherwise, V denotes an arbitrary vector space, although still over the (characteristic-zero) field k.

1.5.1 Young symmetrisers

Definitions 1.18. [11, 4.1],[22, I] To any (descending) partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of a natural number r, we can uniquely define a subset $\{(i, j) | 1 \leq i \leq l(\lambda), 1 \leq j \leq \lambda_i\}$

of $\mathbb{N} \times \mathbb{N}$, called the *Young diagram* corresponding to λ. We will identify each partition λ with its corresponding Young diagram. The $(i, j) \in \lambda$ are called the *boxes* of λ. A *(Young) tableau* of shape λ (λ a partition of $r \in \mathbb{N}$) is a mapping $T : \lambda \to \mathbb{N}$. A tableau is called an *r-tableau* if its entries are the numbers $1, \dots, r$ (so the entries must be distinct).

The following construction appears in [11, 4.1]. For a given partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of r, pick a Young tableau T over the Young diagram $Y(\lambda)$, and then we can define the following two subgroups of $\mathfrak{S}_r$:

$$R_\lambda := \{\sigma \in \mathfrak{S}_r | \, \sigma \text{ preserves each row of } T\},$$

$$C_\lambda := \{\sigma \in \mathfrak{S}_r | \, \sigma \text{ preserves each column of } T\},$$

called the *row stabiliser* and *column stabiliser*, respectively, of the Young tableau T, or equivalently of the Young diagram λ. Now in the group algebra $k[\mathfrak{S}_r] = k\mathfrak{S}_r$, define $a_\lambda := \sum_{\sigma \in R_\lambda} \sigma$, $b_\lambda := \sum_{\sigma \in C_\lambda} \operatorname{sgn}(\sigma)\, \sigma$, called the *row symmetriser* and *column anti-symmetriser* of λ respectively. Now if we consider $\mathfrak{S}_r$ to be acting on the tensor space $V^{\otimes r}$ (where here V may be any vector space) simply by permuting vectors, then the images of these in that action are $\operatorname{Im}(a_\lambda) = S^{\lambda_1} V \otimes \dots \otimes S^{\lambda_k} V$ and $\operatorname{Im}(b_\lambda) = \wedge^{\mu_1} V \otimes \dots \otimes \wedge^{\mu_l} V$, where $\mu = (\mu_1, \dots, \mu_l)$ is the partition *conjugate* to λ (in other words, μ_j is the number of boxes in the j-th column, rather than row, of λ. Notice $\mu_1 = k$ and $l = \lambda_1$).

Now let $c_\lambda := a_\lambda . b_\lambda \in k[\mathfrak{S}_r]$. These c_λ are called *Young symmetrisers*, and each one is idempotent up to a scalar multiple, by which we mean that some scalar multiple of c_λ is idempotent, or equivalently that $c_\lambda^2 = m_\lambda c_\lambda$ for some scalar m_λ. When $\lambda = (r)$, we have $c_\lambda = a_\lambda$ and so the image of c_λ acting on $V^{\otimes r}$ is $S^r V$, and when $\lambda = (1, \dots, 1)$ we have $c_\lambda = b_\lambda$ and $\operatorname{Im}(c_\lambda) = \wedge^r V$.

Theorem 1.19. *Let r be a non-negative integer, then for any partition λ of r, the ideal $k[\mathfrak{S}_r]\, c_\lambda$ (c_λ a Young symmetriser) is an irreducible representation of $\mathfrak{S}_r$; these irreducible representations are mutually non-isomorphic, and every irreducible representation of $\mathfrak{S}_r$ is obtained in this way.*

So, the partitions (and by transitivity, the Young diagrams) of r are in one-to-one correspondence with irreducible representations of the symmetric group $\mathfrak{S}_r$. The irreducible representation of $\mathfrak{S}_r$ corresponding to the partition λ is denoted π_λ.

Definition 1.20. [11, 4.1] The *hook length* of a box x in a Young diagram $Y(\lambda)$ is $\text{hook}(x) := a_\lambda(x) + l_\lambda(x) + 1$, where $a_\lambda(x)$ is the number of boxes to the right, and $l_\lambda(x)$ the number of boxes below, box x (called the *arm length* and *leg length* of x, respectively).

Lemma 1.21. *The dimension of the irreducible representation π_λ of $\mathfrak{S}_r$, (λ a partition of r) is given by the formula*

$$\dim(\pi_\lambda) = \frac{r!}{\prod\limits_{x \in Y(\lambda)} \text{hook}(x)},$$

where $Y(\lambda)$ is the Young diagram for λ.

Example 1.22. The representation $\pi_{(3,2,2)}$ of $\mathfrak{S}_7$ has dimension $\dfrac{5040}{5.4.1.3.2.2.1} = 21$.

1.5.2 Primitive idempotents

Definition 1.23. Let e be idempotent in some ring A. Then e is called *primitive* if there does not exist any pair $f_1, f_2 \in A$ of nonzero idempotents that are orthogonal (that is, $f_1 f_2 = f_2 f_1 = 0$) and that satisfy $e = f_1 + f_2$.

Theorem 1.24. *Let A be a ring, and $e \in A$ be idempotent. Then the A-module Ae is indecomposable if and only if e is primitive. Moreover, if A is a semisimple algebra, then Ae is irreducible if and only if e is primitive.*

Proof. First, let us prove by contrapositive that if Ae is indecomposable then e is primitive. So, assume e is not primitive, that is we have $e = f_1 + f_2$ where f_1 and f_2 are both nonzero idempotents in A and $f_1 f_2 = f_2 f_1 = 0$. Then clearly we have that $Ae = \{ae|a \in A\} = \{af_1 + af_2|a \in A\}$ is a subset of $Af_1 + Af_2 = \{af_1 + bf_2|a, b \in A\}$. Now consider $f_i e = f_i^2 = f_i$ (for both f_i); then for all $af_i \in Af_i$ we have $af_i = af_i e \in Ae$, implying $Af_i \subseteq Ae$ and hence $Ae = Af_1 + Af_2$. Furthermore, this sum is direct since for $x \in Af_1 \cap Af_2$ we have $x = af_1 = bf_2$ for some $a, b \in A$, so $xf_1 = af_1^2 = af_1 = x$ and similarly $xf_2 = x$, but then $x = xf_2 = xf_1 f_2 = 0$, so $Af_1 \cap Af_2 = \{0\}$. Thus we have that Ae is decomposable as $Ae = Af_1 \oplus Af_2$.

Now for the converse, assume Ae is decomposable, that is $Ae = F_1 \oplus F_2$ with $F_1, F_2 \subset Ae$. So $e = f_1 + f_2$ for some $f_1 \in F_1$, $f_2 \in F_2$. Then since for all $x = ae \in Ae$ we have $xe = ae^2 = ae = x$, and since F_1 and F_2 are subsets of Ae, f_1 and f_2 are both in Ae, so

$$f_1 = f_1 e = f_1 (f_1 + f_2) = f_1^2 + f_1 f_2$$

which can be rearranged to get

$$f_1 - f_1^2 = f_1 f_2$$

of which the left-hand side is clearly in F_1 and the right-hand side is clearly in F_2, but it was assumed that $F_1 \cap F_2 = \{0\}$, therefore we must have both $f_1 = f_1^2$ and $f_1 f_2 = 0$. We can do the same for f_2. Now $e = f_1 + f_2$ with f_1 and f_2 orthogonal idempotents in A, thus e is not a primitive idempotent.

The second part of the theorem follows from the first part together with the fact that for a semisimple algebra A, any A-module is irreducible if and only if it is indecomposable as a module over the ring A. □

Lemma 1.25. *Let e be an idempotent in a ring R, and let M be a module over R. Then the space $\mathrm{Hom}_R(Re, M)$ and the submodule eM are isomorphic as abelian groups (if R is an algebra then they are isomorphic as vector spaces): the isomorphism is given by $f \mapsto f(e) = f(e^2) = ef(e)$.*

1.6 Schur-Weyl duality

1.6.1 Double commutant theorem

Lemma 1.26 (Schur's lemma)**.** *[14, Thm. 4.29.1] Let V and W be irreducible representations of a Lie algebra $\mathfrak{g}$, and let $\varphi : V \to W$ be an equivariant map (that is, $g.\varphi(v) = \varphi(g.v)$ for $g \in \mathfrak{g}$ and $v \in V$). Then either $\varphi = 0$ or φ is an isomorphism.*

The following well-known result follows from the above part of Schur's Lemma, and standard properties of the Hom functor; the double commutant theorem, also known as the double centraliser theorem, will follow in turn from this.

Theorem 1.27. *Let A be an algebra over an algebraically closed field k of characteristic 0, and let V be an A-module with decomposition $V \cong \bigoplus_i n_i V_i$, where the V_i are mutually non-isomorphic irreducible A-modules, with $n_i \in \mathbb{Z}_{\geq 0}$. Then the dimension of $\mathrm{Hom}_A(V_j, V)$ is n_j.*

Proof. We have $\mathrm{Hom}_A(V_j, V) = \bigoplus_i n_i \mathrm{Hom}_A(V_j, V_i)$, then by Lemma 1.26 any module homomorphism in $\mathrm{Hom}_A(V_j, V_i)$ is either an isomorphism or zero, so since the V_i are mutually non-isomorphic we have $\mathrm{Hom}_A(V_j, V_i) = \{0\}$ for $i \neq j$. Now, consider the following: for isomorphic A-modules M and N, $\mathrm{Hom}_A(M, N)$ is isomorphic (at least as a vector space) to $\mathrm{End}_A(M)$ (with the isomorphism defined in the following way; a homomorphism θ from M to N maps to $\theta \circ \varphi^{-1}$, where φ is an isomorphism from M to N; the inverse takes an endomorphism η of M to $\eta \circ \varphi$). So, $\dim \mathrm{Hom}_A(V_j, V_j) = \dim \mathrm{End}_A(V_j)$. But by Lemma 1.26 again, everything nonzero in $\mathrm{End}_A(V_j)$ is an isomorphism. Let $f \in \mathrm{End}_A(V_j)$ then f has an eigenvalue μ; the kernel of $f - \mu.\mathrm{id}$ is nonzero but then we must have $f = \mu.\mathrm{id}$, that is, $\mathrm{End}_A(V_j) = k\mathrm{id} \cong k$. Hence, now that we know that $\mathrm{Hom}_A(V_j, V_j)$ is one-dimensional, we have

$$\dim \mathrm{Hom}_A(V_j, V) = 0 + \cdots + 0 + n_j + 0 + \cdots + 0 = n_j.$$

□

Theorem 1.28 (Double commutant theorem)**.** *Now let V be a vector space, and let A and B be subalgebras of* $\mathrm{End}V$*, with A semisimple and $B = C_{\mathrm{End}V} A = \{b \in \mathrm{End}V | ab = ba \ \forall a \in A\}$. Then B is also semisimple, and we have $A = C_{\mathrm{End}V} B$.*

Definition 1.29. With a module V decomposed as a sum of irreducible submodules, some of those submodules will be isomorphic to each other. The sum of all the irreducible submodules in one equivalence class is called an *isotypic component* of that decomposition.

For two subalgebras A and B of $\mathrm{End}V$, where we denote $f.v = f(v) \ \forall f \in A \cup B, v \in V$ and where the two actions always commute, the *joint action* of A and B is the action defining a module structure over $A \otimes B$, given by $(f \otimes g).v := f(g(v))$. (This also equals $g(f(v))$ because of the commuting actions).

Theorem 1.30. *Let A and B be commuting subalgebras of* $\mathrm{End}V$*. Then the following are equivalent;*

1. *A and B are semisimple and each other's centraliser;*

2. *We have* $V \cong \bigoplus_{i=1}^{r} U_i \otimes W_i$ *as* $A \otimes B$*-modules, where the* U_i *are mutually non-isomorphic irreducible* A*-modules and the* W_i *are mutually non-isomorphic irreducible* B*-modules;*

3. V *is semisimple for* A *as well as for* B*, and the* A*-isotypic components of* V *are the same as the* B*-isotypic components.*

Remarks 1.31. The second statement can be restated in a more canonical way: let $U_1, \ldots, U_r$ be the irreducible A-submodules of V up to isomorphism. Then the $\mathrm{Hom}_A(U_i, V)$ are mutually non-isomorphic B-modules and the canonical map $\bigoplus_{i=1}^{r} U_i \otimes \mathrm{Hom}_A(U_i, V) \to V$ is an isomorphism of $A \otimes B$-modules.
Note that each isotypic component in the decomposition of V as an A-module is $\underbrace{U_i \oplus \ldots \oplus U_i}_{d}$ for some $i \in \{1, \ldots, r\}$, where the multiplicity d is the dimension of $\mathrm{Hom}_A(U_i, V)$ by Theorem 1.27.

1.6.2 Representations of GL_n and $\mathfrak{S}_r$

Let $V = k^n$ once again and consider the tensor space $V^{\otimes r}$. We have already seen that $\mathfrak{S}_r$ acts on this space (on the left) by permuting the factors:

$$\sigma(v_1 \otimes \ldots \otimes v_r) = v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(r)}.$$

We can also easily define a left-action of the group GL_n on the space (by simultaneous matrix multiplication):

$$g(v_1 \otimes \ldots \otimes v_r) = gv_1 \otimes \ldots \otimes gv_r.$$

Note that these two actions clearly commute with each other. Now let $R \subseteq \mathrm{End}(V^{\otimes r})$ be the associative algebra generated by all $\rho(g)$, $g \in \mathrm{GL}_n$, and $S \subseteq \mathrm{End}(V^{\otimes r})$ be the associative algebra generated by all $\pi(\sigma)$, $\sigma \in \mathfrak{S}_r$, where π

and ρ are the group representations of $\mathfrak{S}_r$ and GL_n (respectively) defined above. Algebras such as R and S are called *enveloping algebras* of representations.

Theorem 1.32 (Schur-Weyl duality)**.** *With R and S defined as above, we have*

$$R = \left\{a \in \mathrm{End}\left(V^{\otimes r}\right) | as = sa \ \forall s \in S\right\}$$

and

$$S = \left\{a \in \mathrm{End}\left(V^{\otimes r}\right) | aq = qa \ \forall q \in R\right\}.$$

Fulton and Harris's book [11] can be consulted for a proof of this, and their proof includes in it a proof of the double commutant theorem. See also [27, chapter IV].

So, we have a space $V^{\otimes r}$ and we have two semisimple subalgebras (R and S) of $\mathrm{End}\left(V^{\otimes r}\right)$ whose actions on $V^{\otimes r}$ commute with each other; in other words, we have a module to which the double commutant theorem applies. By Theorem 1.30, this means we have a decomposition

$$V^{\otimes r} \cong \bigoplus_{\lambda} U_\lambda \otimes \mathrm{Hom}_S\left(U_\lambda, V^{\otimes r}\right),$$

where the U_λ are mutually non-isomorphic irreducible S-modules and the $\mathrm{Hom}_S\left(U_\lambda, V^{\otimes r}\right)$ are mutually non-isomorphic irreducible R-modules. By Theorem 1.19, all irreducible $\mathfrak{S}_r$ modules are of the form $k\left[\mathfrak{S}_r\right] c_\lambda$ where c_λ is the Young symmetriser for a partition λ of r. So, we should have that the irreducible GL_n-modules in $V^{\otimes r}$ are of the form $\mathrm{Hom}_S\left(k\left[\mathfrak{S}_r\right] c_\lambda, V^{\otimes r}\right)$. By Lemma 1.25, we have a module isomorphism from this space to $c_\lambda V^{\otimes r}$, given by $f \mapsto f\left(c_\lambda\right)$ (note $f\left(c_\lambda\right) = f\left(c_\lambda^2\right) = c_\lambda f\left(c_\lambda\right)$) for any $f \in \mathrm{Hom}_S\left(k\left[\mathfrak{S}_r\right] c_\lambda, V^{\otimes r}\right)$. Thus, the spaces $c_\lambda V^{\otimes r}$, c_λ Young symmetrisers, are irreducible modules for GL_n. Only the partitions $\lambda = \left(\lambda_1, \ldots, \lambda_k\right)$ with $k \leq n$ (Young diagrams with no more than n rows) show up here.

Similarly, we know (Theorem 1.4) that the irreducible modules $V(\mu)$ of GL_n are characterised by their highest weights, and by Lemma 1.5 we have $\mathrm{Hom}_{\mathrm{GL}_n}(V(\mu), V) \cong V_\mu^{U+}$, thus the spaces of highest weight vectors $(V^{\otimes r})_\mu^{U+}$ are irreducible modules of $\mathfrak{S}_r$, for all partitions μ with length less than or equal to r.

1.6.3 Other instances of Schur-Weyl duality

The Schur-Weyl duality stated above, between the symmetric and general linear groups, is what is usually called *classical* Schur-Weyl duality. In general, Schur-Weyl duality states for two particular groups or algebras with commuting actions on a vector space that their enveloping algebras are each other's centraliser. Other pairs of objects where this occurs include;

- GL_n and $B_{r,s}(n)$ both acting on $V^{\otimes r} \otimes V^{*\otimes s}$, $V = k^n$, where $B_{r,s}(n)$ is the walled Brauer algebra with parameters r, s and n;
- $Sp_n, B_r(-n) \circlearrowright V^{\otimes r}$, where Sp_n is the symplectic group and $B_r(-n)$ is the Brauer algebra with parameters $r, -n$;
- $O_n, B_r(n) \circlearrowright V^{\otimes r}$, where O_n is the orthogonal group and $B_r(n)$ is the Brauer algebra with parameters r, n.

Of course, these are even more instances of the Schur-Weyl duality than these ([13] discusses a few). There is also the related *Howe duality*, where both of the groups are continuous, rather than one being discrete (the symmetric group or Brauer algebra in the above examples) as with the Schur-Weyl duality.

Chapter 2

Highest weight vectors in arbitrary characteristic

2.1 Preliminaries

Let k be an algebraically closed field of arbitrary characteristic, let GL_n be the group of invertible $n \times n$ matrices with entries in k and let T_n and U_n be the subgroups of diagonal matrices and of upper uni-triangular matrices respectively. The group $\mathrm{GL}_r \times \mathrm{GL}_s$ acts on the k-vector space Mat_{rs}^m of m-tuples of $r \times s$ matrices with entries in k via $((A,B)\cdot \underline{X})_i = AX_iB^T$, where $\underline{X} = (X_1,\ldots,X_m) \in \mathrm{Mat}_{rs}^m$ and B^T is the transpose of B, and on the coordinate ring $k[\mathrm{Mat}_{rs}^m]$ via $((A,B)\cdot f)(\underline{X}) = f((A^T,B^T)\cdot \underline{X}) = f((A^TX_iB)_{1\le i\le m})$. For (μ,λ) a character of $T_r \times T_s$, the space of highest weight vectors will be denoted $k[\mathrm{Mat}_{rs}^m]^{U_r\times U_s}_{(\mu,\lambda)}$. It consists of the functions $f \in k[\mathrm{Mat}_{rs}^m]$ with $(A,B)\cdot f = f$ for all $(A,B) \in U_r \times U_s$ and $(A,B)\cdot f = \mu(A)\lambda(B)f$ for all $(A,B) \in T_r \times T_s$.

Note that $k[\mathrm{Mat}_{rs}^m]$ is the polynomial algebra over k in the variables $x(l)_{ij}$,

$1 \leq l \leq m$, $1 \leq i \leq r$, $1 \leq j \leq s$, where $x(l)_{ij}$ is the entry in the i-th row and j-column of the l-th matrix. If $m = 1$ we write x_{ij} instead of $x(1)_{ij}$. The $\mathrm{GL}_r \times \mathrm{GL}_s$-module $k[\mathrm{Mat}_{rs}^m]$ is multigraded by tuples of integers ≥ 0 (not necessarily partitions) of length m. We denote the set of such tuples with coordinate sum t by Σ_t. So the elements in the piece of multidegree $\nu \in \Sigma_t$ have total degree t.

2.1.1 Skew Young diagrams and tableaux

In this subsection we introduce some more combinatorics related to Young diagrams and tableaux, that we will need in Section 2.2 and which originates from [18], [28, 29], and [8]. In Subsection 2.1.2 we discuss interpretations in terms of representation theory.

For λ a partition of n we denote the length of λ by $l(\lambda)$ and its coordinate sum by $|\lambda|$. As in Chapter 1, we will identify each partition λ with the corresponding Young diagram $\{(i, j) \,|\, 1 \leq i \leq l(\lambda), 1 \leq j \leq \lambda_i\}$. Recall that the $(i, j) \in \lambda$ are called the *boxes* of λ.

More generally, if λ, μ are partitions with $\lambda \supseteq \mu$, then we denote the diagram λ with the boxes of μ removed (or the set theoretic difference of λ and μ considered as subsets of $\mathbb{N} \times \mathbb{N}$) by λ/μ, and call it the *skew Young diagram* associated to the pair (λ, μ). Of course the skew diagram λ/μ does not determine λ and μ. For a (skew) diagram E, we will denote the transpose by E' and the number of boxes by $|E|$. The group of permutations of the boxes of E will be denoted by $\mathfrak{S}_E$, and the column stabiliser of E in $\mathfrak{S}(E)$, that is, the product of the groups of permutations of each column of E, will be denoted by C_E. By *diagram mapping* we mean a bijection between two diagrams as subsets of $\mathbb{N} \times \mathbb{N}$.

Definitions 2.1. Let E be a skew diagram with t boxes. A *skew tableau* of shape E is a mapping $T : E \to \mathbb{N} = \{1, 2, \ldots\}$. A skew tableau of shape E is called *ordered* if its entries are weakly increasing along rows and weakly increasing down columns, *(column) semi-standard* if its entries are weakly increasing along rows and strictly increasing down columns, or *row-semistandard* if its entries are strictly increasing along rows and weakly increasing down columns. As with ordinary Young tableau (see Chapter 1), a tableau with t boxes is called a *t-tableau* if its entries are the numbers $1, \ldots, t$. A t-tableau whose entries are strictly increasing along both columns and rows is called *standard*. If m is the biggest integer occurring in a tableau T, then the *weight* of T is the m-tuple whose i-th component is the number of occurrences of i in T. Sometimes we will also consider the weight of T as an m'-tuple for some $m' \geq m$ by extending it with zeros.

Definition 2.2. Let P be an ordered tableau, of a shape E and weight $\nu = (\nu_1, \ldots, \nu_m) \in \Sigma_t$. Then we say a t-tableau T of shape E *belongs to* P if for all $i \in \{1, \ldots, m\}$ we have

$$T^{-1}\left(\left\{ \sum_{j=1}^{i-1} \nu_j + b \,\middle|\, b \in \{1, \ldots, \nu_i\} \right\}\right) = P^{-1}(i) \subseteq E.$$

Definition 2.3. For a skew shape E with t boxes, we define the *canonical* skew tableau S_E by filling the boxes in the i-th row with i's, and we define the tableau T_E by filling in the numbers $1, \ldots, t$ row by row from left to right and top to bottom. So S_E is semi-standard, and T_E is a t-tableau which is standard. The *standard enumeration* of a tableau T of shape E is the t-tuple obtained from T by reading its entries row by row from left to right and top to bottom.

Definition 2.4. A semi-standard tableau S of a shape F is called *(E-)special* for a skew diagram E if $S = S_E \circ \alpha$ for some diagram mapping $\alpha : F \to E$ such that for any $a, b \in F$, if $\alpha(b)$ occurs strictly below $\alpha(a)$ in the same column, then b occurs in a strictly lower row than a. We then say that α *represents* S. We call α *admissible* if for any $a, b \in F$, if $\alpha(b)$ occurs strictly below $\alpha(a)$ in the same column, then b occurs in a strictly lower row than a *and* in a column to the left of a or in the same column. A *special* (semi-standard) tableau also refers to any semi-standard tableau that is E-special for some E. By [26, Lem. 6], every special semi-standard tableau has a representative that is admissible. An admissible mapping α is called *special* if additionally, for any $a, b \in F$ with $\alpha(b)$ in a column strictly to the left of $\alpha(a)$, b occurs;

(1) in a column strictly to the right of a and in a row above a or in the same row, if $\alpha(b)$ is in the same row as $\alpha(a)$;

(2) in a column strictly to the right of a or in a strictly lower row, if $\alpha(b)$ is in a strictly lower row than $\alpha(a)$;

It follows from (1) and the admissibility condition that if $\alpha(b)$ is strictly above and to the left of $\alpha(a)$ then b should be strictly above and to the right of a. An alternative way to visualise the concept of special diagram mappings is the following, due to Zelevinsky [28, 29], in which they are called “pictures”. Define two orderings $\leq$ and $\preceq$ on $\mathbb{N} \times \mathbb{N}$ as follows: $(p, q) \leq (r, s)$ if and only if $p \leq r$ and $q \leq s$, and $(p, q) \preceq (r, s)$ if and only if $p < r$ or ($p = r$ and $q \geq s$). Note that $\preceq$ is a linear ordering. Recall that skew Young diagrams are by definition subsets of $\mathbb{N} \times \mathbb{N}$. A diagram mapping $\alpha : F \to E$ is called *special* if $\alpha : (F, \leq) \to (E, \preceq)$ and $\alpha^{-1} : (E, \leq) \to (F, \preceq)$ are order-preserving. It is clear from this second definition that a diagram mapping α is special if and only if α^{-1} is special.

Furthermore, we have that the tableau $S_E \circ \alpha$ is semi-standard (and therefore, a special semi-standard tableau) whenever α is special [28, p155-159], and that every special tableau has a unique special representative [26, Thm. 3].

Examples 2.5. Let $F = (2,2)$ and $E = (3,2)/(1)$ be skew diagrams. Since each has four boxes, we can construct a bijective map $\alpha_1 : F \to E$ between the two shapes. To demonstrate the examples, we first give F the standard enumeration (denoted T_F as a tableau), and then give a tableau S on E such that $T_F = S \circ \alpha_1$, that is, the tableau where each box $a \in E$ is labelled with the same number that its preimage $\alpha_1^{-1}(a) \in F$ is labelled with by T_F, which will be a t-tableau.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \xrightarrow{\alpha_1} \quad \begin{array}{c|c|c|}\cline{2-3} & 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$$

Then α_1 is not admissible, since 4 is below 1 in the same column of E, but it occurs in a column strictly to the right of 1 in F. We now form the canonical tableau S_E on E and pull this numbering back to F via α_1 to obtain the tableau $S = S_E \circ \alpha_1$:

$$S = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} \quad \xrightarrow{\alpha_1} \quad \begin{array}{c|c|c|}\cline{2-3} & 1 & 1 \\ \hline \multicolumn{1}{|c|}{2} & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} = S_E$$

Clearly S is semi-standard, and we also have that for all $a, b \in F$, b occurs in a strictly lower row than a whenever $\alpha_1(b)$ occurs strictly below $\alpha_1(a)$ in the same column. So S is a special semi-standard tableau; in particular it is E-special. Now define $\alpha_2, \alpha_3 : F \to E$ by

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \xrightarrow{\alpha_2} \quad \begin{array}{c|c|c|}\cline{2-3} & 1 & 2 \\ \hline \multicolumn{1}{|c|}{4} & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$$

and

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \xrightarrow{\alpha_3} \begin{array}{ccc} & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{1} \\ \cline{1-3} \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{3} & \\ \cline{1-2} \end{array}$$

Then $S_E \circ \alpha_2 = S_E \circ \alpha_3 = S$, α_2 is admissible, but not special, and α_3 is special. The inverse of α_3 is also special and is therefore the unique special representative of the F-special semi-standard tableau $T = S_F \circ \alpha_3^{-1}$ on E:

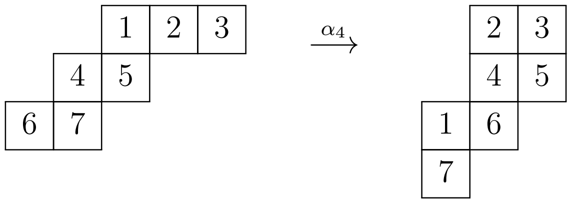
$$T = S_F \circ \alpha_3^{-1} = \begin{array}{ccc} & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} \\ \cline{1-3} \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \\ \cline{1-2} \end{array} \xrightarrow{\alpha_3^{-1}} \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} = S_F$$

Besides T there is one other semi-standard tableau $\tilde{T}$ of shape E and weight $(2, 2)$:

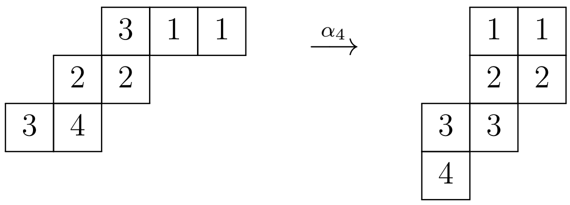
$$\tilde{T} = \begin{array}{ccc} & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\ \cline{1-3} \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \\ \cline{1-2} \end{array} \xrightarrow{\beta} \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}$$

This tableau is not F-special: if $\beta : E \to F$ is a diagram mapping with $\tilde{T} = S_F \circ \beta$ and b is the rightmost box in the top row of E, then there must be a box a of E such that $\beta(a)$ is directly above $\beta(b)$ in the same column in F, but a cannot occur in a higher row than b in E.

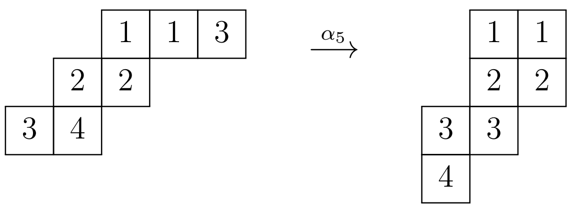
Examples 2.6. For the next example, let $F' = (5, 3, 2)/(2, 1)$ and $E' = (3, 3, 2, 1)$ $/(1, 1)$. Again these are diagrams with the same number of boxes, so we can construct a diagram mapping between them, and again We define the diagram mapping $\alpha_4 : F' \to E'$ by numbering the boxes of E' with a t-tableau and declaring that each box $a \in F'$ is mapped by α_4 to the box of E' which has the same number below:



This time, when we apply the canonical tableau to E' and let $T_1 = S_{E'} \circ \alpha_4$, we see that α_4 is not a representative of a special tableau because the tableau T_1 on F' is not semistandard:



Now define $\alpha_5 : F' \to E'$ as the mapping such that $\alpha_5(a_i) = \alpha_4(a_{4-i})$ for a_i the i-th box in the first row of F' and $\alpha_5(a) = \alpha_4(a)$ otherwise, then the tableau $T_2 = S_{E'} \circ \alpha_5$ is special, and α_5 represents it (in fact α_5 is an admissible representative):



α_5 is still not a special diagram mapping though and hence not a special representative of T_2 because it fails to satisfy condition (1) on the boxes in the second row. However, since T_2 is special, we know that it has a (unique) special representative, and in fact if we define $\alpha_6 : F' \to E'$ to be the mapping $\alpha_6(a_i) = \alpha_5(a_{3-i})$ for a_i

the i-th box in the second row of F', $\alpha_6(a) = \alpha_5(a)$ otherwise, then we find that α_6 is the special representative of $T_2 = S_{E'} \circ \alpha_6 = S_{E'} \circ \alpha_5$. Since α_6 is special, its inverse is also special and is therefore the unique special representative of the special semistandard tableau $S_F \circ \alpha_6^{-1}$ on E:

$$S_F \circ \alpha_3^{-1} = \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 1 \\ \cline{2-3} \multicolumn{1}{c|}{} & 2 & 2 \\ \hline 1 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \quad \xrightarrow{\alpha_6^{-1}} \quad \begin{array}{|c|c|c|c|c|} \cline{3-5} \multicolumn{2}{c|}{} & 1 & 1 & 1 \\ \cline{2-5} \multicolumn{1}{c|}{} & 2 & 2 & \multicolumn{2}{c}{} \\ \cline{1-3} 3 & 3 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array} = S_F$$

As the examples demonstrate, some diagram mappings may be representatives of special semi-standard tableaux by the above definition, without being admissible; however, from now on, when we talk about representatives of special tableaux we will insist that they are admissible, as this is needed for the linear independence in the proof of this chapter's main result, Theorem 2.23, later on, where we will be using a refinement of the above combinatorics.

We need to cut F and E into pieces labelled by certain integers and then we work with certain diagram mappings α which map each piece of F into the piece of E of the same label. We then apply the above combinatorics to the restrictions of α to these pieces.

Definition 2.7. Now once again let E and F be skew diagrams each with t boxes. Let P and Q be ordered tableaux of shapes E and F, both of weight $\nu \in \Sigma_t$. Then a diagram mapping $\alpha : F \to E$ with $P \circ \alpha = Q$ determines an m-tuple of tableaux $(S_{P^{-1}(1)} \circ \alpha_1, \ldots, S_{P^{-1}(m)} \circ \alpha_m)$ (*), where $\alpha_i : Q^{-1}(i) \to P^{-1}(i)$ is the restriction of α to $Q^{-1}(i)$. We will say that α *represents* (*). Notice that the m-tuples (*), for varying α, all have the same tuple of shapes and the same tuple of weights. We express this by saying that the tuple of tableaux has *shapes determined by Q and weights determined by P*. When the tableaux $S_{P^{-1}(i)} \circ \alpha_i$ are special semi-standard, we require the α_i to be admissible.

Example 2.8. Take $F = (4,4,3)/(1)$ and $E = (4,3,3)$ be skew diagrams, take $\nu = (4,6)$ and define Q and P as indicated below.

$$Q = \begin{array}{|c|c|c|c|} \hline & 1 & 1 & 2 \\ \hline 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 2 & \\ \hline \end{array} \quad \xrightarrow{\alpha} \quad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 1 & 1 & 2 & \\ \hline 2 & 2 & 2 & \\ \hline \end{array} = P$$

Then α_1 goes between the "1-pieces" of Q and P and α_2 goes between the "2-pieces" of Q and P. We also indicate the canonical numberings on the pieces of E and certain special semi-standard numberings on the pieces of F which can be obtained by pulling back the canonical numberings along suitable α_i. In particular:

$$\begin{array}{|c|c|c|} \hline & 1 & 1 \\ \hline 2 & 2 & \\ \hline \end{array} \quad \xrightarrow{\alpha_1} \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}$$

and

$$\begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 1 & 2 \\ \hline 3 & 3 & 3 & \\ \hline \end{array} \quad \xrightarrow{\alpha_2} \quad \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline & & 2 & \\ \hline 3 & 3 & 3 & \\ \hline \end{array}$$

It can be shown that the tableau $\begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 2 & 3 \\ \hline 1 & 3 & 3 & \\ \hline \end{array}$ is the only other $P^{-1}(2)$-special semi-standard tableau of shape $Q^{-1}(2)$ and weight $(2,1,3)$ other than

$S_{P^{-1}(2)} \circ \alpha_2$ above. For this particular P and Q, therefore, we see that the number of special diagram mappings from $Q^{-1}(1)$ to $P^{-1}(1)$ is 1, and the number of special diagram mappings from $Q^{-1}(2)$ to $P^{-1}(2)$ is 2.

2.1.2 Bideterminants and skew Schur and Specht modules

In this section we will review some facts from the representation theory of the general linear group as well as the symmetric group. Bideterminants are introduced in [7] and their skew versions in [1]. Other sources are [5, 21, 3, 4, 12]. In the latter three their application to the representations of the symmetric group is also discussed. The representation theory of the symmetric group will not be used here, but it may help to understand the combinatorics we use. It was also used in [26, Thm. 4] to obtain a version in characteristic 0 of the corollary 2.25 to Theorem 2.23 below.

Let E be a skew diagram with t boxes. Let S and T be (skew) tableaux of the same shape E, with S having entries $\leq r$ and T having entries $\leq s$. Then we define the *bideterminant* $(S \,|\, T) \in k[\mathrm{Mat}_{rs}]$ associated to this pair of tableaux by

$$(S \,|\, T) = \prod_{i=1}^{n} \det((x_{S(a),T(b)})_{a,b \in E^i}),$$

where E^i is the i-th column of E and n is the number of columns in E. Note that we have

$$(S \,|\, T) = \sum_{\pi \in C_E} \mathrm{sgn}(\pi) \prod_{a \in E} x_{S(\pi(a)),T(a)} = \sum_{\pi \in C_E} \mathrm{sgn}(\pi) \prod_{a \in E} x_{S(a),T(\pi(a))}\,,$$

where $C_E \leq \mathfrak{S}_E$ is the column stabiliser of E.

Example 2.9. Let $E = (3,2)/(1)$, $S = \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 2 \\ \hline 1 & 2 & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array}$, $T = \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 2 & 3 \\ \hline 1 & 3 & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array}$. Then

$(S \,|\, T) \in k[\mathrm{Mat}_{2,3}]$ is given by

$$(S \,|\, T) \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = a.\det \begin{pmatrix} b & c \\ e & f \end{pmatrix}.f = abf^2 - acef\,.$$

Definition 2.10. The *skew Specht module* $S(E) = S_t(E) = S_{t,k}(E)$ for the group algebra $A = A_{t,k} = k\mathfrak{S}_t$ of the symmetric group $\mathfrak{S}_t$ on $\{1, \ldots, t\}$ is defined just as in the case of an ordinary Young diagram (see Definition 3.9): $S(E) = Ae_1e_2$, where e_1 is the column anti-symmetriser of T_E and e_2 is the row symmetriser of T_E. The module $M(E) = M_{t,k}(E) = Ae_2$ is called the *permutation module* associated to E.

Let E and F be skew tableaux both with t boxes and let μ be the tuple of row lengths of E. If k is of characteristic 0, then we have as in [26, Sect. 3] that the special semi-standard tableaux of shape F and weight μ form a basis of the space $\mathrm{Hom}_{\mathfrak{S}_t}(S(E), S(F)) \cong (S(E) \otimes S(F))^{\mathfrak{S}_t} \cong (S(E) \otimes S(F))_{\mathfrak{S}_t}$, where $N_{\mathfrak{S}_t}$ denotes the space of coinvariants of an A-module N, i.e. the quotient of N by the span of the elements $x - g \cdot x$, $x \in N$, $g \in \mathfrak{S}_t$; hence the number of such special semi-standard tableaux is equal to the dimension of this space. In particular, if we let P and Q be ordered tableaux of shapes E and F, both of weight $\nu \in \Sigma_t$, then [26, Sect. 3] also gives us that the number of m-tuples of semi-standard tableaux with shapes determined by Q and weights determined by P is equal to the dimension of $\mathrm{Hom}_{\mathfrak{S}_\nu}(S(E), S(F)) \cong (S(E) \otimes S(F))^{\mathfrak{S}_\nu} \cong (S(E) \otimes S(F))_{\mathfrak{S}_\nu}$, where $\mathfrak{S}_\nu \leq \mathfrak{S}_t$ is the Young subgroup associated to ν (for more details, see also [28]).

Definition 2.11. The *skew Schur module* associated to a shape E, denoted by $\nabla_{\mathrm{GL}_r}(E)$, is the span in $k[\mathrm{Mat}_{rs}]$, $s \geq$ the number of rows of E, of all the bideterminants $(S \,|\, S_E)$ where S is a tableau of shape E and with entries $\leq r$. In particular, for λ a dominant weight, the Schur module associated to the (ordinary) Young diagram corresponding to the partition λ is also known as the *induced module*.

The skew Schur module $\nabla_{\mathrm{GL}_r}(E)$ will be nonzero if and only if r is greater than or equal to the length of each column of E. $\nabla_{\mathrm{GL}_r}(E)$ is stable under the action of GL_r, and the set of bideterminants $(S \mid S_E)$ with S as above and in addition semi-standard, form a basis (see [5, Thm. 3.3]). It is also possible to define $\nabla_{\mathrm{GL}_r}(E)$ as the span in $k[\mathrm{Mat}_{sr}]$ of all the bideterminants $(S_E \mid T)$ where T is a tableau of shape E and with entries $\leq r$, with the action of GL_r coming from the right multiplication rather than from the left multiplication.

Note that the Specht module $S_{t,k}(E)$ can also be defined as the weight space $\nabla_{\mathrm{GL}_r}(E)_{1^t}$ for any $r \geq t$ (where 1^t denotes the partition of t into ones, whose Young diagram is a single column). This weight space is indeed stable under $\mathfrak{S}_t \leq \mathrm{GL}_r$, where $\mathfrak{S}_t$ is identified with the group of permutation matrices whose nonzero off-diagonal entries are restricted to the first t rows (and columns).

Definition 2.12. [20, II.4.16] Let V be a GL_r-module. An ascending chain $0 = V_0 \subset V_1 \subset \ldots$ of submodules of V is called a *good filtration* if $V = \bigcup_{i\geq 0} V_i$ and if each V_i/V_{i-1} is isomorphic to an induced GL_r-module $\nabla_{\mathrm{GL}_r}(\lambda)$ for some dominant weight λ.

The elements $(S \mid T)$ with S and T both standard, and where the entries of S are $\leq r$ and those of T are $\leq s$, form a basis of $k[\mathrm{Mat}_{rs}]$ [7]. In fact one can use bideterminants to construct explicit good filtrations of $k[\mathrm{Mat}_{rs}]$ as a $\mathrm{GL}_r \times \mathrm{GL}_s$-module [5, Thm. 2.1].

Definition 2.13. The *co-Schur* or *Weyl module* $\Delta_{\mathrm{GL}_r}(E)$ associated to a shape E

is the contravariant dual $\nabla_{\mathrm{GL}_r}(E)^\circ$, which is the dual of the vector space $\nabla_{\mathrm{GL}_r}(E)$ with GL_r acting via the transpose: $(g \cdot f)(v) = f(g^T \cdot v)$.

Lemma 2.14. *[20, Prop. II.4.13] Let G be an algebraic group, and let E and F be skew diagrams. Then we have*

$$\mathrm{Hom}_G\left(\Delta_G(E), \nabla_G(F)\right) = \mathrm{Hom}_G\left(G, \Delta_G(E)^* \otimes \nabla_G(F)\right) = \begin{cases} k & \text{if F = E,} \\ 0 & \text{otherwise.} \end{cases}$$

Once again let E and F be skew diagrams both with t boxes and let μ be the tuple of row lengths of E. Then the number of special semi-standard tableaux of shape F and weight μ is equal to $\dim \mathrm{Hom}_{\mathrm{GL}_r}(\Delta_{\mathrm{GL}_r}(F), \nabla_{\mathrm{GL}_r}(E))$ whenever r is greater than or equal to the smaller of the number of rows of E and the number of rows of F. This can be seen by reducing to the case that k has characteristic 0, using the above lemma together with the fact that $\nabla_{\mathrm{GL}_r}(E)$ has a good filtration: [22, I.5] gives us that, for such an r, $\mathrm{Hom}_{\mathrm{GL}_r}(\Delta_{\mathrm{GL}_r}(F), \nabla_{\mathrm{GL}_r}(E)) \cong \mathrm{Hom}_{\mathfrak{S}_t}(S(E), S(F))$ as vector spaces, and then by the characterisation of the dimension of the latter above, and by [26, Rem. 4.2], we have the desired result.

Assume $r = r_1 + \cdots + r_m$ for certain integers $r_i > 0$, let $\nu \in \Sigma_t$ and let $\mathfrak{S}_\nu \leq \mathfrak{S}_t$ be the Young subgroup associated to ν. If k has characteristic 0, then we have an isomorphism $S_t(E) \cong \bigoplus_P \bigotimes_{i=1}^m S_{\nu_i}(P^{-1}(i))$ of $\mathfrak{S}_\nu$-modules, where the sum is over all ordered tableau P of shape E and weight ν. For k arbitrary, there exists a $(\prod_{i=1}^m \mathrm{GL}_{r_i})$-module filtration of the piece of multidegree ν of $\nabla_{\mathrm{GL}_r}(E)$ with sections in some order isomorphic to the modules $\bigotimes_{i=1}^m \nabla_{\mathrm{GL}_{r_i}}(P^{-1}(i))$, P an ordered tableau of shape E and weight ν. Here we can omit the P's for which $P^{-1}(i)$ has a column of length $> r_i$ for some i.

Remark 2.15. Let λ and μ be partitions with $\mu \subseteq \lambda$. Let r, r_1, s be integers ≥ 0 with $r_1, s \geq l(\lambda)$ and $r_1 \geq l(\mu) + r$ and put $r' = r_1 - r$. We embed $\mathrm{GL}_{r'} \times \mathrm{GL}_r$

in GL_{r_1} such that GL_r fixes the first r' basis vectors. Then one can embed $\nabla_{\mathrm{GL}_r}(\lambda/\mu)$ as a GL_r-submodule in $\nabla_{\mathrm{GL}_{r_1}}(\lambda)$. Indeed one can deduce from [9, 2.3] that $\nabla_{\mathrm{GL}_r}(\lambda/\mu) \cong \mathrm{Hom}_{\mathrm{GL}_{r'}}(\Delta_{\mathrm{GL}_{r'}}(\mu), \nabla_{\mathrm{GL}_{r_1}}(\lambda)) \cong \nabla_{\mathrm{GL}_{r_1}}(\lambda)_\mu^{U_{r'}}$, where μ is considered as a weight for $T_{r'}$. One can also construct an explicit isomorphism as follows. Let $E \in \mathrm{Mat}_{r's}$ be the matrix whose first $\min(r', s)$ rows are those of the $s \times s$ identity matrix followed by $r' - s$ zero rows if $r' > s$. Then the comorphism of the morphism $A \mapsto \left[\begin{smallmatrix} E \\ A \end{smallmatrix}\right] : \mathrm{Mat}_{rs} \to \mathrm{Mat}_{r_1 s}$ maps $\nabla_{\mathrm{GL}_{r_1}}(\lambda)^{U_{r'}}$ isomorphically onto $\nabla_{\mathrm{GL}_r}(\lambda/\mu)$. Combinatorially this is easy to understand: $\nabla_{\mathrm{GL}_{r_1}}(\lambda)^{U_{r'}}$ has a basis labelled by semi-standard tableaux of shape λ with entries $\leq r_1$ in which the entries $\leq r'$ occupy the boxes of μ and form the canonical tableau S_μ. These tableaux are clearly in one-to-one correspondence with the semi-standard tableaux of shape λ/μ with entries $\leq r$: just remove the μ-part and subtract r' from the entries of the resulting tableau of shape λ/μ.

We can now consider the following useful lemma from [1, Thm. II.4.11] (see also the successive Theorem 1.4, Remark 2, Theorem 1.5 and Claim 2 in [21]), giving a good filtration of a skew Schur module over a direct product of algebraic groups where the induced modules that the quotients (or sections) of the filtration are isomorphic to are direct products of induced modules for the groups in the direct sum, which by the above remark will indeed be induced modules themselves for the direct sum.

Lemma 2.16. *Let G and H be algebraic groups, and E be a skew diagram. Then the induced module $\nabla_{G\oplus H}(E)$ over the group $G \oplus H$ has a good filtration with sections isomorphic to*

$$\nabla_G(E_1) \otimes \nabla_H(E_2),$$

with E_1, E_2 running through every pair of skew diagrams with $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \emptyset$ as subsets of $\mathbb{N} \times \mathbb{N}$.

2.2 The action of $\mathrm{GL}_r \times \mathrm{GL}_s$ on several $r \times s$-matrices

2.2.1 Characteristic 0 case where one shape is a single column

Let r, s, t and m be positive integers, and let λ and μ be partitions of t with $l(\mu) \leq r$ and $l(\lambda) \leq s$. Let $\nu = (\nu_1, \ldots, \nu_m) \in \Sigma_t$ (that is, an m-tuple of non-negative integers with $\sum_i \nu_i = t$), let P and Q be ordered tableaux of shape λ and μ respectively and both of weight ν, and let $\alpha : \mu \to \lambda$ be a representative of an m-tuple of diagram mappings $\alpha_i : Q^{-1}(i) \to P^{-1}(i)$ between the skew diagrams consisting of the boxes of λ and μ assigned the same integer by P or Q respectively, or in other words let $\alpha : \mu \to \lambda$ be a mapping such that $P \circ \alpha = Q$. Then [26, Thm. 4] defines

$$u_{P,Q,\alpha} = \sum_{\pi \in C_\mu, \sigma \in C_\lambda} \mathrm{sgn}(\pi)\mathrm{sgn}(\sigma) \prod_{a \in \mu} x\left(Q(a)\right)_{\pi(a)_1, \sigma(\alpha(a))_1}, \tag{2.1}$$

where for a box b in a diagram, b_1 denotes the row index of b in that diagram, and where $x(\eta)_{i,j} \in k\left[\mathrm{Mat}_{rs}^m\right]$ for $\eta \in \{1, \ldots, m\}$ is the function picking out the (i,j)-th entry in the η-th matrix component. [26, Thm. 4] then states that the $u_{P,Q,\alpha}$, where for each P and Q as above α runs through a set of admissible representatives for the m-tuples of special tableaux with shapes determined by Q and weights determined by P, form a basis of the vector space $k\left[\mathrm{Mat}_{rs}^m\right]_{(\mu,\lambda)}^{U_r \times U_s}$ when k has characteristic 0. In fact, for the action of $\mathrm{GL}_r \times \mathrm{GL}_s$ in [26], the inverse of the right-hand matrix is used rather than the transpose as we have here, which explains why we have $\pi(a)_1$ in the formula for the basis elements rather than $r - \pi(a)_1 + 1$, and why we consider the weight μ for the highest weight vectors rather than $-\mu^{\mathrm{rev}}$; We can obtain the highest weight vectors for one action from those for the other by acting with the $r \times r$-matrix with ones on the anti-diagonal and zeros elsewhere.

For a general field k with no restriction on characteristic, [26] also gives (Theorem 2) a basis for $k\left[\mathrm{Mat}_{rs}^{m}\right]_{(\mu,\lambda)}^{U_r \times U_s}$ where either λ or μ is a column: in the case where λ is a column of length $t \leq s$ and μ is some partition of t with $l(\mu) \leq r$ and $\mu_1 \leq m$, then this basis comprises the elements $v_T \in k\left[\mathrm{Mat}_{rs}^{m}\right]$ for the row-semistandard tableaux T of shape μ with entries $\leq m$, with each v_T defined by

$$(A_1, \ldots, A_m) \mapsto$$

$$\sum_S \det\left(A'_{S_{1,1}} e_1 \middle| \ldots \middle| A'_{S_{1,\mu_1}} e_1 \middle| \ldots \middle| A'_{S_{l(\mu),1}} e_{l(\mu)} \middle| \ldots \middle| A'_{S_{l(\mu),\mu_{l(\mu)}}} e_{l(\mu)}\right)_{\lceil t},$$

where the sum is over the orbit of T under C_μ, e_i are the standard basis elements of k^r, A'_i denotes the transpose of the matrix A_i, and $M_{\lceil t}$ is the matrix M with all but the first t rows removed.

From now on let P be the ordered tableau of weight $\nu = (\nu_1, \ldots, \nu_m)$ on a column of length $t \leq s$ and let Q be a fixed ordered tableau of shape $\mu \vdash t$ with $l(\mu) \leq r$, and also of weight ν. Then we have the following results.

Lemma 2.17. *If Q is row-semistandard, exactly one $\alpha : \mu \to 1^t$ representing an m-tuple of admissible diagram mappings between P and Q exists; if Q is not row-semistandard, then no such mappings exist.*

Proof. Consider the skew diagrams $P^{-1}(i)$ and $Q^{-1}(i)$ for one of the $1 \leq i \leq m$ occurring more than once in the tableaux. Since the former is a column, then for any two boxes there, one will necessarily be directly below the other. Pick two boxes in the column and label the box in $Q^{-1}(i)$ mapped by α_i to the one in a higher row by a, and the box mapped by α_i to the lower box of $P^{-1}(i)$ by b. Then for α_i to be admissible, by definition it is necessary for b to occur in a strictly lower row than a. Since this is true for any pair of boxes and their images in α_i, it must be that $Q^{-1}(i)$ does not contain more than one box in any one row, i.e. we must have Q row-semistandard. If this is the case, then any α_i that does not preserve in the column $P^{-1}(i)$ the order of the rows in which the boxes of $Q^{-1}(i)$ occur cannot be admissible by definition. □

Theorem 2.18. *Let k be a field of characteristic* 0. *Then the basis elements $u_{P,Q,\alpha}$ and v_Q, as defined above, of the space $k\left[\mathrm{Mat}_{rs}^m\right]_{(\mu,1^t)}^{U_r\times U_s}$, are equivalent up to multiplication by an integer.*

Proof. Since P is a column tableau it is completely determined by its weight ν, the weight of Q, and then by the above Lemma the suitable α is unique, thus the elements $u_{P,Q,\alpha}$ certainly can vary only with Q.
The column stabiliser C_{1^t} of a column of length t is clearly isomorphic to the symmetric group $\mathfrak{S}_t$ on t letters, so for $\sigma \in C_{1^t}$ we can denote by $\tilde{\sigma}$ the corresponding element of $\mathfrak{S}_t$. Furthermore, for each $j \in \{1, \ldots, t\}$, denote by a_j the box of μ that is mapped to the j-th box in 1^t by α. Now the basis element $u_{P,Q,\alpha}$ can be written

$$u_Q = \sum_{\pi\in C_\mu,\, \tilde{\sigma}\in\mathfrak{S}_t} \mathrm{sgn}(\pi)\mathrm{sgn}\left(\tilde{\sigma}\right) \prod_{j=1}^{t} x\left(Q\left(a_j\right)\right)_{\pi(a_j)_1,\, \tilde{\sigma}(j)},$$

which we can see is now of the form

$$\sum_{\pi \in C_\mu} \mathrm{sgn}(\pi) \det\left(M^\pi\right),$$

for the $t \times t$ matrices M^π given by $M^\pi_{kl} = x\left(Q\left(a_k\right)\right)_{\pi(a_k)_1, l}$. Now for each $\pi \in C_\mu$ define $S_\pi = Q \circ \pi^{-1}$, and denote by $\tilde{\pi}$ the permutation of $\{1, \ldots, t\}$ such that $a_{\tilde{\pi}(k)} = \pi\left(a_k\right)$ for $1 \leq k \leq t$, then if we consider the $t \times t$ matrix N^π formed by permuting the rows of M^π by $\tilde{\pi} \in \mathfrak{S}_t$, that is

$$N^\pi_{kl} = M^\pi_{\tilde{\pi}^{-1}(k)l} = x\left(Q\left(\pi^{-1}\left(a_k\right)\right)\right)_{(a_k)_1, l} = x\left(S_\pi\left(a_k\right)\right)_{(a_k)_1, l},$$

then $\det\left(N^\pi\right) = \mathrm{sgn}\left(\tilde{\pi}\right) \det\left(M^\pi\right) = \mathrm{sgn}(\pi) \det\left(M^\pi\right)$ so the basis element becomes

$$u_Q = \sum_{\pi \in C_\mu} \det\left(N^\pi\right).$$

Comparing this to

$$v_Q = \sum_{S \in C_\mu . Q} \det\left(Y^S\right),$$

with the matrices Y^S defined to be

$$\left(\begin{array}{c|c|c|c|c|c|c} x\left(S_{1,1}\right)_{1,1} & & x\left(S_{1,\mu_1}\right)_{1,1} & & x\left(S_{l(\mu),1}\right)_{l(\mu),1} & & x\left(S_{l(\mu),\mu_{l(\mu)}}\right)_{l(\mu),1} \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ x\left(S_{1,1}\right)_{1,t} & & x\left(S_{1,\mu_1}\right)_{1,t} & & x\left(S_{l(\mu),1}\right)_{l(\mu),t} & & x\left(S_{l(\mu),\mu_{l(\mu)}}\right)_{l(\mu),t} \end{array}\right),$$

(it can be easily checked that these v_Q are the same as those above), we see that there is a fixed $\tau \in \mathfrak{S}_t$ (i.e. dependent on α but not on π) such that for all $\pi \in C_\mu$, the rows of N^π can be permuted by τ to obtain the transpose of Y^{S_π}. (In particular, τ is the permutation for which $a_{\tau^{-1}(i)}$ is the i-th box of μ in the standard numbering, that is going through μ row by row from left to right and top to bottom.) Furthermore, the summands $\det\left(N^{\pi_1}\right)$ and $\det\left(N^{\pi_2}\right)$ of u_Q are equal whenever $\pi_1(Q) = \pi_2(Q)$, so u_Q is divisible by the size of the stabiliser of Q inside the column stabiliser of μ (we shall denote this subgroup by $C_\mu(Q)$), and in particular

$$v_Q = \operatorname{sgn}(\tau)\frac{u_Q}{|C_\mu(Q)|}.$$

□

2.2.2 Field of arbitrary characteristic and arbitrary skew shapes

More generally, we can consider for E and F skew diagrams each with t boxes, P and Q tableaux of shapes E and F respectively, both of weight $\nu \in \Sigma_t$, $\alpha : F \to E$ a diagram mapping such that $P \circ \alpha = Q$, S a tableau of shape F with entries $\leq r$ and T a tableau of shape E with entries $\leq s$, the sum

$$\sum_{(\pi,\sigma)\in C_F \times C_E} \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma) \prod_{a\in F} x(Q(a))_{S(\pi(a)),\,T(\sigma(\alpha(a)))}\,. \tag{2.2}$$

Note that we obtain (2.1) from (2.2) by taking S and T the canonical tableaux S_F and S_E.

We will now show that (2.2) is in $\mathbb{Z}[\mathrm{Mat}^m_{rs}] = \mathbb{Z}[(x(l)_{ij})_{lij}]$ divisible by the order of the subgroup

$$C_{P,Q,\alpha} = \{(\tau,\rho) \in C_F(Q) \times C_E(P) \,|\, \alpha \circ \tau \circ \alpha^{-1} = \rho\}$$

of $C_F \times C_E$, where $C_F(Q)$ and $C_E(P)$ are defined similarly to $C_\mu(Q)$ above. Note that

$$C_{P,Q,\alpha} \cong C_F(Q) \cap \alpha^{-1} C_E(P)\alpha \leq \mathfrak{S}_F \cong \prod_{i=1}^m C_{Q^{-1}(i)} \cap \alpha_i^{-1} C_{P^{-1}(i)}\alpha_i$$

and that

$$C_{P,Q,\alpha} \cong \alpha C_F(Q)\alpha^{-1} \cap C_E(P) \leq \mathfrak{S}_E \cong \prod_{i=1}^m \alpha_i C_{Q^{-1}(i)}\alpha_i^{-1} \cap C_{P^{-1}(i)}\,.$$

In each of the two lines above one may omit "(Q)" in $C_F(Q)$ or "(P)" in $C_E(P)$, but not both.

Theorem 2.19. *Each summand in (2.2) only depends on the left coset of (π, σ) modulo $C_{P,Q,\alpha}$.*

Proof. Let $(\pi_1, \sigma_1), (\pi_2, \sigma_2) \in C_F \times C_E$ and suppose $(\pi_2, \sigma_2) = (\pi_1 \circ \tau, \sigma_1 \circ \rho)$ for some $(\tau, \rho) \in C_{P,Q,\alpha}$. Then $\mathrm{sgn}(\pi_1)\mathrm{sgn}(\sigma_1) = \mathrm{sgn}(\pi_2)\mathrm{sgn}(\sigma_2)$, since $\mathrm{sgn}(\tau) = \mathrm{sgn}(\rho)$. Furthermore,

$$\begin{aligned}\prod_{a \in F} x\left(Q(a)\right)_{S(\pi_2(a)),\, T(\sigma_2(\alpha(a)))} &= \prod_{a \in F} x\left(Q(a)\right)_{S(\pi_1(\tau(a))),\, T(\sigma_1(\rho(\alpha(a))))} \\ &= \prod_{a \in F} x\left(Q(a)\right)_{S(\pi_1(\tau(a))),\, T(\sigma_1(\alpha(\tau(a))))} \\ &= \prod_{a \in F} x\left(Q\left(\tau^{-1}(a)\right)\right)_{S(\pi_1(a)),\, T(\sigma_1(\alpha(a)))} \\ &= \prod_{a \in F} x\left(Q(a)\right)_{S(\pi_1(a)),\, T(\sigma_1(\alpha(a)))} .\end{aligned}$$

□

Now define the *twisted bideterminant* $(S\,|^m_{P,Q,\alpha}\, T) \in k[\mathrm{Mat}^m_{rs}]$ by

$$(S\,|^m_{P,Q,\alpha}\, T) = \sum_{(\pi,\sigma)} \mathrm{sgn}(\pi)\mathrm{sgn}(\sigma) \prod_{a \in F} x(Q(a))_{S(\pi(a)),\, T(\sigma(\alpha(a)))} , \tag{2.3}$$

where the sum is over a set of representatives of the left cosets of $C_{P,Q,\alpha}$ in $C_F \times C_E$. Clearly, if k has characteristic 0, then $(S\,|^m_{P,Q,\alpha}\, T)$ equals (2.2) divided by $|C_{P,Q,\alpha}|$. Note that the product in (2.3) can also be written as

$$\prod_{a \in E} x(P(a))_{S(\pi(\alpha^{-1}(a))),\, T(\sigma(a))} .$$

In case $m = 1$, P and Q are constant equal to 1 and they play no role. We then omit P, Q and the superscript m in our notation and instead of $x(1)_{ij}$ we write x_{ij}. So

$$(S\,|_{\alpha}\,T) = \sum_{(\pi,\sigma)} \mathrm{sgn}(\pi)\mathrm{sgn}(\sigma) \prod_{a\in F} x_{S(\pi(a)),\,T(\sigma(\alpha(a)))}\,, \tag{2.4}$$

where the sum is over a set of representatives of the left cosets of $C_\alpha = \{(\tau,\rho) \in C_F \times C_E \,|\, \alpha\circ\tau\circ\alpha^{-1} = \rho\}$ in $C_F \times C_E$. Note that if $m = 1$, $E = F$ and $\alpha = \mathrm{id}$ we get the ordinary bideterminant.

Lemma 2.20. *If X is a set of representatives for the left cosets of $\alpha C_F(Q)\alpha^{-1} \cap C_E(P)$ in C_E, then $C_F \times X$ is a set of representatives for the left cosets of $C_{P,Q,\alpha}$ in $C_F \times C_E$. If we concatenate all matrices in an m-tuple column-wise, then we obtain an isomorphism between $k[\mathrm{Mat}_{rs}^m]$ and $k[\mathrm{Mat}_{r,ms}]$ as rings, which maps $x(l)_{ij}$ to $x_{i,(l-1)s+j}$. Now we have*

$$(S\,|_{P,Q,\alpha}^m\,T) = \sum_{\sigma\in X} \mathrm{sgn}(\sigma)(S\,|\,T^{\alpha,\sigma})\,,$$

where $T^{\alpha,\sigma}(a) = T(\sigma(\alpha(a))) + (Q(a)-1)s$ for $a \in F$. Of course we could also work with a set $\tilde{X}$ of representatives for the left cosets of $C_F(Q) \cap \alpha^{-1}C_E(P)\alpha$ in $\tilde{C}_F = \alpha^{-1}C_E\alpha$. Then the above sum would be over $\sigma \in \tilde{X}$ with $T^{\alpha,\sigma}(a) = T(\alpha(\sigma(a))) + (Q(a)-1)s$ for $a \in F$.

Similarly, if X is a set of representatives for the left cosets of $C_F(Q)\cap\alpha^{-1}C_E(P)\alpha$ in C_F, then $X \times C_E$ is a set of representatives for the left cosets of $C_{P,Q,\alpha}$ in $C_F \times C_E$. If we concatenate all matrices in an m-tuple row-wise, then we obtain an isomorphism $k[\mathrm{Mat}_{rs}^m] \cong k[\mathrm{Mat}_{mr,s}]$ which maps $x(l)_{ij}$ to $x_{(l-1)r+i,j}$. Then we have

$$(S\,|_{P,Q,\alpha}^m\,T) = \sum_{\pi\in X} \mathrm{sgn}(\pi)(S^{\alpha,\pi}\,|\,T)\,,$$

where $S^{\alpha,\pi}(a) = S(\pi(\alpha^{-1}(a))) + (P(a)-1)r$ for $a \in E$. With $\tilde{X}$ a set of representatives for the left cosets of $\alpha C_F(Q)\alpha^{-1} \cap C_E(P)$ in $\tilde{C}_E = \alpha C_F \alpha^{-1}$, the above sum would be over $\pi \in \tilde{X}$ with $S^{\alpha,\pi}(a) = S(\alpha^{-1}(\pi(a))) + (P(a) - 1)r$ for $a \in E$.

Remark 2.21. In the case of the twisted bideterminants for a single matrix (i.e. $m = 1$), P and Q play no role, so $C_F(Q)$ and $C_E(P)$ can be replaced by C_F and C_E, and in the definitions of $T^{\alpha,\sigma}$ and $S^{\alpha,\pi}$ the terms containing Q or P may be omitted. In [4, Sect. 8-11], the twisted bideterminants $(S \,|_\alpha\, T)$ are known as "shuffle-products", and moving from the single matrix version of the first expression above to that of the second is called "overturn of the P-shuffle product onto the L-side".

Example 2.22. Take $F = (2, 2)$ and $E = (2, 1, 1)$ and Q, P and α as indicated below.

$$Q = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \xrightarrow{\alpha} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \quad = P$$

Here the second tableau of shape F has been given the standard numbering and α maps each box of F to the box of E with the same number. Clearly, $P \circ \alpha = Q$. Note that

$$\alpha_1 : \begin{array}{|c|}\hline 1 \\ \hline \end{array} \longrightarrow \begin{array}{|c|}\hline 1 \\ \hline \end{array} \quad \text{and } \alpha_2 : \begin{array}{c|c|}\cline{2-2} & 2 \\ \hline \multicolumn{1}{|c|}{3} & 4 \\ \hline \end{array} \longrightarrow \begin{array}{c|c|}\cline{2-2} & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array}$$

are special.

Now we consider certain twisted bideterminants in $k[\mathrm{Mat}_{23}^2] \cong k[\mathrm{Mat}_{43}]$. For S a tableau of shape F with entries ≤ 2, T a tableau of shape E with entries ≤ 3, and α, P, Q as above we have

$$(S\,|^2_{P,Q,\alpha}\,T) = \left(\begin{array}{cc|c} s_{11} & s_{21}+2 & \\ s_{12}+2 & & T \\ s_{22}+2 & & \end{array} \right) - \left(\begin{array}{cc|c} s_{21} & s_{11}+2 & \\ s_{12}+2 & & T \\ s_{22}+2 & & \end{array} \right).$$

This can be seen by applying Lemma 2.20 to the set of representatives $X = \langle(1,3)\rangle \leq C_F$ for the left cosets of $C_F(Q) \cap \alpha^{-1}C_E(P)\alpha = \langle(2,4)\rangle$ in C_F.

Let k be an algebraically closed field, then the coordinate ring $k[\mathrm{Mat}^m_{rs}]$ is $\mathbb{N}_0^m$-graded. Fix a multidegree $\nu \in \Sigma_t$. Then one can construct a good filtration

$$M_1 \supseteq M_2 \supseteq \cdots \supseteq M_{q+1} = 0$$

of the graded piece M_1 of degree ν of $k[\mathrm{Mat}^m_{rs}]$ as follows.

Let X be the set of triples (P, Q, α), with P and Q ordered tableaux of weight ν with shapes λ of length $\leq s$ and μ of length $\leq r$ and $\alpha : \mu \to \lambda$ an admissible representative for the m-tuples of special tableaux with shapes determined by Q and weights determined by P (see Section 2.1.1). Then for a numbering

$$(P_1, Q_1, \alpha^1), (P_2, Q_2, \alpha^2), \ldots, (P_q, Q_q, \alpha^q)$$

of X, define the space M_i for each $i \leq q$ to be the span in $k[\mathrm{Mat}^m_{rs}]$ of all twisted bideterminants $(S\,|^m_{P_j,Q_j,\alpha^j}\,T)$ with $j \geq i$, where for λ^j the shape of each P_j and μ^j the shape of the Q_j, S is a tableau of shape μ^j with entries $\leq r$, and T is a tableau of shape λ^j with entries $\leq s$.

Theorem 2.23. *There is a numbering $(P_1, Q_1, \alpha^1), \ldots, (P_q, Q_q, \alpha^q)$ of the set X of triples as defined above such that for each $i \leq q$, the space M_i is* $\mathrm{GL}_r \times \mathrm{GL}_s$-

stable and we have an isomorphism

$$\begin{aligned}(S\,|\,S_{\mu^i}) \times (T\,|\,S_{\lambda^i}) &\mapsto (S\,|^m_{P_i,Q_i,\alpha^i}\,T) \bmod M_{i+1}\\ \nabla_{\mathrm{GL}_r}\big(\mu^i\big)\otimes\nabla_{\mathrm{GL}_s}\big(\lambda^i\big) &\xrightarrow{\sim} M_i/M_{i+1}\,.\end{aligned}$$

Furthermore, the twisted bideterminants $(S\,|^m_{P_j,Q_j,\alpha^j}\,T)$*,* $1 \le j \le q$*,* S *and* T *as above and in addition semi-standard, form a basis of the graded piece of degree* ν *of* $k[\mathrm{Mat}^m_{rs}]$*.*

Proof. We use the isomorphism $k[\mathrm{Mat}^m_{rs}] \cong k[\mathrm{Mat}_{mr,s}]$, see Lemma 2.20. Let t be an integer ≥ 0. We start with the good $\mathrm{GL}_{mr} \times \mathrm{GL}_s$-filtration (see [1, II.4]) of the piece of degree t of $k[\mathrm{Mat}_{mr,s}]$ with sections isomorphic to

$$\nabla_{\mathrm{GL}_{mr}}(\lambda^i)\otimes\nabla_{\mathrm{GL}_s}(\lambda^i). \tag{2.5}$$

Here the λ^i are the partitions of t of length $\le \min(mr, s)$. The isomorphisms to the sections of the filtration are given by

$$(S\,|\,S_{\lambda^i})\otimes(T\,|\,S_{\lambda^i}) \mapsto (S\,|\,T) \text{ modulo the } (i+1)\text{-th filtration space.}$$

After restricting the left multiplication action to GL^m_r we can decompose the above filtration according to the multidegree in $\mathbb{N}^m_0$. From now on we focus on the piece of multidegree $\nu \in \Sigma_t$. By considering GL_{mr} as a direct sum of m copies of GL_r, we can repeatedly apply Lemma 2.16 to $\nabla_{\mathrm{GL}_{mr}}(\lambda^i)$, until we have "refined" the above filtration to a filtration with sections isomorphic to

$$\Big(\bigotimes_{j=1}^{m}\nabla_{\mathrm{GL}_r}(P_i^{-1}(j))\Big)\otimes\nabla_{\mathrm{GL}_s}(\lambda^i), \tag{2.6}$$

where the group GL^m_r acts on the first factor, and the P_i go through all ordered tableaux of shape λ^i with weight ν, which will make the corresponding m-tuples of diagrams $P_i^{-1}(j)$ go through all the m-tuples of tableaux with numbers of boxes determined by ν such that the union of the diagrams in the tuple equals λ^i, for each i. Now we restrict the first factor of (2.6) to the diagonal copy of GL_r in GL_r^m and we have

$$\bigotimes_{j=1}^{m} \nabla_{GL_r}\left(P_i^{-1}(j)\right) \cong \nabla_{GL_r}\left(E_{P_i}\right), \tag{2.7}$$

where for P an ordered tableau with entries $\leq m$ we define $E_P = E_{(P^{-1}(1),\ldots,P^{-1}(m))}$ and for an m-tuple $(D_1, \ldots, D_m)$ of skew Young diagrams

$$E_{(D_1,\ldots,D_m)} = \begin{matrix} & & D_1 \\ & \cdot^{\cdot^{\cdot}} & \\ D_m & & \end{matrix}$$

where each row or column contains boxes from at most one skew tableau D_j. Now we apply [21, Thm. 1.5] and we can refine our previous filtration to a filtration with sections

$$\nabla_{GL_r}(\mu^i) \otimes \nabla_{GL_s}(\lambda^i)\,.$$

Here the μ^i have length $\leq r$. Furthermore, the labelling is coming from triples $(P, \mu, \overline{\alpha})$ where P is an ordered tableau of weight ν, μ a partition of t and $\overline{\alpha} : \mu \to E_P$ goes through a set of admissible representatives for the special tableaux of shape μ and weight the tuple of row lengths of E_P. These triples are in one-to-one correspondence with the triples (P, Q, α) mentioned earlier.

We now have to check that our filtration is indeed given by spans of twisted bideterminants. From Lemma 2.20 it is clear that under the section-isomorphism (2.5) the element $(S \,|^m_{P,Q,\alpha}\, S_{\lambda^i}) \otimes (T \,|\, S_{\lambda^i})$, S of shape μ with entries $\leq r$, $\alpha : \mu \to \lambda^i$, T of shape λ^i with entries $\leq s$, is mapped to $(S \,|^m_{P,Q,\alpha}\, T)$ modulo the $(i+1)$-th filtration space. So it now suffices to show that at "stage (2.7)" the elements $(S \,|^m_{P,Q,\alpha}\, S_{\lambda^i})$ correspond under the section isomorphisms (2.7) and (2.6) to the elements defining the filtration of $\nabla_{\mathrm{GL}_r}(E_{P_i})$ from [21, Thm. 1.5].

For this we focus on one particular i which we suppress in the notation. If $\alpha : \mu \to \lambda$ is an admissible representative of an m-tuple of special tableaux, then the diagram mapping $\overline{\alpha} : \mu \to E_P$ whose restrictions : $Q^{-1}(j) \to P^{-1}(j)$ are the same as those of α, is an admissible representative of the special tableau $T = S_{E_P} \circ \overline{\alpha}$ of shape μ. The elements defining the filtration of $\nabla_{\mathrm{GL}_r}(E_P)$ from the proof of [21, Thm. 1.5] are $(S\,|_{\overline{\alpha}}\, S_{E_P})$, S of shape μ with entries $\leq r$. Here one should bear in mind that in [21] the bideterminants are formed row-wise rather than column-wise, and that there $\overline{\alpha}^{-1}$ is used rather than $\overline{\alpha}$: the map f_T on page 93 of [21] satisfies (after transposing) $T \circ f_T = S_{E_P}$, and it corresponds to the inverse of our $\overline{\alpha}$. Note that actually the $\overline{\alpha}$ corresponding to the f_T from [21] are the (unique) special representatives of the special tableaux T of shape μ and weight the tuple of row lengths of E_P, but it is clear that the arguments there work for any choice of admissible representatives $\overline{\alpha}$. Furthermore, it is clear from the proof of [21, Claim 2 (p94)] that the filtration of $\nabla_{\mathrm{GL}_r}(E_P)$ does not depend on the choice of representing $\overline{\alpha}$.

Now by Lemma 2.20 we have

$$(S\,|_{\overline{\alpha}}\, S_{E_P}) = \sum_{\pi \in X} \mathrm{sgn}(\pi)(S^{\overline{\alpha},\pi} \,|\, S_{E_P})\,,$$

where X is a set of representatives for the left cosets of $C_\mu \cap \overline{\alpha}^{-1} C_{E_P} \overline{\alpha}$ in C_μ and $S^{\overline{\alpha},\pi}(a) = S(\pi(\overline{\alpha}^{-1}(a)))$ for $a \in E_P$. Now we have $C_\mu \cap \overline{\alpha}^{-1} C_{E_P} \overline{\alpha} = C_\mu(Q) \cap \alpha^{-1} C_\lambda(P) \alpha$, so, by Lemma 2.20 we have for the same set X

$$(S\,|^m_{P,Q,\alpha}\, S_\lambda) = \sum_{\pi \in X} \mathrm{sgn}(\pi)(S^{\alpha,\pi} \,|\, S_\lambda)\,,$$

where $S^{\alpha,\pi}(a) = S(\pi(\alpha^{-1}(a))) + (P(a)-1)r$ for $a \in \lambda$. Under the section isomorphisms (2.7) and (2.6), $S^{\overline{\alpha},\pi}$ corresponds to $S^{\alpha,\pi}$, that is, $(S^{\overline{\alpha},\pi} \,|\, S_{E_P})$ is mapped to $(S^{\alpha,\pi} \,|\, S_\lambda)$ modulo the filtration space labelled by "the next P". So, by the above two equations, $(S \,|_{\overline{\alpha}}\, S_{E_P})$ is mapped to $(S \,|^m_{P,Q,\alpha}\, S_\lambda)$ modulo the filtration space labelled by the next P. □

Corollary 2.24. *Let λ, μ be partitions of t with $l(\mu) \leq r$ and $l(\lambda) \leq s$ and let $\nu \in \Sigma_t$. Then the elements $(S_\mu \,|^m_{P,Q,\alpha}\, S_\lambda)$, P, Q ordered tableaux of shapes λ and μ, both of weight ν, and α in a set of representatives for the m-tuples of special tableaux with shapes determined by Q and weights determined by P, form a basis of the piece of degree ν of $k[\mathrm{Mat}^m_{rs}]^{U_r \times U_s}_{(\mu,\lambda)}$.*

Proof. Using Lemma 2.20, we get that the elements $(S_\mu \,|^m_{P,Q,\alpha}\, S_\lambda)$ are highest weight vectors of the given weight. Furthermore, they are linearly independent by Theorem 2.23. On the other hand, it follows from Lemma 2.14 that the dimension of $k[\mathrm{Mat}^m_{rs}]^{U_r \times U_s}_{(\mu,\lambda)}$ is equal to the number of sections $\nabla_{\mathrm{GL}_r}(\mu^i) \otimes \nabla_{\mathrm{GL}_s}(\lambda^i)$ with $(\lambda_i, \mu_i) = (\lambda, \mu)$ in a good filtration of $k[\mathrm{Mat}^m_{rs}]$, since for each such section there will be a copy of k while for sections with $\lambda \neq \mu$ we will get the zero vector space. But then this dimension is equal to the number of elements of our linearly independent set. □

Finally we give a version for the above corollary for the $\mathrm{GL}_r \times \mathrm{GL}_s$-action on $k[\mathrm{Mat}^m_{rs}]$ defined by $((A, B) \cdot f)(\underline{X}) = f((A^{-1} X_i B)_{1 \leq i \leq m})$, that is, we twist the GL_r-action we considered previously with the inverse transpose. We define the *anti-canonical tableau* $\tilde{S}_\mu$ of shape μ by $\tilde{S}_\mu(a) = r - a_1 + 1$, for $a \in \mu$ where a_1 is the row index of a. For a tuple μ of integers of length $\leq r$ we denote by μ^{rev} the reverse of the r-tuple obtained from μ by extending it with zeros.

Corollary 2.25. *Let λ, μ be partitions of t with $l(\mu) \leq r$ and $l(\lambda) \leq s$ and let $\nu \in \Sigma_t$. Then the elements $(\tilde{S}_\mu\,|^m_{P,Q,\alpha}\,S_\lambda)$, P, Q ordered tableaux of shapes λ and μ, both of weight ν, and α in a set of representatives for the m-tuples of special tableaux with shapes determined by Q and weights determined by P, form a basis of the piece of degree ν of $k[\mathrm{Mat}^m_{rs}]^{U_r\times U_s}_{(-\mu^{\mathrm{rev}},\lambda)}$.* □

Remarks 2.26. 1. We now extract from the proof of Theorem 2.23 how the triples (P, Q, α) are enumerated. First we order the P's by identifying each P with the tuple of Young diagrams (i.e. partitions) $P^{-1}(\{1, \ldots, m-i\})_{0\leq i\leq m-1}$ and ordering these lexicographically, where the partitions are themselves also ordered lexicographically. For a fixed P we order the pairs (Q, α) as follows. For each i we let S_i be the tableau obtained by shifting the entries of $S_{P^{-1}(i)} \circ \alpha_i$ by $\sum_{j=0}^{i-1} r_j$, where r_j is the number of rows of $P^{-1}(j)$. Here the α_i are defined as in Section 2.1.1. Let $S_{Q,\alpha}$ be the tableau of the same shape as Q obtained by piecing the S_i together according to Q. Then we say that $(Q^1, \alpha^1) > (Q^2, \alpha^2)$ if the standard enumeration of S_{Q^1,α^1} is lexicographically less than that of S_{Q^2,α^2}. Now we order the triples (P, Q, α) lexicographically by first comparing the P-component and then the (Q, α)-component. Finally, we enumerate the triples (P, Q, α) in decreasing order.

2. We can now give a characteristic-free version of [26, Thm. 3]. Let E and F be skew Young diagrams with t boxes. Let r be $\geq$ the number of rows of F and let s be $\geq$ the number of rows of E, then the twisted bideterminants $(S_F\,|_{P,Q,\alpha}\,S_E) \in k[\mathrm{Mat}_{rs}]$ where α goes through a set of admissible representatives of special tableaux of shape F and weight the tuple of row lengths of E (and P and $Q = P \circ \alpha$ vary together with α), are linearly independent.

This can be deduced from [21] as follows. Write $F = \mu/\tilde{\mu}$ and take $\overline{E}$ to be

E with $\tilde{\mu}$ above and to the right of it in such a way that they have no rows or columns in common. We use the definition of Schur modules that uses the right multiplication action. If we combine this with Remark 2.15 we obtain an isomorphism $\nabla_{\mathrm{GL}_s}(F) \xrightarrow{\sim} \nabla_{\mathrm{GL}_{s_1}}(\mu)^{U_{s'}}_{\tilde{\mu}}$ where $s_1 = s' + s$. By Lemma 2.20 this isomorphism maps $(S_F\,|_{P,Q,\alpha}\,S_E)$ to $(S_\mu\,|_{\overline{\alpha}}\,S_{\overline{E}})$ where $\overline{\alpha} : \mu \to \overline{E}$ is given by $\overline{\alpha}|_F = \alpha$ and $\alpha|_{\tilde{\mu}} = \mathrm{id}$. For α as above, $\overline{\alpha}$ goes through a set of representatives for the special tableaux of shape μ and weight the tuple of row lengths of $\overline{E}$. Since the elements $(S_\mu\,|_{\overline{\alpha}}\,S_{\overline{E}})$ are linearly independent by the proof of [21, Thm. 1.5], the result follows.

3. Assume $r = r_1 + \cdots + r_m$ for certain integers $r_j > 0$. By similar arguments as in the proof of Theorem 2.23 one can construct a good $\left(\prod_{j=1}^m \mathrm{GL}_{r_j}\right) \times \mathrm{GL}_s$-filtration of the degree ν piece of $k[\mathrm{Mat}_{rs}]$ using a spanning set labelled by triples $(\lambda, (\mu^1, \ldots, \mu^m), \alpha)$, where λ is a partition of $t = |\nu|$ of length $\leq s$, $(\mu^1, \ldots, \mu^m)$ is an m-tuple of partitions with μ^j of length $\leq r_j$ and $|\mu^1| + \cdots + |\mu^m| = t$, and where $\alpha : E_{(\mu^1,\ldots,\mu^m)} \to \lambda$ goes through a set of admissible representatives for the special tableaux of shape $E_{(\mu^1,\ldots,\mu^m)}$ and weight λ. These triples are in one-to-one correspondence with the triples $(P, (\mu^1, \ldots, \mu^m), (\alpha_1, \ldots, \alpha_m))$, where P is an ordered tableau of weight ν, $(\mu^1, \ldots, \mu^m)$ is an m-tuple of partitions with μ_j of length $\leq r_j$ and $|\mu^1| + \cdots + |\mu^m| = t = |\nu|$, and each $\alpha_j : \mu_j \to P^{-1}(j)$ goes through a set of admissible representatives for the special tableaux of shape μ_j and weight the tuple of row lengths of $P^{-1}(j)$. The filtration spaces are spanned by twisted bideterminants $(S\,|_{P,Q,\alpha}\,T)$, where S is of shape $E_{(\mu^1,\ldots,\mu^m)}$ with entries $\leq r$, satisfying $S^{-1}((\sum_{l=1}^{j-1} r_l + \{1, \ldots, r_j\}) = \mu^j \subseteq E_{(\mu^1,\ldots,\mu^m)}$ for all j, T is of shape λ with entries $\leq s$ and $\alpha : E_{(\mu^1,\ldots,\mu^m)} \to \lambda$ is as above.

2.3 Highest weight vectors for the conjugation action of GL_n on polynomials

Firstly, let us introduce some further notation. For n a natural number and λ, μ partitions with $l(\lambda)+l(\mu)\le n$, define the descending n-tuple

$$[\lambda,\mu] := (\lambda_1,\dots,\lambda_{l(\lambda)},0,\dots,0,-\mu_{l(\mu)},\dots,-\mu_1).$$

The group GL_n acts on Mat_n via the conjugation action, given by $S\cdot A = SAS^{-1}$ and therefore on the coordinate ring $k[\mathrm{Mat}_n]$ via $(S\cdot f)(A) = f(S^{-1}AS)$. Note that the nilpotent cone $\mathcal{N}_n = \{A\in \mathrm{Mat}_n \,|\, A^n = 0\}$ is under this action a GL_n-stable closed subvariety of Mat_n. We denote the algebra of invariants of $k[\mathrm{Mat}_n]$ under the action of GL_n by $k[\mathrm{Mat}_n]^{\mathrm{GL}_n}$, as in Chapter 1. It is well-known that this is the polynomial algebra in the traces of the exterior powers of the matrix.

Now let r,s be integers ≥ 0 with $r+s\le n$. We let $\mathrm{GL}_r\times\mathrm{GL}_s$ act on $k[\mathrm{Mat}_{rs}^m]$ as at the end of Section 2.2: we use the inverse rather than the transpose to define the action of GL_r. For a matrix M denote by $M_{r\rfloor\lfloor s}$ the lower left $r\times s$ corner of M. For m an integer ≥ 2 we define the map $\varphi_{r,s,n,m} : \mathrm{Mat}_n \to \mathrm{Mat}_{rs}^m$ by

$$\varphi_{r,s,n,m}(X) = \left(X_{r\rfloor\lfloor s}\,, \left(X^2\right)_{r\rfloor\lfloor s}\,,\dots,\left(X^m\right)_{r\rfloor\lfloor s}\right).$$

The restriction of this map to the nilpotent cone $\mathcal{N}_n$ will be denoted by the same symbol. In [26] the following result was proved.

Theorem 2.27 ([26, Thm. 1])**.** *Let $\chi = [\lambda,\mu]$ be a dominant weight in the root lattice, $l(\mu)\le r$, $l(\lambda)\le s$, $r+s\le n$. Then the pull-back map*

$$k[\mathrm{Mat}_{rs}^{n-1}]^{U_r\times U_s}_{(-\mu^{\mathrm{rev}},\lambda)} \to k[\mathcal{N}_n]^{U_n}_{\chi}$$

along $\varphi_{r,s,n,n-1} : \mathcal{N}_n \to \mathrm{Mat}_{rs}^{n-1}$ is surjective.

Combining this with Corollary 2.25 we obtain

Lemma 2.28. *Let $\chi = [\lambda, \mu]$ be a dominant weight in the root lattice, $l(\mu) \leq r$, $l(\lambda) \leq s$, $|\lambda| = |\mu| = t$, $r + s \leq n$. Then the pull-backs of the elements $(\tilde{S}_\mu \,|^m_{P,Q,\alpha}\, S_\lambda)$, ν, P, Q, α as in Corollary 2.25, along $\varphi_{r,s,n,n-1} : \mathcal{N}_n \to \mathrm{Mat}^{n-1}_{rs}$ span the vector space $k[\mathcal{N}_n]^{U_n}_\chi$.*

Next we recall the following instance of the graded Nakayama Lemma from [26].

Lemma 2.29 ([26, Lem. 1])**.** *Let $f_1, \ldots, f_l \in k[\mathrm{Mat}_n]^{U_n}_\chi$ be homogeneous. If the restrictions $f_1|_{\mathcal{N}_n}, \ldots, f_l|_{\mathcal{N}_n}$ span $k[\mathcal{N}_n]^{U_n}_\chi$, then $f_1, \ldots, f_l$ span $k[\mathrm{Mat}_n]^{U_n}_\chi$ as a $k[\mathrm{Mat}_n]^{\mathrm{GL}_n}$-module. The same holds with "span" replaced by "form a basis of".*

Combining Lemmas 2.28 and 2.29 we finally obtain

Theorem 2.30. *Let $\chi = [\lambda, \mu]$ be a dominant weight in the root lattice, $l(\mu) \leq r$, $l(\lambda) \leq s$, $|\lambda| = |\mu| = t$, $r + s \leq n$. Then the pull-backs of the elements $(\tilde{S}_\mu \,|^m_{P,Q,\alpha}\, S_\lambda)$, ν, P, Q, α as in Corollary 2.25, along $\varphi_{r,s,n,n-1} : \mathrm{Mat}_n \to \mathrm{Mat}^{n-1}_{rs}$ span the $k[\mathrm{Mat}_n]^{\mathrm{GL}_n}$-module $k[\mathrm{Mat}_n]^{U_n}_\chi$.*

Remarks 2.31. 1. Note that pulling the $(\tilde{S}_\mu \,|^m_{P,Q,\alpha}\, S_\lambda)$ back just amounts to interpreting $x(Q(a))_{ij}$ as the (i,j)-th entry of the $Q(a)$-th matrix power and replacing $r - a_1 + 1$ by $n - a_1 + 1$. In particular, these pulled-back functions don't depend on the choice of r and s.

2. One obtains a bigger, "easier" spanning set by allowing arbitrary P, Q of weight ν and arbitrary bijections $\alpha : \mu \to \lambda$ with $P \circ \alpha = Q$.

Chapter 3

Highest weight vectors for the symplectic group

Let k be an algebraically closed field, and throughout this chapter assume $\operatorname{char}(k) = 0$. Let Sp_n be the symplectic group, and $\mathfrak{sp}_n$ its Lie algebra. Let U_n denote the subgroup consisting of the unipotent elements in a Borel subgroup. We are interested in describing highest weight vectors in $k\left[\mathfrak{sp}_n\right]$ with respect to the adjoint action of Sp_n on $\mathfrak{sp}_n$ (given by $U.X = UXU^{-1}$). That is, we wish to give a finite spanning set, if not a basis, in terms of λ for the vector space $k\left[\mathfrak{sp}_n\right]_\lambda^{U_n}$ of the highest weight vectors of weight λ, as a module over the ring $k\left[\mathfrak{sp}_n\right]^{\mathrm{Sp}_n}$ of invariants. The aim is to use similar techniques as those used in Chapter 2 for the general linear group, that is, reducing the problem to describing highest weight vectors for an "easier" variety, and pulling these back along a morphism of varieties, applying the comorphism.

3.1 The symplectic group

Let m be a non-negative integer, $n = 2m$ and let J be an $n \times n$ matrix with $J^T = -J = J^{-1}$, e.g. $J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$. Let Sp_n denote the subgroup $\left\{U \in \mathrm{Mat}_n \,\middle|\, UJU^T = J\right\}$ of GL_n and $\mathfrak{sp}_n$ its Lie algebra, the space $\left\{X \in \mathrm{Mat}_n \,\middle|\, JX^T + XJ = 0\right\}$. Note that for $U \in \mathrm{Sp}_n$ we have $U^{-1} = -JU^TJ$ and for $X \in \mathfrak{sp}_n$ we have $X^T = JXJ \in \mathfrak{sp}_n$ and $J\left(X^i\right)^T - (-X)^i J = 0$ for $i \in \mathbb{N}$.

From now on, we will assume J to be the matrix $\begin{pmatrix} 0 & \Xi_m \\ -\Xi_m & 0 \end{pmatrix}$, where for a non-negative integer q, Ξ_q is the $q \times q$-matrix which has ones everywhere on the anti-diagonal and zeros everywhere else.

Let U_n denote the subgroup of Sp_n consisting of the upper uni-triangular symplectic matrices; for our choice of J we find that, as in Chapter 2, these will be the unipotent elements in a Borel subgroup of Sp_n. Similarly, let T_n again denote the subgroup of the diagonal matrices in Sp_n; these are the diagonal matrices $\begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix}$ that have $t_i = t_{n-i+1}^{-1}$ for all i, and this subgroup is the maximal torus in Sp_n.

The character group of this T_n is isomorphic to the set $\mathbb{Z}^m$ under component-wise addition, with a tuple $(a_1, \ldots, a_m)$ corresponding to the character $\begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \mapsto t_1^{a_1} \ldots t_m^{a_m}$. Since Sp_n is simply connected, we have that the character group equals the weight lattice. The root lattice is then the sublattice of the weights with even co-ordinate sum, that is,

$$\left\{ (a_1, \ldots, a_m) \middle| \sum_{i=1}^{m} a_i = 2j \text{ for some j} \in \mathbb{Z} \right\}.$$

3.1.1 Chevalley restriction theorem

We can describe the invariants in $k[\mathfrak{sp}_n]$ under the adjoint (that is, conjugation) action of Sp_n using the following result:

Theorem (Chevalley theorem)**.** *[2, Ch. VIII, 8.3] Let G be a Lie group whose Lie algebra $\mathfrak{g}$ is a semisimple Lie algebra over k, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathscr{W}$ the corresponding Weyl group. Then the restriction $k[\mathfrak{g}] \to k[\mathfrak{h}]$ gives an isomorphism*

$$k[\mathfrak{g}]^G \cong k[\mathfrak{h}]^W.$$

Now let $\mathfrak{h}$ be the subalgebra of diagonal matrices in $\mathfrak{sp}_n$, with a basis

$$\{H_i | H_i(i,i) = 1,\ H_i(n-i+1, n-i+1) = -1,\ H(j,d) = 0 \text{ otherwise}\}.$$

Then the algebra $k[\mathfrak{h}]^W$ is generated [2, Ch. VIII, 13.3.VI] by the algebraically independent elementary symmetric polynomials $e_1, \ldots, e_m$ in the squares of the functions $x_{ii} \in k[\mathfrak{h}]$ with $i \leq m$, where each x_{ij} sends a matrix to its (i,j)-th entry. For the matrix X of the functions x_{ij}, $1 \leq i,j \leq n$, denote the characteristic polynomial by

$$f(z) = \det(zI_n - X) = z^n + f_1(X)z^{n-1} + \cdots + f_{n-1}z + f_n;$$

then when restricted to $\mathfrak{h}$, the coefficients f_i for odd i become zero, while the restrictions of the even coefficients $f_2, f_4, \ldots, f_{n-2}, f_n$ are, up to sign, the aforementioned elementary symmetric polynomials in the squares of the x_{ii} for $i \leq m$.

So we have that the functions $f_2, \dots, f_n$ generate the algebra $k[\mathfrak{sp}_n]^{\mathrm{Sp}_n}$. And since elementary symmetric polynomials are algebraically independent, this gives us that $\{f_2, \dots, f_n\}$ is in fact an algebraically independent generating set of the algebra of invariant polynomials on the Lie algebra.

3.1.2 Reduction to the nilpotent cone

In general, for an algebra of invariants, if $k[\mathfrak{g}]^G = k[s_1, ..., s_q]$ for some family of functions $s_1, \dots, s_q \in k[\mathfrak{g}]$, then the vanishing ideal of the nilpotent cone $\mathcal{N}$ in the Lie algebra $\mathfrak{g}$ is (see [19, Sect. 7] and [25, Prop. 1(i)]) the ideal generated by the $s_1, \dots, s_q$. The co-ordinate algebra $k[\mathcal{N}]$ is then obtained from $k[\mathfrak{g}]$ by reducing the invariants to scalars: $k[\mathcal{N}] = k[\mathfrak{g}]/Mk[\mathfrak{g}] = k \otimes_{k[\mathfrak{g}]^G} k[\mathfrak{g}]$, where M is the maximal ideal of $k[\mathfrak{g}]^G$ generated by the s_i.

From now on, $\mathcal{N}$ will denote the nilpotent cone in $\mathfrak{sp}_n$, that is, the Sp_n-variety $\{X \in \mathfrak{sp}_n \,|X^n = 0\}$. A refinement of the above gives a useful result for the $k[\mathfrak{g}]^G$-modules of highest weight vectors, which we can apply to the symplectic case to obtain another instance of the Nakayama Lemma (compare Lemma 2.29 for the general linear case).

Lemma 3.1. *Let $s_1, \dots, s_q \in k[\mathfrak{sp}_n]_\lambda^{U_n}$ be homogeneous. If the restrictions $s_1|_{\mathcal{N}}, \dots, s_q|_{\mathcal{N}}$ span $k[\mathcal{N}]_\lambda^{U_n}$, then $s_1, \dots, s_q$ span $k[\mathfrak{sp}_n]_\lambda^{U_n}$ as a $k[\mathfrak{sp}_n]^{\mathrm{Sp}_n}$-module.*

After this reduction, we see that we only have to find homogeneous finite spanning sets for certain graded vector spaces, rather than for $k[\mathfrak{sp}_n]^{\mathrm{Sp}_n}$-modules.

3.1.3 Transmutation

The technique we will use to find the desired spanning sets will be *transmutation*, analogous to that used in [26], which was carried forward to Chapter 2. We will move from the conjugation action of the symplectic group Sp_n to a simpler action of GL_r for some $r \in \mathbb{Z}^+$ in such a way that spanning sets for the spaces of highest weight vectors on the transmuted variety, which are easy enough to find, map in a known way to spanning sets for the highest weight vectors in the variety we are interested in. The method derives from the following fact.

Lemma 3.2. *[26] Let G, H be reductive groups and let Y be an affine $G \times H$-variety such that $k[Y] = \bigoplus_{i \in I} L_i^* \otimes N_i$ where the L_i are mutually non-isomorphic G-modules and the N_i are mutually non-isomorphic H-modules. Then, if V is an affine G-variety, then $W = Y \times^G V := \mathrm{Spec}\left(k[Y \times V]^G\right)$ is an H-variety and the irreducible H-modules that show up in $k[W]$ are the N_i, with multiplicities the same as the corresponding L_i in $k[V]$.*

We recall the following from Chapter 2 and [26]. Let $G = \mathrm{GL}_n$ and $V = \mathcal{N}_{n,m} := \{A \in \mathcal{N}_n \,|A^{m+1} = 0\} \leq \mathrm{Mat}_n$. For the purpose of transmutation, choose $r, s \in \mathbb{Z}^+$ such that $r + s \leq n$ and then let $H = \mathrm{GL}_r \times \mathrm{GL}_s$ and $Y = \{(A, B) \in \mathrm{Mat}_{rn} \times \mathrm{Mat}_{ns} |\, AB = 0\}$, and we get

$$W \cong \left\{ (0, AXB, \ldots, AX^m B) \in \mathrm{Mat}_{rs}^{m+1} \middle|\, AB = 0,\, X \in \mathcal{N}_{n,m} \right\}.$$

In this chapter, we have instead $G = \mathrm{Sp}_n$. Fix $r \in \mathbb{Z}^+$ with $2r \leq n$, and let $H = \mathrm{GL}_r$. Define a map $Q : \mathrm{Mat}_{nr} \to \mathrm{Mat}_r$ by $A \mapsto A^T JA$ and let $Y := Q^{-1}(0)$ with a $G \times H$-action on Y given by $(U, V) \cdot A = UAV^T$. Then we have the following theorem from [15].

Theorem 3.3. *[15, Thm. 3.8.6.2], $k[Y]$ decomposes as a $G \times H$-module into*

$$\sum_{\lambda:\, l(\lambda) \leq r} \tau^\lambda \otimes \rho^\lambda$$

where the τ^λ and ρ^λ are the representations of Sp_n and GL_r respectively generated by the $\mathrm{Sp}_n \times \mathrm{GL}_r$-highest weight vectors

$$\delta^\lambda = \prod_{j=1}^{l(\lambda)} \delta_j^{\lambda_j - \lambda_{j+1}},$$

where for a (symmetric) matrix A with entries a_{tu}, δ_j denotes the j-th leading minor, that is, the function

$$A \mapsto \det\left(A_{\lfloor j}\right) = \det \begin{pmatrix} a_{11} & \cdots & a_{1j} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jj} \end{pmatrix}$$

Following the above result, we now have that in order to calculate finite spanning sets for the spaces of highest weight vectors, we need to describe the space $Y \times^{\mathrm{Sp}_n} \mathfrak{sp}_n$.

Definition 3.4. [23, Ch. 3] Let G be an algebraic group, X be an affine G-variety, and Z be an affine variety; give Z the trivial G-action. Then a G-invariant morphism $\varphi : X \to Z$ is a *quotient morphism* for G (also known as a G-quotient morphism) if φ induces an isomorphism $\mathrm{Im}\varphi \cong \mathrm{Spec}\left(k[X]^G\right)$, i.e. if the image of its co-morphism $\varphi^{\mathrm{co}} : k[Z] \to k[X]$ equals the ring of invariants $k[X]^G$. Note that the inclusion $\varphi^{\mathrm{co}}(k[Z]) \subseteq k[X]^G$ follows immediately from the fact that φ is G-invariant.

From the above definition comes the following fact.

Lemma 3.5. *Let G be an algebraic group and X an affine G-variety. A set of G-invariant functions $f_1, \ldots, f_s$ on X generate the ring $k[X]^G$ if and only if the map $\varphi : X \to k^r$, $x \mapsto (f_1(x), \ldots, f_s(x))$ is a quotient morphism.*

For $l < n$, let $\mathcal{N}_l$ denote the Sp_n-variety $\left\{X \in \mathcal{N} \,\middle|\, X^{l+1} = 0\right\}$ and let $\varphi : Y \times \mathcal{N}_l \to \mathrm{Mat}_r^l$ be the morphism $(A, X) \mapsto \left(A^T JXA, \ldots, A^T JX^l A\right)$, where the GL_r-action on Mat_r is given by $g \cdot A = gAg^T$. Since $\mathcal{N} \subseteq \mathfrak{sp}_n$, we have for $(A, X) \in Y \times \mathcal{N}_l$, $\left(A^T JX^i A\right)^T = A^T \left(X^T\right)^i J^T A = -A^T J \left(-X\right)^i A = (-1)^{i+1} A^T JX^i A$ and hence the image of the morphism φ lies inside the variety $M_1 \times \ldots \times M_l$, where $M_i = \{$symmetric $r \times r$-matrices$\}$ for i odd and $\{$skew symmetric $r \times r$-matrices$\}$ for i even.

Theorem 3.6. *Every polynomial function on r n-vectors $a_1, \ldots, a_r$ and one matrix X in the space $\mathfrak{sp}_n$ that is invariant under the action of* Sp_n *by conjugation, is a linear combination of the functions* $\mathrm{tr}\left(X^i\right)$ *with $0 < i < n$ and the functions picking out the entries of $A^T JX^i A$ with $i < n$ where A is the $n \times r$-matrix with the a_j as its columns.*

Proof. We have a spanning set for the multi-linear invariants from the first theorem in [24, Sect. 10]; then, as in [24, Thm. 1.3], since we can fully polarise an invariant to obtain a multi-linear one and then recover the original invariant by identifying the variables, this gives us that the spanning set for the multi-linear invariants will in fact cover all symplectic invariants. Then from this (see also its corollary [24, Thm. 10.1]) we get that the Sp_n-invariants of several vectors $a_1, \ldots, a_r$ and several matrices $X_1, \ldots, X_s \in \mathrm{Mat}_n$ are generated by the functions $\mathrm{tr}(M)$ and the functions picking out the entries of $A^T JMA$, where A is the $n \times r$-matrix with the vectors as its columns, and M is any monomial (including I_n) in the X_j and X_j^*, where for a matrix $B \in \mathrm{Mat}_n$, B^* is defined as $B^* := -JB^T J$. In our case we have only one matrix X, that is, $s = 1$, and furthermore we have that $X \in \mathfrak{sp}_n$, so $XJ = -JX^T$, hence $X^* = -X$. So, the M become simply monomials in X, that is, powers of X. Finally, if $X^n = 0$ then we can assume all the powers of X in the functions in generating set are strictly less than the dimension n, so at least this is true for $X \in \mathcal{N}$ by definition; however by the Cayley-Hamilton theorem, which states that every matrix is a root of its own characteristic polynomial, we can indeed neglect higher powers of any matrix X in the functions for the generating set. □

Corollary 3.7. *The above map φ is a quotient morphism for the Sp_n-action.*

Now consider the matrix $I \in \mathrm{Mat}_{nr}$ consisting of the $r \times r$ identity matrix in the first r rows followed by $n-r$ rows of zeros below. For any $X \in \mathrm{Mat}_n$, the product $I^T XI \in \mathrm{Mat}_r$ will be the upper-left $r \times r$ corner of X. So, because $r \leq m$, the matrix $I^T JI$ is the zero matrix, and hence $I \in Y$. We will consider φ to be a map $\mathcal{N}_l \to \mathrm{Mat}_r^l$ by letting

$$\varphi(X) = \varphi(I, X) = \Big((JX)_{\lfloor r}, \ldots, \big(JX^l\big)_{\lfloor r}\Big)$$

where, for a matrix B, $B_{\lfloor r}$ denotes the B-submatrix consisting of the upper-left $r \times r$ corner of B.

Theorem 3.8. *Let r, l be positive integers with $l < n$ and $2r < n$, then let λ be a partition with length $\leq r$, and let M_i, $1 \leq i \leq l$ be the subspaces of Mat_r consisting of symmetric matrices for i odd and skew symmetric matrices for i even. Then the pull-back map*

$$k[\mathrm{Mat}_r^l]_\lambda^{U_r} \to k[\mathcal{N}_l]_\lambda^{U_n}$$

along $\varphi : \mathcal{N}_l \to \mathrm{Mat}_r^l$ is surjective.

Proof. For an $n \times n$-matrix A, let $A_{\lfloor r}$ denote the $r \times r$-matrix consisting of the entries in the upper-left $r \times r$ corner of A. Then, if A is upper-triangular or upper uni-triangular in Mat_n, $A_{\lfloor r}$ will be upper-triangular or upper uni-triangular in Mat_r, and so we have

$$\varphi(AXA^T) = A_{\lfloor r}\varphi(X)A_{\lfloor r},$$

for all $X \in \mathfrak{sp}_n$. Hence, the highest weight vectors in $k[\mathrm{Mat}_r^l]$ will be mapped to highest weight vectors of the corresponding weights in $k[\mathcal{N}_l]$ by this pull-back map.

Furthermore, a consequence of Weyl's theorem on complete reducibility (that every finite-dimensional module over a semisimple Lie algebra over a field of characteristic zero is itself semisimple as a module) is that a surjection of G-modules will always induce a surjection on the U-invariants of any given weight, for U the unipotent elements in a Borel subgroup of G; hence, the restriction map from Mat_r^l to $\varphi(\mathcal{N}_l)$ induces a surjection from $k[\mathrm{Mat}_r^l]_\lambda^{U_r}$ to $k[\varphi(\mathcal{N}_l)]_\lambda^{U_r}$.

Let $I \in \mathrm{Mat}_{nr}$ be as above, and consider the orbit of I in Y under the $\mathrm{Sp}_n \times \mathrm{GL}_r$-action, which is the subset $\left\{ U_{n\lfloor r}V^T \,\middle|\, U \in \mathrm{Sp}_n, V \in \mathrm{GL}_r \right\}$, where for an $n \times n$-matrix B, $B_{n\lfloor r} \in \mathrm{Mat}_{nr}$ denotes the matrix consisting of the first r columns of B. It can be checked that an $n \times r$-matrix of the form $U_{n\lfloor r}V^T$ for U symplectic and V invertible itself consists of the first r columns of a symplectic matrix UV_n^T, where for an $r \times r$-matrix L, L_n denotes the element

$$\begin{pmatrix} L & 0 & 0 \\ & \vdots & \\ 0\cdots & I_{n-2r} & \cdots 0 \\ & \vdots & \\ 0 & 0 & (L^{-1})^T \end{pmatrix}$$

in Sp_n. So, the orbit is equal to $\left\{ U_{n\lfloor r} \,\middle|\, U \in \mathrm{Sp}_n \right\}$. Since $\mathrm{Sp}_n \times Y$ is an irreducible variety, and we have a surjective morphism $\mathrm{Sp}_n \times Y \to Y$ given by $(g, A) \mapsto gA$, it follows that Y is irreducible, and so the described orbit is dense in Y; therefore, the union of the $\mathrm{Sp}_n \times \mathrm{GL}_r$-conjugates of $\{I\} \times \mathcal{N}_l$ (which will have the same φ-image as $\{I\} \times \mathcal{N}_l$) is dense in $Y \times \mathcal{N}_l$. By the fact that φ is continuous, it then follows that the subset $\varphi(\mathcal{N}_l) = \varphi(\{I\} \times \mathcal{N}_l)$ is dense in $\varphi(Y \times \mathcal{N}_l)$, which means that we have $k[\varphi(\mathcal{N}_l)] = k[\varphi(Y \times \mathcal{N}_l)]$, where $k[S]$ denotes the subalgebra of restrictions of the polynomial functions on Mat_r^l to the subset S. So $k[\varphi(\mathcal{N}_l)]\lambda^{U_r} = k[\varphi(Y \times \mathcal{N}_l)]\lambda^{U_r}$. But by Theorem 3.3 and Lemma 3.2, the dimension of the space $k[\varphi(Y \times \mathcal{N}_l)]\lambda^{U_r}$ is the same as that of $k[\mathcal{N}_l]_\lambda^{U_n}$. So the pull-back map $k[\varphi(\mathcal{N}_l)]\lambda^{U_r} \to k[\mathcal{N}_l]\lambda^{U_n}$, which is clearly injective, is an isomorphism. Therefore, the pull-back map $k[\mathrm{Mat}_r^l]\lambda^{U_r} \to k[\mathcal{N}_l]\lambda^{U_n}$ is surjective. □

3.2 Spaces of highest weight vectors

The following subsection details results on multiplicity-free decompositions of, and highest weight vectors in, the spaces $k[M_1]$ and $k[M_2]$, cited primarily from [15]. These would be of interest in particular for defining bases for the spaces of highest weight vectors on the product $M_1 \times \ldots \times M_l$. In Subsection 3.2.2 however, we will return to the larger variety Mat_r^l instead, and thus we end up defining a somewhat larger spanning set when we pull the highest weight vectors back to the un-transmuted variety in the final subsection.

3.2.1 Symmetric and skew symmetric matrices

We have an inclusion $\mathrm{Mat}_r \hookleftarrow M_1 = \{\text{symmetric matrices}\}$, so the map $k\,[\mathrm{Mat}_r] \to k\,[M_1]$ is surjective. We have a GL_r-action on M_1 given by $g \cdot A = gAg^T$, and a corresponding one on $k\,[M_1]$. If we restrict the $\mathrm{GL}_r \times \mathrm{GL}_r$-action on Mat_r (given by $(g,h) \cdot A = gAh^T$) to the diagonal copy of GL_r in $\mathrm{GL}_r \times \mathrm{GL}_r$, then the surjection $k\,[\mathrm{Mat}_r] \to k\,[M_1]$ is equivariant.

In [15, Thm. 3.1], Howe gave a multiplicity-free decomposition

$$k\,[M_1] \cong \sum_{\lambda:\, l(\lambda)\leq r} \rho^{2\lambda}$$

for the GL_r-action given above, where for a partition μ, ρ^{μ} denotes the representation of highest weight μ, and the partition 2λ is created from the partition λ simply by doubling each integer in the tuple. In the proof of that theorem Howe shows that the highest weight vectors of these weights are the products of the of the leading minors δ_j for $1 \leq j \leq r$.

We recall the formula given in [26, Thm. 4] for the basis elements for the space of highest weight vectors in $k[\mathrm{Mat}_{rs}^m]$ of a given weight $(-\mu^{\mathrm{rev}}, \lambda)$. Under the $\mathrm{GL}_r \times \mathrm{GL}_r$-action above however, as in Chapter 2, we let the second GL_r act via the transpose rather than the inverse, which means that we can replace $-\mu^{\mathrm{rev}}$ by μ. Furthermore, since we are now considering the variety $k[\mathrm{Mat}_r]$, we set $s = r$, and so the weights are of the form (λ, λ), that is we also set $\mu = \lambda$. Then, putting the m from that paper to 1, the only tableau $\lambda \to \mathbb{N}$ applicable is the one that is constant equal to 1, hence $P = Q$, $\alpha = \mathrm{id}$ and these can all be omitted from the notation, and we find that the basis reduces to the single element

$$u_\lambda = \sum_{\pi,\sigma\in C_\lambda} \mathrm{sgn}\,(\pi)\,\mathrm{sgn}\,(\sigma) \prod_{a\in\lambda} x_{\pi(a)_1,\sigma(a)_1},$$

where C_λ is the column stabiliser of λ (as a Young diagram) and where b_1 is the row-index of a box b in λ; that is, this space of highest weight vectors is one-dimensional. Since $A \in T_r$ if and only if $(A, A) \in T_r \times T_r$ and $B \in U_r$ if and only if $(B, B) \in U_r \times U_r$, the highest weight vectors for the action of

GL_r on $k[M_1]$ can be the obtained by restricting to M_1 certain highest weight vectors for the $\mathrm{GL}_r \times \mathrm{GL}_r$-action on $k[\mathrm{Mat}_r]$. A highest weight vector with weight (λ, λ) in the full action will have weight 2λ for the restricted action: if $(A, A).f = \lambda(A)\lambda(A)f$ for all $A \in T_r$ then $A.\varphi(f) = (2\lambda)(A)\varphi(f)$ for all $A \in T_r$ $((A, A).f : X \mapsto f\left(A^T X A\right) = (\lambda(A))^2 f(X) = (2\lambda)(A)f(X))$. Therefore, the highest weight vectors are the scalar multiples of the element u_λ, restricted to the symmetric matrices, for each weight 2λ, λ a partition with $l(\lambda) \leq r$. Furthermore, the multiplicity-free decomposition stated above implies that all the highest weight vectors can be obtained in this way.

Now let i be even so that $M_i = \{\text{skew symmetric matrices}\}$ (we can denote this simply by M_2). In this case we can follow a similar procedure as above with symmetric matrices. The decomposition for this action was given in [15, Thm. 3.8.1]:

$$k[M_2] \cong \sum_{\lambda} \rho^{(2\lambda)^T},$$

where the sum is now over diagrams with row lengths $\leq \frac{r}{2}$, and where μ^T represents the transpose of a diagram μ. The highest weight vectors we are looking for here, again given in the discussion of this theorem in [15], are the products of the Pfaffians of the leading $j \times j$-minors for $j \leq r$ even. The Pfaffian of a skew symmetric $r \times r$-matrix A is defined in [27] as

$$\mathrm{pf}(A) = \frac{1}{\left(\frac{r}{2}\right)!\, 2^{r/2}} \sum_{\sigma \in \mathfrak{S}_r} \mathrm{sgn}(\sigma) \prod_{i=1}^{r/2} a_{\sigma(2i-1),\sigma(2i)},$$

where a_{tu} are the entries of A. It is well-known that the Pfaffian of an even skew symmetric matrix is the square-root of its determinant.

3.2.2 The highest weight vectors in the coordinate ring $k[\mathrm{Mat}_r^l]$ as a direct sum of coinvariant spaces

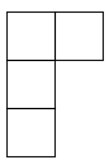
From now on, let $\mathrm{Sym}_r := M_1$ denote the set of $r \times r$-symmetric matrices, and $\mathrm{Skew}_r := M_2$ denote the set of $r \times r$-skew symmetric matrices. For H a group and V an H-module, let V_H denote the set of coinvariants, that is, the quotient of V by the span of $\{gx - x | g \in H, x \in V\}$. Now let $V = k^r$, then we can consider both the spaces $\mathrm{Sym}_r \cong S^2V$ and $\mathrm{Skew}_r \cong \bigwedge^2 V$ as spaces of coinvariants of the tensor square $V \otimes V$ by slightly different actions of the group $\mathbb{Z}_2 \cong \{1, \sigma\}$. In particular, we have $S^2V \cong (V \otimes V)_{\mathbb{Z}_2}$ with the action given by $\sigma.(u \otimes v) = v \otimes u$, and $\bigwedge^2 V \cong (V \otimes V)_{\mathbb{Z}_2}$ with the action given by $\sigma.(u \otimes v) = -v \otimes u$.

Now fix t a non-negative integer and λ a partition of t. Denote by T_λ the standard t-tableau of shape λ, with the numbers $1, \dots, t$ entered into the boxes of λ in order from left to right and then top to bottom.

Definition 3.9. Let $A = k\mathfrak{S}_t$, and let R_λ, $C_\lambda \subseteq \mathfrak{S}_t$ denote the row-stabiliser and column-stabiliser of λ respectively (or of T_λ, to continue the use of a tableau definition of column stabilisers) as in the previous chapters. Denote the column anti-symmetriser of λ in A by $e_{1,\lambda}$. The *Young symmetriser* of λ is the element

$$e_\lambda := \left(\sum_{\pi \in R_\lambda} \pi \right) . \left(\sum_{\sigma \in C_\lambda} \mathrm{sgn}(\sigma).\sigma \right),$$

and the module Ae_λ is called the *Specht module* associated to λ.

Example 3.10. The Young symmetriser associated to the diagram  is the element

$$\begin{aligned} e_{(2,1,1)} &= (12).\left((134)+(143)-(13)-(14)-(34)\right) \\ &= (1342)+(1432)-(132)-(142)-(12)(34) \end{aligned}$$

of $k\mathfrak{S}_4$.

Definition 3.11. Let T be a tableau of shape λ, and Ae_λ the Specht module of λ. Then the *polytabloid* associated to T is the element $[T] \in Ae_\lambda$ given by $[T] := g_T e_\lambda$ where g_T is the unique element of A satisfying $g_T \circ T_\lambda = T$.

Remark 3.12. It is a well-known result (see e.g. [17, Thm. 8.4] for a proof) that for a (skew) diagram E, the polytabloids $[T]$, with T a standard tableau of shape E, form a basis of the (skew) Specht module Ae_E.

Considering the GL_r-action on Mat_r that uses the transpose rather than the inverse, we have the GL_r-module isomorphisms $\mathrm{Mat}_r \cong V \otimes V \cong V^* \otimes V^*$. This gives

$$k\left[\mathrm{Mat}_r^l\right] \cong \bigoplus_{t\geq 0} S^t\left((V\otimes V)^l\right).$$

Now for t and l integers, define as before $\Sigma_t = \{(\nu_i)_{1\leq i\leq l} \in \mathbb{Z}_{\geq 0} | \Sigma_i \nu_i = t\}$, then for $\nu \in \Sigma_t$ define $S^\nu(U) := S^{\nu_1}(U) \otimes \ldots \otimes S^{\nu_l}(U)$. Then we have:

$$\begin{aligned} k\left[\mathrm{Mat}_r^l\right] &\cong \bigoplus_{\substack{t\geq 0\\ \nu\in\Sigma_t}} S^\nu(V\otimes V) \\ &= \bigoplus_{\substack{t\geq 0\\ \nu\in\Sigma_t}} \left(V^{\otimes 2t}\right)_{\mathfrak{S}_\nu}, \end{aligned}$$

where for $\nu \in \Sigma_t$, $\mathfrak{S}_\nu$ can be embedded in $\mathfrak{S}_{2t}$ in the following way. If σ is a permutation in $\mathfrak{S}_t$, then we can consider a corresponding permutation $\sigma' \in \mathfrak{S}_{2t}$ given by $\sigma'(2i-1) = 2\sigma(i)-1$, $\sigma'(2i) = 2\sigma(i)$, for $1 \leq i \leq t$; the group $\mathfrak{S}_\nu \leq \mathfrak{S}_{2t}$ is then defined as the subgroup consisting of all such permutations σ' with $\sigma \in \mathfrak{S}_\nu \leq \mathfrak{S}_t$.

Note that all weights λ of the maximal toral subgroup of diagonal matrices of GL_r on $k[\mathrm{Mat}_r^l]$ must have even coordinate sum; that is, considering weights as Young diagrams, the weights that appear will necessarily be diagrams with an even number of boxes. Indeed the weights of the matrix coordinates all have coordinate sum 2. So, let $\lambda \vdash 2t$ be a weight, and then by the above we have that the space $k\left[\mathrm{Mat}_r^l\right]_\lambda^{U_r}$ of highest weight vectors of weight λ is isomorphic to

$$\bigoplus_{\nu \in \Sigma_t} \left(\left(V^{\otimes 2t}\right)_\lambda^{U_r}\right)_{\mathfrak{S}_\nu},$$

where $(V^{\otimes 2t})_\lambda^{U_r}$ is isomorphic to the Specht module $k\mathfrak{S}_{2t}e_\lambda$, via the map $[T] \mapsto v_T e_{1,T}$ between basis elements, where for a tableau T, v_T is the tensor in $V^{\otimes 2t}$ with v_i in the positions that occur as entries in the i-th row of T, and $e_{1,T}$ is the column anti-symmetriser associated to T. Then a finite spanning set for the Specht module, when pulled back along this isomorphism and then mapped into the space of coinvariants, will provide a finite spanning set for that space. By Remark 3.12, the set of polytabloids corresponding to standard tableaux form a basis of the Specht module. This has led to the following result.

Theorem 3.13. *For an element $v_T e_{1,T}$ in the Specht module and $\mathfrak{S}_\nu \leq \mathfrak{S}_{2t}$ a Young subgroup, denote the canonical image in the set of coinvariants by $\pi_{\mathfrak{S}_\nu}(v_T e_{1,T})$. Then we have that the set*

$$\left\{ \sum_{\nu \in \Sigma_t} \pi_{\mathfrak{S}_\nu}(v_T e_{1,T}) \,\middle|\, T \text{ standard tableau of shape } \lambda \right\}$$

of sums of canonical images of such elements for the standard tableaux T of shape λ, for varying $\mathfrak{S}_\nu$, is a spanning set for the space of highest weight vectors *of weight λ.*

3.2.3 Highest weight vectors in $k\left[\mathfrak{sp}_n\right]$

Now we only need to translate the above results back to the un-transmuted variety $\mathfrak{sp}_n$ to get the highest weight vectors for the conjugation action of Sp_n.

Let λ be a dominant weight in the root lattice, that is, a partition with $l(\lambda) \leq m$ and even co-ordinate sum, then let $t = \frac{1}{2}\sum_{i=1}^{l(\lambda)} \lambda_i$, $\Sigma_t = \{(\nu_i)_{1\leq i\leq n-1} \in \mathbb{Z}_{\geq 0} | \Sigma_i \nu_i = t\}$ and $r \in \mathbb{Z}^+$ with $l(\lambda) \leq r \leq m$, with e_λ and $\mathfrak{S}_\nu \leq \mathfrak{S}_{2t}$ as above, and let $\pi_{\mathfrak{S}_\nu}(f)$ denote the canonical image of an element $f \in k\mathfrak{S}_{2t}e_\lambda$ of the Specht module in the set of coinvariants with respect to the subgroup $\mathfrak{S}_\nu$. Then we have the following theorem.

Theorem 3.14. *The pull-backs along the map* $X \mapsto \left((JX)_{\lfloor r}, \ldots, (JX^{n-1})_{\lfloor r}\right)$, $\mathrm{Mat}_n \to \mathrm{Mat}_r^{n-1}$ *of the functions* $\sum_{\nu\in\Sigma_t} \pi_{\mathfrak{S}_\nu}(v_T e_{1,T})$*, with* T *a standard tableau of shape* λ*, form a spanning set for the space* $k[\mathfrak{sp}_n]_\lambda^{U_n}$ *of highest weight vectors for the conjugation action of* Sp_n *on* $\mathfrak{sp}_n$*, as a* $k[\mathfrak{sp}_n]^{\mathrm{Sp}_n}$*-module.*

Proof. The result follows directly from Theorem 3.8, Lemma 3.1 and from the highest weight vectors for the transmuted variety given in Theorem 3.13. □

Bibliography

[1] K. Akin, D. A. Buchsbaum, J. Weyman, *Schur functors and Schur complexes*, Adv. Math. **44** (1982), no. 3, 207-278.

[2] N. Bourbaki, *Lie groups and Lie algebras, Chapters 7-9*, Springer, 2005.

[3] M. Clausen, *Letter place algebras and a characteristic-free approach to the representation theory of the general linear and symmetric groups. I.*, Adv. Math. **33** (1979), no. 2, 161-191.

[4] M. Clausen, *Letter place algebras and a characteristic-free approach to the representation theory of the general linear and symmetric groups. II.*, Adv. Math. **38** (1980), no. 2, 152-177.

[5] C. De Concini, D. Eisenbud, C. Procesi, *Young diagrams and determinantal varieties*, Invent. Math. **56** (1980), no. 2, 129-165.

[6] A. D. J. Dent, R. H. Tange, *Bases for spaces of highest weight vectors in arbitrary characteristic*, 2017, to appear in Algebras and Representation Theory (https://doi.org/10.1007/s10468-018-9815-3).

[7] J. Désarménien, J. P. S. Kung, G-C. Rota, *Invariant theory, Young bitableaux, and combinatorics*, Adv. Math. **27** (1978), no. 1, 63-92.

[8] I. F. Donin, *Decompositions of tensor products of representations of a symmetric group and of symmetric and exterior powers of the adjoint representation of* $\mathfrak{gl}(N)$, Soviet Math. Dokl. **38** (1989), no. 3, 654-658.

[9] S. Donkin, *Skew modules for reductive groups*, J. Algebra **113** (1988), no. 2, 465-479.

[10] S. Donkin, *On tilting modules for algebraic groups*, Math. Z. **212** (1993), no. 1, 39-60.

[11] W. Fulton, J. Harris, *Representation theory: A first course*, Springer, 1991.

[12] J. A. Green, *Polynomial representations of* GL_n, Second corrected and augmented edition, Lecture Notes in Mathematics **830**, Springer, Berlin, 2007.

[13] R. Goodman, N. R. Wallach, *Symmetry, representations, and invariants*, Springer, 2009.

[14] B. Hall, *Lie groups, Lie algebras, and representations: an elementary introduction*, Springer, 2003.

[15] R. Howe, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, Israel Mathematical Conference proceedings **8** (1995), 1-182.

[16] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer, 1972.

[17] G. D. James, *The representation theory of the symmetric groups*, Lecture Notes in Mathematics **682**, Springer, Berlin, 1978.

[18] G. D. James, M. H. Peel, *Specht series for skew representations of symmetric groups*, J. Algebra **56** (1979), no. 2, 343-364.

[19] J. C. Jantzen, *Nilpotent orbits in representation theory*, Lie theory, 1-211, Progr. Math., 228, Birkhäuser Boston, Boston, MA, 2004.

[20] J. C. Jantzen, *Representations of algebraic groups*, Second edition, American Mathematical Society, Providence, RI, 2003.

[21] F. Kouwenhoven, *Schur and Weyl functors*, Adv. Math. **90** (1991), no. 1, 77-113.

[22] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Second edition, Oxford University Press, New York, 1995.

[23] P. E. Newstead, *Lectures on Introduction to moduli problems and orbit spaces*, Springer-Verlag, 1978.

[24] C. Procesi, *Invariant theory of* $n \times n$ *matrices*, Adv. Math. **19** (1976), 306-381.

[25] R. H. Tange, *Highest weight vectors for the adjoint action of* GL_n *on polynomials*, Pac. J. Math. **258** (2012), no. 2, 497-510.

[26] R. H. Tange, *Highest weight vectors and transmutation*, 2015, to appear in Transformation Groups (https://doi.org/10.1007/s00031-018-9474-9).

[27] H. Weyl, *The classical groups: Their invariants and representations*, Princeton University Press, 1946.

[28] A. V. Zelevinsky, *A generalization of the Littlewood-Richardson rule and the Robinson-Schensted-Knuth correspondence*, J. Algebra **69** (1981), no. 1, 82-94.

[29] A. V. Zelevinsky, *Representations of finite classical groups. A Hopf algebra approach*, Lecture Notes in Mathematics **869**, Springer-Verlag, Berlin-New York, 1981.