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Published in: European Journal of Control

DOI: 10.1016/j.ejcon.2019.06.001

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Document Version Publisher's PDF, also known as Version of record

Publication date: 2019

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA): Cheng, X., Scherpen, J. M. A., & Zhang, F. (2019). Model reduction of synchronized homogeneous Lur'e networks with incrementally sector-bounded nonlinearities. *European Journal of Control, 50*, 11-19. https://doi.org/10.1016/j.ejcon.2019.06.001

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European Journal of Control



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Model reduction of synchronized homogeneous Lur'e networks with incrementally sector-bounded nonlinearities^{*}



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ARTICLE INFO

Article history: Received 13 September 2018 Revised 4 April 2019 Accepted 1 June 2019 Available online 6 June 2019

Recommended by T. Parisini

Keywords: Model reduction Lur'e system Network synchronization Balanced truncation

ABSTRACT

This paper proposes a model order reduction scheme that reduces the complexity of diffusively coupled homogeneous Lur'e systems. We aim to reduce the dimension of each subsystem and meanwhile preserve the synchronization property of the overall network. Using the Laplacian spectral radius, we characterize the robust synchronization of the Lur'e network by a linear matrix inequality (LMI), whose solutions then are treated as generalized Gramians for the balanced truncation of the linear component of each Lur'e subsystem. It is verified that, with the same communication topology, the resulting reduced-order network system is still robustly synchronized, and an *a priori* bound on the approximation error is guaranteed to compare the behaviors of the full-order and reduced-order Lur'e subsystems.

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1. Introduction

Nowadays, booming technologies such as the Internet of things are connecting an enormous number of industrial robots, home appliances, and electronic products embedded with sensors and controllers. There is a clear trend that future systems are becoming more interconnected and complex. In the system and control community, the research on network systems, or multi-agent systems, has received compelling attention [7,29,33]. Such systems are composed of multiple interacting dynamical agents, and each agent is recognized as a subsystem whose input depends on the outputs of its neighboring agents. Thus, the behavior of a network system is determined not only by individual subsystems but also by the way how they are interconnected. To capture the overall behavior of a complex network, a high-dimensional differential model is usually required, which however is difficult for prediction, transient analysis, and controller design, etc.. Therefore, this paper investigates a model reduction technique for network systems,

which aims to generate smaller-sized models to approximate the input-output relation of the original ones.

There are a variety of techniques for dimension reduction of linear or nonlinear systems. These techniques can be roughly classified into two categories: Krylov-subspace methods (also known as moment matching) and singular value decomposition (SVD) based approaches [1]. The schemes in the first category can be found in e.g., [2,3,21,23]. However, these methods generally do not guarantee the stability of the reduced-order model and the bound on the approximation error. In contrast, the techniques in the latter category, based on theories of balancing and Hankel operator, are well-known for their properties of stability preservation and error boundedness, see e.g., [1,18,20,27,32] for an overview on stable linear systems. In the linear case, the controllability and observability energy functionals of the system are analyzed, whose concepts are then extended to nonlinear balancing, see [5,17,35] and the references therein. However, implementing nonlinear balancing is rather expensive, as it requires the solutions of nonlinear partial differential equations, namely the Hamilton-Jacobi equations. Furthermore, as the other methods for model reduction of nonlinear systems, the truncated model from nonlinear balancing lacks an error bound on the approximation.

For a network, the complexity can be cut down from two directions. The first one is to reduce the number of agents in the network. Typical methods are based on graph clustering and generalized balanced truncation, see e.g., [6,11–13,24,30]. The other direction is to reduce the dimension of each subsystem,

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^{*} The work of F. Zhang was partly supported by the State Key Laboratory of Intelligent Control and Decision of Complex Systems, the National Natural Science Foundation of China (Grant No. 61703099), and the China Postdoctoral Science Foundation (Grant No. 2017M621589).

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https://doi.org/10.1016/j.ejcon.2019.06.001

which is of particular interest in this paper. Preliminary results in [31,34] provide methods to reduce network systems composed of linear subsystems. Especially, in [31], the stability and synchronization property of the overall network are preserved after reducing the dimension of each subsystem. In [12], networked passive linear systems are considered, and generalized balanced truncation is applied to reduce the subsystem dynamics and the network topology simultaneously.

In this paper, we consider nonlinear network systems, whose subsystems are identical and cast in the Lur'e-type form. Note that Lur'e systems refer to an important class of nonlinear systems that can be represented as a feedback connection of a linear dynamical system and a nonlinear element [8,25]. Examples can be found in e.g., Chua's circuits, robotic arms with flexible joints, and some hyper-chaotic systems, where the nonlinearity of the control system is in the form of a relay or actuator/sensor nonlinearity. Consensus networked Lur'e systems are modeled when we consider e.g., interconnected robotic arms or Chua's circuits, see the applications in e.g., [14,15,38]. For a Lur'e system, one may apply linear model reduction techniques to the linear component such that both stability and the error bound for the reduced-order model are guaranteed. A pioneering work can be found in [4], where balanced truncation is suggested to reduce the linear part of an absolutely stable Lur'e system. However, for networked Lur'e systems, directly applying this method to each Lur'e subsystem may lose the properties of the overall network, such as the robust synchronization property. In the preliminary version of this work [10], we provide a method based on generalized balanced truncation for reducing Lur'e subsystems with slope-restricted nonlinearity. The current paper, compared to [10], provides two major improvements. First, we consider a more generic nonlinearity, i.e., the incrementally sector-bounded nonlinear feedback in the Lur'e subsystems. This extension results in a different synchronization characterization of Lur'e networks as well as different approximation error analysis of the Lur'e subsystems. Second, this paper provides a different strategy for balancing the linear part of the Lur'e subsystems. Instead of using the minimum and maximum solution of an LMI as generalized Gramians, we balance the subsystem using only one solution of the LMI and a standard Gramian. This strategy potentially provides a lower bound on the approximation error.

This paper is organized as follows. Section 2 introduces the necessary preliminaries of the generalized balanced truncation method and provides a sufficient condition of the robust synchronization of a Lur'e network with incrementally sector-bounded nonlinearities. Section 3 then presents the main result of the paper, which describes the method for synchronization preserving model reduction of Lur'e networks. Section 4 analyzes the approximation error on each subsystem, and Section 5 illustrates the proposed method by an example. Finally, Section 6 concludes the paper.

Notation: The symbol \mathbb{R} denotes the set of real numbers, and \mathbb{R}_+ denotes the set of nonnegative real numbers. I_n represents the identity matrix of size n, and $\mathbf{1}_n$ represents a vector in \mathbb{R}^n of all ones. A symmetric matrix $A > \mathbf{0}$ ($A < \mathbf{0}$) means it is positive (negative) definite, while $A \ge \mathbf{0}$ ($A \le \mathbf{0}$) means it is positive (negative) semidefinite. The trace and of A is denoted by trace(A). The Kronecker product of matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ is denoted by $A \otimes B \in \mathbb{R}^{mp \times nq}$. Furthermore, the \mathcal{L}_2 -norm of a signal $u(t) : [\mathbf{0}, \infty) \mapsto \mathbb{R}^n$ is defined by

$$||u(t)||_2 = \left(\int_0^\infty u(t)^T u(t) dt\right)^{\frac{1}{2}},$$

and the \mathcal{H}_{∞} -norm of a transfer function G(s) is denoted by $\|G(s)\|_{\mathcal{H}_{\infty}} = \sup_{\omega \in \mathbb{R}} \tilde{\sigma}[G(j\omega)],$

where $\bar{\sigma}$ denotes the largest singular value, and *j* is the imaginary unit.

2. Preliminaries

2.1. Generalized balanced truncation

From [1,16], we recapitulate some basic facts on model reduction by using generalized balanced truncation. Consider a linear time-invariant system in a state space representation

$$\Sigma:\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases}$$
(1)

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, and $C \in \mathbb{R}^{q \times n}$, whose transfer function is given by $G(s) := C(sI_n - A)^{-1}B$. Suppose the system Σ is asymptotically stable and minimal, namely, A is Hurwitz, the pair (A, B) is controllable, and the pair (C, A) is observable. Note that for a system (1) that is not minimal, we can always eliminate the uncontrollable or unobservable states in the model (1) such that a minimal state-space realization is achieved with an equivalent transfer function as G(s).

For such a system Σ , a pair of appropriately chosen positive definite matrices, *P* and *Q*, are called the *generalized controllability and observability Gramians*, respectively, if they satisfy

$$AP + PA^T + BB^T \le \mathbf{0},\tag{2a}$$

$$A^T Q + Q A + C^T C \le \mathbf{0}. \tag{2b}$$

Balancing the system in (1) amounts to find a nonsingular matrix $T \in \mathbb{R}^{n \times n}$ such that *P* and *Q* are simultaneously diagonalized in the following way:

$$TPT^{T} = T^{-T}QT^{T} = \Sigma, (3)$$

where $\Sigma := \operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$, and the diagonal entries $\sigma_1 \ge \sigma_2 \ge \dots \ge \sigma_n > 0$ are called the *generalized Hankel singular values* (GHSVs) of the system Σ . Using *T* as a coordinate transformation, we obtain the balanced realization of Σ , in which the state components corresponding to the smaller GHSVs are relatively difficult to reach and observe and thus have less influences on the input–output behavior. Denote $\hat{\Sigma}$ with dimension *r* (*r* \ll *n*) as a reduced-order model, which is acquired by truncating the states with smallest GHSVs in the balanced system. Then, the upper bound of the model reduction error can be measured by the neglected GHSVs, i.e.,

$$\|\mathbf{\Sigma} - \mathbf{\hat{\Sigma}}\|_{\mathcal{H}_{\infty}} \le 2\sum_{i=r+1}^{n} \sigma_{i},\tag{4}$$

which is *a priori* error bound on the approximation. Moreover, the reduced order model $\hat{\Sigma}$ is also asymptotically stable.

2.2. Graph theory

The interconnection topology of a network can be represented by a weighted graph \mathcal{G} that consists of a finite and nonempty node set $\mathcal{V} := \{1, 2, ..., N\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Then, the weighted adjacency matrix \mathcal{W} associated with the graph \mathcal{G} is defined such that the (i, j)th entry of \mathcal{W} , denoted by w_{ij} , is positive if there exists a directed edge from node j to node i, i.e., $(j, i) \in \mathcal{E}$, and $w_{ij} = 0$ otherwise. The following definitions are provided [19,29].

The graph \mathcal{G} is *undirected* if $w_{ij} = w_{ij}$, $\forall i, j \in \mathcal{V}$. An undirected graph \mathcal{G} is *simple*, if \mathcal{G} does not contain self-loops (i.e., $w_{ii} = 0$, $\forall i \in \mathcal{V}$), and there exists only one undirected edge between any two distinct nodes. Furthermore, an undirected path connecting nodes i_0 and i_n is a sequence of undirected edges of the form $(i_{k-1}, i_k), k = 1, ..., n$. Then, an undirected graph \mathcal{G} is *connected* if there is an undirected path between any pair of distinct nodes.



Fig. 1. The illustration of a Lur'e subsystem.

The Laplacian matrix $L \in \mathbb{R}^{N \times N}$ of the graph \mathcal{G} then is introduced, whose (i, j)th entry is given by

$$L_{ij} = \begin{cases} \sum_{j=1, j \neq i}^{n} w_{ij}, & i = j \\ -w_{ij}, & \text{otherwise.} \end{cases}$$
(5)

Note that if \mathcal{G} is an undirected connected simple graph, the associated Laplacian matrix *L* is symmetric and positive semidefinite, whose nullspace is characterized by $\operatorname{span}(\mathbf{1}_N)$. Denote $0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_N$ as the eigenvalues of *L*, where the largest eigenvalue λ_N is called the *Laplacian spectral radius* of \mathcal{G} .

2.3. Lur'e networks

In this subsection, we present the dynamical model of a Lur'e network. The nodal dynamics in such a network are described by identical nonlinear Lur'e-type systems as illustrated in Fig. 1, which is a feedback connection of a linear dynamical system and a nonlinear element. The state space model of each Lur'e subsystem is given by

$$\Sigma_{i}:\begin{cases} \dot{x}_{i} = Ax_{i} + B(u_{i} + z_{i}) \\ y_{i} = Cx_{i} \\ z_{i} = -\phi(y_{i}) \end{cases}, \quad i = 1, 2, \cdots, N,$$
(6)

where $x_i \in \mathbb{R}^n$ is the state vector of node *i*, and $u_i, y_i, z_i \in \mathbb{R}^m$ represent the input, output and internal feedback signal. We denote the linear part of the Lur'e subsystem as

$$\boldsymbol{\Sigma}_{i}^{lin}:\begin{cases} \dot{x}_{i}=Ax_{i}+B\tilde{u}_{i}\\ y_{i}=Cx_{i} \end{cases}, \quad i=1,2,\ldots,N,$$

$$(7)$$

with $\tilde{u}_i := u_i + z_i$.

Assumption 1. We assume that the linear system \sum_{i}^{lin} is asymptotically stable and minimal, i.e., the matrix *A* is Hurwitz, and (*A*, *B*) is controllable, (*A*, *C*) is observable.

The uncertain feedback nonlinearity $\phi(\cdot) : \mathbb{R}^m \mapsto \mathbb{R}^m$ in (6) is memoryless, possibly time-varying, and locally Lipschitz in y_i . The following assumption is made for the nonlinear component $\phi(\cdot)$.

Assumption 2. The nonlinear function $\phi(\cdot)$ is incrementally sectorbounded within the sector $[S_1, S_2]$, with $S_1, S_2 \in \mathbb{R}^{m \times m}$ and $S_2 > S_1$. More precisely, $\phi(\cdot)$ satisfies

$$[\phi(y_i) - \phi(\tilde{y}_i) - S_1(y_i - \tilde{y}_i)]^T [\phi(y_i) - \phi(\tilde{y}_i) - S_2(y_i - \tilde{y}_i)] \le 0,$$
(8)

for all $y_i, \tilde{y}_i \in \mathbb{R}^m$, where $\phi(0) = 0$.

We refer to [9,38] and the references therein for the definitions of incremental sector-boundedness. For the SISO case, i.e. $u_i, z_i, y_i \in \mathbb{R}, S_1$ and S_2 are scalars such that the incremental sector-boundedness condition in (8) becomes a slope-restrictedness condition as in [10].

Furthermore, the finite-gain \mathcal{L}_2 stability of a nonlinear system is characterized by the following lemma.

Lemma 1 [22,37]. Consider the time-invariant system

$$\dot{x} = f(x) + g(x)u, \ x(0) = x_0$$

 $y = h(x),$
(9)

where $f(\cdot)$ is locally Lipschitz, $g(\cdot)$, $h(\cdot)$ are continuous, and f(0) = 0, h(0) = 0. Then, the following statements are equivalent.

- 1. The system is finite-gain \mathcal{L}_2 stable.
- 2. The \mathcal{L}_2 gain of the system is less than or equal to $\gamma > 0$, i.e.,

$$\|y\|_{2} \le \gamma \|u\|_{2} \tag{10}$$

3. There is positive scalar $\gamma > 0$ and a C^1 , positive semidefinite function V(x) such that the following Hamilton–Jacobi inequality is satisfied.

$$\frac{\partial V}{\partial x}f(x) + \frac{1}{2\gamma^2}g(x)g(x)^T \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2}h(x)^T h(x) \le 0.$$
(11)

We make a remark about Lemma 1 in the linear case. Thereby, the system Σ in (1) is considered, and we say Σ is *bounded real*, i.e., $||y||_2 \le \gamma ||u||_2$, or equivalently $||\Sigma||_{\mathcal{H}_{\infty}} < \gamma$, if and only if there exists a positive definite matrix *P* such that the following Riccati inequality holds.

$$A^{T}Q + QK + \frac{1}{\gamma^{2}}KBB^{T}K + C^{T}C < 0, \qquad (12)$$

which can be seen as the Hamilton–Jacobi inequality in the linear case.

All the Lur'e subsystems in the network are interconnected according to the following diffusive coupling protocol.

$$u_i = \sum_{j=1}^{N} w_{ij} (y_i - y_j), \ i = 1, 2, \dots, N,$$
(13)

where $w_{ij} \in \mathbb{R}_+$ is the (i, j)th entry of weighted adjacency matrix of the underlying graph and stands for the intensity of the coupling between subsystem *i* and *j*. Note that the output-feedback protocol in (13) means that the underlying weighted graph is static. Combining (13) and (6) leads to a compact form of the Lur'e dynamical network as

$$\Sigma:\begin{cases} \dot{x} = (I_N \otimes A - L \otimes BC)x - (I_N \otimes B)\Phi(y), \\ y = (I_N \otimes C)x, \end{cases}$$
(14)

where $\Phi(y) := [\phi(y_i)^T, \phi(y_2)^T, \dots, \phi(y_N)^T]^T$, and $x := [x_1^T, x_2^T, \dots, x_N^T]^T$, $y := [y_1^T, y_2^T, \dots, y_N^T]^T$ are the collections of the states and outputs of the *N* subsystems. The matrix *L* is the graph Laplacian of the underlying network as defined in (5).

Assumption 3. We assume throughout this paper that the Lur'e network is defined on an undirected connected simple graph.

In the context of networks, *synchronization* is one of the most important properties, which substantially means that the states of the subsystems can achieve a common value. For a Lur'e network system in (14), the definition of *robust synchronization* is given as follows.

Definition 1 [33,38]. A Lur'e network system in form of (14) is called robustly synchronized if

$$\lim_{t \to \infty} (x_i(t) - x_j(t)) = 0, \ \forall \ i, j = 1, 2, \dots, N,$$

for all initial conditions and all uncertain nonlinearities $\phi(\cdot)$ satisfying (8).

Moreover, a sufficient condition for robust synchronization of the Lur'e network as in (14) is obtained, where the spectra of the Laplacian matrix *L* are used. The proof can be found in [38].

Lemma 2. Consider the Lur'e network Σ as in (14) with an incrementally sector-bounded nonlinear function $\phi(\cdot)$. If there exists a matrix K > 0 such that

$$\begin{bmatrix} (A - \lambda_i BC)^T K + K(A - \lambda_i BC) \\ -C^T (S_1 S_2 + S_2 S_1)C \\ \hline -B^T K + (S_1 + S_2)C \\ \hline -2I_m \end{bmatrix} < \mathbf{0}, \quad (15)$$

for all i = 2, ..., N, then Σ robustly synchronizes. In (15), λ_i are the eigenvalues of the Laplacian matrix L.

Generally, applying the above lemma to verify the synchronization of Σ may be difficult, as one needs to check the feasibility of N-1 LMI's in (15) for all the nonzero eigenvalues of *L*. Furthermore, when the topology information of the network is uncertain, e.g., the spectra of the Laplacian matrix is unknown, Lemma 2 is not applicable.

3. Synchronization preserving model reduction

In this section, we exploit a model reduction strategy that reduces the dimension of each Lur'e subsystem such that the resulting reduced-order Lur'e network can preserve the robust synchronization property.

Before proceeding, we provide a new condition for the robust synchronization of the original Lur'e network Σ when only an upper bound of the Laplacian spectral radius is known. In many applications, due to the uncertainties and time variations of the topology and coupling strengths, it may be difficult to acquire the full knowledge of the interconnection structure in a network. Instead, some feature values of the graph, such as the Laplacian spectral radius, may be estimated [26,36]. Thus, in this paper, we will use an upper bound of the largest eigenvalue of the Laplacian matrix *L* to characterize the robust synchronization of nonlinear Lur'e networks.

Let $\rho > 0$ such that $\rho \ge \lambda_i$, $\forall i = 1, 2, ..., N$. Then, in the following lemma, we propose a new synchronization condition.

Lemma 3. If there exists a scalar $\tau > 0$ and a symmetric matrix $\mathcal{K} > 0$ such that

$$\begin{bmatrix} A^{T}\mathcal{K} + \mathcal{K}A + C^{T}S_{\tau}C & \mathcal{K}B - \tau C^{T}S_{\Sigma} & \rho \mathcal{K}B \\ B^{T}\mathcal{K} - \tau S_{\Sigma}C & -2\tau I_{m} & \mathbf{0} \\ \rho B^{T}\mathcal{K} & \mathbf{0} & -I_{m} \end{bmatrix} < \mathbf{0},$$
(16)

where $S_{\Sigma} := S_1 + S_2$, $S_{\tau} := I_m - \tau (S_1S_2 + S_2S_1)$, and ρ is the upper bound of the Laplacian spectral radius, then the Lur'e network Σ robustly synchronizes.

Proof. Consider the Schur complement of the matrix in (16), which is equivalent to

$$\Xi := A^{T} \mathcal{K} + \mathcal{K}A + C^{T} S_{\tau} C + \rho^{2} \mathcal{K}BB^{T} \mathcal{K} + \frac{1}{2\tau} (\mathcal{K}B - \tau C^{T} S_{\Sigma}) (B^{T} \mathcal{K} - \tau S_{\Sigma} C) < \mathbf{0}.$$
(17)

Since $(\lambda_i B^T \mathcal{K} + C)^T (\lambda_i B^T \mathcal{K} + C) \ge 0$ holds for all λ_i , we relax the above inequality as

$$\Xi < (\lambda_i B^T \mathcal{K} + C)^T (\lambda_i B^T \mathcal{K} + C), \tag{18}$$

Notice that

$$\begin{aligned} &(\lambda_i B^T \mathcal{K} + C)^T (\lambda_i B^T \mathcal{K} + C) \\ &= \lambda_i (C^T B^T \mathcal{K} + \mathcal{K} B C) + \lambda_i^2 \mathcal{K} B B^T \mathcal{K} + C^T C \\ &\leq \lambda_i (C^T B^T \mathcal{K} + \mathcal{K} B C) + \rho^2 \mathcal{K} B B^T \mathcal{K} + C^T C, \end{aligned}$$
(19)

for any λ_i , i = 2, 3, ..., N, as ρ is an upper bound of the Laplacian spectral radius. Thus, combining (18) and (19), we obtain

$$\begin{aligned} & + \mathcal{K}A - \tau C^{T}(S_{1}S_{2} + S_{2}S_{1})C - \lambda_{i}(B^{T}C^{T}\mathcal{K} + \mathcal{K}BC) \\ & + \frac{1}{2\tau}(\mathcal{K}B - \tau C^{T}S_{\Sigma})(B^{T}\mathcal{K} - \tau S_{\Sigma}C) \\ & = \tau [(A - \lambda_{i}BC)^{T}K + K(A - \lambda_{i}BC) - C^{T}(S_{1}S_{2} + S_{2}S_{1})C \\ & + \frac{1}{2}(KB - C^{T}S_{\Sigma})(KB - C^{T}S_{\Sigma})^{T}] < \mathbf{0}, \end{aligned}$$

where $K := \tau^{-1} \mathcal{K}$ with $\tau > 0$. Then, the last inequality is equivalent to (15). Consequently, there exist solutions of $\mathcal{K} > \mathbf{0}$ and $\tau > 0$ such that the LMI in Lemma 2 is feasible, i.e., Σ is robustly synchronized. \Box

The feasibility of the LMI in (16) can be tested easily using some exisiting LMI solvers, such as YALMIP, LMILAB (a toolbox of MATLAB), etc.. Note that Lemma 3 provides a sufficient condition of Lemma 2, meaning that it may be more conservative. However, it is convenient to apply Lemma 3 to check the robust synchronization of the Lur'e network Σ , especially when the size of the network is large, since it only verifies the existence of a positive definite solution \mathcal{K} in (16) rather than the solutions of (15) for all λ_i with i = 2, ..., N. Furthermore, Lemma 3 does not require to know all the eigenvalues of the Laplacian matrix but an upper bound of the Laplacian spectral radius, it thus particularly useful when the detailed topology of the network is unavailable. More importantly, the solution of (16) is suitable for the model reduction of each subsystem with a guaranteed an *a prior* error bound, but the solution of (15) is not.

Remark 1. We observe that the LMI in (16) can be rewrite as

$$\Xi := \left(A - \frac{1}{2}BS_{\Sigma}C\right)^{T} \mathcal{K} + \mathcal{K}\left(A - \frac{1}{2}BS_{\Sigma}C\right) + C^{T}R(\tau)^{2}C + \left(\rho^{2} + \frac{1}{2\tau}\right)\mathcal{K}BB^{T}\mathcal{K} < \mathbf{0},$$
(20)

where the matrix $R(\tau) > \mathbf{0}$ is defined as

.....

 (\mathcal{D})

$$R(\tau)^2 = \frac{\tau}{2}(S_2 - S_1)^2 + I_m = \frac{\tau}{2}S_{\Sigma}^2 - 2(S_1S_2 + S_2S_1) + I_m.$$
(21)

By (12), the synchronization condition of the multi-agent system Σ in Lemma 2 coincides with the *bounded realness* of the following auxiliary linear system

$$\boldsymbol{\Gamma} : \begin{cases} \dot{\boldsymbol{\xi}} = \left(A - \frac{1}{2}BS_{\Sigma}C\right)\boldsymbol{\xi} + \sqrt{\rho^2 + \frac{1}{2\tau}B\nu}, \\ \eta = R(\tau)C\boldsymbol{\xi}, \end{cases}$$
(22)

where $\xi \in \mathbb{R}^n$, $\eta, \nu \in \mathbb{R}^m$, and ρ is larger than the spectral radius of *L*. More precisely, if $\|\Gamma\|_{\mathcal{H}_{\infty}} < 1$, then the Lur'e network Σ robustly synchronizes. This connection is useful for proving the robust synchronization of the reduced-order Lur'e network, see Theorem 1.

Next, we select a pair of generalized Gramians to reduce the Lur'e subsystems such that the resulting reduced-order Lur'e network model preserves the robust synchronization property.

First, we compute \mathcal{P} , which is the solution of the following optimization problem.

$$\begin{aligned} \text{find} & \quad \text{trace}(\mathcal{P}) \\ \text{s.t.} & \quad \tau > 0, \ CS_{\tau}C^{T} > \mathbf{0}, \\ & \quad A\mathcal{P} + \mathcal{P}A^{T} + \frac{2}{\rho^{2}}\mathcal{P}C^{T}S_{\tau}C\mathcal{P} + BB^{T} \\ & \quad + \frac{1}{2\tau\rho^{2}}(B - \tau\mathcal{P}C^{T}S_{\Sigma})(B^{T} - \tau S_{\Sigma}C\mathcal{P}) < \mathbf{0}, \end{aligned}$$

$$(23)$$

where $S_{\tau} = I_m - \tau (S_1S_2 + S_2S_1)$. Notice that the above inequality is not linear, but it is equivalent to the LMI in (16), where $\mathcal{P} = \rho^2 \mathcal{K}^{-1}$ and max trace(\mathcal{K}) is the objective instead. Note that

in Lemma 3, the sufficient condition for the robust synchronization of the network only requires the existence of \mathcal{K} , while to compute a generalized Gramian matrix for the subsequent model reduction, maxtrace(\mathcal{K}) is our objective. Besides, we may also use a solution of (15) for the later balanced truncation, and the resulting reduced-order network still guarantees the synchronization property. However, such a solution, unlike \mathcal{P} in (23), is not a generalized controllability Gramian of Σ_i^{lin} . As a result, an error bound cannot be obtained.

Let Q be the observability Gramian of the linear system in (7), i.e., the solution of the following Lyapunov equation

Remark 2. Note that we can also use the minimum and maximum solution of the LMI in (16) as the generalized Gramians [10], which are regarded as a kind of symmetric type of balancing. However, in this paper, we suggest applying the solutions of (23) and (24) as the generalized Gramians, since (24) is less conservative than the LMI (16), and consequently we can obtain a smaller error bound for the approximation of the linear component of each Lur'e subsystems.

We now use the pair \mathcal{P} and \mathcal{Q} for the balanced truncation of the linear system Σ_i^{lin} . Denote $T \in \mathbb{R}^{n \times n}$ as a nonsingular coordinate transformation matrix such that

$$T\mathcal{P}T^{T} = T^{-T}\mathcal{Q}T^{-1} = \Sigma = \operatorname{diag}(\sigma_{1}, \sigma_{2}, \cdots, \sigma_{n}),$$
(25)

with $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n > 0$ are the corresponding generalized Hankel singular values. Therefore, we define the parameter matrices of the balanced system by

$$\bar{A} := TAT^{-1}, \ \bar{B} := TB, \ \text{and} \ \bar{C} := CT^{-1}.$$
 (26)

Suppose that $\sigma_k >> \sigma_{k+1}$. We therefore can truncate the n-k states corresponding to the smallest σ_i of the balanced system. Consider the following matrix partitions.

$$\bar{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \bar{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \bar{C} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \Sigma = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix},$$
(27)

where $A_{11} \in \mathbb{R}^{k \times k}$, $B_1 \in \mathbb{R}^{k \times m}$, $C_1 \in \mathbb{R}^{m \times k}$, and $\Sigma_1 := \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_k)$. Hereafter, denote

$$\hat{A} = A_{11}, \ \hat{B} = B_1, \ \text{and} \ \hat{C} = C_1.$$
 (28)

Consequently, the reduced matrices of the Lur'e subsystems are also obtained. By substituting the truncated matrices \hat{A} , \hat{B} , and \hat{C} to the Lur'e form in (6), we construct the reduced-order dynamics of each agent as follows.

$$\hat{\Sigma}_{i} : \begin{cases}
\dot{\hat{x}}_{i} = \hat{A}\hat{x}_{i} + \hat{B}(u_{i} + \hat{z}_{i}) \\
\hat{y}_{i} = \hat{C}\hat{x}_{i} , \quad i = 1, 2, \dots, N, \\
\hat{z}_{i} = -\phi(\hat{y}_{i})
\end{cases}$$
(29)

with $\hat{x}_i \in \mathbb{R}^k$, and $\hat{z}_i, \hat{y}_i \in \mathbb{R}^m$. Furthermore, it leads to reduced-order Lur'e network dynamics as

$$\widehat{\boldsymbol{\Sigma}}: \begin{cases} \hat{\boldsymbol{x}} = (\boldsymbol{I}_N \otimes \hat{\boldsymbol{A}} - \boldsymbol{L} \otimes \hat{\boldsymbol{B}}\hat{\boldsymbol{C}})\hat{\boldsymbol{x}} - (\boldsymbol{I}_N \otimes \hat{\boldsymbol{B}})\Phi(\hat{\boldsymbol{y}}), \\ \hat{\boldsymbol{y}} = (\boldsymbol{I}_N \otimes \hat{\boldsymbol{C}})\hat{\boldsymbol{x}}, \end{cases}$$
(30)

comparing to (14). In the model (30), L is an unknown Laplacian matrix. The following theorem then shows that the robust synchronization property is preserved in the approximation of the Lur'e network.

Theorem 1. Consider the full-order Lur'e network $\hat{\Sigma}$ in (14) and its reduced-order model $\hat{\Sigma}$ in (30). If Σ is robustly synchronized by the condition in Lemma 3, then $\hat{\Sigma}$ is also robustly synchronized.

Proof. Recall that the feasibility of (16) in Lemma 3 is equivalent to the auxiliary system Γ being bounded real, i.e., (23) holds. Therefore, to show the robust synchronization of the reduced-order Lur'e network $\hat{\Sigma}$, it is sufficient to prove the bounded realness of the reduced-order auxiliary system $\hat{\Gamma}$, which is formulated as

$$\hat{\mathbf{\Gamma}}:\begin{cases} \dot{\hat{\xi}} = \left(\hat{A} - \frac{1}{2}\hat{B}S_{\Sigma}\hat{C}\right)\hat{\xi} + \sqrt{\rho^2 + \frac{1}{2\tau}\hat{B}\nu},\\ \hat{\eta} = R(\tau)\hat{C}\hat{\xi}.\end{cases}$$
(31)

Multiplying T and T^{T} to the left and right sides of (17) leads to

$$\left(\bar{A} - \frac{1}{2} \bar{B} S_{\Sigma} \bar{C} \right)^{T} \Sigma^{-1} + \Sigma^{-1} \left(\bar{A} - \frac{1}{2} \bar{B} S_{\Sigma} \bar{C} \right) + \bar{C}^{T} R(\tau) \bar{C}$$

$$+ \left(\rho^{2} + \frac{1}{2\tau} \right) \Sigma^{-1} \bar{B} \bar{B}^{T} \Sigma^{-1} < \mathbf{0},$$

$$(32)$$

where $R(\tau)$ is defined in (21), and Σ is the diagonal Gramian in (25). \overline{A} , \overline{B} , \overline{E} , and \overline{C} are parameter matrices of the balanced system in (26). Consider the partitions as in (27). Note that the matrix $\hat{A} - \frac{1}{2}\hat{E}S_{\Sigma}\hat{C}$ is the *k*th order principal submatrix of $\overline{A} - \frac{1}{2}\overline{E}S_{\Sigma}\overline{C}$ in the balanced system. Thus, for the reduced-order model $\hat{\Gamma}$, we obtain

$$\left(\hat{A} - \frac{1}{2} \hat{B} S_{\Sigma} \hat{C} \right)^{I} \Sigma_{1}^{-1} + \Sigma_{1}^{-1} \left(\hat{A} - \frac{1}{2} \hat{B} S_{\Sigma} \hat{C} \right) + \hat{C}^{T} R(\tau) \hat{C} \Sigma_{1} + \left(\rho^{2} + \frac{1}{2\tau} \right) \Sigma_{1}^{-1} \hat{B} \hat{B}^{T} \Sigma_{1}^{-1} < \mathbf{0},$$
 (33)

which implies that the reduced auxiliary system in (31) is bounded real. Therefore, the reduced-order model $\hat{\Sigma}$ is robustly synchronized. \Box

Theorem 1 implies that using the proposed condition in (16), we can select a piar of generalized Gramians to reduce the order of Lur'e subsystems, regardless of the number of subsystems and their interconnection topology.

4. Error analysis

In this section, we analyze the input-output approximation error caused by the model reduction procedure in the last section. From (29), the linear part of the reduced Lur'e subsystem is given by

$$\hat{\Sigma}_{i}^{lin}:\begin{cases} \dot{\hat{x}}_{i} = \hat{A}\hat{x}_{i} + \hat{B}\hat{u}\\ \hat{y}_{i} = \hat{C}\hat{x}_{i} \end{cases}, \quad i = 1, 2, \cdots, N,$$
(34)

where $\hat{u} := u_i + \hat{z}_i$. Denote the transfer functions of Σ_i^{lin} and $\hat{\Sigma}_i^{lin}$ as

$$G(s) := C(sI_n - A)^{-1}B,$$
(35a)

$$\hat{G}(s) := \hat{C}(sI_k - \hat{A})^{-1}\hat{B}.$$
 (35b)

We present the following lemma to guarantee that the described model order reduction technique preserves stability properties.

Lemma 4. Suppose that the original Lur'e network Σ in (14) satisfies the robust synchronization condition in (16), and $S_{\tau} = I_m - \tau (S_1S_2 + S_2S_1) > 0$. Then, both the full-order and reduced-order nonlinear Lur'e subsystems are finite-gain \mathcal{L}_2 stable, i.e., the system $\hat{\Sigma}_i$ in (29) preserves the finite-gain \mathcal{L}_2 stability, if

$$\rho > \mu/\mu_{\tau},\tag{36}$$

where $\mu^2 := \bar{\sigma} \left(\frac{S_1^2 + S_2^2}{2} \right)$, and $\mu_{\tau}^2 := \bar{\sigma}(S_{\tau})$.

Proof. First, we show that (36) indicates the input-output stability of the original Lur'e subsystem. By the assumption that $S_{\tau} > \mathbf{0}$, there exists a scalar $\mu_{\tau} > 0$ such that $S_{\tau} \ge \mu_{\tau} I_m$. Here, we choose $\mu_{\tau} = \bar{\sigma}(S_{\tau})$ to yield

$$A^{T}\mathcal{K} + \mathcal{K}A + \rho^{2}\mathcal{K}BB^{T}\mathcal{K} + \mu_{\tau}C^{T}C$$

$$\leq A^{T}\mathcal{K} + \mathcal{K}A + \rho^{2}\mathcal{K}BB^{T}\mathcal{K} + C^{T}S_{\tau}C < \mathbf{0}, \qquad (37)$$

where the latter inequality is given by (17). Then, by Lemma 1, we obtain

$$\|G(s)\|_{\mathcal{H}_{\infty}} < (\rho\mu_{\tau})^{-1} \tag{38}$$

Moreover, we verify that the incremental sector-boundedness of $\phi(\cdot)$ implies that $\phi(\cdot)$ is globally Lipschitz. From (8), we have

$$\begin{aligned} &(z_i - \tilde{z}_i)^T (z_i - \tilde{z}_i) + (y_i - \tilde{y}_i)^T S_1 S_2 (y_i - \tilde{y}_i) \\ &\leq (z_i - \tilde{z}_i)^T (S_1 + S_2) (y_i - \tilde{y}_i) \\ &\leq \frac{1}{2} (z_i - \tilde{z}_i)^T (z_i - \tilde{z}_i) + \frac{1}{2} (y_i - \tilde{y}_i)^T (S_1 + S_2)^2 (y_i - \tilde{y}_i), \end{aligned}$$

which yields

$$(z_i - \tilde{z}_i)^T (z_i - \tilde{z}_i) \le (y_i - \tilde{y}_i)^T \left(\frac{S_1^2 + S_2^2}{2}\right) (y_i - \tilde{y}_i).$$

Thus,

$$\|z_i - \tilde{z}_i\|_2 \le \mu \|y_i - \tilde{y}_i\|_2.$$
(39)

Taking the Laplace transform of the differential equation in (6), we obtain

$$Y_i(s) = G(s)[U_i(s) + Z_i(s)],$$
(40)

where Y(s), $U_i(s)$, and $Z_i(s)$ are the Laplace transforms of the signals y(t), $u_i(t)$, and $z_i(t)$, respectively. Therefore, the following inequality holds.

$$\begin{aligned} \|y_{i}(t)\|_{2} &\leq \|G(s)\|_{\mathcal{H}_{\infty}}(\|u_{i}(t)\|_{2} + \|z_{i}(t)\|_{2}) \\ &\leq (\rho\mu_{\tau})^{-1}(\|u_{i}(t)\|_{2} + \mu\|y_{i}(t)\|_{2}). \end{aligned}$$
(41)

If (36) is satisfied, i.e., $\rho\mu_{\tau} - \mu > 0$, (41) becomes

$$\|y_i(t)\|_2 \le \frac{1}{\rho\mu_{\tau} - \mu} \|u_i(t)\|_2.$$
(42)

Thus, the Lur'e subsystem Σ_i is finite-gain \mathcal{L}_2 stable due to Lemma 1.

Similarly, we have

$$\hat{A}^{T} \Sigma_{1}^{-1} + \Sigma_{1}^{-1} \hat{A} + \rho^{2} \Sigma_{1}^{-1} \hat{B} \hat{B}^{T} \Sigma_{1}^{-1} + \mu_{\tau} \hat{C}^{T} \hat{C} < \mathbf{0},$$
(43)

for the reduced-order Lur'e subsystem $\hat{\Sigma}_i$. The following bound therefore holds for the linear component of $\hat{\Sigma}_i$:

$$\|\hat{G}(s)\|_{\mathcal{H}_{\infty}} < (\rho\mu_{\tau})^{-1}.$$
(44)

Following the same reasoning line, we show that $\|\hat{y}_i(t)\|_2 \le (\rho\mu_{\tau} - \mu)^{-1} \|u_i(t)\|_2$, i.e., the reduced-order Lur'e system $\hat{\Sigma}_i$ is finite-gain \mathcal{L}_2 stable. \Box

The physical interpretation of condition (36) is illustrated. Consider a special incrementally sector-bounded nonlinearity, namely, a slope-restricted nonlinear feedback [10], with $S_1 = 0$, and $S_2 = s$, which is a positive scalar. Then, (36) becomes $\rho > \frac{s}{\sqrt{2}}$. Clearly, this condition relates the graph Laplacian spectral radius ρ with the size of the sector *s*. More precisely, to guarantee the finite-gain \mathcal{L}_2 stability, ρ should be larger as the sector size becomes larger. If there is more uncertainty on the nonlinearity $\phi(\cdot)$, we then require the underlying network to be stronger connected, i.e., denser interconnections or larger edge weights, in order to achieve the finite-gain \mathcal{L}_2 stability.

Remark 3. The choice of the parameter τ in (16) is discussed. Generally, τ is selected such that $\tau > 0$ and $CS_{\tau}C^{T} > 0$ hold. Specifically, to compute the controllability Gramian \mathcal{P} in (23), a value of τ satisfying the constraint and the LMI (16) is selected such that maxtrace(\mathcal{K}) is achieved and can be regarded as a generalized controllability Gramian of Σ_{i}^{lin} . To further guarantee the finite gain \mathcal{L}_{2} stability of the reduced-order Lur'e subsystem as in Lemma 4, we select a τ fulfilling the constraints $\tau > 0$ and $I_{m} > \tau (S_{1}S_{2} + S_{2}S_{1})$ such that maxtrace(\mathcal{K}) of (16) is achieved.

Now we are ready to explore an *a priori* error bound for the reduction of Lur'e subsystems Σ_i .

Theorem 2. Consider a robustly synchronized Lur'e network Σ , i.e., the condition in (16) holds. If (36) is satisfied, then, the error between the outputs of the full-order and reduced-order Lur'e subsystems, Σ_i and $\hat{\Sigma}_i$, is bounded by

$$\|y_{i}(t) - \hat{y}_{i}(t)\|_{2} \leq \frac{\rho^{2} \mu_{\tau}^{2} \epsilon}{(\rho \mu_{\tau} - \mu)^{2}} \|u_{i}(t)\|_{2},$$
(45)

where $\epsilon := 2 \sum_{k=r+1}^{n} \sigma_k$ with σ_k the generalized Hankel singular values in (25). ρ is the upper bound of the Laplacian spectral radius, and μ , μ_{τ} are positive scalars defined in Lemma 4.

Proof. From (29), we first obtain

$$\hat{Y}_i(s) = \hat{G}(s)[U_i(s) + \hat{Z}_i(s)],$$
(46)

by the Laplace transform, where $\hat{Y}(s)$ and $\hat{Z}_i(s)$ are the Laplace transforms of the signals $\hat{y}(t)$ and $\hat{z}_i(t)$, respectively. Thus, the output error in Laplace domain is presented as

$$Y_i(s) - \hat{Y}_i(s) = \left[G(s) - \hat{G}(s)\right] U_i(s) + G(s) Z_i(s) - \hat{G}(s) \hat{Z}_i(s), \quad (47)$$

which leads to the following inequality

$$\begin{aligned} \|y_{i}(t) - \hat{y}_{i}(t)\|_{2} &\leq \|G(s) - \hat{G}(s)\|_{\mathcal{H}_{\infty}}(\|u_{i}(t)\|_{2} + \|z_{i}(t)\|_{2}) \\ &+ \|\hat{G}(s)\|_{\mathcal{H}_{\infty}}\|z_{i}(t) - \hat{z}_{i}(t)\|_{2}. \end{aligned}$$
(48)

Hereafter, we analyze the bound for each component in (48) as follows.

Note that \mathcal{P} and \mathcal{Q} can be regarded as the generalized Gramians of the linear system Σ_i^{lin} . Therefore, the approximation error is bounded by

$$\|G(s) - \hat{G}(s)\|_{\mathcal{H}_{\infty}} \le 2\sum_{k=r+1}^{n} \sigma_k := \epsilon.$$
(49)

From (39), the incremental sector-boundedness of the uncertain function $\phi(\cdot)$ leads to

$$\|z_i(t)\|_2 \le \mu \|y_i(t)\|_2 \le \frac{\mu}{\rho\mu_\tau - \mu} \|u_i(t)\|_2,$$
(50)

where the inequality (42) is used. Furthermore, we have

$$\|\hat{G}(s)\|_{\mathcal{H}_{\infty}}\|z_{i}(t) - \hat{z}_{i}(t)\|_{2} \le \frac{\mu}{\rho\mu_{\tau}}\|y_{i}(t) - \hat{y}_{i}(t)\|_{2}.$$
(51)

Now, substitution of (49), (50) and (51) to (48) leads to

$$\begin{aligned} \|y_{i}(t) - \hat{y}_{i}(t)\|_{2} \\ &\leq \epsilon \left(1 + \frac{\mu}{\rho\mu_{\tau} - \mu}\right) \|u_{i}(t)\|_{2} + \frac{\mu}{\rho\mu_{\tau}} \|y_{i}(t) - \hat{y}_{i}(t)\|_{2} \\ &= \frac{\rho\mu_{\tau}\epsilon}{\rho\mu_{\tau} - \mu} \|u_{i}(t)\|_{2} + \frac{\mu}{\rho\mu_{\tau}} \|y_{i}(t) - \hat{y}_{i}(t)\|_{2}. \end{aligned}$$
(52)

Since (36) holds, $1 - \frac{\mu}{\rho\mu\tau} > 0$. Thus, (52) gives the explicit bound as in (45). \Box

Remark 4. The error bound in (45) may be conservative, which is determined by several factors. ρ indicates the interconnection property of the network, and the bound may be more conservative



Fig. 2. (a) The state and output trajectories of the full-order Lur'e network; (b) The trajectories of reduced-order Lur'e network $\hat{\Sigma}_1$ obtained by the proposed method; (c) The trajectories of reduced-order Lur'e network $\hat{\Sigma}_2$ obtained by the IRKA algorithm. The initial states are set to random values for both systems. It shows that $\hat{\Sigma}_1$ preserves the robust synchronization, while $\hat{\Sigma}_2$ does not.

if we have a larger and denser network. Furthermore, conservativeness of (45) is also affected by the tightness of the error bound ϵ on the approximation of the linear part. The denominator of (45), i.e., $(\rho\mu_{\tau} - \mu)^2$ is strictly positive when the condition for finite-gain \mathcal{L}_2 stability of the reduced-order subsystem $\hat{\Sigma}_i$ is guaranteed. As a result, the error bound is finite. Nevertheless, the bound in (45) may be conservative if the bound on the \mathcal{L}_2 gain in (42) of the original Lur'e subsystems is conservative.

5. Illustrative example

The feasibility of the proposed method is illustrated through two numerical examples. In the first example, we consider a network of 4 Lur'e subsystems, and we compare the result obtained by the proposed method and the IRKA-based algorithm. In the second example, we validate the proposed method in the application of a networked robotic flexible arms.

5.1. Example 1

Considers a network consisting of 4 Lur'e subsystems, where the interconnection topology is characterized by an unknown unweighted simple graph. From [26], we know that the Laplacian spectral radius is greater or equal to 4. Thus, we take $\rho = 4$ for the following simulation. Suppose that the dynamics of Lur'e

subsystems Σ_i in (6) are given by matrices

	Γ0		0	0	()	1	0	0	ך 0	
<i>A</i> =	0		0	0	()	0	0.25	0	0	
	0		0	0	()	0	0	0.5	0	
	0		0	0	()	0	0	0	1	
	-1		0	0	0		-3	0.25	0	0	,
	0		-1	0	0		1	-1	1	0	
	0		0	-1	()	0	0.5	-2	1	
	0		0	0	-1		0	0	0.5	-2_	
<i>B</i> = [0	0	0	0	0	1	-2	2] ^{<i>T</i>} ,			
C = [1	0	0	0	1	0	0	0],			

and a nonlinearity

$$\phi(y) = |y+1| - |y-1|.$$

Note that $\phi(y)$ is incrementally sector-bounded with $S_1 = 0$ and $S_2 = 2$ by the definition in (8). We use the YALMIP toolbox to check that the LMI in (16) is feasible, and a solution is computed with the maximal trace of \mathcal{K} and $\tau = 0.7044$. Thus, by Lemma 3, the original Lur'e network in form of (14) synchronizes under the unknown interconnection topology.

We use (23) and (24) as the generalized controllability and observability Gramians, which can be simultaneously diagonalized. Using the balanced truncation procedure in Section 3 to eliminate the last four states with the smallest GHSVs in the balanced



Fig. 3. The trajectories of the states and outputs of the full-order Lur'e network (in the left figure) and reduced-order network (in the right figure). It shows that both the full-order and reduced-order Lur'e networks are robustly synchronized over time.

system, leading to the following reduced matrices.

$$\hat{A} = \begin{bmatrix} -0.3846 & -0.2147 & 0.0241 & 0.5101 \\ 0.2120 & -0.0001 & 0.0007 & 0.0054 \\ 0.0224 & -0.0006 & -0.0020 & -0.5836 \\ -0.4847 & 0.0030 & 0.5828 & -0.2972 \end{bmatrix},$$
$$\hat{B} = \begin{bmatrix} -0.0985 & 0.0011 & 0.0040 & -0.0462 \end{bmatrix}^{T},$$
$$\hat{C} = \begin{bmatrix} -0.4036 & -0.0066 & 0.0164 & 0.1926 \end{bmatrix}.$$

The approximation error on each Lur'e subsystem is analyzed. The reduction error of the linear part Σ_i^{lin} is measured as $\|G(s) - \hat{G}(s)\|_{\mathcal{H}_{\infty}} \approx 0.0077$, which indicates that the approximation error on the linear part of each Lur'e subsystem is small. The inequality in (49) shows that the approximation error bound for the linear part is $\epsilon = 0.0124$. Note that in this example, $S_{\tau} = I > \mathbf{0}$ and $\mu = \sqrt{2}$, and $\mu_{\tau} = 1$. Thus, we have $\rho > \mu/\mu_{\tau}$, which implies from Lemma 4 that both the original and reduced-order Lur'e subsystems are finite-gain \mathcal{L}_2 bounded. As a result, Theorem 2 can be applied to provide *a priori* error bound on the approximation of each nonlinear subsystem: $\|y_i(t) - \hat{y}_i(t)\|_2 \le 0.0297 \cdot \|u_i(t)\|_2$.

Using the reduced matrices, the reduced-order Lur'e subsystems $\hat{\Sigma}_i$ are constructed and reconnected to form a lower-dimensional Lur'e network in the form of (30). Note that both original and reduced Lur'e network in (14) and (30) are autonomous. To investigate the synchronization phenomenon in both systems, we stimulate both systems by assigning random values as their initial states. The trajectories of the states and outputs of both networks are then plotted in Fig. 2a and Fig. 2b. We can see that, by the proposed model reduction scheme, the reduced-order Lur'e network preserves the robust synchronization property. For comparison, we apply the IRKA algorithm [21] to reduce the linear component of each Lur'e subsystem. However, the obtained matrices of the reduced-order subsystem do not preserve the finite-gain \mathcal{L}_2 stability. Furthermore, the obtained reduced-order Lur'e network model does not preserve the robust synchronization property. It can be seen from Fig. 2c, in which the state and output trajectories diverge over time.

5.2. Example 2

In this example, we take the nodal dynamics as the model of a flexible link robotic arm, whose graphical illustration is shown in [8,28]. Each controlled robotic arm is a Lur'e subsystem in the form of (6) with matrices [28]

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{\alpha}{\mathcal{I}} & -\frac{\beta_1}{\mathcal{I}} & \frac{\alpha}{\mathcal{I}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{\alpha}{\mathcal{J}} - k_p & -k_d & -\frac{\kappa}{\mathcal{J}} & -\frac{\beta_2}{\mathcal{J}} & k_i \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and a nonlinearity

 $\phi(y) = \sin y + y.$

The input is a reference signal for the torque induced by an electric motor, which is used to control the states of the robotic arm, and the output is the rotation angle of the robotic arm. The system parameters are given as follows: $\mathcal{J} = 0.5 \text{ kg} \cdot \text{m}^2$, $\mathcal{I} = 25.0 \text{ kg} \cdot \text{m}^2$, $\beta_1 = \beta_2 = 1.0 \text{ Nm s/rad}$, $\kappa = 50.0 \text{ Nm/rad}$, $k_p = 120$, $k_i = 10$ and $k_d = 70$. Note that $\phi(y)$ is incrementally sector-bounded with $S_1 = 0$ and $S_2 = 2$ [8]. We consider a network of 10 flexible robotic arms, where interconnection topology is characterized by a circle graph with the weight of each edge to be 0.1. Thus, we take $\rho = 4$. The reduced-order subsystem is obtained with

$$\hat{A} = \begin{bmatrix} -0.3991 & 1.2248 & -1.0472 \\ -0.9649 & -0.3443 & 0.5106 \\ 0.7675 & 0.4746 & -0.7458 \end{bmatrix}, \hat{B} = \begin{bmatrix} -0.5194 \\ -0.2408 \\ 0.3526 \end{bmatrix}$$
$$\hat{C}^{T} = \begin{bmatrix} -0.5781 \\ 0.4329 \\ -0.5189 \end{bmatrix}.$$

It is verified that each reduced-order Lur'e subsystem is finite-gain \mathcal{L}_2 stable, and the *a priori* bound on the approximation of subsystems is computed as $||y_i(t) - \hat{y}_i(t)||_2 \le 0.3684 \cdot ||u_i(t)||_2$. To show the synchronization property of the reduced-order Lur'e network, we plot the trajectories of the states and outputs of both full-order and reduced-order networks in Fig. 3. We stimulate both systems by adding the signals $w(t) = 0.2 - 0.3 \cos(\frac{t}{2\pi})$ as an external input to each subsystem. It is shown that the proposed model reduction scheme preserves the robust synchronization property in the reduced-order Lur'e network.

6. Conclusion

In this paper, we have proposed a model order reduction scheme for uncertain Lur'e networks that are composed of identical nonlinear Lur'e-type subsystems. Using the incrementally sector-bounded uncertain nonlinearity and the upper bound of the graph Laplacian spectral radius, an LMI condition can be provided to sufficiently identify the robust synchronization of the Lur'e network. The solution of the LMI is regarded as generalized Gramians, which are employed for the balanced truncation of the linear part of each Lur'e subsystems. The complexity of the overall network is reduced as the dimension of each subsystem is lowered. It is shown that the reduced-order Lur'e network preserves the robust synchronization property. Moreover, in the time domain, an *a priori* bound on the input-output error between the full-order and reduced-order Lur'e subsystems is acquired.

For future work, there two interesting directions to extend the current results. The first is to reduce the Laplacian matrix, i.e., the coupling protocol of the Lur'e network, such that the dynamics of each subsystem and the network topology can be simultaneously simplified. The second direction is to investigate heterogeneous networks, i.e., the networks that are composed of different nonlinear subsystems.

Author Declaration

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome. We confirm that the manuscript has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. We further confirm that the order of authors listed in the manuscript has been approved by all of us.

We confirm that we have given due consideration to the protection of intellectual property associated with this work and that there are no impediments to publication, including the timing of publication, with respect to intellectual property. In so doing we confirm that we have followed the regulations of our institutions concerning intellectual property.

We understand that the Corresponding Author, Dr. Xiaodong Cheng, is the sole contact for the Editorial process (including Editorial Manager and direct communications with the office). He is responsible for communicating with the other authors about progress, submissions of revisions and final approval of proofs. We confirm that we have provided a current, correct email address which is accessible by the Corresponding Author and which has been configured to accept email from x.cheng@tue.nl

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