A REMARK ON SETS WITH FEW DISTANCES IN $\mathbb{R}^d$

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Abstract. A celebrated theorem due to Bannai-Bannai-Stanton says that if A is a set of points in $\mathbb{R}^d$, which determines s distinct distances, then

$$|A| \leq \binom{d+s}{s}.$$

In this note, we give a new simple proof of this result by combining Sylvester's Law of Inertia for quadratic forms with the proof of the so-called Croot-Lev-Pach Lemma from additive combinatorics.

1. Introduction

Given a positive integer s, a finite subset A in a metric space M is called an s-distance set in M if there are s positive real numbers $d_1, \ldots, d_s$ such that all the pairwise distances determined by the points in M are among these numbers, and each d_i is realized. Upper bounding the size of such sets is a famous problem in combinatorial geometry, with a lot of activity around the various possible variants. See for instance [5] and the references therein. When M is $\mathbb{R}^d$, with the usual Euclidean distance, the classical result in the area is the following result due to Bannai, Bannai and Stanton [1] from 1983.

Theorem 1.1. *If A is an s-distance subset in $\mathbb{R}^d$, then*

$$|A| \leq \binom{d+s}{s}.$$

The proof of Theorem 1 from [1] builds upon the linear independence argument introduced for this problem by Larman, Rogers and Seidel in [3]. In [3], the authors proved that when $s = 2$, the inequality $|A| \leq (d+1)(d+4)/2$ follows from the fact that to each a point in A one can associate a polynomial $f_a \in \mathbb{R}[x_1, \ldots, x_d]$ such that $\{f_a, a \in A\}$ is a set of linearly independent polynomials over the reals, which also happens to lie in a subspace of $\mathbb{R}[x_1, \ldots, x_d]$ of dimension $(d+1)(d+4)/2$. This argument was later amplified by Blokhuis [2] who showed that one can further add a list of $d+1$ other polynomials to $\{f_a : a \in A\}$ and get an even larger list of linearly independent polynomials that lie in the same vector space of dimension $(d+1)(d+4)/2$. This led to $|A| \leq (d+1)(d+4)/2 - (d+1) = \binom{d+2}{2}$, which established the important first case $s = 2$ of Theorem 1.1. This story was successfully generalized by Bannai-Bannai-Stanton in [1], but for larger s the argument to show that one can add a new list of (higher degree) polynomials to the old list and still get a set of linearly independent elements in the same vector space is significantly more technical.

In this paper, we give a new simple proof of Theorem 1.1 via a slightly improved version of the so-called Croot-Lev-Pach Lemma [4, Lemma 1] over the reals, which may be of independent interest. We state this in a general form, which captures the original version of the Croot-Lev-Pach Lemma as well.

Theorem 1.2. *Let V be a finite-dimensional vector space over a field $\mathbb{F}$ and $A \subset V$ be a finite set. Let s be a nonnegative integer and let $p(\overrightarrow{x}, \overrightarrow{y})$ be a $2 \cdot \dim V$-variate polynomial wih coefficients in $\mathbb{F}$ and of degree at most $2s+1$. Consider the matrix $M_{p,A}$ with rows and columns indexed by A and entries $p(\cdot, \cdot)$. It corresponds to a (not necessary symmetric) bilinear form on $\mathbb{F}^A$ by a formula*

$$\Phi_p(f, g) = \sum_{a,b \in A} p(\overrightarrow{a}, \overrightarrow{b}) f(a) g(b), \text{ for } f, g : A \to \mathbb{F},$$

which in turn defines a quadratic form $\Phi_p(f, f)$. Denote by $\operatorname{rank}(p, A)$ the rank of matrix $M_{p,A}$; if $\mathbb{F} = \mathbb{R}$ denote also by $r_+(p), r_-(p)$ the inertia indices of the quadratic form $\Phi_p(f, f)$. Finally, denote by $\dim_s(A)$ the dimension of the space of polynomials of degree at most s considered as functions on A. Then:

1) $\operatorname{rank}(p, A) \leqslant 2\dim_s(A)$.

2) if $\mathbb{F} = \mathbb{R}$, then $\max\{r_+(p, A), r_-(p, A)\} \leqslant \dim_s(A)$.

In the next section, we will first prove Theorem 1.2, and then we will use it to deduce Theorem 1.1. We will need only part 2) of the Lemma above, since part 1) is more or less the original Croot-Lev-Pach lemma in disguise (which doesn't help directly), but we will include nonetheeless a quick new proof of part 1) as well since it motivated part 2).

2. Proof of Theorem 1.2

Proof. Endow the space $\mathbb{F}^A$ with a natural inner product $\langle f, g \rangle = \sum_{a \in A} f(a) g(a)$.

Consider the space $\Omega \subset \mathbb{F}^A$ of functions f on A satisfying $\langle f, \phi \rangle = 0$ for all polynomials ϕ of degree at most s. It is easy to see that the dimension of Ω as a vector space over $\mathbb{F}$ is at least $|A| - \dim_s(A)$.

The key observation is that $\Phi_p(f, g) = 0$ whenever $f, g \in \Omega$. Indeed, for any monomial $x^\alpha y^\beta$ in the polynomial $p(\overrightarrow{x}, \overrightarrow{y})$ (here α, β are multi-indices with sum of degrees at most $2s+1$) we have

$$\sum_{a,b \in A} a^\alpha b^\beta f(a) g(b) = \left(\sum_{a \in A} a^\alpha f(a) \right) \cdot \left(\sum_{b \in B} b^\beta g(b) \right) = 0,$$

since either α or β have degree at most s and f, g are choosing from Ω.

We will now prove both claims of Theorem 1.2 by using dimension arguments.

Indeed, the bilinear form $\Phi_p[\cdot,\cdot]$ on $\mathbb{F}^A$ takes zero values on $\Omega\times\Omega$, thus all non-zero entries of its matrix in appropriate basis (which includes the basis of Ω and any other $|A|-\dim\Omega$ basis vectors) may be covered by $|A|-\dim\Omega$ rows and $|A|-\dim\Omega$ columns. This implies that every minor of $M_{p,A}$ of dimension at least $2(|A|-\dim\Omega)+1$ must vanish. Therefore,

$$\mathrm{rank}(p,A)\leqslant 2(|A|-\dim\Omega)\leqslant 2\mathrm{dim}_s(A).$$

This proves the first claim of Theorem 1.2.

If $\mathbb{F}=\mathbb{R}$, by Sylvester's Law of Inertia, we may choose a subspace $Y\subset\mathbb{F}^A$ of dimension $r_+(p,A)$ such that the quadratic form $\Phi_p(f,f)$ restricted to Y is positive definite. If $f\in Y\cap\Omega$ and $f\neq 0$, we have $0=\Phi_p(f,f)>0$, which is impossible. Therefore, $Y\cap\Omega=\{0\}$ and $\dim Y+\dim\Omega\leqslant|A|$, which yields that $r_+(p,A)=\mathrm{dim}Y\leqslant|A|-\dim\Omega\leqslant\mathrm{dim}_s(A)$. Analogously, we also have that $r_-(p,A)\leqslant\mathrm{dim}_s(A)$. This completes the proof of Theorem 1.2. □

We now deduce Theorem 1.1 from Theorem 1.2.

If A is an s-distance subset in $\mathbb{R}^d$ and S is the set of distinct distances it determines, consider the $2d$-variate polynomial p with real coefficients defined by

$$p(\overrightarrow{x},\overrightarrow{y})=\prod_{d\in S}(d^2-\|x-y\|^2).$$

The matrix $M_{p,A}$ from Theorem 1.2 is then a positive scalar matrix for this polynomial; therefore, $r_+(p,A)=|A|$, and so part 2) of Theorem 1.2 implies that

$$|A|=r_+(p,A)\leqslant\mathrm{dim}_s(A)\leqslant\mathrm{dim}_s(\mathbb{R}^d)=\binom{s+d}{d}.$$

This completes the proof of Theorem 1.1.

References

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