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New results on the q -generalized Bernoulli polynomials of level m

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Abstract: This paper aims to show new algebraic properties from the q -generalized Bernoulli polynomials $B_n^{[m-1]}(x; q)$ of level m , as well as some other identities which connect this polynomial class with the q -generalized Bernoulli polynomials of level m , as well as the q -gamma function, and the q -Stirling numbers of the second kind and the q -Bernstein polynomials.

Keywords: q -generalized Bernoulli polynomials, q -gamma function, q -Stirling numbers, q -Bernstein polynomials

MSC 2010: 33E12

1 Introduction

Fix a fixed $m \in \mathbb{N}$, the generalized Bernoulli polynomials of level m are defined by means of the following generating function [1]

$$\frac{z^m e^{xz}}{e^z - \sum_{l=0}^{m-1} \frac{z^l}{l!}} = \sum_{n=0}^{\infty} B_n^{[m-1]}(x) \frac{z^n}{n!}, \quad |z| < 2\pi, \quad (1.1)$$

where the generalized Bernoulli numbers of level m are defined by $B_n^{[m-1]} := B_n^{[m-1]}(0)$, for all $n \geq 0$. We can say that if $m = 1$ in (1.1), then we obtain the definition via a generating function, of the classical Bernoulli polynomials $B_n(x)$ and classical Bernoulli numbers, respectively, i.e. $B_n(x) = B_n^{[0]}(x)$ and $B_n = B_n^{[0]}$, respectively.

The q -analogue of the classical Bernoulli numbers and polynomials were initially investigated by Carlitz [2]. More recently, J. Choi, T. Ernst, D. Kim, S. Nalci, C.S. Ryoo [3–8] defined the q -Bernoulli polynomials using different methods and studied their properties. There are numerous recent investigations on q -generalizations of this subject by many other authors; see [9–17]. More recently, Mahmudov et al. [18] used the q -Mittag-Leffler function

$$E_{1,m+1}(z; q) := \frac{z^m}{e_q^z - \sum_{h=0}^{m-1} \frac{z^h}{[h]_q!}}, \quad m \in \mathbb{N},$$

to define the generalized q -Apostol Bernoulli numbers and q -Apostol Bernoulli polynomials in x, y of order α and level m using the following generating functions, respectively

$$\left(\frac{z^m}{\lambda e_q^z + T_{m-1,q}(z)} \right)^\alpha = \sum_{n=0}^{\infty} B_{n,q}^{[m-1,\alpha]}(\lambda) \frac{z^n}{[n]_q!},$$

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$$\left(\frac{z^m}{\lambda e_q^z - T_{m-1,q}(z)}\right)^\alpha e_q^{xz} E_q^{yz} = \sum_{n=0}^\infty B_{n,q}^{[m-1,\alpha]}(x, y; \lambda) \frac{z^n}{[n]_q!}, \tag{1.2}$$

where $T_{m-1,q}(z) = \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!}$, $\alpha, \lambda, q \in \mathbb{C}$, $m \in \mathbb{N}$ and $0 < |q| < 1$.

In the present work, we introduce some algebraic properties from the polynomials given in [18] when $\alpha = 1$ and $\lambda = 1$, called q -generalized Bernoulli $B_n^{[m-1]}(x; q)$ of level m , and to research some relations between the q -generalized Bernoulli polynomials of level m and q -gamma function, the q -Stirling numbers of the second kind and the q -Bernstein polynomials.

The paper is organized as follows. Section 2 contains the basic backgrounds about the q -analogue of the generalized Bernoulli polynomials of level m , and some other auxiliary results which we will use throughout the paper. In the Section 3, we introduce some relevant algebraic and differential properties of the q -generalized Bernoulli polynomials of level m . Finally, in Section 4, we show the corresponding relations between q -generalized Bernoulli polynomials of level m and the q -gamma function, as well as the q -Stirling numbers of the second of the kind and the q -Bernstein polynomials.

2 Previous definitions and notations

In this paper, we denote by $\mathbb{N}, \mathbb{N}_0, \mathbb{R}, \mathbb{R}^+$, and \mathbb{C} the sets of natural, nonnegative integer, real, positive real and complex numbers, respectively. The following q -standard definitions and properties can be found in [19–23]. The q -numbers and q -factorial numbers are defined respectively by

$$[z]_q = \frac{1 - q^z}{1 - q} = \frac{q^z - 1}{q - 1}, \quad z \in \mathbb{C}, \quad q \in \mathbb{C} \setminus \{1\}, \quad q^z \neq 1,$$

$$[n]_q! = \prod_{k=1}^n [k]_q = [1]_q [2]_q [3]_q \dots [n]_q, \quad [0]_q! = 1, \quad n \in \mathbb{N}.$$

The q -shifted factorial is defined as

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{j=0}^{n-1} (1 - q^j a), \quad n \in \mathbb{N},$$

$$(a; q)_\infty = \prod_{j=0}^\infty (1 - q^j a), \quad a, q \in \mathbb{C}; \quad |q| < 1.$$

The q -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}, \quad (n, k \in \mathbb{N}_0; 0 \leq k \leq n).$$

The q -analogue of the function $(x + y)^n$ is defined by

$$(x + y)_q^n := \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} x^{n-k} y^k, \quad n \in \mathbb{N}_0,$$

$$(1 - a)_q^n = (a; q)_n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} (-1)^k a^k = \prod_{j=0}^{n-1} (1 - q^j a).$$

The q -derivative of a function $f(z)$ is defined by

$$D_q f(z) = \frac{d_q f(z)}{d_q z} = \frac{f(qz) - f(z)}{(q-1)z}, \quad 0 < |q| < 1, \quad Q = z \in \mathbb{C}.$$

The q -analogue of the exponential function is defined in two ways

$$e_q^z = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1 - (1-q)q^k z)}, \quad 0 < |q| < 1, \quad |z| < \frac{1}{|1-q|}, \tag{2.1}$$

$$E_q^z = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)} z^n}{[n]_q!} = \prod_{k=0}^{\infty} (1 + (1-q)q^k z), \quad 0 < |q| < 1, \quad z \in \mathbb{C}.$$

In this sense, we can see that

$$e_q^z \cdot E_q^{-z} = 1, \\ e_q^z \cdot E_q^w = e_q^{z+w}.$$

Therefore,

$$D_q e_q^z = e_q^z, \quad D_q E_q^z = E_q^{qz}.$$

Definition 2.1. For any $t > 0$

$$\Gamma_q(t) = \int_0^{\infty} x^{t-1} E_q^{-qx} d_q x$$

is called the q -gamma function.

The Jackson’s q -gamma function is defined in [20, 24] as follows

$$\Gamma_q(x) = \frac{(q; q)_{\infty}}{(q^x; q)_{\infty}} (1-q)^{1-x}, \quad 0 < |q| < 1,$$

replacing x by $n + 1$ we have

$$\Gamma_q(n + 1) = \frac{(q; q)_{\infty}}{(q^{n+1}; q)_{\infty}} (1-q)^{-n} = (q; q)_n (1-q)^{-n} = [n]_q!, \quad n \in \mathbb{N}.$$

Furthermore, it satisfies the following relations

$$\Gamma_q(1) = 1, \quad \Gamma_q(n) = [n-1]_q!, \quad \Gamma_q(x+1) = [x]_q \Gamma_q(x).$$

Definition 2.2. [25] For $\alpha, \beta, \gamma \in \mathbb{C}, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0, \operatorname{Re}(\gamma) > 0$ and $|q| < 1$ the function $E_{\alpha, \beta}^{\gamma}(z; q)$ is defined as

$$E_{\alpha, \beta}^{\gamma}(z; q) = \sum_{n=0}^{\infty} \frac{(q^{\gamma}; q)_n}{(q; q)_n} \frac{z^n}{\Gamma_q(\alpha n + \beta)}.$$

Note that when $\gamma = 1$ the equation above is expressed as

$$E_{\alpha, \beta}(z; q) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(\alpha n + \beta)}. \tag{2.2}$$

From (2.2), setting $\alpha = 1$ and $\beta = m + 1$, we can deduce that

$$\sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(n+m+1)} = \sum_{n=0}^{\infty} \frac{z^n}{[n+m]_q!} = \frac{1}{z^m} \sum_{h=m}^{\infty} \frac{z^h}{[h]_q!} = \frac{\left(e_q^z - \sum_{h=0}^{m-1} \frac{z^h}{[h]_q!} \right)}{z^m}. \tag{2.3}$$

The q -Stirling number of the first kind $s(n, k)_q$ and the q -Stirling number of the second kind $S(n, k)_q$ are the coefficients in the expansions, (see [26, p.173])

$$(x)_{n;q} = \sum_{k=0}^n s(n, k)_q x^k,$$

$$x^n = \sum_{k=0}^n S(n, k)_q (x)_{k,q}, \tag{2.4}$$

where

$$(x)_{k,q} = \prod_{n=0}^{k-1} (x - [n]_q).$$

Let $C[0, 1]$ denote the set of continuous functions on $[0, 1]$. For any $f \in C[0, 1]$, the q - $\mathbb{B}_n(f; x)$ is called the q -Bernstein operator of order n for f and is defined as (see [15, p.3 Eq. (28)])

$$\mathbb{B}_n(f; x) = \sum_{r=0}^n f_r \begin{bmatrix} n \\ r \end{bmatrix}_q x^r \prod_{s=0}^{n-r-1} (1 - q^s x) = \sum_{r=0}^n f_r b_{n,r}(x),$$

where $f_r = f([r]_q/[n]_q)$. The q -Bernstein polynomials of degree n or a q -Bernstein basis are defined by

$$b_{n,r}(x) = \begin{bmatrix} n \\ r \end{bmatrix}_q x^r \prod_{s=0}^{n-r-1} (1 - q^s x).$$

We know that $\sum_{k=0}^{n-j} b_{n-j,k}(x) = 1$, and so

$$x^j = \sum_{k=0}^{n-j} \begin{bmatrix} n-j \\ k \end{bmatrix}_q x^{j+k} \prod_{t=0}^{n-j-k-1} (1 - q^t x).$$

By using the identity

$$\begin{bmatrix} n-j \\ k-j \end{bmatrix}_q = \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q}{\begin{bmatrix} n \\ j \end{bmatrix}_q},$$

we have

$$x^j = \sum_{k=j}^n \frac{\begin{bmatrix} k \\ j \end{bmatrix}_q}{\begin{bmatrix} n \\ j \end{bmatrix}_q} b_{n,k}(x). \tag{2.5}$$

Otherwise, setting $\alpha = \lambda = 1$ in the equation (1.2), we have the following definition:

Definition 2.3. Let $m \in \mathbb{N}$, $q, z \in \mathbb{C}$, $0 < |q| < 1$. The q -generalized Bernoulli polynomials $B_n^{[m-1]}(x; q)$ of level m are defined in a suitable neighborhood of $z = 0$ by means of the generating function

$$\left(\frac{z^m}{e_q^z - \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!}} \right) e_q^{xz} E_q^{yz} = \sum_{n=0}^{\infty} B_n^{[m-1]}(x + y; q) \frac{z^n}{[n]_q!}, \quad |z| < 2\pi, \tag{2.6}$$

where the q -generalized Bernoulli numbers of level m are defined by

$$B_n^{[m-1]}(q) := B_n^{[m-1]}(0; q).$$

Furthermore,

$$B_n^{[m-1]}(x, y; q) := B_n^{[m-1]}(x + y; q),$$

$$B_n^{[m-1]}(x, 0; q) := B_n^{[m-1]}(x; q),$$

$$B_n^{[m-1]}(0, y; q) := B_n^{[m-1]}(y; q).$$

The first three q -generalized Bernoulli polynomials of level m (cf. [18, p.7]) are

$$\begin{aligned} B_0^{[m-1]}(x; q) &= [m]_q!, \\ B_1^{[m-1]}(x; q) &= [m]_q! \left(x - \frac{1}{[m+1]_q} \right), \\ B_2^{[m-1]}(x; q) &= [m]_q! \left(x^2 - \frac{[2]_q x}{[m+1]_q} + \frac{[2]_q q^{m+1}}{[m+2]_q [m+1]_q^2} \right). \end{aligned}$$

Also, the first three q -generalized Bernoulli numbers of level m are

$$\begin{aligned} B_0^{[m-1]}(q) &= [m]_q!, \\ B_1^{[m-1]}(q) &= -\frac{[m]_q!}{[m+1]_q}, \\ B_2^{[m-1]}(q) &= \frac{[2]_q [m]_q! q^{m+1}}{[m+2]_q [m+1]_q^2}. \end{aligned}$$

Definition 2.4. [14] Let $q, \alpha \in \mathbb{C}, 0 < |q| < 1$. The q -Bernoulli polynomials in x, y of order α are defined by means of the generating function

$$\left(\frac{z}{e^z - 1} \right)^\alpha e^{xz} E_q^{yz} = \sum_{n=0}^\infty B_n^{(\alpha)}(x, y; q) \frac{z^n}{[n]_q!}, \quad |z| < 2\pi, \tag{2.7}$$

where the q -Bernoulli numbers of order α are defined by

$$B_n^{(\alpha)}(q) := B_n^{(\alpha)}(0, 0; q).$$

Furthermore

$$\begin{aligned} B_n^{(\alpha)}(x, q) &:= B_n^{(\alpha)}(x, 0; q), \\ B_n^{(\alpha)}(y, q) &:= B_n^{(\alpha)}(0, y; q). \end{aligned}$$

3 Properties of the q -generalized Bernoulli polynomials of level m

In this section, we show some properties of the q -generalized Bernoulli polynomials $B_n^{[m-1]}(x; q)$ of level m . We demonstrated the facts for one of them. Obviously, by applying a similar technique, other ones can be determined. The following proposition summarizes some properties of the polynomials $B_n^{[m-1]}(x; q)$. We will show in details the proofs to (2), (5) and (7).

Proposition 3.1. Let a fixed $m \in \mathbb{N}, n, k \in \mathbb{N}_0$ and $q \in \mathbb{C}, 0 < |q| < 1$. Let $\{B_n^{[m-1]}(x; q)\}_{n=0}^\infty$ be the sequence of q -generalized Bernoulli polynomials of level m . Then the following statements hold.

1. *Summation formula.* For every $n \geq 0$

$$B_n^{[m-1]}(x; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q B_k^{[m-1]}(q) x^{n-k}. \tag{3.1}$$

2. For $n \geq 1$

$$\sum_{k=0}^n \begin{bmatrix} n+m \\ k \end{bmatrix}_q B_k^{[m-1]}(q) = 0.$$

3. Addition formulas

$$B_n^{[m-1]}(x + y; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}(n-k)(n-k-1)} B_k^{[m-1]}(x; q) y^{n-k}, \tag{3.2}$$

$$B_n^{[m-1]}(x + y; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}(n-k)(n-k-1)} B_k^{[m-1]}(y; q) x^{n-k}, \tag{3.3}$$

$$B_n^{[m-1]}(x + y; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q B_k^{[m-1]}(q) (x + y)_q^{n-k},$$

$$B_n^{[m-1]}(x + y; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q B_k^{[m-1]}(y; q) x^{n-k}.$$

4. Inversion formulas

$$x^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[k]_q!}{[k + m]_q!} B_{n-k}^{[m-1]}(x; q), \tag{3.4}$$

$$y^n = \frac{[n]_q!}{q^{\frac{1}{2}n(n-1)} [n + m]_q!} \sum_{k=0}^n \begin{bmatrix} n + m \\ k \end{bmatrix}_q B_k^{[m-1]}(y; q), \tag{3.5}$$

$$x^n = \sum_{k=0}^n \frac{[n]_q! B_k^{[m-1]}(x; q)}{[k]_q! \Gamma_q(n - k + m + 1)}. \tag{3.6}$$

5. Difference equations

$$B_n^{[m-1]}(1 + y; q) - B_n^{[m-1]}(y; q) = [n]_q \sum_{k=0}^{n-1} \begin{bmatrix} n - 1 \\ k \end{bmatrix}_q B_k^{[m-1]}(y, q) B_{n-1-k}^{(-1)}(q).$$

6. Differential relations. For $m \in \mathbb{N}$ and $n, j \in \mathbb{N}_0$, where $0 \leq j \leq n$, we have

$$D_q B_{n+1}^{[m-1]}(x; q) = [n + 1]_q B_n^{[m-1]}(x; q), \tag{3.7}$$

$$D_q^{(j)} B_n^{[m-1]}(x; q) = \frac{[n]_q!}{[n - j]_q!} B_{n-j}^{[m-1]}(x; q).$$

7. Integral formulas

$$\int_{x_0}^{x_1} B_n^{[m-1]}(x; q) d_q x = \frac{B_{n+1}^{[m-1]}(x_1; q) - B_{n+1}^{[m-1]}(x_0; q)}{[n + 1]_q}, \quad n \in \mathbb{N}_0, \tag{3.8}$$

$$B_n^{[m-1]}(x; q) = [n]_q \int_0^x B_{n-1}^{[m-1]}(x; q) d_q x + B_n^{[m-1]}(q), \quad n \in \mathbb{N},$$

$$\int_{x_0}^{x_1} B_n^{[m-1]}(x; q) d_q x = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q B_k^{[m-1]}(q) \left(\frac{x_1^{n+1-k} - x_0^{n+1-k}}{[n + 1 - k]_q} \right).$$

Proof. To prove (2), we start with (2.1) and (2.6), from which it follows that

$$z^m = \left(\sum_{n=0}^{\infty} B_n^{[m-1]}(q) \frac{z^n}{[n]_q!} \right) \left(\sum_{h=m}^{\infty} \frac{z^h}{[h]_q!} \right) = \left(\sum_{n=0}^{\infty} B_n^{[m-1]}(q) \frac{z^n}{[n]_q!} \right) \left(\sum_{j=0}^{\infty} \frac{z^{j+m}}{[j + m]_q!} \right),$$

and therefore

$$1 = \left(\sum_{n=0}^{\infty} B_n^{[m-1]}(q) \frac{z^n}{[n]_q!} \right) \left(\sum_{j=0}^{\infty} \frac{z^j}{[j + m]_q!} \right)$$

$$\begin{aligned} &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{B_k^{[m-1]}(q)}{[k]_q!} \cdot \frac{z^n}{[n-k+m]_q!} \\ &= \frac{B_0^{[m-1]}(q)}{[m]_q!} + \sum_{n=1}^{\infty} \sum_{k=0}^n \frac{B_k^{[m-1]}(q)}{[k]_q!} \cdot \frac{z^n}{[n-k+m]_q!}. \end{aligned}$$

By comparing coefficients of $\frac{z^n}{[n]_q!}$, we have

$$1 = \frac{B_0^{[m-1]}(q)}{[m]_q!} \Rightarrow B_0^{[m-1]}(q) = [m]_q!$$

and

$$\sum_{k=0}^n \frac{B_k^{[m-1]}(q)}{[k]_q! [n-k+m]_q!} = 0.$$

By multiplying $[n+m]_q!$ on both sides of the equation above, we have

$$\sum_{k=0}^n \frac{B_k^{[m-1]}(q) [n+m]_q!}{[k]_q! [n-k+m]_q!} = 0 \Rightarrow \sum_{k=0}^n \begin{bmatrix} n+m \\ k \end{bmatrix}_q B_k^{[m-1]}(q) = 0.$$

□

Proof. Proof of (5). Considering the expression $B_n^{[m-1]}(1+y; q) - B_n^{[m-1]}(y; q)$ and using the generating functions (2.6) and (2.7), we have

$$\begin{aligned} I &:= \sum_{n=0}^{\infty} B_n^{[m-1]}(1+y; q) \frac{z^n}{[n]_q!} - \sum_{n=0}^{\infty} B_n^{[m-1]}(y; q) \frac{z^n}{[n]_q!} = \left(\frac{z^m}{e_q^z - \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!}} \right) E_q^{yz} (e_q^z - 1) \\ &= z \sum_{n=0}^{\infty} B_n^{[m-1]}(y; q) \frac{z^n}{[n]_q!} \sum_{n=0}^{\infty} B_n^{(-1)}(q) \frac{z^n}{[n]_q!}. \end{aligned}$$

Therefore

$$\begin{aligned} I &= \sum_{n=0}^{\infty} B_n^{[m-1]}(y; q) \frac{z^n}{[n]_q!} \sum_{n=1}^{\infty} B_{n-1}^{(-1)}(q) \frac{z^n}{[n-1]_q!} \\ &= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} B_k^{[m-1]}(y; q) \frac{z^k}{[k]_q!} B_{n-1-k}^{(-1)}(q) \frac{z^{n-k}}{[n-1-k]_q!} \\ &= \sum_{n=1}^{\infty} [n]_q \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q B_k^{[m-1]}(y; q) B_{n-1-k}^{(-1)}(q) \frac{z^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} [n]_q \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q B_k^{[m-1]}(y; q) B_{n-1-k}^{(-1)}(q) \frac{z^n}{[n]_q!}. \end{aligned}$$

By comparing coefficients of $\frac{z^n}{[n]_q!}$ on both sides we obtain the result. □

Proof. Proof of (7). From (3.7) we have

$$B_n^{[m-1]}(x; q) = \frac{1}{[n+1]_q} D_q B_{n+1}^{[m-1]}(x; q).$$

Now, by integrating on both sides of the equation above, we get

$$\int_{x_0}^{x_1} B_n^{[m-1]}(x; q) d_q x = \frac{1}{[n+1]_q} \int_{x_0}^{x_1} D_q B_{n+1}^{[m-1]}(x; q) d_q x$$

$$\begin{aligned}
 &= \frac{1}{[n+1]_q} B_{n+1}^{[m-1]}(x; q) \Big|_{x_0}^{x_1} \\
 &= \frac{B_{n+1}^{[m-1]}(x_1; q) - B_{n+1}^{[m-1]}(x_0; q)}{[n+1]_q}.
 \end{aligned}$$

Setting $x_0 = 0$ and $x_1 = x$ in (3.8), we have

$$\int_0^x B_n^{[m-1]}(x; q) d_q x = \frac{B_{n+1}^{[m-1]}(x; q) - B_{n+1}^{[m-1]}(q)}{[n+1]_q},$$

and so

$$\int_0^x B_{n-1}^{[m-1]}(x; q) d_q x = \frac{B_n^{[m-1]}(x; q) - B_n^{[m-1]}(q)}{[n]_q}.$$

Finally, we get

$$B_n^{[m-1]}(x; q) = [n]_q \int_0^x B_{n-1}^{[m-1]}(x; q) d_q x + B_n^{[m-1]}(q).$$

□

4 Some connection formulas for the polynomials

$B_n^{[m-1]}(x + y; q)$

From identities (2.4), (2.5) and Proposition 3.1 we can deduce some interesting algebraic relations between the q -generalized Bernoulli polynomials of level m with the q -gamma function, the q -Stirling numbers of the second kind and the q -Bernstein polynomials.

Proposition 4.1. For $n, j, k \in \mathbb{N}_0$, $q \in \mathbb{C}$ where $0 < |q| < 1$ and where $m \in \mathbb{N}$, the q -generalized Bernoulli polynomials of level m are related with the q -gamma function by the means of the following identity

$$B_n^{[m-1]}(x + y; q) = [n]_q! \sum_{k=0}^n \sum_{j=0}^{n-k} \frac{B_k^{[m-1]}(x; q)}{[k]_q!} \frac{q^{\frac{1}{2}(n-j)(n-j-1)}}{[j]_q! \Gamma_q(n-j-k+m+1)} B_j^{[m-1]}(y; q). \tag{4.1}$$

Proof. By substituting (3.6) in (3.3), we have

$$\begin{aligned}
 B_n^{[m-1]}(x + y; q) &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\frac{1}{2}(n-j)(n-j-1)} B_j^{[m-1]}(y; q) \sum_{k=0}^{n-j} \frac{[n-j]_q! B_k^{[m-1]}(x; q)}{[k]_q! \Gamma_q(n-j-k+m+1)} \\
 &= \sum_{k=0}^n \sum_{j=0}^{n-k} \frac{B_k^{[m-1]}(x; q)}{[k]_q!} \frac{[n]_q! q^{\frac{1}{2}(n-j)(n-j-1)}}{[j]_q! \Gamma_q(n-j-k+m+1)} B_j^{[m-1]}(y; q) \\
 &= [n]_q! \sum_{k=0}^n \sum_{j=0}^{n-k} \frac{B_k^{[m-1]}(x; q)}{[k]_q!} \frac{q^{\frac{1}{2}(n-j)(n-j-1)}}{[j]_q! \Gamma_q(n-j-k+m+1)} B_j^{[m-1]}(y; q).
 \end{aligned}$$

Corollary 4.1. For $n, j, k \in \mathbb{N}_0$ and $m \in \mathbb{N}$, we have

□

$$B_n^{[m-1]}(x; q) = [n]_q! \sum_{k=0}^n \sum_{j=0}^{n-k} \frac{B_j^{[m-1]}(q) B_k^{[m-1]}(x; q)}{[k]_q! [j]_q! \Gamma_q(n-j-k+m+1)}. \tag{4.2}$$

Proof. By replacing equation (3.6) in (3.1), we obtain

$$\begin{aligned} B_n^{[m-1]}(x; q) &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q B_j^{[m-1]}(q) \sum_{k=0}^{n-j} \frac{[n-j]_q! B_k^{[m-1]}(x; q)}{[k]_q! \Gamma_q(n-j-k+m+1)} \\ &= \sum_{k=0}^n \sum_{j=0}^{n-k} \frac{B_k^{[m-1]}(x; q)}{[k]_q!} \frac{[n]_q! B_j^{[m-1]}(q)}{[j]_q! \Gamma_q(n-j-k+m+1)} \\ &= [n]_q! \sum_{k=0}^n \sum_{j=0}^{n-k} \frac{B_j^{[m-1]}(q) B_k^{[m-1]}(x; q)}{[k]_q! [j]_q! \Gamma_q(n-j-k+m+1)}. \end{aligned}$$

Corollary 4.2. For $n, j, k \in \mathbb{N}_0$ and $m \in \mathbb{N}$, we have □

$$B_n^{[m-1]}(x; q) = \sum_{k=0}^n \sum_{j=0}^{n-k} \frac{[n]_q! B_j^{[m-1]}(q) B_{n-j-k}^{[m-1]}(x; q)}{[k+m]_q! [n-j-k]_q! [j]_q!}. \tag{4.3}$$

Proof. By substituting (3.4) in equation (3.1), we obtain

$$\begin{aligned} B_n^{[m-1]}(x; q) &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q B_j^{[m-1]}(q) \sum_{k=0}^{n-j} \begin{bmatrix} n-j \\ k \end{bmatrix}_q \frac{[k]_q!}{[k+m]_q!} B_{n-j-k}^{[m-1]}(x; q) \\ &= \sum_{k=0}^n \sum_{j=0}^{n-k} \frac{[k]_q!}{[k+m]_q!} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n-j \\ k \end{bmatrix}_q B_j^{[m-1]}(q) B_{n-j-k}^{[m-1]}(x; q) \\ &= \sum_{k=0}^n \sum_{j=0}^{n-k} \frac{[n]_q! B_j^{[m-1]}(q) B_{n-j-k}^{[m-1]}(x; q)}{[k+m]_q! [n-j-k]_q! [j]_q!}. \end{aligned}$$

Corollary 4.3. For $n, j, k \in \mathbb{N}_0$ and $m \in \mathbb{N}$ □

$$B_n^{[m-1]}(x+y; q) = [n]_q! \sum_{j=0}^n \sum_{k=0}^{n-j} \frac{B_j^{[m-1]}(x; q) B_k^{[m-1]}(y; q)}{[j]_q! [n-j+m-k]_q! [k]_q!}. \tag{4.4}$$

Proof. By substituting the equation (3.5) in (3.2), we obtain

$$\begin{aligned} B_n^{[m-1]}(x+y; q) &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\frac{1}{2}(n-j)(n-j-1)} B_j^{[m-1]}(x; q) \frac{[n-j]_q!}{q^{\frac{1}{2}(n-j)(n-j-1)} [n-j+m]_q!} \sum_{k=0}^{n-j} \begin{bmatrix} n-j+m \\ k \end{bmatrix}_q B_k^{[m-1]}(y; q) \\ &= \sum_{j=0}^n \sum_{k=0}^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{[n-j]_q!}{[n-j+m]_q!} \begin{bmatrix} n-j+m \\ k \end{bmatrix}_q B_j^{[m-1]}(x; q) B_k^{[m-1]}(y; q) \\ &= [n]_q! \sum_{j=0}^n \sum_{k=0}^{n-j} \frac{B_j^{[m-1]}(x; q) B_k^{[m-1]}(y; q)}{[j]_q! [n-j+m-k]_q! [k]_q!}. \end{aligned}$$

Proposition 4.2. For $n, j, k \in \mathbb{N}_0$, $q \in \mathbb{C}$ where $0 < |q| < 1$ and where $m \in \mathbb{N}$, the q -generalized Bernoulli polynomials of level m are related with the q -Stirling numbers of the second kind $S(n, k; q)$ by means of the following identities

$$B_n^{[m-1]}(x+y; q) = \sum_{k=0}^n \sum_{j=0}^{n-k} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\frac{1}{2}(n-j)(n-j-1)} B_j^{[m-1]}(y; q) S(n-j, k; q)(x)_{q,j}, \tag{4.5}$$

$$B_n^{[m-1]}(x; q) = \sum_{j=0}^n \sum_{k=0}^j \begin{bmatrix} n \\ j \end{bmatrix}_q B_{n-j}^{[m-1]}(q) S(j, k; q)(x)_{q,k}. \tag{4.6}$$

Proof. Proof of (4.5). By replacing (2.4) in (3.3), we have

$$\begin{aligned} B_n^{[m-1]}(x+y; q) &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\frac{1}{2}(n-j)(n-j-1)} B_j^{[m-1]}(y; q) \sum_{k=0}^{n-j} S(n-j, k; q)(x)_{q;k} \\ &= \sum_{k=0}^n \sum_{j=0}^{n-k} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\frac{1}{2}(n-j)(n-j-1)} B_j^{[m-1]}(y; q) S(n-j, k; q)(x)_{q;k}. \end{aligned}$$

□

Proof. Proof of (4.6). By substituting (2.4) in (3.1), we have

$$\begin{aligned} B_n^{[m-1]}(x; q) &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q B_{n-j}^{[m-1]}(q) \sum_{k=0}^j S(j, k; q)(x)_{q;k} \\ &= \sum_{j=0}^n \sum_{k=0}^j \begin{bmatrix} n \\ j \end{bmatrix}_q B_{n-j}^{[m-1]}(q) S(j, k; q)(x)_{q;k}. \end{aligned}$$

Corollary 4.4. For $n, k \in \mathbb{N}_0$ and $m \in \mathbb{N}$, we obtain

□

$$\sum_{k=0}^n S(n, k)_q(x)_{k,q} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[k]_q!}{[k+m]_q!} B_{n-k}^{[m-1]}(x; q).$$

Proposition 4.3. For $n, j, k \in \mathbb{N}_0, q \in \mathbb{C}$ where $0 < |q| < 1$ and where $m \in \mathbb{N}$ the q -generalized Bernoulli polynomials of level m are related with the q -Bernstein polynomials $b_{n,k}(x; q)$ by means of the following identities

$$B_n^{[m-1]}(x, q) = \sum_{j=0}^n \sum_{k=0}^{n-j} \begin{bmatrix} k+j \\ j \end{bmatrix}_q B_{n-j}^{[m-1]}(q) b_{n,k+j}(x; q), \tag{4.7}$$

$$B_n^{[m-1]}(x+y; q) = \sum_{j=0}^n \sum_{k=0}^{n-j} \begin{bmatrix} k+j \\ j \end{bmatrix}_q q^{\frac{1}{2}(j)(j-1)} B_{n-j}^{[m-1]}(y; q) b_{n,k+j}(x; q). \tag{4.8}$$

Proof. Proof of (4.7). By replacing (2.5) in (3.1), we have

$$\begin{aligned} B_n^{[m-1]}(x, q) &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q B_{n-j}^{[m-1]}(q) \sum_{k=j}^n \frac{\begin{bmatrix} k \\ j \end{bmatrix}_q}{\begin{bmatrix} n \\ j \end{bmatrix}_q} b_{n,k}(x; q) \\ &= \sum_{j=0}^n \sum_{k=j}^n B_{n-j}^{[m-1]}(q) \begin{bmatrix} k \\ j \end{bmatrix}_q b_{n,k}(x; q) \\ &= \sum_{j=0}^n \sum_{k=0}^{n-j} \begin{bmatrix} k+j \\ j \end{bmatrix}_q B_{n-j}^{[m-1]}(q) b_{n,k+j}(x; q). \end{aligned}$$

Proof. Proof of (4.8). By replacing (2.5) in Equation (3.3), we obtain

□

$$\begin{aligned} B_n^{[m-1]}(x+y; q) &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\frac{1}{2}(j)(j-1)} B_{n-j}^{[m-1]}(y; q) \sum_{k=j}^n \frac{\begin{bmatrix} k \\ j \end{bmatrix}_q}{\begin{bmatrix} n \\ j \end{bmatrix}_q} b_{n,k}(x; q) \\ &= \sum_{j=0}^n \sum_{k=j}^n \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\frac{1}{2}(j)(j-1)} B_{n-j}^{[m-1]}(y; q) b_{n,k}(x; q) \\ &= \sum_{j=0}^n \sum_{k=0}^{n-j} \begin{bmatrix} k+j \\ j \end{bmatrix}_q q^{\frac{1}{2}(j)(j-1)} B_{n-j}^{[m-1]}(y; q) b_{n,k+j}(x; q). \end{aligned}$$

□

Corollary 4.5. For $n, j \in \mathbb{N}_0$ and $x \in [0, 1]$, we have

$$\sum_{k=j}^n \frac{\begin{bmatrix} k \\ j \end{bmatrix}_q}{\begin{bmatrix} n \\ j \end{bmatrix}_q} b_{n,k}(x; q) = \sum_{k=0}^j \begin{bmatrix} j \\ k \end{bmatrix}_q \frac{[k]_q!}{[k+m]_q!} B_{j-k}^{[m-1]}(x; q).$$

Proposition 4.4. For $n, j, k \in \mathbb{N}_0$ and $n \geq j \geq k \geq 0$, we have

$$b_{n,k}(x; q) = x^k \sum_{j=0}^{n-k} \begin{bmatrix} n-j \\ k \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{[j]_q! B_{n-k-j}^{[m-1]}(1-x; q)}{\Gamma_q(j+m+1)}, \tag{4.9}$$

$$b_{n,k}(x; q) = x^k \sum_{j=0}^{n-k} \begin{bmatrix} n-j \\ k \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{[j]_q! B_{n-k-j}^{[m-1]}(1-x; q)}{[j+m]_q!}. \tag{4.10}$$

Proof. To prove (4.9), we used the following equality [14, Theorem 19, p. 10]

$$\frac{x^k z^k}{[k]_q!} e_q^z E_q^{-xz} = \sum_{n=k}^{\infty} b_{n,k}(x; q) \frac{z^n}{[n]_q!}.$$

We see that

$$\frac{x^k z^k}{[k]_q!} e_q^z E_q^{-xz} = \frac{x^k z^k}{[k]_q!} \frac{\left(e_q^z - \sum_{h=0}^{m-1} \frac{z^h}{[h]_q!} \right)}{z^m} \frac{z^m}{\left(e_q^z - \sum_{h=0}^{m-1} \frac{z^h}{[h]_q!} \right)} e_q^z E_q^{-xz}.$$

Next, by using the equations (2.3) and (2.6), we get

$$\begin{aligned} \frac{x^k z^k}{[k]_q!} e_q^z E_q^{-xz} &= \frac{x^k z^k}{[k]_q!} \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(n+m+1)} \sum_{j=0}^{\infty} B_j^{[m-1]}(1-x; q) \frac{z^j}{[j]_q!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^n \frac{x^k z^k}{[k]_q!} \frac{z^j}{\Gamma_q(j+m+1)} B_{n-j}^{[m-1]}(1-x; q) \frac{z^{n-j}}{[n-j]_q!} \\ &= \sum_{n=k}^{\infty} \sum_{j=0}^{n-k} \frac{x^k}{[k]_q!} \frac{B_{n-k-j}^{[m-1]}(1-x; q)}{\Gamma_q(j+m+1)} \frac{z^n}{[n-k-j]_q!} \\ &= \sum_{n=k}^{\infty} x^k \sum_{j=0}^{n-k} \begin{bmatrix} n-j \\ k \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{[j]_q! B_{n-k-j}^{[m-1]}(1-x; q)}{\Gamma_q(j+m+1)} \frac{z^n}{[n]_q!}. \end{aligned}$$

By comparing coefficients of $\frac{z^n}{[n]_q!}$ on both sides we obtain

$$b_{n,k}(x; q) = x^k \sum_{j=0}^{n-k} \begin{bmatrix} n-j \\ k \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{[j]_q! B_{n-k-j}^{[m-1]}(1-x; q)}{\Gamma_q(j+m+1)}.$$

To demonstrate (4.10) we used the identity $\Gamma_q(j+m+1) = [j+m]_q!$ and Equation (4.9). Continuing this process, we get

$$b_{n,k}(x; q) = x^k \sum_{j=0}^{n-k} \begin{bmatrix} n-j \\ k \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{[j]_q! B_{n-k-j}^{[m-1]}(1-x; q)}{[j+m]_q!}.$$

□

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