Interaction Free Measurements and Weinberg's Soft Photons Theorem

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and

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Resumo

É apresentada uma descrição e intrepretação do Teorema de Weinberg para Fotões Moles, usado para a regularização das divergências no infravermelho em Eletrodinâmica Quântica. Este esquema pode ser estendido de modo a incluir a descrição de qualquer fotão de suficientemente baixa energia, independentemente do grau de detetabilidade, deduzindo assim, os fatores corretivos e probabilidade de emissão destes fotões, com esta última a seguir a probabilidade de Poisson esperada semiclassicamente. Um cálculo explícito das probabilidades para o caso de dispersão eletrão-fotão permite concluir que, mesmo no caso mais extremo de fermiões ultrarrelativísticos, a taxa de emissão destes fotões é extremamente baixa, caso o limite mínimo permitido às suas energias seja não-nulo. Qualquer detetor de fotões com um limiar energético de deteção inferior não-nulo teria apenas associado um termo corretivo desprezável. Detetores perfeitos são considerados impossíveis na discussão que se segue. Finalmente, apresento uma descrição de Teoria Quântica de Campo das medidas sem interação e argumento que muitos dos conceitos paradoxais em Mecânica Quântica, envolvendo estas interações, são satisfatoriamente explicadas nesse contexto.

Palavras-chave: Medidas Sem Interação; Divergências Infravermelhas; Fotões moles; Teorema de Weinberg para Fotões Moles

Abstract

I present a detailed mathematical description and interpretation of Weinberg's Soft Photon Theorem for regularising infrared divergences in Quantum Electrodynamics. The theorem can easily be extended to any low-energy photons, regardless of the degree of detectability, thus deducing the correction factors and probability of emission, the latter mirroring the semiclassical Poisson distribution, for these photons. An explicit calculation of those probabilities for the case of electron-photon scattering leads to the conclusion that, even in the most extreme case of ultrarrelativistic fermions, these photons have an extremely low emission rate if the lower limit allowed to their energies to be nonvanishing. Any photon detector with nonzero minimal threshold for detection energy would only get a negligible correction factor. Perfect detectors are deemed impossible in the subsequent discussion. Finally, I give a Quantum Field description of Interaction Free Measurements, and argue that many of the paradoxical concepts about these interactions in Quantum Mechanics have a satisfactory explanation in this framework.

Keywords: Interaction Free Measurements; Infrared divergences; Soft photons; Weinberg's Soft Photons Theorem

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Abbreviations

IFM Interaction Free Measurement

IR Infrared

QED Quantum Electrodynamics

QFT Quantum Field Theory

QM Quantum Mechanics

Conventions

Natural units, $c = \hbar = 1$, are adopted throughout this work. The signature for the Minkowski metric, $\eta_{\mu\nu}$, used is $(-,+,+,+)$. The symbols for particles and propagators are the standard ones in contemporary literature, see Ref. [1, 2]. Feynman diagrams, drawn using TikZ-Feynman [3], follow the left to right convention for the flow of time. Einstein's summation convention is generally used. The following simplified notations were introduced [4],

$$\sum_{\vec{p}} \equiv \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}}, \quad \delta_{\vec{p}-\vec{q}} \equiv (2\pi)^3 2E_{\vec{p}}\,\delta(\vec{p}-\vec{q})\,.$$

Introduction

The original objective of this thesis was to learn more about Elitzur-Vaidman Interaction Free Measurements. This thought experiment [5], better discussed in Chapter 2 and related Appendix B, proposes a way to obtain information about the location of an object without interacting with it. To showcase this possibility, the authors propose a bomb, infinitely sensible to any photon, and claim to be able to localise without exploding it. This raised objections by some authors on pragmatic grounds [6]. The claim about using very low-energy photons, called soft, lead to the consideration of relating the experiment with Weinberg's Soft Photons Theorem, used to regularise infrared divergences in Quantum Electrodynamics [1, 7]. The aim of understanding in-depth this theorem was in the root of a long detour through its mathematical details, displayed in Chapter 1 and the associated Appendix A.

This theorem allows for the computation of probabilities of emission of photons with low energies, which, if detectable, indicate the impossibliness of the bomb. In order to relate this theorem with Interaction Free Measurements, a Quantum Field description of the Elitzur-Vaidman thought experiment, better suited for discussion of ranges of interaction, is presented. This formalism allows for a seemingly less paradoxical interpretation than the purely quantum mechanical case, plagued by the particle-wave duality.

1. Weinberg's Soft Photons Theorem

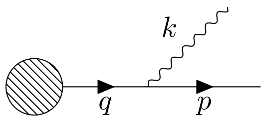
In this chapter, I analyse the soft photon approximation and its role in solving the infrared (IR) divergences of Quantum Electrodynamics (QED). The derivation will mostly follow the one developed by Weinberg [1, 7], but similar deductions can be seen in Refs. [2, 8, 9], and in the original paper by Bloch and Nordsieck in Ref. [10].

Section 1.1 treats the corrections, due to low-energy photons with specific energy, detectable or not, introduced to the M matrix; Section 1.2 resolves the virtual infrared divergences with the introduction of virtual soft photons; Section 1.3 deals with the full correction to the transition rates, by further introducing the modifications due to real soft photons, and provides the probability of emission, valid for low-energy photons, both detectable or not. This chapter has an associated appendix, Appendix A, where some of the calculations are developed, and some explanations of physical quantities are provided.

1.1. Emission and absorption of low-energy photons

Consider a QED process, $\alpha \to \beta$, involving an arbitrary number of charged particles. When interacting, charged particles can emit and absorb photons; some of them, weakly energetic. The most weakly energetic can even evade any physical detector. Such low energy photons have energies much smaller than the typical rest masses of the charged particles involved in the process. Let us, then, compute how the $\alpha \to \beta$ process is altered by the presence of those photons.

The emission of a low-energy photon from a fermion of charge e in β,



,

implies, at first order, the additional factor in the M matrix,

$$
\begin{aligned}
& ie\int \frac{d^4q}{(2\pi)^4}\bar{u}_\sigma(\vec{p})\gamma^\mu S_{\mathrm{F}}(q)\epsilon_\mu^{\lambda *}\left(\vec{k}\right)(2\pi)^4\delta(p+k-q) \\
= & ie\bar{u}_\sigma(\vec{p})\gamma^\mu \frac{i\left(\not{p}+\not{k}-m\right)}{(p+k)^2+m^2-i\varepsilon}\epsilon_\mu^{\lambda *}\left(\vec{k}\right).
\end{aligned}
$$

Noticing that

$$
\gamma^\mu\left(\not{p}-m\right)=\left(-\not{p}+m\right)\gamma^\mu-2p^\mu-2m\gamma^\mu,
$$

using the closing relation,

$$
\sum_\sigma u_\sigma(\vec{p})\,\bar{u}_\sigma(\vec{p})=-\not{p}+m,
$$

expanding the denominator and cancelling the terms on-shell, in the limit of small energies,[1] we have,

$$
-e\bar{u}_\sigma(\vec{p})\,\frac{\sum_{\sigma'}u_{\sigma'}(\vec{p})\,\bar{u}_{\sigma'}(\vec{p})\,\gamma^\mu-2p^\mu-2m\gamma^\mu}{2p\cdot k-i\varepsilon}\epsilon_\mu^{\lambda *}\left(\vec{k}\right).
$$

Simplifying the numerator,

$$
\sum_{\sigma'}\bar{u}_\sigma(\vec{p})u_{\sigma'}(\vec{p})\bar{u}_{\sigma'}(\vec{p})=\sum_{\sigma'}2m\delta_{\sigma\sigma'}\bar{u}_{\sigma'}(\vec{p})=2m\bar{u}_\sigma(\vec{p})\,,
$$

we get

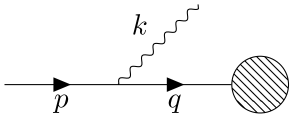
$$
-e\bar{u}_\sigma(\vec{p})\frac{2m\gamma^\mu-2p^\mu-2m\gamma^\mu}{2p\cdot k-i\varepsilon}\epsilon_\mu^{\lambda *}\left(\vec{k}\right).
$$

The $\bar{u}_\sigma\left(\vec{p}\right)$ term was already part of the original $M_{\alpha\to\beta}$ matrix, so we reintroduce it there, yielding the correction factor

$$
\frac{ep^\mu}{p\cdot k-i\varepsilon}\epsilon_\mu^{\lambda *}\left(\vec{k}\right).
$$

[1] Equivalently we can simply neglect the square of the the 4-momentum as being of second order. When on-shell, k^2 is identically null, but for the cases that will arise later, where the soft photon is off-shell and k^2 does not simply cancel out, we will use that argument to still ignore the term.

In the case where the photon is emitted from an initial leg, in α,



,

the element in the M matrix is

$$
\begin{aligned}
& ie\int \frac{d^4q}{(2\pi)^4} S_{\mathrm{F}}(q)\,\gamma^\mu \epsilon_\mu^{\lambda *}\left(\vec{k}\right) u_\sigma(\vec{p})(2\pi)^4\delta(q+k-p) \\
&= ie\frac{i\left(\not{p}-\not{k}-m\right)}{(p-k)^2+m^2-i\varepsilon}\gamma^\mu\epsilon_\mu^{\lambda *}\left(\vec{k}\right)u_\sigma(\vec{p})\,.
\end{aligned}
$$

The previous argument still works, leading to the additional factor

$$
\frac{ep^\mu}{-p\cdot k-i\varepsilon}\epsilon_\mu^{\lambda *}\left(\vec{k}\right) = -\frac{ep^\mu}{p\cdot k+i\varepsilon}\epsilon_\mu^{\lambda *}\left(\vec{k}\right).
$$

We arrive at the conclusion that the emission of a single low-energy photon of 4-momentum k from a branch of 4-momentum p leads to the correction

$$
M_{\alpha\to\beta} \to M_{\alpha\to\beta}\left(\frac{\xi e p^\mu \epsilon_\mu^{\lambda *}\left(\vec{k}\right)}{p\cdot k - i\xi\varepsilon}\right),
$$

where $\xi = 1$ or $\xi = -1$, whether the photon is emitted from a final leg or an initial one, respectively.

Since

$$
p\cdot k = \vec{p}\cdot\vec{k} - E_f E_\gamma = E_\gamma\left(|\vec{p}|\cos(\theta) - \sqrt{|\vec{p}|^2+m^2}\right),
$$

we can easily verify that the only divergence in the denominator occurs when $E_\gamma \to 0$, for nonzero values of m. Therefore, we can now drop the prescription $i\varepsilon$.

The transition probability is altered to

$$
|iM_{\alpha\to\beta}|^2 \to |iM_{\alpha\to\beta}|^2\left|\frac{e^2p^\mu p^\nu\eta_{\mu\nu}}{(p\cdot k)^2}\right| = |iM_{\alpha\to\beta}|^2\left|\frac{em}{p\cdot k}\right|^2,
$$

where we have performed a polarisation sum to obtain the $\eta_{\mu\nu}$ factor.

Of course, if R charged particles of charges e_n intervene in the process, either at the beginning or the end, we have R possible sources for the emitted photon. All possibilities of emission are accounted for with a sum over their probabilities.

The correction to $M_{\alpha\to\beta}$ then takes the form,

$$M_{\alpha\to\beta} \to M_{\alpha\to\beta} \sum_{n=1}^{R} \left(\frac{\xi_n e_n p_n^\mu}{p_n \cdot k} \right) \epsilon_\mu^{\lambda *}\left(\vec{k}\right).$$

If we consider

$$\mathcal{M}^\mu(k) = M_{\alpha\to\beta} \sum_n \left(\frac{\xi_n e_n p_n^\mu}{p_n \cdot k} \right)$$

which satisfies Ward's identity for on-shell photons, $k_\mu \mathcal{M}^\mu = 0$, we have

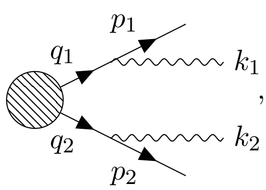
$$\sum_n \left(\frac{\xi_n e_n p_n \cdot k}{p_n \cdot k} \right) M_{\alpha\to\beta} = \left(\sum_n \xi_n e_n \right) M_{\alpha\to\beta} = 0.$$

For a non-zero transition process,

$$\sum_n \xi_n e_n = 0 \Leftrightarrow \sum_f e_f = \sum_i e_i,$$

which is just stating that total charge is conserved in the process.

We now want to consider the corrections due to the emission of more than one low-energy photon. When two emissions originate in two different branches, the corrections simply stack. The diagram, where the legs emitting the photons can be either initial or final,



,

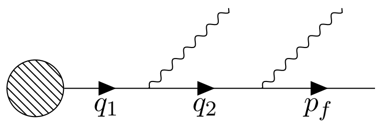
has a matrix element,

$$
\begin{aligned}
&(2\pi)^8 \int \frac{d^4 q_1}{(2\pi)^4} \Big(i e_1 \bar{u}_{\sigma_1}(\vec{p_1})\, \gamma^\mu S_{\mathrm{F}}(q_1)\, \epsilon_\mu^{\lambda_1 *}\Big(\vec{k_1}\Big)\, \delta(p_1 + k_2 - q_1) \Big) \\
&\times \int \frac{d^4 q_2}{(2\pi)^4} \Big(i e_2 \bar{u}_{\sigma_2}(\vec{p_2})\, \gamma^\nu S_{\mathrm{F}}(q_2)\, \epsilon_\nu^{\lambda_2 *}\Big(\vec{k_2}\Big)\, \delta(p_2 + k_2 - q_2) \Big) \\
&= \bar{u}_{\sigma_1}(\vec{p_1})\, \bar{u}_{\sigma_2}(\vec{p_2}) \left(\frac{\xi_1 e_1 p_1^\mu \epsilon_\mu^{\lambda_1 *}\Big(\vec{k_1}\Big)}{p_1 \cdot k_1 - i \xi_1 \varepsilon} \right) \left(\frac{\xi_2 e_2 p_2^\mu \epsilon_\mu^{\lambda_2 *}\Big(\vec{k_2}\Big)}{p_2 \cdot k_2 - i \xi_2 \varepsilon} \right),
\end{aligned}
$$

corresponding to the correction,

$$
M_{\alpha \to \beta} \to M_{\alpha \to \beta} \left(\frac{\xi_1 e_1 p_1^\mu \epsilon_\mu^{\lambda_1 *}\Big(\vec{k_1}\Big)}{p_1 \cdot k_1} \right) \left(\frac{\xi_2 e_2 p_2^\nu \epsilon_\nu^{\lambda_2 *}\Big(\vec{k_2}\Big)}{p_2 \cdot k_2} \right).
$$

That was quite straightforward. The contentious case is when the photons are emitted from the same leg. In that case, the correction from



corresponds to the matrix element

$$
\begin{aligned}
&(ie)^2 \bar{u}_\sigma(\vec{p_f}) \iint d^4 q_1 d^4 q_2 \gamma^\mu S_{\mathrm{F}}(q_2)\, \gamma^\nu S_{\mathrm{F}}(q_1)\, \epsilon_\mu^{\lambda_2 *}\Big(\vec{k_2}\Big)\, \epsilon_\nu^{\lambda_1 *}\Big(\vec{k_1}\Big) \delta(q_2 + k_2 - q_1) \delta(p_f + k_2 - q_2) \\
&= (ie)^2 \bar{u}_\sigma(\vec{p_f})\, \gamma^\mu \epsilon_\mu^{\lambda_2 *}\Big(\vec{k_1}\Big) \frac{i\Big(\not{p}_f + \not{k}_2 - m\Big)}{(p_f + k_2)^2 + m^2 - i\varepsilon} \gamma^\nu \epsilon_\nu^{\lambda_1 *}\Big(\vec{k_1}\Big) \frac{i\Big(\not{p}_f + \not{k}_1 + \not{k}_2 - m\Big)}{(p_f + k_1 + k_2)^2 + m^2 - i\varepsilon} \\
&= \bar{u}_\sigma(\vec{p_f}) \left(\frac{e p_f^\mu \epsilon_\mu^{\lambda_2 *}\Big(\vec{k_2}\Big)}{p_f \cdot k_2 - i\varepsilon} \right) \left(\frac{e p_f^\nu \epsilon_\nu^{\lambda_1 *}\Big(\vec{k_1}\Big)}{p_f \cdot (k_1 + k_2) + k_1 \cdot k_2 - i\varepsilon} \right).
\end{aligned}
$$

Again, we wish to sum over all possible schemes of emission for two photons, in order to get a sum of all possible corrections. For the two photons emitted from a single leg, we ought to add the contributions

$$\left(\frac{ep_f^{\mu}\epsilon_{\mu}^{\lambda_2 *}\left(\vec{k_2}\right)}{p_f\cdot k_2 - i\varepsilon}\right)\left(\frac{ep_f^{\nu}\epsilon_{\nu}^{\lambda_1 *}\left(\vec{k_1}\right)}{p_f\cdot(k_1+k_2)+k_1\cdot k_2 - i\varepsilon}\right)+\left(\frac{ep_f^{\mu}\epsilon_{\mu}^{\lambda_1 *}\left(\vec{k_1}\right)}{p_f\cdot k_1 - i\varepsilon}\right)\left(\frac{ep_f^{\nu}\epsilon_{\nu}^{\lambda_2 *}\left(\vec{k_2}\right)}{p_f\cdot(k_1+k_2)+k_1\cdot k_2 - i\varepsilon}\right).$$

Reducing this expression to the same denominator,

$$\left(\frac{ep_f^{\mu}\epsilon_{\mu}^{\lambda_2 *}\left(\vec{k_2}\right)}{p_f\cdot k_2 - i\varepsilon}\right)\left(\frac{ep_f^{\mu}\epsilon_{\mu}^{\lambda_1 *}\left(\vec{k_1}\right)}{p_f\cdot k_1 - i\varepsilon}\right)\frac{p_f\cdot(k_1+k_2)-2i\epsilon}{p_f\cdot(k_1+k_2)+k_1\cdot k_2 - i\varepsilon},$$

and neglecting $k_1 \cdot k_2$ as a second order term, yields

$$\left(\frac{\xi ep_f^{\mu}\epsilon_{\mu}^{\lambda_1 *}\left(\vec{k_1}\right)}{p_f\cdot k_1}\right)\left(\frac{\xi ep_f^{\mu}\epsilon_{\mu}^{\lambda_2 *}\left(\vec{k_2}\right)}{p_f\cdot k_2}\right),$$

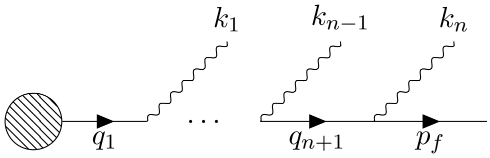
which reproduces the result we would obtain if the photons were emitted from two different legs. We conclude that two corrections can simply be added as independent factors to the $M_{\alpha\to\beta}$ matrix.

1.1.1. General case

To derive the general case, we shall firstly assume N low-energy photons emitted from a single leg, initial or final, contributing with a correction to $M_{\alpha\to\beta}$ as

$$M_{\alpha\to\beta} \to M_{\alpha\to\beta}\prod_{i=1}^{N}\left(\frac{\xi ep_f^{\mu_i}\epsilon_{\mu_i}^{\lambda_i *}\left(\vec{k_i}\right)}{p_f\cdot k_i}\right). \tag{1.1}$$

It is clear the formula works for $N = 1$. For N soft photons emitted from a single leg, we will first consider the following diagram,



.

Of course, this only works for a particular ordering. As for the case $n = 2$, in general, we have to account for all permutations of the k_i's. We get, by summing over all permutations and extracting the factor whose denominator has the most terms in each,

$$\sum_{s=1}^{n+1} \frac{\xi e p_f^{\mu_s} \epsilon_{\mu_s}^{\lambda_s *}\left(\vec{k}_s\right)}{p_f \cdot \sum_{j=1}^{n+1} k_j} \prod_{\substack{i=1 \\ i \neq s}}^{n+1} \frac{\xi e p_f^{\mu_i} \epsilon_{\mu_i}^{\lambda_i *}\left(\vec{k}_i\right)}{p_f \cdot \sum_{j=i}^{n+1} k_j}.$$

The product only has n factors. If we assume the validity of the simplification for n photons, we can write

$$\begin{aligned} \sum_{s=1}^{n+1} \frac{\xi e p_f^{\mu_s} \epsilon_{\mu_s}^{\lambda_s *}\left(\vec{k}_s\right)}{p_f \cdot \sum_{j=1}^{n+1} k_j} \prod_{\substack{i=1 \\ i \neq s}}^{n+1} \left(\frac{\xi e p_f^{\mu_i} \epsilon_{\mu_i}^{\lambda_i *}\left(\vec{k}_i\right)}{p_f \cdot k_i} \right) &= \frac{1}{p_f \cdot \sum_{j=1}^{n+1} k_j} \sum_{s=1}^{n+1} (p_f \cdot k_s) \prod_{i=1}^{n+1} \left(\frac{\xi e p_f^{\mu_i} \epsilon_{\mu_i}^{\lambda_i *}\left(\vec{k}_i\right)}{p_f \cdot k_i} \right) \\ &= \prod_{i=1}^{n+1} \left(\frac{\xi e p_f^{\mu_i} \epsilon_{\mu_i}^{\lambda_i *}\left(\vec{k}_i\right)}{p_f \cdot k_i} \right), \end{aligned}$$

which, by induction, completes the proof, leading to Eq. (1.1). We see that the corrections for n low-energy photons stack the same way, whether they are emitted from the same leg or from different ones. Of course, in the case of a process with several charged legs, we need to sum over all of them, as well as over the polarisations, to get the most complete correction. The $M_{\alpha \to \beta}$ matrix changes, due to the presence of N emitted soft photons from a process with R branches, as

$$M_{\alpha \to \beta} \to M_{\alpha \to \beta} \prod_{j=1}^{N} \left[\sum_{\lambda_j} \sum_{n=1}^{R} \left(\frac{\xi_n e_n p_n^{\mu_j}}{p_n \cdot k_j} \right) \epsilon_{\mu}^{*\lambda_j} \left(\vec{k}_j\right) \right].$$

Finally, noticing that the absorption of a photon is identical to an emission, as long as one changes $k_j \to -k_j$ and the respective polarisation, we can include the absorption of P photons by writing

$$\prod_{j=1}^{N} \left[\sum_{\lambda_j} \sum_{n=1}^{R} \left(\frac{\xi_n e_n p_n^{\mu_j}}{p_n \cdot k_j} \right) \epsilon_{\mu_j}^{*\lambda_j} \left(\vec{k}_j\right) \right] \prod_{l=1}^{P} \left[\sum_{\lambda_l} \sum_{m=1}^{R} \left(\frac{-\xi_m e_m p_m^{\mu_l}}{p_m \cdot k_l} \right) \epsilon_{\mu_l}^{\lambda_l} \left(\vec{k}_l\right) \right]. \tag{1.2}$$

Summary

Assumptions:

- In the low-energy regime, the energies of the photons are neglected compared with the rest mass of the charged particles ($\left|\vec{k}\right| \ll m \implies k^\mu \ll m$) (cf. Ref. [8]). This leads to $\mathcal{O}\left(\left|\vec{k}\right|/m\right) \Leftrightarrow \mathcal{O}\left(\left|\vec{k}\right|\right) \Rightarrow \mathcal{O}\left(\left|\vec{k}\right|^2\right)$.

- Scalar products of 4-momenta of soft photons can also be neglected, $k_i \cdot k_j \approx 0$.

- We are only taking into account divergent corrections. Photons emitted or absorbed by virtual particles can be ignored since denominators in the form $p^2 + m^2 \pm 2p \cdot k - i\varepsilon$ diverge, on the assumptions above, only when the emitting particle is real (see Ref. [9]). In fact, said expression would be null for

$$p^0 = \left|\vec{k}\right| \pm \sqrt{\left(\vec{p} - \vec{k}\right)^2 + m^2} \approx \sqrt{\vec{p}^2 + m^2},$$

  since, when $\vec{p}^2 \ll m^2$, m^2 is clearly dominant and, when $\vec{p}^2 \gg m^2$, the $\vec{p}^2$ part is; when $|\vec{p}| \approx m$, we can still ignore the term $2\vec{p} \cdot \vec{k}$ against $\vec{p}^2$.

Conclusions:

- We have deduced the full corrective factors due to N low-energy photons emitted and P absorbed by real charged particles involved in a process: see Eq. (1.2).

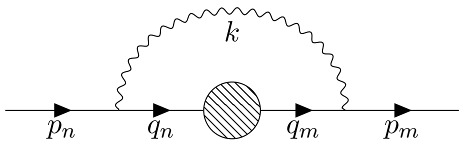
- Low-energy photons imply charge conservation through Ward's Identity.

1.2. Virtual soft photons

Our objective in this section is to remove the infrared divergences that arise due to the emission and absorption of virtual soft photons between external charged lines. We will define a virtual soft photon to be any virtual photon with energy and module of the spatial momentum lower than a threshold Λ. As long as much smaller than the rest mass of any external charged particles, the value for Λ is purely arbitrary.

The very first step is to strip the matrix $M_{\alpha\to\beta}$ from all the virtual soft photons, removing the virtual IR divergences from it at the same time. We want to reintroduce the virtual soft photons as corrections. Let us recall the results in Eq. (1.2), replacing the sums of the polarisation vectors, for emission and absorption, $\epsilon_\mu^{\lambda *}\left(\vec{k}\right)$, $\epsilon_\nu^{\lambda}\left(\vec{k}\right)$, by photonic propagators $\frac{-i\eta_{\mu\nu}}{k^2-i\epsilon}$. There are two cases to pay attention to: when the virtual photon links two final (or two initial) legs and when it links a final and an initial one.

Analysing first the second case,



,

we extract the external spinorial terms from $M_{\alpha\to\beta}$, obtaining the contribution

$$
\begin{aligned}
&-e_n e_m \iiint \frac{d^4k}{(2\pi)^4}\frac{d^4q_n}{(2\pi)^4}\frac{d^4q_m}{(2\pi)^4}\frac{-i\eta_{\mu\nu}}{k^2-i\varepsilon}\bar{u}_\sigma(\vec{p}_n)\,\gamma^\mu S_{\mathrm{F}}(\vec{q}_n)\,M_{\alpha\to\beta}S_{\mathrm{F}}(\vec{q}_m)\,\gamma^\nu u_{\sigma'}(\vec{p}_m)\\
&\times (2\pi)^8\delta(q_n+k-p_n)\,\delta(q_m+k-p_m)\\
&= -e_n e_m \bar{u}_\sigma(\vec{p}_n)\,\gamma^\mu\left[\int\frac{d^4k}{(2\pi)^4}\frac{-i\eta_{\mu\nu}}{k^2-i\varepsilon}S_{\mathrm{F}}\left(\vec{p}_n-\vec{k}\right)M_{\alpha\to\beta}S_{\mathrm{F}}\left(\vec{p}_m-\vec{k}\right)\right]\gamma^\nu u_{\sigma'}(\vec{p}_m)\,.
\end{aligned}
$$

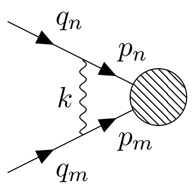
Using the same trick as in the previous section,[2] we have

$$e_n e_m \left[\int \frac{d^4k}{(2\pi)^4} \frac{-i\eta_{\mu\nu}}{k^2 - i\varepsilon} \frac{p_n^\mu}{p_n \cdot k + i\varepsilon} \bar{u}_\sigma(\vec{p}_n)\, M_{\alpha\to\beta}\, u_{\sigma'}(\vec{p}_m)\, \frac{p_m^\nu}{p_m \cdot k + i\varepsilon} \right].$$

Reintroducing the spinorial terms in the matrix, we obtain the correction due to a single virtual soft photon linking a charged particle in α with one in β,

$$\frac{-ie_n e_m (p_n \cdot p_m)}{(2\pi)^4} \left[\int \frac{d^4k}{(k^2 - i\varepsilon)\,(p_n \cdot k + i\varepsilon)\,(p_m \cdot k + i\varepsilon)} \right].$$

Note that since the photons are not on-shell, we ought to keep the $i\varepsilon$ factor for the time being. The case where the photon links two initial legs (analogous if both of them are final),



,

implies the term

$$\begin{aligned}
& (M_{\alpha\to\beta})_{ab} \iiint \frac{d^4k}{(2\pi)^4} \frac{d^4q_n}{(2\pi)^4} \frac{d^4q_m}{(2\pi)^4} S_F(q_n)_{ca} (\gamma^\mu)_{ec}\, \bar{u}_\sigma(\vec{p}_n)_e\, S_F(q_m)_{db} (\gamma^\nu)_{fd}\, \bar{u}_{\sigma'}(\vec{p}_m)_f \\
\times \quad & \Pi_{\mu\nu}(k)(-e_n e_m)(2\pi)^8 \delta(q_n + k - p_n)\delta(k + p_m - q_m) \\
= \quad & (-e_n e_m) \int \frac{d^4k}{(2\pi)^4} \frac{-i\eta_{\mu\nu}}{k^2 - i\varepsilon} (\bar{u}_\sigma(\vec{p}_n)\, \gamma^\mu S_F(p_n - k))_a (M_{\alpha\to\beta})_{ab} (\bar{u}_{\sigma'}(\vec{p}_m)\, \gamma^\nu S_F(p_m + k))_b \\
= \quad & (-e_n e_m) \int \frac{d^4k}{(2\pi)^4} \frac{-i\eta_{\mu\nu}}{k^2 - i\varepsilon} \frac{-ip_n^\mu}{-p_n \cdot k - i\varepsilon} \bar{u}_\sigma(\vec{p}_n)_a (M_{\alpha\to\beta})_{ab}\, \bar{u}_{\sigma'}(\vec{p}_m)_b\, \frac{-ip_m^\nu}{p_m \cdot k - i\varepsilon} \\
= \quad & \frac{-ie_n e_m (p_n \cdot p_m)}{(2\pi)^4} \int \frac{d^4k}{(k^2 - i\varepsilon)(p_n \cdot k + i\varepsilon)(-p_m \cdot k + i\varepsilon)} \left[\bar{u}_\sigma(\vec{p}_n)_a (M_{\alpha\to\beta})_{ab}\, \bar{u}_{\sigma'}(\vec{p}_m)_b \right],
\end{aligned}$$

[2] Virtual photons are not on-shell, so the condition $k^2 = 0$ is not true anymore. The conditions, $k^0, \left|\vec{k}\right| < \Lambda \ll m$, imply that terms in $k_i \cdot k_j = \vec{k}_i \cdot \vec{k}_j - k_i^0 k_j^0 \ll m^2$ are negligible, in particular for $i = j$, reproducing the desired result. This form for the propagators can also be derived from the first condition alone, since $k^2 - 2p \cdot k - i\varepsilon = -\left(k^0\right)^2 + \vec{k}^2 + 2k^0 E - 2\vec{p} \cdot \vec{k} - i\varepsilon = 0$ yields the solutions $k^0 = E\left(1 \pm \sqrt{1 - 2\frac{\vec{v}\cdot\vec{k}}{E} - i\varepsilon}\right)$, neglecting $\left|\vec{k}\right|^2 / E^2$. Retaining only the first terms in the series expansion of the square root, we have the approximate solutions $\vec{v} \cdot \vec{k} + i\varepsilon$ and $2E$. Since $\left|k^0\right| \ll m \leq E$, the last solution is discarded. So, what only remains in the denominator is $\vec{p} \cdot \vec{k} - Ek^0 - i\varepsilon = p \cdot k - i\varepsilon$.

and we arrive at the correction

$$\frac{-ie_n e_m(p_n\cdot p_m)}{(2\pi)^4}\int\frac{d^4k}{(k^2-i\varepsilon)\,(p_n\cdot k+i\varepsilon)\,(-p_m\cdot k+i\varepsilon)}.$$

Making use of the factors ξ_n, we can write these two corrections in a single formula,

$$\begin{aligned}&\frac{-ie_n e_m(p_m\cdot p_n)}{(2\pi)^4}\left[\int\frac{d^4k}{(k^2-i\varepsilon)(-\xi_n p_n\cdot k+i\varepsilon)(\xi_m p_m\cdot k+i\varepsilon)}\right]\\ =&\frac{-ie_n e_m\xi_n\xi_m(p_m\cdot p_n)}{(2\pi)^4}\left[\int\frac{d^4k}{(k^2-i\varepsilon)(-p_n\cdot k+i\xi_n\varepsilon)\,(p_m\cdot k+i\xi_m\varepsilon)}\right].\end{aligned}$$

To compute the integral we first integrate the time component, k^0, which we write as

$$\int\frac{dk^0 d\vec{k}}{\left(\left|\vec{k}\right|^2-(k^0)^2-i\varepsilon\right)\left(E_n k^0-\vec{p}_n\cdot\vec{k}+i\xi_n\varepsilon\right)\left(\vec{p}_m\cdot\vec{k}-E_m k^0+i\xi_m\varepsilon\right)}.$$

It is important to note that the integration in k^0 runs from $-\Lambda$ to Λ. The module of $\vec{k}$ is subject to a similar constraint. Since

$$\left(k^0-\left|\vec{k}\right|+i\varepsilon\right)\left(k^0+\left|\vec{k}\right|-i\varepsilon\right)=\left((k^0)^2-\left|\vec{k}\right|^2+i\varepsilon\right),$$

we make an analytic extension to the complex plane in k^0, identifying four poles,

$$\begin{array}{ll} k_1^0\equiv\left|\vec{k}\right|-i\varepsilon & k_3^0\equiv\vec{v}_n\cdot\vec{k}-i\xi_n\varepsilon \\ k_2^0\equiv-\left|\vec{k}\right|+i\varepsilon & k_4^0\equiv\vec{v}_m\cdot\vec{k}+i\xi_m\varepsilon \end{array},$$

where $\vec{v}_n\equiv\frac{\vec{p}_n}{E_n}$ is the velocity of the particle n on the given frame of reference. It is clear that $|\vec{v}_n|\leq 1$, with the equality only holding when the fermions are massless. This ensures that the real part of all poles is caught inside the range $]-\Lambda,\Lambda[$.

When $\xi_n=-\xi_m=1$, the imaginary parts obey $\Im(k_1^0),\Im(k_3^0),\Im(k_4^0)<0$, while $\Im(k_2^0)>0$. Therefore, we can simply draw a rectangular integration path with vertices $-\Lambda$, Λ, $-\Lambda+iL$ and $\Lambda+iL$, where L is to be taken to infinity, enclosing the single pole on the upper semiplane.

By an elementary calculation in Residue Calculus, we can solve[3] the integration in k^0,

$$
\begin{aligned}
&-\int \frac{dk^0 d\vec{k}}{\left(k^0-\left|\vec{k}\right|+i\varepsilon\right)\left(k^0+\left|\vec{k}\right|-i\varepsilon\right)\left(E_n k^0-\vec{p}_n\cdot\vec{k}+i\varepsilon\right)\left(\vec{p}_m\cdot\vec{k}-E_m k^0-i\varepsilon\right)}\\
&=-\int \frac{2\pi i d\vec{k}}{2\left(-\left|\vec{k}\right|+i\varepsilon\right)\left(-\left|\vec{k}\right|E_n-\vec{p}_n\cdot\vec{k}+i\varepsilon\right)\left(\vec{p}_m\cdot\vec{k}+E_m\left|\vec{k}\right|-i\varepsilon\right)}\\
&\underset{\varepsilon\to 0}{\to} -\int \frac{i\pi d\vec{k}}{\left|\vec{k}\right|^3\left(E_n+\vec{p}_n\cdot\hat{k}\right)\left(E_m+\vec{p}_m\cdot\hat{k}\right)} \underset{\vec{k}\to-\vec{k}}{=} \frac{-i\pi}{E_n E_m}\int \frac{d\vec{k}}{\left|\vec{k}\right|^3\left(1-\vec{v}_n\cdot\hat{k}\right)\left(1-\vec{v}_m\cdot\hat{k}\right)}.
\end{aligned}
$$

In the last step we rotated $\vec{k}$ (as we are free to do, since it is an unitary transformation), so that the result mirrors the one in literature (cf. Ref. [1]).

The case of $-\xi_n=\xi_m=1$ is equivalent to the previous one, but this time the lone pole, k_1^0, is in the lower semiplane. That is the plane we ought to draw our contour on, simply conjugating our complex vertices of the rectangle. The integral then simplifies[4] to

$$
\begin{aligned}
&\int \frac{dk^0 d\vec{k}}{\left(k^0-\left|\vec{k}\right|+i\varepsilon\right)\left(k^0+\left|\vec{k}\right|-i\varepsilon\right)\left(E_n k^0-\vec{p}_n\cdot\vec{k}-i\varepsilon\right)\left(\vec{p}_m\cdot\vec{k}-E_m k^0+i\varepsilon\right)}\\
&=\int \frac{2\pi i d\vec{k}}{2\left(\left|\vec{k}\right|-i\varepsilon\right)\left(E_n\left|\vec{k}\right|-\vec{p}_n\cdot\vec{k}-i\varepsilon\right)\left(\vec{p}_m\cdot\vec{k}-E_m\left|\vec{k}\right|+i\varepsilon\right)}\\
&\underset{\epsilon\to 0}{\to} \frac{-i\pi}{E_n E_m}\int \frac{d\vec{k}}{\left|\vec{k}\right|^3\left(1-\vec{v}_n\cdot\hat{k}\right)\left(1-\vec{v}_m\cdot\hat{k}\right)},
\end{aligned}
$$

which equates to the previous result. These two cases, photons connecting an initial and a final leg ($\xi_n\xi_m=-1$), simplify the correction to

$$
\frac{\pi e_n e_m (p_m\cdot p_n)}{(2\pi)^4 E_n E_m}\int \frac{d\vec{k}}{\left|\vec{k}\right|^3\left(1-\vec{v}_n\cdot\hat{k}\right)\left(1-\vec{v}_m\cdot\hat{k}\right)}.
$$

In the cases where the photon connects either two initial legs or two final ones, $\xi_n=\xi_m$, we get two poles in the upper semiplane and another two in the lower one. We ought to sum

[3] It is easy to verify that the integration on the path around the upper plane, in the infinite limit, vanishes.

[4] In order to preserve the definition of the residue dependent on the (positive) orientation of the contour, this integral, from $-\Lambda$ to $+\Lambda$ on the real line, is affected by an extra negative sign.

two residues this time. When $\xi_n = \xi_m = 1$, integrating in the path closing the upper plane (and enclosing the poles k_2^0, k_4^0), the integral is

$$
\begin{aligned}
&-\int \frac{dk^0 d\vec{k}}{\left(k^0 - \left|\vec{k}\right| + i\varepsilon\right)\left(k^0 + \left|\vec{k}\right| - i\varepsilon\right)\left(E_n k^0 - \vec{p}_n\cdot\vec{k} + i\varepsilon\right)\left(\vec{p}_m\cdot\vec{k} - E_m k^0 + i\varepsilon\right)} \\
&= \frac{-i\pi}{E_n E_m}\left[\int \frac{d\vec{k}}{\left|\vec{k}\right|^3\left(1 - \vec{v}_n\cdot\hat{k}\right)\left(1 - \vec{v}_m\cdot\hat{k}\right)} + \int \frac{2d\vec{k}}{\left(\vec{v}_m\cdot\vec{k} - \left|\vec{k}\right|\right)\left(\vec{v}_m\cdot\vec{k} + \left|\vec{k}\right|\right)\left(\vec{v}_m - \vec{v}_n\right)\cdot\vec{k}}\right] \\
&= \frac{-i\pi}{E_n E_m}\left[\int \frac{d\vec{k}}{\left|\vec{k}\right|^3\left(1 - \vec{v}_n\cdot\hat{k}\right)\left(1 - \vec{v}_m\cdot\hat{k}\right)} - \int \frac{2d\vec{k}}{\left|\vec{k}\right|^3\left(1 - \vec{v}_m\cdot\hat{k}\right)\left(1 + \vec{v}_m\cdot\hat{k}\right)\left(\vec{v}_m - \vec{v}_n\right)\cdot\hat{k}}\right] \\
&= \frac{-i\pi}{E_n E_m}\left[\int \frac{d\vec{k}}{\left|\vec{k}\right|^3\left(1 - \vec{v}_n\cdot\hat{k}\right)\left(1 - \vec{v}_m\cdot\hat{k}\right)} - \int \frac{d\vec{k}}{\left|\vec{k}\right|^3}\left(\frac{1}{1 - \vec{v}_m\cdot\hat{k}} + \frac{1}{1 + \vec{v}_m\cdot\hat{k}}\right)\frac{1}{\left(\vec{v}_m - \vec{v}_n\right)\cdot\hat{k}}\right].
\end{aligned}
$$

Once again, reflecting $\hat{k}$ in relation to the origin on the second term gives

$$
\begin{aligned}
&\frac{-i\pi}{E_n E_m}\left[\int \frac{d\vec{k}}{\left|\vec{k}\right|^3\left(1 - \vec{v}_n\cdot\hat{k}\right)\left(1 - \vec{v}_m\cdot\hat{k}\right)} - \int \frac{d\vec{k}}{\left|\vec{k}\right|^3}\left(\frac{1}{1 - \vec{v}_m\cdot\hat{k}} - \frac{1}{1 - \vec{v}_m\cdot\hat{k}}\right)\frac{1}{\left(\vec{v}_m - \vec{v}_n\right)\cdot\hat{k}}\right] \\
&= \frac{-i\pi}{E_n E_m}\int \frac{d\vec{k}}{\left|\vec{k}\right|^3\left(1 - \vec{v}_n\cdot\hat{k}\right)\left(1 - \vec{v}_m\cdot\hat{k}\right)}.
\end{aligned}
$$

Equivalently, for $\xi_n = \xi_m = -1$, integrating once more on the upper semiplane,

$$
\begin{aligned}
&-\int \frac{dk^0 d\vec{k}}{\left(k^0 - \left|\vec{k}\right| + i\varepsilon\right)\left(k^0 + \left|\vec{k}\right| - i\varepsilon\right)\left(E_n k^0 - \vec{p}_n\cdot\vec{k} - i\varepsilon\right)\left(\vec{p}_m\cdot\vec{k} - E_m k^0 - i\varepsilon\right)} \\
&= \frac{-i\pi}{E_n E_m}\left[\int \frac{d\vec{k}}{\left|\vec{k}\right|^3\left(1 - \vec{v}_n\cdot\hat{k}\right)\left(1 - \vec{v}_m\cdot\hat{k}\right)} - \int \frac{2d\vec{k}}{\left|\vec{k}\right|^3\left(1 - \vec{v}_n\cdot\hat{k}\right)\left(1 + \vec{v}_n\cdot\hat{k}\right)\left(\vec{v}_m - \vec{v}_n\right)\cdot\hat{k}}\right] \\
&= \frac{-i\pi}{E_n E_m}\int \frac{d\vec{k}}{\left|\vec{k}\right|^3\left(1 - \vec{v}_n\cdot\hat{k}\right)\left(1 - \vec{v}_m\cdot\hat{k}\right)},
\end{aligned}
$$

by the same reasoning as before.

The integrations in these two other cases are equal to the ones from the previous two. We can write the correction to $M_{\alpha\to\beta}$ due to a single virtual soft photon linking any two possible charged external legs as a sum over all such charged external legs,

$$\sum_{n,m}\frac{-\pi e_n e_m \xi_n \xi_m}{(2\pi)^4 E_n E_m}(p_n\cdot p_m)\int\frac{d\vec{k}}{\left|\vec{k}\right|^3\left(1-\vec{v}_n\cdot\hat{k}\right)\left(1-\vec{v}_m\cdot\hat{k}\right)},$$

where the integral is subject to the constraint $\left|\vec{k}\right| < \Lambda$. Writing the integral in spherical coordinates makes an integration in the radial component a fairly simple step,

$$\int\frac{d\vec{k}}{\left|\vec{k}\right|^3\left(1-\vec{v}_n\cdot\hat{k}\right)\left(1-\vec{v}_m\cdot\hat{k}\right)}=\ln\left(\frac{\Lambda}{\lambda}\right)B(\vec{v}_n,\vec{v}_m)\,,$$

where we have defined

$$B(\vec{v}_n,\vec{v}_m)\equiv\int\frac{d\Omega}{(1-\vec{v}_n\cdot\hat{r})(1-\vec{v}_m\cdot\hat{r})}$$

to be the integral in the spherical surface part. This integral, solved in Section A.1, can be written as

$$B(\vec{v}_n,\vec{v}_m)=\frac{4\pi}{(1-\vec{v}_n\cdot\vec{v}_m)\beta_{mn}}\text{arctanh}(\beta_{mn})=\frac{-4\pi E_n E_m}{(p_n\cdot p_m)\beta_{mn}}\text{arctanh}(\beta_{mn})$$

with

$$\beta_{mn}\equiv\sqrt{1-\frac{(1-\vec{v}_n^2)(1-\vec{v}_m^2)}{(1-\vec{v}_n\cdot\vec{v}_m)^2}}=\sqrt{1-\frac{m_n^2 m_m^2}{(p_n\cdot p_m)^2}}.$$

The factor β_{mn} is simply the module of the velocity of one of the particles in the reference frame of the other, as discussed in Section A.2.

Plugging everything in the correction to $M_{\alpha\to\beta}$, written above, we get

$$\sum_{n,m}\frac{e_n e_m \xi_n \xi_m}{(2\pi)^2\beta_{mn}}\text{arctanh}(\beta_{mn})\ln\left(\frac{\Lambda}{\lambda}\right).$$

Let

$$A(\alpha\to\beta)\equiv-\sum_{n,m}\frac{e_n e_m \xi_n \xi_m}{(2\pi)^2\beta_{mn}}\text{arctanh}(\beta_{mn})\,,$$

so that the correction due to a single soft photon between any two branches is simply

$$-A(\alpha \to \beta) \ln\left(\frac{\Lambda}{\lambda}\right).$$

We are finally prepared to derive the correction due to N virtual soft photons that connect any two charged external legs. In the previous section, it was derived that, in the soft photon limit, the correction due to N soft photons was just the product of the N individual soft photon corrections. In this case, when the photon connects two different legs, the stacking of corrections continues to occur, replacing each pair of polarisations by a photonic propagator. In the case where the soft photon connects two nodes in the same leg, the diagram can simply be dealt with by renormalising the mass of the fermion. We conclude that it is still possible to independently stack one-photon corrections, in the same way it was done in the previous section.

Accounting for the Feynman rules, we shall divide the result by the symmetry factors: $N!$ due to the permutations between the N virtual soft photons, and 2^N due to the exchange of both ends of each photon line. The correction for N photons is, then,

$$\frac{1}{2^N N!}\left[-A(\alpha \to \beta) \ln\left(\frac{\Lambda}{\lambda}\right)\right]^N.$$

The corrective factor to $M_{\alpha\to\beta}$ should include the corrections not only from exactly N soft photons, but from any number of them, so an infinite sum is in order,

$$\sum_{N=1}^{\infty} \frac{1}{N!}\left[-\frac{1}{2}A(\alpha \to \beta) \ln\left(\frac{\Lambda}{\lambda}\right)\right]^N = \left(\frac{\Lambda}{\lambda}\right)^{-\frac{1}{2}vA(\alpha\to\beta)},$$

that is,

$$M_{\alpha\to\beta} \to M^{\gamma_v}_{\alpha\to\beta} \equiv M_{\alpha\to\beta}\left(\frac{\lambda}{\Lambda}\right)^{\frac{1}{2}A(\alpha\to\beta)},$$

defining $M^{\gamma_v}_{\alpha\to\beta}$ as the transition matrix with the virtual soft photons reincluded. Given that $A(\alpha \to \beta)$ is a positive exponent (see Section A.3), we can already see, since Λ is a finite nonvanishing constant and taking $\lambda \to 0$ implies $M^{\gamma_v}_{\alpha\to\beta} \to 0$, that $M_{\alpha\to\beta}$ has no infrared divergences.

Let us just calculate the transition rate, as it will be useful later. By Fermi's Golden Rule, we have the differential transition [11],

$$d\Gamma^{\gamma_v}_{\alpha\to\beta} = \left(\prod_i \frac{1}{2E_{\vec{p_i}}}\right)|M_{\alpha\to\beta}|^2 \left(\frac{\lambda}{\Lambda}\right)^{A(\alpha\to\beta)} (2\pi)^4 \delta\left(\sum_i p_i - \sum_f p_f\right) = d\Gamma_{\alpha\to\beta}\left(\frac{\lambda}{\Lambda}\right)^{A(\alpha\to\beta)} ,$$

which, after integrating on the final momenta, yields

$$\Gamma^{\gamma_v}_{\alpha\to\beta} = \Gamma_{\alpha\to\beta}\left(\frac{\lambda}{\Lambda}\right)^{A(\alpha\to\beta)} .$$

1.2.1. Comment on the approximation

The condition that both $\left|k^0\right|$ and $\left|\vec{k}\right|$ are below a certain $\Lambda \ll m$ is not very satisfying. The cylindrical integration domain on 4-momentum space might seem a bit arbitrary. An alternative condition is to consider an integration in a 3-ball, with the condition that $\left|k^0\right|^2 + \left|\vec{k}\right|^2 \leq \Lambda^2$, where a Wick rotation is supposed to have been executed. The integrand decays quickly on k^0 so that the validity of it should hold [2]. Not only would we be introducing a more symmetric supposition, but it would also resemble the upper bound regularisation. From these assumptions, a regularisation scheme, like Pauli-Villars, which allows for the interpretation of Λ as a mass of a fictitious massive photon (taking $\Lambda \sim m$ would introduce an upper cutoff similar to that in Ref. [2]), should not be a great step to take. However, I do not wish to carry it out here. Instead, I will argue that the difference between the cylindrical integration and the spherical one is negligible.

In fact, the difference can be written as an integration in k^0 in the regions limited by $\sqrt{\Lambda^2 - \vec{k}^2} \leq k^0 \leq \Lambda$, and another for $\vec{k}$ in the full 2-sphere. This, remember, already with a

Wick rotation,

$$\iint_{\sqrt{\Lambda^2-\vec{k}^2}}^{\Lambda} \frac{dk^0 d\vec{k}}{\left(\left|\vec{k}\right|^2 + (k^0)^2\right)\left(k^0 + \vec{v}_n \cdot \vec{k}\right)\left(k^0 + \vec{v}_m \cdot \vec{k}\right)}$$
$$\leq \frac{1}{\Lambda^2} \int \frac{\left(\Lambda - \sqrt{\Lambda^2 - \vec{k}^2}\right) d\vec{k}}{\left(\sqrt{\Lambda^2 - \vec{k}^2} + \vec{v}_n \cdot \vec{k}\right)\left(\sqrt{\Lambda^2 - \vec{k}^2} + \vec{v}_m \cdot \vec{k}\right)} \sim \frac{C}{\Lambda^3}.$$

So the corrections induced between the two integration domains are, at best, of order Λ^{-3}, much lower than the logarithmic leading order. Thus, they should not alter the result significantly.

Summary

Definitions:

- Virtual soft photon (of energy Λ): A virtual photon with $\left|k^0\right|, \left|\vec{k}\right|$ below an arbitrary value Λ, much smaller than the rest mass of any external charged particle. Note that the value of Λ is frame-dependent, therefore, a conversion to another frame requires the appropriate Lorentz factor. This assumption is again mirrored in Ref. [8].

Conclusions:

- The IR divergence due to virtual soft photons travelling between two charged external legs cancels out when the full 'sea of virtual photons' is taken into account.

- The definition can be replaced by a more symmetrical one without introducing relevant corrections, bridging this derivation with some of the modern ones, like in Ref. [2].

1.3. Real soft photons

We, finally, want to readd the real soft photons that are emitted by our external charged particles. Such photons, albeit hitting our detectors, are of such low energy that do not make them click. Let us assume a 'hard' energy value, E, such that particles with energy lower than it cannot be detected. Since all the charged particles in our $\alpha \to \beta$ process are detectable, E is lower than their rest masses, being in line with the assumption made in the first section.

There is another restriction we ought to keep in mind: conservation of energy is assumed in the process $\alpha \to \beta$, but we are now adding a term $\sum_f \left|\vec{k}_f\right| - \sum_i \left|\vec{k}_i\right|$ to the final energy due to the emission and absorption of soft photons. In order to rescue energy conservation, this new term needs to be within the uncertainty energy of the detector, E_T, that is $\sum_f \left|\vec{k}_f\right| - \sum_i \left|\vec{k}_i\right| \leq E_T$.[5]

With all this in mind, we just square the module of Eq. (1.2),

$$
\begin{aligned}
&|M_{\alpha\to\beta}|^2 \prod_{j=1}^{N} \left| \sum_{\lambda_j} \sum_{n=1}^{R} \left(\frac{\xi_n e_n p_n^{\mu_j}}{p_n \cdot k_j} \right) \epsilon_{\mu_j}^{\lambda_j *}\left(\vec{k}_j\right) \right|^2 \prod_{l=1}^{P} \left[\sum_{\lambda_l} \sum_{m=1}^{R} \left(\frac{-\xi_m e_m p_m^{\mu_l}}{p_m \cdot k_l} \right) \epsilon_{\mu_l}^{\lambda_l}\left(\vec{k}_l\right) \right]^2 \\
&= |M_{\alpha\to\beta}|^2 \prod_j \left[\left(\sum_n \frac{\xi_n e_n p_n^{\mu_j}}{p_n \cdot k_j} \right) \left(\sum_{n'} \frac{\xi_{n'} e_{n'} p_{n'}^{\nu_j}}{p_{n'} \cdot k_j} \right) \sum_{\lambda_j} \epsilon_{\mu_j}^{\lambda_j *}\left(\vec{k}_j\right) \sum_{\lambda'_j} \epsilon_{\nu_j}^{\lambda'_j}\left(\vec{k}_j\right) \right] \cdot \\
&\quad \cdot \prod_l \left[\left(\sum_m \frac{-\xi_m e_m p_m^{\mu_l}}{p_m \cdot k_l} \right) \left(\sum_{m'} \frac{-\xi_{m'} e_{m'} p_{m'}^{\nu_l}}{p_{m'} \cdot k_l} \right) \sum_{\lambda_l} \epsilon_{\mu_l}^{\lambda_l *}\left(\vec{k}_l\right) \sum_{\lambda'_l} \epsilon_{\nu_l}^{\lambda'_l}\left(\vec{k}_l\right) \right] \\
&\to |M_{\alpha\to\beta}|^2 \prod_j \left[\sum_{n,n'} \frac{\xi_n e_n \xi_{n'} e_{n'} (p_n \cdot p_{n'})}{(p_n \cdot k_j)(p_{n'} \cdot k_j)} \right] \prod_l \left[\sum_{m,m'} \frac{\xi_m e_m \xi_{m'} e_{m'} (p_m \cdot p_{m'})}{(p_m \cdot k_l)(p_{m'} \cdot k_l)} \right],
\end{aligned}
$$

where the polarisations were simplified as

$$
\sum_{\lambda_j} \epsilon_{\mu_j}^{\lambda_j *}\left(\vec{k}_j\right) \sum_{\lambda'_j} \epsilon_{\nu_j}^{\lambda'_j}\left(\vec{k}_j\right) = \sum_{\lambda_j} \epsilon_{\mu_j}^{\lambda_j *}\left(\vec{k}_j\right) \epsilon_{\nu_j}^{\lambda_j}\left(\vec{k}_j\right) = \eta_{\mu_j \nu_j}.
$$

[5] Despite the two restrictions over the energy of the real soft photons, they have a common origin. A more precise detector lowers the threshold to detect a soft photon, E, leading to the detection of more soft photons and lowering the value of E_T.

Also, since the corrections for absorbed or emitted soft photons are equal, we can update our j index to include all real soft photons, irrespectively of if they are incomming or outgoing in the process. We might also consider that we are adding the real soft photons to the matrix, $M^{\gamma_v}_{\alpha\to\beta}$, with the virtual photons already present. In that light, we have the correction

$$\left|M^{\gamma_v}_{\alpha\to\beta}\right|^2 \to \left|M^{\gamma_v}_{\alpha\to\beta}\right|^2 \prod_j \left[\sum_{n,m} \frac{\xi_n \xi_m e_n e_m (p_n \cdot p_m)}{(p_n \cdot k_j)(p_m \cdot k_j)}\right].$$

Once again, invoking Fermi's Golden Rule, we have that the differential transition without added real soft photons [11],

$$d\Gamma^{\gamma_v}_{\alpha\to\beta} = \left(\prod_i \frac{1}{2E_{\vec{p_i}}}\right)\left(\prod_f \frac{d^3\vec{p_f}}{2(2\pi)^3 E_{\vec{p_f}}}\right)\left|M^{\gamma_v}_{\alpha\to\beta}\right|^2 (2\pi)^4 \delta\left(\sum_i p_i - \sum_f p_f\right),$$

will now be

$$\begin{aligned} d\Gamma^{\gamma_r}_{\alpha\to\beta} &= \left(\prod_i \frac{1}{2E_{\vec{p_i}}}\right)\left(\prod_f \frac{d^3\vec{p_f}}{2(2\pi)^3 E_{\vec{p_f}}}\right)\left(\prod_j \frac{d^3\vec{k_j}}{2(2\pi)^3\left|\vec{k_j}\right|}\right)(2\pi)^4\delta\left(\sum_i p_i - \sum_f p_f - \sum_j k_j\right) \times \\ &\times \left|M^{\gamma_v}_{\alpha\to\beta}\right|^2 \prod_j \left[\sum_{n,m} \frac{\xi_n \xi_m e_n e_m (p_n \cdot p_m)}{(p_n \cdot k_j)(p_m \cdot k_j)}\right] \\ &= d\Gamma^{\gamma_v}_{\alpha\to\beta} \prod_j \left[\frac{d^3\vec{k_j}}{(2\pi)^3 2\left|\vec{k_j}\right|} \sum_{n,m} \frac{\xi_n \xi_m e_n e_m (p_n \cdot p_m)}{(p_n \cdot k_j)(p_m \cdot k_j)}\right], \end{aligned}$$

noting that $\delta\left(\sum_i p_i - \sum_f p_f - \sum_j k_j\right) \approx \delta\left(\sum_i p_i - \sum_f p_f\right)$. Given our state of ignorance about the momenta of the absorbed soft photons, the integration is extended to such particles and not only to the final ones, like it is usually done in Fermi's Golden Rule.

Integrating over the final momenta,

$$\Gamma^{\gamma_r}_{\alpha\to\beta} = \Gamma^{\gamma_v}_{\alpha\to\beta}\prod_j\left[\sum_{n,m}\xi_n\xi_m e_n e_m (p_n\cdot p_m)\int\frac{d^3\vec{k_j}}{(2\pi)^3 2\left|\vec{k_j}\right|(p_n\cdot k_j)(p_m\cdot k_j)}\right]$$
$$= \Gamma^{\gamma_v}_{\alpha\to\beta}\prod_j\left[\sum_{n,m}\frac{\xi_n\xi_m e_n e_m (p_n\cdot p_m)}{(2\pi)^3 2E_nE_m}\int\frac{d^3\vec{k_j}}{\left|\vec{k_j}\right|^3\left(1-\vec{v}_n\cdot\hat{k_j}\right)\left(1-\vec{v}_m\cdot\hat{k_j}\right)}\right],$$

where $\vec{v}_n = \frac{\vec{p}_n}{E_n}$ are, again, the velocities of the particles in a given reference frame. This integral has now some subtleties that the equivalent in the previous section had not. The restriction on the module of the momentum is still there, now with the maximum value of E, but now it also ought to respect the condition $\sum_s\left|\vec{k}_s\right| \leq E_T$. We shall write this last restriction as a Heaviside step function, $\Theta\left(E_T - \sum_s\left|\vec{k}_s\right|\right)$,[6] so that, making use of the Fourier transform [12],

$$\Theta(x) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\frac{e^{ixt}}{t-i\varepsilon}dt$$

we get

$$\left(\prod_j\int_{\lambda\leq\left|\vec{k_j}\right|\leq E}\frac{d^3\vec{k_j}}{\left|\vec{k_j}\right|^3}\frac{1}{\left(1-\vec{v}_n\cdot\hat{k_j}\right)\left(1-\vec{v}_m\cdot\hat{k_j}\right)}\right)\Theta\left(E_T-\sum_s\left|\vec{k}_s\right|\right)$$
$$=\left(\prod_j\int_{\lambda\leq\left|\vec{k_j}\right|\leq E}\frac{d^3\vec{k_j}}{\left|\vec{k_j}\right|^3}\frac{1}{\left(1-\vec{v}_n\cdot\hat{k_j}\right)\left(1-\vec{v}_m\cdot\hat{k_j}\right)}\right)\frac{1}{2\pi i}\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}dt\frac{e^{it\left(E_T-\sum_s\left|\vec{k}_s\right|\right)}}{t-i\varepsilon}$$
$$=\left(\prod_j\int_{\lambda\leq\left|\vec{k_j}\right|\leq E}\frac{d^3\vec{k_j}}{\left|\vec{k_j}\right|^3}\frac{1}{\left(1-\vec{v}_n\cdot\hat{k_j}\right)\left(1-\vec{v}_m\cdot\hat{k_j}\right)}\right)\frac{1}{2\pi i}\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}dt\frac{e^{itE_T}\prod_j e^{-it\left|\vec{k}_j\right|}}{t-i\varepsilon}$$
$$=\frac{1}{2\pi i}\lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty}dt\frac{e^{itE_T}}{t-i\varepsilon}\prod_j\left[\int_{\lambda\leq\left|\vec{k_j}\right|\leq E}\frac{d^3\vec{k_j}}{\left|\vec{k_j}\right|^3}\frac{e^{-it\left|\vec{k}_j\right|}}{\left(1-\vec{v}_n\cdot\hat{k_j}\right)\left(1-\vec{v}_m\cdot\hat{k_j}\right)}\right].$$

[6] $\Theta(x) = \begin{cases} 1 & x\geq 0 \\ 0 & x<0\end{cases}$

To solve the integrals on the momenta, we use spherical coordinates once again,

$$
\begin{aligned}
&\int_{\lambda\leq|\vec{k}_j|\leq E} \frac{d^3\vec{k}_j}{\left|\vec{k}_j\right|^3} \frac{e^{-it|\vec{k}_j|}}{\left(1-\vec{v}_n\cdot\hat{k}_j\right)\left(1-\vec{v}_m\cdot\hat{k}_j\right)} \\
=&\int_\lambda^E \frac{r^2dr}{r^3}e^{-itr}\int \frac{d\Omega}{\left(1-\vec{v}_n\cdot\hat{k}_j\right)\left(1-\vec{v}_m\cdot\hat{k}_j\right)} \\
=&\left[\int_\lambda^E \frac{dr}{r}e^{-itr}\right]B(\vec{v}_n,\vec{v}_m) \\
=&\left[\sum_{n=1}^{\infty}\int_\lambda^E dr\frac{(-it)^n}{n!}r^{n-1}+\int_\lambda^E\frac{dr}{r}\right]B(\vec{v}_n,\vec{v}_m) \\
=&\left[\sum_{n=1}^{\infty}\frac{(-it)^n}{n!n}(E^n-\lambda^n)+\ln\!\left(\frac{E}{\lambda}\right)\right]B(\vec{v}_n,\vec{v}_m)\,.
\end{aligned}
$$

This does not depend on the index j, so, for N soft photons,

$$
\begin{aligned}
&\prod_j\left(\left[\sum_{n=1}^{\infty}\frac{(-it)^n}{n!n}(E^n-\lambda^n)+\ln\!\left(\frac{E}{\lambda}\right)\right]B(\vec{v}_n,\vec{v}_m)\right) \\
=&\frac{1}{N!}\left[\sum_{n=1}^{\infty}\frac{(-it)^n}{n!n}(E^n-\lambda^n)+\ln\!\left(\frac{E}{\lambda}\right)\right]^N B^N(\vec{v}_n,\vec{v}_m) \\
=&\frac{1}{N!}\sum_{j=0}^{N}\binom{N}{j}\left[\sum_{n=1}^{\infty}\frac{(-it)^n}{n!n}(E^n-\lambda^n)\right]^j\left[\ln\!\left(\frac{E}{\lambda}\right)\right]^{N-j} B^N(\vec{v}_n,\vec{v}_m)\,,
\end{aligned}
$$

where the division by $N!$ is needed since soft photons are indistinguishable and we want to avoid overcounting them. The integral in t, then takes the form of a sum of terms like

$$
C_r\lim_{\epsilon\to0^+}\int_{-\infty}^{\infty}dt\frac{e^{itE_T}}{t-i\varepsilon}t^r,
$$

where C_r are constants containing factors such as powers of $\ln\left(\frac{E}{\lambda}\right)$. This can be solved through Calculus of Residues, with the respective analytic extension to the complex plane. The integrand has a pole in $i\varepsilon$, corresponding to the residue $e^{-\varepsilon E_T}\,(i\varepsilon)^r$. The integration over the upper semiplane vanishes on the infinite limit, so the integral is simply

$$\int_{-\infty}^{\infty} dt \frac{e^{itE_T}}{t-i\epsilon} t^r = 2\pi i e^{-\varepsilon E_T} (i\varepsilon)^r \underset{\varepsilon\to 0}{\to} 0,$$

for $r \geq 1$. In the case where $r=0$, the integral becomes

$$\frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dt \frac{e^{itE_T}}{t-i\epsilon}\left[\ln\left(\frac{E}{\lambda}\right)\right]^N = \left[\ln\left(\frac{E}{\lambda}\right)\right]^N \Theta(E_T)\,.$$

Of course, $\Theta(E_T) = 1$[7] and, recalling the definition,

$$A(\alpha\to\beta) \equiv \sum_{n,m} \frac{\xi_n \xi_m e_n e_m (p_n \cdot p_m)}{(2\pi)^3 2 E_n E_m} B(\vec{v}_n, \vec{v}_m) = -\sum_{n,m} \frac{e_n e_m \xi_n \xi_m}{(2\pi)^2 \beta_{mn}} \operatorname{arctanh}(\beta_{mn})\,,$$

allows us to write the transition rate, with N real soft photons included, in a neater way as

$$\Gamma^{\gamma_r}_{\alpha\to\beta} = \Gamma^{\gamma_v}_{\alpha\to\beta} \frac{A(\alpha\to\beta)^N}{N!}\left[\ln\left(\frac{E}{\lambda}\right)\right]^N\,.$$

Again, given the undetectability of the soft photons, a process can emit or absorb 1, 2, 6, 42, 756... soft photons and we would be unable to notice it. The final transition rate ought to include a sum over all possibilities of emission and absorption,

$$\begin{aligned}
\Gamma^{\gamma_r}_{\alpha\to\beta} &= \Gamma^{\gamma_v}_{\alpha\to\beta} \sum_{N=0}^{\infty} \frac{A(\alpha\to\beta)^N}{N!}\left[\ln\left(\frac{E}{\lambda}\right)\right]^N \\
&= \Gamma^{\gamma_v}_{\alpha\to\beta}\left(\frac{E}{\lambda}\right)^{A(\alpha\to\beta)} \\
&= \Gamma^{\gamma_v}_{\alpha\to\beta}\left(\frac{E}{\Lambda}\right)^{A(\alpha\to\beta)}\left(\frac{\Lambda}{\lambda}\right)^{A(\alpha\to\beta)} \\
&= \Gamma_{\alpha\to\beta}\left(\frac{E}{\Lambda}\right)^{A(\alpha\to\beta)}\,.
\end{aligned} \tag{1.3}$$

Thus, we have obtained the transition rate of a process $\alpha\to\beta$, $\Gamma^{\gamma_r}_{\alpha\to\beta}$, where α and β are hard particles, in the presence of soft photons, virtual and real, that is, with all the photons

[7] $E_T = 0$ would be a perfect detector with no photon being able to escape it. We will just consider we are always in the case were $E_T > 0$. This constant drops now from the calculations.

reintroduced. All terms in the right hand side of the expression (1.3) are finite, therefore so is $\Gamma^{\gamma_r}_{\alpha\to\beta}$: the infrared divergences of virtual and real soft photons cancel out in the full transition rate. Of course, $\Gamma^{\gamma_r}_{\alpha\to\beta}$ cannot depend on Λ, hence this dependence is buried in $\Gamma_{\alpha\to\beta}$ by the definition of the extracted virtual soft photons.

Finally, as we have computed the transition rate, we can also compute the probability of emission of N soft photons[8] with energies comprised in the interval $[E_-, E_+]$. We proceed to ignore the corrections from virtual soft photons, including them instead in a constant K that will be dealt with by normalisation. Since the total energy of the N emitted soft photons is still lower than E_T, we can just reuse the results obtained previously to get the probability,

$$\begin{aligned}
\mathrm{P}(N; E_- \leq E \leq E_+) &= K\frac{A(\alpha\to\beta)^N}{N!}\left[\ln\left(\frac{E_+}{E_-}\right)\right]^N \\
&= \frac{A(\alpha\to\beta)^N}{N!}\left[\ln\left(\frac{E_+}{E_-}\right)\right]^N\left(\frac{E_-}{E_+}\right)^{A(\alpha\to\beta)},
\end{aligned}$$

with $\sum_{N=0}^{\infty}\mathrm{P}(N; E_- \leq E \leq E_+) = 1$. Writing $\mu \equiv A(\alpha\to\beta)\ln\left(\frac{E_+}{E_-}\right)$, we can easily see that the probability,

$$\mathrm{P}(N;\mu) = \frac{\mu^N}{N!}e^{-\mu},$$

follows a Poisson distribution, with μ as the mean value of emitted photons. Fixing E_+ (possibly as E) and making $E_- = \lambda\to 0$, we see that μ grows to infinity and the probability of emitting a finite number of soft photons vanishes [10]. This is the so called soft photon cloud that a charged particle emits. The same results allow to further compute the mean energy each of the N photons emits, integrating in $\left|\vec{k_j}\right| d^3\vec{k_j}$, instead. The integration in the energies gives

$$\int_{E_-}^{E_+} dr e^{-itr} = (E_+ - E_-) + \sum_{n=1}^{\infty}\frac{(-it)^n}{(n+1)!}\left(E_+^{n+1} - E_-^{n+1}\right),$$

with all the terms with n higher than 1, as before, vanishing, when integrating in $\int_{-\infty}^{\infty}\frac{dt}{t-i\epsilon}$

[8] Since, after normalisation, this probability will only depend on the integrations on the momenta of the real photons, as long as we assume perturbative regime ($E_T \ll m$) and low-energy photons ($\left|\vec{k_i}\right| \ll m$), it still suffices to describe the probability of emission of low-energy, yet detectable, photons.

and taking the limit $\epsilon \to 0$. This yields $\frac{A(\alpha\to\beta)^N}{N!}(E_+ - E_-)^N \left(\frac{E_-}{E_+}\right)^{A(\alpha\to\beta)}$, when multiplying it by the normalisation. Summing over N gives the mean energy each emitted soft photon carries, $e^{A(\alpha\to\beta)(E-\lambda)}\left(\frac{\lambda}{E}\right)^{A(\alpha\to\beta)} \to 0$ as $\lambda \to 0$. This is not surprising, since any other finite value would mean an infinite amount of emitted energy due to the emission of an infinite number of soft photons.

Summary

Definitions:

- Real soft photon (of energy E): A real photon with $\left|\vec{k}\right|$ below the detection energy E. Such energy is assumed to be much smaller than the rest mass of any external charged particle.

Conclusions:

- The IR divergence due to real soft photons emitted or absorbed by charged external legs disappears since the transition rate can be written as a finite quantity, depending on the capabilities of the external detectors.

- The emission of low-energy photons follows a Poisson distribution, which was already expected from the semiclassical results, with the mean value of emission equating $A(\alpha\to\beta)\ln\left(\frac{E_+}{E_-}\right)$. We will see in Section 2.2 that, for nonzero E_-, this factor is extremely small, so the emission of these photons is incredibly rare. The energy carried by each soft photon, in the limit $\lambda\to 0$, vanishes.

2. Interaction Free Measurements

2.1. Introduction

Avshalom C. Elitzur and Lev Vaidman proposed, in 1993, a method to obtain information about the location of an object without interacting with it, which they called an Interaction Free Measurement (IFM) [5].

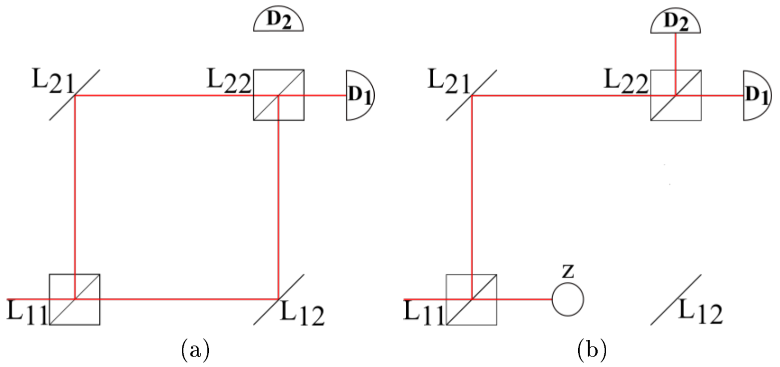


Figure 2.1.: Interferometer without (a) and with (b) a bomb, in z, blocking the lower path.

The method, in its simpler form, consists of a Mach-Zehnder interferometer with a bomb placed on one of its arms (vide Figure 2.1). It is assumed that an incoming particle on the boobytrapped path will always interact with and trigger the bomb.[1] This interaction is local and the arms of the interferometer are so far apart that particles in the other path would never interact with the bomb. Whereas without it, every particle thrown into the interferometer would be detected in a single detector, D_1, with the bomb, both detectors share a quarter

[1] If it did not, the probabilities would be slightly changed, but the general argument would be maintained, given that this case would count as a no detection or explosion.

of probability of detection, with a further half being the probability of triggering the bomb. The conclusion of this experiment is that, whenever the second detector, D_2, ticks, there is an object blocking one of the paths of the interferometer.

Note that the experiment can be executed with any opaque object, but its usefulness derives from the fact that it allows us to infer the existence of unstable states without interacting with them. Such states are modelled by the bomb and its propensity to explode once it interacts. More complex configurations can improve the rate of detection of the bomb without exploding it, assuming it is placed inside a cavity, possibly up to a theoretical limit of 100% [13, 14, 15], but the simpler scheme is enough to make an analysis.

The proposed nomenclature raised some opposition and discussion about the possibility of having an infinitely sensitive bomb [6, 16]. Indeed, an infinitely sensitive bomb would be physically unreasonable: quantum fluctuations or, based on the considerations of Chapter 1, soft photon that reaches the bomb would be able to detonate it, and they should be infinite in number, would do the same.[2]. This should not be a cause for concern, though: not only the real usefulness of the Elitzur and Vaidman's setup is not on measuring the position of nonexistent objects, but also because the nonexistence of an infinitely sensitive bomb does not imply a transference of momentum from the incoming object to the bomb, when D_2 ticks. This is recognised in Ref. [6]. The possible transfer of momentum, they argue, would occur due to the wave function collapse, but this does not invalidate or support the claim that there is no interaction before that moment. The best we shall be able to do, even with QFT, is to say that there is a superposition of explosion and no explosion of the bomb before the collapse.

[2] Note that if the detonation is, in any way, detectable, those soft photons would not be soft by the definition given, but merely low-energy. This should not alter the argument, but I find important to highlight this caveat in order to maintain semantic consistency.

2.1.1. Quantum field theory description

Given the proposed sensibility of the bomb, it should be considered a quantum mechanical object. Let us make the mathematical description in terms of quantum field theory. The standard QM derivation is available in Ref. [5]. Instead of considering the evolution of a given external state, we will consider the evolution of the field itself, in the presence of an interferometer, and consider the probability of taking a particular path. Without a bomb, let us have the fields

$$\phi(x) = \sum_{\vec{p}} \left[a(\vec{p}) e^{ip\cdot x} + a^\dagger(\vec{p}) e^{-ip\cdot x} \right], \qquad \pi(x) = \sum_{\vec{p}} (-iE_{\vec{p}}) \left[a(\vec{p}) e^{ip\cdot x} - a^\dagger(\vec{p}) e^{-ip\cdot x} \right],$$

where x is a point in the Minkowski space. Now consider a mirror placed on z, described as

$$\diagup_z[\phi(x)] = \begin{cases} \phi(x) & \vec{z} \neq \vec{x} \\ V^\dagger \phi(x) V & \vec{z} = \vec{x} \end{cases},$$

with $V = \exp\left(\frac{i\pi}{2} \sum_{\vec{p}} a^\dagger(\mathbf{R}\cdot\vec{p}) a(\vec{p}) \right)$ and $\mathbf{R} = \mathbf{1} - 2\,(\hat{n} \otimes \hat{n})$.[3] Explicitly writing the transformations of the field and its reflected counterpart, $\phi'(\mathrm{R}\cdot x)$, (see Ref. [18]), in the same reference frame of $\phi(x)$,

$$\begin{aligned} V^\dagger \phi(x) V &= i \sum_{\vec{p}} a(\mathbf{R}\cdot\vec{p}) e^{ip\cdot x} - a^\dagger(\mathbf{R}\cdot\vec{p}) e^{-ip\cdot x} \equiv \phi'(\mathrm{R}\cdot x) \\ V^\dagger \phi'(\mathrm{R}\cdot x) V &= -\sum_{\vec{p}} a(\vec{p}) e^{ip\cdot x} + a^\dagger(\vec{p}) e^{-ip\cdot x} = -\phi(x) \end{aligned}$$

Note that, fortunately,[4]

$$\left[V^\dagger \phi(x) V, V^\dagger \pi(y) V \right] = V^\dagger [\phi(x), \pi(y)] V = i\delta(x-y)\,.$$

[3] This Householder matrix simply indicates the action of the mirror on the momenta [17]. It has the property $\mathbf{R}^\dagger = \mathbf{R} = \mathbf{R}^{-1}$. Its extension to Minskowski space is merely R, with the time component kept undisturbed. The global phase $e^{\pm i\pi/2} = \pm i$ arises from the angle $\pi/2$ between the incoming and outgoing paths [18]. Ultimately, it is this phase shift that leads to the interference, independently of the description used.

[4] Further note that the momentum of the field $\phi'(\mathrm{R}\cdot x)$ is the reflected momentum of $\phi(x)$, whilst the energy remains unchanged (cf. Section B.1).

The mirror is not more than a local interaction, but since we want to compute the probabilities of a path going through it, we can just introduce it as a function of the field. We can define the beamsplitter in a similar fashion,

$$\boxslash_z[\phi(x)] = \begin{cases} \phi(x) & \vec{z} \neq \vec{x} \\ U^\dagger \phi(x) U & \vec{z} = \vec{x} \end{cases},$$

with $U = \exp\Big(\frac{i\pi}{4}\sum_{\vec{p}} a^\dagger(\mathbf{R}\cdot\vec{p})a(\vec{p})\Big)$. Its action on the field can be explicitly written as

$$U^\dagger \phi(x) U = \frac{1}{\sqrt{2}}\sum_{\vec{p}} \left\{a(\vec{p})e^{ip\cdot x} + a^\dagger(\vec{p})e^{-ip\cdot x}\right\} + \frac{i}{\sqrt{2}}\sum_{\vec{p}}\left\{a(\mathbf{R}\cdot\vec{p})e^{ip\cdot x} - a^\dagger(\mathbf{R}\cdot\vec{p})e^{-ip\cdot x}\right\}.$$

This corresponds to a split of the field in two superimposed components $\frac{1}{\sqrt{2}}\phi\,(x)$ and $\frac{1}{\sqrt{2}}\phi'(\mathrm{R}\cdot x)$. The momentum field splits in a similar way, in components $\frac{1}{\sqrt{2}}\pi(x)$ and $\frac{1}{\sqrt{2}}\pi'(\mathrm{R}\cdot x)$. The following commutating relations hold:

$$[\phi(x), \pi(y)] = \left[\phi'(x), \pi'(y)\right] = i\delta(x-y)\,,$$
$$\left[\phi(x), \pi'(y)\right] = \left[\phi'(x), \pi(y)\right] = i\delta(x-y)\,.$$

If the fields are well localised, we shall consider this last condition null, thus well separable fields commute.

Defining the vertices of a square in Minkowski space, L_{11}, L_{12}, L_{21} and L_{22}, we want to compute the probability amplitude of propagating a field from a point a to the beamsplitter in L_{11}, then propagate $\frac{1}{\sqrt{2}}\phi(x)$ to L_{12} and $\frac{1}{\sqrt{2}}\phi'(\mathrm{R}\cdot x)$ to L_{21}. In those two points, two mirrors reflect the fields. The fields are finally propagated to the last beamsplitter in L_{22}, where they are recombined and propagated to a point b. The evolution inside the interferometer goes as

$$\phi(L_{11}) \overset{\boxslash_{L_{11}}}{\rightarrow} \frac{1}{\sqrt{2}}\big(\phi(L_{11}) + \phi'(\mathrm{R}\cdot L_{11})\big) \underset{/_{L_{21}}}{\overset{/_{L_{12}}}{\rightarrow}} \frac{1}{\sqrt{2}}\big(\phi'(\mathrm{R}\cdot L_{12}) - \phi(L_{21})\big) \overset{\boxslash_{L_{22}}}{\rightarrow} -\phi(L_{22})\,,$$

with the full path corresponding to a probability amplitude, written in terms of two-point correlation function, $G(\cdot,\cdot)$,

$$G(a,L_{11})G(L_{11},L_{12})G(L_{11},L_{12})G(L_{12},L_{22})G(L_{12},L_{22})G(L_{22},b)\,.$$

We have written the field as if it were scalar, but since the transformation acts only on the ladder operators, we can use a photon field instead, interacting with a fermionic bomb.[5] Let us assume the interaction happens in the path between L_{11} and L_{12},[6] then

$$G(L_{11},L_{12})\to G(L_{11},z-\epsilon)\langle\Omega|A^1_\mu(z-\epsilon)A^2_\nu(z+\epsilon)\bar{\psi}(y)\psi(x)|\Omega\rangle G(z+\epsilon,L_{12})\,,$$

for an interaction range of radius $|\vec{\epsilon}|$ (with $\epsilon=(0,\vec{\epsilon})$).[7] We can now apply perturbation theory to compute the $\langle\Omega|\cdot|\Omega\rangle$ term, but there is no need to do it explicitly as it will eventually correspond to a scattering matrix and the claim that the incoming state is lost can be retrieved. Indeed, as long as the theories considered have a finite effective interaction range, but the force carriers have infinite range, such as QED, we can always build the interferometer such that there is no interaction between the superimposed field and the bomb. In fact, we get the exact same result as in QM: before collapse, we have a superposition of interaction and detection in D_2 (and another in D_1 that we do not consider).

Finally note that, although the point is to localise an object by interacting with it, the experiment is often linked with the which-path problem [18]. Since the interference is obviously destroyed by the bomb, this problem is not relevant (see Ref. [19]). In fact, considering perturbations of the field with sufficiently small dispersion, allowing for the identification of both superimposed fields with each perturbation, as long as we can indicate which path was obstructed by the bomb, we can state which field gave the detection in D_2.

[5] The unitary transformations should be amended to include transformations of the external factors. A detailed derivation is presented in Section B.2.

[6] Any other of the four internal paths could have been chosen. It is important to require the interaction to be unique. We are assuming only one of the superimposed fields interacts with the bomb. This can be done with very localised perturbations of the fields and assuming that long-range correlations are negligible.

[7] For QED, $|\vec{\epsilon}|\sim 1/m$, where m is the mass of the lightest fermion [2].

2.2. Pollution by low-energy photons

Considering, for instance, the bomb as a fermion, we can examine the possibility of altering the standard probabilities of the setup due to the emission of low-energy photons.

On the assumptions of Chapter 1, we have a Poisson probability of emission of a low-energy photon with energy comprised between E_- and E_+, $\mathrm{P}(N;\mu) = \frac{\mu^N}{N!}e^{-\mu}$, with the mean value μ given by $A(f \to f')\ln\left(\frac{E_+}{E_-}\right)$. $A(f \to f')$ is the Weinberg's factor, which depends only on the charges and velocities of the particles in the process: in this case, a fermion possibly transforming into another one. One of the easiest ways to model the bomb is to consider a scattering of a photon by an electron. There is no privileged direction of integration, so the emitted photons have equal probability of leaving in any direction.

We are focusing solely on detectable photons, therefore, we can just take E_- to be our detection threshold. If it is nonvanishing, μ is finite. In fact, μ grows very slowly, since it is logarithmic in E_- and, even though $A(f \to f')$ diverges for $\beta \to 1$, we have

$$A\big(f \to f'\big) = \frac{2e_f^2}{(2\pi)^2}\left[\frac{1}{\beta}\mathrm{arctanh}(\beta) - 1\right] \leq \frac{2e_f^2}{\beta}\mathrm{arctanh}(\beta) \approx 10e_f^2,$$

for a velocity at 99.99% the speed of light. In these conditions, the number of emitted detectable photons is so low that it is very unlikely that one will be emitted in the direction of the incoming photon.

Only for the detection limit, $E_- \to 0$, which is physically difficult to justify, we would get, as in the case of soft photons, a divergent μ and, therefore, a cloud of detectable photons, in particular, with a beam going in the direction of the splitter. So, when decreasing the detection threshold to extremely low values, the detectors would tick, even after an interaction. Hence, knowing how low the detection energy would have to be, allows us to conclude that the predictions of Elitzur and Vaidman are safe from pollution by these photons.

Conclusions

Fields can be written as superpositions, and well localised perturbations can be treated as separate and independent fields, as long as long-range interactions are negligible. This is the case of QED. In fact, for this type of theories, the Elitzur-Vaidman results for IFM can be derived in terms of QFT. Thus, the scattering and finite interaction ranges imposed ad hoc in a QM description arise naturally in this formalism as a Scattering Matrix and interaction ranges that depend on the normalisation parameters.

From the discussion on the Weinberg's Soft Photon Theorem, a Poisson probability for low-energy photons can be derived. The mean value of the number of emitted photons, $A(\alpha \to \beta) \ln\left(\frac{E_+}{E_-}\right)$, depends on Weinberg's $A(\alpha \to \beta)$ factor. This factor was computed in the case of photon-electron scattering. Despite divergent in the limit when the relative velocity of the incomming and outgoing electron is the speed of light, its value is, for typical velocities, very low. For example, for 99.99% the speed of light, this factor is lower than ten times the square of the electron charge. Only when the emitted photons are allowed to take arbitrarily low energies, the mean value grows to infinity and the 4-momenta of the emitted photons covers the whole 4-momentum space. In other words, there is no sea of low-energy photons if a sensible lower limit for the energies of detection is imposed. In this case, the emission of these photons can be safely ignored. In particular, the IFM probabilities only acquire a neglible correction.

A. Appendix to Chapter 1

A.1. Solution of the integral of Section 1.2

We want to solve

$$\int \frac{d^3\vec{r}}{|\vec{r}|^3(1-\vec{a}\cdot\hat{r})\left(1-\vec{b}\cdot\hat{r}\right)},$$

where $\vec{a}$, $\vec{b}$ are two constant vectors satisfying $|\vec{a}|, \left|\vec{b}\right| \leq 1$.

The obvious first step is to rewrite the integral in spherical coordinates. The integration on the radius is quite straightforward,

$$\ln\left(\frac{\Lambda}{\lambda}\right)\int \frac{d\Omega}{(1-\vec{a}\cdot\hat{r})\left(1-\vec{b}\cdot\hat{r}\right)}.$$

The frame of reference will be oriented in such a way that $\vec{a}$ is aligned with the axis $\mathcal{O}z$ and the projection of $\vec{b}$ in the plane $\mathcal{O}xy$ is aligned with the axis $\mathcal{O}x$. Writing $\hat{r} = \sin(\theta)\cos(\phi)\hat{x} + \sin(\theta)\sin(\phi)\hat{y} + \cos(\theta)\hat{z}$, we have $\vec{a}\cdot\hat{r} = a\cos(\theta)$ and, defining θ' as the angle between $\vec{a}$ and $\vec{b}$, we further have $\vec{b}\cdot\hat{r} = b\cos(\theta')\cos(\theta) + b\sin(\theta')\sin(\theta)\cos(\phi)$. Rewriting the integral in the spherical surface in this fashion corresponds to

$$\int \frac{\sin(\theta)d\theta d\phi}{(1-a\cos(\theta))(1-b\cos(\theta)\cos(\theta')-b\sin(\theta)\sin(\theta')\cos(\phi))}.$$

The integral in ϕ is of the form

$$\int_0^{2\pi}\frac{d\phi}{(w-u\cos(\phi))},$$

with $w = 1 - b\cos(\theta)\cos(\theta')$ and $u = b\sin(\theta)\sin(\theta')$. These constants satisfy the condition $w \geq u$.[1] Since the integrand is even, we can reduce it to

$$\int_{-\pi/2}^{\pi/2} \frac{2d\phi}{(w + u\sin(\phi))}.$$

Given that $w^2 \geq u^2$, the indefinite integral has the solution (2.551-3 of Ref. [20])

$$\frac{4}{\sqrt{w^2 - u^2}} \arctan\left(\frac{w\tan\left(\frac{\phi}{2}\right) + u}{\sqrt{w^2 - u^2}}\right).$$

Since there are no divergences in the interval $]-\pi/2, \pi/2[$, after a few simplifications, we get $\frac{2\pi}{\sqrt{w^2-u^2}}$. Writing the terms under the square root as,

$$\begin{aligned}
&w^2 - u^2 \\
&= \left(1 - b\cos(\theta)\cos\left(\theta'\right)\right)^2 - \left(b\sin(\theta)\sin\left(\theta'\right)\right)^2 \\
&= 1 - 2b\cos(\theta)\cos\left(\theta'\right) + b^2\cos^2(\theta)\cos^2\left(\theta'\right) - b^2\left(1 - \cos^2(\theta)\right)\sin^2\left(\theta'\right) \\
&= 1 - b^2\sin^2\left(\theta'\right) - 2b\cos(\theta)\cos\left(\theta'\right) + b^2\cos^2(\theta) \\
&= A + B\cos(\theta) + C\cos^2(\theta),
\end{aligned}$$

where we have defined

$$\begin{aligned}
A &= 1 - b^2\sin^2\left(\theta'\right), \\
B &= -2b\cos\left(\theta'\right), \\
C &= b^2,
\end{aligned}$$

[1]It suffices to note $w - u = 1 - b\cos(\theta)\cos(\theta') - b\sin(\theta)\sin(\theta') = 1 - b\cos(\theta - \theta') \geq 0$.

and so, the last integral to solve takes the form

$$\begin{aligned}
&\int_0^{\pi} \frac{2\pi \sin(\theta) d\theta}{(1-a\cos(\theta))\sqrt{A+B\cos(\theta)+C\cos^2(\theta)}} \\
=&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2\pi \cos(\theta) d\theta}{(1+a\sin(\theta))\sqrt{A-B\sin(\theta)+C\sin^2(\theta)}} \\
=&\int_{-1}^{1} \frac{2\pi dx}{(1+ax)\sqrt{A-Bx+Cx^2}}.
\end{aligned}$$

Changing the variable to $y = ax+1$,

$$\begin{aligned}
&\frac{2\pi}{a}\int_{1-a}^{1+a} \frac{dy}{y\sqrt{A-\frac{B}{a}(y-1)+\frac{C}{a^2}(y-1)^2}} \\
=&\,2\pi\int_{1-a}^{1+a} \frac{dy}{y\sqrt{(Aa^2+Ba+C)-(Ba+2C)y+Cy^2}} \\
=&\frac{2\pi}{\sqrt{Aa^2+Ba+C}}\int_{1-a}^{1+a} \frac{dy}{y\sqrt{1-\frac{Ba+2C}{Aa^2+Ba+C}y+\frac{C}{Aa^2+Ba+C}y^2}}.
\end{aligned}$$

Substituting back the A, B, C constants, using

$$\begin{aligned}
Aa^2+Ba+C &= \left(1-b^2\sin^2\left(\theta'\right)\right)a^2 - 2ab\cos\left(\theta'\right)+b^2 \\
&= a^2-a^2b^2+a^2b^2\cos^2\left(\theta'\right)-2ab\cos\left(\theta'\right)+b^2 \\
&= a^2\left(1-b^2\right)-\left(1-b^2\right)+1-2\left(\vec{a}\cdot\vec{b}\right)+\left(\vec{a}\cdot\vec{b}\right)^2 \\
&= \left(1-\vec{a}\cdot\vec{b}\right)^2-\left(1-a^2\right)\left(1-b^2\right) \\
&= \left(1-\vec{a}\cdot\vec{b}\right)^2\beta^2,
\end{aligned}$$

and defining

$$\beta = \sqrt{1-\frac{(1-\vec{a}^2)\left(1-\vec{b}^2\right)}{\left(1-\vec{a}\cdot\vec{b}\right)^2}},$$

we can write the integral as

$$\frac{2\pi}{\left(1-\vec{a}\cdot\vec{b}\right)\beta}\int_{1-a}^{1+a}\frac{dy}{y\sqrt{1-\frac{2b^2-2ab\cos\left(\theta'\right)}{\left(1-\vec{a}\cdot\vec{b}\right)^2\beta^2}y+\frac{b^2}{\left(1-\vec{a}\cdot\vec{b}\right)^2\beta^2}y^2}}$$

$$=\frac{2\pi}{\left(1-\vec{a}\cdot\vec{b}\right)\beta}\int_{1-a}^{1+a}\frac{dy}{y\sqrt{1-2\frac{b^2-\vec{a}\cdot\vec{b}}{\left(1-\vec{a}\cdot\vec{b}\right)^2\beta^2}y+\frac{b^2}{\left(1-\vec{a}\cdot\vec{b}\right)^2\beta^2}y^2}}.$$

The indefinite integral is (2.266 of Ref. [20])

$$\int\frac{dy}{y\sqrt{1-2\frac{b^2-\vec{a}\cdot\vec{b}}{\left(1-\vec{a}\cdot\vec{b}\right)^2\beta^2}y+\frac{b^2}{\left(1-\vec{a}\cdot\vec{b}\right)^2\beta^2}y^2}}$$

$$=\ln(y)-\ln\left[2-2\frac{b^2-\vec{a}\cdot\vec{b}}{\left(1-\vec{a}\cdot\vec{b}\right)^2\beta^2}y+2\sqrt{1-\frac{2\left(b^2-\vec{a}\cdot\vec{b}\right)y-b^2y^2}{\left(1-\vec{a}\cdot\vec{b}\right)^2\beta^2}}\right].$$

Evaluating it in the limits, it is simply

$$\ln\left(\frac{1+a}{1-a}\right)-\ln\left[\frac{1-\frac{b^2-\vec{a}\cdot\vec{b}}{\left(1-\vec{a}\cdot\vec{b}\right)^2\beta^2}(1+a)+\sqrt{1-\frac{2\left(b^2-\vec{a}\cdot\vec{b}\right)(1+a)-b^2(1+a)^2}{\left(1-\vec{a}\cdot\vec{b}\right)^2\beta^2}}}{1-\frac{b^2-\vec{a}\cdot\vec{b}}{\left(1-\vec{a}\cdot\vec{b}\right)^2\beta^2}(1-a)+\sqrt{1-\frac{2\left(b^2-\vec{a}\cdot\vec{b}\right)(1-a)-b^2(1-a)^2}{\left(1-\vec{a}\cdot\vec{b}\right)^2\beta^2}}}\right].$$

The square roots can be simplified to

$$\sqrt{1-\frac{2\left(b^2-\vec{a}\cdot\vec{b}\right)(1\pm a)-b^2(1\pm a)^2}{\left(1-\vec{a}\cdot\vec{b}\right)^2\beta^2}}=\frac{a\pm\vec{a}\cdot\vec{b}}{\left(1-\vec{a}\cdot\vec{b}\right)\beta},$$

since $a \geq \vec{a}\cdot\vec{b}$. This is, followed by a series of algebraic steps,

$$
\begin{aligned}
&\ln\left[\left(\frac{1+a}{1-a}\right)\left(\frac{1-\frac{b^2-\vec{a}\cdot\vec{b}}{\left(1-\vec{a}\cdot\vec{b}\right)^2\beta^2}(1-a)+\frac{a-\vec{a}\cdot\vec{b}}{\left(1-\vec{a}\cdot\vec{b}\right)\beta}}{1-\frac{b^2-\vec{a}\cdot\vec{b}}{\left(1-\vec{a}\cdot\vec{b}\right)^2\beta^2}(1+a)+\frac{a+\vec{a}\cdot\vec{b}}{\left(1-\vec{a}\cdot\vec{b}\right)\beta}}\right)\right]\\
&=\ln\left[\left(\frac{1+a}{1-a}\right)\frac{\left(1-\vec{a}\cdot\vec{b}\right)^2\beta^2-\left(b^2-\vec{a}\cdot\vec{b}\right)(1-a)+\left(a-\vec{a}\cdot\vec{b}\right)\left(1-\vec{a}\cdot\vec{b}\right)\beta}{\left(1-\vec{a}\cdot\vec{b}\right)^2\beta^2-\left(b^2-\vec{a}\cdot\vec{b}\right)(1+a)+\left(a+\vec{a}\cdot\vec{b}\right)\left(1-\vec{a}\cdot\vec{b}\right)\beta}\right]\\
&=\ln\left[\left(\frac{1+a}{1-a}\right)\frac{\left(1-\vec{a}\cdot\vec{b}\right)^2-\left(1-a^2\right)\left(1-b^2\right)-\left(b^2-\vec{a}\cdot\vec{b}\right)(1-a)+\left(a-\vec{a}\cdot\vec{b}\right)\left(1-\vec{a}\cdot\vec{b}\right)\beta}{\left(1-\vec{a}\cdot\vec{b}\right)^2-(1-a^2)(1-b^2)-\left(b^2-\vec{a}\cdot\vec{b}\right)(1+a)+\left(a+\vec{a}\cdot\vec{b}\right)\left(1-\vec{a}\cdot\vec{b}\right)\beta}\right]\\
&=\ln\left[\left(\frac{1+a}{1-a}\right)\frac{a^2+\left(\vec{a}\cdot\vec{b}\right)\left(\vec{a}\cdot\vec{b}-a\right)-\left(\vec{a}\cdot\vec{b}\right)-a^2b^2+ab^2+\left(a-\vec{a}\cdot\vec{b}\right)\left(\beta-\beta\left(\vec{a}\cdot\vec{b}\right)\right)}{a^2+\left(\vec{a}\cdot\vec{b}\right)\left(\vec{a}\cdot\vec{b}+a\right)-\left(\vec{a}\cdot\vec{b}\right)-a^2b^2-ab^2+\left(a+\vec{a}\cdot\vec{b}\right)\left(\beta-\beta\left(\vec{a}\cdot\vec{b}\right)\right)}\right]\\
&=\ln\left[\left(\frac{1+a}{1-a}\right)\frac{a^2-a-a^2b^2+ab^2+\left(a-\vec{a}\cdot\vec{b}\right)\left(\beta+1-(\beta+1)\left(\vec{a}\cdot\vec{b}\right)\right)}{a^2+a-a^2b^2-ab^2+\left(a+\vec{a}\cdot\vec{b}\right)\left(\beta-1-(\beta-1)\left(\vec{a}\cdot\vec{b}\right)\right)}\right]\\
&=\ln\left[\left(\frac{1+a}{1-a}\right)\frac{a\left(1-b^2\right)(a-1)+\left(a-\vec{a}\cdot\vec{b}\right)(\beta+1)\left(1-\vec{a}\cdot\vec{b}\right)}{a(1-b^2)(a+1)+\left(a+\vec{a}\cdot\vec{b}\right)(\beta-1)\left(1-\vec{a}\cdot\vec{b}\right)}\right]\\
&=\ln\left[\frac{-a\left(1-b^2\right)\left(1-a^2\right)+(1+a)\left(a-\vec{a}\cdot\vec{b}\right)(\beta+1)\left(1-\vec{a}\cdot\vec{b}\right)}{a(1-b^2)(1-a^2)+(1-a)\left(a+\vec{a}\cdot\vec{b}\right)(\beta-1)\left(1-\vec{a}\cdot\vec{b}\right)}\right]\\
&=\ln\left[\frac{-a\frac{\left(1-b^2\right)\left(1-a^2\right)}{\left(1-\vec{a}\cdot\vec{b}\right)^2}+\frac{(1+a)\left(a-\vec{a}\cdot\vec{b}\right)}{\left(1-\vec{a}\cdot\vec{b}\right)}(\beta+1)}{a\frac{(1-b^2)(1-a^2)}{\left(1-\vec{a}\cdot\vec{b}\right)^2}+\frac{(1-a)\left(a+\vec{a}\cdot\vec{b}\right)}{\left(1-\vec{a}\cdot\vec{b}\right)}(\beta-1)}\right]\\
&=\ln\left[\frac{a\left(\beta^2-1\right)+\frac{(1+a)\left(a-\vec{a}\cdot\vec{b}\right)}{\left(1-\vec{a}\cdot\vec{b}\right)}(\beta+1)}{a(1-\beta^2)+\frac{(1-a)\left(a+\vec{a}\cdot\vec{b}\right)}{\left(1-\vec{a}\cdot\vec{b}\right)}(\beta-1)}\right]\\
&=\ln\left[\frac{a(\beta-1)\left(1-\vec{a}\cdot\vec{b}\right)+(1+a)\left(a-\vec{a}\cdot\vec{b}\right)}{a(\beta+1)\left(1-\vec{a}\cdot\vec{b}\right)-(1-a)\left(a+\vec{a}\cdot\vec{b}\right)}\left(\frac{1+\beta}{1-\beta}\right)\right]\\
&=\ln\left[\frac{a\beta\left(1-\vec{a}\cdot\vec{b}\right)-\vec{a}\cdot\vec{b}+a^2}{a\beta\left(1-\vec{a}\cdot\vec{b}\right)-\vec{a}\cdot\vec{b}+a^2}\left(\frac{1+\beta}{1-\beta}\right)\right]=\ln\left[\frac{1+\beta}{1-\beta}\right].
\end{aligned}
$$

We get the final result,

$$\int \frac{d^3\vec{r}}{|\vec{r}|^3(1-\vec{a}\cdot\hat{r})\left(1-\vec{b}\cdot\hat{r}\right)} = \frac{2\pi}{\left(1-\vec{a}\cdot\vec{b}\right)\beta}\ln\left[\frac{1+\beta}{1-\beta}\right]\ln\left(\frac{\Lambda}{\lambda}\right),$$

with the definition

$$\beta \equiv \sqrt{1-\frac{(1-\vec{a}^2)\left(1-\vec{b}^2\right)}{\left(1-\vec{a}\cdot\vec{b}\right)^2}}.$$

It is noteworthy to mention that the result is invariant by interchange of $\vec{a}$ and $\vec{b}$, as expected.

Evaluating the extreme values inside the square root, one concludes that, when $\vec{a}$ and $\vec{b}$ are either parallel or antiparallel,

$$\frac{|a-b|}{1-ab} \leq \beta \leq \frac{a+b}{1+ab},$$

and, in particular, we have that $0 \leq \beta \leq 1$, with second equality holding when and only when at least one of the vectors has module 1. In fact, for such case, $\beta = 1$ for every angle between the two vectors. This will be obvious, physically, when we finally identify β with the module of a velocity in natural units.

Finally, taking into account the limiting values of β, we can simply note that the integral on the spherical surface can be written as

$$\int \frac{d^2\Omega}{(1-\vec{a}\cdot\hat{r})\left(1-\vec{b}\cdot\hat{r}\right)} = \frac{4\pi}{\left(1-\vec{a}\cdot\vec{b}\right)\beta}\text{arctanh}(\beta)\,,$$

leading to the final result,

$$\int \frac{d^3\vec{r}}{|\vec{r}|^3\,(1-\vec{a}\cdot\hat{r})\left(1-\vec{b}\cdot\hat{r}\right)} = \frac{4\pi}{\left(1-\vec{a}\cdot\vec{b}\right)\beta}\text{arctanh}(\beta)\text{ln}l\left(\frac{\Lambda}{\lambda}\right).$$

A.2. Interpretation of β_{mn}

Consider the factor

$$\beta_{mn} = \sqrt{1 - \frac{m_n^2 m_m^2}{(p_n \cdot p_m)^2}} = \sqrt{1 - \frac{(1-\vec{v}_n^2)(1-\vec{v}_m^2)}{(1-\vec{v}_n \cdot \vec{v}_m)^2}},$$

with $\vec{v}_n \equiv \vec{p}_n/E_n$, $|\vec{v}_n| \leq 1$ and $p_n = (E_n, \vec{p}_n)$. The momenta of both particles is measured in the same reference frame. Call it $\mathcal{O}$. Let us assume $\mathcal{O}$ is the reference frame where the particle with label m is at rest. Then

$$\beta_{mn} = |\vec{v}_n| = \frac{|\vec{p}_n|}{E_n},$$

that is, β_{mn} equals the module of the velocity of the particle n measured in the reference frame $\mathcal{O}$.

Let us now consider that the particles with labels m and n are collinear and measured in the same reference frame $\mathcal{O}$. Then, β_{mn} can be simplified to

$$\beta_{mn} = \frac{|v_n - v_m|}{1 - v_n v_m}.$$

If we interpret one of the velocities of a particle as the velocity of a reference frame $\mathcal{O}'$, it becomes clear that β_{mn} is the module of the velocity, obtained through the velocity addition formula, of the other particle in the frame of reference of the first (see Refs. [1, 10]).

Expecting this interpretation to hold in several dimensions, let us check that

$$\beta_{mn} = \sqrt{\left(\beta_{mn}^{\|}\right)^2 + (\beta_{mn}^{\perp})^2}.$$

We assume that $\beta_{mn}^{\|}$, $\beta_{mn}^{\perp}$ are the parallel and perpendicular components, respectively, of the velocity of one particle measured in the reference frame of the other. To ease the computations, we will just orient $\mathcal{O}$ such that the x-axis coincides with the velocity of the particle m, and $\vec{v}_n$ is on the $\mathcal{O}xy$ plane,

$$\begin{aligned}\vec{v}_m &= v_m\hat{x}\\ \vec{v}_n &= v_n\cos(\theta)\hat{x} + v_n\sin(\theta)\hat{y}.\end{aligned}$$

So $\left(\beta^{\|}_{mn}\right)^2$ takes the form

$$\left(\beta^{\|}_{mn}\right)^2 = \left(\frac{v_n\cos(\theta) - v_m}{1 - \vec{v}_m\cdot\vec{v}_n}\right)^2.$$

The perpendicular part is, from the velocity addition formula,

$$\left(\beta^{\perp}_{mn}\right)^2 = \left(\frac{v_n\sin(\theta)\sqrt{1 - v_m^2}}{1 - \vec{v}_m\cdot\vec{v}_n}\right)^2.$$

Computing β_{mn},

$$\begin{aligned}\beta_{mn} &= \frac{1}{1 - \vec{v}_m\cdot\vec{v}_n}\sqrt{(v_n\cos(\theta) - v_m)^2 + \left(v_n\sin(\theta)\sqrt{1 - v_m^2}\right)^2}\\ &= \frac{1}{1 - \vec{v}_m\cdot\vec{v}_n}\sqrt{v_n^2 + v_m^2 - 2v_nv_m\cos(\theta) - v_m^2v_n^2\sin^2(\theta)}\\ &= \frac{1}{1 - \vec{v}_m\cdot\vec{v}_n}\sqrt{v_n^2 + v_m^2 - 2\vec{v}_n\cdot\vec{v}_m + (\vec{v}_n\cdot\vec{v}_m)^2 - v_m^2v_n^2}\\ &= \frac{1}{1 - \vec{v}_m\cdot\vec{v}_n}\sqrt{(1 - \vec{v}_m\cdot\vec{v}_n)^2 - (1 - v_n^2)(1 - v_m^2)}\\ &= \sqrt{1 - \frac{(1 - \vec{v}_n^2)(1 - \vec{v}_m^2)}{1 - \vec{v}_m\cdot\vec{v}_n}}.\end{aligned}$$

This is exactly what we want to prove: β_{mn} is the module of the velocity of one particle in the reference frame of the other. This can be written as

$$\beta_{mn} = \left|\vec{\beta}_{mn}\right| = |\vec{v}_m \oplus (-\vec{v}_n)| = \sqrt{1 - \frac{(1 - \vec{v}_n^2)(1 - \vec{v}_m^2)}{1 - \vec{v}_m\cdot\vec{v}_n}},$$

where $\oplus$ stands for the relativistic sum of velocities. In particular, writing in the frame of reference of the particle 1, we have $\vec{\beta}_{mn} = \vec{\beta}_{m1} \oplus \left(-\vec{\beta}_{n1}\right)$.

A.3. Positiveness of $A(\alpha\to\beta)$

We can simplify

$$A(\alpha\to\beta) = -\sum_{n,m}\frac{e_n e_m \xi_n \xi_m}{(2\pi)^2 \beta_{mn}}\operatorname{arctanh}(\beta_{mn})$$

to

$$A(\alpha\to\beta) = -\sum_{n,m}\frac{\xi'_n \xi'_m}{\beta_{mn}}\operatorname{arctanh}(\beta_{mn})\,,$$

where the $\xi'_n = \frac{e_n \xi_n}{2\pi}$ satisfy the conservation of charge, $\sum_n \xi'_n = 0$. Since β_{mn} is the modulus of a vector $\vec{\beta}_{mn} = \vec{\beta}_{m1} \oplus \left(-\vec{\beta}_{n1}\right)$, as we saw in the previous section, we can write $A(\alpha\to\beta)$ in terms of the rapidity,

$$\vec{\zeta}_{mn} = \hat{\beta}_{mn}\operatorname{arctanh}\left(\left|\vec{\beta}_{mn}\right|\right),$$

and, thus,

$$A(\alpha\to\beta) = -\sum_{n,m}\xi'_n \xi'_m \frac{\left|\vec{\zeta}_{mn}\right|}{\tanh\left|\vec{\zeta}_{mn}\right|}.$$

Through the property of the sum of rapidities, we have,

$$\vec{\zeta}_1 + \vec{\zeta}_2 = \frac{\vec{\beta}_1 \oplus \vec{\beta}_2}{\left|\vec{\beta}_1 \oplus \vec{\beta}_2\right|}\operatorname{arctanh}\left(\left|\vec{\beta}_1 \oplus \vec{\beta}_2\right|\right),$$

so

$$\vec{\zeta}_{mn} = \frac{\vec{\beta}_{mn}}{\left|\vec{\beta}_{mn}\right|}\operatorname{arctanh}\left(\left|\vec{\beta}_{mn}\right|\right) = \frac{\vec{\beta}_{m1} \oplus \left(-\vec{\beta}_{n1}\right)}{\left|\vec{\beta}_{m1} \oplus \left(-\vec{\beta}_{n1}\right)\right|}\operatorname{arctanh}\left(\left|\vec{\beta}_{m1} \oplus \left(-\vec{\beta}_{n1}\right)\right|\right) = \vec{\zeta}_{m1} - \vec{\zeta}_{n1}$$

and therefore,

$$A(\alpha\to\beta) = -\sum_{n,m}\frac{\xi'_n \xi'_m \left|\vec{\zeta}_{m1} - \vec{\zeta}_{n1}\right|}{\tanh\left|\vec{\zeta}_{m1} - \vec{\zeta}_{n1}\right|}.$$

Defining $f(\vec{x},\vec{y}) = \frac{|\vec{x}-\vec{y}|}{\tanh|\vec{x}-\vec{y}|}$, $\vec{x},\vec{y} \in \mathbb{R}^n$, it then suffices to show that

$$\sum_{n,m} \xi_n' \xi_m' f\left(\vec{\zeta}_{m1}, \vec{\zeta}_{n1}\right) \leq 0.$$

If $f(\vec{x},\vec{y}) + f(\vec{y},\vec{z}) \geq f(\vec{x},\vec{z})$, we have

$$\begin{aligned}
\sum_{n,m} \xi_n' \xi_m' f(\zeta_{1m}, \zeta_{1n}) &\leq \sum_{n,m} \xi_n' \xi_m' \left(f\left(\vec{\zeta}_{1m}, \vec{\zeta}_{1k}\right) + f\left(\vec{\zeta}_{1n}, \vec{\zeta}_{1k}\right)\right) \\
&= \sum_n \xi_n' \sum_m \xi_m' f\left(\vec{\zeta}_{1m}, \vec{\zeta}_{1k}\right) + \sum_m \xi_m' \sum_n \xi_n' f\left(\vec{\zeta}_{1n}, \vec{\zeta}_{1k}\right) \\
&= 0 + 0 = 0,
\end{aligned}$$

like we want to prove. We need, then, to prove that inequality. In fact, it suffices to show that $g(x) \equiv \frac{x}{\tanh x}$ is subaddictive for $x \in \mathbb{R}_0^+$, that is, $g(x+y) \leq g(x) + g(y)$, and, then, that $g(x)$ is monotonically increasing in the same interval.

Proving subaddictivity first,

$$\begin{aligned}
g(x+y) &= \frac{x+y}{\tanh(x+y)} \\
&= (x+y)\frac{1+\tanh(x)\tanh(y)}{\tanh(x)+\tanh(y)} \\
&= \left(\frac{x}{\tanh(x)}\right)\frac{\tanh(x)+\tanh^2(x)\tanh(y)}{\tanh(x)+\tanh(y)} + (x \leftrightarrow y) \\
&\leq \frac{x}{\tanh(x)} + \frac{y}{\tanh(y)} = g(x) + g(y)
\end{aligned}$$

since

$$\frac{\tanh(x)+\tanh^2(x)\tanh(y)}{\tanh(x)+\tanh(y)} \leq 1$$

$$\tanh^2(x)\tanh(y) \leq \tanh(y)$$

$$\frac{\tanh(y)}{\cosh^2(x)} \geq 0$$

which is always true for $y \geq 0$.

That $g(x)$ is monotonically increasing comes from differentiating $g(x)$ and noting the signal remains positive for positive x, since

$$g'(x) = \frac{1}{\tanh(x)} - \frac{x}{\cosh^2(x)} = \frac{\cosh^3(x) - x\sinh(x)}{\sinh(x)\cosh^2(x)}$$

and the numerator,

$$\begin{aligned}
&\cosh^3(x) - x\sinh(x) \\
\geq &\cosh(x)\big(1+\sinh^2(x)\big) - x\sinh(x) \\
\geq &\sinh(x)(\cosh(x)\sinh(x) - x) \\
\geq &\sinh(x)(\sinh(x) - x) \geq 0,
\end{aligned}$$

given that $\sinh(x) \geq x$. Finally, since $|\vec{x}-\vec{z}| \leq |\vec{x}-\vec{y}| + |\vec{y}-\vec{z}|$, putting everything together, we have

$$g(|\vec{x}-\vec{y}|) + g(|\vec{x}-\vec{z}|) \geq g(|\vec{x}-\vec{y}| + |\vec{y}-\vec{z}|) \geq g(|\vec{x}-\vec{z}|)\,.$$

B. Appendix to Chapter 2

B.1. Energies and momenta of the transformed fields

From the formula for the 4-momentum of the field [2],

$$P^{\mu} = \int T^{0\mu} d^3x,$$

we see that the momentum of the reflected field, $\phi'(x)$, is given by

$$\begin{aligned}
P'_i &= -\int d^3x \big(\pi'(x)\, \partial_i \phi'(x)\big) \\
&= -ii \sum_{\vec{p},\vec{q}} \int d^3x \Big[-iE_{\vec{p}} \Big(a(\mathbf{R}\cdot\vec{p}) e^{ip\cdot x} + a^{\dagger}(\mathbf{R}\cdot\vec{p}) e^{-ip\cdot x} \Big) q_i i \Big(a(\mathbf{R}\cdot\vec{q}) e^{iq\cdot x} + a^{\dagger}(\mathbf{R}\cdot\vec{q}) e^{-iq\cdot x} \Big) \Big] \\
&= \frac{1}{2} \sum_{\vec{p}} p_i \Big(-a(\mathbf{R}\cdot\vec{p}) a(-\mathbf{R}\cdot\vec{p}) e^{-i2E_{\vec{p}}t} - a^{\dagger}(\mathbf{R}\cdot\vec{p}) a^{\dagger}(-\mathbf{R}\cdot\vec{p}) e^{i2E_{\vec{q}}t} \Big) \\
&+ \frac{1}{2} \sum_{\vec{p}} p_i \Big(a^{\dagger}(\mathbf{R}\cdot\vec{p})\, a(\mathbf{R}\cdot\vec{p}) + a(\mathbf{R}\cdot\vec{p}) a^{\dagger}(\mathbf{R}\cdot\vec{p}) \Big) \\
&= \frac{1}{2} \sum_{\vec{p}} p_i \Big(a^{\dagger}(\mathbf{R}\cdot\vec{p}) a\,(\mathbf{R}\cdot\vec{p}) + a(\mathbf{R}\cdot\vec{p}) a^{\dagger}(\mathbf{R}\cdot\vec{p}) \Big) \\
&= \frac{1}{2} \sum_{\vec{p}} R_{ij} p_j \Big(a^{\dagger}(\vec{p})\, a(\vec{p}) + a(\vec{p}) a^{\dagger}(\vec{p}) \Big) = R_{ij} P_j,
\end{aligned}$$

that is, the momentum of the reflected field, $\phi'(x)$, is the reflected momentum of $\phi(x)$.

As for the energy,

$$H' = \frac{1}{2} \int d^3x \Big(\pi'^2(x) + \big(\nabla\phi'(x)\big)^2 + m^2 \phi'^2(x) \Big),$$

we can compute each term separately,

$$
\begin{aligned}
\int d^3x\, \pi'^2(x) &= \sum_{\vec p,\vec q} E_{\vec p}E_{\vec q}\int d^3x\Big(a(\mathbf{R}\cdot\vec p)e^{ip\cdot x} + a^\dagger(\mathbf{R}\cdot\vec p)e^{-ip\cdot x}\Big)\Big(a(\mathbf{R}\cdot\vec q)e^{iq\cdot x} + a^\dagger(\mathbf{R}\cdot\vec q)e^{-iq\cdot x}\Big)\\
&= \frac{1}{2}\sum_{\vec p} E_{\vec p}\Big(a(\mathbf{R}\cdot\vec p)a(-\mathbf{R}\cdot\vec p)e^{-2iE_{\vec p}t} + a^\dagger(\mathbf{R}\cdot\vec p)a^\dagger(-\mathbf{R}\cdot\vec p)e^{2iE_{\vec p}t}\Big)\\
&+ \frac{1}{2}\sum_{\vec p} E_{\vec p}\Big(a^\dagger(\mathbf{R}\cdot\vec p)a(\mathbf{R}\cdot\vec p) + a(\mathbf{R}\cdot\vec p)a^\dagger(\mathbf{R}\cdot\vec p)\Big)\,,\\
\int d^3x\big(\nabla\phi'(x)\big)^2 &= -\sum_{\vec p,\vec q} iip_jq^j\int d^3x\Big(a(\mathbf{R}\cdot\vec p)e^{ip\cdot x} + a^\dagger(\mathbf{R}\cdot\vec p)e^{-ip\cdot x}\Big)\Big(a(\mathbf{R}\cdot\vec q)e^{iq\cdot x} + a^\dagger(\mathbf{R}\cdot\vec q)e^{-iq\cdot x}\Big)\\
&= \frac{1}{2}\sum_{\vec p}\left(E_{\vec p} - \frac{m^2}{E_{\vec p}}\right)\Big(-a(\mathbf{R}\cdot\vec p)a(-\mathbf{R}\cdot\vec p)e^{-2iE_{\vec p}t} - a^\dagger(\mathbf{R}\cdot\vec p)a^\dagger(-\mathbf{R}\cdot\vec p)e^{2iE_{\vec p}t}\Big)\\
&+ \frac{1}{2}\sum_{\vec p}\left(E_{\vec p} - \frac{m^2}{E_{\vec p}}\right)\Big(a^\dagger(\mathbf{R}\cdot\vec p)a(\mathbf{R}\cdot\vec p) + a(\mathbf{R}\cdot\vec p)a^\dagger(\mathbf{R}\cdot\vec p)\Big)\,,\\
\int d^3x\, m^2\phi'^2(x) &= -m^2\sum_{\vec p,\vec q}\int d^3x\Big(a(\mathbf{R}\cdot\vec p)e^{ip\cdot x} - a^\dagger(\mathbf{R}\cdot\vec p)e^{-ip\cdot x}\Big)\Big(a(\mathbf{R}\cdot\vec q)e^{iq\cdot x} - a^\dagger(\mathbf{R}\cdot\vec q)e^{-iq\cdot x}\Big)\\
&= \frac{1}{2}\sum_{\vec p}\frac{m^2}{E_{\vec p}}\Big(-a(\mathbf{R}\cdot\vec p)a(-\mathbf{R}\cdot\vec p)e^{-2iE_{\vec p}t} - a^\dagger(\mathbf{R}\cdot\vec p)a^\dagger(-\mathbf{R}\cdot\vec p)e^{2iE_{\vec p}t}\Big)\\
&+ \frac{1}{2}\sum_{\vec p}\frac{m^2}{E_{\vec p}}\Big(a^\dagger(\mathbf{R}\cdot\vec p)\,a(\mathbf{R}\cdot\vec p) + a(\mathbf{R}\cdot\vec p)a^\dagger(\mathbf{R}\cdot\vec p)\Big)\,.
\end{aligned}
$$

Summing the three terms yields,

$$
H' = \frac{1}{2}\sum_{\vec p} E_{\vec p}\Big(a^\dagger(\mathbf{R}\cdot\vec p)a(\mathbf{R}\cdot\vec p) + a(\mathbf{R}\cdot\vec p)a^\dagger(\mathbf{R}\cdot\vec p)\Big) = H,
$$

that is, the transformation leaves the energy unchanged.

B.2. Transformations of vector fields

Like we have done for scalar fields, we wish to show that the transformations for vector fields hold. Since not only the internal indices but the components of nonscalar fields should be altered, we ought to generalise the unitary transformations derived in Chapter 2.

With that objective in sight, let $V(\alpha)$ be a unitary operator, parameterised by α, defined as

$$V(\alpha) = \exp\left(i\alpha R^{\mu\nu} \sum_{\vec{p}} \sum_{\lambda,\lambda'} \epsilon_{\mu}^{*\lambda}(\mathbf{R}\cdot\vec{p}) a_{\lambda}^{\dagger}(\mathbf{R}\cdot\vec{p}) \epsilon_{\nu}^{\lambda'}(\vec{p}) a_{\lambda'}(\vec{p}) \right),$$

where R is the Householder matrix on Minkowski space defined in Chapter 2. We want to consider the following transformation, $V^{\dagger}(\alpha) \sum_{\kappa} \epsilon_{\alpha}^{\kappa}\left(\vec{k}\right) a_{\kappa}\left(\vec{k}\right) V(\alpha)$.

To make use of the Baker-Hausdorff Lemma,[1] with

$$A = R^{\mu\nu} \sum_{\vec{p}} \sum_{\lambda,\lambda'} \epsilon_{\mu}^{*\lambda}(\mathbf{R}\cdot\vec{p}) a_{\lambda}^{\dagger}(\mathbf{R}\cdot\vec{p}) \epsilon_{\nu}^{\lambda'}(\vec{p}) a_{\lambda'}(\vec{p}), \quad B = \sum_{\kappa} \epsilon_{\alpha}^{\kappa}\left(\vec{k}\right) a_{\kappa}\left(\vec{k}\right),$$

we need to compute the following commutators,

$$\begin{aligned}
[A,B] &= \left[R^{\mu\nu} \sum_{\vec{p}} \sum_{\lambda,\lambda'} \epsilon_{\mu}^{*\lambda}(\mathbf{R}\cdot\vec{p}) a_{\lambda}^{\dagger}(\mathbf{R}\cdot\vec{p}) \epsilon_{\nu}^{\lambda'}(\vec{p}) a_{\lambda'}(\vec{p}), \sum_{\kappa} \epsilon_{\alpha}^{\kappa}\left(\vec{k}\right) a_{\kappa}\left(\vec{k}\right) \right] \\
&= R^{\mu\nu} \sum_{\vec{p}} \sum_{\lambda,\lambda',\kappa} \epsilon_{\mu}^{*\lambda}(\mathbf{R}\cdot\vec{p}) \epsilon_{\nu}^{\lambda'}(\vec{p}) \epsilon_{\alpha}^{\kappa}\left(\vec{k}\right) \left[a_{\lambda}^{\dagger}(\mathbf{R}\cdot\vec{p}), a_{\kappa}\left(\vec{k}\right) \right] a_{\lambda'}(\vec{p}) \\
&= -R^{\mu\nu} \sum_{\vec{p}} \sum_{\lambda,\lambda',\kappa} \epsilon_{\mu}^{*\lambda}(\mathbf{R}\cdot\vec{p}) \epsilon_{\nu}^{\lambda'}(\vec{p}) \epsilon_{\alpha}^{\kappa}\left(\vec{k}\right) \delta\left(\mathbf{R}\cdot\vec{p} - \vec{k}\right) \delta_{\lambda\kappa}\, a_{\lambda'}(\vec{p}) \\
&= -R^{\mu\nu} \sum_{\lambda,\lambda'} \epsilon_{\mu}^{\lambda *}\left(\vec{k}\right) \epsilon_{\nu}^{\lambda'}\left(\mathbf{R}^{-1}\cdot\vec{k}\right) \epsilon_{\alpha}^{\lambda}\left(\vec{k}\right) a_{\lambda'}\left(\mathbf{R}^{-1}\cdot\vec{k}\right) \\
&= -R^{\mu\nu} \eta_{\mu\alpha} \sum_{\lambda'} \epsilon_{\nu}^{\lambda'}\left(\mathbf{R}^{-1}\cdot\vec{k}\right) a_{\lambda'}\left(\mathbf{R}^{-1}\cdot\vec{k}\right) \\
&= -R_{\alpha}^{\nu} \sum_{\kappa} \epsilon_{\nu}^{\kappa}\left(\mathbf{R}\cdot\vec{k}\right) a_{\kappa}\left(\mathbf{R}\cdot\vec{k}\right),
\end{aligned}$$

[1] That is, $e^{i\alpha A} B e^{-i\alpha A} = B - i\alpha[A,B] + \frac{1}{2!}(-i\alpha)^2 [A,[A,B]] + \ldots$ (cf. Ref. [18]).

where we used, in the last step, that $\mathbf{R} = \mathbf{R}^{-1}$, and

$$[A,[A,B]] = R^{\nu}_{\alpha} R^{\mu}_{\nu} \sum_{\kappa} \epsilon^{\kappa}_{\mu} \Big(\big(\mathbf{R}^{-1}\big)^2 \cdot \vec{k} \Big) a_{\kappa} \Big(\big(\mathbf{R}^{-1}\big)^2 \cdot \vec{k} \Big) = \sum_{\kappa} \epsilon^{\kappa}_{\alpha} \Big(\vec{k} \Big) a_{\kappa} \Big(\vec{k} \Big) .$$

Therefore,

$$\begin{aligned} e^{i\alpha A} \sum_{\kappa} \epsilon^{\kappa}_{\alpha} \Big(\vec{k} \Big) a_{\kappa} \Big(\vec{k} \Big) e^{-i\alpha A} &= \sum_{\kappa} \epsilon^{\kappa}_{\alpha} \Big(\vec{k} \Big) a_{\kappa} \Big(\vec{k} \Big) + i\alpha R^{\nu}_{\alpha} \sum_{\kappa} \epsilon^{\kappa}_{\nu} \Big(\mathbf{R} \cdot \vec{k} \Big) a_{\kappa} \Big(\mathbf{R} \cdot \vec{k} \Big) \\ &+ \frac{1}{2} (i\alpha)^2 \sum_{\kappa} \epsilon^{\kappa}_{\alpha} \Big(\vec{k} \Big) a_{\kappa} \Big(\vec{k} \Big) + \ldots \\ &= \cos(\alpha) \sum_{\kappa} \epsilon^{\kappa}_{\alpha} \Big(\vec{k} \Big) a_{\kappa} \Big(\vec{k} \Big) + i \sin(\alpha) R^{\nu}_{\alpha} \sum_{\kappa} \epsilon^{\kappa}_{\nu} \Big(\mathbf{R} \cdot \vec{k} \Big) a_{\kappa} \Big(\mathbf{R} \cdot \vec{k} \Big) . \end{aligned}$$

For the mirror operator, $V \equiv V(\pi/2)$ and

$$V^{\dagger} \sum_{\kappa} \epsilon^{\kappa}_{\alpha} \Big(\vec{k} \Big) a_{\kappa} \Big(\vec{k} \Big) V = i \sum_{\kappa} \epsilon^{\kappa}_{\nu} \Big(\mathbf{R} \cdot \vec{k} \Big) a_{\kappa} \Big(\mathbf{R} \cdot \vec{k} \Big) ,$$

while for the beamsplitter, $U \equiv V(\pi/4)$ and

$$U^{\dagger} \sum_{\kappa} \epsilon^{\kappa}_{\alpha} \Big(\vec{k} \Big) a_{\kappa} \Big(\vec{k} \Big) U = \frac{1}{\sqrt{2}} \left[\sum_{\kappa} \epsilon^{\kappa}_{\alpha} \Big(\vec{k} \Big) a_{\kappa} \Big(\vec{k} \Big) + i \sum_{\kappa} \epsilon^{\kappa}_{\nu} \Big(\mathbf{R} \cdot \vec{k} \Big) a_{\kappa} \Big(\mathbf{R} \cdot \vec{k} \Big) \right] .$$

The transformations for $\epsilon^{*\kappa}_{\alpha} \Big(\vec{k} \Big) a^{\dagger}_{\kappa} \Big(\vec{k} \Big)$ are obtained by taking the hermitian conjugate of the previous expressions. With these results, we finally have the general transformation of the photonic field, written as

$$\begin{aligned} V^{\dagger}(\alpha) A_{\mu}(x) V(\alpha) = \cos(\alpha) &\sum_{\vec{p},\lambda} \Big[\epsilon^{\lambda}_{\mu}(\vec{p}) a_{\lambda}(\vec{p}) e^{ip\cdot x} + a^{\dagger}_{\lambda}(\vec{p}) e^{-ip\cdot x} \Big] \\ &+ i \sin(\alpha) R^{\nu}_{\mu} \sum_{\vec{p},\lambda} \Big[\epsilon^{\lambda}_{\nu}(\mathbf{R}\cdot\vec{p}) a_{\lambda}(\mathbf{R}\cdot\vec{p}) e^{ip\cdot x} - \epsilon^{*\lambda}_{\nu}(\mathbf{R}\cdot\vec{p}) a^{\dagger}_{\lambda}(\mathbf{R}\cdot\vec{p}) e^{-ip\cdot x} \Big] \\ &= \cos(\alpha) A_{\mu}(x) + \sin(\alpha) A'_{\mu}(R\cdot x) . \end{aligned}$$

Taking α as $\pi/2$ or $\pi/4$ yields the desired transformations due to the mirror and beamsplitter, respectively.

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