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Identification of an unknown spatial load distribution in a vibrating beam or plate from the final state

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Abstract. The theoretical and numerical determination of a space-dependent load distribution in a simply supported non-homogeneous Euler-Bernoulli beam and Kirchhoff-Love plate is investigated. The uniqueness of a solution to this inverse source problem is proved, whilst counterexamples are constructed to discuss the conditions under which uniqueness holds. A convergent and stable iterative algorithm is proposed for the recovery of the unknown load source and a stopping criterion is also given. Several one-dimensional numerical experiments are considered to investigate the properties of the proposed iterative procedure.

Keywords. spatial load identification, beams, plates, inverse problems, iterative regularization, finite element method.

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1 Introduction

In the presented work, the problem of identifying a spatial load distribution in a vibrating beam or plate from the deflection u in z direction at final time $t = T$ is studied. The additional information is given by

$$u(\cdot, T) = \xi_T(\cdot). \tag{1.1}$$

This problem belongs to the class of inverse source problems (ISPs). The domain $\Omega \subset \mathbb{R}^d$ is a thin beam (1d elastic structure in the 3d space) or a rectangular thin plate (2d elastic structure in the 3d space) with boundary Γ. The dynamic vibration of a simply supported non-homogeneous Euler-Bernoulli beam ($d = 1$) and Kirchhoff-Love plate ($d = 2$) is governed by the following forward problem

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(for small deflection):

$$\begin{cases} \partial_{tt}(\rho u) + \mu\partial_t u + \Delta(k\Delta u) - T^r\Delta u &= f(\mathbf{x})h(t) \quad \text{in } Q_T, \\ u &= 0 \quad \text{on } \Sigma_T, \\ k\Delta u &= 0 \quad \text{on } \Sigma_T, \\ u(\mathbf{x},0) &= \tilde{u}_0(\mathbf{x}) \quad \mathbf{x} \in \Omega, \\ \partial_t u(\mathbf{x},0) &= \tilde{v}_0(\mathbf{x}) \quad \mathbf{x} \in \Omega, \end{cases} \tag{1.2}$$

where $Q_T := \Omega \times (0,T)$ and $\Sigma_T := \Gamma \times (0,T)$. Here, u is the displacement function from the equilibrium position $u \equiv 0$, $\tilde{u}_0$ is the initial deflection, $\tilde{v}_0$ is the initial velocity, $f(\mathbf{x})$ (unknown) and $h(t)$ (given) are the spatial and temporal load distributions. Further, $\rho(\mathbf{x},t)$ is the mass density, $\mu(\mathbf{x},t)$ is the damping coefficient and $T_r(\mathbf{x},t)$ is the traction force. If $d = 1$, then $k(\mathbf{x},t) = EI(\mathbf{x},t)$ where E is the elasticity modulus and I is the moment of inertia, whilst k is the bending stiffness of the plate if $d = 2$, e.g. $k(\mathbf{x},t) = \frac{h^3 E(\mathbf{x},t)}{12(1-\nu(\mathbf{x},t)^2)}$ for a rectangular uniform plate of thickness h where E is the Young modulus and ν is the Poisson ratio.

1.1 Literature overview

Vibration problems related to the static and dynamic response of beams and plates have applications in building science, mechanical and aircraft engineering, in earth science and engineering [8]. For the simplest Euler-Bernoulli and Kirchhoff equations, the coefficient identification problems have attracted a great deal of attention in inverse problems, cf. [3–5, 11, 16, 21, 23, 28–30, 37]. In these papers, the determination of spatial coefficients based on additional measured data is studied using methods based on spectral theory and on observations (input-output mappings).

This contribution focuses on the determination of a unknown spatial load distribution $f(\mathbf{x})$ from the final in time deflection in the nonhomogeneous Euler-Bernoulli beam and Kirchhoff-Love plate equations with arbitrary but separable source term. Other source identification problems for Euler-Bernoulli equations from boundary or final in time observations can be found in [12–15, 20, 24, 32] and for the Kirchhoff-Love equation in [10]. In [32], using spectral theory, the point source $a(x)$ is uniquely determined in the constant coefficient dynamic Euler-Bernoulli equation $\ddot{u} + u'''' = \lambda(t)a(x)$ where $\lambda \in \mathrm{C}^1([0,T])$ is given and $x \in (0,1)$. The missing information is compensated by the following boundary measurements: respectively $u'(0,t)$ or $u''(0,t)$ for all $t \in (0,T)$. This source identification problem has been reconsidered in [20] for more general Euler-Bernoulli equation, which includes a constant damping and a constant traction force. An effective combination of the Lie-group adaptive method and the differential quadrature method is proposed in [24] to recover an unknown space and time dependent load in a constant coefficient Euler-Bernoulli beam vibration equation. For the variable coefficient Euler-Bernoulli equation $\rho(x)\ddot{u} + (k(x)u'')'' = F(x,t)$ with $(x,t) \in (0,L)\times(0,T)$, using the least-square method (or quasi-solution approach) combined with the adjoint problem approach, Hasanov [12] considered the inverse problems of determining the unknown source term $F(x,t)$ from the measured data $u(x,T)$ or $u_t(x,T)$. Uniqueness of the solution can be obtained when a positivity condition holds on the solution [12, Lemma 7.2] . The theory developed in this article is then applied in [13] to the problem of determining the unknown spatial load $f(x)$ from the final displacement observation in a cantilever beam of the form $\rho(x)\ddot{u} + (k(x)u'')'' = f(x)h(t)$. In [15], two inverse source problems of identifying asynchronously distributed spatial loads governed in the Euler-Bernoulli beam equation $\rho(x)\ddot{u} + \mu(x)\dot{u} + (k(x)u'')'' - T_r u'' = \sum_{m=1}^{M} h_m(t) f_m(x)$ with hinged-clamped ends are studied. In the first identification problem, $(f_1, \dots, f_M)$ is determined from the measured deflection in a neighborhood of a finite set of points. In the second identification problem, $(f_1, \dots, f_M)$ is determined from the measured slope in a neighborhood of the same set of points. Solution to the ISPs are obtained by using Tikhonov regularization (thus by minimizing a cost functional). In [14], the problem of identifying the temporal load distribution $h(t)$ and the problem of identifying the spatial load distribution $f(x)$ in a vibrating beam $\rho(x)\ddot{u} + (k(x)u'')'' = f(x)h(t)$ from the boundary observation $u'(0,t)$ (i.e. the slope at $x = 0$) is investigated. The approach in that article is based on the weak solution theory of PDEs and the quasi-solution method for inverse problems. Finally, in [10], an inverse source problem for the Kirchhoff-Love plate equation $\partial_{tt} u + k\Delta^2 u = h(t) f(\mathbf{x})$ is studied. The load source $f(\mathbf{x})$ is determined from a measurement of the displacement at one or more discrete points over a time interval under the assumption that $h(t)$ is a harmonic load. The uniqueness theorem for this problem is stated, and the fundamental solution to the plate equation is derived. The fundamental solution method and the Tikhonov regularization technique are used to calculate the source term.

1.2 Discussion and outline

In this contribution, the recovery of the unknown source $f(\mathbf{x})$ is not achieved by minimizing a cost functional or by using a spectral method or an adjoint problem. First, a variational approach is used to obtain the uniqueness of a solution to the ISP, which is a new result. The conditions under which uniqueness holds are examined by explicitly constructing counterexamples, that is by constructing more than one solution in the case when the conditions for uniqueness are violated. Moreover, a Landweber-Fridman type iterative regularization method is developed to obtain an approximation of the unknown load source. This approach is already successfully applied for the heat conduction equation [6, 18] and for thermoelastic systems [35,36]. Note that the coefficients in problem (2.1) are also time-dependent, which is not the case in the other papers referenced before.

The paper is organized as follows. First, the corresponding forward problem is studied in Section 2. Then, the uniqueness of a solution to the inverse problem under consideration is established in Section 3. Also counterexamples are given. Afterwards, in Section 4, an algorithm for the recovery of the unknown spatial load function is proposed. This method is based on a sequence of well-posed direct problems, which can be numerically solved at each iteration step by using the finite element method. The instability of this inverse source problem is overcome by stopping the iterations using the discrepancy principle. Finally, some numerical experiments are developed in Section 5 and some concluding remarks and ideas for future work are presented in Section 6.

Remark 1.1 (Other boundary conditions). The approach given in this article is also applicable in the case of other boundary conditions arising in engineering applications. More specifically, the results above remain valid, when instead of the hinged-hinged end conditions $u = k\Delta u = 0$ on Σ_T in (2.1), the following conditions are given [8]: clamped-clamped ends $u = \nabla u \cdot \boldsymbol{\nu} = 0$ on Σ_T ($\boldsymbol{\nu}$ is the unit normal outward vector); hinged-clamped ends $u(0,t) = u''(0,t) = 0$, $u(L,t) = u'(L,t) = 0$ ($d = 1$); sliding-clamped ends $u(0,t) = (ku'')'(0,t) = 0$, $u(L,t) = u'(L,t) = 0$ ($d = 1$); clamped-free ends (cantilever beam) $u(0,t) = u'(0,t) = 0$, $u''(L,t) = (ku'')'(L,t) = 0$ ($d = 1$); hinged-free ends $u(0,t) = u''(0,t) = 0$, $u''(L,t) = (ku'')'(L,t) = 0$ ($d = 1$).

Remark 1.2 (Notations). Denote by $(\cdot,\cdot)$ the standard inner product in $\mathrm{L}^2(\Omega)$ and by $\|\cdot\|$ its induced norm. The norm $|\cdot|_{\mathrm{e}}$ is the Euclidean norm. Consider an abstract Banach space X with norm $\|\cdot\|_X$. Let $p \geqslant 1$. The space $\mathrm{L}^p\left((0,T),X\right)$ consists of functions $u : [0,T] \to X$ such that

$$\|u\|_{\mathrm{L}^p((0,T),X)} = \left(\int_0^T \|u(t)\|_X^p \,\mathrm{d}t\right)^{1/p} < \infty.$$

The space $\mathrm{C}\left([0,T],X\right)$ consists of continuous functions $u : [0,T] \to X$ satisfying

$$\|u\|_{\mathrm{C}([0,T],X)} = \max_{[0,T]} \|u(t)\|_X < \infty.$$

The space $\mathrm{L}^\infty\left((0,T),X\right)$ consists of all measurable functions $u : (0,T) \to X$ that are essentially bounded. The space $\mathrm{H}^1((0,T),X)$ consists of functions u :

$[0, T] \to X$ such that

$$\|u\|_{\mathrm{H}^1((0,T),X)} = \left(\int_0^T \|u(t)\|_X^2 + \left\|u'(t)\right\|_X^2 \, \mathrm{d}t \right)^{\frac{1}{2}} < \infty.$$

The symbol X^* stands for the dual space of X. Moreover, the values C, ε and C_ε are generic and positive constants independent of the discretization parameter τ. The value ε is small and $C_\varepsilon \leqslant C(\varepsilon^{-1})$.

Remark 1.3. The analysis made in this contribution is valid for a Lipschitz domain $\Omega \subset \mathbb{R}^d$ with $d \in \mathbb{N}$.

2 The forward problem

Let $V := \mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega)$. The norms $\sum_{|\boldsymbol{\alpha}| \leqslant 2} \|D^{\boldsymbol{\alpha}} u\|^2$ and $\|\Delta u\|^2$ are equivalent in V, see [9, Theorem 1]. In the left-hand side (LHS) of the governing PDE, an additional term λu is added, and moreover a not necessarily separable source in the right-hand side (RHS) is considered, i.e.

$$\begin{cases} \rho \partial_{tt} u + \mu \partial_t u + \lambda u + \Delta(k \Delta u) - T^r \Delta u & = f & \text{in } Q_T, \\ u & = 0 & \text{on } \Sigma_T, \\ k \Delta u & = 0 & \text{on } \Sigma_T, \\ u(\mathbf{x}, 0) & = \tilde{u}_0(\mathbf{x}) & \mathbf{x} \in \Omega, \\ \partial_t u(\mathbf{x}, 0) & = \tilde{v}_0(\mathbf{x}) & \mathbf{x} \in \Omega. \end{cases} \tag{2.1}$$

Remark 2.1. This problem is equivalent with (1.2) when the following replacements are made:

$$\mu \to \mu + 2\partial_t \rho \quad \text{and} \quad \lambda \to \partial_{tt} \rho. \tag{2.2}$$

Further one, the reader needs to keep these replacements in mind when interpreting the assumptions on the data μ and ρ in problem (1.2).

After two times application of Green's theorem, the variational formulation of problem (2.1) reads as follows:

Find $u(t) \in V$ with $\partial_t u(t) \in V$ and $\partial_{tt} u(t) \in \mathrm{L}^2(\Omega)$ such that

$$\begin{aligned} (\rho(t)\partial_{tt} u(t), \varphi) + (\mu(t)\partial_t u(t), \varphi) + (\lambda(t) u(t), \varphi) + (k(t)\Delta u(t), \Delta\varphi) \\ - (T^r(t)\Delta u(t), \varphi) = (f(t), \varphi)\,, \end{aligned} \tag{2.3}$$

for all $\varphi \in V$ and a.a. $t \in (0, T]$.

Remark 2.2. The variational formulation is well-defined when $\partial_t u(t) \in \mathrm{L}^2(\Omega)$ and $\rho(t)\partial_{tt} u(t) \in V^*$ for a.a. $t \in (0,T]$. However, the requirement $\partial_t u(t) \in V$ is necessary to be able to prove the uniqueness of a solution to the inverse source problem, see Theorem 3.1. This also implies that $\partial_t u$ can be used as test function in the proof of uniqueness of a solution to the direct problem, see Theorem 2.3.

To obtain the existence of a weak solution to problem (2.1) for given $f : [0,T] \to \mathrm{L}^2(\Omega)$ in the required function spaces, it is assumed that the functions ρ, μ, λ, k, T_r, $\tilde{u}_0$, $\tilde{v}_0$ and f satisfy the following conditions:

$$\begin{aligned}
0 < \tilde{\rho}_0 \leqslant \rho(\mathbf{x},t) &\leqslant \tilde{\rho}_1, \quad (\mathbf{x},t) \in \Omega \times [0,T],\\
|\partial_t \rho(\mathbf{x},t)| &\leqslant \tilde{\rho}_2,\\
|\mu(\mathbf{x},t)| &\leqslant \tilde{\mu}_1,\\
|\partial_t \mu(\mathbf{x},t)| &\leqslant \tilde{\mu}_2,\\
|\lambda(\mathbf{x},t)| &\leqslant \tilde{\lambda}_1,\\
|\partial_t \lambda(\mathbf{x},t)| &\leqslant \tilde{\lambda}_2,\\
0 < \tilde{k}_0 \leqslant k(\mathbf{x},t) &\leqslant \tilde{k}_1,\\
|\partial_t k(\mathbf{x},t)| &\leqslant \tilde{k}_2,\\
|\partial_{tt} k(\mathbf{x},t)| &\leqslant \tilde{k}_3,\\
|\nabla k(\mathbf{x},0)|_{\mathrm{e}} &\leqslant \tilde{k}_4,\\
|\Delta k(\mathbf{x},0)| &\leqslant \tilde{k}_5,\\
|T^r(\mathbf{x},t)| &\leqslant \tilde{T}_1,\\
|\partial_t T^r(\mathbf{x},t)| &\leqslant \tilde{T}_2,\\
f &\in \mathrm{H}^1\left((0,T), \mathrm{L}^2(\Omega)\right),\\
\tilde{u}_0 &\in \mathrm{H}^4(\Omega) \cap \mathrm{H}^1_0(\Omega),\\
k(\mathbf{x},0)\Delta\tilde{u}_0(\mathbf{x}) &= 0, \quad \mathbf{x} \in \Gamma,\\
\tilde{v}_0 &\in V.
\end{aligned} \tag{2.4}$$

Theorem 2.3 (Uniqueness of solution to the direct problem). *Let the conditions* (2.4) *be fulfilled. Then, there exists at most one solution satisfying problem* (2.3).

Proof. Problem (2.3) has a unique weak solution when $\tilde{u}_0 = \tilde{v}_0 = f = 0$ results in $u = 0$. In (2.3), we put for a.a. $t \in (0,T)$ the testfunction equal to $\partial_t u(t) \in V$ and then we integrate the sequence of problems in time over $(0,\eta) \subset (0,T)$ to arrive at

$$\int_0^\eta (\rho(t)\partial_{tt}u(t), \partial_t u(t))\,\mathrm{d}t + \int_0^\eta (\mu(t)\partial_t u(t), \partial_t u(t))\,\mathrm{d}t + \int_0^\eta (\lambda(t)u(t), \partial_t u(t))\,\mathrm{d}t + \int_0^\eta (k(t)\Delta u(t), \Delta\partial_t u(t))\,\mathrm{d}t - \int_0^\eta (T_r(t)\Delta u(t), \partial_t u(t))\,\mathrm{d}t = 0.$$

Using integration by parts, we readily obtain that

$$\int_0^\eta (\rho(t)\partial_{tt}u(t), \partial_t u(t))\,\mathrm{d}t = \frac{1}{2}\left\|\sqrt{\rho(\eta)}\partial_t u(\eta)\right\|^2 - \frac{1}{2}\int_0^\eta \left(\partial_t\rho(t), (\partial_t u(t))^2\right)\mathrm{d}t,$$

and

$$\int_0^\eta (k(t)\Delta u(t), \Delta\partial_t u(t))\,\mathrm{d}t = \frac{1}{2}\left\|\sqrt{k(\eta)}\Delta u(\eta)\right\|^2 - \frac{1}{2}\int_0^\eta \left(\partial_t k(t), (\Delta u(t))^2\right)\mathrm{d}t.$$

We derive the following estimates

$$\left|\int_0^\eta \left(\partial_t\rho(t), (\partial_t u(t))^2\right)\mathrm{d}t\right| \leqslant \tilde{\rho}_2\int_0^\eta \|\partial_t u(t)\|^2\,\mathrm{d}t,$$
$$\left|\int_0^\eta \left(\partial_t k(t), (\Delta u(t))^2\right)\mathrm{d}t\right| \leqslant \tilde{k}_2\int_0^\eta \|\Delta u(t)\|^2\,\mathrm{d}t.$$

Applying the Cauchy and Young inequalities (remember $\tilde{u}_0 = 0$), it holds that

$$\left|\int_0^\eta (\mu(t)\partial_t u(t), \partial_t u(t))\,\mathrm{d}t\right| \leqslant \tilde{\mu}_1\int_0^\eta \|\partial_t u(t)\|^2\,\mathrm{d}t,$$

$$\left|\int_0^\eta (\lambda(t)u(t), \partial_t u(t))\,\mathrm{d}t\right| \leqslant \frac{\tilde{\lambda}_1^2}{2}\int_0^\eta \left\|\int_0^t \partial_t u(s)\ \mathrm{d}s\right\|^2\mathrm{d}t + \frac{1}{2}\int_0^\eta \|\partial_t u(t)\|^2\,\mathrm{d}t \leqslant \left(\frac{1+\tilde{\lambda}_1^2T^2}{2}\right)\int_0^\eta \|\partial_t u(t)\|^2\,\mathrm{d}t$$

and

$$\left|\int_0^\eta (T_r(t)\Delta u(t), \partial_t u(t))\,\mathrm{d}t\right| \leqslant \frac{\tilde{T}_1^2}{2}\int_0^\eta \|\Delta u(t)\|^2\,\mathrm{d}t + \frac{1}{2}\int_0^\eta \|\partial_t u(t)\|^2\,\mathrm{d}t.$$

After collecting all estimates above, we arrive at

$$\begin{aligned}
&\frac{\tilde{\rho}_0}{2}\left\|\partial_t u(\eta)\right\|^2+\frac{\tilde{k}_0}{2}\left\|\Delta u(\eta)\right\|^2\\
&\leqslant\left(1+\tilde{\mu}_1+\frac{\tilde{\rho}_2}{2}+\frac{\tilde{\lambda}_1^2T^2}{2}\right)\int_0^\eta\left\|\partial_t u(t)\right\|^2\mathrm{d}t+\left(\frac{\tilde{T}_1^2}{2}+\frac{\tilde{k}_2}{2}\right)\int_0^\eta\left\|\Delta u(t)\right\|^2\mathrm{d}t.
\end{aligned}$$

An application of Grönwall's lemma [2] gives $\partial_t u=0$ a.e. in Q_T. Due to $\tilde{u}_0=0$, it holds that $u=0$ a.e. in Q_T. □

Theorem 2.4 (Well-posedness of the direct problem). *Let the conditions* (2.4) *be fulfilled. Then, there exists a unique weak solution to problem* (2.3) *satisfying*

$$u\in \mathrm{C}\left([0,T],V\right)$$

with

$$\partial_t u\in \mathrm{C}\left([0,T],\mathrm{L}^2(\Omega)\right)\cap \mathrm{L}^2\left((0,T),V\right)$$

and

$$\partial_{tt}u\in \mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right).$$

In the special situation that $\tilde{u}_0=0$, $\tilde{v}_0=0$ and $f=f(\mathbf{x})$, there exists a positive constant C such that the following estimate is valid

$$\max_{t\in[0,T]}\left\|\partial_t u(t)\right\|^2+\max_{t\in[0,T]}\left\|\Delta u(t)\right\|^2\leqslant C\left\|f\right\|^2. \tag{2.5}$$

Proof. The derivation of the estimate follows the same lines as the proof of uniqueness of a solution except that $f\neq 0$. In the case that $f=f(\mathbf{x})$, the corresponding term can be handled as follows

$$\left|\int_0^\eta\left(f,\partial_t u(t)\right)\mathrm{d}t\right|\leqslant\frac{T}{2}\left\|f\right\|^2+\frac{1}{2}\int_0^\eta\left\|\partial_t u(t)\right\|^2\mathrm{d}t.$$

To address the existence of a solution to the variational problem (2.3), the semidiscretization in time is employed [19]. First, the interval $[0,T]$ is divided into $n\in\mathbb{N}$ equidistant subintervals $[t_{i-1},t_i]$ with the time step $\tau=\frac{T}{n}<1$, thus $t_i=i\tau, i=0,\ldots,n$. With the standard notation for the discretized fields, for any function z

$$z_i\approx z(t_i),\quad \partial_t z(t)\approx\delta z_i=\frac{z_i-z_{i-1}}{\tau}$$

and

$$\partial_{tt}z(t)\approx\delta^2 z_i=\frac{z_i-z_{i-1}}{\tau^2}-\frac{\delta z_{i-1}}{\tau},$$

the following linear recurrent scheme is proposed to approximate the original problem (2.3) for $i = 1, \dots, n$ (for all $\varphi \in V$):

$$\left(\rho_i \delta^2 u_i, \varphi\right) + (\mu_i \delta u_i, \varphi) + (\lambda_i u_i, \varphi) + (k_i \Delta u_i, \Delta\varphi) - (T_i^r \Delta u_i, \varphi) = (f_i, \varphi), \quad (2.6)$$

with $u_0 = \tilde{u}_0$ and $\delta u_0 = \tilde{v}_0$, which is equivalent to solving

$$\begin{aligned}\left(\left(\frac{\rho_i}{\tau^2} + \frac{\mu_i}{\tau} + \lambda_i\right) u_i, \varphi\right) + (k_i \Delta u_i, \Delta\varphi) - (T_i^r \Delta u_i, \varphi) \\ = (f_i, \varphi) + \left(\left(\frac{\rho_i}{\tau^2} + \frac{\mu_i}{\tau}\right) u_{i-1}, \varphi\right) + \left(\frac{\mu_i}{\tau} \delta u_{i-1}, \varphi\right). \quad (2.7)\end{aligned}$$

The Lax-Milgram lemma gives the existence and uniqueness of a solution $u_i \in V$ to (2.7) if $\tilde{u}_0, \tilde{v}_0 \in \mathrm{L}^2(\Omega)$.

Next, a priori estimates are derived, which serve as uniform bounds to prove the convergence of the semidiscrete scheme (2.6). The following version of Abel's summation rule will be frequently used: for any sequences of real numbers $\{z_i\}_{i=1}^n$ and $\{w_i\}_{i=1}^n$, it holds that

$$\begin{aligned}&\sum_{i=1}^n z_i \omega_i (\omega_i - \omega_{i-1}) \\ &= z_n \omega_n^2 - z_0 \omega_0^2 - \sum_{i=1}^n \left(z_i \omega_i - z_{i-1} \omega_{i-1}\right) \omega_{i-1} \\ &= \frac{1}{2} z_n \omega_n^2 - \frac{1}{2} z_0 \omega_0^2 - \frac{1}{2} \sum_{i=1}^n (z_i - z_{i-1}) \omega_{i-1}^2 + \frac{1}{2} \sum_{i=1}^n z_i (\omega_i - \omega_{i-1})^2. \quad (2.8)\end{aligned}$$

We set $\varphi = \delta u_i \tau$ in (2.6) and sum up these equations for $1 \le i \le j$, with $1 \leqslant j \leqslant n$. We obtain that

$$\begin{aligned}\sum_{i=1}^j \left(\rho_i \delta^2 u_i, \delta u_i\right) \tau + \sum_{i=1}^j (\mu_i \delta u_i, \delta u_i) \tau + \sum_{i=1}^j (\lambda_i u_i, \delta u_i) \tau \\ + \sum_{i=1}^j (k_i \Delta u_i, \Delta \delta u_i) \tau - \sum_{i=1}^j (T_i^r \Delta u_i, \delta u_i) \tau = \sum_{i=1}^j (f_i, \delta u_i) \tau. \quad (2.9)\end{aligned}$$

Using (2.8), the first and fourth term on the LHS can be estimated from below as follows

$$
\begin{aligned}
&\sum_{i=1}^{j} \left(\rho_i \delta^2 u_i, \delta u_i\right) \tau \\
&= \frac{1}{2} \left\|\sqrt{\rho_j} \delta u_j\right\|^2 - \frac{1}{2} \left\|\sqrt{\rho_0} \tilde{v}_0\right\|^2 \\
&\qquad - \frac{1}{2} \sum_{i=1}^{j} \int_\Omega \delta \rho_i (\delta u_{i-1})^2 \tau + \frac{1}{2} \sum_{i=1}^{j} \int_\Omega \rho_i \left(\delta u_i - \delta u_{i-1}\right)^2 \\
&\geqslant \frac{\tilde{\rho}_0}{2} \left\|\delta u_j\right\|^2 - \frac{\tilde{\rho}_1}{2} \left\|\tilde{v}_0\right\|^2 - \frac{1}{2} \sum_{i=1}^{j} \int_\Omega \delta \rho_i (\delta u_{i-1})^2 \tau + \frac{\tilde{\rho}_0}{2} \sum_{i=1}^{j} \left\|\delta u_i - \delta u_{i-1}\right\|^2,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=1}^{j} \left(k_i \Delta u_i, \Delta \delta u_i\right) \tau \\
&\geqslant \frac{\tilde{k}_0}{2} \left\|\Delta u_j\right\|^2 - \frac{\tilde{k}_1}{2} \left\|\Delta \tilde{u}_0\right\|^2 - \frac{1}{2} \sum_{i=1}^{j} \int_\Omega \delta k_i (\Delta u_{i-1})^2 \tau + \frac{\tilde{k}_0}{2} \sum_{i=1}^{j} \left\|\Delta u_i - \Delta u_{i-1}\right\|^2.
\end{aligned}
$$

We have that

$$
\begin{aligned}
\left|\sum_{i=1}^{j} \int_\Omega \delta \rho_i (\delta u_{i-1})^2 \tau\right| &\leqslant \tilde{\rho}_2 \sum_{i=1}^{j} \left\|\delta u_{i-1}\right\|^2 \tau \leqslant \tilde{\rho}_2 \left\|\tilde{v}_0\right\|^2 + \tilde{\rho}_2 \sum_{i=1}^{j-1} \left\|\delta u_i\right\|^2 \tau, \\
\left|\sum_{i=1}^{j} \int_\Omega \delta k_i (\Delta u_{i-1})^2 \tau\right| &\leqslant \tilde{k}_2 \sum_{i=1}^{j} \left\|\Delta u_{i-1}\right\|^2 \tau \leqslant \tilde{k}_2 \left\|\Delta \tilde{u}_0\right\|^2 + \tilde{k}_2 \sum_{i=1}^{j-1} \left\|\Delta u_i\right\|^2 \tau.
\end{aligned}
$$

The other terms in (2.9) are estimated by Cauchy's and Young's inequality, i.e.

$$\left|\sum_{i=1}^{j}(\mu_i\delta u_i,\delta u_i)\,\tau\right| \leqslant \tilde{\mu}_1\sum_{i=1}^{j}\|\delta u_i\|^2\,\tau,$$

$$\begin{aligned}\left|\sum_{i=1}^{j}(\lambda_i u_i,\delta u_i)\,\tau\right| &\leqslant \frac{\tilde{\lambda}_1^2}{2}\sum_{i=1}^{j}\left\|\tilde{u}_0+\sum_{k=1}^{i}\delta u_k\tau\right\|^2\tau+\frac{1}{2}\sum_{i=1}^{j}\|\delta u_i\|^2\,\tau\\ &\leqslant \frac{\tilde{\lambda}_1^2}{2}\|\tilde{u}_0\|^2+\left(\frac{1+\tilde{\lambda}_1^2T^2}{2}\right)\sum_{i=1}^{j}\|\delta u_i\|^2\,\tau,\end{aligned}$$

$$\left|\sum_{i=1}^{j}(T_i^r\Delta u_i,\delta u_i)\,\tau\right| \leqslant \frac{\tilde{T}_1^2}{2}\sum_{i=1}^{j}\|\Delta u_i\|^2\,\tau+\frac{1}{2}\sum_{i=1}^{j}\|\delta u_i\|^2\,\tau,$$

$$\left|\sum_{i=1}^{j}(f_i,\delta u_i)\,\tau\right| \leqslant \frac{1}{2}\sum_{i=1}^{j}\|f_i\|^2\,\tau+\frac{1}{2}\sum_{i=1}^{j}\|\delta u_i\|^2\,\tau.$$

Collecting all the results above and applying Grönwall's lemma give for sufficiently small τ that

$$\max_{0\leqslant i\leqslant n}\left\{\|\delta u_i\|^2+\|\Delta u_i\|^2\right\} + \sum_{i=1}^{n}\left(\|\delta u_i-\delta u_{i-1}\|^2+\|\Delta u_i-\Delta u_{i-1}\|^2\right)\leqslant C_1, \quad (2.10)$$

with $C_1 = C_1\left(T, \|f\|_{\mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right)}, \|\tilde{u}_0\|_V, \|\tilde{v}_0\|_{\mathrm{L}^2(\Omega)}\right)$. From

$$u_i = \tilde{u}_0+\tau\sum_{j=1}^{i}\delta u_j$$

and (2.10), it follows that

$$\max_{0\leqslant i\leqslant n}\|u_i\| \leqslant \|\tilde{u}_0\|+\sqrt{C_1}T =: C_2. \quad (2.11)$$

For the following a priori estimate, we need that the discrete variational formulation (2.6) is well-defined for $i=0$. We can define

$$\begin{aligned}&\delta^2u_0(\mathbf{x}) := \partial_{tt}u(\mathbf{x},0)\\ &= \frac{1}{\rho(\mathbf{x},0)}\left(f(\mathbf{x},0)-\mu\tilde{v}_0(\mathbf{x})-\lambda\tilde{u}_0(\mathbf{x})-\Delta(k(\mathbf{x},0)\Delta\tilde{u}_0(\mathbf{x}))+T^r(\mathbf{x},0)\Delta\tilde{u}_0(\mathbf{x})\right)\\ &\in \mathrm{L}^2(\Omega),\end{aligned}$$

if the following compatibility conditions are satisfied

$$\begin{cases} \tilde{u}_0 & \in \mathrm{H}^4(\Omega) \cap \mathrm{H}_0^1(\Omega), & \\ k(\mathbf{x},0)\Delta\tilde{u}_0(\mathbf{x}) & = 0, & \mathbf{x} \in \Gamma, \\ |\nabla k(\mathbf{x},0)|_{\mathrm{e}} & \leqslant \tilde{k}_4, & \mathbf{x} \in \Omega, \\ |\Delta k(\mathbf{x},0)| & \leqslant \tilde{k}_5, & \mathbf{x} \in \Omega. \end{cases}$$

Then, we are able to replace i by $i-1$ in (2.6) and to subtract it from (2.6). Next, we put $\varphi = \delta^2 u_i$ and we sum the result up for $i = 1, 2, \ldots, j$ with $1 \leqslant j \leqslant n$. Using the rule

$$\delta(a_i b_i) = b_i \delta a_i + a_{i-1}\delta b_i,$$

we obtain that

$$\begin{aligned} &\sum_{i=1}^{j} \left(\rho_i \delta^3 u_i + \delta\rho_i \delta^2 u_{i-1}, \delta^2 u_i\right)\tau + \sum_{i=1}^{j} \left(\mu_i \delta^2 u_i + \delta\mu_i \delta u_{i-1}, \delta^2 u_i\right)\tau \\ &\quad + \sum_{i=1}^{j} \left(\lambda_i \delta u_i + \delta\lambda_i u_{i-1}, \delta^2 u_i\right)\tau + \sum_{i=1}^{j} \left(k_i \Delta\delta u_i + \delta k_i \Delta u_{i-1}, \Delta\delta^2 u_i\right)\tau \\ &\qquad - \sum_{i=1}^{j} \left(T_i^r \Delta\delta u_i + \delta T_i^r \Delta u_{i-1}, \delta^2 u_i\right)\tau = \sum_{i=1}^{j} \left(\delta f_i, \delta^2 u_i\right)\tau. \end{aligned} \tag{2.12}$$

The crucial estimates in the LHS are (using (2.8))

$$\begin{aligned} &\sum_{i=1}^{j} \left(\rho_i \delta^3 u_i, \delta^2 u_i\right)\tau \\ &\geqslant \frac{\tilde{\rho}_0}{2}\left\|\delta^2 u_j\right\|^2 - \frac{\tilde{\rho}_1}{2}\left\|\delta^2 u_0\right\|^2 - \frac{1}{2}\sum_{i=1}^{j}\int_\Omega \delta\rho_i(\delta^2 u_{i-1})^2\tau + \frac{\tilde{\rho}_0}{2}\sum_{i=1}^{j}\left\|\delta^2 u_i - \delta^2 u_{i-1}\right\|^2, \end{aligned}$$

$$\begin{aligned} &\sum_{i=1}^{j} \left(k_i \Delta\delta u_i, \Delta\delta^2 u_i\right)\tau \\ &\geqslant \frac{\tilde{k}_0}{2}\left\|\Delta\delta u_j\right\|^2 - \frac{\tilde{k}_1}{2}\left\|\Delta\tilde{v}_0\right\|^2 - \frac{1}{2}\sum_{i=1}^{j}\int_\Omega \delta k_i(\Delta\delta u_{i-1})^2\tau + \frac{\tilde{k}_0}{2}\sum_{i=1}^{j}\left\|\Delta\delta u_i - \Delta\delta u_{i-1}\right\|^2. \end{aligned}$$

We have that

$$\left|\sum_{i=1}^{j}\int_{\Omega}\delta\rho_i(\delta^2 u_{i-1})^2\tau\right| \leqslant \tilde{\rho}_2\left\|\delta^2 u_0\right\|^2+\tilde{\rho}_2\sum_{i=1}^{j-1}\left\|\delta^2 u_i\right\|^2\tau,$$
$$\left|\sum_{i=1}^{j}\int_{\Omega}\delta k_i(\Delta\delta u_{i-1})^2\tau\right| \leqslant \tilde{k}_2\left\|\Delta\tilde{v}_0\right\|^2+\tilde{k}_2\sum_{i=1}^{j-1}\left\|\Delta\delta u_i\right\|^2\tau.$$

Using

$$\sum_{i=1}^{j} b_i\delta a_i = a_j b_j - a_0 b_0 - \sum_{i=1}^{j} a_{i-1}\delta b_i, \qquad u_{-1}=\tilde{u}_0-\tau\tilde{v}_0,$$

we obtain that

$$\sum_{i=1}^{j}\left(\delta k_i\Delta u_{i-1},\Delta\delta^2 u_i\right)\tau$$
$$=\left(\delta k_j\Delta u_{j-1},\Delta\delta u_j\right)\tau-\left(\delta k_0\Delta u_{-1},\Delta\tilde{v}_0\right)\tau-\sum_{i=1}^{j}\left(\delta\left(\delta k_i\Delta u_{i-1}\right),\Delta\delta u_{i-1}\right)\tau.$$

We separately estimate each term on the RHS of the previous equality as follows

$$\left|\left(\delta k_j\Delta u_{j-1},\Delta\delta u_j\right)\tau\right| \leqslant C_\varepsilon\tilde{k}_2^2 C_1+\varepsilon\left\|\Delta\delta u_j\right\|^2,$$
$$\left|\left(\delta k_0\Delta u_{-1},\Delta\tilde{v}_0\right)\tau\right| \leqslant \tilde{k}_2\left\|\Delta(\tilde{u}_0-\tau\tilde{v}_0)\right\|\left\|\Delta\tilde{v}_0\right\|,$$

and

$$\left|\sum_{i=1}^{j}\left(\delta^2 k_i\Delta u_{i-1}+\delta k_{i-1}\Delta\delta u_{i-1},\Delta\delta u_{i-1}\right)\tau\right|$$
$$\leqslant \frac{\tilde{k}_3^2}{2}\left\|\Delta\tilde{u}_0\right\|^2+\frac{\tilde{k}_3^2}{2}\sum_{i=1}^{j-1}\left\|\Delta u_i\right\|^2\tau+\left(\frac{1}{2}+\tilde{k}_2\right)\left\|\Delta\tilde{v}_0\right\|^2+\left(\frac{1}{2}+\tilde{k}_2\right)\sum_{i=1}^{j-1}\left\|\Delta\delta u_i\right\|^2\tau.$$

The other terms in (2.12) are handled as follows

$$\left|\sum_{i=1}^{j}\left(\delta\rho_i\delta^2 u_{i-1},\delta^2 u_i\right)\tau\right| \leqslant \frac{\tilde{\rho}_2^2}{2}\left\|\delta^2 u_0\right\|^2 + \left(\frac{\tilde{\rho}_2^2}{2}+\frac{1}{2}\right)\sum_{i=1}^{j}\left\|\delta^2 u_i\right\|^2\tau,$$

$$\left|\sum_{i=1}^{j}\left(\mu_i\delta^2 u_i+\delta\mu_i\delta u_{i-1},\delta^2 u_i\right)\tau\right| \leqslant \frac{\tilde{\mu}_2^2 C_1 T}{2} + \left(\frac{1}{2}+\tilde{\mu}_1\right)\sum_{i=1}^{j}\left\|\delta^2 u_i\right\|^2\tau,$$

$$\left|\sum_{i=1}^{j}\left(\lambda_i\delta u_i+\delta\lambda_i u_{i-1},\delta^2 u_i\right)\tau\right| \leqslant \frac{\tilde{\lambda}_1^2 C_1 T + \tilde{\lambda}_2^2 C_2^2 T}{2} + \sum_{i=1}^{j}\left\|\delta^2 u_i\right\|^2\tau,$$

$$\left|\sum_{i=1}^{j}\left(T_i^r\Delta\delta u_i,\delta^2 u_i\right)\tau\right| \leqslant \frac{\tilde{T}_1^2}{2}\sum_{i=1}^{j}\left\|\Delta\delta u_i\right\|^2\tau + \frac{1}{2}\sum_{i=1}^{j}\left\|\delta^2 u_i\right\|^2\tau,$$

$$\left|\sum_{i=1}^{j}\left(\delta T_i^r\Delta u_{i-1},\delta^2 u_i\right)\tau\right| \leqslant \frac{\tilde{T}_2^2 C_1 T}{2} + \frac{1}{2}\sum_{i=1}^{j}\left\|\delta^2 u_i\right\|^2\tau,$$

$$\left|\sum_{i=1}^{j}\left(\delta f_i,\delta^2 u_i\right)\tau\right| \leqslant \frac{1}{2}\sum_{i=1}^{j}\left\|\delta f_i\right\|^2\tau + \frac{1}{2}\sum_{i=1}^{j}\left\|\delta^2 u_i\right\|^2\tau.$$

Collecting all the results above, fixing ε sufficiently small and applying Grönwall's lemma give for sufficiently small τ that

$$\max_{0\leqslant i\leqslant n}\left\{\left\|\delta^2 u_i\right\|^2+\left\|\Delta\delta u_i\right\|^2\right\} + \sum_{i=1}^{n}\left(\left\|\delta^2 u_i-\delta^2 u_{i-1}\right\|^2+\left\|\Delta\delta u_i-\Delta\delta u_{i-1}\right\|^2\right)\leqslant C_3, \quad (2.13)$$

with $C_3 = C_3\left(T, \|f\|_{\mathrm{H}^1\left((0,T),\mathrm{L}^2(\Omega)\right)}, \|\tilde{u}_0\|_{\mathrm{H}^4(\Omega)\cap\mathrm{H}_0^1(\Omega)}, \|\tilde{v}_0\|_V, \left\|\delta^2 u_0\right\|\right)$.

Finally, the existence of a solution can be proven. Now, we further introduce the following piecewise linear in time functions $u_n : [0,T] \to \mathrm{L}^2(\Omega)$ and $v_n : [0,T] \to \mathrm{L}^2(\Omega)$

$$\begin{aligned}
&u_n(0) = \tilde{u}_0\\
&u_n(t) = u_{i-1} + (t-t_{i-1})\delta u_i, \quad t\in(t_{i-1},t_i], \quad 1\le i\le n;\\
&v_n(0) = \tilde{v}_0\\
&v_n(t) = \delta u_{i-1} + (t-t_{i-1})\delta^2 u_i \quad t\in(t_{i-1},t_i], \quad 1\le i\le n,
\end{aligned}$$

and the piecewise constant in time functions $\overline{u}_n : [0,T] \to \mathrm{L}^2(\Omega)$ and $\overline{v}_n : [0,T] \to \mathrm{L}^2(\Omega)$

$$\overline{u}_n(0) = \tilde{u}_0, \quad \overline{u}_n(t) = u_i, \quad t \in (t_{i-1}, t_i], \quad 1 \le i \le n;$$
$$\overline{v}_n(0) = \tilde{v}_0, \quad \overline{v}_n(t) = \delta u_i, \quad t \in (t_{i-1}, t_i], \quad 1 \le i \le n.$$

Using these so-called *Rothe's functions*, the variational formulation (2.6) can be rewritten, for all $\varphi \in V$ and a.a. $t \in [0,T]$, as

$$(\overline{\rho}_n(t)\partial_t v_n(t), \varphi) + (\overline{\mu}_n(t)\partial_t u_n(t), \varphi) + \left(\overline{\lambda}_n(t)\overline{u}_n(t), \varphi\right) + \left(\overline{k}_n(t)\Delta\overline{u}_n(t), \Delta\varphi\right) - \left(\overline{T^r}_n(t)\Delta\overline{u}_n(t), \varphi\right) = \left(\overline{f}_n(t), \varphi\right). \quad (2.14)$$

From $\tilde{u}_0 \in V$ and (2.10), it follows that

$$\max_{t\in[0,T]} \left\{ \|\overline{u}_n(t)\|_V^2 + \|\partial_t u_n(t)\|^2 \right\} \leqslant C.$$

Moreover, it holds that $V \hookrightarrow\hookrightarrow \mathrm{L}^2(\Omega)$. The conditions of [19, Lemma 1.3.13] are satisfied and, therefore, there exist a function $u \in \mathrm{C}\left([0,T], \mathrm{L}^2(\Omega)\right) \cap \mathrm{L}^\infty\left((0,T), V\right)$ and a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ (denoted by the same symbol yet again) such that

$$\begin{cases} u_n \to u & \text{in } \mathrm{C}\left([0,T], \mathrm{L}^2(\Omega)\right), \\ u_n(t) \rightharpoonup u(t) & \text{in } V, \text{ for all } t \in [0,T], \\ \overline{u}_n(t) \rightharpoonup u(t) & \text{in } V, \text{ for all } t \in [0,T], \\ \partial_t u_n \rightharpoonup \partial_t u & \text{in } \mathrm{L}^2\left((0,T), \mathrm{L}^2(\Omega)\right). \end{cases}$$

The a priori estimate (2.10) implies that

$$\int_0^T \|u_n(t) - \overline{u}_n(t)\|^2 \, \mathrm{d}t = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|(t - t_i)\delta u_i\|^2 \, \mathrm{d}t \leqslant \tau^2 C_1 T,$$

i.e. $\{u_n\}$ and $\{\overline{u}_n\}$ have the same limit in $\mathrm{L}^2\left((0,T), \mathrm{L}^2(\Omega)\right)$. Moreover, estimate (2.13) implies that

$$\partial_t u_n \rightharpoonup \partial_t u \quad \text{in } \mathrm{L}^2\left((0,T), V\right)$$

and

$$\partial_t v_n \rightharpoonup \partial_{tt} u \quad \text{in } \mathrm{L}^2\left((0,T), \mathrm{L}^2(\Omega)\right)$$

by the reflexivity of these spaces. Therefore,

$$u \in \mathrm{C}\left([0,T], V\right) \quad \text{and} \quad \partial_t u \in \mathrm{C}\left([0,T], \mathrm{L}^2(\Omega)\right),$$

cf. [33, Lemma 7.3]. It holds that $\overline{\rho}_n \to \rho$, $\overline{\mu}_n \to \mu$, $\overline{\lambda}_n \to \lambda$, $\overline{k}_n \to k$ and $\overline{T^r}_n \to T^r$ in $\mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right)$ as $n \to \infty$. Now, we integrate (2.14) in time and pass to the limit for $\tau \to 0$ using the preceding convergence results. Afterwards, we differentiate the result with respect to the time variable to arrive at (2.3). The convergence of Rothe's functions towards the weak solution has been shown for a subsequence. However, taking into account the uniqueness of a solution, the entire Rothe's sequence converges towards the solution. □

Remark 2.5. As already noted in Remark 2.2, the well-posedness of the forward problem can also be obtained under lower regularity assumptions on the data. When

$$
\begin{gathered}
0 < \tilde{\rho}_0 \leqslant \rho(\mathbf{x},t) \leqslant \tilde{\rho}_1, |\partial_t \rho(\mathbf{x},t)| \leqslant \tilde{\rho}_2, |\mu(\mathbf{x},t)| \leqslant \tilde{\mu}_1, |\lambda(\mathbf{x},t)| \leqslant \tilde{\lambda}_1, \\
0 < \tilde{k}_0 \leqslant k(\mathbf{x},t) \leqslant \tilde{k}_1, |\partial_t k(\mathbf{x},t)| \leqslant \tilde{k}_2, |T^r(\mathbf{x},t)| \leqslant \tilde{T}_1, \\
f \in \mathrm{L}^2\left((0,T),V^*\right), \tilde{u}_0 \in V, \tilde{v}_0 \in \mathrm{L}^2(\Omega),
\end{gathered}
$$

then a priori estimates (2.10) and (2.11) are satisfied. Moreover,

$$
\sum_{i=1}^{n} \left\| \delta^2 u_i \right\|_{V^*}^2 \tau \leqslant C,
$$

when $|\nabla \rho(\mathbf{x},t)|_{\mathrm{e}} \leqslant \tilde{\rho}_3$ and $|\Delta \rho(\mathbf{x},t)| \leqslant \tilde{\rho}_4$. These additional estimate is sufficient to do the convergence analysis and to obtain the existence of a weak solution satisfying

$$
u \in \mathrm{C}\left([0,T],\mathrm{L}^2(\Omega)\right) \cap \mathrm{L}^2\left((0,T),V\right)
$$

with

$$
\partial_t u \in \mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right)
$$

and

$$
\partial_{tt} u \in \mathrm{L}^2\left((0,T),V^*\right).
$$

When $|\partial_{tt}\rho(\mathbf{x},t)| \leqslant \tilde{\rho}_5$ (note that this is already satisfied if $\lambda = \partial_{tt}\rho$), then the solution is also unique. Remember that problem (2.3) has a unique weak solution when $\tilde{u}_0 = \tilde{v}_0 = f = 0$ results in $u = 0$. Now, we first integrate (2.3) in time over $t \in (0,\xi) \subset (0,T)$, then we put for a.a. $\xi \in (0,T)$ the testfunction equal to $u(\xi) \in V$ and afterwards we integrate again in time over $\xi \in (0,\eta) \subset (0,T)$ to obtain that

$$\int_0^\eta \left(\int_0^\xi \rho(t)\partial_{tt}u(t)\,\mathrm{d}t, u(\xi)\right)\mathrm{d}\xi + \int_0^\eta \left(\int_0^\xi \mu(t)\partial_t u(t)\,\mathrm{d}t, u(\xi)\right)\mathrm{d}\xi$$
$$+ \int_0^\eta \left(\int_0^\xi \lambda(t)u(t)\,\mathrm{d}t, u(\xi)\right)\mathrm{d}\xi + \int_0^\eta \left(\int_0^\xi k(t)\Delta u(t)\,\mathrm{d}t, \Delta u(\xi)\right)\mathrm{d}\xi$$
$$- \int_0^\eta \left(\int_0^\xi T_r(t)\Delta u(t)\,\mathrm{d}t, u(\xi)\right)\mathrm{d}\xi = 0.$$

A careful handling of all the terms (especially multiple times partial integration) is needed to get that

$$\|u(\eta)\|^2 + \left\|\int_0^\eta \Delta u(t)\,\mathrm{d}t\right\|^2 = 0.$$

3 Uniqueness of a solution to the ISP

In this section, the uniqueness of a solution to the ISP (determining $f(\mathbf{x})$ from ψ_T) is discussed. First, the governing partial differential equation (PDE) in (2.1) is divided by the known (given) function $h \in \mathrm{C}^1([0,T])$. In doing this, it is assumed that $h \neq 0$, i.e. $h(t) > 0$ (or $h(t) < 0$) for all $t \in [0,T]$. Let

$$v(\mathbf{x},t) = \frac{u(\mathbf{x},t)}{h(t)} \quad \text{and} \quad \alpha(t) = \frac{h'(t)}{h(t)},$$

then

$$\frac{\partial_t u}{h} = \partial_t v + v\alpha,$$
$$\frac{\partial_{tt} u}{h} = \partial_{tt} v + (2\partial_t v + v\alpha)\,\alpha + v\alpha'.$$

It follows that the PDE in (2.1) can be rewritten in terms of the unknown v as follows

$$\rho\partial_{tt}v + (\mu + 2\rho\alpha)\,\partial_t v + \left(\lambda + \mu\alpha + \rho\alpha^2 + \rho\alpha'\right)v + \Delta(k\Delta v) - T_r\Delta v = f(\mathbf{x}), \tag{3.1}$$

with

$$\begin{cases} v &= 0 & \text{on } \Sigma_T, \\ k\Delta v &= 0 & \text{on } \Sigma_T, \\ v(\mathbf{x},0) &= \frac{\tilde{u}_0(\mathbf{x})}{h(0)} & \mathbf{x} \in \Omega, \\ \partial_t v(\mathbf{x},0) &= \frac{\tilde{v}_0(\mathbf{x})}{h(0)} - \frac{\tilde{u}_0(\mathbf{x})}{h(0)}\alpha(0) & \mathbf{x} \in \Omega, \\ v(\mathbf{x},T) &= \frac{\xi_T(\mathbf{x})}{h(T)} & \mathbf{x} \in \Omega. \end{cases} \tag{3.2}$$

Thus, problem (1.1)-(1.2) is transformed in problem (3.1)-(3.2) wherein the right-hand side solely depends on the space variable. The variational formulation of the forward problem corresponding with (3.1) is given by: find $v(t) \in V$ with $\partial_t v(t) \in V$ and $\partial_{tt} v(t) \in \mathrm{L}^2(\Omega)$ such that

$$\begin{aligned} (\rho(t)\partial_{tt}v(t),\varphi) &+ ((\mu(t)+2\rho(t)\alpha(t))\,\partial_t v(t),\varphi) \\ &+ \big(\big(\lambda(t)+\mu(t)\alpha(t)+\rho(t)\alpha(t)^2+\rho(t)\alpha'(t)\big)\,v(t),\varphi\big) \\ &+ (k(t)\Delta v(t),\Delta\varphi) - (T_r(t)\Delta v(t),\varphi) = (f,\varphi)\,, \end{aligned} \tag{3.3}$$

for all $\varphi \in V$ and a.a. $t \in (0,T]$. Following Theorem 2.4, this formulation is well-posed for given $f \in \mathrm{L}^2(\Omega)$ if the conditions (2.4) are satisfied and

$$|\alpha(t)| \leqslant \alpha_1, \quad \left|\alpha'(t)\right| \leqslant \alpha_2, \quad \left|\alpha''(t)\right| \leqslant \alpha_3, \quad t \in [0,T], \tag{3.4}$$

which is satisfied if next to $h \in \mathrm{C}^1([0,T])$ also holds that

$$\left|h''(t)\right| \leqslant h_2, \quad \left|h'''(t)\right| \leqslant h_3, \quad t \in [0,T].$$

In the following theorem, the uniqueness of a solution to the ISP is investigated.

Theorem 3.1 (Uniqueness). *Let the conditions* (2.4) *and* (3.4) *be satisfied. Moreover, assume that T_r is solely time dependent with*

$$T_r'(t) \leqslant 0, \quad t \in [0,T],$$

and

$$\xi_T \in \mathrm{L}^2(\Omega), \quad \partial_t\rho \leqslant 0, \quad \mu \geqslant \mu_0 > 0, \quad \partial_t k \leqslant 0,$$

and

$$\alpha(t) \geqslant 0, \quad \partial_t\big(\lambda+\mu\alpha+\rho\alpha^2+\rho\alpha'\big) \leqslant 0.$$

Then, there exists at most one $f \in \mathrm{L}^2(\Omega)$ such that problem (2.1) *together with condition* (1.1) *holds.*

Proof. A classical variational approach is used to establish the uniqueness of a solution. The proof is by contradiction. Suppose that there are two solutions $\langle u_1, f_1\rangle$ and $\langle u_2, f_2\rangle$ to (1.1)-(1.2). Set $u = u_1 - u_2$, $v = v_1 - v_2$ and $f = f_1 - f_2$. Then $u(\mathbf{x},0) = 0$, $u(\mathbf{x},T) = 0$ and $\partial_t u(\mathbf{x},0) = 0$. Therefore, also $v(\mathbf{x},0) = 0$, $v(\mathbf{x},T) = 0$ and $\partial_t v(\mathbf{x},0) = 0$. First, we prove that $v = 0$ (thus $u = 0$) and then we show that $f = 0$. In doing this, we subtract the variational formulation (3.3) for the corresponding solution $\langle v_2, f_2\rangle$ from the variational formulation for

$\langle v_1, f_1\rangle$. We choose $\varphi = \partial_t v(t)$ as testfunction and integrate in time over $(0,T)$ to obtain that

$$\begin{aligned}\int_0^T (\rho(t)\partial_{tt}v(t), \partial_t v(t))\,\mathrm{d}t + \int_0^T ((\mu(t) + 2\rho(t)\alpha(t))\,\partial_t v(t), \partial_t v(t))\,\mathrm{d}t \\ + \int_0^T \left(\left(\lambda(t) + \mu(t)\alpha(t) + \rho(t)\alpha^2(t) + \rho(t)\alpha'(t)\right) v(t), \partial_t v(t)\right)\mathrm{d}t \\ + \int_0^T (k(t)\Delta v(t), \Delta\partial_t v(t))\,\mathrm{d}t - \int_0^T (T_r(t)\Delta v(t), \partial_t v(t))\,\mathrm{d}t \\ = \int_0^T (f, \partial_t v(t))\,\mathrm{d}t = (f, v(T) - v(0)) = 0.\end{aligned}$$

The first four terms in the LHS can be handled as follows:

$$\int_0^T (\rho(t)\partial_{tt}v(t), \partial_t v(t))\,\mathrm{d}t = \frac{1}{2}\left\|\sqrt{\rho(T)}\partial_t v(T)\right\|^2 - \frac{1}{2}\int_0^T \left(\partial_t\rho, (\partial_t v)^2\right)\mathrm{d}t \geqslant 0,$$

$$\int_0^T ((\mu + 2\rho\alpha)\,\partial_t v(t), \partial_t v(t))\,\mathrm{d}t \overset{\alpha\geqslant 0}{\geqslant} \mu_0 \int_0^T \|\partial_t v(t)\|^2\,\mathrm{d}t,$$

$$\begin{aligned}&\int_0^T \left(\left(\lambda + \mu\alpha + \rho\alpha^2 + \rho\alpha'\right) v(t), \partial_t v(t)\right)\mathrm{d}t \\ &\qquad = \frac{1}{2}\int_0^T \left(\lambda + \mu\alpha + \rho\alpha^2 + \rho\alpha', \partial_t\left(v^2\right)\right)\mathrm{d}t \\ &\qquad\qquad = -\frac{1}{2}\int_0^T \left(\partial_t(\lambda + \mu\alpha + \rho\alpha^2 + \rho\alpha'), v^2\right)\mathrm{d}t \geqslant 0,\end{aligned}$$

$$\begin{aligned}&\int_0^T (k(t)\Delta v(t), \Delta\partial_t v(t))\,\mathrm{d}t \\ &\qquad = -\frac{1}{2}\int_0^T \left(\partial_t k(t), (\Delta v)^2(t)\right)\mathrm{d}t + \frac{1}{2}\left\|\sqrt{k(T)}\Delta v(T)\right\|^2 \geqslant 0.\end{aligned}$$

The traction term is the most difficult term to handle. If T_r is solely time dependent, then (using $\partial_t v = 0$ on Σ_T and $\nabla v(\cdot, T) = \nabla v(\cdot, 0) = \mathbf{0}$ in Ω)

$$\begin{aligned}-\int_0^T T_r(t)\,(\Delta v(t), \partial_t v(t))\,\mathrm{d}t &= \frac{1}{2}\int_0^T T_r(t)\int_\Omega \partial_t\,|\nabla v(\mathbf{x}, t)|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &= -\frac{1}{2}\int_0^T T_r'(t)\,\|\nabla v(t)\|^2\,\mathrm{d}t \geqslant 0.\end{aligned}$$

From this, we get that

$$\mu_0 \int_0^T \|\partial_t v(t)\|^2 \,\mathrm{d}t \leqslant 0.$$

Therefore, $v = 0$ a.e. in Q_T. Substituting $v = 0$ in (3.3) gives

$$(f, \varphi) = 0, \qquad \forall \varphi \in V.$$

We conclude by [38, Proposition 18.2] that $f = 0$ in $\mathrm{L}^2(\Omega)$. □

Remark 3.2. Note that in terms of the original problem (1.2), the assumption $\mu \geqslant \mu_0 > 0$ means that $\mu + 2\partial_t \rho \geqslant \mu_0 > 0$, see the replacements (2.2). Moreover, $\partial_t\big(\lambda + \mu\alpha + \rho\alpha^2 + \rho\alpha'\big) \leqslant 0$ is equivalent with $\partial_t\big(\partial_{tt}\rho + \mu\alpha + \rho\alpha^2 + \rho\alpha'\big) \leqslant 0$.

Remark 3.3. The assumption $\alpha(t) \geqslant 0$ is equivalent with $h'(t) \geqslant 0$ if $h(t) > 0$ for all $t \in [0, T]$ and with $h'(t) \leqslant 0$ if $h(t) < 0$ for all $t \in [0, T]$.

In the following subsection, the 'changing sign' assumptions made in the proof to guarantee the uniqueness of a solution are examined.

3.1 Counterexamples uniqueness

Example 3.4 ($h(t)$ is changing sign). Consider the following one-dimensional ISP for $x, t \in (0, \pi)$ with zero final time data

$$\begin{cases} u_{tt} + u_t + u_{xxxx} - u_{xx} &= h(t)f(x) & (x,t) \in Q_T, \\ u(0,t) = u(\pi,t) &= 0 & t \in (0,\pi), \\ u_{xx}(0,t) = u_{xx}(\pi,t) &= 0 & t \in (0,\pi), \\ u(x,0) &= 0 & x \in (0,\pi), \\ u_t(x,0) &= 0 & x \in (0,\pi), \end{cases} \tag{3.5}$$

where $h(t) = (t+1)\sin(t) + (t+2)\cos(t)$ in $[0, \pi]$. Besides the trivial solution $(u, f) = (0, 0)$ to (3.5), also the following non-trivial one exists

$$u(x,t) = \sin(x)\sin(t)t,$$
$$f(x) = \sin(x).$$

Example 3.5 ($\alpha(t)$ is changing sign). Consider the following one-dimensional ISP for $x \in (0, \pi)$, $t \in (0, 4)$ with zero final time data

$$\begin{cases} u_{tt} + u_t + u_{xxxx} - u_{xx} & = h(t)f(x) \quad (x,t) \in Q_T, \\ u(0,t) = u(\pi,t) & = 0 \quad t \in (0,4), \\ u_{xx}(0,t) = u_{xx}(\pi,t) & = 0 \quad t \in (0,4), \\ u(x,0) & = 0 \quad x \in (0,\pi), \\ u_t(x,0) & = 0 \quad x \in (0,\pi), \end{cases} \tag{3.6}$$

where $h(t) = A + t\cos(t) > 0$ in $[0, 4]$ with

$$A := \frac{2\exp(-2)\sin\left(2\sqrt{7}\right)\sqrt{7} + 14\exp(-2)\cos\left(2\sqrt{7}\right) + 14\,\cos(4) + 21\,\sin(4)}{-7 + \exp(-2)\sin\left(2\sqrt{7}\right)\sqrt{7} + 7\exp(-2)\cos\left(2\sqrt{7}\right)}.$$

Besides the trivial solution $(u, f) = (0, 0)$ to (3.6), also the following non-trivial one exists

$$f(x) = \sin(x),$$
$$u(x,t) = f(x)\Phi(t),$$

with

$$\Phi(t) = -\frac{1}{14}\exp\left(-\frac{t}{2}\right)\sin\left(\frac{\sqrt{7}}{2}t\right)(-2 + A)\sqrt{7} + \exp\left(-\frac{t}{2}\right)\cos\left(\frac{\sqrt{7}}{2}t\right)\left(1 - \frac{A}{2}\right) + \frac{1}{2}\,(t-2)\cos(t) + \frac{1}{2}\,(t-1)\sin(t) + \frac{A}{2}.$$

Example 3.6 ($T_r'(t)$ is changing sign). Consider the following one-dimensional ISP for $x, t \in (0, \pi)$ with zero final time data

$$\begin{cases} u_{tt} + u_t + u_{xxxx} - \dfrac{50}{\sin(t)}u_{xx} & = h(t)f(x) \quad (x,t) \in Q_T, \\ u(0,t) = u(\pi,t) & = 0 \quad t \in (0,\pi), \\ u_{xx}(0,t) = u_{xx}(\pi,t) & = 0 \quad t \in (0,\pi), \\ u(x,0) & = 0 \quad x \in (0,\pi), \\ u_t(x,0) & = 0 \quad x \in (0,\pi), \end{cases} \tag{3.7}$$

where $h(t) = 50t + (t+2)\cos(t) + \sin(t) > 0$, $\alpha(t) \geqslant 0$ and $\left(\alpha + \alpha^2 + \alpha'\right)' \leqslant 0$ in $[0, \pi]$. Besides the trivial solution $(u, f) = (0, 0)$ to (3.7), also the following non-trivial one exists

$$\begin{aligned} u(x,t) &= \sin(x)\sin(t)t, \\ f(x) &= \sin(x). \end{aligned}$$

These examples show that the 'changing sign' conditions in Theorem 3.1 cannot be removed without violating uniqueness of the ISP (1.1)-(1.2).

4 Reconstruction of the spatial load distribution

In this section, an algorithm for finding the spatial load distribution f is described. It is assumed that the assumptions (2.4), (3.4) and the assumptions of Theorem 3.1 are satisfied. The solution $\langle u, f\rangle$ to problem (1.1)-(1.2) is given by $\langle hv_* + hv_{**}, f\rangle$ where v_{**} is the unique solution (see Theorem 2.4) to

$$\begin{aligned} \rho\partial_{tt}v_{**} + (\mu + 2\rho\alpha)\,\partial_t v_{**} + \left(\lambda + \mu\alpha + \rho\alpha^2 + \rho\alpha'\right) v_{**} \\ + \Delta(k\Delta v_{**}) - T_r\Delta v_{**} = 0, \end{aligned} \tag{4.1}$$

with

$$\begin{cases} v_{**} &= 0 & \text{on } \Sigma_T, \\ k\Delta v_{**} &= 0 & \text{on } \Sigma_T, \\ v_{**}(\mathbf{x}, 0) &= \frac{\tilde{u}_0(\mathbf{x})}{h(0)} & \mathbf{x} \in \Omega, \\ \partial_t v_{**}(\mathbf{x}, 0) &= \frac{\tilde{v}_0(\mathbf{x})}{h(0)} - \frac{\tilde{u}_0(\mathbf{x})}{h(0)}\alpha(0) & \mathbf{x} \in \Omega, \end{cases} \tag{4.2}$$

and $\langle v_*, f\rangle$ is the unique solution (see Theorem 3.1) to

$$\begin{aligned} \rho\partial_{tt}v_* + (\mu + 2\rho\alpha)\,\partial_t v_* + \left(\lambda + \mu\alpha + \rho\alpha^2 + \rho\alpha'\right) v_* \\ + \Delta(k\Delta v_*) - T_r\Delta v_* = f(\mathbf{x}), \end{aligned} \tag{4.3}$$

with

$$\begin{cases} v_* &= 0 & \text{on } \Sigma_T, \\ k\Delta v_* &= 0 & \text{on } \Sigma_T, \\ v_*(\mathbf{x}, 0) &= 0 & \mathbf{x} \in \Omega, \\ \partial_t v_*(\mathbf{x}, 0) &= 0 & \mathbf{x} \in \Omega, \end{cases} \tag{4.4}$$

and

$$v_*(\cdot, T) = \frac{\xi_T(\cdot)}{h(T)} - v_{**}(\cdot, T) =: \widetilde{\xi}_T(\cdot). \tag{4.5}$$

Therefore, solving problem (1.1)-(1.2) is equivalent with solving problem (4.3)-(4.4)-(4.5).

4.1 Algorithm for finding the source term

The algorithm for solving problem (1.1)-(1.2) is based on the Landweber-Fridman iterative regularization method [7, 22]. Let $v \in \mathrm{C}([0,T], V)$ the unique solution to (4.3)-(4.4) for given f. The corresponding input-output operator $M_T \in \mathcal{L}\left(\mathrm{L}^2(\Omega), \mathrm{L}^2(\Omega)\right)$ is defined by

$$M_T f = v(\cdot, T).$$

The boundedness of M_T follows from (2.5). Finding a solution to the ISP is then equivalent to solving the following operator equation

$$M_T f = \widetilde{\xi}_T,$$

or equivalent to solving the fixed point equation

$$f = f + \omega M_T \left(\widetilde{\xi}_T - M_T f\right), \qquad \omega > 0,$$

due to the linearity of the operator M_T. The parameter ω is called a relaxation parameter. The method of successive approximations can be applied to this latter equation as follows ($k \in \mathbb{N}$)

$$f_k := f_{k-1} - \omega M_T \left(M_T f_{k-1} - \widetilde{\xi}_T\right),$$

with an initial guess $f_0 \in \mathrm{L}^2(\Omega)$. This all gives rise to the procedure presented below for the reconstruction of the solution u and the source term f in problem (1.1)-(1.2). The procedure is similar to the one presented in [6,17,18,36] and reads as follows:

(i) Solve problem (4.1)-(4.2) and determine the transformed final overdetermination $\widetilde{\xi}_T$, cf. (4.5). Denote the solution by v_{**};

(ii) Choose an initial guess $f_0 \in \mathrm{L}^2(\Omega)$. Let v_0 be the solution to (4.3)-(4.4) with $f = f_0$;

(iii) Assume that f_{k-1} and v_{k-1} have been constructed. Let w_{k-1} solve (4.3)-(4.4) with $f(\mathbf{x}) = v_{k-1}(\mathbf{x}, T) - \widetilde{\xi}_T(\mathbf{x})$;

(iv) Define

$$f_k(\mathbf{x}) = f_{k-1}(\mathbf{x}) - \omega w_{k-1}(\mathbf{x}, T), \quad \text{a.a. } \mathbf{x} \in \Omega,$$

where $\omega > 0$, and let v_k solve (4.3)-(4.4) with $f = f_k$;

(v) Repeat steps (iv) and (v) until a desired level of accuracy is achieved, see Section 4.2;

(vi) Suppose that the algorithm is stopped after $\tilde{k}$ iterations. Denote the corresponding solution by $\langle v_{\tilde{k}}, f_{\tilde{k}} \rangle$. Then, the approximating solution to the original problem (1.1)-(1.2) is given by $\langle h(v_{**} + v_{\tilde{k}}), f_{\tilde{k}} \rangle$.

An application of [34, Theorem 3] gives the convergence of the proposed algorithm.

Theorem 4.1. *Assume that the assumptions of Theorem* 3.1 *are satisfied and suppose that the relaxation parameter* ω *satisfies* $0 < \omega < \|M_T\|^2_{\mathcal{L}\left(\mathrm{L}^2(\Omega), \mathrm{L}^2(\Omega)\right)}$. *Denote by* (u, f) *the unique solution to the original inverse problem* (1.1)-(1.2). *Let* (u_k, f_k) *the kth approximation in the iterative algorithm of Subsection* 4.1. *Then*

$$\lim_{k \to \infty} \left\{ \|f_k - f\| + \|u_k - u\|_{\mathrm{C}\left([0,T], \mathrm{L}^2(\Omega)\right)} \right\} = 0,$$

for every $f_0 \in \mathrm{L}^2(\Omega)$.

4.2 Stopping criteria

In order to simulate errors present in practical experiments, it is considered that there is some error in the additional measurement (1.1), i.e.

$$\|\xi_T - \xi_T^e\| \leqslant e,$$

with $e > 0$. This implies that also $\widetilde{\xi}_T$ is perturbed, see (4.5). The perturbed function is denoted by $\widetilde{\xi}_T^e$. The functions f_k^e and v_k^e are obtained by using the algorithm with no noise on the initial data. In this contribution, the discrepancy principle by Morozov [31] is used to obtain a stopping criterion for the algorithm. This principle suggests to finish the iterations at the smallest index $k = k(e, \omega)$ for which

$$E_k := \left\| v_k^e(\cdot, T) - \widetilde{\xi}_T^e \right\| \leqslant \tau_0 e. \tag{4.6}$$

for some $\tau_0 > 1$ (typically between 1 and 1.2).

The following section discusses the results of numerical experiments. These experiments indicate that the proposed scheme can be successfully applied on the inverse problem studied in this contribution.

5 Numerical experiments

5.1 Setting

In the experiments, a simply supported homogeneous steel beam (ASTM-A36) is considered of length l, width b and height h such that $l \gg b, h$. It is assumed that $b = h = 0.01$ m such that the vibrations of the beam can be described in a one-dimensional setting. The governing equations in (1.1)-(1.2) can be rearranged for constant coefficients in the following nondimensionalized form (it is assumed that $h \equiv 1$ in the experiments)

$$\begin{cases} \ddot{U} + \dot{U} + U^{(iv)} - \sqrt{\frac{\rho}{k}}\frac{T_r}{\mu}U'' & = F, & \text{in } \left(0, \frac{l}{\ell}\right) \times \left(0, \frac{cT}{\ell}\right], \\ U(0,\tilde{t}) = U\left(\frac{l}{\ell},\tilde{t}\right) & = 0, & \tilde{t} \in \left(0, \frac{cT}{\ell}\right], \\ U''(0,\tilde{t}) = U''\left(\frac{l}{\ell},\tilde{t}\right) & = 0, & \tilde{t} \in \left(0, \frac{cT}{\ell}\right], \\ U(\tilde{x},0) & = U_0(\tilde{x}), & \tilde{x} \in (0, \frac{l}{\ell}), \\ \dot{U}(\tilde{x},0) & = V_0(\tilde{x}), & \tilde{x} \in (0, \frac{l}{\ell}), \\ U(\tilde{x},T) & = \Upsilon(\tilde{x}) & \tilde{x} \in (0, \frac{l}{\ell}), \end{cases} \tag{5.1}$$

by using the following nondimensional variables:

$$\tilde{x} = \frac{1}{\ell}x, \quad \tilde{t} = \frac{c}{\ell}t, \quad U(\tilde{x},\tilde{t}) = \frac{1}{\tilde{u}}u(x,t), \quad F(\tilde{x},\tilde{t}) = \frac{1}{\tilde{f}}f(x,t),$$
$$\Upsilon(\tilde{x}) = \frac{\xi_T(\ell\tilde{x})}{\tilde{u}}, \quad U_0(\tilde{x}) = \frac{1}{\tilde{u}}u_0(\ell\tilde{x}), \quad V_0(\tilde{x}) = \frac{\ell}{c\tilde{u}}v_0(\ell\tilde{x}),$$

where $c = \frac{k^{1/4}\mu^{1/2}}{\rho^{3/4}}$ (SI unit: $\mathrm{m/s}$), $\ell = \frac{\rho c}{\mu}$ (SI unit: m), $\tilde{u} = \ell$ and $\tilde{f} = \frac{\rho c^2}{\ell}$ (SI unit: $\mathrm{kg/s^2}$). Now, F is rewritten as

$$F(\tilde{x},\tilde{t}) = \mathcal{F}(\tilde{x}) + \mathcal{G}(\tilde{x},\tilde{t}),$$

with $\mathcal{F}(\tilde{x})$ unknown. The dimensionless variables are used in the following experiments, however, in the sequel, the tilde is dropped in order to simplify the notations employed.

In the numerical experiments, the exact displacement is prescribed as follows

$$U(x,t) = 0.01(1+t)^2x^2(x-4)^4, \quad x \in [0,4], \quad t \in [0,1].$$

The moment of inertia (SI-unit: m^4) of the rectangular solid cross section of the beam is given by $I = \frac{bh^3}{12}$. The elasticity modulus E is 200×10^9 $\mathrm{N/m^2}$ such that the flexural rigidity k approximately equals $167\,\mathrm{Nm}^2$. The other material coefficients are given by: $\rho = 8\,{}^{\mathrm{kg}}\!/\!{}_{\mathrm{m}}$, $T_r = 6 \times 10^3\,\mathrm{N}$ and $\mu = 6 \times 10^2\,{}^{\mathrm{kg}}\!/\!{}_{\mathrm{s}\times\mathrm{m}}$. This setting corresponds with a beam of length $l \approx 1\,\mathrm{m}$ and with final time $T \approx 0.0133\,\mathrm{s}$. The exact sources used in the experiments are given by

$$\begin{aligned}
\mathcal{F}^1(x) &= 0.25 \times x(x-4),\\
\mathcal{F}^2(x) &= 0.1 \times x(x-4)^2,\\
\mathcal{F}^3(x) &= \sin\left(\frac{\pi x}{2}\right),\\
\mathcal{F}^4(x) &= \exp\left(-2(x-2)^2\right),\\
\mathcal{F}^5(x) &= \begin{cases} 0 & 0 \leqslant x \leqslant 1 \\ x-1 & 1 \leqslant x \leqslant 2 \\ -x+3 & 2 \leqslant x \leqslant 3 \\ 0 & 3 \leqslant x \leqslant 4 \end{cases}
\end{aligned}$$

and

$$\mathcal{F}^6(x) = \begin{cases} 0 & 0 \leqslant x < 1 \\ 1 & 1 \leqslant x \leqslant 3 \\ 0 & 3 < x \leqslant 4. \end{cases}$$

This implies that a wide range of sources is considered, namely symmetric ($\mathcal{F}^{1,4}$) and non-symmetric ($\mathcal{F}^{2,3}$) continuous piecewise smooth load sources and symmetric load fields with discontinuities ($\mathcal{F}^{5,6}$).

The solution to the inverse source problem is found by applying the algorithm proposed in Subsection 4.1. The forward problems in this procedure are discretized in time according to the backward Euler method. It is assumed that the time step for the equidistant time partitioning is chosen to be 0.001. To solve the problem (2.1) using Lagrange finite element basis functions, the equation is split into two second-order equations. Then, at each time step, the resulting elliptic mixed problems are solved numerically by the finite element method using second order (P2–FEM) Lagrange polynomials for the space discretization (the number of finite elements is taken to be equal to 200). The finite element library DOLFIN [26, 27] from the FEniCS project [1, 25] is used to solve the forward problems.

In the following subsection, the exact value for the source is compared with its corresponding numerically retrieved value $f_{\tilde{k}}$, obtained when the algorithm is stopped after a finite number of $\tilde{k}$ iterations. The value for the relaxation parameter ω equals 10 in the experiments. Next to the Morozov stopping criterion (with $\tau_0 = 1.1$ in (4.6)), the algorithm has also the following stopping criteria:

- The maximum number of iterations is set equal to 10000;

- $$\frac{|E_k - E_{k-1}|}{E_k} < 1 \times 10^{-4}.$$

The initial guess f_0 is chosen to be equal to 0. Moreover, a randomly generated uncorrelated noise with magnitude $\tilde{e}$ is added to the additional condition in order to simulate the errors present in real measurements.

5.2 Results

The numerical results for different noise levels $\tilde{e}$ are depicted in Figure 1 and 3. The accurate approximations of the sources show the stability of the numerical procedure. In Figure 2, the results for the experiments related to $\mathcal{F}^4$, $\mathcal{F}^5$ and $\mathcal{F}^6$ are presented when exact data is used such that the iterative process is continued indefinitely. From these figures, it can be seen that the numerical solution converges to its corresponding exact solution as k increases. However, this process is time consuming as each numerical experiment requires about 50 (25) hours for 10000 (5000) iterations (Intel® Core$^{\text{TM}}$ i7-4810MQ Processor). The CPU time (in minutes) and the stopping iteration index $\tilde{k}$ for the experiments with noise can be found in Table 1 and Table 2. As can be noticed from Table 2, a drawback of this method is that the process is time consuming for $\mathcal{F}^i$, $i = 4, 5, 6$, which gives a limitation of this method.

The obtained results are in accordance with the numerical experiments performed for the heat conduction equation in [6, 18] and for thermoelasticity in [35, 36]:

- The attainability of the stopping criteria becomes faster if $\tilde{e}$ increases (see Table 1 and 2);
- Also for larger noise level for $\mathcal{F}^1$ and $\mathcal{F}^3$, an accurate approximation for the sources is obtained (see Figure 1);
- The algorithm is sensitive to the amount of noise in the experiment (see Figure 3).

6 Conclusions and further research

In this paper, an ISP associated with the dynamic vibration of a simply supported beam and rectangular plate was considered (nevertheless also other boundary conditions can be considered). More precisely, the theoretical and numerical determination of a spatial load distribution was studied from the knowledge of a supplementary measurement of the deflection at the final time. First, the well-posedness

Table 1. The stopping iteration number $\tilde{k}$, the CPU time (mins) and the value of the relaxation parameter at $\tilde{k}$ for $\mathcal{F}^i, i = 1, 2, 3$, with $\tilde{e} = 0.05\%, 0.1\%$ and $\tilde{e} = 1\%$.

$\tilde{e} = 0.05\%$	$\tilde{k}$	time	$\tilde{e} = 0.1\%$	$\tilde{k}$	time	$\tilde{e} = 1\%$	$\tilde{k}$	time
$\mathcal{F}^1$	135	54	$\mathcal{F}^1$	2	1	$\mathcal{F}^1$	2	1
$\mathcal{F}^2$	362	163	$\mathcal{F}^2$	20	5	$\mathcal{F}^2$	9	3
$\mathcal{F}^3$	29	12	$\mathcal{F}^3$	25	10	$\mathcal{F}^3$	15	4

Table 2. The stopping iteration number $\tilde{k}$, the CPU time (mins) and the value of the relaxation parameter at $\tilde{k}$ for $\mathcal{F}^i, i = 4, 5, 6$, with $\tilde{e} = 0.01\%$ and $\tilde{e} = 0.1\%$.

$\tilde{e} = 0.01\%$	$\tilde{k}$	time	$\tilde{e} = 0.1\%$	$\tilde{k}$	time
$\mathcal{F}^4$	5714	1888	$\mathcal{F}^4$	1642	421
$\mathcal{F}^5$	6693	2140	$\mathcal{F}^5$	1962	804
$\mathcal{F}^6$	7458	2321	$\mathcal{F}^6$	2247	918

of the corresponding forward problem is investigated. Afterwards, the uniqueness of a solution to the inverse problem is proved. The conditions under which uniqueness holds were examined by constructing counterexamples.

An iterative algorithm of Landweber-Fridman type was proposed for the recovery of the unknown load source and a stopping criterion was also given. The one-dimensional numerical experiments carried out herein were implemented using the FEM and validated the stability of the proposed iterative procedure. In these experiments, it was showed that the procedure proposed herein is applicable to the reconstruction of symmetric and non-symmetric continuous piecewise smooth load sources and symmetric load fields with discontinuities. The main disadvantage was that the process is time consuming for the symmetric load fields with discontinuities and the exponential load field.

A direction for future research concerns the comparison of the results with faster iterative methods such as the conjugate gradient method.

Bibliography

[1] M. S. Alnæs, J. Blechta, J. Hake, A. Johansson, B. Kehlet, A. Logg, C. Richardson, J. Ring, M. E. Rognes and G. N. Wells, The FEniCS Project Version 1.5, *Archive of Numerical Software* **3** (2015).

[2] D. Bainov and P. Simeonov, *Integral inequalities and applications*, Mathematics and Its Applications. East European Series. 57. Dordrecht, Kluwer Academic Publishers, 1992.

[3] C. Cavaterra and M. Grasselli, AN INVERSE PROBLEM FOR THE LINEAR VISCOELASTIC KIRCHHOFF PLATE, *Quarterly of Applied Mathematics* **53** (1995), 9–33.

[4] J.-D. Chang and B.-Z. Guo, Identification of variable spacial coefficients for a beam equation from boundary measurements, *Automatica* **43** (2007), 732–737.

[5] F. Colombo and D. Guidetti, An inverse problem for the beam equation with memory with nonhomogeneous boundary conditions, *Inverse Problems* **24** (2008), 065015.

[6] S. D'haeyer, B. T. Johansson and M. Slodička, Reconstruction of a spacewise-dependent heat source in a time-dependent heat diffusion process, *IMA Journal of Applied Mathematics* **79** (2014), 33–53.

[7] V. M. Fridman, Method of successive approximations for a Fredholm integral equation of the 1st kind, *Uspekhi Mat. Nauk* **11** (1956), 233–234 (Russian).

[8] G.M.L. Gladwell, *Inverse problems in vibration*, Mechanics: Dynamical Systems, Springer Netherlands, 2012.

[9] M. Grimmonprez and M. Slodička, Reconstruction of an unknown source parameter in a semilinear parabolic problem, *J. Comput. Appl. Math.* **289** (2015), 331–345.

[10] Z.-j. Gu and Y.-j. Tan, Fundamental solution method for inverse source problem of plate equation, *Applied Mathematics and Mechanics* **33** (2012), 1513–1532.

[11] B.Z. Guo, On the Boundary Control of a Hybrid System with Variable Coefficients, *Journal of Optimization Theory and Applications* **114** (2002), 373–395.

[12] A. Hasanov, Identification of an unknown source term in a vibrating cantilevered beam from final overdetermination, *Inverse Problems* **25** (2009), 115015.

[13] A. Hasanov and O. Baysal, Identification of an unknown spatial load distribution in a vibrating cantilevered beam from final overdetermination., *J. Inverse Ill-Posed Probl.* **23** (2015), 85–102 (English).

[14] A. Hasanov and O. Baysal, Identification of unknown temporal and spatial load distributions in a vibrating Euler-Bernoulli beam from Dirichlet boundary measured data, *Automatica* **71** (2016), 106–117.

[15] A. Hasanov and A. Kawano, Identification of unknown spatial load distributions in a vibrating Euler-Bernoulli beam from limited measured data, *Inverse Problems* **32** (2016), 055004.

[16] M. Ikehata, An Inverse Problem for the Plate in the Love-Kirchhoff Theory, *SIAM Journal on Applied Mathematics* **53** (1993), 942–970.

[17] B. T. Johansson and D. Lesnic, A procedure for determining a spacewise dependent heat source and the initial temperature, *Applicable Analysis* **87** (2008), 265–276.

[18] T. Johansson and D. Lesnic, Determination of a Spacewise Dependent Heat Source, *J. Comput. Appl. Math.* **209** (2007), 66–80.

[19] J. Kačur, *Method of Rothe in evolution equations*, Teubner Texte zur Mathematik 80, Teubner, Leipzig, 1985.

[20] A. Kawano, Uniqueness in the identification of asynchronous sources and damage in vibrating beams, *Inverse Problems* **30** (2014), 065008.

[21] M. Krstic, B.-Z. Guo, A. Balogh and A Smyshlyaev, Control of a Tip-Force Destabilized Shear Beam by Observer-Based Boundary Feedback, *SIAM Journal on Control and Optimization* **47** (2008), 553–574.

[22] L. Landweber, An Iteration Formula for Fredholm Integral Equations of the First Kind, *American Journal of Mathematics* **73** (1951), 615–624 (English).

[23] D. Lesnic, L. Elliott and D. B. Ingham, Analysis of coefficient identification problems associated to the inverse Euler-Bernoulli beam theory, *IMA Journal of Applied Mathematics* **62** (1999), 101–116.

[24] C.-S. Liu, A Lie-group adaptive differential quadrature method to identify an unknown force in an Euler-Bernoulli beam equation, *Acta Mechanica* **223** (2012), 2207–2223.

[25] A. Logg, K-A. Mardal, G. N. Wells et al., *Automated Solution of Differential Equations by the Finite Element Method*, Springer, Berlin, Heidelberg, 2012.

[26] A. Logg and G. N. Wells, DOLFIN: Automated Finite Element Computing, *ACM Trans. Math. Software* **37** (2010), 28.

[27] A. Logg, G. N. Wells and J. Hake, *DOLFIN: a C++/Python Finite Element Library*, ch. 10, Springer, Berlin, Heidelberg, 2012.

[28] A. Maciag and A. Pawinska, Solution of the direct and inverse problems for beam, *Computational and Applied Mathematics* **35** (2016), 187–201.

[29] T. T. Marinov and R. S. Marinova, An inverse problem for estimation of bending stiffness in Kirchhoff-Love plates, *Computers & Mathematics with Applications* **65** (2013), 512 – 519, Efficient Numerical Methods for Scientific Applications.

[30] T. T. Marinov and A. S. Vatsala, Inverse problem for coefficient identification in the Euler-Bernoulli equation, *Computers & Mathematics with Applications* **56** (2008), 400 –410.

[31] V. A. Morozov, On the solution of functional equations by the method of regularization, *Soviet Math. Dokl.* **7** (1966), 414–417.

[32] S. Nicaise and O. Zair, Determination of point sources in vibrating beams by boundary measurements: Identifiability, stability, and reconstruction results., *Electronic Journal of Differential Equations (EJDE) [electronic only]* **2004** (2004), Paper No. 20, 17 p., electronic only–Paper No. 20, 17 p., electronic only (eng).

[33] T. Roubíček, *Nonlinear partial differential equations with applications*, ISNM 153, Birkhäuser Verlag, Basel, 2005.

[34] M. Slodička and V. Melicher, An iterative algorithm for a Cauchy problem in eddy-current modelling., *Appl. Math. Comput.* **217** (2010), 237–246 (English).

[35] K. Van Bockstal and L. Marin, Recovery of a space-dependent vector source in anisotropic thermoelastic systems, *Computer Methods in Applied Mechanics and Engineering* **321** (2017), 269–293.

[36] K. Van Bockstal and M. Slodička, Recovery of a space-dependent vector source in thermoelastic systems, *Inverse Problems Sci. Eng.* **23** (2015), 956–968.

[37] G. Yuan and M. Yamamoto, Lipschitz stability in inverse problems for a Kirchhoff plate equation, **53** (2007).

[38] E. Zeidler, *Nonlinear Functional Analysis and its Applications II/A: Linear Monotone Operators*, Springer, 1990.

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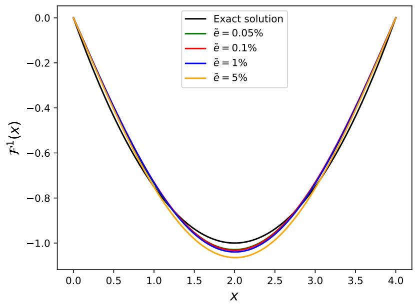

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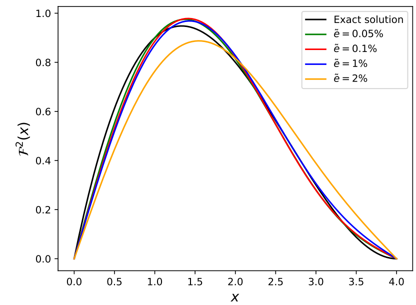

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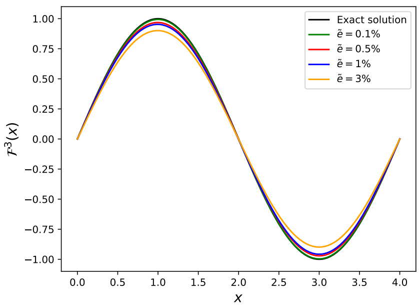

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Figure 1. The exact sources $\mathcal{F}^1$, $\mathcal{F}^2$ and $\mathcal{F}^3$ and its corresponding numerical solution, retrieved using various levels of noise in the additional measurement (a,b,c).

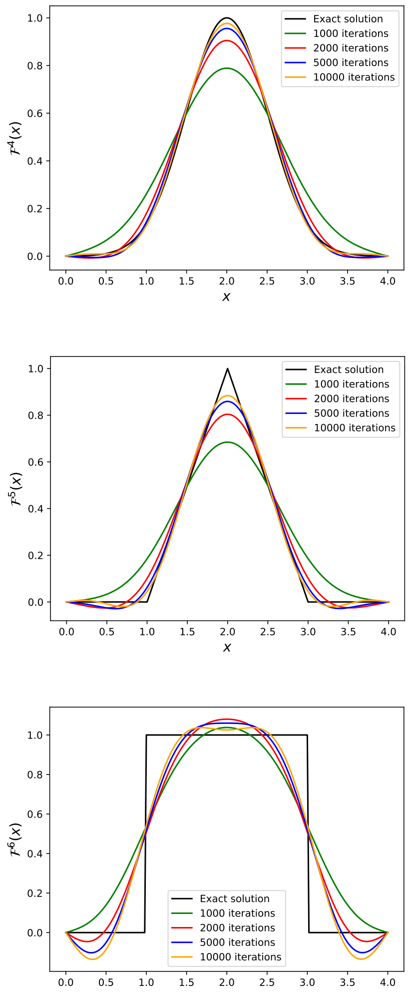

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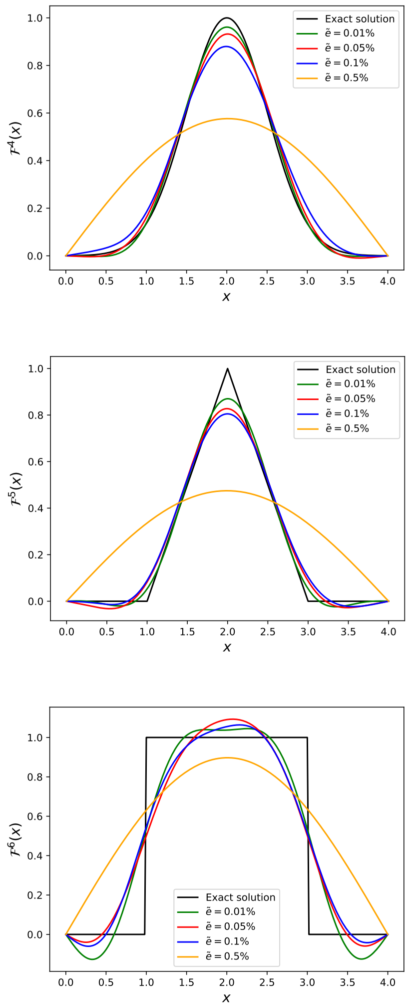
Figure 2. The exact sources $\mathcal{F}^4$, $\mathcal{F}^5$ and $\mathcal{F}^6$ and its corresponding numerical solution, retrieved without noise on the measurement (a,b,c).



(a)

(b)

(c)

Figure 3. The exact sources $\mathcal{F}^4$, $\mathcal{F}^5$ and $\mathcal{F}^6$ and its corresponding numerical solution, retrieved using various levels of noise in the additional measurement (a,b,c).