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Truncated-exponential-based Frobenius–Euler polynomials

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Abstract

In this paper, we first introduce a new family of polynomials, which are called the truncated-exponential based Frobenius–Euler polynomials, based upon an exponential generating function. By making use of this exponential generating function, we obtain their several new properties and explicit summation formulas. Finally, we consider the truncated-exponential based Apostol-type Frobenius–Euler polynomials and their quasi-monomial properties.

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1 Introduction and preliminaries

Various families of polynomials play a key role in applied mathematics due to the fact that they can be described in many different ways, for example, by orthogonality conditions, by generating functions, as solutions to differential equations, by integral transforms, by recurrence relations, by operational formulas, and so on. In light of their many important properties, their extensions and generalizations with applications are also considered by the researchers in mathematical and physical sciences. The resulting formulas are very important and potentially useful, because they include expansions for many transcendent expressions of mathematical physics in series of the classical orthogonal polynomials. The developments bear heavily upon the work of many researchers who have earlier studied the special polynomials with applications to p-adic analysis, q-analysis, umbral analysis, and so on (see, for example, the recent work [3–22] and [23]).

Frobenius [10] (see also [4]) studied the polynomials $F_n(x|u)$ in great detail by means of the following exponential generating function:

$$\sum_{n=0}^{\infty} F_n(x|u)\frac{t^n}{n!} = \frac{1-u}{e^t-u}e^{xt} \quad \bigl(u \in \mathbb{C} \setminus \{1\}\bigr). \tag{1.1}$$

Several identities and characterizations of the Frobenius polynomials $F_n(x|u)$ can be found in the works by Kim *et al.* [14–17]. In the case when $u = -1$ in (1.1), it reduces to the following relationship with the Euler polynomials $E_n(x)$:

$$F_n(x|-1) = E_n(x),$$

which are given in (1.4) below. Owing to their important properties, and in the honor of Frobenius, the polynomials $F_n(x|u)$ are called the Frobenius–Euler polynomials.

These polynomials are expressed recursively, in terms of the Frobenius–Euler numbers defined by

$$F_n(u) := F_n(0|u),$$

as follows:

$$F_n(x|u) = \sum_{k=0}^{n} \binom{n}{k} F_k(u) x^{n-k} \quad (n \geqq 0), \tag{1.2}$$

where the Frobenius–Euler numbers $F_n(u)$ satisfy the following recurrence relation:

$$F_0(u) = 1 \quad \text{and} \quad \bigl(F(u) + 1\bigr)^n - F_n(u) = (1-u)\delta_{n,0}, \tag{1.3}$$

by simply replacing $F^n(u)$ by $F_n(u)$, $\delta_{n,k}$ being the Kronecker delta.

The classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$ are analogous to the Frobenius–Euler polynomials $F_n(x|u)$. They are specified by the following exponential generating functions:

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{xt} \quad \text{and} \quad \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2}{e^t + 1} e^{xt}. \tag{1.4}$$

The two-variable special polynomials from application viewpoint are very important as they allow the descent of a bunch of handy, advantageous and pragmatic identities in a fairly simple way. They also prove to be handy in originating new clan of special polynomials. The two-variable families of the Appell polynomials were originated by Bretti *et al.* [3] with the usage of an iterated isomorphism. The two-variable truncated-exponential, Hermite, Legendre and Laguerre polynomials along their extensions are investigated and examined in [2, 5–7, 23] by several authors.

The properties of the truncated-exponential polynomials (TEP) are comparatively little known, despite the fact that these polynomials prove to be very handy in solving many problems of quantum mechanics and optics. The main definition of TEP [1] is given as follows:

$$e_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}. \tag{1.5}$$

It is noteworthy here that

$$\lim_{n\to\infty} e_n(x) = e^x.$$

The comprehensive investigation and examination for the first time of certain properties of $e_n(x)$ was made by Dattoli *et al.* [6].

The most remarkable properties of these polynomials can be established by using (1.5). An integral representation of these polynomials is given by

$$e_n(x) = \frac{1}{n!}\int_0^\infty e^{-\xi}(x+\xi)^n \,\mathrm{d}\xi, \tag{1.6}$$

which is a notable consequence of the following well-flourished expression [1]:

$$n! = \int_0^\infty e^{-\xi}\xi^n \,\mathrm{d}\xi. \tag{1.7}$$

The TEP can also be written in terms of the ordinary generating function as follows [6]:

$$\sum_{n=0}^\infty e_n(x)t^n = \frac{e^{xt}}{1-t} \quad \bigl(t \in \mathbb{C}; |t| < 1\bigr). \tag{1.8}$$

A further extension of the TEP $e_n(x)$ to two variables was given by Dattoli *et al.* [6]. The TEP has shown to play a vital and key role in evaluating integrals containing products of special functions. They also emerge in numerous problems of quantum mechanics and optics, but their properties are not known in a way they should be.

Recalling that the two-variable TEP $e_n(x,y)$ are determined by means of the generating relation (see [6])

$$\sum_{n=0}^\infty {}_{[2]}e_n(x,y)t^n = \frac{e^{xt}}{1-yt^2} \tag{1.9}$$

and possess the following series definition:

$${}_{[2]}e_n(x,y) = \sum_{k=0}^{[\frac{n}{2}]} \frac{y^k x^{n-2k}}{(n-2k)!}. \tag{1.10}$$

Recalling also that the higher-order two-variable TEP $e_n(x,y)$ are determined by the generating relation given by (see [6])

$$\sum_{n=0}^\infty {}_{[s]}e_n(x,y)t^n = \frac{e^{xt}}{1-yt^s}, \tag{1.11}$$

which satisfy the following formula:

$${}_{[s]}e_n(x,y) = \sum_{k=0}^{[\frac{n}{s}]} \frac{y^k x^{n-sk}}{(n-sk)!}. \tag{1.12}$$

In view of Eqs. (1.8), (1.9) and (1.11), we find that

$${}_{[2]}e_n(x,y) := e_n^{(2)}(x,y) \quad \text{and} \quad e_n(x) := e_n^{(1)}(x,1).$$

We note that

$$U_n(y) = {}_{[2]}e_n(0, y), \tag{1.13}$$

where $U_n(y)$ represents the Chebyshev polynomials of the second kind, which is determined by the following ordinary generating relation [1]:

$$\sum_{n=0}^{\infty} U_n(x)t^n = \frac{1}{1 - 2xt + t^2} \quad \bigl(|t| < 1; x \leqq 1\bigr). \tag{1.14}$$

Furthermore, under the operation of the multiplicative operator $\widehat{\mathcal{M}}$ and the derivative operator $\widehat{\mathcal{M}}$, we get

$$\widehat{\mathcal{M}}_{e^{(s)}} = x + syD_y yD_x^{s-1} \tag{1.15}$$

and

$$\widehat{\mathcal{P}}_{e^{(s)}} = D_y, \tag{1.16}$$

respectively. It follows from (1.15) and (1.16) that the higher-order two-variable TEP ${}_{[s]}e_n(x, y)$ are *quasi-monomial* ([24] and [23]).

The idea of the monomiality principle traces back to the year 1941, when Steffenson [25] introduced the concept and method of poweroid. Subsequently, this method was modified by Dattoli [5]. According to the hypothesis of monomiality, the operators $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{P}}$ occur and perform as multiplicative and derivative operators for a given polynomial set $\{q_n(x)\}_{n\in\mathbb{N}}$, that is, they satisfy the following relations:

$$q_{n+1}(x) = \widehat{\mathcal{M}}\bigl\{q_n(x)\bigr\} \tag{1.17}$$

and

$$nq_{n-1}(x) = \widehat{\mathcal{P}}\bigl\{q_n(x)\bigr\}. \tag{1.18}$$

The set $\{q_n(x)\}_{n\in\mathbb{N}}$ operated upon by the multiplicative and derivative operators is then called a quasi-monomial set and must obey the following relation:

$$[\widehat{\mathcal{P}}, \widehat{\mathcal{M}}] = \widehat{\mathcal{P}}\widehat{\mathcal{M}} - \widehat{\mathcal{M}}\widehat{\mathcal{P}} = \widehat{1}, \tag{1.19}$$

which obviously exhibits a structure of the Weyl group.

If the underlying set $\{q_n(x)\}_{n\in\mathbb{N}}$ is quasi-monomial, its properties can be obtained from those of the operators $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{P}}$. Specifically, the following properties hold true:

(i) $q_n(x)$ exhibits the differential equation given by

$$\widehat{\mathcal{M}}\widehat{\mathcal{P}}\bigl\{q_n(x)\bigr\} = nq_n(x) \tag{1.20}$$

if $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{P}}$ have differential realizations.

(ii) $q_n(x)$ can be explicitly formulated as follows:

$$q_n(x) = \widehat{\mathcal{M}}^n\{1\} \tag{1.21}$$

with the initial condition $q_0(x) = 1$.

(iii) The exponential generating relation of $q_n(x)$ can be put in the following form:

$$e^{t\widehat{\mathcal{M}}}\{1\} = \sum_{n=0}^{\infty} q_n(x)\frac{t^n}{n!} \quad \bigl(|t| < \infty\bigr) \tag{1.22}$$

by using of the identity (1.21) (see, for details, [5, 6] and [23]).

There is ongoing use of the above-mentioned operational methods in such fields of research as classical optics, quantum mechanics and many areas of mathematical physics. Thus, clearly, these methods provide efficient and powerful means of investigation of various families of polynomials.

This article is organized as follows. In Sect. 2, the truncated-exponential based Frobenius–Euler polynomials are introduced and their several interesting properties are obtained. In Sect. 3, summation formulas are established for these types of polynomials. In the last section (Sect. 4), the truncated-exponential based Apostol-type Frobenius–Euler polynomials are introduced and their quasi-monomial properties are derived.

2 Truncated-exponential based Frobenius–Euler polynomials

With a view to generating the truncated-exponential based Frobenius–Euler polynomials (TEFEPs) denoted by ${}_{e^{(s)}}F_n(x,y|u)$, we first prove the following result.

Theorem 2.1 *The exponential generating function for the TEFEP* ${}_{e^{(s)}}F_n(x,y|u)$ *is given by*

$$\sum_{n=0}^{\infty} {}_{e^{(s)}}F_n(x,y|u)\,\frac{t^n}{n!} = \frac{1-u}{(e^t-u)(1-yt^s)}e^{xt}. \tag{2.1}$$

Proof Upon replacing x in Eq. (1.1) by $\widehat{\mathcal{M}}_{e^{(s)}}$, that is, by the multiplicative operator of the polynomials ${}_{[s]}e_n(x,y)$, we have

$$\frac{1-u}{e^t-u}\exp(\widehat{\mathcal{M}}_{e^{(s)}}t)\{1\} = \sum_{n=0}^{\infty} F_n(\widehat{\mathcal{M}}_{e^{(s)}}|u)\frac{t^n}{n!}. \tag{2.2}$$

Now, if we first make use of the expression for $\widehat{\mathcal{M}}_{e^{(s)}}$ given by (1.15) and then decouple the exponential term in the left-hand side of the resulting equation by means of the Crofton identity:

$$f\left(z + m\mu\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\right)\{1\} = \exp\left(\mu\frac{\mathrm{d}^m}{\mathrm{d}z^m}\right)\bigl\{f(y)\bigr\}, \tag{2.3}$$

we get

$$\frac{1-u}{e^t-u}e^{(yD_yyD_x^s)}e^{xt} = \sum_{n=0}^{\infty} F_n\bigl(x+syD_yyD_x^{s-1}|u\bigr)\frac{t^n}{n!} \quad \left(D_z := \frac{\mathrm{d}}{\mathrm{d}z}\right). \tag{2.4}$$

Denoting the TEFEP in the right-hand side of Eq. (2.4) by ${}_{e(s)}F_n(x,y|u)$, we find that

$$F_n\bigl(x + syD_y yD_x^{s-1}|u\bigr) = {}_{e(s)}F_n(x,y|u). \tag{2.5}$$

Also, upon expanding the first exponential in the left-hand side of Eq. (2.4) by using the following binomial expansion:

$$(1-z)^{-\lambda} = \sum_{\ell=0}^{\infty} \frac{(\lambda)_\ell}{\ell!} z^\ell, \tag{2.6}$$

the assertion (2.1) is established, $(\lambda)_\ell$ being the familiar Pochhammer symbol. □

The next result is proved in order to frame the TEFEP ${}_{e(s)}F_n(x,y|u)$ in the context of the monomiality hypothesis.

Theorem 2.2 *The following succeeding derivative and multiplicative operators for the TEFEP* ${}_{e(s)}F_n(x,y|u)$ *hold true*:

$$\widehat{\mathcal{P}}_{e(s)F} = D_x \tag{2.7}$$

and

$$\widehat{\mathcal{M}}_{e(s)F} = x + syD_y yD_x^{s-1} - \frac{e^{D_x}}{e^{D_x} - u}, \tag{2.8}$$

respectively.

Proof Differentiating both sides (2.2) with respect to t partially, we have

$$\left(\widehat{\mathcal{M}}_{e(s)} - \frac{e^t}{e^t - u}\right) \frac{1-u}{(e^t - u)} \exp(\widehat{\mathcal{M}}_{e(s)} t) = \sum_{n=0}^{\infty} F_{n+1}(\widehat{\mathcal{M}}_{e(s)}|u) \frac{t^n}{n!}. \tag{2.9}$$

If we first substitute from (1.15) and (2.5) into both sides of Eq. (2.9) and then use the following identity:

$$D_x\left\{\frac{1-u}{(e^t-u)(1-yt^s)} e^{xt}\right\} = t\left\{\frac{1-u}{(e^t-u)(1-yt^s)} e^{xt}\right\} \tag{2.10}$$

in the resulting equation, we get

$$\left(x + syD_y yD_x^{s-1} - \frac{e^{D_x}}{e^{D_x} - u}\right) \sum_{n=0}^{\infty} {}_{e(s)}F_n(x,y|u) \frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_{e(s)}F_{n+1}(x,y|u) \frac{t^n}{n!}. \tag{2.11}$$

Equating the coefficients of like powers of t on both sides of Eq. (2.11), we are led to the first result (2.8) asserted by Theorem 2.2.

Next, by using Eq. (2.1) on both sides of the identity (2.10), we have

$$D_x\left\{\sum_{n=0}^{\infty} {}_{e(s)}F_n(x,y|u) \frac{t^n}{n!}\right\} = \left\{\sum_{n=0}^{\infty} {}_{e(s)}F_{n-1}(x,y|u) \frac{t^n}{(n-1)!}\right\}. \tag{2.12}$$

Comparing the coefficients of like powers of t on both sides of the above equation (2.12), we obtain the second result (2.7) asserted by Theorem 2.2. □

Remark 2.1 Using Eqs. (2.7) and (2.8) in Eq. (1.15), we find the following differential equation for the TEFEP ${}_{e^{(s)}}F_n(x,y|u)$:

$$\left(xD_x + syD_y yD_x^s - \frac{e^{D_x}}{e^{D_x}-u}D_x - n\right){}_{e^{(s)}}F_n(x,y|u) = 0. \tag{2.13}$$

We next turn to the series definitions of the TEFEP ${}_{e^{(s)}}F_n(x,y|u)$ by proving Theorem 2.3.

Theorem 2.3 *The following expansion*:

$$\frac{{}_{e^{(s)}}F_n(x,y|u)}{n!} = \sum_{k=0}^{n} {}_{[s]}e_{n-k}(x,y)\frac{F_k(u)}{k!} \tag{2.14}$$

or, equivalently,

$$\frac{{}_{e^{(s)}}F_n(x,y|u)}{n!} = \sum_{k=0}^{n} U_{n-k}(y)\frac{F_k(x|u)}{k!} \tag{2.15}$$

holds true for the TEFEP.

Proof Using Eq. (1.1) with $x = 0$ and (1.11) in the left-hand side of Eq. (2.1), we find that

$$\sum_{n=0}^{\infty} {}_{e^{(s)}}F_n(x,y|u)\frac{t^n}{n!} = \left(\sum_{k=0}^{\infty} F_k(u)\frac{t^k}{k!}\right)\left(\sum_{n=0}^{\infty} {}_{[s]}e_n(x,y)t^n\right). \tag{2.16}$$

Thus, by using the Cauchy product rule in the right-hand side of Eq. (2.16), the assertion (2.14) is established.

Similarly, by using Eqs. (1.1) and (1.11) with $x = 0$ in the left-hand side of Eq. (2.1), we get

$$\sum_{n=0}^{\infty} {}_{e^{(s)}}F_n(x,y|u)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} F_n(x|u)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty} U_k(y)t^k\right). \tag{2.17}$$

Thus, if we apply the Cauchy product rule in the right-hand side of the above equation (2.17), the assertion (2.15) is established. □

3 Summation formulas

In this section, we give several implicit summation formulas for the TEFEP.

Theorem 3.1 *The following addition property for the TEFEP holds true*:

$${}_{e^{(s)}}F_n(x+v,y|u) = \sum_{k=0}^{n} \binom{n}{k} {}_{e^{(s)}}F_{n-k}(x,y|u)v^k. \tag{3.1}$$

Proof Upon setting $x \mapsto x + v$ in Eq. (2.1), we find that

$$\frac{1-u}{(e^t-u)(1-yt^s)}e^{(x+v)t} = \sum_{n=0}^{\infty} {}_{e(s)}F_n(x+v,y|u)\frac{t^n}{n!}. \tag{3.2}$$

Expanding the exponential term in Eq. (3.2) and then on using the Cauchy product rule in the resulting equation, we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}v^k {}_{e(s)}F_{n-k}(x,y|u)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_{e(s)}F_n(x+v,y|u)\frac{t^n}{n!}. \tag{3.3}$$

Exchanging the sides and comparing the coefficients of like powers of t in the resulting equation, the assertion (3.1) is established. □

Upon setting $v = 1$ in (3.1), we get the following corollary.

Corollary 3.1 *It is asserted that*

$${}_{e(s)}F_n(x+1,y|u) = \sum_{k=0}^{n}\binom{n}{k} {}_{e(s)}F_{n-k}(x,y|u). \tag{3.4}$$

Theorem 3.2 *The TEFEPs ${}_{e(s)}F_n(x,y|u)$ satisfy the following implicit summation formula*:

$${}_{e(s)}F_{n+k}(\eta,y|u) = \sum_{l=0}^{n}\sum_{m=0}^{k}\binom{n}{l}\binom{k}{m}(\eta-x)^{l+m} {}_{e(s)}F_{n+k-l-m}(x,y|u). \tag{3.5}$$

Proof By setting $t \mapsto t + w$ in Eq. (2.1), we find that

$$\frac{1-u}{(e^{t+w}-u)(1-y(t+w)^s)}e^{x(t+w)} = \sum_{n=0}^{\infty} {}_{e(s)}F_n(x,y|u)\frac{(t+w)^n}{n!}. \tag{3.6}$$

Using

$$\sum_{M=0}^{\infty} f(M)\frac{(u+v)^M}{M!} = \sum_{i,j=0}^{\infty} f(i+j)\frac{u^i}{i!}\frac{v^j}{j!} \tag{3.7}$$

in Eq. (3.6) and shifting the exponential term to the right-hand side in the resulting equation, we get

$$\frac{1-u}{(e^{t+w}-u)(1-y(t+w)^s)} = e^{-x(t+w)}\sum_{n,k=0}^{\infty} {}_{e(s)}F_{n+k}(x,y|u)\frac{t^n}{n!}\frac{w^k}{k!}. \tag{3.8}$$

Upon letting $x = \eta$ in Eq. (3.8) and then comparing the resulting equation with Eq. (3.8) itself, we obtain

$$\sum_{n,k=0}^{\infty} {}_{e(s)}F_{n+k}(\eta,y|u)\frac{t^n}{n!}\frac{w^k}{k!} = e^{(\eta-x)(t+w)}\sum_{n,k=0}^{\infty} {}_{e(s)}F_{n+k}(x,y|u)\frac{t^n}{n!}\frac{w^k}{k!}. \tag{3.9}$$

In view of Eq. (3.7), by expanding the exponential term in Eq. (3.9), we have

$$\sum_{n,k=0}^{\infty} {}_{e^{(s)}}F_{n+k}(\eta,y|u)\frac{t^n}{n!}\frac{w^k}{k!} = \sum_{l,m=0}^{\infty}(\eta-x)^{l+m}\frac{t^l}{l!}\frac{w^m}{m!}\sum_{n,k=0}^{\infty} {}_{e^{(s)}}F_{n+k}(x,y|u)\frac{t^n}{n!}\frac{w^k}{k!}. \tag{3.10}$$

Finally, if we apply the Cauchy product rule in the right-hand side of Eq. (3.10) and compare the coefficients of like powers of t in the resulting equation, the assertion (3.5) is established. □

For $n = 0$ in (3.5), we deduce the following corollary.

Corollary 3.2 *It is asserted that*

$$ {}_{e^{(s)}}F_k(\eta,y|u) = \sum_{m=0}^{k}\binom{k}{m}(\eta-x)^m {}_{e^{(s)}}F_{k-m}(x,y|u).$$

Replacing η by $\eta + x$ and setting $y = 0$ in (3.5), we get the following corollary.

Corollary 3.3 *It is asserted that*

$$ {}_{e^{(s)}}F_k(\eta+x,0|u) = \sum_{l=0}^{n}\sum_{m=0}^{k}\binom{n}{l}\binom{k}{m}\eta^{m+l} {}_{e^{(s)}}F_{n+k-l-m}(x|u).$$

For $\eta = 0$ in (3.5), we get the following corollary.

Corollary 3.4 *It is asserted that*

$$ {}_{e^{(s)}}F_k(0,y|u) = \sum_{l,m=0}^{n,k}\binom{n}{l}\binom{k}{m}(-x)^{l+m} {}_{e^{(s)}}F_{n+k-l-m}(x,y|u).$$

4 Concluding remarks and observation

Various other allied families of the Apostol-type polynomials are investigated by several researchers in an organized way (see, for example, [9, 20] and [21]). We recall here the Apostol-type Frobenius–Euler polynomials (ATFEPs) $\mathcal{F}_n(x|u,\lambda)$ which are given by the following definition.

Definition 4.1 The ATFEPs $\mathcal{F}_n(x|u,\lambda)$ are determined by the following generating relation:

$$\frac{1-u}{\lambda e^t - u}e^{xt} = \sum_{n=0}^{\infty}\mathcal{F}_n(x|u,\lambda)\frac{t^n}{n!} \quad (u \in \mathbb{C}; u \neq 1), \tag{4.1}$$

which, upon setting $x = 0$, reduces as follows:

$$\frac{1-u}{\lambda e^t - u} = \sum_{n=0}^{\infty}\mathcal{F}_n(u,\lambda)\frac{t^n}{n!} \tag{4.2}$$

for the Apostol-type Frobenius–Euler numbers $\mathcal{F}_n(u|\lambda)$.

Putting $u = -1$, the ATFEP gives the Apostol–Euler polynomials $\mathfrak{E}_n(x;\lambda)$ (see [18]), which (for $\lambda = 1$) reduces to (1.4). Also, for $\lambda = 1$ and $u = -1$, the ATFEPs reduce to the classical Euler polynomials $E_n(x)$ in (1.1).

Here, in this last section, we examine the truncated-exponential based Apostol-type Frobenius–Euler polynomials (TEAFEPs) represented by ${}_{e^{(s)}}F_n(x,y|\lambda,u)$ by demonstrating the following result.

Theorem 4.1 *The TEAFEPs satisfy the following exponential generating function*:

$$\sum_{n=0}^{\infty} {}_{e^{(s)}}F_n(x,y|\lambda,u)\frac{t^n}{n!} = \frac{1-u}{(\lambda e^t - u)(1-yt^s)}e^{xt}. \tag{4.3}$$

Proof Changing x in Eq. (4.1) by the multiplicative operator of the two-variable truncated polynomials ${}_{[s]}e_n(x,y)$, we get

$$\frac{1-u}{(\lambda e^t - u)}\exp(\widehat{\mathcal{M}}_{e^{(r)}}t)\{1\} = \sum_{n=0}^{\infty} F_n(\widehat{\mathcal{M}}_{e^{(r)}}|\lambda,u)\frac{t^n}{n!}. \tag{4.4}$$

By applying the expression for $\widehat{\mathcal{M}}_{e^{(s)}}$ given by (1.15) and decoupling the exponential term in the resulting equation by means of the identity (2.3), we get

$$\frac{1-u}{(\lambda e^t - u)}e^{(yD_y yD_x^s)}e^{xt} = \sum_{n=0}^{\infty} {}_{e^{(s)}}F_n\bigl(x + syD_y yD_x^{s-1}|\lambda,u\bigr)\frac{t^n}{n!}. \tag{4.5}$$

Now, if we denote the TEAFEP in the right-hand side of Eq. (4.5) by ${}_{e^{(s)}}F_n(x,y|\lambda,u)$, we obtain

$${}_{e^{(s)}}F_n\bigl(x + syD_y yD_x^{s-1}|\lambda,u\bigr) = {}_{e^{(s)}}F_n(x,y|\lambda,u). \tag{4.6}$$

Also, in view of Eq. (2.6), by expanding the first exponential in the left-hand side of Eq. (4.5), we get the assertion (4.3). □

In order to frame the TEAFEP ${}_{e^{(s)}}F_n(x,y|\lambda,u)$ in the context of the monomiality hypothesis, we demonstrate the following result.

Theorem 4.2 *For the TEAFEP* ${}_{e^{(s)}}F_n(x,y|\lambda,u)$, *the following relationships involving the derivative and multiplicative operators hold true*:

$$\widehat{\mathcal{P}}_{e^{(s)}F} = D_x \tag{4.7}$$

and

$$\widehat{\mathcal{M}}_{e^{(s)}F} = x + syD_y yD_x^{s-1} - \frac{\lambda e^{D_x}}{\lambda e^{D_x} - u}, \tag{4.8}$$

respectively.

Proof The proof of Theorem 4.2 can be given as in Theorem 2.2. So we omit the details involved. □

Remark 4.1 Substituting from Eqs. (4.7) and (4.8) into Eq. (1.15), we arrive at the following differential equation:

$$\left(xD_x + syD_yyD_x^s - \frac{\lambda e^{D_x}}{\lambda e^{D_x} - u}D_x - n\right)_{e^{(s)}}F_n(x,y|u) = 0.$$

Remark 4.2 By using an approach as that already applied in this paper, we can derive several summation formulas, symmetry identities, recurrence relations and other types results for the TEAFEP ${}_{e^{(r)}}F_n(x,y|\lambda,u)$. We leave the details involved as an exercise for the interested reader.

Before finalizing this paper, we give the following definition which seems to be a multi-dimensional version of the TEAFEP ${}_{e^{(r)}}F_n(x,y|\lambda,u)$.

Definition 4.2 Multi-dimensional (or multivariable) of truncated based exponential based Apostol-type Frobenius–Euler polynomials are determined by the generating series:

$$\sum_{n=0}^{\infty} {}_{e^{(s)}}F_n^r(\overrightarrow{X},y|\lambda,u)\frac{t^n}{n!} = \left(\frac{1-u}{\lambda e^t - u}\right)^r \frac{e^{t\sum_{i=1}^r x_i}}{1-yt^s}, \tag{4.9}$$

where $\overrightarrow{X} = (x_1, x_2, \ldots, x_r)$.

In the case $\lambda = 1$, we get a multi-dimensional version of the TEFEP ${}_{e^{(r)}}F_n(x,y|u)$ as follows:

$$\sum_{n=0}^{\infty} {}_{e^{(s)}}F_n^r(\overrightarrow{X},y|u)\frac{t^n}{n!} = \left(\frac{1-u}{e^t - u}\right)^r \frac{e^{t\sum_{i=1}^r x_i}}{1-yt^s}.$$

Corollary 4.1 *Taking* $r = 1$ *in* (4.9) *reduces to Eq.* (2.1).

In our forthcoming investigation, we plan to establish further results and properties associated with some generalized forms of the above-mentioned families of polynomials.

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Authors' contributions
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