EXISTENTIALLY CLOSED BROUWERIAN SEMILATTICES

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Abstract. The variety of Brouwerian semilattices is amalgamable and locally finite, hence by well-known results [19], it has a model completion (whose models are the existentially closed structures). In this paper, we supply a finite and rather simple axiomatization of the model completion.

§1. Introduction. In algebraic logic some attention has been paid to the class of existentially closed structures in varieties coming from the algebraization of common propositional logics. In fact, there are relevant cases where such classes are elementary: this includes, besides the easy case of Boolean algebras, also Heyting algebras [10, 11], diagonalizable algebras [17, 11] and some universal classes related to temporal logics [9],[8]. However, very little is known about the related axiomatizations, with the remarkable exception of the case of the amalgamable varieties of Heyting algebras recently investigated in [6] and [5], and of the simpler cases of posets and semilattices studied in [1]. In this paper, we use a methodology similar to [6] (relying on classifications of minimal extensions) in order to investigate the case of Brouwerian semilattices, i.e. the algebraic structures corresponding to the implication-conjunction fragment of intuitionistic logic. We obtain the finite axiomatization reported below, which is similar in spirit to the axiomatizations from [6] (in the sense that we also have kinds of 'density' and 'splitting' conditions). The main technical problem we must face for this result (making axioms formulation slightly more complex and proofs much more involved) is the lack of joins in the language of Brouwerian semilattices.

1.1. Statement of the main result. The first researcher to consider Brouwerian semilattices as algebraic objects in their own right was W. C. Nemitz in [15]. A *Brouwerian semilattice* is a poset $(P, \leq)$ having a greatest element (which we denote with 1), inf's of pairs (the inf of $\{a, b\}$ is called 'meet' of a and b and denoted with $a \wedge b$) and relative pseudo-complements (the relative pseudo-complement of a and b is denoted with $a \to b$). $a \to b$ is also called the implication of a and b. We recall that $a \to b$ is characterized by the following property: for every $c \in P$ we have

$$c \leq a \to b \quad \text{iff} \quad c \wedge a \leq b.$$

Brouwerian semilattices can also be defined in an alternative way as algebras over the signature $1, \wedge, \to$, subject to the following equations

$$\begin{aligned} a \wedge a &= a & \qquad a \wedge (a \to b) &= a \wedge b \\ a \wedge b &= b \wedge a & b \wedge (a \to b) &= b \\ a \wedge (b \wedge c) &= (a \wedge b) \wedge c & a \to (b \wedge c) &= (a \to b) \wedge (a \to c) \\ a \wedge 1 &= a & a \to a &= 1 \end{aligned}$$

In case this equational axiomatization is adopted, the partial order $\leq$ is recovered via the definition $a \leq b$ iff $a \wedge b = a$.

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By a result due to Diego and McKay [7, 14], Brouwerian semilattices are locally finite (meaning that all finitely generated Brouwerian semilattices are finite); since they are also amalgamable, it follows [19, 13] that the theory of Brouwerian semilattices has a model completion. We prove that such a model completion is given by the above set of axioms for the theory of Brouwerian semilattices together with the three additional axioms (Density1, Density2, Splitting) below.

We use the shorthand $a \ll b$ to mean that $a \leq b$ and $b \to a = a$. Note that $a \ll a$ iff $a = 1$.

[**Density 1**] For every c there exists an element b different from 1 such that $b \ll c$.

[**Density 2**] For every c, a_1, a_2, d such that $a_1, a_2 \neq 1$, $a_1 \ll c$, $a_2 \ll c$ and $d \to a_1 = a_1$, $d \to a_2 = a_2$ there exists an element b different from 1 such that:

$$\begin{aligned} a_1 &\ll b \\ a_2 &\ll b \\ b &\ll c \\ d \to b &= b \end{aligned}$$

[**Splitting**] For every a, b_1, b_2 such that $1 \neq a \ll b_1 \wedge b_2$ there exist elements a_1 and a_2 different from 1 such that:

$$\begin{aligned} &b_1 \geq a_1,\ b_2 \geq a_2 \\ &a_2 \to a = a_1 \\ &a_1 \to a = a_2 \\ &a_2 \to b_1 = b_2 \to b_1 \\ &a_1 \to b_2 = b_1 \to b_2 \end{aligned}$$

As an evidence of the interest of the above axiomatization, we mention some easy consequences that can be drawn from it: in an existentially closed Brouwerian semilattice (i) there is no bottom element; (ii) there are no joins of pairwise incomparable elements; (iii) there are no meet-irreducible elements.

The paper is structured as follows: Section 2 gives the basic notions and definitions. In particular, it describes the finite duality and characterizes the existentially closed structures by means of embeddings of finite extensions of finite sub-structures. In Section 3 we investigate the minimal finite extensions and use them to give an intermediate characterization of the existentially closed structures. Section 4 focuses on the axiomatization, it is split into two subsections: the first about the Splitting axiom and the second about the Density axioms.[1]

§2. Preliminary Background.

[1] The paper contains full proofs. However, in few cases, when proofs are just routine execises or when the statement to be proved is known from the literature, we preferred to omit straightforward details in order to facilitate reading. In any case, such omitted details are available too in the online arXiv version at the link `http://arxiv.org/abs/1702.08352`

Remark 2.1. The following is a list of identities holding in any Brouwerian semilattice that might be used without explicit mention.

$$a \to 1 = 1 \qquad 1 \to a = a$$
$$a \wedge (a \to b) = a \wedge b \qquad b \wedge (a \to b) = b$$
$$(a \to b) \wedge ((a \to b) \to b) = b \qquad ((a \to b) \to b) \to b = a \to b$$
$$a \to (b_1 \wedge \cdots \wedge b_n) = (a \to b_1) \wedge \cdots \wedge (a \to b_n)$$
$$(a_1 \wedge \cdots \wedge a_n) \to b = a_1 \to (\cdots \to (a_n \to b))$$

In particular

$$a \to (b \to c) = b \to (a \to c)$$

Furthermore, in any Brouwerian semilattice:

$$a \leq b \quad \text{iff} \quad a \to b = 1$$
$$\text{if } b \leq c \text{ then } a \to b \leq a \to c \text{ and } c \to a \leq b \to a$$

Proposition 2.2. *Any finite Brouwerian semilattice is a Heyting algebra.*

Proof. It is sufficient to show that any finite Brouwerian semilattice is a distributive lattice. Any finite semilattice is complete, so it is a lattice. Furthermore, the map $a \wedge (-)$ preserves suprema because it has a right adjoint given by $a \to (-)$. Thus the distributive laws hold. ⊣

Definition 2.3. Let A, B be Brouwerian semilattices. A map $f : A \to B$ is a *Brouwerian semilattice homomorphism* if it preserves 1, the meet and relative pseudo-complement of any two elements of A.

Notice that such a morphism f is an order preserving map because, for any a, b elements of a Brouwerian semilattice, we have $a \leq b$ iff $a \wedge b = a$.

Remark 2.4. Every finite Brouwerian semilattice is a Heyting algebra but it is not true that every Brouwerian semilattice morphism among finite Brouwerian semilattices is a Heyting algebra morphism.

Definition 2.5. Let L be a Brouwerian semilattice.
We say that $m \in L$ is *meet-irreducible* iff for every $n \geq 0$ and $b_1, \ldots, b_n \in L$, we have that

$$m = b_1 \wedge \ldots \wedge b_n \quad \text{implies} \quad m = b_i \text{ for some } i = 1, \ldots, n.$$

Notice that by taking $n = 0$ we obtain that meet-irreducibles are different from 1.

Proposition 2.6. *Let L be a Brouwerian semilattice and $m \in L$. Then the following conditions are equivalent:*

1. *m is meet-irreducible;*
2. *$m \neq 1$ and for any $b_1, b_2 \in L$ we have that $m = b_1 \wedge b_2$ implies $m = b_1$ or $m = b_2$;*
3. *For every $n \geq 0$ and $b_1, \ldots, b_n \in L$ we have that $b_1 \wedge \ldots \wedge b_n \leq m$ implies $b_i \leq m$ for some $i = 1, \ldots, n$;*
4. *$m \neq 1$ and for any $b_1, b_2 \in L$ we have that $b_1 \wedge b_2 \leq m$ implies $b_1 \leq m$ or $b_2 \leq m$;*
5. *$m \neq 1$ and for any $a \in L$ we have that $a \to m = 1$ or $a \to m = m$.*

Proof. The implications $1 \Leftrightarrow 2$, $3 \Leftrightarrow 4$ and $3 \Rightarrow 1$ are straightforward. For the remaining ones see Lemma 2.1 in [12]. Note that 3 implies $m \neq 1$ by taking $n = 0$. ⊣

Remark 2.7. In a finite Brouwerian semilattice, m is meet-irreducible iff it has a unique successor, i.e. a minimal element among the elements strictly greater than m. In that case, we denote the successor by m^+ and it is equal to $\bigwedge_{m<a} a$.

Definition 2.8. Let L be a Brouwerian semilattice and $a \in L$.
A *meet-irreducible component* of a is a minimal element among the meet-irreducibles of L that are greater than or equal to a.

Remark 2.9. Let L be a finite Brouwerian semilattice. For any $a \in L$ we have

$$a = \bigwedge\{\text{meet-irreducible components of } a\}.$$

Hence, for any $a, b \in L$, condition 5 of Proposition 2.6 implies that

$$a \to b = \bigwedge\{m \mid m \text{ is a meet-irreducible component of } b \text{ such that } a \nleq m\}.$$

Recall that $a \ll b$ means $a \leq b$ and $b \to a = a$. Thus, in any finite Brouwerian semilattice, $a \ll b$ if and only if $a \leq b$ and there is no meet-irreducible component of a that is greater than or equal to b. Finally, if m is meet-irreducible then $m \ll m^+$.

This last remark implies the following lemma.

Lemma 2.10. *A finite Brouwerian semilattice is generated as a meet-semilattice with* 1 *by its meet-irreducible elements. Moreover, its Brouwerian semilattice structure is completely determined by the poset of its meet-irreducible elements.*

This correspondence between finite Brouwerian semilattices and the posets of their meet-irreducible elements gives rise to a duality first presented by Köhler in [12].

2.1. Finite duality.

Definition 2.11. Let $(P, \leq)$ be a poset. For any $a \in P$ we define $\uparrow a = \{p \in P \mid a \leq p\}$ and for any $A \subseteq P$ we define $\uparrow A = \bigcup_{a \in A} \uparrow a$. A subset $U \subseteq P$ such that $U = \uparrow U$ is called an *upset*, i.e. an upward closed subset, of P. The upsets $\uparrow a$ and $\uparrow A$ are called the *upsets generated by* a and A. An upset is *principal* if it is generated by an element of P, i.e. it is of the form $\uparrow a$ for some $a \in P$. The set consisting of the upsets of P is denoted by $\mathcal{U}(P)$. The analogous notations $\downarrow a$ and $\downarrow A$ are used for downsets.

Remark 2.12. $\mathcal{U}(P)$ ordered by reverse inclusion has naturally a structure of Brouwerian semilattice. Meets coincide with the union of subsets and the top element with the empty subset. It turns out that the implication of the upsets A and B, i.e. $A \to B$, is given by $\uparrow(B \setminus A)$. Suppose now that P is finite. Clearly $\mathcal{U}(P)$ is finite as well. If A, B are two upsets of P then A is generated by the set of its minimal elements and $A \to B$ is the upset generated by the minimal elements of B that are not in A. The meet-irreducibles of $\mathcal{U}(P)$ are exactly the principal upsets. Therefore, the meet-irreducible components of an upset A are the principal upsets generated by the minimal elements of A. Notice that this is not always the case when P is infinite. When P is finite, $A, B \in \mathcal{U}(P)$ satisfy $A \ll B$ if and only if $B \subseteq A$ and B does not contain any minimal element of A.

The following theorem states the finite duality due to Köhler.

Theorem 2.13. *There is a dual equivalence between the category* $\mathbf{BS}_{fin}$ *of finite Brouwerian semilattices and the category* $\mathbf{P}$ *whose objects are finite posets and whose morphisms are partial mappings* $\alpha : P \to Q$ *satisfying:*

(i) $\forall p, q \in dom\ \alpha$ *if* $p < q$ *then* $\alpha(p) < \alpha(q)$*;*

(ii) $\forall p \in \mathrm{dom}\,\alpha$ *and* $\forall q \in Q$ *if* $q < \alpha(p)$ *then* $\exists r \in \mathrm{dom}\,\alpha$ *such that* $r < p$ *and* $\alpha(r) = q$.

Proof. The proof can be found in [12]. We just recall how the equivalence works. To a finite poset P it is associated the Brouwerian semilattice $\mathcal{U}(P)$ of upsets of P ordered by reverse inclusion. On the other hand, to a finite Brouwerian semilattice L it is associated its sub-poset $\mathcal{M}(L)$ given by its meet-irreducible elements. The isomorphism $P \cong \mathcal{M}(\mathcal{U}(P))$ is given by the mapping $p \mapsto \uparrow p$. The map $U \mapsto \bigwedge U$ gives an isomorphism $\mathcal{U}(\mathcal{M}(L)) \cong L$ whose inverse is $a \mapsto \{m \in \mathcal{M}(L) \mid a \leq m\}$.
To a **P**-morphism among finite posets it is associated the Brouwerian semilattice homomorphism that maps an upset to the upset generated by its preimage. More explicitly, to a **P**-morphism $f : P \to Q$ is associated the morphism that maps an upset U of Q to $\uparrow f^{-1}(U) = \{p \in P \mid \exists p' \leq p \ (p' \in \mathrm{dom} f \ \& \ f(p') \in U)\}$. On the other hand, to a Brouwerian semilattice homomorphism $h : L \to L'$, it is associated the **P**-morphism $f : \mathcal{M}(L') \to \mathcal{M}(L)$ whose domain is given by the $a \in \mathcal{M}(L')$ that are meet-irreducible components in L' of $h(b)$ for some $b \in \mathcal{M}(L)$ and it is defined by $f(a) = b$. ⊣

The following proposition is easily checked:

Proposition 2.14. *Let* P, Q *be finite posets and* $f : P \to Q$ *a* **P***-morphism. Let* $\alpha : \mathcal{U}(Q) \to \mathcal{U}(P)$ *be the associated Brouwerian semilattice homomorphism. Then*

(i) α *is injective if and only if* f *is surjective.*
(ii) α *is surjective if and only if* $\mathrm{dom}\, f = P$ *and* f *is injective.*

Duality results involving all Brouwerian semilattices can be found in the recent paper [2] due to G. Bezhanishvili and R. Jansana. Other dualities are described in [18] and [3].

2.2. Amalgamation property and local finiteness. The variety of Brouwerian semilattices enjoys two properties that will be used extensively throughout the paper: it has the amalgamation property and it is locally finite.

Theorem 2.15. *The theory of Brouwerian semilattices has the amalgamation property.*

The amalgamation property for Brouwerian semilattices is the algebraic counterpart of a syntactic property of the implication-conjunction fragment of intuitionistic propositional logic: the interpolation property. The proof that such a fragment satisfies this property can be found in [16].
Alternatively, it can be shown in a semantic way, using the finite duality, that the theory of Brouwerian semilattices enjoys the amalgamation property. This proof can be found in the online arXiv version of this paper at the link `http://arxiv.org/abs/1702.08352`.

Theorem 2.16. *The variety of Brouwerian semilattices is locally finite.*

Proof. We just sketch the proof first presented in [14]. A Brouwerian semilattice L is subdirectly irreducible iff $L \setminus \{1\}$ has a greatest element, or equivalently L has a single co-atom, i.e. a maximal element distinct from 1.
Let L be subdirectly irreducible and u be the greatest element of $L \setminus \{1\}$. Then $L \setminus \{u\}$ is a Brouwerian sub-semilattice of L. This implies that any generating set of L must contain u.
Moreover, if L is generated by n elements, then $L \setminus \{u\}$ can be generated by $n-1$ elements. It follows that the cardinality of subdirectly irreducible Brouwerian semilattices generated by n elements is bounded by $\#F_{n-1}+1$ where F_m is the free Brouwerian semilattice on m generators. Since $\#F_0 = 1$, by induction we obtain that F_m is finite for any m because it is a subdirect product of a

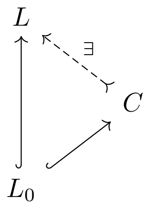

FIGURE 1. The property characterizing the existentially closed Brouwerian semilattices

finite family of subdirectly irreducible Brouwerian semilattices generated by m elements. ⊣

Computing the cardinality of F_m is a hard task. It is known that $\#F_0 = 1, \#F_1 = 2, \#F_2 = 18$ and $\#F_3 = 623,662,965,552,330$. The size of F_4 is still unknown. In [12] it is proved that the number of meet-irreducible elements of F_4 is $2,494,651,862,209,437$. This shows that although the cardinality of the free Brouwerian semilattice on a finite number of generators is always finite, it grows very rapidly.

2.3. Existentially closed Brouwerian semilattices. In this subsection we want to characterize the existentially closed Brouwerian semilattices using the finite extensions of their finite Brouwerian sub-semilattices.

DEFINITION 2.17. Let T be a first order theory and $\mathcal{A}$ a model of T. $\mathcal{A}$ is said to be *existentially closed for* T if for every model $\mathcal{B}$ of T such that $\mathcal{A} \subseteq \mathcal{B}$ every existential sentence in the language extended by names for elements of $\mathcal{A}$ which holds in $\mathcal{B}$ also holds in $\mathcal{A}$.

The following proposition is well-known from textbooks [4].

PROPOSITION 2.18. *Let T be a universal theory. If T has a model completion T^*, then the class of models of T^* is the class of models of T which are existentially closed for T.*

Thanks to the local finiteness and the amalgamability, by an easy model-theoretic reasoning we obtain the following characterization of the existentially closed Brouwerian semilattices.

THEOREM 2.19. *Let L be a Brouwerian semilattice. L is existentially closed iff for any finite Brouwerian sub-semilattice $L_0 \subseteq L$ and for any finite extension $C \supseteq L_0$ there exists an embedding $C \to L$ fixing L_0 pointwise (see Figure 1).*

§3. Minimal finite extensions.

In this section we focus on the finite extensions of Brouwerian semilattices. In particular, we are interested in the minimal ones since any finite extension can be decomposed into a finite chain of minimal extensions. We will study minimal finite extensions by describing the properties of some elements which generate them. This investigation will lead us to another characterization of the existentially closed Brouwerian semilattices.

DEFINITION 3.1. Let A and B subsets of a poset P. We say that $A \leq B$ iff there exist $a \in A$ and $b \in B$ such that $a \leq b$.

PROPOSITION 3.2. *Surjective* **P***-morphisms with domain P are determined, up to isomorphism, by pairs $(P_0, \mathcal{F})$ where P_0 is a subset of P and $\mathcal{F}$ is a partition of P_0 such that:*

1. *for all $A, B \in \mathcal{F}$ if $A \leq B$ and $B \leq A$ then $A = B$,*
2. *for all $A, B \in \mathcal{F}$ and $a \in A$ if $B \leq A$ then there exists $b \in B$ such that $b \leq a$,*

3. for all $A \in \mathcal{F}$ all the elements of A are two-by-two incomparable.

Proof. Given a surjective **P**-morphism $f : P \to Q$, take the pair $(\text{dom } f, \mathcal{F})$ where $\mathcal{F}$ is the collection of the fibers of f. On the other hand, given a partition $\mathcal{F}$ of a subset P_0 of P satisfying the conditions 1, 2 and 3, we obtain a poset Q by taking the quotient set of P_0 given by $\mathcal{F}$ with the order as in Definition 3.1. The projection onto the quotient $\pi : P \to Q$ with domain P_0 is a surjective **P**-morphism.
It is routine to check that a surjective **P**-morphism $f : P \to Q$ differs by an isomorphism from the projection onto the quotient defined by the partition given by the fibers of f. ⊣

Definition 3.3. Let P, Q be finite posets and $f : P \to Q$ a surjective **P**-morphism (or equivalently: let $\mathcal{F}$ satisfy conditions 1, 2 and 3 of Proposition 3.2). We say that f (or $\mathcal{F}$) is minimal if $\#P = \#Q + 1$.

Remark 3.4. If $\mathcal{F}$ is minimal, then at most one element of $\mathcal{F}$ is not a singleton.

Theorem 3.5. *Let $f : P \to Q$ be a surjective* **P***-morphism between finite posets. Let $n = \#P - \#Q$. Then there exist $Q_0, \dots, Q_n$ with $Q_0 = P$, $Q_n = Q$ and $f_i : Q_{i-1} \to Q_i$ which are minimal surjective* **P***-morphisms for $i = 1, \dots, n$ such that $f = f_n \circ \cdots \circ f_1$.*

Proof. Let $R = \text{dom } f$, we can decompose $f = f'' \circ f'$ where $f'' : R \to Q$ is just the restriction of f on its domain and $f' : P \to R$ is the partial morphism with domain R that acts as the identity on R.
f'' is a total surjective **P**-morphism. We prove, by induction on $\#R - \#Q$, that it can be decomposed into a finite chain of minimal surjective **P**-morphisms. Suppose $\#R - \#Q > 1$ and let us consider the partition $\mathcal{F}$ of R given by the fibers of f''. Let $x \in P$ be maximal among the elements of R that are not in a singleton of $\mathcal{F}$ and let G be the element of $\mathcal{F}$ containing x. Denote with Q_{n-1} the quotient of R defined by the refining of $\mathcal{F}$ in which G is substituted by $\{x\}$ and $G \backslash \{x\}$. It is straightforward to check that our choice of x implies that the projection onto the quotient $\pi : R \to Q_{n-1}$ is a total surjective **P**-morphism and the map $f_n : Q_{n-1} \to Q$ induced by f'' is a minimal surjective **P**-morphism. Therefore, we obtain the decomposition applying the induction hypothesis on π.
It remains to decompose f'. To do this, just enumerate the elements of $P \setminus R = \{p_1, \dots, p_k\}$ with $k = n - (\#R - \#Q)$. Let $f'_1 : R \cup \{p_1\} \to R$ be the partial morphism with domain R that acts as the identity on R. Then construct $f'_2 : R \cup \{p_1, p_2\} \to R \cup \{p_1\}$ in the same way and so on until p_k. ⊣

Definition 3.6. We say that a proper extension $L_0 \subseteq L$ of finite Brouwerian semilattices is minimal if there is no intermediate proper extension $L_0 \subsetneq L_1 \subsetneq L$.

The following proposition is an immediate consequence of Proposition 3.2 and Theorem 3.5.

Proposition 3.7. *An extension $L_0 \subseteq L$ of finite Brouwerian semilattices is minimal if and only if the surjective* **P***-morphism that is dual to the inclusion is minimal.*

It follows immediately from Definition 3.3 that there are two different kinds of minimal surjective **P**-morphisms between finite posets: of addition type and of decomposition type.

Definition 3.8. We call a minimal surjective **P**-morphism *of addition type* when there is exactly one element outside its domain. In this case, the restriction of such a map on its domain is an isomorphism of posets. Indeed, any bijective **P**-morphism is an isomorphism of posets. Not every morphism of this type is dual to a Heyting algebra homomorphism.

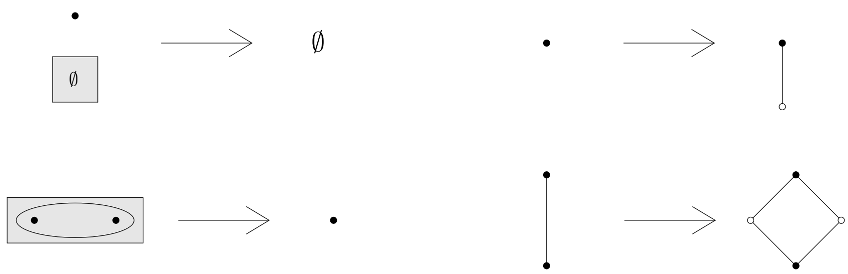

Figure 2. Simplest examples of minimal extensions and their duals; on the left are shown two minimal surjective **P**-morphisms and on the right the corresponding minimal extensions of Brouwerian semilattices. The domain is denoted by a rectangle and the partition into fibers is represented by the encircled sets of points. The white points represents the elements outside the images of the inclusions. Notice that the inclusion on the top right is not a Heyting algebra homomorphism.

We call a minimal surjective **P**-morphism *of decomposition type* when it is total, i.e. there are no elements outside its domain. In this case there is exactly a single fiber which is not a singleton and it contains exactly two elements. All the minimal surjective **P**-morphisms of decomposition type are dual to Heyting algebra embeddings.
We call a finite minimal extension of Brouwerian semilattices either of addition type or of decomposition type if the corresponding minimal surjective **P**-morphism is respectively of addition type or of decomposition type.
Figures 2 and 3 show some examples of minimal surjective **P**-morphisms and the relative extensions of Brouwerian semilattices.

Remark 3.9. A finite minimal extension of Brouwerian semilattices of addition type preserves the meet-irreducibility of all the meet-irreducibles in the domain. Indeed, since the corresponding **P**-morphism is an isomorphism when restricted on its domain, we have that the upset generated by the preimage of a principal upset is still principal.
A finite minimal extension of Brouwerian semilattices of decomposition type preserves the meet-irreducibility of all the meet-irreducibles in the domain except one which becomes the meet of the two new meet-irreducible elements in the codomain. Indeed, the corresponding **P**-morphism is total and all its fibers are singletons except one. Hence, the preimage of any principal upset is principal except for one whose preimage is an upset generated by two elements.

It turns out that we can characterize the finite minimal extensions of Brouwerian semilattices by means of their generators.

Definition 3.10. Let L_0 be a finite Brouwerian semilattice and L an extension of L_0. We call an element $x \in L$ *primitive* over L_0 if the following conditions are satisfied:

1. $x \notin L_0$

and for any a meet-irreducible of L_0:

2. $x \to a \in L_0$,
3. $a \to x = x$ or $a \to x = 1$.

Lemma 3.11. *Let L_0 be a finite Brouwerian semilattice, L a (not necessarily finite) extension of L_0 and $x \in L$ primitive over L_0. Then the two following properties hold for all $a \in L_0$:*

(i) $x \to a \in L_0$,

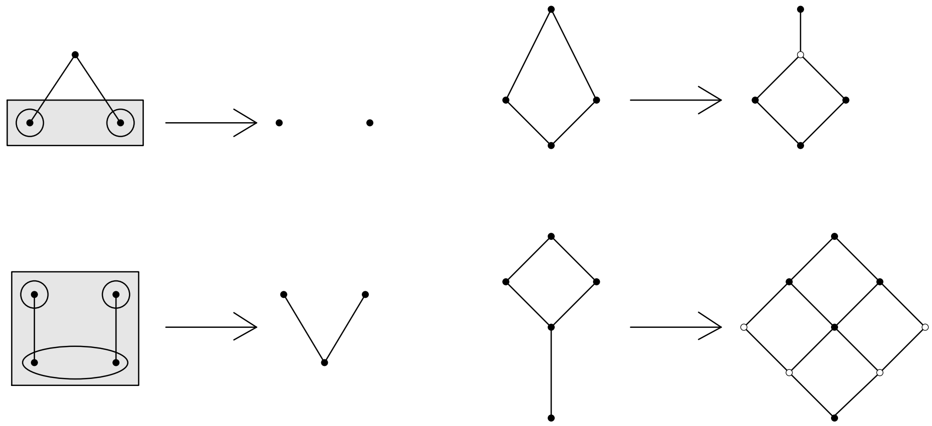

FIGURE 3. More complex examples of minimal extensions and their duals.

(ii) $a \to x = x$ *or* $a \to x = 1$.

PROOF. Let $a \in L_0$ and $a_1, \ldots, a_n$ be its meet-irreducible components in L_0. Since L_0 is finite, we have $a = a_1 \wedge \cdots \wedge a_n$. To prove (i) observe that

$$x \to a = (x \to a_1) \wedge \cdots \wedge (x \to a_n)$$

which is an element of L_0 because it is meet of elements of L_0 as a consequence of 2 of Definition 3.10.
Furthermore, to prove (ii) notice that

$$a \to x = (a_1 \wedge \cdots \wedge a_n) \to x = a_1 \to (\cdots \to (a_n \to x))$$

and that 3 of Definition 3.10 implies that there are two possibilities: $a_i \to x = x$ for any $i = 1, \ldots, n$ or $a_i \to x = 1$ for some i. In the former case, we have $a \to x = x$. In the latter, suppose that i is the greatest index such that $a_i \to x = 1$ then

$$a \to x = a_1 \to (\cdots \to (a_i \to x)) = a_1 \to (\cdots \to 1) = 1.$$

$\dashv$

In the rest of the paper, given a Brouwerian sub-semilattice L_0 of L and $x_1, \ldots, x_n \in L$, we denote by $L_0\langle x_1, \ldots, x_n\rangle$ the Brouwerian sub-semilattice of L generated by $x_1, \ldots, x_n$ over L_0, i.e. the one generated by $L_0 \cup \{x_1, \ldots, x_n\}$. Note that, if L_0 is finite, then $L_0\langle x_1, \ldots, x_n\rangle$ is finite by local finiteness.

THEOREM 3.12. *Let L_0 be a finite Brouwerian semilattice and L an extension of L_0. If $x \in L$ is primitive over L_0, then the Brouwerian sub-semilattice $L_0\langle x\rangle$ of L generated by x over L_0 is a finite minimal extension of L_0 of addition type.*

PROOF. As an easy consequence of Lemma 3.11, $\{a, a \wedge x \mid a \in L_0\}$, i.e. the meet-subsemilattice of L generated by L_0 and x, coincides with $L_0\langle x\rangle$. We want to show that the meet-irreducibles of $L_0\langle x\rangle$ are exactly the meet-irreducibles of L_0 together with x. This implies that $L_0\langle x\rangle$ is a minimal extension of L_0 of addition type. In the following, a is always assumed to be an element of L_0.

- *x is meet-irreducible in $L_0\langle x\rangle$:*
  Suppose that $b \wedge c \leq x$ with $b, c \in L_0\langle x\rangle$ and $b, c \not\leq x$. Then b and c must be elements of L_0 because they cannot be of the form $a \wedge x$. It follows from Lemma 3.11 (ii) and $b, c \not\leq x$ that $b \to x = c \to x = x$. Hence $1 = (b \wedge c) \to x = b \to (c \to x) = b \to x = x$, contradicting $x \notin L_0$.
- *The meet-irreducibles of L_0 are still meet-irreducible in $L_0\langle x\rangle$:*
  It is sufficient to show that for any meet-irreducible m in L_0: if $a \wedge x \leq m$, then $a \leq m$ or $x \leq m$. Note that $m = (x \to m) \wedge ((x \to m) \to m)$ by Remark 2.1

and $x \to m, (x \to m) \to m \in L_0$ by Definition 3.10. Thus m being meet-irreducible in L_0 implies that either $m = x \to m$ or $m = (x \to m) \to m$. In the former case, $a \to m = a \to (x \to m) = (a \wedge x) \to m = 1$, so $a \leq m$. In the latter case, $x \to m = ((x \to m) \to m) \to m = m \to m = 1$ which implies $x \leq m$.

- *There are no other meet-irreducibles in $L_0\langle x\rangle$:*
  Clearly, neither elements that are not meet-irreducible in L_0 nor elements of the form $a \wedge x$ distinct from a and x can be meet-irreducible in $L_0\langle x\rangle$.

$\dashv$

Definition 3.13. Let L_0 be a finite Brouwerian semilattice and L a (not necessarily finite) extension of L_0. We call a pair (x_1, x_2) of elements of L *primitive* over L_0 if the following conditions are satisfied:

1. $x_1, x_2 \notin L_0$ and $x_1 \neq x_2$

and there exists m meet-irreducible element of L_0 such that:

2. $x_1 \to m = x_2$ and $x_2 \to m = x_1$,
3. for any meet-irreducible element a of L_0 such that $m < a$ we have $x_i \to a \in L_0$ for $i = 1, 2$.

Remark 3.14. m in Definition 3.13 is univocally determined by (x_1, x_2) because $m = x_1 \wedge x_2$.
Indeed, by condition 2 of Definition 3.13, we have $m \leq x_1$, $m \leq x_2$ and also $(x_1 \wedge x_2) \to m = x_1 \to (x_2 \to m) = x_1 \to x_1 = 1$ which implies $x_1 \wedge x_2 \leq m$.

Lemma 3.15. *Let L_0 be a finite Brouwerian semilattice, L an extension of L_0 and $(x_1, x_2) \in L^2$ primitive over L_0. Then the two following properties hold for all $a \in L_0$:*

(i) $x_i \to a \in L_0$ or $x_i \to a = b \wedge x_j$ with $b \in L_0$ for $\{i, j\} = \{1, 2\}$;
(ii) $a \to x_i = x_i$ or $a \to x_i = 1$ for $i = 1, 2$.

Proof. Let $m = x_1 \wedge x_2$. We first prove that if $a \neq m$ is meet-irreducible in L_0, then $x_i \to a \in L_0$ for $i = 1, 2$. By condition 3 of Definition 3.13 we can assume $m \nleq a$. Condition 2 of Definition 3.13 implies that $m \leq x_j \to m = x_i$ where $j \neq i$. Thus $a \leq x_i \to a \leq m \to a = a$ by the meet-irreducibility of a. Therefore $x_i \to a = a \in L_0$. Let now a be any element of L_0 and $a_1, \dots, a_n$ be its meet-irreducible components in L_0, then

$$x_i \to a = (x_i \to a_1) \wedge \cdots \wedge (x_i \to a_n).$$

By what we showed at the beginning of the proof, if $a_k \neq m$ then $x_i \to a_k \in L_0$ for any k. Thus, if m is not a meet-irreducible component of a, we have $x_i \to a \in L_0$. Otherwise, if e.g. $a_n = m$, then $x_i \to a_n = x_i \to m = x_j$ with $j \neq i$. Thus $x_i \to a = b \wedge x_j$ for some $b \in L_0$. This proves (i).
We now prove (ii). If $a \leq m$, then $a \leq x_i$ which is equivalent to $a \to x_i = 1$. Otherwise, since m is meet-irreducible in L_0, $a \to m = m$. Thus, if $i \neq j$

$$a \to x_i = a \to (x_j \to m) = x_j \to (a \to m) = x_j \to m = x_i.$$

$\dashv$

Theorem 3.16. *Let L_0 be a finite Brouwerian semilattice and L an extension of L_0. If (x_1, x_2) is primitive over L_0 then the Brouwerian sub-semilattice $L_0\langle x_1, x_2\rangle$ of L is a finite minimal extension of L_0 of decomposition type.*

Proof. By Lemma 3.15 and the fact that $x_1 \wedge x_2 = m \in L_0$, the meet-subsemilattice of L generated by L_0 and $\{x_1, x_2\}$, i.e. $\{a, a \wedge x_1, a \wedge x_2 \mid a \in L_0\}$, coincides with $L_0\langle x_1, x_2\rangle$. We want to show that the meet-irreducibles of $L_0\langle x_1, x_2\rangle$ are exactly x_1, x_2 and the meet-irreducibles of L_0 different from m. This implies that $L_0\langle x_1, x_2\rangle$ is a minimal extension of L_0 of decomposition type. In the following, a is always assumed to be an element of L_0.

- x_1, x_2 *are meet-irreducible in* $L_0\langle x_1, x_2\rangle$:
  Suppose $x_1 = b \wedge c$ with $b, c \in L_0\langle x_1, x_2\rangle$ and $x_1 \neq b, c$. Then b and c must be either elements of L_0 or of the form $a \wedge x_2$. Since $x_1 \notin L_0$, one of b and c is of the form $a \wedge x_2$, so $x_1 \leq x_2$. Hence, by Definition 3.13, $1 = x_1 \to x_2 = x_1 \to (x_1 \to m) = x_1 \to m = x_2$ which contradicts $x_2 \notin L_0$. The meet-irreducibility of x_1 is proved analogously.
- m *is not meet-irreducible in* $L_0\langle x_1, x_2\rangle$:
  $m = x_1 \wedge x_2$ and $x_1, x_2 \neq m$ because $x_1, x_2 \notin L_0$.
- *All the meet-irreducibles of* $L_0\langle x_1, x_2\rangle$ *are either* x_1, x_2 *or meet-irreducible in* L_0:
  Clearly neither elements of L_0 that are not meet-irreducible in L_0 nor elements of the form $a \wedge x_1$ or $a \wedge x_2$ distinct from a, x_1, x_2 can be meet-irreducible in $L_0\langle x_1, x_2\rangle$.
- *All the meet-irreducibles of* L_0 *except* m *are still meet-irreducible in* $L_0\langle x_1, x_2\rangle$:
  Let $b \in L_0$ be meet-irreducible in L_0 but not in $L_0\langle x_1, x_2\rangle$. Let $y_1, \dots, y_r$ be the meet-irreducible components of b in $L_0\langle x_1, x_2\rangle$. The y_i's are in $L_0 \cup \{x_1, x_2\}$. Since b is meet-irreducible in L_0 and not in $L_0\langle x_1, x_2\rangle$, at least one of the y_i's is not in L_0. We can suppose $y_1 = x_1$, so $b \leq x_1$. One among $y_2, \dots, y_r$ has to be equal to x_2 because otherwise $y_2 \wedge \cdots \wedge y_r \in L_0$, which implies, since the y_i's are the meet-irreducible components of b, that $x_1 = (y_2 \wedge \cdots \wedge y_r) \to b$ contradicting $x_1 \notin L_0$. Hence $b \leq m$. If $b < m$ then $m \to b = b$ because b is meet-irreducible in L_0. But in this case $y_2 \wedge \cdots \wedge y_r = x_1 \to b \leq m \to b = b \leq x_1 = y_1$ and this is not possible because the y_i's are the meet-irreducible components of b. Therefore $b = m$.

⊣

Theorem 3.17. *Let* L_0 *be a finite Brouwerian semilattice and* L *a finite minimal extension of* L_0*, then* L *is generated over* L_0 *either by a primitive element or by a primitive pair over* L_0.

Proof. Let $f : P \to Q$ be the surjective minimal **P**-morphism dual to the inclusion of L_0 into L. Recall that P and Q are the posets $\mathcal{M}(L)$ and $\mathcal{M}(L_0)$ of the meet-irreducibles of L and L_0, respectively. Consider two cases:
- f *is of addition type.*
  Then dom $f \neq P$, there exists only one element $p \in P \setminus \text{dom } f$ and the restriction of f on its domain is an isomorphism of posets. It turns out that p is a primitive element over L_0.
- f *is of decomposition type.*
  Then dom $f = P$ and only two elements p_1, p_2 have the same image by f (recall that p_1, p_2 are incomparable). It turns out that (p_1, p_2) is a primitive pair over L_0.

It is easy to check that p and (p_1, p_2) are primitive over L_0 using that, by finite duality, any meet-irreducible in a Brouwerian semilattice corresponds to the upset generated by itself in the dual poset. ⊣

Definition 3.18. Let L_0 be a finite Brouwerian semilattice.
We call a pair (h, M) a *signature of addition type* in L_0 if $h \in L_0$ and M is a set of two-by-two incomparable meet-irreducible elements of L_0 such that $m < h$ for all $m \in M$. We allow M to be empty.
We call a triple (h_1, h_2, m) a *signature of decomposition type* in L_0 if $h_1, h_2 \in L_0$, m is a meet-irreducible element of L_0 such that $h_1 \wedge h_2 = m^+$. Recall that m^+ is the unique successor of m in L_0. To keep the notation simple, we consider the signatures (h_1, h_2, m) and (h_2, h_1, m) to be equal.

Theorem 3.19. *Let* L_0 *be a finite Brouwerian semilattice. Then*

1. *to give a signature of addition type in* L_0 *is equivalent to give a minimal extension of addition type of* L_0*, up to isomorphism over* L_0*;*

2. to give a signature of decomposition type in L_0 is equivalent to give a minimal extension of decomposition type of L_0, up to isomorphism over L_0.

Proof. In the following, L is a minimal extension of L_0.

- *To any minimal extension of addition type it is associated a signature of addition type.*
  Let $L_0 \subseteq L$ be of addition type. Let x be the unique element of $\mathcal{M}(L) \setminus \mathcal{M}(L_0)$. Thus x is primitive over L_0. Define $h := x^+ \in L$ and M to be the set of maximal elements in $\{m \in \mathcal{M}(L_0) \mid m < x\}$. We showed in the proof of Theorem 3.12 that L is generated as a meet-semilattice by L_0 and x. So any element above x is in L_0. In particular $h = x^+ \in L_0$. Therefore (h, M) is a signature of addition type.
- *Any signature of addition type is the signature associated to a unique, up to isomorphism over L_0, minimal finite extension of addition type of L_0.*
  Let (h, M) be a signature of addition type of L_0. Then h corresponds to an upset U of $\mathcal{M}(L_0)$ and M is an antichain in $\mathcal{M}(L_0)$ such that $U \subseteq {\uparrow}m$ for any $m \in M$. Define $P = \mathcal{M}(L_0) \sqcup \{x\}$ and define an order on P by extending the one on $\mathcal{M}(L_0)$. Let $q < x$ iff $q \in {\downarrow}M$ and $x < q$ iff $q \in U$ for any $q \in \mathcal{M}(L_0)$. Take $\operatorname{dom} f = \mathcal{M}(L_0) \subset P$ and f as the identity on its domain. It is easy to prove that $f : P \to \mathcal{M}(L_0)$ is a minimal surjective **P**-morphism of addition type. If $f' : P' \to \mathcal{M}(L_0)$ is another minimal surjective **P**-morphism of addition type whose dual induces the same signature on L_0 then it is straightforward to define an isomorphism of posets $\varphi : P_1 \to P_2$ such that $f_2 \circ \varphi = f_1$.
- *To any minimal extension of decomposition type it is associated a signature of decomposition type.*
  Let $L_0 \subseteq L$ be of decomposition type. Let $\{x_1, x_2\} = \mathcal{M}(L) \setminus \mathcal{M}(L_0)$. Thus (x_1, x_2) is primitive over L_0. Define $h_1 := x_1^+ \in L$ and $h_2 := x_2^+ \in L$. We showed in the proof of Theorem 3.16 that L is generated as a meet-semilattice by L_0 and x_1, x_2. So any element above x_1 or x_2 is in L_0. In particular $h_1 = x_1^+, h_2 = x_2^+ \in L_0$. Let $m = x_1 \wedge x_2$ which is in $\mathcal{M}(L_0)$ by Remark 3.14. It remains to prove that $m^+ = h_1 \wedge h_2$. Suppose $m < a$ for some $a \in L_0$. Let $a_1, \ldots, a_n$ be the meet-irreducible components of a in L. For each i, $x_1 \wedge x_2 < a_i$, thus a_i is either above x_1 or above x_2. Note that $a_i \neq x_1$, otherwise $a \to m \geq x_1 \to m = x_2 > m$, contradicting the meet-irreducibility of m and $m < a$. Hence $a_i \in L_0$ and $h_1 = x_1^+ \leq a_i$ or $h_2 = x_2^+ \leq a_i$. Therefore $h_1 \wedge h_2 \leq a$. So $m^+ = h_1 \wedge h_2$ and (h_1, h_2, m) is a signature of decomposition type.
- *Any signature of decomposition type is the signature associated to a unique, up to isomorphism over L_0, minimal finite extension of decomposition type of L_0.*
  Let (h_1, h_2, m) be a signature of decomposition type of L_0. Then h_1, h_2 correspond to upsets U_1, U_2 of $\mathcal{M}(L_0)$ such that $U_1 \cup U_2 = {\uparrow}m \setminus \{m\}$. Let $P = \mathcal{M}(L_0) \setminus \{m\} \sqcup \{x_1, x_2\}$ where $x_1 \neq x_2$. Define an order on P by extending the one on $\mathcal{M}(L_0) \setminus \{m\}$. Set $x_i < q$ iff $q \in U_i$ and $q < x_i$ iff $q < m$ for any $q \in \mathcal{M}(L_0)$ for $i = 1, 2$. Take $\operatorname{dom} f = P$ and f such that it maps x_1, x_2 into m and acts as the identity on $\mathcal{M}(L_0) \setminus \{m\}$. It is easy to prove that $f : P \to \mathcal{M}(L_0)$ is a minimal surjective **P**-morphism of decomposition type. If $f' : P' \to \mathcal{M}(L_0)$ is another minimal surjective **P**-morphism of decomposition type whose dual induces the same signature on L_0 then it is straightforward to define an isomorphism of posets $\varphi : P_1 \to P_2$ such that $f_2 \circ \varphi = f_1$.

$\dashv$

Therefore signatures inside a finite Brouwerian semilattice L_0 are like 'footprints' left by the minimal finite extensions of L_0: any minimal finite extension of L_0 leaves a 'footprint' inside L_0 given by the corresponding signature. On *the other hand, given a signature inside L_0 we can reconstruct a unique (up to isomorphism over L_0) minimal extension of L_0 corresponding to that signature. By Theorems 3.12, 3.16 and 3.17, minimal finite extension of a finite Brouwerian semilattice L_0 are exactly the ones generated over L_0 either by a primitive element or by a primitive pair. Thus, to any primitive element or pair we can associate a unique signature in L_0. This is what we did in the proof of Theorem 3.19.*

Definition 3.20. Let L_0 be a finite Brouwerian semilattice and L an extension of L_0.
We say that $x \in L$, a primitive element over L_0, induces a signature (h, M) of addition type in L_0 if

- $h = x^+$ in $L_0\langle x\rangle$;
- M is the set of maximal elements of $\{m \in \mathcal{M}(L_0) \mid m < x\}$.

We say that $(x_1, x_2) \in L^2$, a primitive pair over L_0, induces a signature (h_1, h_2, m) of decomposition type in L_0 if

- $h_1 = x_1^+$ and $h_2 = x_2^+$ in $L_0\langle x_1, x_2\rangle$;
- $m = x_1 \wedge x_2$.

Corollary 3.21. *Let L_0 be a finite Brouwerian semilattice and L an extension of L_0.
A primitive element $x \in L$ induces a signature (h, M) iff the extension $L_0 \subseteq L_0\langle x\rangle$ corresponds to that signature.
A primitive pair $(x_1, x_2) \in L^2$ induces a signature (h_1, h_2, m) iff the extension $L_0 \subseteq L_0\langle x_1, x_2\rangle$ corresponds to that signature.*

Proof. Follows immediately from Theorem 3.19 and its proof. ⊣

Proposition 3.22. *Let L_0 be a finite Brouwerian semilattice and L an extension of L_0.
A primitive element $x \in L$ over L_0 induces a signature of addition type (h, M) in L_0 if and only if for any a meet-irreducible of L_0 we have that*

$$x < a \text{ iff } h \leq a \quad \text{and} \quad a < x \text{ iff } a \leq m \text{ for some } m \in M.$$

A primitive pair $(x_1, x_2) \in L^2$ over L_0 induces a signature of decomposition type (h_1, h_2, m) in L_0 if $m = x_1 \wedge x_2$ and for any a meet-irreducible of L_0 we have that

$$x_i < a \text{ iff } h_i \leq a \quad \text{for } i = 1, 2.$$

Proof. This follows from Corollary 3.21 by Lemma 2.10 and the fact that $\mathcal{M}(L_0\langle x\rangle) = \mathcal{M}(L_0) \cup \{x\}$ and $\mathcal{M}(L_0\langle x_1, x_2\rangle) = (\mathcal{M}(L_0) \setminus \{m\}) \cup \{x_1, x_2\}$. ⊣

We have thus finally obtained an intermediate characterization of existentially closed Brouwerian semilattices:

Theorem 3.23. *A Brouwerian semilattice L is existentially closed iff for any finite Brouwerian sub-semilattice $L_0 \subseteq L$ we have:*

1. *Any signature of addition type in L_0 is induced by a primitive element $x \in L$ over L_0.*
2. *Any signature of decomposition type in L_0 is induced by a primitive pair $(x_1, x_2) \in L^2$ over L_0.*

Proof. By the characterization of the existentially closed Brouwerian semilattices given in Theorem 2.19 we have that a Brouwerian semilattice L is existentially closed iff for any finite Brouwerian sub-semilattice L_0 and for any finite extension L_0' of L_0 we have that L_0' embeds into L fixing L_0 pointwise. Since any finite extension of L_0 can be decomposed into a chain of minimal extensions, we can restrict to the case in which L_0' is a minimal finite extension of L_0. Then the claim follows from Theorem 3.19 and Corollary 3.21. ⊣

Thanks to Theorem 3.23 and Proposition 3.22 we already get an axiomatization for the class of the existentially closed Brouwerian semilattices, indeed the quantification over the finite Brouwerian sub-semilattice L_0 can be expressed elementarily using an infinite number of axioms. But this axiomatization is clearly unsatisfactory: other than being infinite, it is not conceptually clear.

§4. Axioms. In this section we prove the main theorem of this paper:

THEOREM 4.1. *A Brouwerian semilattice is existentially closed if and only if it satisfies the Splitting, Density 1 and Density 2 axioms.*

The result will follow from Theorems 4.3, 4.14, 4.15, 4.16 and 4.19 by using the characterization of existentially closed Brouwerian semilattices described in Theorem 3.23. Subsection 4.1 focuses on the Splitting axiom and subsection 4.2 on the Density axioms.

To show the validity of the axioms in any existentially closed Brouwerian semilattice, we will use the following lemma which is the analogue of Lemma 2.3 in [6]. Its proof is straightforward.

LEMMA 4.2. *Let $\theta(\underline{x})$ and $\phi(\underline{x}, \underline{y})$ be quantifier-free formulas in the language of Brouwerian semilattices. Assume that for every finite Brouwerian semilattice L_0 and every tuple $\underline{a}$ of elements of L_0 such that $L_0 \vDash \theta(\underline{a})$, there exists an extension L_1 of L_0 which satisfies $\exists \underline{y}\phi(\underline{a}, \underline{y})$.*
Then every existentially closed Brouwerian semilattice satisfies the following sentence:

$$\forall \underline{x}(\theta(\underline{x}) \longrightarrow \exists \underline{y}\phi(\underline{x}, \underline{y})).$$

4.1. Splitting axiom. [Splitting Axiom] For every a, b_1, b_2 such that $1 \neq a \ll b_1 \wedge b_2$ there exist elements a_1 and a_2 different from 1 such that:

$$b_1 \geq a_1,\ b_2 \geq a_2$$
$$a_2 \to a = a_1$$
$$a_1 \to a = a_2$$
$$a_2 \to b_1 = b_2 \to b_1$$
$$a_1 \to b_2 = b_1 \to b_2$$

THEOREM 4.3. *Any existentially closed Brouwerian semilattice satisfies the Splitting Axiom.*

PROOF. It is sufficient to show, by Lemma 4.2, that for any finite Brouwerian semilattice L_0 and $a, b_1, b_2 \in L_0$ such that $1 \neq a \ll b_1 \wedge b_2$ there exists a finite extension $L_0 \subseteq L$ with $a_1, a_2 \in L$ different from 1 such that:

$$a_2 \to a = a_1 \leq b_1$$
$$a_1 \to a = a_2 \leq b_2$$
$$a_2 \to b_1 = b_2 \to b_1$$
$$a_1 \to b_2 = b_1 \to b_2$$

The following construction is analogous to the one presented in the proof of Lemma 4.2 in [6]. Let $Q = \mathcal{M}(L_0)$ and A, B_1, B_2 be its upsets corresponding to a, b_1, b_2.
We now build a surjective **P**-morphism $\pi : P \to Q$. For $i = 1, 2$ and any $x \in Q$ such that $x \notin B_i$, let $\xi_{x,i}$ be a new symbol. Moreover, for any $x \in Q$ such that $x \in B_1 \cap B_2$ let $\xi_{x,0}$ be a new symbol.
Let P be the set of all these symbols, we define an order on P setting:

$$\xi_{x,j} \leq \xi_{y,i} \Leftrightarrow x \leq y \text{ and } \{i, j\} \neq \{1, 2\}$$

Intuitively P is made of a copy of $B_1 \cup B_2$ and two copies of $Q \backslash (B_1 \cup B_2)$, one of the two copies is placed underneath B_1 and the other underneath B_2.
We define $\pi : P \to Q$ by setting dom $\pi = P$ and $\pi(\xi_{x,i}) = x$.
Let $a_1, \dots, a_r$ be the minimal elements of A, for any i we have $a_i \notin B_1 \cup B_2$ because by hypothesis $A \ll B_1 \cup B_2$. Therefore $\pi^{-1}(\uparrow a_i) = \uparrow \xi_{a_i,1} \cup \uparrow \xi_{a_i,2}$ for $i = 1, \dots, r$.
We take:

$$A_1 = \bigcup_{i=1}^{r} \uparrow \xi_{a_i,1} \qquad \text{and} \qquad A_2 = \bigcup_{i=1}^{r} \uparrow \xi_{a_i,2}$$

We obtain $A_1 \to \pi^{-1}(A) = \uparrow(\pi^{-1}(A) \backslash A_1) = A_2$ and $A_2 \to \pi^{-1}(A) = \uparrow(\pi^{-1}(A) \backslash A_2) = A_1$, they are both nonempty because $r \geq 1$ and A is nonempty.
Furthermore, for any $x \in B_1 \cup B_2$ we have that $a_i \leq x$ for some i. Therefore if $x \in B_1 \backslash B_2$ then $\xi_{a_i,1} \leq \xi_{x,1}$. If $x \in B_2 \backslash B_1$ then $\xi_{a_i,2} \leq \xi_{x,2}$. If $x \in B_1 \cap B_2$ then $\xi_{a_i,1} \leq \xi_{x,0}$ and $\xi_{a_i,2} \leq \xi_{x,0}$. This implies that $\pi^{-1}(B_1) \subseteq A_1$ and $\pi^{-1}(B_2) \subseteq A_2$.
We now show that $A_1 \cap A_2 = \pi^{-1}(B_1) \cap \pi^{-1}(B_2)$.
If $\xi \in \pi^{-1}(B_1) \cap \pi^{-1}(B_2)$ then $\pi(\xi) \in B_1 \cap B_2$, therefore $\xi = \xi_{x,0}$ and $a_i \leq x$ for some i. It implies that $\xi_{a_i,1} \leq \xi_{x,0}$, thus $\xi_{x,0} \in A_1$ and $\xi_{a_i,2} \leq \xi_{x,0}$, therefore $\xi_{x,0} \in A_2$ and $\xi \in A_1 \cap A_2$.
On the other hand, if $\xi \in A_1 \cap A_2$ then there exist i, j such that $\xi_{a_i,1} \leq \xi$ and $\xi_{a_j,2} \leq \xi$. By definition of the order on P it has to be $\xi = \xi_{x,0}$ with $x \in B_1 \cap B_2$, therefore $\xi \in \pi^{-1}(B_1) \cap \pi^{-1}(B_2)$.
Since $\pi^{-1}(B_1) \subseteq A_1$ and $\pi^{-1}(B_2) \subseteq A_2$, we have:

$$\pi^{-1}(B_1) \cap \pi^{-1}(B_2) \subseteq \pi^{-1}(B_1) \cap A_2 \subseteq A_1 \cap A_2 = \pi^{-1}(B_1) \cap \pi^{-1}(B_2).$$

Therefore

$$\begin{aligned} &A_2 \to \pi^{-1}(B_1) = (\pi^{-1}(B_1) \cap A_2) \to \pi^{-1}(B_1) \\ &= (\pi^{-1}(B_1) \cap \pi^{-1}(B_2)) \to \pi^{-1}(B_1) = \pi^{-1}(B_2) \to \pi^{-1}(B_1). \end{aligned}$$

Analogously we can show

$$A_1 \to \pi^{-1}(B_2) = \pi^{-1}(B_1) \to \pi^{-1}(B_2).$$

Thus, by taking the embedding $L_0 \hookrightarrow L$ dual to π and $a_1, a_2 \in L$ corresponding to A_1, A_2, we have obtained what we were looking for. ⊣

LEMMA 4.4. *If L is a Brouwerian semilattice generated by a finite subset X then any meet-irreducible element of L is a meet-irreducible component in L of some element of X.*

PROOF. It follows by an easy induction that any term in the language of Brouwerian semilattices is equivalent to a term of the form $x_1 \wedge \cdots \wedge x_n$ with $x_1, \dots x_n$ containing only the implication symbol and variables. Notice that, if an element $x_1 \wedge \cdots \wedge x_n$ with $x_1, \dots x_n \in L$ is meet-irreducible, then it coincides with x_i for some $i = 1, \dots, n$; thus any meet-irreducible element m of L is the interpretation of a term t over the variables X containing only the implication symbol. This implies that m is the meet of some meet-irreducible components of the interpretation of the rightmost variable in t. Indeed, this can be proved by induction on the complexity of the term as, by Remark 2.9, any meet-irreducible component of $a \to b$ is a meet-irreducible component of b. Thus m is the meet of the meet-irreducible components of some $x \in X$. Then, since it is meet-irreducible, it is a meet-irreducible component of x. ⊣

REMARK 4.5. Lemma 4.4 is not true for Heyting algebras.
Indeed, consider the inclusion $L_0 \hookrightarrow L_1$ of Heyting algebras described by Figure 4. L_1 is generated by L_0 and a but $b = a \vee (a \to 0)$ is meet-irreducible in L_1 and it is not a meet-irreducible component of any element of L_0 or a.

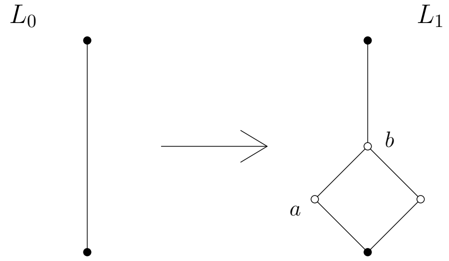

FIGURE 4. The inclusion $L_0 \hookrightarrow L_1$

LEMMA 4.6. *Let L_0 be a finite Brouwerian sub-semilattice of L and let L be generated by L_0 and $a_1, \dots, a_n \in L$.*
If $a_1, \dots, a_n$ are meet-irreducible components in L of elements of L_0, then the surjective $\mathbf{P}$-morphism $\varphi : \mathcal{M}(L) \to \mathcal{M}(L_0)$ dual to the inclusion $L_0 \hookrightarrow L$ is such that dom $\varphi = \mathcal{M}(L)$. In particular, the inclusion is also a Heyting algebra morphism, i.e. it preserves joins and 0.

PROOF. By Lemma 4.4, all the meet-irreducible elements of L are meet-irreducible components in L of elements of L_0. Indeed, by hypothesis, $a_1, \dots, a_n$ are meet-irreducible components of elements of L_0. Suppose there is $x \in \mathcal{M}(L) \setminus \text{dom } \varphi$, then x cannot be a meet-irreducible component of any element of L_0. Indeed, x cannot be a minimal element of $\uparrow\varphi^{-1}(U)$ for any U upset of $\mathcal{M}(L_0)$. ⊣

LEMMA 4.7. *Let L be a Brouwerian semilattice and L_0 a finite Brouwerian sub-semilattice of L, m be meet-irreducible in L_0 and $y_1, y_2 \in L$ be elements different from 1 such that*

$$y_1 \to m = y_2$$
$$y_2 \to m = y_1$$

Let $L_0\langle y_1, y_2\rangle$ be the Brouwerian sub-semilattice of L generated by L_0 and $\{y_1, y_2\}$. We have that:

1. *$m = y_1 \wedge y_2$, $y_1 \neq y_2$ and $y_1, y_2 \in L \setminus L_0$,*
2. *any meet-irreducible a of L_0 such that $m \not\leq a$ is still meet-irreducible in $L_0\langle y_1, y_2\rangle$,*
3. *y_1, y_2 are the meet-irreducible components of m in $L_0\langle y_1, y_2\rangle$.*

PROOF. We prove the three statements separately.

1. *We have $m = y_1 \wedge y_2$, $y_1 \neq y_2$ and $y_1, y_2 \in L \setminus L_0$.*
The identity $y_1 \wedge y_2 = m$ holds because $m \leq y_1$ and $m \leq y_2$ and
$$(y_1 \wedge y_2) \to m = y_1 \to (y_2 \to g) = y_1 \to y_1 = 1.$$
Furthermore $y_1, y_2 \notin L_0$. Indeed, suppose that $y_1 \in L_0$, then $y_2 = y_1 \to m \in L_0$. Since m is meet-irreducible in L_0 and $m = y_1 \wedge y_2$, we have that $m = y_1$ or $m = y_2$. It follows respectively that $y_2 = 1$ or $y_1 = 1$, in both cases we have a contradiction because $y_1, y_2 \neq 1$. Similarly, we obtain that $y_2 \notin L_0$.
We also have that $y_1 \neq y_2$. Indeed, suppose $y_1 = y_2$, then $y_1 \to m = y_1$ implies that $m = y_1 = 1$ and this is absurd.
2. *Any meet-irreducible a of L_0 such that $m \not\leq a$ is still meet-irreducible in $L_0\langle y_1, y_2\rangle$.*
Let $i : L_0 \hookrightarrow L_0\langle y_1, y_2\rangle$ be the inclusion map and $g : L_0\langle y_1, y_2\rangle \to L_0\langle y_1, y_2\rangle/\uparrow m$ be the projection onto the quotient of $L_0\langle y_1, y_2\rangle$ over the filter $\uparrow m$. Then the homomorphism $f = g \circ i$ is surjective because $L_0\langle y_1, y_2\rangle$

is generated over L_0 by y_1, y_2 which are both in the filter $\uparrow m$. By Proposition 2.14, surjective homomorphisms map meet-irreducibles to meet-irreducibles or to 1. Thus, $f(a)$ is meet-irreducible in $L_0\langle y_1, y_2\rangle/\uparrow m$ because $a \notin \uparrow m$ by hypothesis. Note that, since $m \not\leq a$ and a is meet-irreducible in L_0, we have $m \to a = a$. To show that a is still meet-irreducible in $L_0\langle y_1, y_2\rangle$, we prove that for any $x \in L_0\langle y_1, y_2\rangle$ either $x \to a = 1$ or $x \to a = a$. Since $f(a) = g(a)$ is meet-irreducible, $g(x \to a) = g(x) \to g(a)$ is either 1 or $g(a)$. Hence, either $m \wedge (x \to a) = m$ or $m \wedge (x \to a) = m \wedge a$. In the former case, $m \leq x \to a$ and so $x \leq m \to a = a$. Thus $x \to a = 1$. In the latter, $m \wedge (x \to a) \leq a$, so $x \to a \leq m \to a = a$, which implies $x \to a = a$. Therefore a is meet-irreducible in $L_0\langle y_1, y_2\rangle$.

3. *y_1, y_2 are the meet-irreducible components of m in $L_0\langle y_1, y_2\rangle$.*
We show first that y_1 is meet-irreducible, for y_2 it is analogous.
Let $i : L_0 \hookrightarrow L_0\langle y_1, y_2\rangle$ be the inclusion map and $k : L_0\langle y_1, y_2\rangle \to L_0\langle y_1, y_2\rangle/\uparrow y_2$ be the projection onto the quotient of $L_0\langle y_1, y_2\rangle$ over the filter $\uparrow y_2$. Then the homomorphism $h = k \circ i$ is surjective because $L_0\langle y_1, y_2\rangle$ is generated over L_0 by y_1, y_2 with $k(y_2) = 1$ and $k(y_1) = k(y_2 \to m) = k(y_2) \to k(m) = 1 \to k(m) = k(m) = h(m)$. Thus, since h is onto and $h(m) \neq 1$ because $m \notin \uparrow y_2$, we have that $k(y_1) = h(m)$ is meet-irreducible in $L_0\langle y_1, y_2\rangle/\uparrow y_2$. Note that $y_2 \to y_1 = y_2 \to (y_2 \to m) = y_2 \to m = y_1$. To show that y_1 is meet-irreducible in $L_0\langle y_1, y_2\rangle$, we prove that for any $x \in L_0\langle y_1, y_2\rangle$ either $x \to y_1 = 1$ or $x \to y_1 = y_1$. Since $k(y_1)$ is meet-irreducible, $k(x \to y_1) = k(x) \to k(y_1)$ is either 1 or $k(y_1)$. Hence, either $y_2 \wedge (x \to y_1) = y_2$ or $y_2 \wedge (x \to y_1) = y_2 \wedge y_1$. In the former case, $y_2 \leq x \to y_1$ and so $x \leq y_2 \to y_1 = y_1$. Thus $x \to y_1 = 1$. In the latter, $y_2 \wedge (x \to y_1) \leq y_1$, so $x \to y_1 \leq y_2 \to y_1 = y_1$, which implies $x \to y_1 = y_1$. Therefore y_1 is meet-irreducible in $L_0\langle y_1, y_2\rangle$. Finally, to prove that y_1, y_2 are the meet-irreducible components of m in $L_0\langle y_1, y_2\rangle$, we simply have to notice that $y_1 \not\leq y_2$ and $y_2 \not\leq y_1$. Just observe that if $y_1 \leq y_2$ then $m = y_1 \wedge y_2 = y_1 \notin L_0$ which is absurd. Analogously, it cannot be $y_2 \leq y_1$.

⊣

We now prove a series of lemmas which will lead to the main theorem of this subsection.

LEMMA 4.8. *Let L be a Brouwerian semilattice and $L_0 \subseteq L$ a finite Brouwerian sub-semilattice. Let (h, h, m) be a signature of decomposition type in L_0. If $x_1, x_2 \in L$ are different from 1 and satisfy*

$$\begin{aligned} x_1 \to m = x_2 \leq h \\ x_2 \to m = x_1 \leq h \end{aligned} \tag{1}$$

then (x_1, x_2) is a primitive pair over L_0 inducing the signature (h, h, m).

PROOF. We prove the result in two steps.

- *(x_1, x_2) is a primitive pair.*
Lemma 4.7 shows that $x_1 \neq x_2$ and $x_1, x_2 \notin L_0$. The hypotheses say that $x_1 \to m = x_2$ and $x_2 \to m = x_1$. Furthermore, for any a meet-irreducible element of L_0, we have that $m < a$ implies $x_i \to a = 1 \in L_0$ because $x_i \leq h = m^+$ for $i = 1, 2$.
- *(x_1, x_2) induces the signature (h, h, m).*
We use the Proposition 3.22. By Lemma 4.7, $m = x_1 \wedge x_2$. Let a be meet-irreducible in L_0 and $i \in \{1, 2\}$. If $x_i < a$ then $m < a$ because $m \leq x_i$. Thus $h = m^+ \leq a$. On the other hand, $h \leq a$ implies $x_i < a$. Indeed, $x_i \notin L_0$ and $x_i \leq h$.

⊣

Definition 4.9. Let L_0 be a finite Brouwerian semilattice and $h_1, h_2 \in L_0$. We define $\mathrm{ht}_{L_0}(h_1, h_2)$ to be the maximum length of chains of meet-irreducible elements of L_0

$$k_1 < k_2 < \cdots < k_n$$

such that $h_1 \leq k_1$ and $h_2 \not\leq k_n$. Equivalently, since L_0 is a Heyting algebra because it is finite, $h_1 \leq k_1$ and $h_1 \vee h_2 \not\leq k_n$ with the join taken inside L_0.
We call $\mathrm{ht}_{L_0}(h_1, h_2)$ the *height of* h_1 *relative to* h_2 *in* L_0.
We define the *relative height of* (h_1, h_2) *in* L_0, which we denote by $\mathrm{H}_{L_0}(h_1, h_2)$, as

$$\mathrm{H}_{L_0}(h_1, h_2) := \mathrm{ht}_{L_0}(h_1, h_2) + \mathrm{ht}_{L_0}(h_2, h_1)$$

Intuitively, $\mathrm{H}_{L_0}(h_1, h_2)$ measures how much $h_1 \vee h_2$ is bigger than h_1 and h_2 in L_0.
Note that $\mathrm{H}_{L_0}(h_1, h_2) = 0$ if and only if $h_1 = h_2$.

Lemma 4.10. *Let L be a Brouwerian semilattice and $L_0 \subseteq L$ a finite Brouwerian sub-semilattice. Let (h_1, h_2, m) be a signature of decomposition type in L_0. If $y_1, y_2 \in L$ are different from 1 and satisfy*

$$\begin{aligned} y_1 \to m &= y_2 \leq h_2 \\ y_2 \to m &= y_1 \leq h_1 \\ y_1 \to h_2 &= h_1 \to h_2 \\ y_2 \to h_1 &= h_2 \to h_1 \end{aligned} \tag{2}$$

then:

1. *$(h_1,\ h_2 \vee y_1,\ y_1)$ and $(h_1 \vee y_2,\ h_2,\ y_2)$ are signatures of decomposition type in $L_0\langle y_1, y_2\rangle$, where the joins are taken inside $L_0\langle y_1, y_2\rangle$;*
2. *$ht_{L_0\langle y_1,y_2\rangle}(h_1, h_2) = ht_{L_0}(h_1, h_2)$ and $ht_{L_0\langle y_1,y_2\rangle}(h_2, h_1) = ht_{L_0}(h_2, h_1)$;*
3. *If $h_1 \not\leq h_2$ then $ht_{L_0\langle y_1,y_2\rangle}(h_2 \vee y_1, h_1) < ht_{L_0}(h_2, h_1)$.*
   If $h_2 \not\leq h_1$ then $ht_{L_0\langle y_1,y_2\rangle}(h_1 \vee y_2, h_2) < ht_{L_0}(h_1, h_2)$.

Proof. We prove the three statements separately.

1. *$(h_1,\ h_2 \vee y_1,\ y_1)$ and $(h_1 \vee y_2,\ h_2,\ y_2)$ are signatures of decomposition type in $L_0\langle y_1, y_2\rangle$.*
   By Lemma 4.7, $y_1, y_2 \notin L_0$ are the meet-irreducible components of m in $L_0\langle y_1, y_2\rangle$.
   Moreover, in $L_0\langle y_1, y_2\rangle$ we have that:

$$\begin{aligned} h_1 \wedge (h_2 \vee y_1) &= y_1^+ \\ (h_1 \vee y_2) \wedge h_2 &= y_2^+ \end{aligned} \tag{3}$$

   Indeed

$$h_1 \wedge (h_2 \vee y_1) = (h_1 \wedge h_2) \vee (h_1 \wedge y_1) = (h_1 \wedge h_2) \vee y_1 = m^+ \vee y_1$$

   which coincides with y_1^+, the successor of y_1 in $L_0\langle y_1, y_2\rangle$. To show this, observe that, as a consequence of Lemma 4.6, the inclusion $L_0 \hookrightarrow L_0\langle y_1, y_2\rangle$ is dual to a total surjective $\mathbf{P}$-morphism $\varphi : \mathcal{M}(L_0\langle y_1, y_2\rangle) \to \mathcal{M}(L)$. Recall (see the proof of Theorem 2.13) that the preimage of an element of $\mathcal{M}(L)$ under φ consists of the meet-irreducible components of such an element inside $L_0\langle y_1, y_2\rangle$. Then $\varphi^{-1}(m) = \{y_1, y_2\}$ because y_1, y_2 are the meet-irreducible components of m in $L_0\langle y_1, y_2\rangle$.
   As a consequence of the surjectivity and totality of φ we have:

$$\uparrow\varphi^{-1}(\uparrow m \setminus \{m\}) = \varphi^{-1}(\uparrow m \setminus \{m\}) = (\uparrow y_1 \cup \uparrow y_2) \setminus \{y_1, y_2\}.$$

Therefore

$$\begin{aligned}\uparrow\varphi^{-1}(\uparrow m \setminus \{m\}) \cap \uparrow y_1 &= (\uparrow y_1 \cup \uparrow y_2) \setminus \{y_1, y_2\} \cap \uparrow y_1 \\ &= \uparrow y_1 \setminus \{y_1, y_2\} = \uparrow y_1 \setminus \{y_1\}.\end{aligned}$$

Which means $m^+ \vee y_1 = y_1^+$. That $(h_1 \vee y_2) \wedge h_2 = y_2^+$ is proved similarly.

2. *$ht_{L_0\langle y_1,y_2\rangle}(h_1, h_2) = ht_{L_0}(h_1, h_2)$ and $ht_{L_0\langle y_1,y_2\rangle}(h_2, h_1) = ht_{L_0}(h_2, h_1)$.*
Suppose there exists a chain of meet-irreducibles in $L_0\langle y_1, y_2\rangle$

$$k_1 < k_2 < \cdots < k_r$$

such that $h_1 \leq k_1$ and $h_2 \not\leq k_r$. Let, as above, $\varphi : \mathcal{M}(L_0\langle y_1, y_2\rangle) \to \mathcal{M}(L)$ be the surjective total **P**-morphism dual to the inclusion $L_0 \hookrightarrow L_0\langle y_1, y_2\rangle$. Then

$$\varphi(k_1) < \varphi(k_2) < \cdots < \varphi(k_r)$$

is a chain of meet-irreducibles in L_0 such that $h_1 \leq \varphi(k_1)$ and $h_2 \not\leq \varphi(k_r)$. Indeed, **P**-morphisms preserve the strict order.
On the other hand, a chain of meet-irreducibles in L_0

$$b_1 < b_2 < \cdots < b_r$$

such that $h_1 \leq b_1$ and $h_2 \not\leq b_r$ can be lifted to a chain of meet-irreducibles of $L_0\langle y_1, y_2\rangle$

$$k_1 < k_2 < \cdots < k_r$$

such that $\varphi(k_s) = b_s$ for $s = 1, \ldots, r$ using the fact that φ is a surjective **P**-morphism. We have that $h_1 \leq k_1$ and $h_2 \not\leq k_r$.
Therefore $\mathrm{ht}_{L_0\langle y_1,y_2\rangle}(h_1, h_2) = \mathrm{ht}_{L_0}(h_1, h_2)$. That $\mathrm{ht}_{L_0\langle y_1,y_2\rangle}(h_2, h_1) = \mathrm{ht}_{L_0}(h_2, h_1)$ is shown analogously.

3. *If $h_1 \not\leq h_2$ then $ht_{L_0\langle y_1,y_2\rangle}(h_2 \vee y_1, h_1) < ht_{L_0}(h_2, h_1)$.*
Let $n_2 = \mathrm{ht}_{L_0}(h_2, h_1)$. Note that $n_2 \neq 0$ because $h_1 \not\leq h_2$. Suppose there exists a chain in $\mathcal{M}(L_0\langle y_1, y_2\rangle)$

$$k_1 < k_2 < \cdots < k_{n_2}$$

such that $h_2 \vee y_1 \leq k_1$ and $h_1 \not\leq k_{n_2}$. We have that k_1 is not a meet-irreducible component of h_2 in $L_0\langle y_1, y_2\rangle$. Indeed, $y_1 \leq k_1$ and the meet-irreducible components of h_2 that are greater than or equal to y_1 are the same that are greater than or equal to h_1 because $y_1 \to h_2 = h_1 \to h_2$, but $h_1 \not\leq k_1$. Thus there would exist a continuation of such a chain given by k_0 meet-irreducible component of h_2 in $L_0\langle y_1, y_2\rangle$, but this is absurd because we have proved above that n_2 is the maximum length of those chains.
Symmetrically, if $h_2 \not\leq h_1$ then $\mathrm{ht}_{L_0\langle y_1,y_2\rangle}(h_1 \vee y_2, h_2) < \mathrm{ht}_{L_0}(h_1, h_2)$.

$\dashv$

LEMMA 4.11. *Let L_0, L, (h_1, h_2, m) and (y_1, y_2) as in Lemma 4.10. Let $(y_{11}, y_{12}) \in L^2$ and $(y_{21}, y_{22}) \in L^2$ be primitive pairs over $L_0\langle y_1, y_2\rangle$ inducing the signatures $(h_1, h_2 \vee y_1, y_1)$ and $(h_1 \vee y_2, h_2, y_2)$, respectively. Then the extension of finite Brouwerian semilattices $L_0\langle y_1, y_2, y_{11}, y_{12}\rangle \subseteq L_0\langle y_{ij} \mid i, j = 1, 2\rangle$ is minimal of decomposition type. This implies that any meet-irreducible of $L_0\langle y_1, y_2\rangle$ different from y_1, y_2 is still meet-irreducible in $L_0\langle y_{ij} \mid i, j = 1, 2\rangle$.*

PROOF. By the hypotheses we have that:

1. $y_{11} \neq y_{12}$ and $y_{11}, y_{12} \notin L_0\langle y_1, y_2\rangle$,
2. $y_{11} \to y_1 = y_{12}$ and $y_{12} \to y_1 = y_{11}$

and for any a meet-irreducible of $L_0\langle y_1, y_2\rangle$:

3. if $y_1 < a$ then $y_{1i} \to a \in L_0\langle y_1, y_2\rangle$ for $i = 1, 2$,
4. $y_{11} < a$ iff $h_1 \leq a$ and $y_{12} < a$ iff $(h_2 \vee y_1) \leq a$.

furthermore

1. $y_{21} \neq y_{22}$ and $y_{21}, y_{22} \notin L_0\langle y_1, y_2\rangle$,
2. $y_{21} \to y_2 = y_{22}$ and $y_{22} \to y_2 = y_{21}$

and for any a meet-irreducible of $L_0\langle y_1, y_2\rangle$:

3. if $y_2 < a$ then $y_{2i} \to a \in L_0\langle y_1, y_2\rangle$ for $i = 1, 2$,
4. $y_{21} < a$ iff $(h_1 \vee y_2) \leq a$ and $y_{22} < a$ iff $h_2 \leq a$.

Notice that properties 4 of y_{11}, y_{12} and 4 of y_{21}, y_{22} actually hold for any $a \in L_0\langle y_1, y_2\rangle$ since any element in a finite Brouwerian semilattice is the meet of meet-irreducible elements.
First of all, we observe that

$$y_{2i} \to y_1 = y_1 \quad \text{and} \quad y_{1i} \to y_2 = y_2 \qquad \text{for } i = 1, 2. \tag{4}$$

Indeed,

$$y_1 \leq y_{2i} \to y_1 \leq y_2 \to y_1 = y_2 \to (y_2 \to m) = y_2 \to m = y_1. \tag{5}$$

The second equation of (4) is shown analogously. The inequalities (5) and their analogues also imply that $y_2 \to y_1 = y_1$ and $y_1 \to y_2 = y_2$.
Moreover

$$y_{1i} - y_{2j} = y_{1i} \quad \text{and} \quad y_{2i} - y_{1j} = y_{2i} \qquad \text{for } i, j = 1, 2 \tag{6}$$

Indeed,

$$\begin{aligned} y_{11} \leq y_{21} \to y_{11} \leq y_2 \to y_{11} = y_2 \to (y_{12} \to y_1) &= y_{12} \to (y_2 \to y_1) \\ &= y_{12} \to y_1 = y_{11} \end{aligned}$$

and thus $y_{21} \to y_{11} = y_{11}$. The remaining equations of (6) are proved analogously.

- (y_{21}, y_{22}) *is a primitive pair inducing the signature* $(h_1 \vee y_2, h_2, y_2)$.
  As a consequence of Lemma 4.7, y_1, y_2 are meet-irreducible in $L_0\langle y_1, y_2\rangle$, thus y_2 is meet-irreducible in $L_0\langle y_1, y_2, y_{11}, y_{12}\rangle$.
  $y_{21} \neq y_{22}$ by property 1 of y_{21}, y_{22}. Also $y_{21}, y_{22} \notin L_0\langle y_1, y_2, y_{11}, y_{12}\rangle$. Indeed, if $y_{21} \in L_0\langle y_1, y_2, y_{11}, y_{12}\rangle$ then $y_{22} = y_{21} \to y_2 \in L_0\langle y_1, y_2, y_{11}, y_{12}\rangle$ and vice versa. In that case, $y_2 = y_{21} \wedge y_{22} \in L_0\langle y_1, y_2, y_{11}, y_{12}\rangle$ with $y_{21}, y_{22} \neq y_2$ because they are not in $L_0\langle y_1, y_2\rangle$. This is impossible because y_2 is meet-irreducible in $L_0\langle y_1, y_2, y_{11}, y_{12}\rangle$.
  $y_{21} \to y_2 = y_{22}$ and $y_{22} \to y_2 = y_{21}$ by property 2 of y_{21}, y_{22}.
  Since (y_{11}, y_{12}) is a primitive pair inducing the signature $(h_1, h_2 \vee y_1, y_1)$, the meet-irreducibles of $L_0\langle y_1, y_2, y_{11}, y_{12}\rangle$ are y_{11}, y_{12} and the meet-irreducibles of $L_0\langle y_1, y_2\rangle$ except y_1. If a is a meet-irreducible of $L_0\langle y_1, y_2, y_{11}, y_{12}\rangle$ such that $y_2 < a$ then a is meet-irreducible in $L_0\langle y_1, y_2\rangle$ because $a \neq y_{11}, y_{12}$. Indeed, $y_2 \not< y_{11}, y_{12}$ because $y_2 \to y_{1i} = (y_{21} \wedge y_{22}) \to y_{1i} = y_{22} \to (y_{21} \to y_{1i}) = y_{1i} \neq 1$ by (6). Thus $y_{2i} \to a \in L_0\langle y_1, y_2\rangle$ by property 3 of y_{21}, y_{22}.
- *Every meet-irreducible of* $L_0\langle y_1, y_2\rangle$ *different from* y_1, y_2 *is still meet-irreducible in* $L_0\langle y_{ij} \mid i, j = 1, 2\rangle$. We have that

$$\mathcal{M}(L_0\langle y_1, y_2\rangle) \setminus \{y_1, y_2\} \subseteq \mathcal{M}(L_0\langle y_1, y_2, y_{11}, y_{12}\rangle) \setminus \{y_2\} \subseteq \mathcal{M}(L_0\langle y_{ij} \mid i, j = 1, 2\rangle)$$

  because the two extensions involved are both minimal of decomposition type.
$\dashv$

Lemma 4.12. *Let* $L_0, L,$ (h_1, h_2, m) *and* (y_1, y_2) *as in Lemma 4.10. Let* $(y_{11}, y_{12}) \in L^2$ *and* $(y_{21}, y_{22}) \in L^2$ *be primitive pairs over* $L_0\langle y_1, y_2\rangle$ *inducing the signatures* $(h_1, h_2 \vee y_1, y_1)$ *and* $(h_1 \vee y_2, h_2, y_2)$, *respectively. If* $x_1 = y_{11} \wedge y_{21}$ *and* $x_2 = y_{12} \wedge y_{22}$, *then* (x_1, x_2) *is a primitive pair over* L_0 *inducing the signature* (h_1, h_2, m).

PROOF. First of all, we observe that

$$x_1 \to m = x_2 \quad \text{and} \quad x_2 \to m = x_1. \tag{7}$$

Indeed, thanks to equations (4) we have:

$$\begin{aligned} x_1 \to m = (y_{11} \wedge y_{21}) \to (y_1 \wedge y_2) &= (y_{11} \to (y_{21} \to y_1)) \wedge (y_{21} \to (y_{11} \to y_2)) \\ &= (y_{11} \to y_1) \wedge (y_{21} \to y_2) = y_{12} \wedge y_{22} = x_2; \end{aligned}$$

showing the second equation of (7) is analogous.
In the rest of the proof we will refer to the properties of $y_{11}, y_{12}, y_{21}, y_{22}$ listed at the beginning of the proof of Lemma 4.11.

- (x_1, x_2) *is a primitive pair over* L_0.
  $x_1, x_2 \neq 1$ since $y_{11}, y_{12}, y_{21}, y_{22}$ are not in L_0. m is meet-irreducible and equations (7) imply that $m = x_1 \wedge x_2$. Thus if $x_1 = x_2$, then $x_1 = m = 1$ but this is absurd. By equations (7) we get that $x_1 \to m = x_2$, $x_2 \to m = x_1$. Furthermore $x_1, x_2 \notin L_0$. This is because m is meet-irreducible in L_0 and $x_1 \to m = x_2$, $x_2 \to m = x_1$ are different from 1 and m.
  It remains to show that, for any a meet-irreducible element of L_0 and $i = 1, 2$, if $m < a$ then $x_i \to a \in L_0$. We show $x_1 \to a \in L_0$, that $x_2 \to a \in L_0$ is proved analogously.
  Since $m < a$, we have $h_1 \wedge h_2 = m^+ \leq a$. Thus $h_1 \leq a$ or $h_2 \leq a$. We consider these two cases separately. Suppose $h_1 \leq a$. Then $x_1 \to a = 1 \in L_0$ because $x_1 \leq y_{11} \leq h_1$. Suppose $h_2 \leq a$ and $h_1 \nleq a$, we want to prove that $x_1 \to a = a \in L_0$. Note that the meet-irreducible components of a in $L_0\langle y_1, y_2\rangle$ coincide with the meet-irreducible component of a in $L_0\langle y_{ij} \mid i, j = 1, 2\rangle$. Indeed, since a is the meet of its meet-irreducible components in $L_0\langle y_1, y_2\rangle$, it is sufficient to prove that any meet-irreducible component b of a in $L_0\langle y_1, y_2\rangle$ is meet-irreducible in $L_0\langle y_{ij} \mid i, j = 1, 2\rangle$. We have $b \neq y_1, y_2$ because $h_2 \leq b$ and $h_2 \nleq y_1, y_2$. Indeed, if $h_2 \leq y_i$ then $1 = h_2 \to y_i = h_2 \to (y_j \to m) = y_j \to (h_2 \to m) = y_j \to m = y_i$ with $i \neq j$ which is absurd. Thus by Lemma 4.11 we have that b is also meet-irreducible in $L_0\langle y_{ij} \mid i, j = 1, 2\rangle$.
  Since a is meet-irreducible in L_0 and $h_1 \nleq a$, we have $h_1 \to a = a$. For any b meet-irreducible component of a in $L_0\langle y_1, y_2\rangle$ we have $h_1 \nleq b$ because $h_1 \to a = a$ means that h_1 is not smaller than or equal to any meet-irreducible component of a. Since $h_1 \nleq b$ and in particular $h_1 \vee y_2 \nleq b$, then property 4 of y_{11} and property 4 of y_{21} imply that $y_{11}, y_{21} \nleq b$. Therefore $x_1 = y_{11} \wedge y_{21} \nleq b$ because b is meet-irreducible in $L_0\langle y_{ij} \mid i, j = 1, 2\rangle$. This implies that $x_1 \to a = a$ because x_1 is not smaller than or equal to any meet-irreducible component of a in $L_0\langle y_{ij} \mid i, j = 1, 2\rangle$.
- (x_1, x_2) *induces the signature* (h_1, h_2, m).
  We use Proposition 3.22. Let a be meet-irreducible in $L_0\langle y_1, y_2\rangle$.
  If $h_i \leq a$ then $x_i \leq y_{ii} < a$ by property 4 of y_{11} and property 4 of y_{22}
  If $x_1 < a$ then $m < a$ by (7) and $m^+ = h_1 \wedge h_2 \leq a$. Let b be a meet-irreducible component of a in $L_0\langle y_1, y_2\rangle$. We claim that $h_1 \leq b$. We have that b is meet-irreducible in $L_0\langle y_1, y_2\rangle$ and $b \neq y_1, y_2$ because $x_1 < b$ and $x_1 \not< y_1, y_2$. Indeed, by equations (6), we have:

$$\begin{aligned} x_1 \to y_2 &= y_{11} \to (y_{21} \to y_2) = y_{11} \to y_{22} = y_{22} \neq 1 \\ x_1 \to y_1 &= y_{21} \to (y_{11} \to y_1) = y_{21} \to y_{12} = y_{12} \neq 1 \end{aligned} \tag{8}$$

  Suppose $h_1 \nleq b$, then by property 4 of y_{11} we would get $y_{11} \not< b$. Furthermore, $h_1 \vee y_2 \nleq b$ and by property 4 of y_{21} we would get $y_{21} \not< b$. Then b would also be meet-irreducible in $L_0\langle y_{ij} \mid i, j = 1, 2\rangle$ by Lemma 4.11. Therefore $x_1 = y_{11} \wedge y_{21} \not< b$ but this is absurd. Thus for any b meet-irreducible component of a we have $h_1 \leq b$ and hence $h_1 \leq a$. For x_2 the reasoning is analogous.

⊣

Lemma 4.13. *Let* L_0, L, (h_1, h_2, m) *and* (y_1, y_2) *as in Lemma 4.10 with* $h_2 < h_1$. *Let* $(y_{11}, y_{12}) \in L^2$ *be a primitive pair over* $L_0\langle y_1, y_2\rangle$ *inducing the signature* $(h_1, h_2 \vee y_1, y_1)$. *If* $x_1 = y_{11}$ *and* $x_2 = y_{12} \wedge y_2$, *then* (x_1, x_2) *is a primitive pair over* L_0 *inducing the signature* (h_1, h_2, m).

Proof. By equations (3), we have $y_2^+ = (h_1 \vee y_2) \wedge h_2 = h_2$ in $L_0\langle y_1, y_2\rangle$. We will refer to the properties 1, 2, 3, 4 of y_{11}, y_{12} listed at the beginning of the proof of Lemma 4.11. We have

$$x_1 \to m = x_2 \quad \text{and} \quad x_2 \to m = x_1. \tag{9}$$

Indeed,

$$\begin{aligned} x_1 \to m &= y_{11} \to (y_1 \wedge y_2) = (y_{11} \to y_1) \wedge (y_{11} \to y_2) = y_{12} \wedge y_2 = x_2 \\ x_2 \to m &= (y_{12} \wedge y_2) \to (y_1 \wedge y_2) = (y_{12} \to (y_2 \to y_1)) \wedge (y_{12} \to (y_2 \to y_2)) \\ &= y_{12} \to y_1 = y_{11} = x_1 \end{aligned}$$

We have used that $y_{11} \to y_2 = y_2$, this is proven in the same way as (4) in Lemma 4.12.

- (x_1, x_2) *is a primitive pair over* L_0.
  Equations (9) imply $m = x_1 \wedge x_2$. Moreover, if $x_1 = x_2$ then $x_1 = m = 1$ but this is absurd because $x_1, x_2 \neq 1$ since $y_{11}, y_{12}, y_2 \notin L_0$. Thus $x_1 \neq x_2$. We have $x_1, x_2 \notin L_0$ because m is meet-irreducible in L_0 and $x_1 \to m = x_2$ and $x_2 \to m = x_1$ are different from 1 and m.
  By equations (9) we have $x_1 \to m = x_2$ and $x_2 \to m = x_1$.
  It remains to show that, for any a meet-irreducible element of L_0 and $i = 1, 2$, if $m < a$ then $x_i \to a \in L_0$.
  If $m < a$ then $h_2 = h_1 \wedge h_2 = m^+ \leq a$. Thus $h_1 \leq a$ or $h_2 \leq a$. We consider these two cases separately. Suppose $h_1 \leq a$. Then $x_1 \to a = 1 \in L_0$ because $y_{11} \leq h_1 = x_1$, moreover $x_2 < y_2 \leq h_2 < h_1 \leq a$ imply $x_2 \to a = 1 \in L_0$. Suppose $h_2 \leq a$ and $h_1 \not\leq a$. Clearly $x_2 \to a = 1$ because $x_2 \leq y_2 \leq h_2 \leq a$. We want to prove that $x_1 \to a = a \in L_0$. Note that the meet-irreducible components of a in $L_0\langle y_1, y_2\rangle$ coincide with the meet-irreducible components of a in $L_0\langle y_1, y_2, y_{11}, y_{12}\rangle$. Indeed, since a is the meet of its meet-irreducible components in $L_0\langle y_1, y_2\rangle$, it is sufficient to prove that any meet-irreducible component b of a in $L_0\langle y_1, y_2\rangle$ is meet-irreducible in $L_0\langle y_1, y_2, y_{11}, y_{12}\rangle$. We have $b \neq y_1$ because $h_2 \leq b$ and $h_2 \not\leq y_1$. Indeed, $h_2 \not\leq y_1$ holds because $h_2 \leq y_1$ would imply $1 = h_2 \to y_1 = h_2 \to (y_2 \to m) = y_2 \to (h_2 \to m) = y_2 \to m = y_1$ which is absurd. Then we have that b is also meet-irreducible in $L_0\langle y_1, y_2, y_{11}, y_{12}\rangle$. This follows from the fact that (y_{11}, y_{12}) is a primitive pair over $L_0\langle y_1, y_2\rangle$ inducing the signature $(h_1, h_2 \vee y_1, y_1)$ which implies that $\mathcal{M}(L_0\langle y_1, y_2\rangle) \setminus \{y_1\} \subseteq \mathcal{M}(L_0\langle y_1, y_2, y_{11}, y_{12}\rangle)$. Since a is meet-irreducible of L_0 and $h_1 \not\leq a$ it is $h_1 \to a = a$. For any b meet-irreducible component of a in $L_0\langle y_1, y_2\rangle$ we have $h_1 \not\leq b$ because $h_1 \to a = a$ means that h_1 is not smaller than or equal to any meet-irreducible component of a. Since $h_1 \not\leq b$, by property 4 of y_{11}, we have that $x_1 = y_{11} \not< b$, therefore $x_1 \not\leq b$ because $b \neq y_{11}$ since $y_{11} \notin L_0\langle y_1, y_2\rangle$. This implies that $x_1 \to a = a$ because x_1 is not smaller than or equal to any meet-irreducible component of a in $L_0\langle y_1, y_2, y_{11}, y_{12}\rangle$.
- (x_1, x_2) *induces the signature* (h_1, h_2, m).
  We use Proposition 3.22. Let a be meet-irreducible in $L_0\langle y_1, y_2\rangle$.
  By property 4 of y_{11} we have that $h_1 \leq a$ iff $x_1 = y_{11} < a$.
  If $h_2 \leq a$ then $x_2 \leq y_2 < h_2 \leq a$ since $h_2 = y_2^+$.
  If $x_2 < a$, since $m = y_1 \wedge y_2 \leq y_{12} \wedge y_2 = x_2$, then $m < a$ and $h_2 = h_1 \wedge h_2 = m^+ \leq a$.

⊣

Theorem 4.14. *Let L be a Brouwerian semilattice satisfying the Splitting Axiom.*
Then for any finite Brouwerian sub-semilattice $L_0 \subseteq L$ and for any signature (h_1, h_2, m) of decomposition type in L_0 there exists a primitive pair $(x_1, x_2) \in L^2$ over L_0 inducing that signature.

Proof. We prove the theorem by induction on $\mathrm{H}_{L_0}(h_1, h_2)$.
Base case of induction: $\mathrm{H}_{L_0}(h_1, h_2) = 0$.
In this case we have $h_1 = h_2 = m^+$. We denote $h_1 = h_2$ by h.
Since $m \ll h$, we can apply the splitting axiom to m, h, h. Hence there exist elements $x_1, x_2 \in L$ different from 1 such that:

$$\begin{aligned} x_1 \to m = x_2 \leq h \\ x_2 \to m = x_1 \leq h. \end{aligned} \tag{10}$$

By Lemma 4.8, we have that (x_1, x_2) is a primitive pair inducing the signature (h_1, h_2, m). *Inductive step.*
Assume the statement of the theorem be true for any pair (h_1, h_2) of relative height smaller than n, we show it is true for $\mathrm{H}_{L_0}(h_1, h_2) = n$.
Since $m \ll m^+ = h_1 \wedge h_2$, we can apply the splitting axiom to m, h_1, h_2 to find $y_1, y_2 \in L$ different from 1 such that:

$$\begin{aligned} y_1 \to m = y_2 \leq h_2 \\ y_2 \to m = y_1 \leq h_1 \\ y_1 \to h_2 = h_1 \to h_2 \\ y_2 \to h_1 = h_2 \to h_1 \end{aligned} \tag{11}$$

By local finiteness, $L_0\langle y_1, y_2\rangle$ is finite and thus a Heyting algebra. By Lemma 4.10 we have

1. $(h_1,\ h_2 \vee y_1,\ y_1)$ and $(h_1 \vee y_2,\ h_2,\ y_2)$ are signatures of decomposition type in $L_0\langle y_1, y_2\rangle$, where the joins are taken inside $L_0\langle y_1, y_2\rangle$;
2. $\mathrm{ht}_{L_0\langle y_1,y_2\rangle}(h_1, h_2) = \mathrm{ht}_{L_0}(h_1, h_2)$ and $\mathrm{ht}_{L_0\langle y_1,y_2\rangle}(h_2, h_1) = \mathrm{ht}_{L_0}(h_2, h_1)$;
3. If $h_1 \not\leq h_2$ then $\mathrm{ht}_{L_0\langle y_1,y_2\rangle}(h_2 \vee y_1, h_1) < \mathrm{ht}_{L_0}(h_2, h_1)$.
   If $h_2 \not\leq h_1$ then $\mathrm{ht}_{L_0\langle y_1,y_2\rangle}(h_1 \vee y_2, h_2) < \mathrm{ht}_{L_0}(h_1, h_2)$.

We can now apply the inductive hypothesis. To do so we shall consider different cases.
First, we consider the case in which $h_1 \not\leq h_2$ and $h_2 \not\leq h_1$, i.e. h_1, h_2 are incomparable.
In this case $H_{L_0\langle y_1,y_2\rangle}(h_1, h_2 \vee y_1) < H_{L_0}(h_1, h_2)$.
Indeed, since $h_1 \vee (h_2 \vee y_1) = h_1 \vee h_2$ and so $\mathrm{ht}_{L_0\langle y_1,y_2\rangle}(h_1, h_2 \vee y_1) = \mathrm{ht}_{L_0\langle y_1,y_2\rangle}(h_1, h_2)$, we have:

$$\begin{aligned} H_{L_0\langle y_1,y_2\rangle}(h_1, h_2 \vee y_1) &= \mathrm{ht}_{L_0\langle y_1,y_2\rangle}(h_1, h_2 \vee y_1) + \mathrm{ht}_{L_0\langle y_1,y_2\rangle}(h_2 \vee y_1, h_1) \\ &= \mathrm{ht}_{L_0\langle y_1,y_2\rangle}(h_1, h_2) + \mathrm{ht}_{L_0\langle y_1,y_2\rangle}(h_2 \vee y_1, h_1) \\ &< \mathrm{ht}_{L_0}(h_1, h_2) + \mathrm{ht}_{L_0}(h_2, h_1) = H_{L_0}(h_1, h_2). \end{aligned}$$

Analogously, $H_{L_0\langle y_1,y_2\rangle}(h_1 \vee y_2, h_2) < H_{L_0}(h_1, h_2)$.
Therefore we can apply the inductive hypothesis on both the two signatures $(h_1,\ h_2 \vee y_1,\ y_1)$ and $(h_1 \vee y_2,\ h_2,\ y_2)$ considered inside $L_0\langle y_1, y_2\rangle$ to obtain two primitive pairs $(y_{11}, y_{12}) \in L^2$ and $(y_{21}, y_{22}) \in L^2$ of decomposition type over $L_0\langle y_1, y_2\rangle$ which induce the two signatures, respectively.
Let $x_1 = y_{11} \wedge y_{21}$ and $x_2 = y_{12} \wedge y_{22}$. Lemma 4.12 guarantees that (x_1, x_2) is a primitive pair of decomposition type over L_0 inducing the signature (h_1, h_2, m).
Finally, we consider the case in which h_1 and h_2 are comparable.
We assume $h_1 < h_2$. Then, as shown above, $h_2 \not\leq h_1$ implies $H_{L_0\langle y_1,y_2\rangle}(h_1 \vee$

$y_2, h_2) < H_{L_0}(h_1, h_2)$. Thus, we can apply the inductive hypothesis on the signature $(h_1 \vee y_2,\ h_2,\ y_2)$ considered inside $L_0\langle y_1, y_2\rangle$ to obtain the primitive pair $(y_{11}, y_{12}) \in L^2$ over $L_0\langle y_1, y_2\rangle$ which induces that signature. Define $x_1 = y_{11}$ and $x_2 = y_{12} \wedge y_2$. Lemma 4.13 guarantees that (x_1, x_2) is a primitive pair of decomposition type over L_0 inducing the signature (h_1, h_2, m).
The case $h_2 < h_1$ is analogous and the case $h_1 = h_2$ is considered in the base case of the induction. ⊣

4.2. Density axioms. [**Density 1 Axiom**] For every c there exists $b \neq 1$ such that $b \ll c$

THEOREM 4.15. *Any existentially closed Brouwerian semilattice satisfies the Density 1 Axiom.*

PROOF. It is sufficient to show, by Lemma 4.2, that for any finite Brouwerian semilattice L_0 and $c \in L_0$ there exists a finite extension $L_0 \subseteq L$ with $b \in L$ different from 1 such that $b \ll c$.
Let C be the upset of $\mathcal{M}(L_0)$ corresponding to c.
Let P be the poset obtained from $\mathcal{M}(L_0)$ by adding a new least element $l \in P$ such that $l \leq p$ for any $p \in \mathcal{M}(L_0)$. Let $\varphi : P \to \mathcal{M}(L_0)$ be the surjective **P**-morphism such that dom $\varphi = \mathcal{M}(L_0)$ and it is the identity on its domain. Then $\uparrow l \ll C$. Let L be the Brouwerian semilattice dual to P and $b \in L$ corresponding to $\uparrow l$. ⊣

[**Density 2 Axiom**] For every c, a_1, a_2, d such that $a_1, a_2 \neq 1$, $a_1 \ll c$, $a_2 \ll c$ and $d \to a_1 = a_1$, $d \to a_2 = a_2$ there exists an element b different from 1 such that:

$$\begin{aligned} b &\ll c \\ a_1 &\ll b \\ a_2 &\ll b \\ d \to b &= b \end{aligned}$$

THEOREM 4.16. *Any existentially closed Brouwerian semilattice satisfies the Density 2 Axiom.*

PROOF. It is sufficient to show, by Lemma 4.2, that for any finite Brouwerian semilattice L_0 and c, a_1, a_2, d such that $a_1, a_2 \neq 1$, $a_1 \ll c$, $a_2 \ll c$ and $d \to a_1 = a_1$, $d \to a_2 = a_2$ there exists a finite extension $L_0 \subseteq L$ with $b \in L$ different from 1 such that $b \ll c$, $a_1 \ll b$, $a_2 \ll b$ and $d \to b = b$.
Let C, A_1, A_2, D be the upsets of $\mathcal{M}(L_0)$ corresponding to c, a_1, a_2, d.
We proceed in two ways depending on whether C is empty or not.
If $C = \emptyset$ choose two minimal elements α^1, α^2 respectively of A_1 and A_2 and obtain a poset P by adding a new element β to P_0 and setting for any $x \in P$:

- $x \leq \beta$ iff $x = \beta$ or $x \leq \alpha^1$ or $x \leq \alpha^2$.
  If α^1, α^2 are incomparable, they become the only two predecessors of β in P, otherwise if e.g. $\alpha^1 \leq \alpha^2$ then α^2 is the only predecessor of β.
- $\beta \leq x$ iff $x = \beta$, i.e. β is maximal in P.

Define a surjective **P**-morphism $\varphi : P \to \mathcal{M}(L_0)$ taking dom $\varphi = \mathcal{M}(L_0)$ and φ acting as the identity on its domain. Take $B = \uparrow\beta$, we have:

- $B \ll \emptyset = \uparrow\varphi^{-1}(C)$,
- $A_1 \cup \{\beta\} = \uparrow\varphi^{-1}(A_1) \ll B$,
- $A_2 \cup \{\beta\} = \uparrow\varphi^{-1}(A_2) \ll B$,
- $B = \uparrow\varphi^{-1}(D) \to B$.
  Indeed, since $d \to a_1 = a_1$ and $d \to a_2 = a_2$, D does not contain any minimal element of A_1 or A_2, in particular it does not contain α^1 or α^2.

Thus, take L to be the Brouwerian semilattice dual to P and $b \in L$ corresponding to B.

If $C \neq \emptyset$ let $\gamma_1, \dots, \gamma_n$ be the minimal elements of C.
Choose for any $i = 1, \dots, n$ two minimal elements α_i^1, α_i^2 respectively of A_1 and A_2 such that $\alpha_i^1 \leq \gamma_i$ and $\alpha_i^2 \leq \gamma_i$. Notice that they exist and $\gamma_i \neq \alpha_i^1$, $\gamma_i \neq \alpha_i^2$ because $A_1 \ll C$ and $A_2 \ll C$.
Obtain a poset P by adding new elements $\beta_1, \dots, \beta_n$ to P_0 and setting for any $x \in P$:

- $x \leq \beta_i$ iff $x = \beta_i$ or $x \leq \alpha_i^1$ or $x \leq \alpha_i^2$.
  If α_i^1, α_i^2 are incomparable they become the only two predecessors of β_i in P, otherwise if e.g. $\alpha_i^1 \leq \alpha_i^2$ then α_i^2 is the only predecessor of β_i.
- $\beta_i \leq x$ iff $x = \beta$ or $\gamma_i \leq x$,
  i.e. γ_i is the unique successor of β_i in P.

Define a surjective **P**-morphism $\varphi : P \to \mathcal{M}(L_0)$ by taking dom $\varphi = \mathcal{M}(L_0)$ and φ acting as the identity on its domain.
Take $B = \uparrow\beta_1 \cup \dots \cup \uparrow\beta_n$, we have:

- $B \ll \uparrow\varphi^{-1}(C)$,
- $A_1 \cup \{\beta_1, \dots, \beta_n\} = \uparrow\varphi^{-1}(A_1) \ll B$,
- $A_2 \cup \{\beta_1, \dots, \beta_n\} = \uparrow\varphi^{-1}(A_2) \ll B$,
- $B = \uparrow\varphi^{-1}(D) \to B$.
  Indeed D does not contain any minimal element of A_1 or A_2, in particular it does not contain α_i^1 or α_i^2 for any $i = 1, \dots, n$.

Then, take L to be the Brouwerian semilattice dual to P and $b \in L$ corresponding to B. ⊣

Lemma 4.17. *Let L be a Brouwerian semilattice and $L_0 \subseteq L$ a finite Brouwerian sub-semilattice. Let $(h, \emptyset)$ be a signature of addition type in L_0 and 0_{L_0} the least element of L_0. If $1 \neq t \in L$ is such that $t \ll 0_{L_0}$, then:*

1. *$L_1 := L_0 \cup \{t\}$ is a Brouwerian sub-semilattice of L,*
2. *$(h, 0_{L_0}, t)$ is a signature of decomposition type in L_1,*
3. *If (x_1, x_2) is a primitive pair of elements of L over L_1 inducing the signature $(h, 0_{L_0}, t)$, then x_1 is a primitive element of L over L_0 inducing the signature $(h, \emptyset)$.*

Proof. We prove the result in two steps.

- *L_1 is a Brouwerian sub-semilattice of L and $(h, 0_{L_0}, t)$ is a signature of decomposition type in L_1.*
  L_1 is clearly closed under meets. It is also closed under implications. Indeed, for any $a \in L_0$ we have $t < a$ and thus $t \to a = 1$ and $t \leq a \to t \leq 0_{L_0} \to t = t$, therefore $a \to t = t$. This also shows that t is a meet-irreducible of L_1. Moreover, it is clear that the meet-irreducibles of L_1 are the meet-irreducibles of L_0 and t.
  $(h, 0_{L_0}, t)$ is a signature of decomposition type in L_1 because $h \wedge 0_{L_0} = 0_{L_0} = t^+$.

Since (x_1, x_2) is a primitive pair inducing the signature $(h, 0_{L_0}, t)$, we have the following list of properties:

1. $x_1 \neq x_2$ and $x_1, x_2 \notin L_1$,
2. $x_1 \to t = x_2$ and $x_2 \to t = x_1$

and for any c meet-irreducible of L_1:

3. if $m < c$ then $x_i \to c \in L_1$ for $i = 1, 2$,
4. $x_1 < c$ iff $h \leq c$ and $x_2 < c$ iff $0_{L_0} \leq c$.

Recall that Lemma 3.15 implies that for any $c \in L_1$:

(i) $x_i \to c \in L_1$ or $x_i \to c = b \wedge x_j$ for some $b \in L_1$ with $\{i, j\} = \{1, 2\}$.
(ii) $c \to x_i = x_i$ or $c \to x_i = 1$ for $i = 1, 2$.

- *x_1 is a primitive element of L over L_0 inducing the signature $(h, \emptyset)$.*
  $x_1 \notin L_0$ because $x_1 \notin L_1$. Let a be a meet-irreducible of L_0. Then $x_1 \to a \in L_0$. Indeed, by property 4 of x_2, it follows from $0_{L_0} \leq a$ that $x_2 < a$. Thus, by (ii) either $x_1 \to a \in L_1$ or $x_1 \to a = b \wedge x_2$ with $b \in L_1$. The latter is impossible because we would get $x_2 < a \leq x_1 \to a = b \wedge x_2 \leq x_2$. Therefore it has to be $x_1 \to a \in L_1$. Thus, $x_1 \to a \in L_0$ because $t < a \leq x_1 \to a$.
  We have that $a \to x_1 = x_1$ or $a \to x_1 = 1$ by property (ii).
  We use Proposition 3.22 to show that x_1 induces the signature $(h, \emptyset)$.
  $x_1 < a$ if and only if $h \leq a$ by property 4 of x_1. Moreover $a \not< x_1$. Indeed, if $a < x_1$ then $0_{L_0} < x_1$ and so $1 = 0_{L_0} \to x_1 = 0_{L_0} \to (x_2 \to t) = x_2 \to (0_{L_0} \to t) = x_2 \to t = x_1$ which is impossible because $x_1 \notin L_1$.

$\dashv$

Lemma 4.18. *Let L be a Brouwerian semilattice and $L_0 \subseteq L$ a finite Brouwerian sub-semilattice. Let $(h, \{m_1, \dots, m_k\})$ be a signature of addition type in L_0 with $k \geq 1$. Let $y \in L$ be a primitive element over L_0 inducing the signature $(h, \{m_1, \dots, m_{k-1}\})$. Then*

1. *(h, m_k^+, m_k) is a signature of decomposition type in $L_0\langle y\rangle$, where m_k^+ is the unique successor of m_k in $L_0\langle y\rangle$;*
2. *let (m_k', m_k'') be a primitive pair inducing the signature (h, m_k^+, m_k) and $1 \neq x \in L$ such that*

$$x \ll h,\ y \ll x,\ m_k' \ll x \text{ and } d \to x = x \tag{12}$$

   where $d = \bigwedge\{b$ meet-irreducible of L_0 s.t. $b \not\leq m_1, \dots, b \not\leq m_k\}$. Then x is a primitive element inducing the signature $(h, \{m_1, \dots, m_k\})$.

Proof. By Definition 3.10 and Proposition 3.22, we have that for any a meet-irreducible of L_0:

1. $y \notin L_0$,
2. $y \to a \in L_0$,
3. either $a \to y = y$ or $a \to y = 1$,
4. $y < a$ iff $h \leq a$, and $a < y$ iff $a \leq m_i$ for some $i = 1, \dots, k-1$.

Recall that Lemma 3.11 shows that the properties 2 and 3 actually hold for any $a \in L_0$.
Notice that m_k is still meet-irreducible in the Brouwerian sub-semilattice $L_0\langle y\rangle \subseteq L$ generated by L_0 and y since $L_0 \subseteq L_0\langle y\rangle$ is a minimal finite extension of addition type by Theorem 3.12.

- *(h, m_k^+, m_k) is a signature of addition type in $L_0\langle y\rangle$.*
  Indeed, $m_k \ll m_k^+ = h \wedge m_k^+$.

The elements $m_k', m_k'' \in L$ satisfy:

1. $m_k', m_k'' \notin L_0\langle y\rangle$ and $m_k' \neq m_k''$,
2. $m_k' \to m_k = m_k''$ and $m_k'' \to m_k = m_k'$

and for any a meet-irreducible of $L_0\langle y\rangle$:

3. if $m_k < a$ then $m_k' \to a \in L_0\langle y\rangle$ and $m_k'' \to a \in L_0\langle y\rangle$,
4. $m_k' < a$ iff $h \leq a$ and $m_k'' < a$ iff $m_k^+ \leq a$.

Observe that property 4 actually holds for any $a \in L_0\langle y\rangle$ since any element in a finite Brouwerian semilattice is the meet of meet-irreducible elements.

- *x is primitive over L_0.*
  We have $x \notin L_0$. Indeed, if $x \in L_0$ then, by property 4 of y, it would be $h \leq x$ because $y < x$. This is impossible because $x \neq 1$ and $x \ll h$.
  Let a be meet-irreducible in L_0. If $h \leq a$ then $x \to a = 1 \in L_0$ since $x \leq h$ by equations (12). If $h \not\leq a$ then by property 4 of m_k' we have $m_k' \not< a$. We consider two cases:

- If $h \not\leq a$ and $a \neq m_k$, then a is still meet-irreducible in $L_0\langle y, m'_k, m''_k\rangle$ (since $L_0\langle y\rangle \subseteq L_0\langle y, m'_k, m''_k\rangle$ is a minimal finite extension by Theorem 3.16). Hence $m'_k \to a = a$. Therefore $x \to a = a \in L_0$ because $a \leq x \to a \leq m'_k \to a = a$ since $m'_k \leq x$.
- If $a = m_k$, then $m'_k \ll x$ by equations (12) and

$$\begin{aligned} x \to m_k = x \to (m'_k \wedge m''_k) &= (x \to m'_k) \wedge (x \to m''_k) = m'_k \wedge (x \to (m'_k \to m_k)) \\ &= m'_k \wedge ((m'_k \wedge x) \to m_k) = m'_k \wedge (m'_k \to m_k) = m'_k \wedge m''_k = m_k \in L_0. \end{aligned}$$

We also have $a \to x = 1$ or $a \to x = x$. Indeed, we consider again two cases. Suppose $a \leq m_i$ for some $i = 1, \dots, k$. If $i \neq k$ then $a \leq y \leq x$ and $a \to x = 1$ by property 4 of y and equations (12). If $i = k$ then $a \leq m_k \leq m'_k \leq x$ and $a \to x = 1$. Suppose now $a \not\leq m_i$ for any $i = 1, \dots, k$ then, by definition of d, we have $d \leq a$. So $a \to x = x$ because $x \leq a \to x \leq d \to x = x$.

- *x induces the signature (h, M).*

We use Proposition 3.22.
If $x < a$, then $m'_k \leq x < a$ and thus $h \leq a$ by property 4 of m'_k.
If $h \leq a$, then $x < a$ because $x < h$ by (12).
If $a < x$ and $a \not\leq m_1, \dots, a \not\leq m_k$, then $d \leq a$ and $1 = a \to x \leq d \to x = x$ which is absurd. Thus, $m_i \leq a$ for some $i = 1, \dots, k$.
Let $a \leq m_i$ for some $i = 1, \dots, k$. If $i \neq k$, then $a \leq m_i < y < x$ because $m_i < y$ by property 4 of y. Therefore $a < x$. If $i = k$ then $a \leq m_k < m'_k < x$ and thus $a < x$.

⊣

THEOREM 4.19. *Let L be a Brouwerian semilattice satisfying the Splitting, Density 1 and Density 2 Axioms. Then for any finite Brouwerian sub-semilattice $L_0 \subseteq L$ and for any signature (h, M) of addition type in L_0 there exists a primitive element $x \in L$ over L_0 inducing that signature.*

PROOF. Let $M = \{m_1, \dots, m_k\}$, the proof is by induction on k.
Base case: $k = 0$, i.e. $M = \emptyset$.
Let 0_{L_0} be the minimum element of L_0. By Density 1 there exists $1 \neq t \in L$ such that $t \ll 0_{L_0}$.
By Lemma 4.17, $L_1 := L_0 \cup \{t\}$ is a Brouwerian sub-semilattice of L and $(h, 0_{L_0}, t)$ is a signature of decomposition type in L_1. Thanks to the Splitting Axiom, we can apply Theorem 4.14 to the signature $(h, 0_{L_0}, t)$ in L_1 and obtain the existence of a primitive pair $(x_1, x_2) \in L^2$ inducing the signature $(h, 0_{L_0}, t)$. Lemma 4.17 shows that x_1 is a primitive element over L_0 inducing the signature $(h, \emptyset)$.
Inductive step.
Assume $k \geq 1$ and that the statement of the theorem is true for any signature (h, M) with $\#M = k - 1$. By inductive hypothesis there exists a primitive element $y \in L$ over L_0 which induces the signature $(h, \{m_1, \dots, m_{k-1}\})$. By Lemma 4.18, (h, m_k^+, m_k) is a signature of decomposition type in $L_0\langle y\rangle$. Since L satisfies the Splitting Axiom, we can apply Theorem 4.14 to the signature (h, m_k^+, m_k) in $L_0\langle y\rangle$ to obtain a primitive pair (m'_k, m''_k) of elements of L inducing (h, m_k^+, m_k). We want to apply the Density 2 Axiom on h, y, m'_k, d where

$$d = \bigwedge\{b \text{ meet-irreducible of } L_0 \text{ s.t. } b \not\leq m_1, \dots, b \not\leq m_k\}.$$

We need to show that we can apply the axiom. Since y is primitive over L_0 and induces the signature $(h, \{m_1, \dots, m_{k-1}\})$, by Lemma 3.11 and Proposition 3.22, we have the following two properties of y:

1. for any $a \in L_0$, either $a \to y = y$ or $a \to y = 1$,
2. for any b a meet-irreducible of L_0, $y < b$ iff $h \leq b$, and $b < y$ iff $b \leq m_i$ for some $i = 1, \dots, k - 1$.

$y \ll h$ since $y < h$ because $h \in L_0$ and, by property 1 of y, we have $h \to y = y$.
$m'_k \ll h$ since $m'_k < h$ and $h \to m'_k = h \to (m''_k \to m_k) = m''_k \to (h \to m_k) = m''_k \to m_k = m'_k$. Notice that $h \to m_k = m_k$ because m_k is meet-irreducible in L_0 and $m_k < h$.
$d \to y = y$ because for any b meet-irreducible in L_0 such that $b \nleq m_1, \dots, b \nleq m_k$ we have $b \to y = y$. Indeed, otherwise it would be $b \to y = 1$ because y is meet-irreducible in $L_0\langle y\rangle$. So $b < y$ and then, by property 2 of y, we would have $b \leq m_i$ for some $i < k$, which is impossible.
$d \to m'_k = m'_k$. Indeed, since m_k is meet-irreducible in L_0:

$$m_k \leq d \to m_k \leq \bigwedge\{b \text{ meet-irreducible of } L_0 \text{ s.t. } b \nleq m_k\} \to m_k = m_k.$$

So $d \to m'_k = d \to (m''_k \to m_k) = m''_k \to (d \to m_k) = m''_k \to m_k = m'_k$.
Then, by the Density 2 Axiom, there exists $1 \neq x \in L$ such that

$$x \ll h,\ y \ll x,\ m'_k \ll x \text{ and } d \to x = x.$$

Lemma 4.18 shows that x is a primitive element of L inducing the signature (h, M). ⊣

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