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Stationary nonlinear Schrödinger equations in $\mathbb{R}^2$ with potentials vanishing at infinity

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Abstract We deal with a class of 2-D stationary nonlinear Schrödinger equations (NLS) involving potentials V and weights Q decaying to zero at infinity as $(1+|x|^\alpha)^{-1}$, $\alpha \in (0,2)$, and $(1+|x|^\beta)^{-1}$, $\beta \in (2,+\infty)$, respectively, and nonlinearities with exponential growth of the form $\exp \gamma_0 s^2$ for some $\gamma_0 > 0$. Working in weighted Sobolev spaces, we prove the existence of a bound state solution, i.e. a solution belonging to $H^1(\mathbb{R}^2)$. Our approach is based on a weighted Trudinger–Moser-type inequality and the classical mountain pass theorem.

Keywords Nonlinear Schrödinger equation · Bound state · Vanishing potentials · Trudinger–Moser inequality · Exponential growth

Mathematics Subject Classification 35J91 · 35A23 · 35J20

1 Introduction

This paper concerns the existence of solutions of stationary nonlinear Schrödinger equations of the form

$$-\Delta u + V(x)u = Q(x)f(u) \quad \text{in } \mathbb{R}^2 \tag{NLS}$$

in the case when the potential V and the weight Q decay to zero at infinity as $(1+|x|^\alpha)^{-1}$ with $\alpha \in (0,2)$ and $(1+|x|^\beta)^{-1}$ with $\beta \in (2,+\infty)$, respectively, and the nonlinear term $f = f(s)$ has exponential growth.

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Equation (NLS) is a particular case of the following more general class of two-dimensional problems

$$-\Delta u + V(x)u = g(x, u) \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

where $V = V(x)$ is positive and $g = g(x, s)$ has exponential growth at infinity with respect to the variable s, i.e.

$$\lim_{|s|\to+\infty} \frac{|g(x,s)|}{e^{\gamma s^2}} = \begin{cases} 0 & \text{if } \gamma > \gamma_0, \\ +\infty & \text{if } \gamma < \gamma_0, \end{cases}$$

for some $\gamma_0 \geq 0$.

We mention that, for bounded domains $\Omega \subset \mathbb{R}^2$ and nonlinear terms $g = g(x, s)$ with exponential growth at infinity, a lot of work has been devoted to the study of corresponding elliptic equations of the form

$$\begin{cases} -\Delta u = g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We limit ourselves to refer the reader to the following papers [2,3,17,19–22,38].

1.1 Potentials bounded away from zero

In the last decades, considerable attention has been paid to the study of equations of the form (1.1), under various assumptions on the potential V. However, to our knowledge, it is everywhere assumed (with the only exception of [5,27]) that V is bounded away from zero by a positive constant, that is

(V_0) there exists $V_0 > 0$ such that $V(x) \geq V_0$ for any $x \in \mathbb{R}^2$

Assuming, in addition to (V_0),

$$\frac{1}{V} \in L^1(\mathbb{R}^2) \tag{1.2}$$

or

$$\lim_{|x|\to+\infty} V(x) = +\infty \tag{1.3}$$

results concerning the existence of solutions for problem (1.1) can be found in [4,6,24,25,30,40,41]. While, in the case when the potential V is constant

$$V(x) = V_0 \quad x \in \mathbb{R}^2$$

the results available in the literature are [7,16,26,28,29,37].

It is important to point out that (V_0) ensures that the natural space for a variational study of (1.1) is a *complete* subspace E of the classical Sobolev space $H^1(\mathbb{R}^2)$, more precisely

$$E := \left\{ u \in H^1(\mathbb{R}^2) \,\middle|\, \int_{\mathbb{R}^2} V(x)u^2 \,\mathrm{d}x < +\infty \right\}$$

and $E = H^1(\mathbb{R}^2)$ if the potential V is constant. Besides this property of the function space setting, there is a main difference between the above-mentioned classes of problems distinguished by the behavior of the potential at infinity: When the potential V is large at infinity [i.e. (1.2) or (1.3) holds], one gains compact embeddings of the subspace E of $H^1(\mathbb{R}^2)$ in L^p-spaces, while when V is constant, one has to deal with the loss of compact embeddings in $L^p(\mathbb{R}^2)$ given by the unboundedness of the domain $\mathbb{R}^2$.

1.2 Vanishing potentials

The new aspect of the present paper is that we will consider a class of *positive* potentials vanishing at infinity, i.e.,

$$\lim_{|x|\to+\infty} V(x) = 0.$$

Starting from the work by Ambrosetti et al. [8], various types of stationary nonlinear Schrödinger equations involving decaying potentials at infinity have been studied in the higher-dimensional case $N \geq 3$, and we refer the reader to [9–11,13–15,31,32,39] and the references therein, even if these references are far to be exhaustive.

In particular, the analysis developed in [9,13–15,31,39] covers also the two-dimensional case but for nonlinearities with polynomial growth at infinity (more precisely, $g(x, s) = Q(x)s^p$) or asymptotically linear growth. Moreover, with the only exception of [31,39], these results for the 2-D case concern the study of semiclassical states of (3.1). If we replace the operator $-\Delta$ by $-\varepsilon^2\Delta$ in (1.1), then a *semiclassical state* u_ε is a solution with $\varepsilon << 1$ and the authors of [9,13–15] constructed semiclassical states concentrating on some set S (i.e. tending uniformly to zero as $\varepsilon \downarrow 0$, outside of a neighborhood of S) by means of the Lyapunov–Schmidt reduction method or penalization schemes.

As already mentioned, the only results available in the literature for 2-D stationary nonlinear Schrödinger equations with vanishing potentials and exponential growth nonlinearities are [5,27], see Remark 2.2.

2 Main result

Inspired by Ambrosetti et al. [8], we will study the existence of solutions of (NLS) under the following growth conditions on the potential V and the weight Q:

(V) $V \in \mathcal{C}(\mathbb{R}^2)$, there exist $\alpha, a, A > 0$ such that

$$\frac{a}{1+|x|^\alpha} \leq V(x) \leq A$$

and $V(x) \sim |x|^{-\alpha}$ as $|x| \to +\infty$;
(Q) $Q \in \mathcal{C}(\mathbb{R}^2)$, there exist $\beta, b > 0$ such that

$$0 < Q(x) \leq \frac{b}{1+|x|^\beta}$$

and $Q(x) \sim |x|^{-\beta}$ as $|x| \to +\infty$.

In particular, we restrict our attention to the case when α and β satisfy

$$\alpha \in (0,2) \quad \text{and} \quad \beta \in (2,+\infty). \tag{2.1}$$

This choice will be motivated in Sect. 3, more precisely see Theorem 3.1 and Remark 3.4, and it is strictly related with the variational structure of (NLS). In fact, we aim to develop a variational approach to study the existence of solutions of (NLS) via the classical mountain pass theorem. To this aim, we will frame the variational study of (NLS) in the weighted space

$$H^1_V(\mathbb{R}^2) := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^2) \;\middle|\; |\nabla u| \in L^2(\mathbb{R}^2) \text{ and } \int_{\mathbb{R}^2} V(x)u^2\,dx < +\infty \right\}$$

that will be discussed in some details in Sect. 3. When (V) and (Q) hold with $\alpha \in (0, 2)$ and $\beta \in [2, +\infty)$, it turns out that functions belonging to $H^1_V(\mathbb{R}^2)$ satisfy the following weighted exponential integrability condition with weight Q

$$\int_{\mathbb{R}^2} Q(x)(e^{\gamma u^2} - 1)\,\mathrm{d}x < +\infty \quad \text{for any} \;\; u \in H^1_V(\mathbb{R}^2),\; \gamma > 0,$$

and, in Sect. 4, we will obtain the corresponding uniform inequality of Trudinger–Moser type. This motivates the choice of nonlinear terms $f = f(s)$ with exponential growth at infinity of the form $e^{\gamma_0 s^2}$ for some $\gamma_0 > 0$, i.e. there exists $\gamma_0 > 0$ such that

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\gamma s^2}} = \begin{cases} 0 & \text{if } \gamma > \gamma_0, \\ +\infty & \text{if } \gamma < \gamma_0. \end{cases} \tag{f_0}$$

We point out that, due to the presence of a potential V and a weight Q satisfying (V) and (Q), this is the maximal growth which can be treated variationally in the space $H^1_V(\mathbb{R}^2)$ (see Theorem 4.1).

We also assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $f(0) = 0$ and

- there exists $\mu > 2$ such that

$$0 < \mu F(s) := \mu \int_0^s f(t)\,\mathrm{d}t \le s f(s) \quad \text{for any} \;\; s \in \mathbb{R}\setminus\{0\}, \tag{f_1}$$

- there exist $s_0, M_0 > 0$ such that

$$0 < F(s) \le M_0|f(s)| \quad \text{for any} \;\; |s| \ge s_0. \tag{f_2}$$

To guarantee that the mountain pass level is inside the region of compactness of Palais–Smale sequences, we assume an additional growth condition on the nonlinearity f. In particular, we will consider two different type of growth conditions. The first one prescribes an asymptotic behavior at infinity, more precisely

$$\liminf_{|s|\to+\infty} \frac{s f(s)}{e^{\gamma_0 s^2}} = \beta_0 > \mathcal{M}, \tag{f_3}$$

where

$$\mathcal{M} = \mathcal{M}(V, Q) := \inf_{r>0} \frac{4e^{\frac{1}{2}r^2 V_{\max,r}}}{\gamma_0 r^2 Q_{\min,r}}, \quad V_{\max,r} := \max_{|x|\le r} V(x) > 0 \quad \text{and}$$
$$Q_{\min,r} := \min_{|x|\le r} Q(x) > 0.$$

We recall that this condition was introduced in [2] and then refined in [21].

Remark 2.1 It is easy to see that if

$$V(x) = \frac{a}{1 + |x|^\alpha} \quad \text{and} \quad Q(x) = \frac{b}{1 + |x|^\beta}$$

with $\alpha \in (0, 2)$ and $\beta \in (2, +\infty)$ then $\mathcal{M} > 0$. A typical example of nonlinear term satisfying (f_0)–(f_3) is

$$f_\lambda(s) := \lambda s(e^{\gamma_0 s^2} - 1) \quad s \in \mathbb{R} \tag{2.2}$$

with $\lambda > 0$ and $\gamma_0 > 0$.

The second growth condition that we will take into account was introduced in [16] and prescribes the growth of f near the origin:

$$\text{there exists } \ p > 2 \text{ such that } F(s) \geq \frac{\lambda}{p}\,|s|^p \quad \text{for any } \ s \in \mathbb{R}, \tag{f_3'}$$

where

$$\lambda > \left(\frac{\gamma_0}{4\pi}\,\frac{p-2}{p}\right)^{\frac{p-2}{2}} S_{p,V,Q}^{p/2}$$

and

$$S_{p,V,Q} := \inf_{u \in H^1_V(\mathbb{R}^2)\setminus\{0\}} \frac{\int_{\mathbb{R}^2}[\,|\nabla u|^2 + V(x)u^2\,]\,\mathrm{d}x}{\left(\int_{\mathbb{R}^2} Q(x)|u|^p\,\mathrm{d}x\right)^{2/p}}.$$

Note that $S_{p,V,Q} > 0$ for any $p \geq 2$, see Theorem 3.1. An example of nonlinear term satisfying (f_0)–(f_2) and (f_3') is given by the function f_λ defined by (2.2) provided $\lambda > 0$ is sufficiently large. As pointed out in [41, Proposition 2.9], there exist continuous functions such that (f_0)–(f_2) and (f_3') are satisfied but (f_3) is *not* satisfied.

Our main result is the following

Theorem 2.1 *Assume (V) and (Q) hold with α and β in the range (2.1). Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(0) = 0$, (f_0), (f_1) and (f_2). If in addition either (f_3) or (f_3') holds then equation (NLS) admits a nontrivial mountain pass solution $u_0 \in H^1_V(\mathbb{R}^2)$.*

Remark 2.2 We recall that Fei and Yin [27] studied concentration properties of *semiclassical states* of (NLS) in the case when $f(u) := |u|^{p-2}ue^{\gamma_0 u^2}$ with $p > 2$ and $\gamma_0 > 0$, i.e.

$$-\varepsilon^2\,\Delta u + V(x)u = Q(x)|u|^{p-2}ue^{\gamma_0 u^2} \quad \text{in } \mathbb{R}^2, \tag{2.3}$$

under more general assumptions on V and Q. More precisely, it is just required that

- $V(x) \geq \dfrac{a}{1+|x|^2}$ and $Q(x) \leq b(1+|x|^\beta)$ with a, b and $\beta > 0$
- or $V(x) \geq \dfrac{a}{1+|x|^\alpha}$ and $Q(x) \leq be^{\beta|x|^{(2-\alpha)/2}}$ with $a, b, \beta > 0$ and $\alpha \in (0, 2)$

provided there exists a smooth bounded domain $\Lambda \subset \mathbb{R}^2$ such that

$$\max_{x\in\overline{\Lambda}} \frac{V(x)}{Q(x)} < \frac{4\pi}{\gamma_0}^{\frac{p}{2}-1} S_p^{-\frac{p}{2}}, \quad \text{where} \quad S_p := \inf_{u\in H^1(\mathbb{R}^2)\setminus\{0\}} \frac{\int_{\mathbb{R}^2}[\,|\nabla u|^2 + u^2\,]\,\mathrm{d}x}{\left(\int_{\mathbb{R}^2}|u|^p\,\mathrm{d}x\right)^{2/p}},$$

and the ground energy functional associated to the limit problem has local minimum points.

In this framework, the authors of [27] constructed semiclassical states u_ε of (2.3) belonging to $H^1(\mathbb{R}^2)$ and concentrating around some point $x_0 \in \Lambda$ by means of a penalization method. However, it should be pointed out that the existence result in [27] only works for $\varepsilon < 1$ sufficiently small. In the present work, we consider the case $\varepsilon = 1$ and, in fact, we can deal with any fixed $\varepsilon > 0$.

More recently, Albuquerque et al. [5] considered the existence of *radial* solutions of (NLS) when the nonlinear term f has exponential growth at infinity (i.e. f satifies (f_0)) and, V and Q are unbounded or decaying *radial* potentials. Besides the restriction to the *radial* case, the growth conditions on V and Q in [5] are less restrictive than (V) and (Q) with α and β in the range (2.1), but a rigorous interpretation of the function space setting considered in

[5] is needed (see for instance Remark 3.1). With the help of a weighted Trudinger–Moser inequality for *radial* functions, the authors in [5] obtained the existence of a positive *radial* solution in $H^1(\mathbb{R}^2)$ with exponential decay outside of a neighborhood of the origin.

Note that here, we do not require V and Q to be radial and, the vanishing behavior of V seems to prevent a reduction of the problem to the radial case.

Of particular interest are solutions of (NLS) which have finite L^2-norm, i.e. *bound state* solutions. The mountain pass solution $u_0 \in H^1_V(\mathbb{R}^2)$ obtained in Theorem 2.1 is a weak solution of (NLS) in the sense that

$$\int_{\mathbb{R}^2} (\nabla u_0 \cdot \nabla v + V(x) u_0 v)\, \mathrm{d}x - \int_{\mathbb{R}^2} Q(x) f(u_0) v\, \mathrm{d}x = 0 \quad \text{for any } v \in H^1_V(\mathbb{R}^2) \tag{2.4}$$

and we will show that $u_0 \in L^2(\mathbb{R}^2)$, hence $u_0 \in H^1(\mathbb{R}^2)$. In fact, we will prove that any weak solution in the sense expressed by (2.4) is a bound state solution of (NLS).

Proposition 2.2 *Assume* (V) *and* (Q) *hold with* α *and* β *in the range* (2.1)*. Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying* $f(0) = 0$*,* (f_0)*,* (f_1) *and* (f_2)*. If* (NLS) *admits a weak solution* $u_0 \in H^1_V(\mathbb{R}^2)$ *(i.e.* u_0 *satisfies* (2.4)*) then* $u_0 \in L^2(\mathbb{R}^2)$ *and hence* $u_0 \in H^1(\mathbb{R}^2)$*.*

2.1 Open question

Assume V, Q and f satisfy the assumptions of Theorem 2.1. The arguments of the proofs of Theorem 2.1 and Proposition 2.2 can be easily adapted to obtain, for any $\varepsilon > 0$, the existence of a nontrivial mountain pass solution $u_\varepsilon \in H^1_V(\mathbb{R}^2)$ of the problem

$$-\varepsilon^2 \Delta u_\varepsilon + V(x) u_\varepsilon = Q(x) f(u_\varepsilon) \quad \text{in } \mathbb{R}^2,$$

and $u_\varepsilon \in H^1(\mathbb{R}^2)$. To study the concentration behavior of such solutions $\{u_\varepsilon\}_{\varepsilon>0}$ when $\varepsilon \downarrow 0$, some sharp pointwise decay estimates and appropriate bounds of the energy are needed, *uniformly* with respect to $\varepsilon > 0$. This problem is still unsolved.

2.2 Notations

Let $w : \mathbb{R} \to [0, +\infty)$ be a weight function, we denote by $L^p_w(\mathbb{R}^2)$ with $p \in [1, +\infty]$ the corresponding weighted L^p-space, i.e. $L^p_w(\mathbb{R}^2)$ is the space consisting of all measurable functions $u : \mathbb{R}^2 \to \mathbb{R}$ with

$$\int_{\mathbb{R}^2} w(x)|u|^p\, \mathrm{d}x < +\infty \quad \text{when } p \in [1, +\infty)$$

and

$$\inf\{\, C \geq 0 \mid w(x)|u(x)| \leq C \text{ a.e. in } \mathbb{R}^2 \,\} < +\infty \quad \text{when } p = +\infty.$$

We also denote by $B(x, R) \subset \mathbb{R}^2$ the closed ball of radius $R > 0$ centered at $x \in \mathbb{R}^2$ and, to simplify notations, we set

$$B_R := B(0, R) \quad \text{and} \quad B_R^{\mathrm{c}} := \mathbb{R}^2 \setminus B_R.$$

3 The functional space setting

In order to develop a variational approach to study the existence of solutions of (NLS), a key step is to identify a suitable function space setting. Since we are interested in vanishing potentials at infinity, this basic step turns out to be a priori not obvious. The difficulty is due to the peculiar features of the two-dimensional case and can be seen comparing our situation with the higher-dimensional case $N \geq 3$. In fact, let us consider a nonlinear Schrödinger equation of the form

$$-\Delta u + V(x)u = g(x, u) \quad \text{in } \mathbb{R}^N,\ N \geq 2, \tag{3.1}$$

where $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a suitable nonlinear term and V is continuous, positive and vanishing at infinity, i.e.

$$V \in \mathcal{C}(\mathbb{R}^N), \quad V > 0 \text{ in } \mathbb{R}^N \quad \text{and} \quad V(x) \to 0 \text{ as } |x| \to +\infty. \tag{3.2}$$

Since we deal with a potential V which decays to zero at infinity, the variational theory in $H^1(\mathbb{R}^N)$ cannot be used. Moreover, under the above conditions (3.2) on V, the space

$$\left\{ u \in H^1(\mathbb{R}^N) \,\middle|\, \int_{\mathbb{R}^N} V(x)u^2 \,\mathrm{d}x < +\infty \right\}$$

endowed with the norm

$$\|u\|^2 := \|\nabla u\|_2^2 + \int_{\mathbb{R}^N} V(x)u^2 \,\mathrm{d}x$$

is not complete in general. In the higher-dimensional case $N \geq 3$, this leads to frame the variational study of problem (3.1) in the space

$$H_V^1(\mathbb{R}^N) := \mathcal{D}^{1,2}(\mathbb{R}^N) \cap L_V^2(\mathbb{R}^N),\ N \geq 3,$$

which is a Banach space with respect to the norm $\|\cdot\|$.

Remark 3.1 The situation in the two-dimensional case is more delicate, due to the fact that the completion $\mathcal{D}^{1,2}(\mathbb{R}^2)$ of the space of smooth compactly supported functions with respect to the Dirichlet norm $\|\nabla \cdot\|_2$ is not directly comparable with the space $L_V^2(\mathbb{R}^2)$ and it does not make sense to consider the intersection

$$\mathcal{D}^{1,2}(\mathbb{R}^2) \cap L_V^2(\mathbb{R}^2),$$

unless a rigorous interpretation is specified.

In analogy with the higher-dimensional case, when $N = 2$, the natural framework for a variational approach of problem (3.1) is given by the space

$$H_V^1(\mathbb{R}^2) := \left\{ u \in L_V^2(\mathbb{R}^2) \,\middle|\, |\nabla u| \in L^2(\mathbb{R}^2) \right\}.$$

Actually, $H_V^1(\mathbb{R}^2)$ endowed with the norm

$$\|u\|^2 := \|\nabla u\|_2^2 + \int_{\mathbb{R}^2} V(x)u^2 \,\mathrm{d}x \tag{3.3}$$

is a Banach space. In fact, as a consequence of (3.2), we have

$$H_V^1(\mathbb{R}^2) \hookrightarrow H_{\mathrm{loc}}^1(\mathbb{R}^2)$$

and this continuous embedding, together with the definition of Cauchy sequences and Fatou Lemma, enables to show that $(H^1_V(\mathbb{R}^2), \|\cdot\|)$ is complete. Note also that the norm $\|\cdot\|$ comes from the inner product

$$< u, v >:= \int_{\mathbb{R}^2} \big[\nabla u \cdot \nabla v + V(x)uv\big]\,\mathrm{d}x. \tag{3.4}$$

Remark 3.2 If $V \in L^1(\mathbb{R}^2)$ then any constant function $u \equiv c$ in $\mathbb{R}^2$, with $c \in \mathbb{R}$, belongs to $H^1_V(\mathbb{R}^2)$. However, under the assumption (V) and since $\alpha \in (0, 2)$, our potential $V \notin L^1(\mathbb{R}^2)$ and in this case the only constant function that belongs to $H^1_V(\mathbb{R}^2)$ is the trivial one, i.e. $u \equiv 0$ in $\mathbb{R}^2$.

In conclusion, we frame the variational study of (NLS) in the Hilbert space $H^1_V(\mathbb{R}^2)$ with inner product $< \cdot, \cdot >$ and norm $\|\cdot\|$ given respectively by (3.4) and (3.3).

Remark 3.3 In view of (V), the potential V is positive and uniformly bounded on $\mathbb{R}^2$, therefore we have

$$H^1(\mathbb{R}^2) \hookrightarrow H^1_V(\mathbb{R}^2).$$

Moreover, the space $\mathcal{C}^\infty_0(\mathbb{R}^2)$ of smooth compactly supported functions is dense in $(H^1_V(\mathbb{R}^2), \|\cdot\|)$. This can be proved by standard arguments and using, for instance, the property

$$\lim_{|x|\to+\infty} |x|^2 V(x) > 0$$

which follows directly from (V) and the range of α given by (2.1).

Similarly to the higher-dimensional case $N \geq 3$, the vanishing behavior of the potential V (i.e. $V(x) \to 0$ as $|x| \to +\infty$) implies that

$$H^1_V(\mathbb{R}^2) \not\hookrightarrow L^p(\mathbb{R}^2) \quad \text{for any } \ p \in [1, +\infty]. \tag{3.5}$$

As a consequence, this rules out exponential integrability and hence any kind of Trudinger–Moser-type inequality on $H^1_V(\mathbb{R}^2)$, unless one introduces some suitable weight in the target space. This remark justifies the choice a nonlinear term of the form

$$g(x, u) := Q(x) f(u)$$

in equation (NLS). In fact, for a variational study of (NLS) in the function space $H^1_V(\mathbb{R}^2)$, some suitable integrability condition on the nonlinearity is needed: the validity of (3.5) leads to introduce a weight $Q(x)$ and look for appropriate assumptions on $Q(x)$ in such a way that

$$H^1_V(\mathbb{R}^2) \not\hookrightarrow L^p_Q(\mathbb{R}^2) \tag{3.6}$$

at least for some $p \geq 1$. In particular, the vanishing behavior of Q given by assumption (Q) guarantees that the embeddings

$$L^p_Q(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2) \quad \text{for any } \ p \in [1, +\infty) \tag{3.7}$$

do *not* hold. Note that, in view of (3.5), the validity of (3.7) would be against the embedding (3.6).

The embedding (3.6) is a particular case of embeddings of weighted spaces discussed in [34], where the following result is proved.

Theorem 3.1 ([34], Example 20.6) *Suppose that* (V) *and* (Q) *hold with* $\alpha \in (0, 2]$ *and* $\beta \in [2, +\infty)$. *Then*

$$H^1_V(\mathbb{R}^2) \hookrightarrow L^p_Q(\mathbb{R}^2) \quad \textit{for any} \;\; p \in [2, +\infty) \tag{3.8}$$

and there exists $C_p > 0$ *such that*

$$\int_{\mathbb{R}^2} Q(x)|u|^p \,\mathrm{d}x \leq C_p \|u\|^p \quad \textit{for any} \;\; u \in H^1_V(\mathbb{R}^2).$$

Moreover, if $\beta \neq 2$ *then the above embeddings are compact.*

Note that if $V(x) \sim (1 + |x|^\alpha)^{-1}$ with $\alpha \in (0, 2]$ and $Q(x) \sim (1 + |x|^\beta)^{-1}$ then the growth restriction $\beta \in [2, +\infty)$ on the weight Q is a necessary condition for the embedding (3.8), as proved in [34].

Remark 3.4 If (V) and (Q) hold with $\alpha \in (0, 2]$ and $\beta = 2$ then the embeddings (3.8) are continuous but *not* compact. For this reason, we can say that the case $\beta = 2$ should correspond to the *critical* case. Since we confine our attention to the study of problem (NLS) when (V) and (Q) hold with α and β satisfying (2.1), in particular $\beta \neq 2$ and in this respect problem (NLS) can be seen as *subcritical*. Note also that assuming (2.1), we also require that $\alpha \neq 2$: this is just a technical restriction due to the method of proof that we use to obtain the corresponding weighted Trudinger–Moser inequality (see Sect. 4).

In view of Theorem 3.1 and Remark 3.4, in what follows, we will assume that (V) and (Q) hold with α and β satisfying (2.1). In this framework, since

$$H^1_0(B_1) \hookrightarrow H^1(\mathbb{R}^2) \hookrightarrow H^1_V(\mathbb{R}^2), \tag{3.9}$$

we infer that

$$H^1_V(\mathbb{R}^2) \not\hookrightarrow L^\infty_Q(\mathbb{R}^2).$$

In fact, it is well known that there exists $\overline{u} \in H^1_0(B_1)$ such that $\overline{u} \notin L^\infty(B_1)$. Therefore, $\overline{u} \in H^1_V(\mathbb{R}^2)$ but $\overline{u} \notin L^\infty_Q(\mathbb{R}^2)$ and, it is natural to look for a weighted Trudinger–Moser inequality on $H^1_V(\mathbb{R}^2)$. Due to the embedding (3.9) and the uniform boundedness of the weight Q, it turns out to be reasonable to consider an exponential growth function ϕ of the form

$$\phi(t) := e^{\gamma t^2} - 1, \quad \gamma > 0.$$

4 A subcritical Trudinger–Moser-type inequality in weighted spaces

In this Section we will prove the following weighted Trudinger–Moser inequality on the space $(H^1_V(\mathbb{R}^2), \|\cdot\|)$

Theorem 4.1 *Suppose that* (V) *and* (Q) *hold with* $\alpha \in (0, 2)$ *and* $\beta \in [2, +\infty)$. *For any* $\gamma > 0$ *and any* $u \in H^1_V(\mathbb{R}^2)$, *we have*

$$\int_{\mathbb{R}^2} Q(x)\big(e^{\gamma u^2} - 1\big) \,\mathrm{d}x < +\infty. \tag{4.1}$$

Moreover, if we consider the supremum

$$S_\gamma = S_\gamma(V, Q) := \sup_{u \in H^1_V(\mathbb{R}^2),\ \|u\| \le 1} \int_{\mathbb{R}^2} Q(x)\big(e^{\gamma u^2} - 1\big)\,\mathrm{d}x, \quad \gamma > 0,$$

then, for any $\gamma \in (0, 4\pi)$*, there exists a constant* $C = C(\gamma, V, Q) > 0$ *such that*

$$S_\gamma \le C \tag{4.2}$$

and

$$S_\gamma = +\infty \quad \text{for any } \gamma > 4\pi. \tag{4.3}$$

Note that the inequality that we obtain is *subcritical*, in the sense that the range of the exponent is the open interval $(0, 4\pi)$. This is essentially due to the technical difficulties arising from the decay of the potential V at infinity. In fact, the vanishing behavior of V seems to prevent a reduction of the problem to radial case. For instance, it is not possible to apply classical symmetrization methods and this forces to look for a rearrangement-free argument.

Even if our proof does not cover the *critical* case $\gamma = 4\pi$, the *subcritical* inequality expressed by Theorem 4.1 will enable us to obtain the existence of a nontrivial solution for the nonlinear Schrödinger equation (NLS).

To prove (4.2), we will combine the ideas of Kufner and Opic [34] with the argument by Yang and Zhu [42]. More precisely, we will obtain the desired uniform estimate by means of a suitable covering lemma and the classical Trudinger–Moser inequality on balls, i.e.

Theorem 4.2 ([33]) *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain. There exists a constant* $C > 0$ *such that*

$$\sup_{u \in H^1_0(\Omega),\ \|\nabla u\|_2 \le 1} \int_\Omega e^{\alpha u^2}\,\mathrm{d}x \begin{cases} \le C|\Omega| & \text{if } 0 < \alpha \le 4\pi, \\ = +\infty & \text{if } \alpha > 4\pi. \end{cases} \tag{4.4}$$

In particular, inspired by Yang et al. [42], we will mainly make use of the following local estimate that can be derived directly from (4.4) with the aid of the scaling $\tilde{u} := u/\|\nabla u\|_2$

Lemma 4.3 ([42], Lemma 2.1) *There exists a constant* $C > 0$ *such that for any* $y \in \mathbb{R}^2$*,* $R > 0$ *and any* $u \in H^1_0(B(y, R))$ *with* $\|\nabla u\|_2 \le 1$*, we have*

$$\int_{B(y,R)} (e^{4\pi u^2} - 1)\,\mathrm{d}x \le C R^2 \int_{B(y,R)} |\nabla u|^2\,\mathrm{d}x. \tag{4.5}$$

In view of the fact that V and Q are bounded away from zero by positive constants on compact subsets of $\mathbb{R}^2$, the sharpness (4.3) is a direct consequence of the sharpness of the following Trudinger–Moser inequality due to Ruf [36] (see also [4] and [18, Remark 6.1]; in addition, we refer to [1] for a scale invariant form of the result in [36]).

Theorem 4.4 ([36]) *Let* $\Omega \subseteq \mathbb{R}^2$ *be a domain (possibly unbounded) and let* $\tau > 0$*. For any* $\gamma \in [0, 4\pi]$ *there exists a constant* $C_\tau > 0$ *such that*

$$R_\gamma(\tau, \Omega) := \sup_{u \in H^1_0(\Omega),\ \|\nabla u\|_2^2 + \tau \|u\|_2^2 \le 1} \int_\Omega (e^{\gamma u^2} - 1)\,\mathrm{d}x \le C_\tau$$

and the above inequality is sharp, i.e.

$$R_\gamma(\tau, \Omega) = +\infty \quad \text{for any } \gamma > 4\pi.$$

First, we set

$$\tilde{V} := \max_{x\in B_1} V(x) \quad \text{and} \quad \tilde{Q} := \min_{x\in B_1} Q(x).$$

Since V and Q are continuous and positive, we have that $\tilde{V}, \tilde{Q} > 0$. Therefore recalling (3.9), we may estimate

$$S_\gamma \geq \sup_{u\in H_0^1(B_1),\ \|u\|\leq 1} \int_{\mathbb{R}^2} Q(x)\big(e^{\gamma u^2}-1\big)\,\mathrm{d}x \geq \tilde{Q} \sup_{u\in H_0^1(B_1),\ \|u\|\leq 1} \int_{B_1} (e^{\gamma u^2}-1)\,\mathrm{d}x.$$

Inasmuch as

$$\|u\|^2 = \|\nabla u\|_2^2 + \int_{B_1} V(x)u^2\,\mathrm{d}x \leq \|\nabla u\|_2^2 + \tilde{V}\|u\|_2^2 \quad \text{for any } u\in H_0^1(B_1),$$

we get

$$S_\gamma \geq \tilde{Q} \sup_{u\in H_0^1(B_1),\ \|\nabla u\|_2^2+\tilde{V}\|u\|_2^2\leq 1} \int_{B_1} (e^{\gamma u^2}-1)\,\mathrm{d}x = R_\gamma(\tilde{V}, B_1).$$

Hence, for $\gamma > 4\pi$, we have

$$S_\gamma \geq R_\gamma(\tilde{V}, B_1) = +\infty.$$

Next, we will derive (4.1) from (4.2) whose proof will be carried out in essentially two steps. In what follows, $\gamma\in(0, 4\pi)$ is fixed and we set

$$\gamma = 4\pi(1-\varepsilon)$$

for a suitable $\varepsilon\in(0,1)$.

4.1 Uniform estimate on a large ball

Let $u\in H_V^1(\mathbb{R}^2)$ be such that $\|u\|\leq 1$ and let us estimate

$$\int_{B_R} Q(x)\big(e^{\gamma u^2}-1\big)\,\mathrm{d}x$$

for some $R>0$ to be chosen during the proof independently of u. First, note that using (Q) we have

$$\int_{B_R} Q(x)\big(e^{\gamma u^2}-1\big)\,\mathrm{d}x \leq b\int_{B_R} (e^{\gamma u^2}-1)\,\mathrm{d}x = b\int_{B_R} (e^{4\pi(1-\varepsilon)u^2}-1)\,\mathrm{d}x.$$

Next, we follow the argument in [42] and we introduce a cutoff function $\varphi\in\mathcal{C}_0^\infty(B_{2R})$ such that

$$0\leq\varphi\leq 1 \text{ in } B_{2R}, \quad \varphi\equiv 1 \text{ in } B_R \quad \text{and} \quad |\nabla\varphi|\leq\frac{C}{R} \text{ in } B_{2R}$$

for some universal constant $C>0$. Then $\varphi u\in H_0^1(B_{2R})$ and by Young's inequality

$$\begin{aligned}\int_{B_{2R}} |\nabla(\varphi u)|^2\,\mathrm{d}x &\leq (1+\varepsilon)\int_{B_{2R}} \varphi^2|\nabla u|^2\,\mathrm{d}x + \left(1+\frac{1}{\varepsilon}\right)\int_{B_{2R}} |\nabla\varphi|^2 u^2\,\mathrm{d}x\\ &\leq (1+\varepsilon)\int_{B_{2R}} |\nabla u|^2\,\mathrm{d}x + \left(1+\frac{1}{\varepsilon}\right)\frac{C^2}{R^2}\int_{B_{2R}} u^2\,\mathrm{d}x.\end{aligned}$$

In view of (V)

$$V(x) \geq \frac{a}{1+|x|^\alpha} \geq \frac{a}{1+(2R)^\alpha}\,,$$

and hence

$$\int_{B_{2R}} |\nabla(\varphi u)|^2\,\mathrm{d}x \leq (1+\varepsilon)\int_{B_{2R}} |\nabla u|^2\,\mathrm{d}x + \Big(1+\frac{1}{\varepsilon}\Big)\frac{C^2}{a}\frac{1+(2R)^\alpha}{R^2}\int_{B_{2R}} V(x)u^2\,\mathrm{d}x.$$

Since by assumption $\alpha \in (0,2)$, we can choose $\overline{R} > 0$ sufficiently large so that

$$\Big(1+\frac{1}{\varepsilon}\Big)\frac{C^2}{a}\frac{1+(2R)^\alpha}{R^2} \leq 1+\varepsilon \quad \text{for any } \; R \geq \overline{R}.$$

We remark that the choice of $\overline{R}$ is independent of u, $\overline{R} = \overline{R}(\varepsilon, a, \alpha)$, and by construction

$$\int_{B_{2R}} |\nabla(\varphi u)|^2\,\mathrm{d}x \leq (1+\varepsilon)\|u\|^2 \leq 1+\varepsilon.$$

Therefore, if we define

$$v := \sqrt{1-\varepsilon}\,\varphi u \in H_0^1(B_{2R})$$

we have that $\|\nabla v\|_2^2 \leq 1-\varepsilon^2 \leq 1$, and by applying the classical Trudinger–Moser inequality (4.4), we can conclude

$$\int_{B_R} \big(e^{4\pi(1-\varepsilon)u^2}-1\big)\,\mathrm{d}x = \int_{B_R} (e^{4\pi(1-\varepsilon)(\varphi u)^2}-1)\,\mathrm{d}x \leq \int_{B_{2R}} e^{4\pi v^2}\,\mathrm{d}x \leq CR^2.$$

What we proved so far shows the existence of $\overline{R} = \overline{R}(\varepsilon, a, \alpha) > 0$ such that for any $R \geq \overline{R}$ we have

$$\int_{B_R} Q(x)\big(e^{\gamma u^2}-1\big)\,\mathrm{d}x \leq CR^2 \quad \text{for any } \; u \in H_V^1(\mathbb{R}^2) \text{ with } \|u\| \leq 1\,. \tag{4.6}$$

4.2 Uniform estimate in the exterior of a large ball

Let $\tilde{n} >> 1$ to be chosen later during the proof. For any fixed $n \geq \tilde{n}$, we consider the exterior B_n^{c} of the ball B_n and we introduce the covering of B_n^{c} consisting of all annuli A_n^σ with $\sigma > n$ defined by

$$A_n^\sigma := \big\{x \in B_n^{\mathrm{c}} \;\big|\; |x| < \sigma\big\} = \big\{x \in \mathbb{R}^2 \;\big|\; n < |x| < \sigma\big\}.$$

For any $\sigma > \tilde{n}$, in view of the Besicovitch covering lemma (see for instance [23]), there exist a sequence of points $\{x_k\}_k \in A_{\tilde{n}}^\sigma$ and a universal constant $\theta > 0$ such that

- $A_{\tilde{n}}^\sigma \subseteq \bigcup_k U_k^{1/2}$, where $U_k^{1/2} := B\Big(x_k, \frac{1}{2}\,\frac{|x_k|}{3}\Big)$;
- $\sum_k \chi_{U_k}(x) \leq \theta$ for any $x \in \mathbb{R}^2$, where χ_{U_k} is the characteristic function of $U_k := B\Big(x_k, \frac{|x_k|}{3}\Big)$.

Actually, the classical version of the Besicovitch covering lemma states that

$$\sum_k \chi_{U_k^{1/2}}(y) \leq \eta \quad \text{for any } \; y \in \mathbb{R}^2 \tag{4.7}$$

for some universal constant $\eta > 0$, and $U_k^{1/2} \subset U_k$. However, it is possible to show that (4.7) implies

$$\sum_k \chi_{U_k}(y) \le \theta \quad \text{for any } \; y \in \mathbb{R}^2 \tag{4.8}$$

where $\theta = \theta(\eta) > 0$. To prove that (4.8) holds, we recall the statement of Besicovitch covering lemma.

Let E be a subset of $\mathbb{R}^N$. A collection $\mathcal{F}$ of nontrivial closed balls in $\mathbb{R}^N$ is a Besicovitch covering for E if each $x \in E$ is the center of a nontrivial ball belonging to $\mathcal{F}$.

Lemma 4.5 ([12]) *Let E be a bounded subset of $\mathbb{R}^N$ and let $\mathcal{F}$ be a Besicovitch covering for E. There exist a countable collection $\{x_k\}_k$ of points in E and a corresponding collection of balls $\{B_k\}_k$ in $\mathcal{F}$, where $B_k := B(x_k, \rho_k)$, with $E \subset \bigcup_k B_k$. Moreover, there exists a positive integer c_N (depending only on the dimension N and independent of E and the covering $\mathcal{F}$) such that the balls $\{B_k\}_k$ can be organized into at most c_N subcollections $\mathcal{B}_j := \{B_{j_k}\}_k$, $j = 1, 2, \ldots, c_N$ in such a way that the balls $\{B_{j_k}\}_k$ of each subcollection $\mathcal{B}_j$ are disjoint.*

Proof of (4.8) We recall that, by Lemma 4.5, $A_{\tilde{n}}^{\sigma} \subseteq \bigcup_k U_k^{1/2}$ and there exists a positive integer η such that the balls $\{U_k^{1/2}\}_k$ can be organized into at most η subcollections $\mathcal{B}_j := \{U_{j_k}^{1/2}\}_k$, $j = 1, 2, \ldots, \eta$ where the balls $\{U_{j_k}^{1/2}\}_k$ of each subcollection $\mathcal{B}_j$ are disjoint. Then

$$\sum_k \chi_{U_k^{1/2}}(y) \le \eta \quad \text{for any } \; y \in \mathbb{R}^2 .$$

Next, we show that

$$\sum_k \chi_{U_k}(y) \le 196\eta \quad \text{for any } \; y \in \mathbb{R}^2 .$$

Assume that $y \in U_{j_k}$ for some $j \in \{\, 1, 2, \ldots, \eta \,\}$ and $k \ge 1$. Then $\frac{2}{3}|x_{j_k}| < |y| < \frac{4}{3}|x_{j_k}|$ and it follows that

$$U_{j_k}^{1/2} \subset B(0, \frac{7}{4}|y|).$$

Note that the ball $B(0, \frac{7}{4}|y|)$ contains at most 196 disjoint balls $B(x, \frac{1}{2}\frac{|x|}{3})$ with $\frac{3}{4}|y| < |x| < \frac{3}{2}|y|$. Thus, for any $j = 1, 2, \ldots, \eta$,

$$\sum_k \chi_{U_{j_k}}(y) \le 196$$

and

$$\sum_k \chi_{U_k}(y) = \sum_{j=1}^{\eta} \sum_k \chi_{U_{j_k}}(y) \le 196\eta.$$

The proof is completed. □

Let $u \in H_V^1(\mathbb{R}^2)$ be such that $\|u\| \le 1$ and let us estimate the weighted exponential integral of u on A_{3n}^{σ} with $n \ge \tilde{n}$ and $\sigma > n$. To do this, following [34], we introduce the set of indices

$$K_{n,\sigma} := \{k \in \mathbb{N} \,\big|\, U_k^{1/2} \cap B_{3n}^{c} \ne \varnothing\}.$$

From the definition of $K_{n,\sigma}$ and recalling that

$$A_{3n}^{\sigma} \subset A_{\tilde{n}}^{\sigma} \subseteq \bigcup_{k} U_k^{1/2},$$

we deduce that

$$A_{3n}^{\sigma} \subseteq \bigcup_{k\in K_{n,\sigma}} U_k^{1/2}$$

and hence

$$\int_{A_{3n}^{\sigma}} Q(x)\big(e^{\gamma u^2}-1\big)\,\mathrm{d}x \le \sum_{k\in K_{n,\sigma}} \int_{U_k^{1/2}} Q(x)\big(e^{\gamma u^2}-1\big)\,\mathrm{d}x.$$

Next, we estimate the single terms of the series on the right hand side. In this respect, the choice of the balls $U_k^{1/2}$ and U_k will play a crucial role to overcome the difficulties arising from the vanishing behavior of the potential V and the weight Q.

Remark 4.1 We have

$$\frac{2}{3}|x_k| \le |y| \le \frac{4}{3}|x_k| \quad \text{for any } \ y\in U_k.$$

Consequently, in view of the assumptions (V) and (Q), we get

$$V(y) \ge \frac{a}{1+|y|^{\alpha}} \ge \frac{a}{1+C_{\alpha}|x_k|^{\alpha}} \quad \text{for any } \ y\in U_k \tag{4.9}$$

where $C_{\alpha} := (4/3)^{\alpha}$, and

$$Q(y) \le \frac{b}{1+|y|^{\beta}} \le \frac{b}{1+C_{\beta}|x_k|^{\beta}} \quad \text{for any } \ y\in U_k \tag{4.10}$$

where $C_{\beta} := (2/3)^{\beta}$.

Moreover, it is easy to prove that if $U_k \cap B_{3n}^{c} \neq \varnothing$ then $U_k \subset B_n^{c}$ and this entails

$$\bigcup_{k\in K_{n,\sigma}} U_k^{1/2} \subseteq \bigcup_{k\in K_{n,\sigma}} U_k \subseteq B_n^{c} \subseteq B_{\tilde{n}}^{c}. \tag{4.11}$$

Properties (4.9) and (4.10) together with (4.11) will be useful in the proof to obtain some suitable uniform estimates.

Let us fix $k\in K_{n,\sigma}$. In view of (4.10),

$$\int_{U_k^{1/2}} Q(x)\big(e^{\gamma u^2}-1\big)\,\mathrm{d}x \le \frac{b}{1+C_{\beta}|x_k|^{\beta}} \int_{U_k^{1/2}} (e^{\gamma u^2}-1)\,\mathrm{d}x$$

and

$$\int_{U_k^{1/2}} (e^{\gamma u^2}-1)\,\mathrm{d}x = \int_{U_k^{1/2}} (e^{4\pi(1-\varepsilon)u^2}-1)\,\mathrm{d}x.$$

Following [42], the idea is to estimate the integral on the right hand side by means of the local Trudinger–Moser inequality (4.5) on U_k. To this aim, we consider the cutoff function $\varphi_k \in \mathcal{C}_0^{\infty}(U_k)$ satisfying

$$0\le \varphi_k \le 1 \text{ in } U_k, \quad \varphi_k \equiv 1 \text{ in } U_k^{1/2} \quad \text{and} \quad |\nabla\varphi| \le \frac{C}{|x_k|} \ \text{ in } U_k$$

for some universal constant $C > 0$. Then $\varphi_k u \in H_0^1(U_k)$ and we may estimate

$$\int_{U_k} |\nabla(\varphi_k u)|^2 \,\mathrm{d}x \le (1+\varepsilon)\int_{U_k} |\nabla u|^2 \,\mathrm{d}x + \left(1+\frac{1}{\varepsilon}\right)\frac{C^2}{|x_k|^2}\int_{U_k} u^2 \,\mathrm{d}x$$
$$\le (1+\varepsilon)\int_{U_k} |\nabla u|^2 \,\mathrm{d}x + \left(1+\frac{1}{\varepsilon}\right)\frac{C^2}{a}\,\frac{1+C_\alpha|x_k|^\alpha}{|x_k|^2}\int_{U_k} V(x)u^2 \,\mathrm{d}x$$

where we also used (4.9). Recalling that $k \in K_{n,\sigma}$, in view of (4.11), we have that $x_k \in B_{\tilde{n}}^{\mathrm{c}}$. Since $\alpha \in (0,2)$, we can choose $\tilde{n} = \tilde{n}(\varepsilon, a, \alpha)$ sufficiently large so that

$$\left(1+\frac{1}{\varepsilon}\right)\frac{C^2}{a}\,\frac{1+C_\alpha|x_k|^\alpha}{|x_k|^2} \le 1+\varepsilon \quad \text{for any } k \in K_{n,\sigma},\ n \ge \tilde{n}.$$

In this way, we get

$$\int_{U_k} |\nabla(\varphi_k u)|^2 \,\mathrm{d}x \le (1+\varepsilon)\int_{U_k} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x \le (1+\varepsilon). \tag{4.12}$$

If we let

$$v_k := \sqrt{1-\varepsilon}\;\varphi_k \overline{u} \in H_0^1(U_k)$$

then $\|\nabla v_k\|_2^2 \le 1-\varepsilon^2 \le 1$ and we can apply Lemma 4.3 to v_k obtaining

$$\int_{U_k^{1/2}} (e^{4\pi(1-\varepsilon)u^2} - 1)\,\mathrm{d}x = \int_{U_k^{1/2}} (e^{4\pi(1-\varepsilon)(\varphi_k u)^2} - 1)\,dx \le \int_{U_k} (e^{4\pi v_k^2} - 1)\,\mathrm{d}x$$
$$\le C|x_k|^2 \int_{U_k} |\nabla v_k|^2 \,\mathrm{d}x.$$

Finally, from (4.12), we deduce

$$\int_{U_k^{1/2}} (e^{4\pi(1-\varepsilon)u^2} - 1)\,\mathrm{d}x \le C|x_k|^2(1-\varepsilon)\int_{U_k} |\nabla(\varphi_k u)|^2 \,\mathrm{d}x$$
$$\le C|x_k|^2(1-\varepsilon^2)\int_{U_k} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x.$$

Combining the above estimates, we get

$$\int_{A_{3n}^\sigma} Q(x)\big(e^{\gamma u^2} - 1\big)\,\mathrm{d}x \le bC(1-\varepsilon^2)\sum_{k\in K_{n,\sigma}} \frac{|x_k|^2}{1+C_\beta|x_k|^\beta}\int_{U_k} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x$$
$$\le bC(1-\varepsilon^2)\sum_{k\in K_{n,\sigma}} \frac{|x_k|^2}{1+C_\beta|x_k|^\beta}\int_{B_n^{\mathrm{c}}} (|\nabla u|^2 + V(x)u^2)\chi_{U_k}(x)\,\mathrm{d}x$$

where the last inequality follows from (4.11). Using again (4.11), we have

$$\frac{|x_k|^2}{1+C_\beta|x_k|^\beta} \le \mathcal{B}_n := \sup_{x\in B_n^{\mathrm{c}}} \frac{|x|^2}{1+C_\beta|x|^\beta} \quad \text{for any } k \in K_{n,\sigma}.$$

Hence

$$\int_{A_{3n}^\sigma} Q(x)\big(e^{\gamma u^2} - 1\big)\,\mathrm{d}x \le bC(1-\varepsilon^2)\mathcal{B}_n \sum_{k\in K_{n,\sigma}} \int_{B_n^{\mathrm{c}}} (|\nabla u|^2 + V(x)u^2)\chi_{U_k}(x)\,\mathrm{d}x$$

and, in view of the Besicovitch covering lemma,

$$\int_{A_{3n}^{\sigma}} Q(x)\big(e^{\gamma u^2}-1\big)\,\mathrm{d}x \le bC(1-\varepsilon^2)\theta\mathcal{B}_n\int_{B_n^c}(|\nabla u|^2+V(x)u^2)\,\mathrm{d}x.$$

Letting $\sigma\to+\infty$, we can conclude the existence of $\tilde{n}=\tilde{n}(\varepsilon,a,\alpha)>>1$ such that for any $n\ge\tilde{n}$ we have

$$\int_{B_{3n}^c} Q(x)\big(e^{\gamma u^2}-1\big)\,\mathrm{d}x \le bC\theta\mathcal{B}_n\int_{B_n^c}(|\nabla u|^2+V(x)u^2)\,\mathrm{d}x. \tag{4.13}$$

Note that

$$\lim_{n\to+\infty}\mathcal{B}_n=\lim_{n\to+\infty}\frac{n^2}{1+C_\beta n^\beta}=\begin{cases}0 & \text{if } \beta>2\\ 1/(1+C_2) & \text{if } \beta=2\end{cases}$$

therefore, in particular, we have also the following estimate that can be seen as an analogue of [8, Proposition 11] for the two-dimensional case

Proposition 4.6 *Suppose that (V) and (Q) hold with α and β satisfying (2.1), i.e. $\alpha\in(0,2)$ and $\beta\in(2,+\infty)$, and let $0<\gamma<4\pi$. Then for any $\eta>0$ there exists $\tilde{n}=\tilde{n}(\gamma,a,\alpha)>1$ such that for any $n\ge\tilde{n}$*

$$\int_{B_{3n}^c} Q(x)\big(e^{\gamma u^2}-1\big)\,dx \le \eta\int_{B_n^c}(|\nabla u|^2+V(x)u^2)\,dx \quad \textit{for any } u\in H_V^1(\mathbb{R}^2) \textit{ with } \|u\|\le 1.$$

The above Proposition will be useful to prove the existence of a bound state solution of (NLS), see Sect. 6.

4.3 Proof of Theorem 4.1 completed

To conclude the proof of (4.2), it is sufficient to combine (4.6) with (4.13).

Now, we show that (4.1) holds. This follows from (4.2) and the density of $C_0^\infty(\mathbb{R}^2)$ in $H_V^1(\mathbb{R}^2)$ (see Remark 3.3). In fact, let $\gamma>0$ and $u\in H_V^1(\mathbb{R}^2)$. Then by density, there exists $u_0\in C_0^\infty(\mathbb{R}^2)$ such that

$$\|u-u_0\|\le\sqrt{\frac{1}{\gamma}}$$

and, we may estimate

$$u^2=(u-u_0+u_0)^2\le 2(u-u_0)^2+2u_0^2.$$

Let $R>0$ be such that supp $u_0\subseteq B_R$. Recalling the elementary inequality

$$ab-1\le\frac{1}{2}(a^2-1)+\frac{1}{2}(b^2-1)\quad\text{for any } a,b\ge 0$$

we get

$$\begin{aligned}\int_{\mathbb{R}^2} Q(x)(e^{\gamma u^2}-1)\,\mathrm{d}x &\le \int_{\mathbb{R}^2} Q(x)(e^{2\gamma(u-u_0)^2}e^{2\gamma u_0^2}-1)\,\mathrm{d}x\\ &\le \frac{1}{2}\int_{\mathbb{R}^2} Q(x)(e^{4\gamma(u-u_0)^2}-1)\,\mathrm{d}x+\frac{1}{2}\int_{B_R} Q(x)(e^{4\gamma u_0^2}-1)\,\mathrm{d}x\\ &\le \frac{1}{2}S_4+\frac{b}{2}\,|B_R|\,e^{4\gamma\|u_0\|_\infty^2}<+\infty,\end{aligned}$$

which completes the proof of Theorem 4.1.

5 Existence result

This section is devoted to the proof of Theorem 2.1 which is based on the classical mountain pass theorem.

First, we introduce the functional setting for a variational approach to problem (NLS). Since the nonlinear term f satisfies $f(0) = 0$, (f_0) and (f_1), for fixed $\gamma > \gamma_0$, $q \geq 1$ and for any $\sigma > 0$ we have

$$|f(s)| \leq \sigma|s| + C(\gamma, q, \sigma)|s|^{q-1}(e^{\gamma s^2} - 1) \quad \text{for any } s \in \mathbb{R}. \tag{5.1}$$

Hence, the Ambrosetti–Rabinowitz condition (f_1) yields

$$|F(s)| \leq \sigma|s|^2 + C(\gamma, q, \sigma)|s|^q(e^{\gamma s^2} - 1) \quad \text{for any } s \in \mathbb{R}. \tag{5.2}$$

Given $u \in H^1_V(\mathbb{R}^2)$, we can use (5.2) with $\gamma > \gamma_0$, $q \geq 2$ and $\sigma > 0$ to obtain the following estimate

$$\begin{aligned}\int_{\mathbb{R}^2} Q(x)F(u)\,\mathrm{d}x &\leq \sigma \int_{\mathbb{R}^2} Q(x)u^2\,\mathrm{d}x + C(\gamma, q, \sigma)\int_{\mathbb{R}^2} Q(x)u^q(e^{\gamma u^2} - 1)\,\mathrm{d}x\\ &\leq \sigma \int_{\mathbb{R}^2} Q(x)u^2\,\mathrm{d}x\\ &\quad + C(\gamma, q, \sigma)\left(\int_{\mathbb{R}^2} Q(x)u^{qp}\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^2} Q(x)(e^{\gamma p' u^2} - 1)\,\mathrm{d}x\right)^{\frac{1}{p'}}\end{aligned} \tag{5.3}$$

where we also applied Hölder's inequality with $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Since α and β satisfy (2.1), we have the continuous embeddings (3.8) and also the Trudinger–Moser estimate (4.1) and this enables us to conclude that

$$\int_{\mathbb{R}^2} Q(x)F(u)\,\mathrm{d}x < +\infty \quad \text{for any } u \in H^1_V(\mathbb{R}^2). \tag{5.4}$$

Therefore, if we introduce the functional

$$I(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^2} Q(x)F(u)\,\mathrm{d}x$$

from (5.4) it follows that I is well defined on $(H^1_V(\mathbb{R}^2), \|\cdot\|)$. Moreover, I is of class $\mathcal{C}^1$ with

$$I'[u](v) := < u, v > - \int_{\mathbb{R}^2} Q(x)f(u)v\,\mathrm{d}x \quad \text{for any } u, v \in H^1_V(\mathbb{R}^2).$$

In particular, any critical point u_0 of I is a weak solution of (NLS).

Lemma 5.1 *The functional I has a mountain pass geometry on $(H^1_V(\mathbb{R}^2), \|\cdot\|)$. More precisely*

(i) there exist $\tau > 0$ and $\varrho > 0$ such that $I(u) \geq \tau$ provided $\|u\| = \varrho$;
(ii) there exists $e_* \in H^1_V(\mathbb{R}^2)$ with $\|e_*\| > \varrho$ such that $I(e_*) < 0$.

Proof Let $\gamma > \gamma_0$ $q > 2$ and $p > 1$ with $\frac{1}{p} + \frac{1}{p'} = 1$. It is easy to see that (5.3) implies that for any $\sigma > 0$

$$\int_{\mathbb{R}^2} Q(x)F(u)\,\mathrm{d}x \leq C_1\sigma\|u\|^2 + C_2(\gamma, q, \sigma)\|u\|^q \quad \text{for any } u \in H^1_V(\mathbb{R}^2) \text{ with } \|u\| = \varrho$$

where $\varrho > 0$ satisfies

$$\gamma p' \varrho^2 < 4\pi.$$

In fact, due to the choice of α and β in the range (2.1), it suffices to use the continuous embeddings given by Theorem 3.1 and the Trudinger–Moser inequality (4.2).

Therefore, if $u \in H^1_V(\mathbb{R}^2)$ and $\|u\| = \varrho$ then

$$I(u) \geq \left(\frac{1}{2} - C_1\sigma\right)\|u\|^2 - C_2(\gamma, q, \sigma)\|u\|^q = \left(\frac{1}{2} - C_1\sigma\right)\varrho^2 - C_2(\gamma, q, \sigma)\varrho^q$$

and, choosing $\sigma > 0$ sufficiently small,

$$I(u) \geq \tilde{C}_1\varrho^2 - C_2(\gamma, q, \sigma)\varrho^q.$$

Since $q > 2$, for $\varrho > 0$ small enough, there exists $\tau > 0$ such that

$$I(u) \geq \tau \quad \text{for any } u \in H^1_V(\mathbb{R}^2) \text{ with } \|u\| = \varrho.$$

To prove (ii), first note that, from (f_1),

$$F(s) \geq A|s|^\mu - B \quad \text{for any } s \in \mathbb{R}$$

for some $A, B > 0$. If $u \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ with supp $u \subseteq B_R$, for some $R > 0$, then for any $t > 0$

$$\begin{aligned} I(tu) &= \frac{1}{2}t^2\|u\|^2 - \int_{B_R} Q(x)F(tu)\,\mathrm{d}x \\ &\leq \frac{1}{2}t^2\|u\|^2 - At^\mu \int_{\mathbb{R}^2} Q(x)|u|^\mu\,\mathrm{d}x + B\int_{B_R} Q(x)\,\mathrm{d}x \end{aligned}$$

and, since $\mu > 2$, $I(tu) \to -\infty$ as $t \to +\infty$. □

In view of the mountain pass geometry of I on $(H^1_V(\mathbb{R}^2), \|\cdot\|)$, we can consider the mountain pass level

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)) \geq \tau > 0$$

where

$$\Gamma := \{\,\gamma \in \mathcal{C}([0, 1], H^1_V(\mathbb{R}^2)) \mid \gamma(0) = 0 \text{ and } I(\,\gamma(1)\,) < 0\,\}.$$

5.1 Estimate of the mountain pass level

As a consequence of (f_3) or (f_3'), using standard arguments, we will obtain the following upper bound for the mountain pass level

$$c < \frac{2\pi}{\gamma_0}. \tag{5.5}$$

We start assuming that the nonlinear term f satisfies the growth condition at infinity (f_3), i.e.

$$\liminf_{|s|\to+\infty} \frac{sf(s)}{e^{\gamma_0 s^2}} = \beta_0 > \mathcal{M}, \tag{f_3}$$

where

$$\mathcal{M} = \mathcal{M}(V, Q) := \inf_{r>0} \frac{4e^{\frac{1}{2}r^2 V_{\max,r}}}{\gamma_0 r^2 Q_{\min,r}},$$
$$V_{\max,r} := \max_{|x|\le r} V(x) > 0 \quad \text{and} \quad Q_{\min,r} := \min_{|x|\le r} Q(x) > 0.$$

In this case, for fixed $r > 0$, we consider Moser's sequence of functions (see [33])

$$\tilde{w}_n(x) := \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log n} & \text{if } |x| \le \frac{r}{n}, \\ \dfrac{\log \frac{r}{|x|}}{\sqrt{\log n}} & \text{if } \frac{r}{n} \le |x| \le r, \\ 0 & \text{if } |x| \ge r. \end{cases}$$

It is well known that $\tilde{w}_n \in H_0^1(B_r) \subset H_V^1(\mathbb{R}^2)$ and one can easily prove (see for instance [30, Equation (3.5)] or [40, Lemma 3.2]) that

$$1 \le \|\tilde{w}_n\|^2 \le 1 + \frac{d_n(r)}{\log n} V_{\max,r} \tag{5.6}$$

where

$$d_n(r) := \frac{r^2}{4} + o_n(1) \quad \text{and} \quad o_n(1) \to 0 \text{ as } n \to +\infty.$$

Let

$$w_n := \frac{\tilde{w}_n}{\|\tilde{w}_n\|} \in H_0^1(B_r) \subset H_V^1(\mathbb{R}^2)$$

so that $\|w_n\| = 1$ and, in view of (5.6), when $|x| \le \frac{r}{n}$ we have

$$\begin{aligned} w_n^2(x) = \frac{1}{2\pi} \log n \left(\frac{1}{\|\tilde{w}_n\|^2} \pm 1\right) &\ge \frac{1}{2\pi}\left(\log n - \frac{d_n(r)\, V_{\max,r}}{\|\tilde{w}_n\|^2}\right) \\ &\ge \frac{1}{2\pi}\Big(\log n - d_n(r)\, V_{\max,r}\Big). \end{aligned} \tag{5.7}$$

Note that, from (f_3), we deduce the existence of $r > 0$ such that

$$\beta_0 > \frac{4e^{\frac{1}{2}r^2 V_{\max,r}}}{\gamma_0 r^2 Q_{\min,r}} \tag{5.8}$$

and, with this choice of $r > 0$, we will prove the following

Lemma 5.2 *There exists $n \in \mathbb{N}$ such that*

$$\max_{t\ge 0} I(tw_n) < \frac{2\pi}{\gamma_0}.$$

Proof The arguments of the proof are standard (see for instance [30, Lemma 3.6] or [40, Lemma 3.3]) but for the convenience of the reader we will sketch the main steps.

We argue by contradiction assuming that for any $n \in \mathbb{N}$

$$\max_{t\ge 0} I(tw_n) \ge \frac{2\pi}{\gamma_0}.$$

Let $t_n > 0$ be such that

$$I(t_n w_n) = \max_{t \geq 0} I(t w_n)$$

then

$$t_n^2 \geq 2I(t_n w_n) \geq \frac{4\pi}{\gamma_0} \tag{5.9}$$

and, since

$$\frac{\mathrm{d}}{\mathrm{d}t} I(t w_n) \Big|_{t=t_n} = 0,$$

we have also

$$t_n^2 = \int_{\mathbb{R}^2} Q(x) f(t_n w_n) t_n w_n \, \mathrm{d}x. \tag{5.10}$$

Note that, as a consequence of (f_3), for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$s f(s) \geq (\beta_0 - \varepsilon) e^{\gamma_0 s^2} \quad \text{for any } |s| \geq R_\varepsilon. \tag{5.11}$$

Let $x_n \in B_{r/n}$ be the minimum point of the weight Q on $B_{r/n}$, i.e.

$$Q(x_n) = \min_{|x| \leq r/n} Q(x),$$

then

$$\lim_{n \to +\infty} Q(x_n) = Q(0) > 0.$$

Therefore, using (5.10) and recalling (5.7), we get

$$t_n^2 \geq (\beta_0 - \varepsilon) \int_{B_{r/n}} Q(x) e^{\gamma_0 (t_n w_n)^2} \, \mathrm{d}x \geq (\beta_0 - \varepsilon) Q(x_n) \left(\frac{r}{n}\right)^2 e^{\frac{\gamma_0}{2\pi} t_n^2 [\,\log n - d_n(r) V_{\max,r}\,]}$$

and, from this inequality, we deduce not only that the sequence $\{t_n\}_n$ is bounded but, in view of (5.9),

$$\lim_{n \to +\infty} t_n^2 = \frac{4\pi}{\gamma_0}.$$

To reach a contradiction, we try to obtain an estimate of β_0 from above. From (5.10) and (5.11), it follows that

$$\begin{aligned} t_n^2 &\geq \int_{B_r} Q(x) f(t_n w_n) t_n w_n \, \mathrm{d}x \geq Q_{\min,r} \int_{B_r} f(t_n w_n) t_n w_n \, \mathrm{d}x \\ &\geq Q_{\min,r} \Big[(\beta_0 - \varepsilon) \int_{B_r} e^{\gamma_0 (t_n w_n)^2} \, \mathrm{d}x + \int_{\{t_n w_n < R_\varepsilon\}} f(t_n w_n) t_n w_n \, \mathrm{d}x \\ &\qquad - (\beta_0 - \varepsilon) \int_{\{t_n w_n < R_\varepsilon\}} e^{\gamma_0 (t_n w_n)^2} \, \mathrm{d}x \Big]. \end{aligned}$$

Since $w_n \to 0$ a.e. in $\mathbb{R}^2$, we can apply the Lebesgue dominated convergence theorem obtaining

$$\lim_{n \to +\infty} \int_{\{t_n w_n < R_\varepsilon\}} f(t_n w_n) t_n w_n \, \mathrm{d}x = 0 \quad \text{and} \quad \lim_{n \to +\infty} \int_{\{t_n w_n < R_\varepsilon\}} e^{\gamma_0 (t_n w_n)^2} \, \mathrm{d}x = \pi r^2.$$

Moreover, (5.9) yields

$$\int_{B_r} e^{\gamma_0 (t_n w_n)^2}\,\mathrm{d}x \ge \int_{B_{r/n}} + \int_{B_r \setminus B_{r_n}} e^{4\pi w_n^2}\,\mathrm{d}x.$$

On one hand, using (5.7), we get

$$\int_{B_{r/n}} e^{4\pi w_n^2}\,\mathrm{d}x \ge \pi r^2\, e^{-2d_n(r)\, V_{\max,r}}.$$

On the other hand, using the definition of w_n and the change of variable $s = re^{-\|\tilde{w}_n\|\sqrt{\log n}\, t}$,

$$\begin{aligned}
&\int_{B_r \setminus B_{r_n}} e^{4\pi w_n^2}\,\mathrm{d}x \\
&= 2\pi \int_{r/n}^{r} e^{2\frac{\log^2 r/s}{\|\tilde{w}_n\|^2 \log n}}\, s\,\mathrm{d}s = 2\pi r^2 \|\tilde{w}_n\|\sqrt{\log n} \int_0^{\frac{\sqrt{\log n}}{\|\tilde{w}_n\|}} e^{2(\,t^2 - \|\tilde{w}_n\|\sqrt{\log n}\, t\,)}\,\mathrm{d}t \\
&\ge 2\pi r^2 \|\tilde{w}_n\|\sqrt{\log n} \int_0^{\frac{\sqrt{\log n}}{\|\tilde{w}_n\|}} e^{-2\|\tilde{w}_n\|\sqrt{\log n}\, t}\,\mathrm{d}t = \pi r^2\big(1 - e^{-2\log n}\big).
\end{aligned}$$

In conclusion,

$$\frac{4\pi}{\gamma_0} = \lim_{n\to+\infty} t_n^2 \ge (\beta_0 - \varepsilon)\; Q_{\min,r}\; \pi r^2 e^{-\frac{1}{2}\, r^2\, V_{\max,r}}$$

and, from the arbitrary choice of $\varepsilon > 0$, we deduce that

$$\beta_0 \le \frac{4e^{\frac{1}{2}\, r^2\, V_{\max,r}}}{\gamma_0 r^2\; Q_{\min,r}}$$

which contradicts (5.8). □

Next we consider the case when the nonlinear term f satisfies the growth condition (f_3'), i.e.

$$\text{there exists } \; p > 2 \text{ such that } F(s) \ge \frac{\lambda}{p}\,|s|^p \quad \text{for any } \; s \in \mathbb{R}, \qquad (f_3')$$

where

$$\lambda > \left(\frac{\gamma_0}{4\pi}\,\frac{p-2}{p}\right)^{\frac{p-2}{2}} S_{p,V,Q}^{p/2}$$

and

$$S_{p,V,Q} := \inf_{u \in H_V^1(\mathbb{R}^2)\setminus\{0\}} \frac{\|u\|^2}{\left(\int_{\mathbb{R}^2} Q(x)|u|^p\,\mathrm{d}x\right)^{2/p}}.$$

In view of Theorem 3.1, the embedding $H_V^1(\mathbb{R}^2) \hookrightarrow L_Q^p(\mathbb{R}^2)$ is compact and hence, there exists $\overline{u} \in H_V^1(\mathbb{R}^2)$ such that

$$\|\overline{u}\|^2 = S_{p,V,Q} \quad \text{and} \quad \int_{\mathbb{R}^2} Q(x)|\overline{u}|^p\,\mathrm{d}x = 1.$$

Therefore, we may estimate

$$c \le \max_{t\ge 0} I(t\overline{u}) = \max_{t \ge 0}\left\{\frac{1}{2}t^2 S_{p,V,Q} - \int_{\mathbb{R}^2} Q(x)F(t\overline{u})\,\mathrm{d}x\right\}$$

and, using (f_3'), we get

$$c \leq \max_{t\geq 0}\left\{\frac{1}{2}t^2 S_{p,V,Q} - \frac{1}{p}\lambda t^p\right\} = \frac{p-2}{2p}\,\frac{S_{p,V,Q}^{p/(p-2)}}{\lambda^{2/(p-2)}} < \frac{2\pi}{\gamma_0}.$$

5.2 Palais–Smale sequences

Applying the mountain pass theorem without the Palais–Smale compactness condition, we get the existence of a Palais–Smale sequence $\{u_n\}_n \subset H_V^1(\mathbb{R}^2)$ at the level c (for short $(\mathrm{PS})_c$-sequence), i.e.

$$I(u_n) \to c \quad \text{and} \quad I'[u_n] \to 0\,, \quad \text{as} \quad n \to +\infty. \tag{5.12}$$

Lemma 5.3 *Any $(PS)_c$-sequence $\{u_n\}_n$ for I is bounded in $(H_V^1(\mathbb{R}^2), \|\cdot\|)$ and satisfies*

$$\sup_n \int_{\mathbb{R}^2} Q(x) f(u_n) u_n\, dx < +\infty. \tag{5.13}$$

Proof Since $\{u_n\}_n$ is a $(\mathrm{PS})_c$-sequence for I, we have

$$I(u_n) \to c \quad \text{as} \quad n \to +\infty \tag{5.14}$$

and

$$\left| < u_n, v > - \int_{\mathbb{R}^2} Q(x) f(u_n) v\, \mathrm{d}x \right| \leq \varepsilon_n \|v\| \quad \text{for any } \; v \in H_V^1(\mathbb{R}^2) \tag{5.15}$$

where $\varepsilon_n \downarrow 0$ as $n \to +\infty$.

From (5.14), we deduce that $\{I(u_n)\}_n \subset \mathbb{R}$ is bounded and hence, there exists a constant $C > 0$ such that

$$\frac{1}{2}\,\|u_n\|^2 \leq C + \int_{\mathbb{R}^2} Q(x) F(u_n)\, \mathrm{d}x \quad \text{for any } \; n \geq 1.$$

In view of the Ambrosetti–Rabinowitz condition (f_1),

$$\int_{\mathbb{R}^2} Q(x) F(u_n)\, \mathrm{d}x \leq \frac{1}{\mu}\int_{\mathbb{R}^2} Q(x) f(u_n) u_n\, \mathrm{d}x$$

and, using (5.15) with $v = u_n$,

$$\int_{\mathbb{R}^2} Q(x) f(u_n) u_n\, \mathrm{d}x \leq \|u_n\|^2 + \varepsilon_n \|u_n\|. \tag{5.16}$$

Therefore

$$\frac{1}{2}\,\|u_n\|^2 \leq C + \frac{1}{\mu}\,\|u_n\|^2 + \frac{\varepsilon_n}{\mu}\,\|u_n\|$$

and, since $\mu > 2$,

$$0 \leq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 \leq C + \frac{\varepsilon_n}{\mu}\,\|u_n\|$$

from which we deduce that $\{u_n\}_n$ must be bounded in $(H_V^1(\mathbb{R}^2), \|\cdot\|)$.

Finally, the boundedness of $\{u_n\}_n$ in $(H_V^1(\mathbb{R}^2), \|\cdot\|)$ together with (5.16) gives (5.13). □

Without loss of generality, we may assume that

$$u_n \rightharpoonup u_0 \quad \text{in } H^1_V(\mathbb{R}^2). \tag{5.17}$$

Moreover, in view of (5.13), we may apply [21, Lemma 2.1] obtaining

$$Q(x)f(u_n) \to Q(x)f(u_0) \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^2).$$

Hence,

$$< u_0, \varphi > - \int_{\mathbb{R}^2} Q(x)f(u_0)\varphi \, dx = 0 \quad \text{for any } \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$$

and u_0 is a weak solution of (NLS). To prove that $u_0 \neq 0$ and complete the proof of Theorem 2.1, we will use the following convergence result

Lemma 5.4 *If $\{u_n\}_n$ is a $(PS)_c$-sequence for I, with $u_n \rightharpoonup u_0$ in $H^1_V(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} Q(x)F(u_n)\, dx \to \int_{\mathbb{R}^2} Q(x)F(u_0)\, dx \quad \text{as } n \to +\infty.$$

Proof This result is essentially a consequence of the compact embedding

$$H^1_V(\mathbb{R}^2) \hookrightarrow\hookrightarrow L^2_Q(\mathbb{R}^2) \tag{5.18}$$

and the generalized Lebesgue dominated convergence theorem (see for instance [35, Chapter 4, Theorem 17]). Recall that (5.18) holds in view of Theorem 3.1 and the assumptions (V) and (Q) with α and β satisfying (2.1).

First note that from (f_1) and (f_2), it follows that

$$0 \leq \lim_{|s|\to+\infty} \frac{F(s)}{sf(s)} \leq \lim_{|s|\to+\infty} \frac{M_0}{|s|} = 0$$

and for any $\varepsilon > 0$ there exists $\bar{s} = \bar{s}(\varepsilon) > 0$ such that

$$F(s) \leq \varepsilon s f(s) \quad \text{for any } |s| \geq \bar{s}.$$

Since $u_0 \in H^1_V(\mathbb{R}^2)$ and recalling the uniform bound (5.13), we have also

$$\int_{\mathbb{R}^2} Q(x)f(u_0)u_0\, dx \leq C \quad \text{and} \quad \int_{\mathbb{R}^2} Q(x)f(u_n)u_n\, dx \leq C \quad \text{for any } n \geq 1$$

for some constant $C > 0$.

Consequently, for fixed $\varepsilon > 0$, we get

$$\int_{\{|u_0|\geq\bar{s}\}} Q(x)F(u_0)\, dx \leq \varepsilon \int_{\{|u_0|\geq\bar{s}\}} Q(x)f(u_0)u_0\, dx \leq C\varepsilon$$

and

$$\int_{\{|u_n|\geq\bar{s}\}} Q(x)F(u_n)\, dx \leq \varepsilon \int_{\{|u_n|\geq\bar{s}\}} Q(x)f(u_n)u_n\, dx \leq C\varepsilon.$$

Now, we let

$$h_n(x) := Q(x)\chi_{\{|u_n|<\bar{s}\}}F(u_n) \quad \text{and} \quad h(x) := Q(x)\chi_{\{|u_0|<\bar{s}\}}F(u_0).$$

Then $\{h_n\}_n$ is a sequence of measurable functions and

$$h_n(x) \to h(x) \quad \text{for a.e. } x \in \mathbb{R}^2,$$

as a consequence of the fact that $u_n \to u_0$ a.e. in $\mathbb{R}^2$. Using (5.2) with $\gamma > \gamma_0$, $q = 2$ and $\sigma > 0$, we may estimate for any $|s| \le \bar{s}$

$$F(s) \le \sigma s^2 + C(\gamma, \sigma) s^2 (e^{\gamma s^2} - 1) \le C(\gamma, \sigma, \bar{s}) s^2.$$

Then, letting

$$g_n(x) := C(\gamma, \sigma, \bar{s}) Q(x) u_n^2 \quad \text{and} \quad g(x) := C(\gamma, \sigma, \bar{s}) Q(x) u_0^2\,,$$

we get

$$0 \le h_n(x) \le g_n(x) \quad x \in \mathbb{R}^2\,.$$

Note that $\{g_n\}_n$ is a sequence of measurable functions, $g_n(x) \to g(x)$ a.e. in $\mathbb{R}^2$ and, in view of the compact embedding (5.18),

$$\lim_{n\to+\infty} \int_{\mathbb{R}^2} g_n(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} g(x)\,\mathrm{d}x.$$

Therefore, applying the generalized Lebesgue dominated convergence theorem, we get

$$\lim_{n\to+\infty} \int_{\mathbb{R}^2} h_n(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} h(x)\,\mathrm{d}x.$$

In conclusion, for any fixed $\varepsilon > 0$, we have

$$\begin{aligned} L_n &:= \left| \int_{\mathbb{R}^2} Q(x)F(u_n)\,\mathrm{d}x - \int_{\mathbb{R}^2} Q(x)F(u_0)\,\mathrm{d}x \right| \le \int_{\{|u_n|\ge\bar{s}\}} Q(x)F(u_n)\,\mathrm{d}x \\ &\quad + \int_{\{|u_0|\ge\bar{s}\}} Q(x)F(u_0)\,\mathrm{d}x + \left| \int_{\{|u_n|<\bar{s}\}} Q(x)F(u_n)\,\mathrm{d}x - \int_{\{|u_0|<\bar{s}\}} Q(x)F(u_0)\,\mathrm{d}x \right| \\ &\le 2C\varepsilon + \left| \int_{\mathbb{R}^2} h_n(x)\,\mathrm{d}x - \int_{\mathbb{R}^2} h(x)\,\mathrm{d}x \right| \end{aligned}$$

and, passing to the limit as $n \to +\infty$,

$$0 \le \lim_{n\to+\infty} L_n \le 2C\varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily fixed, letting $\varepsilon \downarrow 0$, we obtain the desired convergence result. □

5.3 Nontrivial mountain pass solution

In order to complete the proof of Theorem 2.1, we have simply to show that the weak limit u_0 given by (5.17) is nontrivial, i.e. $u_0 \ne 0$. To this aim, we argue by contradiction assuming that $u_0 = 0$.

Since $\{u_n\}_n$ is a $(\mathrm{PS})_c$-sequence, (5.12) holds. In particular

$$\lim_{n\to+\infty} \|u_n\|^2 = \lim_{n\to+\infty} 2\Big(\int_{\mathbb{R}^2} Q(x)F(u_n)\,\mathrm{d}x + c \Big) \tag{5.19}$$

and

$$\lim_{n\to+\infty} \|u_n\|^2 = \lim_{n\to+\infty} \int_{\mathbb{R}^2} Q(x) f(u_n) u_n\,\mathrm{d}x. \tag{5.20}$$

From the convergence result expressed by Lemma 5.4, we deduce that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^2} Q(x)F(u_n)\,\mathrm{d}x = 0.$$

This together with (5.19) yields

$$\lim_{n\to+\infty} \|u_n\|^2 = 2c > 0. \tag{5.21}$$

In view of (5.5),

$$c < \frac{2\pi}{\gamma_0}$$

and we deduce the existence of $\varepsilon > 0$ and $\overline{n} \geq 1$ such that

$$\|u_n\|^2 \leq \frac{4\pi}{\gamma_0}(1-\varepsilon) \quad \text{for any } \; n \geq \overline{n}.$$

Therefore, we can choose $\gamma > \gamma_0$ sufficiently close to γ_0 and $p > 1$ sufficiently close to 1 in such a way that

$$\gamma p \|u_n\|^2 < 4\pi(1-\varepsilon^4) \quad \text{for any } \; n \geq \overline{n}. \tag{5.22}$$

With this choice of $\gamma > \gamma_0$ and $p > 1$, we apply (5.1) with $q = 2$ and Hölder's inequality with $\frac{1}{p} + \frac{1}{p'} = 1$ obtaining

$$\begin{aligned}
&\int_{\mathbb{R}^2} Q(x) f(u_n) u_n \,\mathrm{d}x \\
&\quad \leq C_1 \int_{\mathbb{R}^2} Q(x) u_n^2 \,\mathrm{d}x + C_2 \int_{\mathbb{R}^2} Q(x) u_n^2 (e^{\gamma u_n^2} - 1) \,\mathrm{d}x \\
&\quad \leq C_1 \int_{\mathbb{R}^2} Q(x) u_n^2 \,\mathrm{d}x + C_2 \Big(\int_{\mathbb{R}^2} Q(x) u_n^{2p'} \,\mathrm{d}x \Big)^{\frac{1}{p'}} \Big(\int_{\mathbb{R}^2} Q(x) (e^{\gamma p u_n^2} - 1) \,\mathrm{d}x \Big)^{\frac{1}{p}}.
\end{aligned}$$

Note that $2p' > 2$ and, in view of Theorem 3.1 and the assumptions (V) and (Q) with α and β in the range (2.1), we have the compact embeddings

$$H_V^1(\mathbb{R}^2) \hookrightarrow\hookrightarrow L_Q^2(\mathbb{R}^2) \quad \text{and} \quad H_V^1(\mathbb{R}^2) \hookrightarrow\hookrightarrow L_Q^{2p'}(\mathbb{R}^2).$$

Moreover, from (5.22),

$$\sup_{n \geq \overline{n}} \int_{\mathbb{R}^2} Q(x)(e^{\gamma p u_n^2} - 1) \,\mathrm{d}x \leq S_{4\pi(1-\varepsilon^4)}(V, Q)$$

where $S_{4\pi(1-\varepsilon^4)}(V, Q) < +\infty$ is the supremum of the Trudinger–Moser inequality given by Theorem 4.1.

Therefore

$$\lim_{n\to+\infty} \int_{\mathbb{R}^2} Q(x) f(u_n) u_n \,\mathrm{d}x = 0$$

and, from (5.20), we get

$$\lim_{n\to+\infty} \|u_n\|^2 = 0$$

which contradicts (5.21).

6 Bound state solutions

This Section is devoted to the proof of Proposition 2.2. In particular, we will prove that if $u_0 \in H^1_V(\mathbb{R}^2)$ is a weak solution of (NLS), i.e.

$$< u_0, v > - \int_{\mathbb{R}^2} Q(x) f(u_0) v \,\mathrm{d}x = 0 \quad \text{for any } \; v \in H^1_V(\mathbb{R}^2)$$

then $u_0 \in L^2(\mathbb{R}^2)$ and hence $u_0 \in H^1(\mathbb{R}^2)$.

We will follow almost the same arguments introduced in [8, Lemma 17 and Lemma 18], see also [31, Section 3].

Lemma 6.1 *Suppose that* (V) *and* (Q) *hold with* α *and* β *satisfying* (2.1)*, i.e.* $\alpha \in (0, 2)$ *and* $\beta \in (2, +\infty)$*. Let* $\gamma > 0$ *and* $u \in H^1_V(\mathbb{R}^2)$*. Then for any* $\varepsilon > 0$ *there exists* $\overline{R} = \overline{R}(u, \gamma, a, \alpha) > 1$ *such that for any* $R \geq \overline{R}$

$$\int_{B_R^c} Q(x)(e^{\gamma u^2} - 1)\,\mathrm{d}x \leq \varepsilon\, \gamma \int_{B_R^c} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x.$$

Proof Let $R > 1$ and let $\tilde{\psi}_R : \mathbb{R}^+ \to [0, 1]$ be a smooth nondecreasing function such that

$$\tilde{\psi}_R(r) := \begin{cases} 0 & \text{if } 0 \leq r \leq R - R^{\alpha/2} \\ 1 & \text{if } r \geq R \end{cases}$$

and

$$|\tilde{\psi}_R'(r)| \leq \frac{2}{R^{\alpha/2}}.$$

In polar coordinates $(r, \theta) \in [0, +\infty) \times \mathbb{S}^1$, we define

$$\tilde{u}_R(r, \theta) := \begin{cases} 0 & \text{if } 0 \leq r \leq R - R^{\alpha/2}, \\ \tilde{\psi}_R(r) u(2R - r, \theta) & \text{if } R - R^{\alpha/2} \leq r \leq R, \\ u(r, \theta) & \text{if } r \geq R. \end{cases}$$

Arguing as in [8, Proposition 11], we can prove the following estimate

$$\int_{A_R} (\,|\nabla \tilde{u}_R|^2 + V(x)\tilde{u}_R^2\,)\,\mathrm{d}x \leq C \int_{B_R^c} (\,|\nabla u|^2 + V(x)u^2\,)\,\mathrm{d}x$$

where A_R is the annulus

$$A_R := \{\, x \in \mathbb{R}^2 \;|\; R - R^{\alpha/2} \leq |x| \leq R \,\}.$$

Recalling that $\tilde{u}_R \equiv 0$ when $|x| \leq R - R^{\alpha/2}$ and $\tilde{u}_R \equiv u$ when $|x| \geq R$, we get

$$\|\tilde{u}_R\|^2 = \int_{B^c_{R-R^{\alpha/2}}} (\,|\nabla \tilde{u}_R|^2 + V(x)\tilde{u}_R^2\,)\,\mathrm{d}x \leq (1 + C) \int_{B_R^c} (\,|\nabla u|^2 + V(x)u^2\,)\,\mathrm{d}x.$$

Since $u \in H^1_V(\mathbb{R}^2)$, there exists $\overline{R} = \overline{R}(u, \gamma) > 1$ such that

$$\int_{B^c_{\overline{R}}} (\,|\nabla u|^2 + V(x)u^2\,)\,\mathrm{d}x \leq \frac{1}{1 + C}\,\frac{1}{\gamma}$$

and in particular

$$\|\sqrt{\gamma}\,\tilde{u}_R\| \le 1 \quad \text{for any} \;\; R \ge \overline{R}.$$

Therefore, we may estimate

$$\int_{B_R^c} Q(x)(e^{\gamma u^2}-1)\,\mathrm{d}x = \int_{B_R^c} Q(x)(e^{\gamma \tilde{u}_R^2}-1)\,\mathrm{d}x \le \int_{B^c_{R-R^{\alpha/2}}} Q(x)(e^{(\sqrt{\gamma}\,\tilde{u}_R)^2}-1)\,\mathrm{d}sx$$
$$\le \eta\|\sqrt{\gamma}\,\tilde{u}_R\|^2 = \eta\,\gamma\,\|\tilde{u}_R\|^2$$

where $\eta > 0$ is arbitrarily fixed and we used Proposition 4.6. This is possible provided $\overline{R} - \overline{R}^{\alpha/2} \ge 3\tilde{n}$ where $\tilde{n} = \tilde{n}(\gamma, a, \alpha) > 1$ is given by Proposition 4.6. □

From now on, $u_0 \in H^1_V(\mathbb{R}^2)$ will denote a weak solution of (NLS).

Lemma 6.2 *There exists $\tilde{R} > 0$ such that for any $n \in \mathbb{N}$ satisfying $R_n := n^{2/(2-\alpha)} \ge \tilde{R}$ we have*

$$\int_{B^c_{R_{n+1}}} (\,|\nabla u_0|^2 + V(x)u_0^2\,)\,\mathrm{d}x \le \frac{3}{4}\int_{B^c_{R_n}} (\,|\nabla u_0|^2 + V(x)u_0^2\,)\,\mathrm{d}x.$$

Proof Let $\chi_n : \mathbb{R}^2 \to [0,1]$ be a piecewise affine function such that

$$\chi_n(x) := \begin{cases} 0 & \text{if } |x| \le R_n, \\ 1 & \text{if } |x| \ge R_{n+1}. \end{cases}$$

Arguing as in [8, Lemma 17], we can prove that

$$|\nabla\chi_n(x)|^2 \le V(x).$$

By construction $\chi_n u_0 \in H^1_V(\mathbb{R}^2)$,

$$\int_{B^c_{R_{n+1}}} (\,|\nabla u_0|^2 + V(x)u_0^2\,)\,\mathrm{d}x \le \int_{B^c_{R_n}} \chi_n(\,|\nabla u_0|^2 + V(x)u_0^2\,)\,\mathrm{d}x$$

and we can compute

$$< u_0, \chi_n u_0 > = \int_{B^c_{R_n}} \chi_n(\,|\nabla u_0|^2 + V(x)u_0^2\,)\,\mathrm{d}x + \int_{B^c_{R_n}} u_0\nabla u_0\cdot\nabla\chi_n\,\mathrm{d}x.$$

Moreover, if we use $\chi_n u_0 \in H^1_V(\mathbb{R}^2)$ as test function, we obtain

$$< u_0, \chi_n u_0 > - \int_{\mathbb{R}^2} Q(x) f(u_0)\chi_n u_0\,\mathrm{d}x = 0.$$

Therefore, we may estimate

$$\begin{aligned}
&\int_{B^c_{R_n}} \chi_n(\,|\nabla u_0|^2 + V(x)u_0^2\,)\,\mathrm{d}x \\
&= \int_{B^c_{R_n}} Q(x) f(u_0)\chi_n u_0\,\mathrm{d}x - \int_{B^c_{R_n}} u_0\nabla u_0\cdot\nabla\chi_n\,\mathrm{d}x \\
&\le \int_{B^c_{R_n}} Q(x) f(u_0)u_0\,\mathrm{d}x + \frac{1}{2}\Big(\int_{B^c_{R_n}} |\nabla u_0|^2\,\mathrm{d}x + \int_{B^c_{R_n}} |\nabla\chi_n|^2 u_0^2\,\mathrm{d}x\,\Big) \\
&\le \int_{B^c_{R_n}} Q(x) f(u_0)u_0\,\mathrm{d}x + \frac{1}{2}\int_{B^c_{R_n}} (\,|\nabla u_0|^2\,\mathrm{d}x + V(x)u_0^2\,)\,\mathrm{d}x.
\end{aligned}$$

To complete the proof, it is sufficient to prove the existence of $\tilde{R} > 0$ such that for any $n \in \mathbb{N}$ with $R_n \geq \tilde{R}$

$$\int_{B^c_{R_n}} Q(x) f(u_0) u_0 \, \mathrm{d}x \leq \frac{1}{4} \int_{B^c_{R_n}} (\, |\nabla u_0|^2 \, dx + V(x) u_0^2 \,) \, \mathrm{d}x.$$

To this aim, arguing as in (5.3), for fixed $\gamma > \gamma_0$ and $\sigma > 0$ we have

$$\int_{B^c_{R_n}} Q(x) f(u_0) u_0 \, \mathrm{d}x \leq \sigma \int_{B^c_{R_n}} Q(x) u_0^2 \, \mathrm{d}x + C(\gamma, \sigma) \left(\int_{B^c_{R_n}} Q(x) u_0^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{B^c_{R_n}} Q(x) (e^{2\gamma u_0^2} - 1) \, \mathrm{d}x \right)^{\frac{1}{2}}.$$

Let $\overline{R} > 1$ be as in Lemma 6.1. If $\tilde{R} \geq \overline{R}$ then, for any $n \in \mathbb{N}$ satisfying $R_n \geq \tilde{R}$, we may apply Lemma 6.1 obtaining

$$\int_{B^c_{R_n}} Q(x) (e^{2\gamma u_0^2} - 1) \, \mathrm{d}x \leq C\gamma \int_{B^c_{R_n}} (\, |\nabla u_0|^2 + V(x) u_0^2 \,) \, \mathrm{d}x. \tag{6.1}$$

Moreover, if $\tilde{R} > 0$ is sufficiently large then, for any $n \in \mathbb{N}$ satisfying $R_n \geq \tilde{R}$, we have

$$\sup_{x \in B^c_{R_n}} \frac{Q(x)}{V(x)} \leq \sup_{x \in B^c_{\tilde{R}}} \frac{Q(x)}{V(x)} \leq \sup_{x \in B^c_{\tilde{R}}} \frac{b}{a} \frac{1 + |x|^\alpha}{1 + |x|^\beta} \leq \frac{b}{a} \frac{1 + |\tilde{R}|^\alpha}{1 + |\tilde{R}|^\beta} =: \mathcal{B}(\tilde{R})$$

where we used assumptions (V) and (Q) with α and β in the range (2.1). Therefore, when $R_n \geq \tilde{R}$,

$$\int_{B^c_{R_n}} Q(x) u_0^2 \, \mathrm{d}x \leq \mathcal{B}(\tilde{R}) \int_{B^c_{R_n}} V(x) u_0^2. \tag{6.2}$$

Combining (6.1) and (6.2), we obtain

$$\int_{B^c_{R_n}} Q(x) f(u_0) u_0 \, \mathrm{d}x \leq \left[\, \sigma \, [\mathcal{B}(\tilde{R})]^{\frac{1}{2}} + \tilde{C}(\gamma, \sigma) \, \right] [\mathcal{B}(\tilde{R})]^{\frac{1}{2}} \int_{B^c_{R_n}} (\, |\nabla u_0|^2 + V(x) u_0^2 \,) \, \mathrm{d}x.$$

Due to the range (2.1) of the parameters α and β, we point out that

$$\lim_{\tilde{R} \to +\infty} \mathcal{B}(\tilde{R}) = \lim_{\tilde{R} \to +\infty} \frac{b}{a} \, \tilde{R}^{\alpha - \beta} = 0$$

and, since $\sigma > 0$ and $\gamma > \gamma_0$ are fixed, we can choose $\tilde{R} > 0$ sufficiently large so that

$$\left[\, \sigma \, [\mathcal{B}(\tilde{R})]^{\frac{1}{2}} + \tilde{C}(\gamma, \sigma) \, \right] [\mathcal{B}(\tilde{R})]^{\frac{1}{2}} \leq \frac{1}{4}.$$

□

Lemma 6.3 *There exists* $\tilde{R} > 0$ *and a constant* $C > 0$ *such that for any* $\varrho > 2\tilde{R}$

$$\int_{B^c_\varrho} (\, |\nabla u_0|^2 + V(x) u_0^2 \,) \, \mathrm{d}x \leq C e^{-\left|\log \frac{3}{4}\right| \varrho^{(2-\alpha)/2}}.$$

Proof Let $\tilde{R}$ and $\{R_n\}_n$ be as in Lemma 6.2 and let $\varrho > 2\tilde{R}$. Then there exist two positive integers $\overline{n} > \tilde{n}$ such that

$$R_{\tilde{n}} \leq \tilde{R} \leq R_{\tilde{n}+1} \quad \text{and} \quad R_{\overline{n}-1} \leq \varrho \leq R_{\overline{n}}$$

and it is easy to see that

$$\overline{n} - \tilde{n} \geq \varrho^{(2-\alpha)/2} - \tilde{R}^{(2-\alpha)/2} > \tilde{R}^{(2-\alpha)/2}(2^{(2-\alpha)/2} - 1) > 2$$

provided $\tilde{R} > 0$ is sufficiently large. Therefore $\overline{n} - \tilde{n} \geq 3$, in particular

$$R_{\overline{n}-2} \geq R_{\tilde{n}+1} \geq \tilde{R}$$

and we may estimate, using Lemma 6.2,

$$\begin{aligned}
\int_{B_\varrho^c} (\,|\nabla u_0|^2 + V(x)u_0^2\,)\,\mathrm{d}x &\leq \int_{B^c_{R_{\overline{n}-1}}} (\,|\nabla u_0|^2 + V(x)u_0^2\,)\,\mathrm{d}x \\
&\leq \left(\frac{3}{4}\right)^{\overline{n}-\tilde{n}-2} \int_{B^c_{\tilde{R}}} (\,|\nabla u_0|^2 + V(x)u_0^2\,)\,\mathrm{d}x \\
&\leq \left(\frac{4}{3}\right)^2 e^{-\left|\log\frac{3}{4}\right|(\varrho^{(2-\alpha)/2} - \tilde{R}^{(2-\alpha)/2})} \int_{B^c_{\tilde{R}}} (\,|\nabla u_0|^2 + V(x)u_0^2\,)\,\mathrm{d}x.
\end{aligned}$$

□

In order to conclude that $u_0 \in L^2(\mathbb{R}^2)$, it is enough to prove that

$$\int_{B_2^c} u_0^2\,\mathrm{d}x < +\infty.$$

First, for fixed $r \geq 2$ and $|y| \geq 2r$, note that

$$\sup_{x\in B(y,r)} \frac{1+|x|^\alpha}{a|y|^\alpha} \leq \frac{1+(r+|y|)^\alpha}{a|y|^\alpha} \leq \frac{1+\left(\frac{3}{2}\,|y|\right)^\alpha}{a|y|^\alpha} \leq \sup_{y\in B_4^c} \frac{1+\left(\frac{3}{2}\,|y|\right)^\alpha}{a|y|^\alpha} =: C(\alpha) < +\infty.$$

Hence, in view of (V), we have

$$\int_{B(y,r)} u_0^2\,\mathrm{d}x \leq \int_{B(y,r)} \frac{1+|x|^\alpha}{a}\, V(x)u_0^2\,\mathrm{d}x \leq C(\alpha)\,|y|^\alpha \int_{B^c_{|y|/2}} V(x)u_0^2\,\mathrm{d}x$$

where we also used the inclusion $B(y,r) \subseteq B^c_{|y|/2}$. If $r > 2\tilde{R}$ then we may apply Lemma 6.3 and get

$$\int_{B(y,r)} u_0^2\,\mathrm{d}x \leq \tilde{C}(\alpha)\,|y|^\alpha\, e^{-\left|\log\frac{3}{4}\right|\left(\frac{|y|}{2}\right)^{(2-\alpha)/2}}. \tag{6.3}$$

Next, let $m \in \mathbb{N}$ and $|y_i| \geq 2$ with $i \in \{1,\ldots,m\}$ be such that

$$B_5 \backslash B_2 \subset \bigcup_{i=1}^m B(y_i, 1)$$

and let $y_{i,k} := 2^k y_i$. If K_0 is a positive integer such that $2^{K_0} > 2\tilde{R}$ then, using (6.3) with $r = 2^k$ and $y = y_{i,k}$,

$$\int_{B(y_{i,k},2^k)} u_0^2\,\mathrm{d}x \leq \tilde{C}(\alpha)\,|y_{i,k}|^\alpha\, e^{-\left|\log\frac{3}{4}\right|\left(\frac{|y_{i,k}|}{2}\right)^{(2-\alpha)/2}} \quad \text{for any } k \geq K_0$$

and

$$\int_{B_2^c} u_0^2 \, dx \le \sum_{k=0}^{+\infty} \int_{2^k B_5 \setminus B_2} u_0^2 \, dx \le \sum_{i=1}^{m} \sum_{k=0}^{+\infty} \int_{B(y_{i,k}, 2^k)} u_0^2 \, dx$$
$$\le \sum_{i=1}^{m} \sum_{k=0}^{K_0-1} \int_{B(y_{i,k}, 2^k)} u_0^2 \, dx + \tilde{C}(\alpha) \sum_{i=1}^{m} \sum_{k=K_0}^{+\infty} |y_{i,k}|^{\alpha} \, e^{-\left|\log \frac{3}{4}\right| \left(\frac{|y_{i,k}|}{2}\right)^{(2-\alpha)/2}} < +\infty$$

since $\alpha \in (0, 2)$. This completes the proof of Proposition 2.2.

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