On Some Asymptotically Equivalence Types for Double Sequences and Relations among Them

Uğur ULUSU[*1], **Erdinç Dündar**[†2]

[1,2]Department of Mathematics, Faculty of Science and Literature, Afyon Kocatepe University, 03200, Afyonkarahisar, TURKEY

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Abstract: In this study, we give definitions of asymptotically lacunary invariant equivalence, strongly asymptotically lacunary invariant equivalence and asymptotically lacunary ideal invariant equivalence for double sequences. We also examine the existence of some relations among these new equivalence definitions.

1. Introduction and Background

Throughout the paper $\mathbb{N}$ denotes the set of natural numbers.
Many authors have studied on the concepts of invariant mean and invariant convergence (see, [10, 11, 13, 14, 19, 20, 25]).
Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies following conditions:

1. $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
2. $\phi(e) = 1$, where $e = (1, 1, 1, \ldots)$ and
3. $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$.
In the case σ is translation mappings $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit.
The concept of lacunary strong σ-convergence was introduced by Savaş [21] and then Pancaroğlu and Nuray [15] defined the concept of lacunary invariant summability.
The idea of $\mathscr{I}$-convergence was introduced by Kostyrko et al. [6] which is based on the structure of the ideal $\mathscr{I}$ of subset of the set $\mathbb{N}$. For more detail, see [7].
A family of sets $\mathscr{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if (i) $\emptyset \in \mathscr{I}$, (ii) For each $A, B \in \mathscr{I}$ we have $A \cup B \in \mathscr{I}$, (iii) For each $A \in \mathscr{I}$ and each $B \subseteq A$ we have $B \in \mathscr{I}$.
An ideal is called non-trivial if $\mathbb{N} \notin \mathscr{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathscr{I}$ for each $n \in \mathbb{N}$.
Recently, the concept of $\sigma\theta$-uniform density of any subset A of the set $\mathbb{N}$ and corresponding the concept of $\mathscr{I}_{\sigma\theta}$-convergence for real sequences were introduced by Ulusu and Nuray [28].
Several convergence concepts for double sequences and some properties of these concepts which are noted following can be seen in [1, 2, 8, 12, 16, 18, 23].
A double sequence $x = (x_{kj})$ is said to be bounded if there exists an $M > 0$ such that $|x_{kj}| < M$ for all k and j, i.e., if $\sup_{k,j} |x_{kj}| < \infty$.
The set of all bounded double sequences will be denoted by ℓ_∞^2.

*ulusu@aku.edu.tr
†edundar@aku.edu.tr

A non-trivial ideal $\mathscr{I}_2$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible ideal if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathscr{I}_2$ for each $i\in N$.
It is evident that a strongly admissible ideal is admissible also.
Let (X,ρ) be a metric space and $\mathscr{I}_2$ be a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$. A double sequence $x=(x_{mn})$ in X is said to be $\mathscr{I}_2$-convergent to $L\in X$ if for every $\varepsilon>0$,

$$A(\varepsilon)=\big\{(m,n)\in\mathbb{N}\times\mathbb{N}:\rho(x_{mn},L)\geq\varepsilon\big\}\in\mathscr{I}_2.$$

It is denoted by $\mathscr{I}_2-\lim_{m,n\to\infty}x_{mn}=L$.
The double sequence $\theta_2=\{(k_r,j_u)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0=0,\ h_r=k_r-k_{r-1}\to\infty\ \text{ and }\ j_0=0,\ \bar{h}_u=j_u-j_{u-1}\to\infty\ \text{ as }\ r,u\to\infty.$$

We use the following notations in the sequel:

$$k_{ru}=k_rj_u,\ \ h_{ru}=h_r\bar{h}_u,\ \ I_{ru}=\big\{(k,j):k_{r-1}<k\leq k_r\ \text{ and }\ j_{u-1}<j\leq j_u\big\}.$$

Recently, the definitions of some invariant convergence for double sequences were presented in a study by Ulusu et al. [27] as below:
Let $\theta_2=\{(k_r,j_u)\}$ be a double lacunary sequence. A double sequence $x=(x_{kj})$ is said to be lacunary invariant convergent to L if

$$\lim_{r,u\to\infty}\frac{1}{h_{ru}}\sum_{k,j\in I_{ru}}x_{\sigma^k(m),\sigma^j(n)}=L,$$

uniformly in $m,n=1,2,\ldots$ and it is denoted by $x_{kj}\to L\big(V_2^{\sigma\theta}\big)$.
A double sequence $x=(x_{kj})$ is said to be strongly lacunary invariant convergent to L if

$$\lim_{r,u\to\infty}\frac{1}{h_{ru}}\sum_{k,j\in I_{ru}}|x_{\sigma^k(m),\sigma^j(n)}-L|=0,$$

uniformly in m,n and it is denoted by $x_{kj}\to L\big([V_2^{\sigma\theta}]\big)$.
Let $\theta_2=\{(k_r,j_u)\}$ be a double lacunary sequence, $A\subseteq\mathbb{N}\times\mathbb{N}$ and

$$s_{ru}=\min_{m,n}\Big|A\cap\big\{\big(\sigma^k(m),\sigma^j(n)\big):(k,j)\in I_{ru}\big\}\Big|,$$

$$S_{ru}=\max_{m,n}\Big|A\cap\big\{\big(\sigma^k(m),\sigma^j(n)\big):(k,j)\in I_{ru}\big\}\Big|.$$

If the following limits exist

$$\underline{V_2^{\theta}}(A)=\lim_{r,u\to\infty}\frac{s_{ru}}{h_{ru}}\quad\text{and}\quad\overline{V_2^{\theta}}(A)=\lim_{r,u\to\infty}\frac{S_{ru}}{h_{ru}},$$

then they are called a lower lacunary σ-uniform density and an upper lacunary σ-uniform density of the set A, respectively. If $\underline{V_2^{\theta}}(A)=\overline{V_2^{\theta}}(A)$, then $V_2^{\theta}(A)=\underline{V_2^{\theta}}(A)=\overline{V_2^{\theta}}(A)$ is called the lacunary σ-uniform density of A.
Denoted by $\mathscr{I}_2^{\sigma\theta}$ the class of all $A\subseteq\mathbb{N}\times\mathbb{N}$ with $V_2^{\theta}(A)=0$.
A double sequence $x=(x_{kj})$ is said to be lacunary $\mathscr{I}_2$-invariant convergent or $\mathscr{I}_2^{\sigma\theta}$-convergent to L if for every $\varepsilon>0$

$$A_\varepsilon=\big\{(k,j)\in I_{ru}:|x_{kj}-L|\geq\varepsilon\big\}\in\mathscr{I}_2^{\sigma\theta},$$

i.e., $V_2^{\theta}(A_\varepsilon)=0$. It is denoted by $\mathscr{I}_2^{\sigma\theta}-\lim x_{kj}=L$ or $x_{kj}\to L\big(\mathscr{I}_2^{\sigma\theta}\big)$.
Marouf [9] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. Then, the concept of asymptotically equivalence has been developed by many researchers (see, [3, 5, 17, 22, 24]).
Two nonnegative sequences $x=(x_k)$ and $y=(y_k)$ are said to be asymptotically equivalent if

$$\lim_k\frac{x_k}{y_k}=1.$$

It is denoted by $x\sim y$.

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Hazarika and Kumar [4] presented some asymptotically equivalence definitions for double sequences as follows: Two nonnegative double sequences $x=(x_{kl})$ and $x=(y_{kl})$ are said to be P-asymptotically equivalent if

$$P-\lim_{k,l}\frac{x_{kl}}{y_{kl}}=1,$$

denoted by $x\sim^{P}y$.
Two nonnegative double sequences $x=(x_{kl})$ and $x=(y_{kl})$ are said to be asymptotically $\mathscr{I}_2$-equivalent of multiple L if for every $\varepsilon>0$

$$\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}\in\mathscr{I}_2,$$

denoted by $x\sim^{\mathscr{I}^L}y$ and simply asymptotically $\mathscr{I}_2$-equivalent if $L=1$.
Recently, Ulusu [26] by defining the concept of lacunary $\mathscr{I}_\sigma$-asymptotically equivalence and the concepts of lacunary σ-asymptotically equivalence for real sequences, studied some relationships among these concepts.

2. Main Results

In this study, we give definitions of asymptotically lacunary invariant equivalence, strongly asymptotically lacunary invariant equivalence and asymptotically lacunary ideal invariant equivalence for double sequences. We also examine the existence of some relations among these new equivalence definitions.

Definition 2.1 Two nonnegative double sequence $x=(x_{kj})$ and $y=(y_{kj})$ are said to be asymptotically lacunary σ_2-equivalent of multiple L if

$$\lim_{r,u\to\infty}\frac{1}{h_{ru}}\sum_{k,j\in I_{ru}}\frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}}=L,$$

uniformly in m and n. In this case, we write $x\overset{N_{2(L)}^{\sigma\theta}}{\sim}y$ and simply asymptotically lacunary σ_2-equivalent if $L=1$.

Definition 2.2 Two nonnegative double sequences $x=(x_{kj})$ and $y=(y_{kj})$ are said to be asymptotically lacunary $\mathscr{I}_2$-invariant equivalent of multiple L if for every $\varepsilon>0$

$$A_\varepsilon^{\sim}:=\left\{(k,j)\in I_{ru}:\left|\frac{x_{kj}}{y_{kj}}-L\right|\geq\varepsilon\right\}\in\mathscr{I}_2^{\sigma\theta},$$

i.e., $V_2^{\theta}(A_\varepsilon^{\sim})=0$. In this case, we write $x\overset{\mathscr{I}_{2(L)}^{\sigma\theta}}{\sim}y$ and simply asymptotically lacunary $\mathscr{I}_2$-invariant equivalent if $L=1$.
The set of all asymptotically lacunary $\mathscr{I}_2$-invariant equivalent of multiple L sequences will be denoted by $\mathfrak{I}_{2(L)}^{\sigma\theta}$.

Theorem 2.3 *Suppose that $x=(x_{kj}),y=(y_{kj})\in\ell_\infty^2$. If x and y are asymptotically lacunary $\mathscr{I}_2$-invariant equivalent of multiple L, then these sequences are asymptotically lacunary σ_2-equivalent of multiple L.*

Proof. Let $m,n\in\mathbb{N}$ be arbitrary and $\varepsilon>0$. Now, we calculate

$$t(\theta_2,m,n):=\left|\frac{1}{h_{ru}}\sum_{k,j\in I_{ru}}\frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}}-L\right|.$$

We have

$$t(\theta_2,m,n)\leq t_1(\theta_2,m,n)+t_2(\theta_2,m,n),$$

where

$$t_1(\theta_2,m,n):=\frac{1}{h_{ru}}\sum_{\substack{k,j\in I_{ru}\\ \left|\frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}}-L\right|\geq\varepsilon}}\left|\frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}}-L\right|$$

and

$$t_2(\theta_2,m,n):=\frac{1}{h_{ru}}\sum_{\substack{k,j\in I_{ru}\\ \left|\frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}}-L\right|<\varepsilon}}\left|\frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}}-L\right|.$$

We get $t_2(\theta_2,m,n)<\varepsilon$, for every $m,n=1,2,\dots$. The boundedness of x and y implies that there exists a $M>0$ such that

$$\left|\frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}}-L\right|\leq M,$$

for all $k,j\in I_{ru}$ and for every m,n. Then, this implies that

$$\begin{aligned} t_1(\theta_2,m,n) &\leq \frac{M}{h_{ru}}\left|\left\{(k,j)\in I_{ru}:\left|\frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}}-L\right|\geq\varepsilon\right\}\right| \\ &\leq M\frac{\max_{m,n}\left|\left\{(k,j)\in I_{ru}:\left|\frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}}-L\right|\geq\varepsilon\right\}\right|}{h_{ru}}=M\frac{S_{ru}}{h_{ru}}, \end{aligned}$$

hence $x \overset{N_{2(L)}^{\sigma\theta}}{\sim} y$. □

The converse of Theorem 2.3 does not hold. For example, $x=(x_{kj})$ and $y=(y_{kj})$ are the sequences defined by following;

$$x_{kj}:=\begin{cases} 2\ , & \text{if } \begin{array}{l} k_{r-1}<k<k_{r-1}+[\sqrt{h_r}], \\ j_{r-1}<j<j_{r-1}+[\sqrt{\bar{h}_u}], \end{array} \text{ and } k+j \text{ is an even integer.} \\ 0\ , & \text{if } \begin{array}{l} k_{r-1}<k<k_{r-1}+[\sqrt{h_r}], \\ j_{r-1}<j<j_{r-1}+[\sqrt{\bar{h}_u}], \end{array} \text{ and } k+j \text{ is an odd integer.} \end{cases}$$

$$y_{kj}:=1.$$

When $\sigma(m)=m+1$ and $\sigma(n)=n+1$, this sequences are asymptotically lacunary σ_2-equivalent but they are not asymptotically lacunary $\mathscr{I}_2$-invariant equivalent.

Definition 2.4 Two nonnegative double sequence $x=(x_{kj})$ and $y=(y_{kj})$ are said to be strongly asymptotically lacunary σ_2-equivalent of multiple L if

$$\lim_{r,u\to\infty}\frac{1}{h_{ru}}\sum_{k,j\in I_{ru}}\left|\frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}}-L\right|=0,$$

uniformly in m and n. In this case, we write $x \overset{[N_{2(L)}^{\sigma\theta}]}{\sim} y$ and simply strongly asymptotically lacunary σ_2-equivalent if $L=1$.

The set of all strongly asymptotically lacunary invariant equivalent of multiple L sequences will be denoted by $[\mathfrak{N}_{2(L)}^{\sigma\theta}]$.

Theorem 2.5 *If double sequences $x=(x_{kj})$ and $y=(y_{kj})$ are strongly asymptotically lacunary σ_2-equivalent of multiple L, then these sequences are asymptotically lacunary $\mathscr{I}_2$-invariant equivalent of multiple L.*

Proof. Let $x \overset{[N_{2(L)}^{\sigma\theta}]}{\sim} y$ and given $\varepsilon>0$. Then, for every $m,n\in\mathbb{N}$ we have

$$\begin{aligned} \sum_{k,j\in I_{ru}}\left|\frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}}-L\right| &\geq \sum_{\substack{k,j\in I_{ru} \\ \left|\frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}}-L\right|\geq\varepsilon}}\left|\frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}}-L\right| \\ &\geq \varepsilon\cdot\left|\left\{(k,j)\in I_{ru}:\left|\frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}}-L\right|\geq\varepsilon\right\}\right| \\ &\geq \varepsilon\cdot\max_{m,n}\left|\left\{(k,j)\in I_{ru}:\left|\frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}}-L\right|\geq\varepsilon\right\}\right| \end{aligned}$$

and so

$$\begin{aligned} \frac{1}{h_{ru}}\sum_{k,j\in I_{ru}}\left|\frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}}-L\right| &\geq \varepsilon\cdot\frac{\max_{m,n}\left|\left\{(k,j)\in I_{ru}:\left|\frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}}-L\right|\geq\varepsilon\right\}\right|}{h_{ru}} \\ &= \varepsilon\cdot\frac{S_{ru}}{h_{ru}} \end{aligned}$$

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This implies that $\lim_{r,u\to\infty} \frac{S_{ru}}{h_{ru}} = 0$ and so $x \overset{\mathscr{I}_{2(L)}^{\sigma\theta}}{\sim} y$. □

Theorem 2.6 *Suppose that $x = (x_{kj}), y = (y_{kj}) \in \ell_\infty^2$. If double sequences x and y are asymptotically lacunary $\mathscr{I}_2$-invariant equivalent of multiple L, then these sequences strongly asymptotically lacunary σ_2-equivalent of multiple L.*

Proof. Suppose that $x, y \in \ell_\infty^2$ and $x \overset{\mathscr{I}_{2(L)}^{\sigma\theta}}{\sim} y$. Let $\varepsilon > 0$. By assumption, we have $V_2^\theta(A_\varepsilon^\sim) = 0$. The boundedness of x and y implies that there exists an $M > 0$ such that

$$\left| \frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}} - L \right| \leq M$$

for all $k, j \in I_{ru}$ and for every m, n. Observe that, for every $m, n \in \mathbb{N}$ we have

$$\begin{aligned} \frac{1}{h_{ru}} \sum_{k,j \in I_{ru}} \left| \frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}} - L \right| &= \frac{1}{h_{ru}} \sum_{\substack{k,j \in I_{ru} \\ \left| \frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}} - L \right| \geq \varepsilon}} \left| \frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}} - L \right| \\ &\quad + \frac{1}{h_{ru}} \sum_{\substack{k,j \in I_{ru} \\ \left| \frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}} - L \right| < \varepsilon}} \left| \frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}} - L \right| \\ &\leq M \frac{\max_{m,n} \left| \left\{ (k,j) \in I_{ru} : \left| \frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}} - L \right| \geq \varepsilon \right\} \right|}{h_{ru}} + \varepsilon \\ &\leq M \frac{S_{ru}}{h_{ru}} + \varepsilon. \end{aligned}$$

Hence, we obtain

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{k,j \in I_{ru}} \left| \frac{x_{\sigma^k(m),\sigma^j(n)}}{y_{\sigma^k(m),\sigma^j(n)}} - L \right| = 0,$$

uniformly in m and n. □

Theorem 2.7

$$\mathfrak{I}_{2(L)}^{\sigma\theta} \cap \ell_\infty^2 = [\mathfrak{N}_{2(L)}^{\sigma\theta}] \cap \ell_\infty^2.$$

Proof. This is an immediate consequence of Theorem 2.5 and Theorem 2.6. □

References

[1] Das, P., Kostyrko, P., Wilczyński, W., Malik, P. $\mathscr{I}$ and $\mathscr{I}^*$-convergence of double sequences. Math. Slovaca, 58(5), (2008), 605-620.

[2] Dündar, E., Altay, B. $\mathscr{I}_2$-convergence and $\mathscr{I}_2$-Cauchy of double sequences. Acta Mathematica Scientia, 34(2), (2014), 343-353.

[3] Esi, A. On asymptotically double lacunary statistically equivalent sequences. Applied Mathematics Letters, 22(12), (2009), 1781-1785.

[4] Hazarika, B., Kumar, V. On asymptotically double lacunary statistical equivalent sequences in ideal context. Journal of Inequlities and Applications, 2013:543, (2013), 1-15.

[5] Hazarika, B. On asymptotically ideal equivalent sequences. Journal of the Egyptian Mathematical Society, 23(1), (2015), 67-72.

[6] Kostyrko, P., Wilczyński, W., Šalát, T., $\mathscr{I}$-Convergence. Real Anal. Exchange, 26(2), (2000), 669-686.

[7] Kostyrko, P., Macaj, M., Šalát, T., Sleziak, M. $\mathscr{I}$-convergence and external $\mathscr{I}$-limit points. Math. Slovaca, 55(4), (2005), 443-464.

[8] Kumar, V. On $\mathscr{I}$ and $\mathscr{I}^*$-convergence of double sequences. Math. Commun. 12(2), (2007), 171-181.

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[9] Marouf, M. Asymptotic equivalence and summability. Int. J. Math. Math. Sci. 16(4), (1993), 755-762.

[10] Mursaleen, M. Invariant means and some matrix transformations. Tamkang J. Math. 10(2), (1979), 183-188.

[11] Mursaleen, M. Matrix transformation between some new sequence spaces. Houston J. Math. 9(4), (1983), 505-509.

[12] Mursaleen, M., Edely, O. H. H. Statistical convergence of double sequences. J. Math. Anal. Appl. 288(1), (2003), 223-231.

[13] Mursaleen, M., Edely, O. H. H. On the invariant mean and statistical convergence. Appl. Math. Lett. 22(11), (2009), 1700-1704.

[14] Nuray, F., Gök, H., Ulusu, U. $\mathscr{I}_\sigma$-convergence. Math. Commun. 16(2), (2011), 531-538.

[15] Pancaroğlu, N., Nuray, F. Statistical lacunary invariant summability. Theoretical Mathematics and Applications, 3(2), (2013), 71-78.

[16] Patterson, R. F., Savaş, E. Lacunary statitistical convergence of double sequences. Math. Commun. 10(1), (2005), 55-61.

[17] Patterson, R. F., Savaş, E. On asymptotically lacunary statistically equivalent sequences, Thai J. Math. 4(2), (2006), 267-272.

[18] Pringsheim, A. Zur theorie der zweifach unendlichen Zahlenfolgen. Math. Ann. 53(3), (1900), 289-321.

[19] Raimi, R. A. Invariant means and invariant matrix methods of summability. Duke Math. J. 30(1), (1963), 81-94.

[20] Savaş, E. Some sequence spaces involving invariant means. Indian J. Math. 31, (1989), 1-8.

[21] Savaş, E. On lacunary strong σ-convergence. Indian J. Pure Appl. Math. 21(4) (1990), 359-365.

[22] Savaş, E., Patterson, R. F. σ-asymptotically lacunary statistical equivalent sequences. Open Mathematics, 4(4), (2006), 648-655.

[23] Savaş, E., Patterson, R. F. Double σ-convergence lacunary statistical sequences. Journal of Computational Analysis and Applications, 11(4), (2009), .

[24] Savaş, E. On $\mathscr{I}$-asymptotically lacunary statistical equivalent sequences. Adv. Difference Equ. 2013:111, (2013), 7 pages. doi:10.1186/1687-1847-2013-111.

[25] Schaefer, P. Infinite matrices and invariant means. Proc. Amer. Math. Soc. 36(1), (1972), 104-110.

[26] Ulusu, U. Lacunary $\mathscr{I}_\sigma$-asimptotic equivalence, AKU J. Sci. Eng. 17, (2017), 899-905. DOI: 10.5578/fmbd.66209

[27] Ulusu, U., Dündar, E., Nuray, F. Lacunary $\mathscr{I}_2$-invariant convergence and some properties. International Journal of Analysis and Applications, 16(3), (2018), 317-327.

[28] Ulusu, U., Nuray, F. Lacunary $\mathscr{I}_\sigma$-convergence, (under review).