$\mathcal{I}_2$-LACUNARY STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES OF SETS

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Abstract

In this paper, we introduce the concepts of the Wijsman $\mathcal{I}_2$-statistical convergence, Wijsman $\mathcal{I}_2$-lacunary statistical convergence and Wijsman strongly $\mathcal{I}_2$-lacunary convergence of double sequences of sets and investigate the relationship between them.

Introduction, Definitions and Notations

The concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets. The one of these such extensions considered in this paper is the concept of Wijsman convergence (see, [2, 4, 11, 25, 27, 28]).
Nuray and Rhoades [17] extended the notion of convergence of set sequences to statistical convergence, and gave some basic theorems.
Ulusu and Nuray [25] defined the Wijsman lacunary statistical convergence of sequence of sets, and considered its relation with Wiijsman statistical convergence, which was defined by Nuray and Rhoades.

Introduction, Definitions and Notations

Recently, Kişi and Nuray [11] introduced a new convergence notion, for sequences of sets, which is called Wijsman $\mathcal{I}$-convergence. The concepts of convergence, statistical convergence and ideal convergence of double sequences of sets were studied by Nuray et. al [19, 20, 21, 22].
Das et al. [5] introduced new notions, namely $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence by using ideal. Recently, Ulusu and Dündar [26] studied the concepts of Wijsman $\mathcal{I}$-statistical convergence, Wijsman $\mathcal{I}$-lacunary statistical convergence and Wijsman strongly $\mathcal{I}$-lacunary convergence of sequences of sets.

Introduction, Definitions and Notations

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

Throughout the paper, we let (X, ρ) be a metric space and A, A_k be any non-empty closed subsets of X.

Introduction, Definitions and Notations

Definition 1

([17]) We say that the sequence $\{A_k\}$ is Wijsman statistical convergent to A if for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case we write $st - \lim_W A_k = A$ or $A_k \to A(WS)$.

Introduction, Definitions and Notations

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Definition 2

([25]) We say that the sequence $\{A_k\}$ is Wijsman lacunary statistically convergent to A, if for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_r \frac{1}{h_r} |k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon| = 0.$$

Introduction, Definitions and Notations

Definition 3

([13]) A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$,
(ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$,
(iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.
An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if
$\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.
Throughout the paper we take $\mathcal{I}_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Introduction, Definitions and Notations

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in N$.
It is evident that a strongly admissible ideal is admissible also.
$\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A)\}$.
Then $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

Introduction, Definitions and Notations

Definition 4

([13]) A family of sets $F \subseteq 2^{\mathbb{N}}$ is a filter if and only if

(i) $\emptyset \notin F$,
(ii) For each $A, B \in F$ we have $A \cap B \in F$,
(iii) For each $A \in F$ and each $B \supseteq A$ we have $B \in F$.

Proposition 2.1

([13]) $\mathcal{I}$ is a non-trivial ideal in $\mathbb{N}$ if and only if

$$F(\mathcal{I}) = \{M \subset \mathbb{N} : (\exists A \in \mathcal{I})(M = \mathbb{N} \backslash A)\}$$

is a filter in $\mathbb{N}$.

Introduction, Definitions and Notations

Definition 5

([6]) We say that an admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j \in \mathcal{I}_2^0$, i.e., $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$ (hence $B_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

Throughout the paper, we let (X, ρ) be a separable metric space, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal, $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal and A, A_{kj} be any non-empty closed subsets of X.

Introduction, Definitions and Notations

Definition 6

([12]) We say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}$-statistical convergent to A or $S(\mathcal{I}_W)$-convergent to A if for each $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \delta \right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A(S(\mathcal{I}_W))$.

Introduction, Definitions and Notations

Definition 7

([12]) Let θ be lacunary sequence. We say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}$-lacunary statistical convergent to A or $S_\theta(\mathcal{I}_W)$-convergent to A if for each $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \{ k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon \} \right| \geq \delta \right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A(S_\theta(\mathcal{I}_W))$.

Introduction, Definitions and Notations

Definition 8

([12]) Let θ be lacunary sequence. We say that the sequence $\{A_k\}$ is said to be Wijsman strongly $\mathcal{I}$-lacunary convergent to A or $N_\theta[\mathcal{I}_W]$-convergent to A if for each $\varepsilon > 0$ and for each $x \in X$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| \geq \varepsilon \right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A\,(N_\theta\,[\mathcal{I}_W])$.

Introduction, Definitions and Notations

Definition 9

([21]) We say that the double sequence $\{A_{kj}\}$ is Wijsman statistically convergent to A if for each $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn}|\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}| = 0,$$

that is,

$$|d(x, A_{kj}) - d(x, A)| < \varepsilon, \quad \text{a.a. (k,j)}.$$

In this case we write $st_2 - \lim_W A_k = A$.

Introduction, Definitions and Notations

The double sequence $\theta = \{(k_r, j_s)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty \quad as \quad r \to \infty$$

and

$$j_0 = 0, \quad \bar{h}_u = j_u - j_{u-1} \to \infty \quad as \quad u \to \infty.$$

We use following notations in the sequel:

$$k_{ru} = k_r j_u, \;\; h_{ru} = h_r \bar{h}_u,$$

$$I_{ru} = \{(k,j) : k_{r-1} < k \leq k_r \;\; and \;\; j_{u-1} < j \leq j_u\},$$

$$q_r = \frac{k_r}{k_{r-1}} \;\; and \;\; q_u = \frac{j_u}{j_{u-1}}.$$

Introduction, Definitions and Notations

Definition 10

([22]) We say that the double sequence $\{A_{kj}\}$ is Wijsman lacunary statistically convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{r,u\to\infty} \frac{1}{h_r \bar{h}_u} |\{(k,j) \in I_{ru} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case we write $st_2 - \lim_{W_\theta} A_{kj} = A$.

Introduction, Definitions and Notations

Definition 11

([19]) We say that the double sequence of sets $\{A_{kj}\}$ is $\mathcal{I}_{W_2}$-convergent to A, if for every $x \in X$ and for every $\varepsilon > 0$,

$$\{(k,j) \in \mathbb{N} \times \mathbb{N} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case we write $\mathcal{I}_{W_2} - \lim\limits_{k,j \to \infty} A_{kj} = A$.

Main Results

In this section, we define the concepts of Wijsman $\mathcal{I}_2$-statistical convergence, Wijsman $\mathcal{I}_2$-lacunary statistical convergence and Wijsman strongly $\mathcal{I}_2$-lacunary convergence of double sequences of sets and investigate the relationship between them.

Main Results

Definition 12

We say that the sequence $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2$-statistical convergent to A or $S(\mathcal{I}_{W_2})$-convergent to A if for each $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$, the set

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}| \geq \delta\right\}$$

belongs to $\mathcal{I}_2$. In this case, we write $A_{kj} \to A(S(\mathcal{I}_{W_2}))$.

The set of Wijsman $\mathcal{I}_2$-statistical convergent double sequences will be denoted by $\{S(\mathcal{I}_{W_2})\}$.

Main Results

Definition 13

Let θ be a double lacunary sequence. We say that the sequence $\{A_{kj}\}$ is said to be Wijsman $\mathcal{I}_2$-lacunary convergent to A or $N_\theta(\mathcal{I}_{W_2})$-convergent to A if for each $\varepsilon > 0$ and for each $x \in X$, the set

$$\left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \left(\frac{1}{h_r \overline{h}_u} \sum_{(k,j) \in I_{ru}} d(x, A_{kj}) - d(x, A) \right) \geq \varepsilon \right\}$$

belongs to $\mathcal{I}_2$. In this case, we write $A_{kj} \to A\left(N_\theta\left(\mathcal{I}_{W_2}\right)\right)$.

Main Results

Definition 14

Let θ be a double lacunary sequence. We say that the sequence $\{A_{kj}\}$ is said to be Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A or $N_\theta[\mathcal{I}_{W_2}]$-convergent to A if for each $\varepsilon > 0$ and for each $x \in X$,

$$\left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \overline{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \to A \left(N_\theta \left[\mathcal{I}_{W_2} \right] \right)$.

The set of Wijsman strongly $\mathcal{I}_2$-lacunary convergent double sequences will be denoted by $\left\{ N_\theta \left[\mathcal{I}_{W_2} \right] \right\}$.

Main Results

Definition 15

Let θ be a double lacunary sequence. We say that the sequence $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2$-lacunary statistical convergent to A or $S_\theta\left(\mathcal{I}_{W_2}\right)$-convergent to A if for each $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \left| \{(k,j) \in I_{ru} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\} \right| \geq \delta\right\}$$

belogs to $\mathcal{I}_2$. In this case, we write $A_{kj} \to A\left(S_\theta\left(\mathcal{I}_{W_2}\right)\right)$.

The set of Wijsman $\mathcal{I}_2$-lacunary statistical convergent double sequences will be denoted by $\left\{S_\theta\left(\mathcal{I}_{W_2}\right)\right\}$.

Main Results

Definition 16

A double sequence $\{A_{kj}\}$ is said to be bounded if there exists a positive real number M such that

$$|d(x, A_{kj})| < M,$$

for each $x \in X$ and for all $k, j \in \mathbb{N}$. That is

$$\sup_{k,j} d(x, A_{kj}) < \infty.$$

The set of all bounded double sequences of sets will be denoted by L^2_∞.

Main Results

Theorem 17

Let θ be a double lacunary sequence. Then,
$A_{kj} \rightarrow A(N_\theta[\mathcal{I}_{W_2}]) \Rightarrow A_{kj} \rightarrow A(S_\theta(\mathcal{I}_{W_2}))$.

Main Results

Proof: Let $A_{kj} \to A(N_\theta[\mathcal{I}_{W_2}])$ and $\varepsilon > 0$. Then, for each $x \in X$ we can write

$$\sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq \sum_{\substack{(k,j)\in I_{ru} \\ |d(x,A_{kj})-d(x,A)|\geq\varepsilon}} |d(x, A_{kj}) - d(x, A)|$$

$$\geq \varepsilon\{(k, j) \in I_{ru} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}$$

and so

Main Results

Proof:

$$\frac{1}{\varepsilon \cdot h_r \overline{h}_u} \sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)|$$
$$\geq \frac{1}{h_r \overline{h}_u} \{(k,j) \in I_{ru} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}$$

Main Results

Proof: Hence, for each $x \in X$ and for any $\delta > 0$,

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \overline{h}_u} \left|\{(k,j) \in I_{ru} : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\}\right| \geq \delta\right\}$$

$$\subseteq \left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \overline{h}_u} \sum_{(k,j) \in I_{ru}} |d(x,A_{kj}) - d(x,A)| \geq \varepsilon \cdot \delta\right\} \in \mathcal{I}_2.$$

This proof is completed.

Main Results

Theorem 18

Let θ be a double lacunary sequence. Then, $\{A_{kj}\} \in L^2_\infty$ and $A_{kj} \to A(S_\theta(\mathcal{I}_{W_2})) \Rightarrow A_{kj} \to A(N_\theta[\mathcal{I}_{W_2}])$.

Main Results

Proof: Suppose that $A_{kj} \to A(S_\theta(\mathcal{I}_{W_2}))$ and $A_{kj} \in L^2_\infty$. Then, there exists an $M > 0$ such that

$$|d(x, A_{kj}) - d(x, A)| \leq M$$

for each $x \in X$ and all $k, j \in \mathbb{N}$.

Main Results

Proof: Given $\varepsilon > 0$, for each $x \in X$ we have

$$\frac{1}{h_r\bar{h}_u}\sum_{(k,j)\in I_{ru}}|d(x,A_{kj})-d(x,A)|$$
$$=\frac{1}{h_r\bar{h}_u}\sum_{\substack{(k,j)\in I_{ru}\\ |d(x,A_{kj})-d(x,A)|\geq\frac{\varepsilon}{2}}}|d(x,A_{kj})-d(x,A)|$$
$$+\frac{1}{h_r\bar{h}_u}\sum_{\substack{(k,j)\in I_{ru}\\ |d(x,A_{kj})-d(x,A)|<\frac{\varepsilon}{2}}}|d(x,A_{kj})-d(x,A)|$$
$$\leq\frac{M}{h_r\bar{h}_u}\{(k,j)\in I_{ru}:|d(x,A_{kj})-d(x,A)|\geq\frac{\varepsilon}{2}\}+\frac{\varepsilon}{2}.$$

Main Results

Proof: Hence, for each $x \in X$ we have

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq \varepsilon \right\} \subseteq$$

$$\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \{(k,j) \in I_{ru} : |d(x, A_{kj}) - d(x, A)| \geq \frac{\varepsilon}{2}\} \geq \frac{\varepsilon}{2M}\} \in \mathcal{I}$$

This proof is completed.

Main Results

We have the following Theorem by Theorem 17 and Theorem 18.

Theorem 19

Let θ be a double lacunary sequence. Then,
$\{S_\theta(\mathcal{I}_{W_2})\} \cap L^2_\infty = \{N_\theta[\mathcal{I}_{W_2}]\} \cap L^2_\infty$.

Theorem 20

Let θ be a double lacunary sequence. If $\liminf_r q_r > 1$ *and* $\liminf_u q_u > 1$ *then,* $A_{kj} \to A(S(\mathcal{I}_{W_2}))$ *implies* $A_{kj} \to A(S_\theta(\mathcal{I}_{W_2}))$.

Main Results

Proof: Assume that $\liminf_r q_r > 1$ and $\liminf_u q_u > 1$, then there exist $\lambda, \mu > 0$ such that

$$q_r \geq 1 + \lambda \quad and \quad q_u \geq 1 + \mu$$

for sufficiently large r, u which implies that

$$\frac{h_r \overline{h}_u}{k_{ru}} \geq \frac{\lambda\mu}{(1+\lambda)(1+\mu)}.$$

Main Results

Proof: If $A_{kj} \to A(S(\mathcal{I}_{W_2}))$, then for every $\varepsilon > 0$, for each $x \in X$ and for sufficiently large r, u, we have

$$\frac{1}{k_r j_u} |\{k \leq k_r, j \leq j_u : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}|$$
$$\geq \frac{1}{k_r j_u} |\{(k, j) \in I_{ru} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}|$$
$$\geq \frac{\lambda\mu}{(1+\lambda)(1+\mu)} . \left(\frac{1}{h_r \overline{h}_u} |\{(k, j) \in I_{ru} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}| \right).$$

Main Results

Proof: Hence, for each $x \in X$ and for any $\delta > 0$ we have

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r \bar{h}_u} \left|\{(k,j) \in I_{ru} : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\}\right| \geq \delta\right\}$$

$$\subseteq \left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{k_r j_u} \left|\{k \leq k_r, j \leq j_u : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\}\right| \geq$$

Hence, $A_{kj} \to A(S_\theta(\mathcal{I}_{W_2}))$.

Main Results

Theorem 21

Let θ be a double lacunary sequence. If $\limsup_r q_r < \infty$ *and* $\limsup_u q_u < \infty$ *then,* $A_{kj} \to A(S_\theta(\mathcal{I}_{W_2}))$ *implies* $A_{kj} \to A(S(\mathcal{I}_{W_2}))$.

Main Results

Proof: If $\limsup_r q_r < \infty$ and $\limsup_u q_u < \infty$, then there is an $M, N > 0$ such that $q_r < M$ and $q_u < N$, for all r, u. Suppose that $A_{kj} \to A(S_\theta(\mathcal{I}_{W_2}))$ and let

$$U_{ru} = U(r,u,x) := \left|\left\{(k,j) \in I_{ru} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\right\}\right|.$$

Since $A_{kj} \to A(S_\theta(\mathcal{I}_{W_2}))$, it follows that for each $x \in X$, for every $\varepsilon > 0$ and $\delta > 0$

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u}\left|\left\{(k,j) \in I_{ru} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\right\}\right| \geq \delta\right\}$$

$$= \left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{U_{ru}}{h_r \bar{h}_u} \geq \delta\right\} \in \mathcal{I}_2.$$

Main Results

Proof: Hence, we can choose a positive integers $r_0, u_0 \in \mathbb{N}$ such that

$$\frac{U_{ru}}{h_r \overline{h}_u} < \delta, \text{ for all } r > r_0, u > u_0.$$

Now let

$$K := \max \left\{ U_{ru} : 1 \leq r \leq r_0, 1 \leq u \leq u_0 \right\}$$

and let t and v be any integers satisfying $k_{r-1} < t \leq k_r$ and $j_{u-1} < v \leq j_u$.

Main Results

Proof: Then we have

$$\frac{1}{tv}\left|\left\{k\le t, j\le v : |d(x,A_{kj})-d(x,A)|\ge\varepsilon\right\}\right|$$

$$\le \frac{1}{k_{r-1}j_{u-1}}\left|\left\{k\le k_r, j\le j_u : |d(x,A_{kj})-d(x,A)|\ge\varepsilon\right\}\right|$$
$$=\frac{1}{k_{r-1}j_{u-1}}\left(U_{11}+U_{12}+U_{21}+U_{22}+\cdots+U_{r_0u_0}+\cdots+U_{ru}\right)$$

$$\le \frac{K}{k_{r-1}j_{u-1}}\cdot r_0u_0+\frac{1}{k_{r-1}j_{u-1}}\left(h_{r_0}\overline{h}_{u_0+1}\frac{U_{r_0,u_0+1}}{h_{r_0}\overline{h}_{u_0+1}}+h_{r_0+1}\overline{h}_{u_0}\frac{U_{r_0+1,u_0}}{h_{r_0+1}\overline{h}_{u_0}}\right.$$

$$\left.+\cdots+h_r\overline{h}_u\frac{U_{ru}}{h_r\overline{h}_u}\right)$$

Main Results

Proof:

$$\leq \frac{r_0 u_0 \cdot K}{k_{r-1} j_{u-1}} + \frac{1}{k_{r-1} j_{u-1}} \left(\sup_{\substack{r>r_0 \\ u>u_0}} \frac{U_{ru}}{h_r \overline{h}_u} \right) \left(h_{r_0} \overline{h}_{u_0+1} + h_{r_0+1} \overline{h}_{u_0} + \cdots + h_r \overline{h}_u \right)$$

$$\leq \frac{r_0 u_0 \cdot K}{k_{r-1} j_{u-1}} + \varepsilon \cdot \frac{(k_r - k_{r_0})(j_u - j_{u_0})}{k_{r-1} j_{u-1}}$$

$$\leq \frac{r_0 u_0 \cdot K}{k_{r-1} j_{u-1}} + \varepsilon \cdot q_r \cdot q_u \leq \frac{r_0 u_0 \cdot K}{k_{r-1} j_{u-1}} + \varepsilon \cdot M \cdot N.$$

Main Results

Proof: Since $k_{r-1}j_{u-1} \to \infty$ as $t, v \to \infty$, it follows that

$$\frac{1}{tv}\Big|\{k \leq t, j \leq v : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}\Big| \to 0$$

and consequently, for any $\delta_1 > 0$, the set

$$\left\{(t, v) \in \mathbb{N}\times\mathbb{N} : \frac{1}{tv}\big|\{k \leq t, j \leq v : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}\big| \geq \delta_1\right\} \in \mathcal{I}$$

This shows that $A_{kj} \to A(S(\mathcal{I}_{W_2}))$.

Main Results

Theorem 22

Let θ be a double lacunary sequence. If
$1 < \liminf_r q_r \leq \limsup_r q_r < \infty$ *and*
$1 < \liminf_u q_u \leq \limsup_u q_u < \infty$ *then* $\{S_\theta(\mathcal{I}_{W_2})\} = \{S(\mathcal{I}_{W_2})\}$.

Proof: This follows from Theorem 20 and Theorem 21.

Main Results

Theorem 23

Let $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal satisfying property (AP2) and $\theta \in \mathcal{F}(\mathcal{I}_2)$. If $\{A_{kj}\} \in \{S(\mathcal{I}_{W_2})\} \cap \{S_\theta(\mathcal{I}_{W_2})\}$, then $S(\mathcal{I}_{W_2}) - \lim A_{kj} = S_\theta(\mathcal{I}_{W_2}) - \lim A_{kj}$.

Main Results

Proof: Assume that $S(\mathcal{I}_{W_2}) - \lim A_{kj} = A$ and $S_\theta(\mathcal{I}_{W_2}) - \lim A_{kj} = B$ and $A \neq B$. Let

$$0 < \varepsilon < \frac{1}{2}|d(x, A) - d(x, B)|$$

for each $x \in X$. Since $\mathcal{I}_2$ satisfies the property $(AP2)$, there exists $M \in \mathcal{F}(\mathcal{I}_2)$ $(i.e., \mathbb{N} \times \mathbb{N} \backslash M \in \mathcal{I}_2)$ such that for each $x \in X$ and for $(m, n) \in M$,

$$\lim_{m,n \to \infty} \frac{1}{mn} |\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}| = 0.$$

Main Results

Proof: Let

$$P = \{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A) \geq \varepsilon|\}$$

and

$$R = \{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, B)| \geq \varepsilon\}.$$

Then, $mn = |P \cup R| \leq |P| + |R|$. This implies that

$$1 \leq \frac{|P|}{mn} + \frac{|R|}{mn}.$$

Since

$$\frac{|R|}{mn} \leq 1 \quad and \quad \lim_{m,n\to\infty} \frac{|P|}{mn} = 0,$$

so we must have

$$\lim_{m,n\to\infty} \frac{|R|}{mn} = 1.$$

Main Results

Proof: Let $M^* = M \cap \theta \in \mathcal{F}(\mathcal{I}_2)$. Then, for $(k_l, j_t) \in M^*$ the $k_l j_t$th term of the statistical limit expression

$$\frac{1}{mn}\left|\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, B)| \geq \varepsilon\}\right|$$

is

$$\frac{1}{k_l j_t}\left|\left\{(k,j) \in \bigcup_{r,u=1,1}^{l,t} I_{ru} : |d(x, A_{kj}) - d(x, B)| \geq \varepsilon\right\}\right|$$

$$= \frac{1}{\sum\limits_{r,u=1,1}^{l,t} h_r \overline{h}_u} \sum_{r,u=1,1}^{l,t} v_{ru} h_r \overline{h}_u, \qquad (3.1)$$

Main Results

Proof: where

$$v_{ru} = \frac{1}{h_r \bar{h}_u} \left| \{(k,j) \in I_{ru} : |d(x, A_{kj}) - d(x, B)| \geq \varepsilon\} \right| \xrightarrow{\mathcal{I}_2} 0$$

because $\{A_{kj}\} \to B(S_\theta(\mathcal{I}_{W_2}))$. Since θ is a double lacunary sequence, (3.1) is a regular weighted mean transform of v_{ru}'s and therefore it is also $\mathcal{I}_2$-convergent to 0 as $l, t \to \infty$, and so it has a subsequence which is convergent to 0 since $\mathcal{I}_2$ satisfies property $(AP2)$.

Main Results

Proof: But since this is a subsequence of

$$\left\{ \frac{1}{mn} \left| \left\{ k \leq m, j \leq n : \left| d(x, A_{kj}) - d(x, B) \right| \geq \varepsilon \right\} \right| \right\}_{(m,n) \in M},$$

we infer that

$$\left\{ \frac{1}{mn} \left| \left\{ k \leq m, j \leq n : \left| d(x, A_{kj}) - d(x, B) \right| \geq \varepsilon \right\} \right| \right\}_{(m,n) \in M}$$

is not convergent to 1. This is a contradiction. Hence the proof is completed.

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