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$\mathcal{I}_2$-CONVERGENCE OF DOUBLE SEQUENCES OF FUZZY NUMBERS

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ABSTRACT. In this paper, we introduce and study the concepts of $\mathcal{I}_2$-convergence, $\mathcal{I}_2^*$-convergence for double sequences of fuzzy real numbers, where $\mathcal{I}_2$ denotes the ideal of subsets of $\mathbb{N} \times \mathbb{N}$. Also, we study some properties and relations of them.

1. Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [9] and Schoenberg [33]. A lot of development have been made in this area after the works of Śalát [27] and Fridy [10, 12]. In general, statistically convergent sequences satisfy many of the properties of ordinary convergent sequences in metric spaces [9, 10, 12, 25]. This concept was extended to the double sequences by Mursaleen and Edely [18] and Tripathy [36] independently. Çakan and Altay [4] presented multidimensional analogues of the results presented by Fridy and Orhan [11].

The concept of ordinary convergence of a sequence of fuzzy real numbers was firstly introduced by Matloka [17] and proved some basic theorems for sequences of fuzzy real numbers. Nanda [19] studied the sequences of fuzzy real numbers and showed that the set of all convergent sequences of fuzzy real numbers form a complete metric space. Recently, Nuray and Savaş [23] defined the concepts of statistical convergence and statistically Cauchy for sequences of fuzzy real numbers. They proved that a sequence of fuzzy real number is statistically convergent if and only if it is statistically Cauchy. Nuray [22] introduced Lacunary statistical convergence of sequences of fuzzy real numbers whereas Savaş [29] studied some equivalent alternative conditions for a sequence of fuzzy real numbers to be statistically Cauchy. A lot of development have been made in this area after the works of Altnok et al. [2], Bede [3], Saadati [26], Savaş [31, 32], Talo and Başar [34], Tripathy and Sarma [37] and many others.

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [13] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers $\mathbb{N}$. Nuray and Ruckle [21] indepedently introduced the same with another name generalized

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statistical convergence. Kostyrko et al. [14] gave some of basic properties of $\mathcal{I}$-convergence and dealt with extremal $\mathcal{I}$-limit points. Das et al. [5] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. Also, Das and Malik [6] introduced the concept of $\mathcal{I}$-limit points, $\mathcal{I}$-cluster points and $\mathcal{I}$-limit superior and $\mathcal{I}$-limit inferior of double sequences. A lot of developments have been made in this area after the works of Kumar [15], Šalát et al. [28], Tripathy and Tripathy [35], Nabiev et al. [20] and many others.

Kumar and Kumar [16] studied the concepts of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence and $\mathcal{I}$-Cauchy sequence for sequences of fuzzy real numbers.

In this paper, we introduce and study the concepts of $\mathcal{I}_2$-convergence and $\mathcal{I}_2^*$-convergence for double sequences of fuzzy real numbers where $\mathcal{I}_2$ denotes the ideal of subsets of $\mathbb{N} \times \mathbb{N}$. Also, we study some properties and relations of them.

2. **Definitions and Notations**

Now, we recall the concept of ideal, convergence, statistical convergence, ideal convergence of sequence, double sequence and fuzzy numbers and some basic definitions (See [1, 5, 8, 9, 13, 18, 24]).

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$, whenever $m, n > N_\varepsilon$. In this case we write

$$\lim_{m,n\to\infty} x_{mn} = L.$$

A double sequence $x = (x_{mn})$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{mn}| < M$, for all $m, n \in \mathbb{N}$. That is,

$$\|x\|_\infty = \sup_{m,n} |x_{mn}| < \infty.$$

Let $K \subset \mathbb{N} \times \mathbb{N}$ and K_{mn} be the number of $(j, k) \in K$ such that $j \le m$, $k \le n$. If the sequence $\{K_{mn}/(mn)\}$ converges in Pringsheim's sense then we say that K has double natural density and is denoted by

$$d_2(K) = \lim_{m,n\to\infty} \frac{K_{mn}}{mn}.$$

A double sequence $x = (x_{mn})$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$, if for any $\varepsilon > 0$ we have $d_2(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - L| \ge \varepsilon\}$.

Let $X \ne \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:

(i) $\emptyset \in \mathcal{I}$,

(ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,

(iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

Let $X \ne \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:

(i) $\emptyset \notin \mathcal{F}$,

(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,
(iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

Lemma 2.1. [13] *If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class*

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.

A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Throughout the paper we take $\mathcal{I}_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$. Details about different types of ideals of $\mathbb{N} \times \mathbb{N}$ is found in Tripathy and Tripathy [35].

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal is also admissible.

Let $\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A)\}$. Then $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

Let (X, ρ) be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$ we have $A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2$ and we write

$$\mathcal{I}_2 - \lim_{m,n \to \infty} x_{mn} = L.$$

If $\mathcal{I}_2$ is a strongly admissible ideal on $\mathbb{N} \times \mathbb{N}$, then usual convergence implies $\mathcal{I}_2$-convergence.

Let (X, ρ) be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ of elements of X is said to be $\mathcal{I}_2^*$-convergent to $L \in X$, if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N} \times \mathbb{N} \backslash M \in \mathcal{I}_2$) such that

$$\lim_{m,n \to \infty} x_{mn} = L,$$

for $(m, n) \in M$ and we write

$$\mathcal{I}_2^* - \lim_{m,n \to \infty} x_{mn} = L.$$

We say that an admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ satisfies property (AP2), if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j \in \mathcal{I}_2^0$, i.e., $A_j \Delta B_j$ is included in a finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$ (hence $B_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

A fuzzy real number is a fuzzy set on the real axis, i.e., a mapping $u : \mathbb{R} \to [0, 1]$ which satisfies the following four conditions:

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.

(ii) u is fuzzy convex, i.e., $u[\lambda x + (1 - \lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0, 1]$.

(iii) u is upper semi-continuous.

(iv) The set $[u]_0 := \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact, (cf. Zadeh [38]), where $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$ denotes the closure of the set $\{x \in \mathbb{R} : u(x) > 0\}$ in the usual topology of $\mathbb{R}$.

We denote the set of all fuzzy real numbers on $\mathbb{R}$ by E^1 and call it as the space of fuzzy real numbers. α-level set $[u]_\alpha$ of $u \in E^1$ is defined by

$$[u]_\alpha := \begin{cases} \{t \in \mathbb{R} : x(t) \geq \alpha\} & , \quad (0 < \alpha \leq 1), \\ \overline{\{t \in \mathbb{R} : x(t) > \alpha\}} & , \quad (\alpha = 0). \end{cases}$$

The set $[u]_\alpha$ is closed, bounded and non-empty interval for each $\alpha \in [0,1]$ which is defined by $[u]_\alpha := [u^-(\alpha), u^+(\alpha)]$. $\mathbb{R}$ can be embedded in E^1, since each $r \in \mathbb{R}$ can be regarded as a fuzzy real number $\overline{r}$ defined by

$$\overline{r}(x) := \begin{cases} 1 & , \quad (x = r) \\ 0 & , \quad (x \neq r) \end{cases} .$$

Theorem 2.2. [8] *Let $[u]_\alpha = [u^-(\alpha), u^+(\alpha)]$ for $u \in E^1$ and for each $\alpha \in [0,1]$. Then the following statements hold:*

(i) *u^- is a bounded and non-decreasing left continuous function on $(0,1]$.*
(ii) *u^+ is a bounded and non-increasing left continuous function on $(0,1]$.*
(iii) *The functions u^- and u^+ are right continuous at the point $\alpha = 0$.*
(iv) *$u^-(1) \leq u^+(1)$.*

Conversely, if the pair of functions u^- and u^+ satisfies the conditions (i)-(iv), then there exists a unique $u \in E^1$ such that $[u]_\alpha := [u^-(\alpha), u^+(\alpha)]$ for each $\alpha \in [0,1]$. The fuzzy real number u corresponding to the pair of functions u^- and u^+ is defined by $u : \mathbb{R} \to [0,1]$, $u(x) := \sup\{\alpha : u^-(\alpha) \leq x \leq u^+(\alpha)\}$.

Let $u, v, w \in E^1$ and $k \in \mathbb{R}$. Then the operations addition, scalar multiplication and product defined on E^1 by

$$\begin{aligned} u + v = w \quad &\Longleftrightarrow \quad [w]_\alpha = [u]_\alpha + [v]_\alpha, \quad \text{for all } \alpha \in [0,1] \\ &\Longleftrightarrow \quad w^-(\alpha) = u^-(\alpha) + v^-(\alpha) \text{ and } w^+(\alpha) = u^+(\alpha) + v^+(\alpha), \\ & \qquad\ [ku]_\alpha = k[u]_\alpha, \quad \text{for all } \alpha \in [0,1] \end{aligned}$$

and

$$uv = w \Longleftrightarrow [w]_\alpha = [u]_\alpha [v]_\alpha, \quad \text{for all } \alpha \in [0,1],$$

where it is immediate that

$$\begin{aligned} w^-(\alpha) &= \min\{u^-(\alpha)v^-(\alpha), u^-(\alpha)v^+(\alpha), u^+(\alpha)v^-(\alpha), u^+(\alpha)v^+(\alpha)\}, \\ w^+(\alpha) &= \max\{u^-(\alpha)v^-(\alpha), u^-(\alpha)v^+(\alpha), u^+(\alpha)v^-(\alpha), u^+(\alpha)v^+(\alpha)\} \end{aligned}$$

for all $\alpha \in [0,1]$.

Let W be the set of all closed bounded intervals A of real numbers with endpoints $\underline{A}$ and $\overline{A}$, i.e. $A := [\underline{A}, \overline{A}]$. Define the relation d on W by

$$d(A,B) := \max\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\}.$$

It can be observed that d is a metric on W and (W, d) is a complete metric space, (cf. Nanda [19]). Now, we may define the metric D on E^1 by means of the Hausdorff metric d as

$$D(u,v) := \sup_{\alpha\in[0,1]} d([u]_\alpha,[v]_\alpha) := \sup_{\alpha\in[0,1]} \max\{|u^-(\alpha)-v^-(\alpha)|,|u^+(\alpha)-v^+(\alpha)|\}.$$

One can see that

$$D(u,\overline{0}) = \sup_{\alpha\in[0,1]} \max\{|u^-(\alpha)|,|u^+(\alpha)|\} = \max\{|u^-(0)|,|u^+(0)|\}. \tag{1}$$

The partial ordering relation $\preceq$ on E^1 is defined as follows:

$$u \preceq v \Leftrightarrow u^-(\alpha) \le v^-(\alpha) \text{ and } u^+(\alpha) \le v^+(\alpha), \quad \text{for all } \ \alpha \in [0,1].$$

Two fuzzy numbers u and v are said to be comparable if $u \preceq v$ or $v \preceq u$ holds.

Now, we may give:

Proposition 2.3. [3] *Let $u, v, w, z \in E^1$ and $k \in \mathbb{R}$. Then,*

(i) *(E^1, D) is a complete metric space.*
(ii) *$D(ku, kv) = |k|D(u, v)$.*
(iii) *$D(u + v, w + v) = D(u, w)$.*
(iv) *$D(u + v, w + z) \le D(u, w) + D(v, z)$.*
(v) *$|D(u,\overline{0}) - D(v,\overline{0})| \le D(u, v) \le D(u,\overline{0}) + D(v,\overline{0})$.*

Following Matloka [17], we give some definitions concerning the sequences of fuzzy real numbers below, which are needed in the text.

A sequence $u = (u_k)$ of fuzzy real numbers is a function u from the set $\mathbb{N}$ into the set E^1. The fuzzy real number u_k denotes the value of the function at $k \in \mathbb{N}$ and is called as the k^{th} term of the sequence. By $w(F)$, we denote the set of all sequences of fuzzy real numbers

A sequence $(u_n) \in w(F)$ is called convergent with limit $u \in E^1$, if for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $D(u_n, u) < \varepsilon$, for all $n \ge n_0$.

Definition 2.4. A double sequence $u = (u_{nk})$ of fuzzy real numbers is defined by a function u from the set $\mathbb{N} \times \mathbb{N}$ into the set E^1. The fuzzy number u_{nk} denotes the value of the function at $(n, k) \in \mathbb{N} \times \mathbb{N}$.

Definition 2.5. [30] A double sequence $u = (u_{mn})$ of fuzzy real numbers is said to be convergent in the Pringsheim's sense or P-convergent if for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $D(u_{mn}, u_0) < \varepsilon$ for all $m, n \ge k$ and is denoted by

$$P - \lim_{m,n\to\infty} u_{mn} = u_0.$$

The fuzzy real number u_0 is called the Pringsheim limit of u.

3. $\mathcal{I}_2$-convergence

Definition 3.1. Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $u = (u_{mn})$ of fuzzy real numbers is said to be $\mathcal{I}_2$-convergent to a fuzzy real number u_0, if for any $\varepsilon > 0$ we have

$$A(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, u_0) \ge \varepsilon\} \in \mathcal{I}_2$$

and is written as

$$\mathcal{I}_2 - \lim_{m,n\to\infty} u_{mn} = u_0.$$

Theorem 3.2. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. If a double sequence $u = (u_{mn})$ of fuzzy real numbers is $\mathcal{I}_2$-convergent to a fuzzy real number u_0, then u_0 determined uniquely.*

Proof. Suppose that $u = (u_{mn})$ is $\mathcal{I}_2$-convergent to two different fuzzy real numbers u_0 and v_0. We first prove that under the assumption of the theorem u_0 and v_0 are comparable. Suppose that u_0 and v_0 are not comparable. Then there exists an $\alpha_0 \in [0,1]$ such that

$$u_0^-(\alpha_0) < v_0^-(\alpha_0) \text{ and } u_0^+(\alpha_0) > v_0^+(\alpha_0) \tag{2}$$

or

$$u_0^-(\alpha_0) > v_0^-(\alpha_0) \text{ and } u_0^+(\alpha_0) < v_0^+(\alpha_0). \tag{3}$$

We prove (2) only, (3) can be analogously proved. Suppose (2) holds. Choose $\varepsilon_1 = v_0^-(\alpha_0) - u_0^-(\alpha_0)$ and $\varepsilon_2 = u_0^+(\alpha_0) - v_0^+(\alpha_0)$, then it is clear that $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. Let $\varepsilon' = \min\{\varepsilon_1, \varepsilon_2\}$. Choose ε such that $0 < \varepsilon < \frac{\varepsilon'}{2}$. Since $u = (u_{mn})$ is $\mathcal{I}_2$-convergent to fuzzy real numbers u_0 and v_0, we have

$$M_1(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, u_0) < \varepsilon\} \in \mathcal{F}(\mathcal{I}_2)$$

and

$$M_2(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, v_0) < \varepsilon\} \in \mathcal{F}(\mathcal{I}_2).$$

Since $\mathcal{F}(\mathcal{I}_2)$ is a filter on $\mathbb{N}\times\mathbb{N}$, so $\emptyset \neq M_1 \cap M_2 \in \mathcal{F}(\mathcal{I}_2)$. Let $(m,n) \in M_1 \cap M_2$, then we have $D(u_{mn}, u_0) < \varepsilon$ and $D(u_{mn}, v_0) < \varepsilon$. This implies that $d([u_{mn}]_\alpha, [u_0]_\alpha) < \varepsilon$ and $d([u_{mn}]_\alpha, [v_0]_\alpha) < \varepsilon$, for each $\alpha \in [0,1]$. Hence we have $d([u_{mn}]_{\alpha_0}, [u_0]_{\alpha_0}) < \varepsilon$ and $d([u_{mn}]_{\alpha_0}, [v_0]_{\alpha_0}) < \varepsilon$. Now definition of d implies that

$$|u_{mn}^-(\alpha_0) - u_0^-(\alpha_0)| < \varepsilon \text{ and } |u_{mn}^-(\alpha_0) - v_0^-(\alpha_0)| < \varepsilon, \tag{4}$$

$$|u_{mn}^+(\alpha_0) - u_0^+(\alpha_0)| < \varepsilon \text{ and } |u_{mn}^+(\alpha_0) - v_0^+(\alpha_0)| < \varepsilon. \tag{5}$$

(4) shows that $u_{mn}^-(\alpha_0) \in \big(u_0^-(\alpha_0) - \varepsilon, u_0^-(\alpha_0) + \varepsilon\big) \cap \big(v_0^-(\alpha_0) - \varepsilon, v_0^-(\alpha_0) + \varepsilon\big) = \emptyset$. In this way we obtain a contradiction. Hence u_0 and v_0 are comparable fuzzy real numbers. We may suppose that $u_0 \preceq v_0$. Take $\varepsilon = D(u_0, v_0)/3 > 0$ such that the neighborhoods

$$A(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, u_0) < \varepsilon\}$$

and

$$B(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, v_0) < \varepsilon\}$$

of u_0 and v_0, respectively, are disjoint. Since (u_{mn}) is $\mathcal{I}_2$-convergent to u_0 and v_0 so by definition of $\mathcal{I}_2$-convergence $A(\varepsilon), B(\varepsilon) \in \mathcal{F}(\mathcal{I}_2)$ and this implies $A(\varepsilon) \cap B(\varepsilon) \neq \emptyset$. In this way we obtain a contradiction to the fact that the neighborhoods $A(\varepsilon)$ and $B(\varepsilon)$ of u_0 and v_0, respectively, are disjoint. Hence, u_0 is unique. □

Theorem 3.3. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, $u = (u_{mn})$ be a double sequence of fuzzy real numbers and u_0 be a fuzzy real number. Then*

$$P - \lim_{m,n\to\infty} u_{mn} = u_0 \textit{ implies } \mathcal{I}_2 - \lim_{m,n\to\infty} u_{mn} = u_0.$$

Proof. Let

$$P - \lim_{m,n\to\infty} u_{mn} = u_0.$$

For every $\varepsilon > 0$ there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that $D(u_{mn}, u_0) < \varepsilon$ for all $m, n \geq k_0$. Then,

$$\begin{aligned} A(\varepsilon) &= \{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, u_0) \geq \varepsilon\} \\ &\subset \big(\mathbb{N}\times\{1,2,\ldots,(k_0-1)\} \cup \{1,2,\ldots,(k_0-1)\}\times\mathbb{N}\big). \end{aligned}$$

Since $\mathcal{I}_2$ is a strongly admissible ideal, so $\big(\mathbb{N}\times\{1,2,\ldots,(k_0-1)\} \cup \{1,2,\ldots,(k_0-1)\}\times\mathbb{N}\big) \in \mathcal{I}_2$ and $A(\varepsilon) \in \mathcal{I}_2$. Hence, we have

$$\mathcal{I}_2 - \lim_{m,n\to\infty} u_{mn} = u_0.$$

□

Theorem 3.4. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, $u = (u_{mn})$, $v = (v_{mn})$ be two double sequences of fuzzy real numbers and u_0, v_0 be two fuzzy real numbers. If $c \in \mathbb{R}$,*

$$\mathcal{I}_2 - \lim_{m,n\to\infty} u_{mn} = u_0 \textit{ and } \mathcal{I}_2 - \lim_{m,n\to\infty} v_{mn} = v_0,$$

then we have

$$(i)\ \mathcal{I}_2 - \lim_{m,n\to\infty} cu_{mn} = cu_0 \textit{ and } (ii)\ \mathcal{I}_2 - \lim_{m,n\to\infty} (u_{mn} + v_{mn}) = u_0 + v_0.$$

Proof. (i) Let $c \in \mathbb{R}$ and $\mathcal{I}_2 - \lim_{m,n\to\infty} u_{mn} = u_0$. If $c = 0$, there is nothing to prove, so we assume that $c \neq 0$. Let $\varepsilon > 0$ be given. Then,

$$\Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : D(cu_{mn}, cu_0) \geq \varepsilon\Big\} \subseteq \Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, u_0) \geq \frac{\varepsilon}{|c|}\Big\} \in \mathcal{I}_2.$$

Hence, we have

$$\mathcal{I}_2 - \lim_{m,n\to\infty} cu_{mn} = cu_0.$$

(ii) Let $\mathcal{I}_2 - \lim_{m,n\to\infty} u_{mn} = u_0$ and $\mathcal{I}_2 - \lim_{m,n\to\infty} v_{mn} = v_0$. We write

$$A\left(\frac{\varepsilon}{2}\right) = \Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, u_0) \geq \frac{\varepsilon}{2}\Big\}$$

and

$$B\left(\frac{\varepsilon}{2}\right) = \Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : D(v_{mn}, v_0) \geq \frac{\varepsilon}{2}\Big\},$$

for every $\varepsilon > 0$. Then, for every $\varepsilon > 0$ we have

$$\Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn} + v_{mn}, u_0 + v_0) \geq \varepsilon\Big\} \subset A\left(\frac{\varepsilon}{2}\right) \cup B\left(\frac{\varepsilon}{2}\right),$$

for $D(u_{mn} + v_{mn}, u_0 + v_0) \leq D(u_{mn}, u_0) + D(v_{mn}, v_0)$ and hence

$$\Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn} + v_{mn}, u_0 + v_0) \geq \varepsilon\Big\} \in \mathcal{I}_2.$$

□

Theorem 3.5. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, $u = (u_{mn})$ and $v = (v_{mn})$ be two double sequences of fuzzy real numbers such that*

(i) $u_{mn} \preceq v_{mn}$ for every $(m,n) \in M \subset \mathbb{N}\times\mathbb{N}$ with $M \in \mathcal{F}(\mathcal{I}_2)$,

(ii) $\mathcal{I}_2 - \lim_{m,n\to\infty} u_{mn} = u_0$ and $\mathcal{I}_2 - \lim_{m,n\to\infty} v_{mn} = v_0$.

Then $u_0 \preceq v_0$.

Proof. By (ii) for each $\varepsilon > 0$

$$A = \{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, u_0) \geq \varepsilon\} \in \mathcal{I}_2$$

and

$$B = \{(m,n) \in \mathbb{N}\times\mathbb{N} : D(v_{mn}, v_0) \geq \varepsilon\} \in \mathcal{I}_2.$$

Suppose that $u_0 \preceq v_0$ is not true, then there exist $\alpha_0 \in [0,1]$ such that $u_0^-(\alpha_0) > v_0^-(\alpha_0)$ or $u_0^+(\alpha_0) > v_0^+(\alpha_0)$. We may suppose that $u_0^-(\alpha_0) > v_0^-(\alpha_0)$, the case for $u_0^+(\alpha_0) > v_0^+(\alpha_0)$ is analogously proved. Take $\varepsilon = \frac{u_0^-(\alpha_0) - v_0^-(\alpha_0)}{3}$. Since $M \in \mathcal{F}(\mathcal{I}_2)$ and $A, B \in \mathcal{I}_2$ so we have $\emptyset \neq M \cap A^c \cap B^c \in \mathcal{F}(\mathcal{I}_2)$. Let $(m,n) \in M \cap A^c \cap B^c$, then we have

$$u_{mn} \preceq v_{mn}, \;\; D(u_{mn}, u_0) < \varepsilon \text{ and } D(v_{mn}, v_0) < \varepsilon. \tag{6}$$

Last two inequalities of (6) give the following

$$|u_{mn}^-(\alpha_0) - u_0^-(\alpha_0)| < \varepsilon \text{ and } |v_{mn}^-(\alpha_0) - v_0^-(\alpha_0)| < \varepsilon \tag{7}$$

and

$$|u_{mn}^+(\alpha_0) - u_0^+(\alpha_0)| < \varepsilon \text{ and } |v_{mn}^+(\alpha_0) - v_0^+(\alpha_0)| < \varepsilon. \tag{8}$$

Thus equation (7) shows that $u_{mn}^-(\alpha_0) > v_{mn}^-(\alpha_0)$. Therefore we obtain a contradiction to $u_{mn} \preceq v_{mn}$ as $(m,n) \in M$. Hence we have $u_0 \preceq v_0$. □

Theorem 3.6. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, $u = (u_{mn})$, $v = (v_{mn})$ and $w = (w_{mn})$ be three double sequences of fuzzy real numbers such that*

(i) $u_{mn} \preceq v_{mn} \preceq w_{mn}$ for every $(m,n) \in M \subset \mathbb{N}\times\mathbb{N}$ with $M \in \mathcal{F}(\mathcal{I}_2)$

(ii) $\mathcal{I}_2 - \lim_{m,n\to\infty} u_{mn} = u_0$ and $\mathcal{I}_2 - \lim_{m,n\to\infty} w_{mn} = u_0$.

Then $\mathcal{I}_2 - \lim_{m,n\to\infty} v_{mn} = u_0$.

Proof. Let $\mathcal{I}_2 - \lim_{m,n\to\infty} u_{mn} = \mathcal{I}_2 - \lim_{m,n\to\infty} w_{mn} = u_0$. For $\varepsilon > 0$ we can take

$$A = \{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, u_0) \geq \varepsilon\} \in \mathcal{I}_2$$

and

$$B = \{(m,n) \in \mathbb{N}\times\mathbb{N} : D(w_{mn}, u_0) \geq \varepsilon\} \in \mathcal{I}_2.$$

Now we define the set $C = \{(m,n) \in \mathbb{N}\times\mathbb{N} : D(v_{mn}, u_0) \geq \varepsilon\}$. We can have either $(m,n) \in M$ or $(m,n) \in M^c$. Assume that $(m,n) \in M$ (as otherwise $C \subset A \cup B \cup M^c$) then we have $u_{mn} \preceq v_{mn} \preceq w_{mn}$. Since

$$D(v_{mn}, u_0) = \sup_{\alpha\in[0,1]} \max\{|v_{mn}^-(\alpha) - u_0^-(\alpha)|, |v_{mn}^+(\alpha) - u_0^+(\alpha)|\} \geq \varepsilon,$$

therefore by definition of supremum, there exists $\alpha_0 \in [0,1]$ such that

$$\max\{|v_{mn}^-(\alpha_0) - u_0^-(\alpha_0)|, |v_{mn}^+(\alpha_0) - u_0^+(\alpha_0)|\} \geq \varepsilon - \varepsilon',$$

for every $0 < \varepsilon' < \varepsilon$. This implies that

$$|v_{mn}^-(\alpha_0) - u_0^-(\alpha_0)| \geq \varepsilon - \varepsilon' \tag{9}$$

or

$$|v_{mn}^+(\alpha_0) - u_0^+(\alpha_0)| \geq \varepsilon - \varepsilon'. \tag{10}$$

Without loss of generality we may assume that (9) holds. Now according to v_{mn} and u_0 are comparable or not we have the following possibilities:

$$v_{mn}^-(\alpha_0) < u_0^-(\alpha_0) \text{ and } v_{mn}^+(\alpha_0) < u_0^+(\alpha_0) \text{ or } v_{mn}^+(\alpha_0) > u_0^+(\alpha_0) \tag{11}$$

and

$$v_{mn}^-(\alpha_0) > u_0^-(\alpha_0) \text{ and } v_{mn}^+(\alpha_0) < u_0^+(\alpha_0) \text{ or } v_{mn}^+(\alpha_0) > u_0^+(\alpha_0). \tag{12}$$

We can suppose that (1) holds. One can analogously prove that (12) holds. Since $u_{mn} \preceq v_{mn} \leq w_{mn}$, we have $u_{mn}^-(\alpha_0) \leq v_{mn}^-(\alpha_0)$. But then (9) implies that $|u_{mn}^-(\alpha_0) - u_0^-(\alpha_0)| \geq \varepsilon - \varepsilon'$. As ε' was chosen arbitrarily so $D(u_{mn}, u_o) \geq \varepsilon$. This shows that $(m,n) \in A$ and therefore $C \subset A \cup B \cup M^c$. Hence, $C \in \mathcal{I}_2$ and we have

$$\mathcal{I}_2 - \lim_{m,n\to\infty} v_{mn} = u_0.$$

□

4. $\mathcal{I}_2^*$-convergence

Definition 4.1. Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $u = (u_{mn})$ of fuzzy real numbers is said to be $\mathcal{I}_2^*$-convergent to $u_0 \in E^1$, if there exists $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $H = \mathbb{N} \times \mathbb{N} \backslash M \in \mathcal{I}_2$) such that

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in M}} u_{mn} = u_0$$

and is written

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} u_{mn} = u_0.$$

Theorem 4.2. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, $u = (u_{mn})$ be a double sequence of fuzzy real numbers and $u_0 \in E^1$. Then,*

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} u_{mn} = u_0 \text{ implies } \mathcal{I}_2 - \lim_{m,n\to\infty} u_{mn} = u_0.$$

Proof. Let $\mathcal{I}_2^* - \lim_{m,n\to\infty} u_{mn} = u_0$. By definition, there exists a $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $H = \mathbb{N} \times \mathbb{N} \backslash M \in \mathcal{I}_2$) such that

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in M}} u_{mn} = u_0.$$

Then, for every $\varepsilon > 0$ there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$ we have $D(u_{mn}, u_0) < \varepsilon$ whenever $m, n \geq k_0$ for $(m,n) \in M$. Now, let $A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : D(u_{mn}, u_0) \geq \varepsilon\}$. Therefore, clearly we have

$$A(\varepsilon) \subset H \cup \big[M \cap \big((\{1, 2, \ldots, (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \ldots, (k_0 - 1)\})\big)\big].$$

Since $\mathcal{I}_2$ is a strongly admissible ideal, so

$$H \cup \big[M \cap \big((\{1,2,\ldots,(k_0-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\ldots,(k_0-1)\})\big)\big] \in \mathcal{I}_2.$$

Hence, we have $A(\varepsilon) \in \mathcal{I}_2$ and consequently

$$\mathcal{I}_2 - \lim_{m,n\to\infty} u_{mn} = u_0.$$

□

Theorem 4.3. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. For any double sequence $u = (u_{mn})$ of fuzzy real numbers, if there exist two sequences $v = (v_{mn})$ and $w = (w_{mn})$ of fuzzy numbers such that*

$$u = v + w, \quad \lim_{m,n\to\infty} v_{mn} = u_0 \ \text{ and } \ supp\, w \in \mathcal{I}_2,$$

where $supp\, w = \{(m,n) \in \mathbb{N}\times\mathbb{N} : w_{mn} \neq \overline{0}\}$ and $\overline{0}$ is the zero element of fuzzy real numbers, then $u = (u_{mn})$ is $\mathcal{I}_2$-convergent to $u_0 \in E^1$.

Proof. Suppose that there exist two sequences $v = (v_{mn})$ and $w = (w_{mn})$ of fuzzy real numbers such that

$$u = v + w, \quad \lim_{m,n\to\infty} v_{mn} = u_0 \ \text{ and } \ supp\, w \in \mathcal{I}_2.$$

Let $M = \{(m,n) \in \mathbb{N}\times\mathbb{N} : w_{mn} = \overline{0}\}$. Since $supp\, w \in \mathcal{I}_2$, so $M \in \mathcal{F}(\mathcal{I}_2)$. Since $u_{mn} = v_{mn}$ for each $(m,n) \in M$ and $\lim_{m,n\to\infty} v_{mn} = u_0$, so it follows that $\lim_{m,n\to\infty} u_{mn} = u_0$. This shows that

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} u_{mn} = u_0.$$

Thus, by Theorem 4.2 we have

$$\mathcal{I}_2 - \lim_{m,n\to\infty} u_{mn} = u_0.$$

□

Theorem 4.4. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal with property (AP2), $u = (u_{mn})$ be a double sequence of fuzzy numbers and u_0 be a fuzzy real number. Then,*

$$\mathcal{I}_2 - \lim_{m,n\to\infty} u_{mn} = u_0 \ \text{ implies } \ \mathcal{I}_2^* - \lim_{m,n\to\infty} u_{mn} = u_0.$$

Proof. Let $\mathcal{I}_2$ satisfy the property $(AP2)$ and $u = (u_{mn})$ be a double sequence of fuzzy real numbers and $u_0 \in E^1$ such that $\mathcal{I}_2 - \lim_{m,n\to\infty} u_{mn} = u_0$. Then for any $\varepsilon > 0$

$$A(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, u_0) \geq \varepsilon\} \in \mathcal{I}_2.$$

Now put

$$\begin{aligned} A_1 &= \{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, u_0) \geq 1\}, \\ A_k &= \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{k} \leq D(u_{mn}, u_0) < \frac{1}{k-1}\right\} \end{aligned}$$

for $k \geq 2$. It is clear that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $A_i \in \mathcal{I}_2$ for each $i \in \mathbb{N}$. By virtue of $(AP2)$ there exists a sequence $\{B_k\}_{k\in\mathbb{N}}$ of sets such that $A_j \bigtriangleup B_j$ is included in finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$.

We prove that

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in M}} u_{mn} = u_0,$$

for $M = \mathbb{N}\times\mathbb{N}\backslash B \in \mathcal{F}(\mathcal{I}_2)$. Let $\delta > 0$ be given. Choose $k \in \mathbb{N}$ such that $1/k < \delta$. Then, we have

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, u_0) \geq \delta\} \subset \bigcup_{j=1}^{k} A_j.$$

Since $A_j \triangle B_j$ are included in finite union of rows and columns for $j \in \{1, 2, \dots, k\}$, there exists $n_0 \in \mathbb{N}$ such that

$$\Big(\bigcup_{j=1}^{k} B_j\Big) \cap \{(m,n) : m \geq n_0 \wedge n \geq n_0\} = \Big(\bigcup_{j=1}^{k} A_j\Big) \cap \{(m,n) : m \geq n_0 \wedge n \geq n_0\}$$

If $m, n \geq n_0$ and $(m,n) \notin B$ then

$$(m,n) \notin \bigcup_{j=1}^{k} B_j \text{ and so } (m,n) \notin \bigcup_{j=1}^{k} A_j.$$

Thus, we have $D(u_{mn}, u_0) < \frac{1}{k} < \delta$. This implies that

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in M}} u_{mn}(x) = u_0.$$

Hence, we have

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} u_{mn} = u_0.$$

□

Definition 4.5. [7] Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence of functions $\{f_{mn}\}$ is said to be $\mathcal{I}_2$-uniformly convergent to f on a set $S \subset \mathbb{R}$ if for every $\varepsilon > 0$

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : |f_{mn}(x) - f(x)| \geq \varepsilon\} \in \mathcal{I}_2, \text{ for each fixed } x \in S$$

and is denoted by $f_{mn} \rightrightarrows_{\mathcal{I}_2} f$. This can be stated as follows : For $\varepsilon > 0$, $\exists H \in \mathcal{I}_2$ such that for all $x \in S$, $|f_{mn}(x) - f(x)| < \varepsilon$, $\forall (m,n) \notin H$.

A double sequence of functions $\{f_{mn}\}$ is said to be $\mathcal{I}_2^*$-uniformly convergent to f on $S \subset \mathbb{R}$ if and only if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N}\times\mathbb{N}\backslash M \in \mathcal{I}_2$) such that for $\varepsilon > 0$

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in M}} f_{mn}(x) = f(x), \text{ for each (fixed) } x \in S$$

and is written $f_{mn} \rightrightarrows_{\mathcal{I}_2^*} f$.

Theorem 4.6. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal with the property (AP2), $u = (u_{mn})$ be a double sequence of fuzzy real numbers and $u_0 \in E^1$. Then, $\mathcal{I}_2 - \lim_{m,n\to\infty} u_{mn} = u_0$ if and only if*

$$u_{mn}^-(\alpha) \rightrightarrows_{\mathcal{I}_2^*} u_0^-(\alpha) \text{ and } u_{mn}^+(\alpha) \rightrightarrows_{\mathcal{I}_2^*} u_0^+(\alpha)$$

on $[0, 1]$.

Proof. Let $\mathcal{I}_2 - \lim_{m,n\to\infty} u_{mn} = u_0$. By definition we have

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : D(u_{mn}, u_0) \geq \varepsilon\} \in \mathcal{I}_2,$$

for every $\varepsilon > 0$. Then, by Theorem 4.4 for every $\varepsilon > 0$ there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ and $N = N(\varepsilon)$ such that

$$D(u_{mn}, u_0) = \sup_{\alpha\in[0,1]} \max\left\{|u^-_{mn}(\alpha) - u^-_0(\alpha)|, |u^+_{mn}(\alpha) - u^+_0(\alpha)|\right\} < \varepsilon,$$

for all $m, n \geq N$ and $(m,n) \in M$. This implies that

$$\max\left\{|u^-_{mn}(\alpha) - u^-_0(\alpha)|, |u^+_{mn}(\alpha) - u^+_0(\alpha)|\right\} < \varepsilon,$$

hence the result follows for all $\alpha \in [0,1]$.

To prove the converse implication, let $\varepsilon > 0$ be fixed. Then by Definition 4.5 and by Theorem 4.4, there exist $M_1 \in \mathcal{F}(\mathcal{I}_2)$ and $N_1 = N_1(\varepsilon) \in \mathbb{N}$ such that

$$|u^-_{mn}(\alpha) - u^-_0(\alpha)| < \varepsilon,$$

for all $m, n \geq N_1$ and $(m,n) \in M_1$, and for each $\alpha \in [0,1]$. Similarly, there exist $M_2 \in \mathcal{F}(\mathcal{I}_2)$ and $N_2 = N_2(\varepsilon) \in \mathbb{N}$ such that

$$|u^+_{mn}(\alpha) - u^+_0(\alpha)| < \varepsilon,$$

for all $m, n \geq N_2$ and $(m,n) \in M_2$, and for each $\alpha \in [0,1]$.

Let $N_3 = \max\{N_1, N_2\}$ and $M_3 = M_1 \cap M_2 \in \mathcal{F}(\mathcal{I}_2)$. Thus, for every $\varepsilon > 0$ there exists $M_3 \in \mathcal{F}(\mathcal{I}_2)$ such that for all $m, n \geq N_3$, $(m,n) \in M_3$,

$$\sup_{\alpha\in[0,1]} \max\left\{|u^-_{mn}(\alpha) - u^-_0(\alpha)|, |u^+_{mn}(\alpha) - u^+_0(\alpha)|\right\} = D(u_{mn}, u_0) < \varepsilon.$$

This implies $\mathcal{I}^*_2 - \lim_{m,n\to\infty} u_{mn} = u_0$. Hence, we have

$$\mathcal{I}_2 - \lim_{m,n\to\infty} u_{mn} = u_0.$$

□

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