Real Function Algebras

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This thesis entitled **Real Banach Algebras and Gleason-Kahane-Zelazko theorem** by D. Anish is approved for the degree of Master of Science from IIT Hyderabad.

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Contents

Chapter 1

Introdution

1.1 Elements of Real Banach Algebras

Definition 1.1.1 (Algebra)**.** *Let $\mathcal{A}$ be a non empty set. Then $\mathcal{A}$ is called an algebra, if*

1. $(\mathcal{A}, +, .)$ *is a vector space over a field* $\mathbb{F}$.
2. $(\mathcal{A}, +, \circ)$ *is a ring.*
3. $(\alpha.a) \circ b = \alpha.(a \circ b) = a \circ (\alpha b)$ *for all* $a,b \in \mathcal{A}$ *and* $\alpha \in \mathbb{F}$.

From here onwards we write ab instead of $a \circ b$ *to denote the multiplication of* $a, b \in \mathcal{A}$.

Note 1. *An algebra $\mathcal{A}$ is called real is algebra if $\mathbb{F} = \mathbb{R}$ the field of real numbers and complex algebra if $\mathbb{F} = \mathbb{C}$, the field of complex numbers.*

Example 1.1.2. $\mathbb{R}(\mathbb{R})$, $\mathbb{C}(\mathbb{R})$ *are algebras with respect to addition and multiplications.*

Definition 1.1.3. *If $\mathcal{A}$ is a algebra and $||.||$ is a norm on $\mathcal{A}$ satisfying* $||ab|| \leq ||a||||b||,\ \forall a, b \in \mathcal{A}$ *then* $(\mathcal{A}, ||.||)$ *is called* ***normed algebra***.

Example 1.1.4. $\mathbb{R}(\mathbb{R})$, $\mathbb{C}(\mathbb{R})$ *where* $||a|| = |a|$.

Definition 1.1.5. *A complete normed algebra is called a* ***Banach algebra***.

Example 1.1.6. *Let X be an compact Hausdorff topological space and*

$$C(X) = \{f : X \to \mathbb{C} \ \ is \ \ continuous\}$$

is the set of all complex valued continuous functions on X. Then $C(X)$ is a complex Banach algebra with the following operations;

$$\begin{aligned}(f+g)(x) &= f(x) + g(x), \quad \forall x \in X,\\ (fg)(x) &= f(x)g(x),\\ ||f|| &= \sup\{|f(x)| : \forall x \in X\}.\end{aligned}$$

We discuss some examples of closed real sub algebras that are not complex sub algebras.
Let Y *be a closed subset of* X*. Consider* $C_Y = \{f \in C(X) : f(Y) \subset \mathbb{R}\}$*. If* $Y = \emptyset$*, then* $C_Y = C(X)$*. For every non empty set* C_Y *is a real commutative Banach algebra.*

Example 1.1.7. *Let* S *be a non empty set. Define*

$$\mathcal{B}(S) = \{f : S \to \mathbb{C} : f \ \text{ is bounded on } \ S\}.$$

Define operations on $B(S)$ *as follows;*

$$\begin{aligned}(f+g)(x) &= f(x) + g(x), \ \forall x \in X\\ (fg)(x) &= f(x)g(x),\\ ||f||_\infty &= \sup\big\{|f(x)| : \forall x \in X\big\}.\end{aligned}$$

With above operators $\mathcal{B}(S)$ *is a commutative Banach algebra.*

Example 1.1.8. *Let* X *be a locally compact Hausdorff space. If*

$$\mathcal{C}_b(X) = \{f \in C(X) : f \text{ is bounded}\},$$

then $\mathcal{C}_b(X)$ *is a commutative Banach algebra.*

Example 1.1.9. *Let* $\mathbf{H} = \big\{a_o + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3 \in R\big\}$ *is real quaternion algebra. Here*

$$\begin{aligned}i^2 = j^2 = k^2 &= -1,\\ i.j = k &= -j.i,\\ j.k = i &= -k.j,\\ k.i = j &= -i.k.\end{aligned}$$

For $a = a_o + a_1 i + a_2 j + a_3 k$ in $\mathbf{H}$. We define $\|a\| = (a_o^2 + a_1^2 + a_2^2 + a_3^2)^{\frac{1}{2}}$.

Example 1.1.10. *Let* $M_n(\mathbb{C})$*, where* $n \geq 2$ *be the set of all* $n \times n$ *matrices with respect to matrix addition and multiplication is an algebra. Define*

$$||A|| = \Big(\sum_{i,j=0}^{n} |a_{ij}|\Big).$$

Then $M_n(\mathbb{C})$ *is a non commutative unital Banach Algebra.*

Example 1.1.11. *Let* W *be the set of all complex valued functions on* $[0, 2\pi]$ *whose fourier series are absolutely convergent, that is, functions of the form*

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp(int), \ t \in [0, 2\pi],$$

where c_n is a complex number for each n and $\sum_{n=-\infty}^{\infty} |c_n|$ is finite. For such functions f, we define

$$||f|| = \sum_{n=-\infty}^{\infty} |c_n|.$$

Under pointwise operations, W is a complex Banach algebra. This is known as the ***Wiener algebra****.*

Definition 1.1.12. *Given a real algebra $\mathcal{A}$, the complexification $\mathcal{B}$ of $\mathcal{A}$ is the set $\mathcal{A}\times\mathcal{A}$ with the operations of addition, multiplication and scalar multiplication define by,*

$$\begin{aligned}(a,b)+(c,d) &= (a+c, b+d),\\ (\alpha+i\beta)(a,b) &= (\alpha a-\beta b, \alpha b+\beta a),\\ (a,b)(c,d) &= (ac-bd, ad+bc),\end{aligned}$$

for all $a,b,c,d \in \mathcal{A}$ and $\alpha,\beta \in \mathbb{R}$.

Definition 1.1.13 (Invertible element)**.** *Let $\mathcal{A}$ be a real or complex algebra with unit 1 and $a \in \mathcal{A}$. An element $b \in \mathcal{A}$ is called inverse of a, if $ab = ba = 1$. If a has inverse, then a is called invertible or regular, otherwise singular.*

Definition 1.1.14 (Spectrum of an element)**.** *Let $\mathcal{A}$ be a complex algebra with unit 1 and $a \in \mathcal{A}$, then spectrum of an element $a \in \mathcal{A}$ is defined by*

$$Sp(a,\mathcal{A}) = \big\{\lambda \in \mathbb{C} : a-\lambda \ \text{ is singular in } \mathcal{A}\big\}.$$

Definition 1.1.15. *Let $\mathcal{A}$ be a real algebra with unit 1. For $a \in A$, the spectrum of $a \in A$ is a subset of $\mathbb{C}$, defined as follows;*

$$Sp(a,\mathcal{A}) = \{s+it : (a-s)^2+t^2 \text{ is singular in } \mathcal{A}\}.$$

Clearly, $s+it \in Sp(a,\mathcal{A})$ if and only if $s-it \in Sp(a,\mathcal{A})$.

Remark 1.1.16. *Note that we have used the symbol λ for the scalar and the element $\lambda.1 \in \mathcal{A}$.*

Remark 1.1.17. *If $\mathcal{A}$ is a complex algebra with unit 1 and if $\mathcal{A}_{\mathbb{R}}$ denotes $\mathcal{A}$ regarded as a real algebra, then*

$$Sp(a,\mathcal{A}_{\mathbb{R}}) = Sp(a,\mathcal{A}) \cup \{\lambda : \lambda \in Sp(a,\mathcal{A})\}.$$

Remark 1.1.18. *Let $\mathcal{A}$ be a real normed algebra and $a \in \mathcal{A}$. The spectral radius $r(a)$ of a defined by $r(a) = \inf\big\{\|a^n\|^{\frac{1}{n}} : n = 1,2,3...\big\}$.*

Lemma 1.1.19. *Let $\mathcal{A}$ be a normed algebra and $a \in \mathcal{A}$. Then $r(a) = \lim \|a^n\|^{\frac{1}{n}}$.*

We know by definition $r(a) = \inf\{\|a^n\|^{\frac{1}{n}} : n = 1, 2, 3...\}$, $r(a) \leq \|a^n\|^{\frac{1}{n}}\ \forall n$. Now for a given $\epsilon > 0$ there exist k such that $r(a) + \epsilon \geq \|a^k\|^{\frac{1}{k}}$. By division algorithm theorem for every natural number n, there exist unique non negative integers p and q such that $n = p.k + q$ and $q < k - 1$. As $n \to \infty$, $\dfrac{q}{n} \to 0$. Hence $\dfrac{p.k}{n} \to 1$. Thus we have

$$\|a^n\|^{\frac{1}{n}} = \|a^{pk+q}\|^{\frac{1}{n}} \quad \leq \|a^{pk}\|^{\frac{1}{n}}\|a^q\|^{\frac{1}{n}} \to \|a^k\|^{\frac{1}{k}} < r(a) + \epsilon r(a) \leq \ \|a^n\|^{\frac{1}{n}} \leq r(a).$$

Hence $\lim \|a^n\|^{\frac{1}{n}} = r(a)$.

Lemma 1.1.20. *Let $\mathcal{A}$ be a real Banach algebra with unity 1. Let $a \in A$ and $s, t \in \mathbb{R}$;*

1. *If $r(a) \leq |s|$, then $a - s$ is regular in $\mathcal{A}$ and*
2. *$(a - s)^{-1} = -\sum_{n=0}^{\infty} \frac{a^n}{s^{n+1}}$.*

Proof. Let $r(a) \leq |s|$, There exist p such that $r(a) \leq p \leq |s|$. Thus $\|a^n\| \leq p^n$ For all sufficiently large n. For such n, we have $\|\frac{a^n}{s^{n+1}}\| \leq \dfrac{p^n}{\mid s \mid^{n+1}}$. So $\sum_{n=0}^{\infty} \dfrac{a^n}{s^{n+1}}$ convergent absolutely. Since A is a Banach algebra, let c $=-\sum_{n=0}^{\infty} \dfrac{a^n}{s^{n+1}}$ and $c_m = -\sum_{n=0}^{m} \dfrac{a^n}{s^{n+1}}$. Then

$$\begin{aligned} c_m(a-s) &= -\Big(\frac{1}{s} + \frac{a}{s^2} + ... + \frac{a^m}{s^{m+1}}\Big)(a-s) \\ &= 1 - \frac{a^{m+1}}{s^{m+1}} \\ &= (a-s)c_m. \end{aligned}$$

Since $(\frac{a}{s})^m$ tend to zero as m tends to infinity, if follows that $c(a - s) = 1 = (a - s)c$, that is c is the inverse of $a - s$. □

Corollary 1.1.21. *Let $\mathcal{A}$ be a complex Banach algebra with unit 1. Let $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. If $r(a) < |\lambda|$, then $a - \lambda$ is regular in A and*

$$(a - \lambda)^{-1} = -\sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}.$$

Proof. Let $\lambda = s + it$. If $t = 0$, the conclusion follows from the above lemma. Now let $t \neq 0$. Note that $(a-s)^2+t^2 = (a-\lambda)(a-\bar{\lambda})$ is invertible and its inverse is $d = \sum_{n=0}^{\infty} b_n$, which is given in the second part of the lemma. Thus $(a-\lambda)(a-\bar{\lambda})d = 1 = (a-\bar{\lambda})d(a-\lambda)$. This implies that $(a-\lambda)$ invertible and its inverse is $(a-\bar{\lambda})d = \sum_{n=0}^{\infty}(a-s).b_n + i\sum_{n=0}^{\infty} tb_n$ the coefficient of a^n in $\sum_{n=0}^{\infty}(a-s)b_n$, is $-[cos(n+1)\theta]/q^{n+1}$ and the coefficient of a^n in $\sum_{n=0}^{\infty} tb_n$ *is* $[\sin(n+1)\theta]/q^{n+1}$, where $q = (s^2 + t^2)^{\frac{1}{2}} = |\lambda|$. Hence

$(a - \lambda)^{-1} = (a - \bar{\lambda})d = -\sum_{n=0}^{\infty} \dfrac{a^n}{\lambda^{n+1}}$. □

Corollary 1.1.22. *Let $\mathcal{A}$ be a real Banach algebra with unit* 1*. Suppose that $a \in A$ is invertible and $b \in \mathcal{A}$ is such that $\|b-a\| \leq \frac{\epsilon}{\|a^{-1}\|}$ with $0 < \epsilon < 1$. Then b is invertible and $\|b^{-1} - a^{-1}\| \leq \frac{\|a^{-1}\|^2\|b-a\|}{1-\epsilon}$.*

Proof. We have

$$r(1 - a^{-1}b) \leq \|1 - a^{-1}b\| = \|a^{-1}a - a^{-1}b\| \leq \|a^{-1}\|\|a - b\| \leq \epsilon < 1.$$

Hence, $a^{-1}b$ is invertible, a is invertible implies b is invertible. Further,

$$\|b^{-1}\|\|a^{-1}\| \leq \|b^{-1} - a^{-1}\| = \|b^{-1} - b^{-1}ba^{-1}\| \leq \|b^{-1}\|\|1 - ba^{-1}\| \leq \epsilon\|b^{-1}\|.$$

Hence, $\|b^{-1}\| \leq \frac{\|a^{-1}\|}{1-\epsilon}$.
We have

$$\begin{aligned}\|b^{-1} - a^{-1}\| &\leq \|b^{-1}\|\|a - b\|\|a^{-1}\| \\ &\leq \frac{\|a^{-1}\|^2\|a - b\|}{1 - \epsilon}.\end{aligned}$$

Hence proved. □

Remark 1.1.23. *Let $\mathcal{A}$ be a Banach algebra with unity* 1*. Let $Inv(\mathcal{A})$ denote the set of all invertible elements in $\mathcal{A}$. Thus $Inv(\mathcal{A})$ group under multiplication. Corollary 1.1.14 says that $Inv(\mathcal{A})$ is an open set in $\mathcal{A}$ and the map $a \to a^{-1}$ is continuous on $Inv(\mathcal{A})$.*

Theorem 1.1.24. *Let $\mathcal{A}$ be a real Banach algebra with unit* 1 *and $a \in \mathcal{A}$. Then*

$$r(a) = \sup\{(s^2 + t^2)^{\frac{1}{2}} : s + it \in Sp(a, \mathcal{A})\}.$$

In particular, $Sp(a, \mathcal{A})$ is non empty.

Proof. Let $\alpha = \sup\{(s^2 + t^2)^{\frac{1}{2}} : s + it \in Sp(a, \mathcal{A})\}$. We know that $r(a) < (s^2 + t^2)^2$. Then$(a - s)^2 + t^2$ is regular in A. $(a - s)^2 + t^2$ is not regular in A implies that $r(a) \geq (s^2 + t^2)^{\frac{1}{2}}$, from this $s + it \in Sp(\mathcal{A})$ then $r(a) \geq (s^2 + t^2)^{\frac{1}{2}}$ Hence, $r(a) \geq \alpha$. We now prove $r(a) < \alpha$. Let $r(a) = 0$. If $0 \notin Sp(a, \mathcal{A})$, then a is invertible. Since a and a^{-1} commute, we have $1 = r(aa^{-1}) = 0$, which is contradiction, hence $\alpha \geq 0 = r(a)$. □

Corollary 1.1.25. *Let $\mathcal{A}$ be a real Banach algebra with unit* 1 *and $a \in \mathcal{A}$. Then $Sp(a, \mathcal{A})$ is a compact subset of $\mathbb{C}$.*

Proof. We know that $(s^2 + t^2)^{\frac{1}{2}} < r(a) < \|a\|$. That is, $Sp(a)$ is bounded. Thus it is enough show that $Sp(a)$ is closed. For this, it is enough to show $\mathbb{C}\backslash Sp(a)$ is open. Suppose $s + it \in \mathbb{C}\backslash Sp(a)$. Then $(a - s)^2 + t^2$ is not singular. That is, $(a - s)^2 + t^2$ is invertible. Hence $(a - s)^2 + t^2 \in Inv(\mathcal{A})$. The map $f : \mathbb{C} \to A$ defined by

$$f(x + iy) = (a - x)^2 + y^2$$

is continuous and $Inv(\mathcal{A})$ is an open neighbourhood of $f(s + it)$. Hence there exist an open neighbourhood U of $s + it$ such that $f(U) \subset Inv(\mathcal{A})$. But then $U \subset \mathbb{C}\backslash Sp(A)$. Thus $\mathbb{C}\backslash Sp(a)$ is open. Hence $Sp(a)$ is closed. □

Definition 1.1.26 (Homomorphism). *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras over a field $\mathbb{F}$ and $\phi : \mathcal{A} \to \mathcal{B}B$ is said to be homomorphism if*

1. *ϕ is linear.*
2. *ϕ is multiplicatives. i.e $\phi(a.b) = \phi(a)\phi(b)$, for all $a, b \in \mathcal{A}$.*

Definition 1.1.27 (Isomorphism). *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras over a field $\mathbb{F}$ and $\phi : \mathcal{A} \to \mathcal{B}$ is said to be isomorphism if*

1. *ϕ is homomorphism.*
2. *ϕ is one one and onto.*

Definition 1.1.28 (Isometry). *Let X and Y be two normed linear spaces over $\mathbb{C}$ or $\mathbb{R}$. Then a linear map $S : X \to Y$ is called isometry if $\|S(x)\| = \|x\|$ for every $x \in X$. An isometrically isomorphism between two normed algebras is a map which is both an isomorphism and an isometry.*

Example 1.1.29. *Let X is real commutative normed algebra. $\mathcal{A} = BL(X)$, the set of all bounded linear functionals on X. Define $T_x : \mathcal{A} \to \mathbb{R}$ by*

$$T_x(f) = f(x) \ for \ x \in X.$$

Then T is a homomorphism.

Example 1.1.30. *We know that $\mathcal{A} = M_{n\times n}(\mathbb{R})$ is a real algebra. X is fixed element of $\mathcal{A}$ with $\det(X) \neq 0$. Define $T : \mathcal{A} \to \mathcal{A}$ by*

$$T(B) = XBX^{-1}, \text{ for all } B \in \mathcal{A}.$$

Then T is a homomorphism.

Corollary 1.1.31. *Let $\mathcal{A}$ and $\mathcal{B}$ be normed algebras with units and $T : \mathcal{A} \to \mathcal{B}$ be a homomorphism. Then $r(T(a)) \leq r(a)$ for every $a \in \mathcal{A}$.*

Proof. We denote the unit of both $\mathcal{A}$ and $\mathcal{B}$ by the same symbol 1. Then $T(1) = 1$. Let $s + it \notin sp(a, \mathcal{A})$ then there is $b \in \mathcal{A}$ such that $[(a-s)^2 + t^2]b = 1 = b[(a-s)^2 + t^2]$. Hence $[(T(a)-s)^2 + t^2]T(b) = 1 = T(b)[(T(a)-s)^2 + t^2]$ $T(a)$ is invertible in $\mathcal{B}$. Hence $s + it \notin Sp(T(a))$. Thus $Sp(T(a), \mathcal{B}) \subset Sp(a, \mathcal{A})$. □

Remark 1.1.32. *If T is an isomorphism, then $r(T(a)) = r(a)$.*

Definition 1.1.33 (Spectrally normed algebra). *Suppose $\mathcal{B}$ has the property $\|b\| \leq k.r(b)$ for every $b \in \mathcal{B}$. Then $\mathcal{B}$ is called spectral normed algebra. Note $\|T(a)\| \leq k.r(T(a)) \leq k.r(a) \leq k\|a\|$ $a \in \mathcal{A}$ thus every homomorphism of $\mathcal{A}$ into a spectrally normed Banach algebra is continuous. In particular, every homomorphism of $\mathcal{A}$ into $\mathbb{R}$ or $\mathbb{C}$ is continuous.*

Theorem 1.1.34 (Mazur-Gelfand theorem). *Let $\mathcal{A}$ be Banach division algebra.*

1. *If $\mathcal{A}$ is complex algebra, then $\mathcal{A}$ isometrically isomorphic to $\mathbb{C}$.*

2. *If $\mathcal{A}$ is real algebra such that $x, y \in \mathcal{A}$ and $x^2 + y^2 = 0$ implies that $x = y = 0$ then $\mathcal{A}$ is isometrically isomorphic to $\mathbb{R}$.*

3. *If $\mathcal{A}$ is commutative real algebra and $\mathcal{A}$ does not satisfy $x, y \in \mathcal{A}$ and $x^2 + y^2 = 0$ implies that $x = y = 0$ then $\mathcal{A}$ is isomorphic to $\mathbb{C}$.*

Proof. 1) Let $a \in \mathcal{A}$ and $\lambda \in Sp(a, \mathcal{A})$ $a - \lambda$ is singular. Hence $a - \lambda = 0$ thus $a = \lambda.1$ and $\|a\| = |\lambda|.\|1\| = |\lambda|$.
2) Let $a \in \mathcal{A}$ and $\lambda \in Sp(a, \mathcal{A})$ $(a-s)^2 + t^2$ is singular. Hence $(a-s)^2 + t^2 = 0$ implies that $a = s$ *and* $t = 0$. Also $\|a\| = |s|$, hence proved.
3) Given that $\mathcal{A}$ is a division algebra and not satisfying the condition $x^2+y^2 = 0$ implies that $x = y = 0$ means that there exist $x, y \in \mathcal{A}$ such that $x^2 + y^2 = 0$ at least one of x and y is nonzero. Let $a \in \mathcal{A}$ and s+it∈Sp(a). Then as above $(a-s)^2 + t^2 = 0$, so that $(a - (s+it))(a - (s-it)) = (a-s)^2 + t^2 = 0$. Hence $a = s + it$ *or* $a = s - it$, as a division algebra. Thus span$\{1, i\}$, $i^2 = -1$,, that is, $\mathcal{A}$ isomorphic to $\mathbb{C}$. $\square$

Definition 1.1.35. *A subset D of $\mathbb{C}$ is said to be symmetric about real axis if $\bar{z} \in D$ for every z in D.*

Let $D \subset \mathbb{C}$ be symmetric about the real axis. By $P_R(D)$ we denote the algebra of all polynomials on D with real coefficients. Note that D is an infinite set, then a polynomial p with complex coefficients belongs to $P_R(D)$ if and only if $p(\bar{z}) = \bar{p}(z)$ for every $z \in D$.

Let $\mathcal{A}$ be a real Banach Algebra with unit and $a \in \mathcal{A}$. Note that $Sp(a, \mathcal{A})$ is symmetric about real axis.

Theorem 1.1.36. *Let $\mathcal{A}$ be a Banach algebra with unit* 1 *and $a \in A$. Let D be an open neighborhood of $Sp(a, \mathcal{A})$ which is symmetric about real axis. Then the mapping $p \mapsto p(a)$ is a homomorphism of $P_R(D)$ into $\mathcal{A}$ which satisfies*

$$Sp(p(a), \mathcal{A}) = \{p(z) : z \in Sp(a, \mathcal{A})\}.$$

Proof. We know that $p \mapsto p(a)$ is a homomorphism of $P_R(D)$ into $\mathcal{A}$. Let $\lambda \in \mathbb{C}$ and p in $P_R(D)$ be of degree n. Then there exist $\beta, \alpha_1, ...\alpha_n$ in $\mathbb{C}$ such that

$$\lambda - p(z) = \beta(\alpha_1 - z)...(\alpha_n - z), \quad \forall z \in D. \tag{1.1}$$

Since $z \in D$ implies that $\bar{z} \in D$, replacing z by $\bar{z}$

$$\lambda - p(\bar{z}) = \beta(\alpha_1 - \bar{z})...(\alpha_n - \bar{z}), \ \forall z \in D. \tag{1.2}$$

Note that $p(\bar{z})$=$\bar{p}(z)$ and taking complex conjugates both sides in 1.2 we get

$$\bar{\lambda} - p(z) = \bar{\beta}(\bar{\alpha}_1 - z)...(\bar{\alpha}_n - z). \tag{1.3}$$

from 1.1 and 1.3 we obtain

$$(\lambda - p(z))(\bar{\lambda} - p(z)) = |\beta|^2(\alpha_1 - z)(\bar{\alpha}_1 - z)...(\alpha_n - z)(\bar{\alpha}_n - z). \tag{1.4}$$

Take $\lambda = s + it$ and $\alpha_k = s_k + it_k,\ k = 1, 2, 3, ...n$. Then the equation 1.4 becomes

$$(s - p(z))^2 + t^2 = |\beta|^2[(s_1 - z)^2 + t^2]...[(s_n - z)^2 + t^2. \tag{1.5}$$

Since $p \mapsto p(a)$ is a homomorphism from $P_R(D)$ into $\mathcal{A}$, we have $(s-p(a))^2+t^2 = |\beta|^2[(s_1-a)^2+t_1^2]....[(s_n-a)^2+t_n^2]$. When p is non constant $\beta \neq 0$, $(s-p(a))^2+t^2$ is singular if and only if $[(s_k-a)^2+t_k^2]$ is singular for some k. Thus $\lambda \in Sp(p(a))$ if and only if $\lambda = p(z)$ for some z in $Sp(a)$. Hence proved. □

Suppose $\mathcal{A}$ is a real Banach algebra with unit 1. Let $a \in \mathcal{A}$ and $Sp(a) \subset D$, where D is a disk symmetric about real axis. In particular, the center of D is real number. Let α be the center of D. Let ρ be the radius of D and $(H_R(D))$ denote the algebra of analytic functions f in D satisfying $f(\bar{z}) = \bar{f}(z)$ for all $z \in D$. Such an f can be expanded in Taylor series around α so that

$$f(z) = \sum_{n=0}^{\infty} \lambda_n (z-\alpha)^n, \text{ for all } z \in D \text{ with } \lambda_n \in R$$

and $\overline{\lim}_{n\to\infty} |\lambda_n| \leq \dfrac{1}{\rho}$. Since Sp(a) $\subset D$, we have $Sp(a-\alpha) \subset \{z : |z| < \rho\}$ so that $r(a-\alpha) < \eta < \rho$. Hence for a large n

$$\|\lambda_n(a-\alpha)^n\| \leq (n/\rho)^n$$

converges as $\sum_{n=0}^{\infty} \lambda_n(a-\alpha)^n$ convergent to an element of $\mathcal{A}$, which is denoted by $f(a)$.

Theorem 1.1.37 (Spectral mapping theorem). *Let $\mathcal{A}$ be a real Banach algebra with unit* 1. *For $a \in \mathcal{A}$, suppose that $Sp(a) \subset D = \{z \in \mathbb{C} : |z-\alpha| < \rho\},\ \alpha \in \mathbb{R}$. Then the mapping $f \mapsto f(a)$ is a homomorphism of $H_R(D)$ into A, satisfying $Sp(f(a), A) = \{f(z) : z \in Sp(a, A)\}$.*

Proof. Observe that the mapping $f \mapsto f(a)$ is a homomorphism. Let $f \in H_R(D)$, then $f(z) = \sum_{n=0}^{\infty} \lambda_n(z-\alpha)^n$. *Consider* $p_m(z) = \sum_{n=0}^{m} \lambda_n(z-\alpha)^n$, m= 0, 1,2, 3, ... λ_n in $\mathbb{R}$, z in D. Then $p_m(z) \in P_R(D)$ for all m=1,2,3,4.... $||p_m(a) - f(a)|| \to 0$ and p_m convergent uniformly to f over each close disc contained in D.

Let $\lambda \in Sp(a), f(\lambda) = s+it$ and $p_m(\lambda) = s_m + it_m$. Since $p_m(a) \to f(a)$ and $s_m + it_m \to s + it$ we have $(s_m - p_m(a))^2 + t_m^2 \to (s - f(a))^2 + t^2$ as $m \to \infty$. If $s + it \notin Sp(f(a))$ *then* $(s - f(a))^2 + t^2$ is invertible and $(s_m - P_m(a))^2 + t^2$ is not invertible. For a large m, hence for such $p_m(\lambda) = s_m + it_m \notin Sp(p_m(a))$. This is contradiction to $\{Sp(p(a), \mathcal{A})\ = \{p(z) : z \in Sp(a, \mathcal{A})\}$.

Hence $Sp(f(a), \mathcal{A}) \subset \{f(z) : z \in Sp(a, \mathcal{A})\}$.

Next, let $z \in Sp(f(a))$ then $f(Sp(a)) := \{f(\lambda) : \lambda \in Sp(a)\}$ is a compact set. Let $\mathcal{A}$ be real Banach algebra with unit 1 and $a \in \mathcal{A}$ *then* $Sp(a, \mathcal{A})$ is a compact subset of $\mathbb{C}$. If $z \notin f(Sp(a))$ then $\delta := \inf\{|z - f(\lambda)| : \lambda \in Sp(a)\} > 0$. Again $Sp(a)$, being compact is contained in a closed disk contained in D. And

$p_n \to f$ uniformly over such disk contained in D.
Hence we have to find m_0 such that for all $m \geq m_0$ and $\lambda \in Sp(a)$, we have $p_m(\lambda) - f(\lambda) < \frac{\delta}{2}$ but then for all λ in $Sp(a)$ and $m \geq m_0$ $|z - p_m(\lambda)| \geq |z - f(\lambda)| - |f(\lambda)p_m(\lambda)| \geq \frac{\delta}{2} > 0$, this implies that z $\notin Sp(P_m(a))$ for all $m \geq m_0$. Let z=s+it and $b_m = (s - p_m(a))^2 + t^2$. Then b_m is invertible for all m$> m_0$ we have $m > m_0$, We have

$$\begin{aligned} r(b^{-1}) &= \sup\{|((s - p_m(\lambda))^2 + t^2)^{-1}| : \lambda \in Sp(a)\} \\ &= \sup\{|((z - p_m(\lambda))|^{-1}|\bar{z} - p_m(\lambda)|^{-1}| : \lambda \in Sp(a)\} \\ &\leq \frac{4}{\delta^2}. \end{aligned}$$

Let $b = (s - f(a))^2 + t^2$, then $b_m \to b$ as $m \to \infty$. Hence we can find $m \geq m_0$ such that $||b_m - b|| < \frac{\delta^2}{4}$. Note that b_m commutes with b.
$r(1 - b^{-1}b) = r(b_m^{-1}(b_m - b)) \leq r(b_m^{-1})r(b_m - b) \leq \frac{4}{\delta^2}||b_m - b|| < 1$. Hence $b_m b$ is invertible and consequently b is invertible. Hence $Sp(f(a)) \subset f(Sp(a))$. $\square$

Remark 1.1.38. *Let $\mathcal{A}$ be a real Banach algebra with unit 1. Since* $\exp(z)$ *is an entire function and* $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ *converges uniformly to* $\exp(z)$ *over every closed disk with center at* 0*. We have* $\exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$ *and we want to show that if a and b are commutes, then* $\exp(a+b) = \exp(a)\exp(b)$*. Hence* $\exp(a)$ *is invertible and its inverse is* $\exp(-a)$*. Now let*

$$\begin{aligned} \mathbb{C}^+ &\equiv \{s + it \in \mathbb{C} : s > 0 \text{ or } t \neq 0\} \\ &= \mathbb{C}\backslash\{t \in R : t \leq 0\}. \end{aligned}$$

For z in $\mathbb{C}^+, z = \exp(i\theta), r > 0, -\pi < \theta < \pi$. We denote $Ln(z)$ the principle branch of the logarithm of z, that is, $Ln(z) = ln\ r + i\theta, -\pi < \theta < \pi$ Then

$$\begin{aligned} ln(\exp(z)) &= z,\ \forall z \in \mathbb{C} \text{ and} \\ \exp(ln(z)) &= z,\ \forall z \in \mathbb{C}^+. \end{aligned}$$

By the spectral mapping theorem, if $Sp(a) \subset \{z \in \mathbb{C} : |z - \alpha| < \rho\} \subset \mathbb{C}^+$ for some $\alpha \in R$ then $Ln(a) \in \mathcal{A}$ and

$$\exp(ln(a)) = a, ln(\exp(a)) = a.$$

Corollary 1.1.39. *Let $\mathcal{A}$ be a real Banach algebra with unit* 1 *and $a \in \mathcal{A}$. If $r(1 - a) < 1$, then*

$$ln(a) = -\sum_{n=1}^{\infty} \frac{(1 - a)^n}{n} \in \mathcal{A}$$

Proof. Since $r(1-a) < 1$, then $Sp(a) \subset \{z \in C : |z-1| < 1\} \subset \mathbb{C}^{+}$. Hence $ln(a) \in A$. Also Taylor series expansion of $ln(z)$ around 1 is $ln(z) = -\sum_{n=1}^{\infty} \dfrac{(1-z)^n}{n}$. Hence proved. □

Lemma 1.1.40. *Let $\mathcal{A}$ be real or complex Banach algebra with unit 1. Then*

1. *$Sp(ab)\backslash\{0\}$ for all a,b in $\mathcal{A}$, so that $r(ab) = r(ba)$.*
2. *For all a in A $r(ab) = r(ba)$, and $||ba|| \leq k||ab||$ for all $a, b \in \mathcal{A}$.*

If there exist $k > 0$ such that $||a||^2 \leq k||a^2||$ for all $a \in \mathcal{A}$, then $\mathcal{A}$ is spectrally normed algebra, in fact, $||a||^2 \leq kr(a)$.

Proof. 1) First, let $\mathcal{A}$ be a real Banach algebra. Let s+it be in $\mathbb{C}$, $s^2 + t^2 = 1$ and $s + it$ be not in Sp(ab). Let c denote the inverse of $(ab-s)^2 + t^2$, then it is followed by direct calculation that $1 + b(2s-ab)ca$ is the inverse of $(ba-s)^2 + t^2$, so that $s + it \notin Sp(ba)$. Hence $r(ba) = r(ab)$.
2) Since $||a||^2 \leq k||a^2||$ for all $a \in \mathcal{A}$. We have by induction, We prove that $||a||^{2^n} \leq k^{2^n-1}||a^{2^n}||$ for all $n = 1, 2, 3\cdots$

$$\begin{aligned}(||a||^{2^{n+1}}) &= (||a||^{2^n})^2 \\ &\leq (k^{2^n-1}||a^n||)^2 \\ &\leq k^{2^{n+1}-2}k||a^{2^{n+1}}|| \\ &\leq k^{2^{n+1}-1}||a^{2^{n+1}}||.\end{aligned}$$

Hence $(||a||^{2^n}) \leq k^{2^n-1}||a^{2^n}||$ *for all* $n = 1, 2, 3...$ We know that $r(a) = \lim_{n\to\infty} ||a^n||^{\frac{1}{n}}$

$$\lim_{n\to\infty}(||a||) \leq \lim_{n\to\infty} k^{1-\frac{1}{2^n}}||a^{2^n}||^{\frac{1}{2^n}}||a|| \leq kr(a).$$

Now it follows from (1) that $||ba|| \leq kr(ba) \leq kr(ab) \leq k||ab||$.. □

Theorem 1.1.41. *Suppose that $\mathcal{A}$ is a Banach algebra with unit 1 satisfying one of the following conditions*

1. *$\mathcal{A}$ is a complex algebra then there exist a positive constant k such that $||a||^2 \leq k||a^2||$ for all $a \in \mathcal{A}$.*
2. *$\mathcal{A}$ is a real algebra then there exist a positive integer k such that $||a||^2 \leq k||a^2 + b^2||$ for all $a, b \in \mathcal{A}$ with $ab = ba$.*

Then $\mathcal{A}$ is commutative.

Proof. In both the cases, $||a||^2 \leq k||a^2||$ for all $a \in \mathcal{A}$.

$$||ba|| \leq ||ab|| \textit{ for all } a, b \in \mathcal{A}.$$

Now ϕ is a continuous linear function on $\mathcal{A}$ and $a, b \in \mathcal{A}$. Suppose $\mathcal{A}$ satisfies the condition (1). Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \phi(\exp(za)b\ \exp(-za))\ for z \in \mathbb{C}.$$

Then f is entire function. Further by(1)

$$\begin{aligned}|f(z)| &\leq ||\phi|| || \exp(za)b\ \exp(-za)|| \\ &\leq k||\phi||\ ||b\ \exp(za)\exp(-za)|| = k||\phi||\ ||b||.\end{aligned}$$

Thus f is bounded and hence it is constant by Liouville's theorem. $f(0) = f(1)$ implies that

$$\phi(b) = \phi[\exp(a)b\ \exp(-a)]. \tag{1.6}$$

Next suppose $\mathcal{A}$ satisfy the condition (2). Define $u : \mathbb{R}^2 \to \mathbb{R}$ by

$$u(x, y) = \phi\{(\exp(xa)[(cosya)b\ cosya + (sinya)b\ sinya]\exp(-xa)\}, \text{for } (x, y) \in \mathbb{R}^2$$

Then $u(x, y)$ is harmonic function in $\mathbb{R}^2$.

$$\begin{aligned}|u(x, y)| &\leq ||\phi||||(\exp(xa)[(cosya)b\ cosya + (sinya)b\ sinya]\exp(-xa)|| \\ &\leq k||\phi||\ ||[(cosya)b\ cosya + (sinya)b\ sinya]|| \\ &\leq k||\phi||\ ||b||\ ||cosya||^2 + ||sinya||^2 \\ &\leq 2k^2||\phi||\ ||b||\ ||cos^2ya + sin^2|| \\ &\leq 2k^2||\phi||\ ||b||.\end{aligned}$$

Thus $u(x, y)$ is bounded harmonic function on $\mathbb{R}^2$. Hence u is constant. Therefore u(0,0)=u(1,0) again implies equation (1.6). Since equation (1.6) for every continuous linear functional ϕ on $\mathcal{A}$, we have

$$b = \exp(a)b\ \exp(-a).$$

That is,

$$b\ \exp(a) = \exp(a)b\ for\ all\ a, b\ in\ \mathcal{A}. \tag{1.7}$$

Let c and d any two elements of $\mathcal{A}$. If $||c|| < 1$ there exist a in A such that $1 + c = \exp(a)$. From (1.7) that is, $d(1 + c) = (1 + c)d$ or $dc = cd$ the restriction $||c|| < 1$ can we removing by considering $\frac{c}{2||c||}$ if $c \neq 0$. □

1.2 Commutative Real Banach Algebras

Definition 1.2.1. *Let $\mathcal{A}$ be a real algebra. The carrier space of $\mathcal{A}$ denoted by $Car(\mathcal{A})$, is the set of all nonzero homomorphisms from $\mathcal{A}$ to $\mathbb{C}$.*

Let $\phi \in Car\mathcal{A}$ and define $\bar{\phi}$ by $\bar{\phi}(a) = \overline{\phi(a)}$ for $a \in \mathcal{A}$. Then it is easy to see that $\bar{\phi} \in Car\mathcal{A}$. Also we have the following

$$\begin{aligned}\overline{\phi}(a+b) &= \overline{\phi(a+b)} \\ &= \overline{\phi(a)} + \overline{\phi(b)} \\ \overline{\phi}(\alpha a) &= \overline{\phi(\alpha a)}, \quad \alpha \in \mathbb{R}. \\ &= \alpha\overline{\phi(a)}\end{aligned}$$

Hence $\bar{\phi}$ is linear.

$$\begin{aligned}\overline{\phi}(a.b) &= \overline{\phi(a.b)} \\ &= \overline{\phi(a).\phi(b)} \\ &= \overline{\phi(a)}.\overline{\phi(b)} \\ &= \bar{\phi}(a).\bar{\phi}(b)\end{aligned}$$

That is, $\bar{\phi}$ is multiplicative. Hence $\bar{\phi} \in \mathcal{A}$.

Definition 1.2.2 (Gelfand topology)**.** *For $a \in \mathcal{A}$, the Gelfand transformation is a map $\hat{a} : Car\mathcal{A} \to \mathbb{C}$, given by $\hat{a}(\phi) = \phi(a)$ for $\phi \in Car\mathcal{A}$. The weakest topology on $Car\mathcal{A}$ that makes $\hat{a}$ continuouns on $Car(\mathcal{A})$ for all $a \in \mathcal{A}$ is called the Gelfand topology on $Car(\mathcal{A})$.*

Note that, if $\mathcal{A}$ has unit 1 and $\phi \in Car(\mathcal{A})$, then $\phi(1) = 1$ because ϕ is a nonzero homomorphism. If a is invertible then $\phi(a^{-1})\phi(a) = 1$. Thus $\phi(a)$ is non zero for every invertible element a.

Lemma 1.2.3. *Let be a real Banach algebra with unity 1 and $\phi \in Car(\mathcal{A})$. Then $\|\phi\| := \sup\{|\phi(a)| : a \in \mathcal{A}, \|a\| \leq 1\} = 1$.*

Proof. We know that ϕ is a homomorphism from $\mathcal{A}$ to $\mathbb{C}$. Also $r(\phi(a)) = |\phi(a)| for\ every\ a \in \mathcal{A}$. We have $|\phi(a)| = r(\phi(a)) \leq r(a) \leq \|a\| \leq 1$ for every $a \in \mathcal{A}$. Hence $\|\phi\| \leq 1$. Since $\phi(1) = 1$, we have $\|\phi\| = 1$.

The following example shows that $Car(\mathcal{A})$ can be empty. Consider the quaternion algebra $\mathbf{H}= \{a_0 + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$ Suppose that $\phi \in Car(\mathbf{H})$ then

$$\begin{aligned}\phi(2k) &= \phi(ij - ji) \\ &= \phi(i)\phi(j) - \phi(j).\phi(i) = 0.\end{aligned}$$

Since $2k$ is invertible, we have $1 = \phi(1) = \phi(2k).\phi((2k)^{-1}) = 0$, which is a contradiction. Hence Car($\mathbf{H}$) is empty. We shall however show in the sequel that if $\mathcal{A}$ is a commutative normed algebra, then $Car(\mathcal{A})$ is non empty. $\square$

Definition 1.2.4 (Ideal)**.** *Let $\mathcal{A}$ be a real or complex algebra with unit. A subspace I of $\mathcal{A}$ is called a left ideal if $a \in I$ and $x \in \mathcal{A}$ imply that $xa \in I$. Similarly, a right ideal is defined. The space I is called an ideal if I is both a left and a right ideal.*

An ideal is called proper if it is different from $\mathcal{A}$. When $\mathcal{A}$ is commutative, every left ideal or right ideal is ideal.

Definition 1.2.5 (Maximal ideal)**.** *A maximal ideal is a proper ideal M which is not contained in any other proper ideal, that is if I is an ideal and $M \subset I \subset \mathcal{A}$ then $M = I$ or $I = \mathcal{A}$.*

Lemma 1.2.6. *Let M be a closed ideal in a normed algebra $\mathcal{A}$ with unit. Let $\frac{\mathcal{A}}{M}$ denote the set of all cosets of the form*

$$M + a = \{x + a : x \in M\}.$$

For a, b in A and $\alpha \in R$, define

$$\begin{aligned}(M + a) + (M + b) &= M + (a + b)\\ (M + a).(M + b) &= M + ab\\ \alpha(M + a) &= M + \alpha a.\end{aligned}$$

Then the operations are well defined and under these operations $\frac{\mathcal{A}}{M}$ is an algebra with unit $M + 1$. Further define $\|M + a\| = \inf\{\|x + a\| : x \in M\}$ for $a \in \mathcal{A}$. Then $\frac{A}{I}$ is a normed algebra with respect to this norm. If $\mathcal{A}$ is Banach algebra, then $\frac{\mathcal{A}}{I}$ is Banach algebra.

Proof. First we need to prove that $\|M + a\| = \inf\{\|x + a\| : x \in M\}$ is well defined, Note that
1) The above norm is clearly non negative.
2) We need to prove second property $\|x\| = 0$ if and only if $x = 0$. If $a+M = M$ implies $\|0 + M\| = 0$. If

$$\|x + M\| = 0 \implies \inf\{x + m : m \in M\} = 0.$$

By the definition of the norm, we assert the existence of a sequence $< x_n >$ in M such that

$$\begin{aligned}\lim_{n\to\infty} \|x + x_n\| = \|x + M\| &\implies \|x + x_n\| \to 0\\ &\implies x + x_n \to 0 \implies x_n \to -x.\end{aligned}$$

Using the fact that M is closed, we say that $-x \in M$ because it is the limit of a Cauchy sequence of points of M which implies $x + M = M$.
3) Let $a \in F$. Then

$$\begin{aligned}\|ax + M\| &= \inf_{m\in M} \|ax + M\|\\ &= \inf_{m\in M} \|ax + am\|\\ &= |a| \inf_{m\in M} \|x + m\|\\ &= |a|\|x + M\|.\end{aligned}$$

4) Since $\|x+y+M\|$ has been defined as greatest lower bound, we can assert the existence of sequences of points $< x_n > \ and < y_n >$ in M such that

$$\lim_{n\to\infty} \|x+x_n\| = \|x+M\| \ and \ \lim_{n\to\infty} \|y+y_n\| = \|y+M\|.$$

For any n, this implies

$$\|x+y+M\| \leq |x+y+x_n+y_n| \leq \|x+x_n\| + \|y+y_n\|.$$

Since the above statement true for any n , it must also be true in the limit. Thus

$$\|x+y+M\| \leq \|x+M\| + \|y+M\|$$

Thus triangle inequality satisfied. Hence $\dfrac{X}{M}$ is a normed linear space.

Suppose that $< x_n + M >$ is a Cauchy sequence of points of $\dfrac{X}{M}$. It follows that if a convergent sequence can be extracted from it then the entire sequence must convergent, and furthermore, it must convergent to same limit as the subsequence.

The sequence $< x_i + M >$ has a subsequence $< y_i + M >$,

$$\|(y_{i+1} + M - (y_i + M)\| = \|(y_{i+1} - y_i) + M\| < 1/2^i.$$

By virtue of the definition of this norm, there existence $h_i \in (y_{i+1} - y_i) + M$ such that $\|h_i\| < 1/2^i$.

Choose $z_1 \in y_1 + M$ and suppose $z_1, z_2, ...z_n$ have been chosen such that $z_{i+1} \in y_{i+1} + M$ and

$$z_{i+1} - z_i = h_i \tag{1.8}$$

for $i = 1, 2, 3, ..., n-1$. We wish to demonstrate that an $(n+1)th$ vector can be add to the list having, the same properties. To this end we write

$$h_n = y_{n+1} - y_n + m \ and \ z_n + m^{'}.$$

Where $m, m^{'} \in M$.

The first equation implies that

$$h_n + y_n = y_{n+1} + m.$$

Adding $m^{'}$ to both sides and letting $m + m^{'} = m^{''}$ this completes the induction process concerning the existence of vector $z_1, z_2, ...$ having the properties (1.7) it will be shown that z_n is Cauchy sequence. suppose $n > m$, and consider

$$\begin{aligned}
\|z_n - z_m\| &\leq \|z_n - z_{n-1}\| + ... + \|z_{m+1} - z_m\| \\
&\leq \|h_{n-1}1\| + \|h_{n-2}\| + ... + \|h_m\| \text{ from (1.7)} \\
&\leq \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + ... + \frac{1}{2^m} \\
&\leq \frac{1}{2^{m-1}}.
\end{aligned}$$

It is clear that $(z_n)_{n=1}^{\infty}$ is a Cauchy sequence of elements from X. Since X is Banach space, most convergent to some point z of X. Further the sequence of elements $< y_i + M >$ from X/M, approaches $(z+M)$ as a limit.For $y_{i+1} + M = z_i + M$ and

$$\|z + M - (y_i + M)\| = \|z + M - (z_i + M)\|. \ But \ \|z - z_i + M\| \leq \|z - z_i\| \to 0.$$

This implies $z_i + M \to z + M$. This proves that the original sequence $\{x_i + M\}_{n=1}^{\infty}$ convergence to $z + M$. Hence X/M is a Banach space. □

Lemma 1.2.7. *Let I be a proper ideal in a (real or complex) commutative Banach algebra $\mathcal{A}$ with unit 1. Then closure J of I is a proper ideal. In particular, every maximal ideal is closed.*

Proof. We claim that closure J of an ideal I is an ideal. Let $x \in \bar{I} = J$ implies $x \in I$ or $x \notin I$. Suppose $x \imath n I$ implies $a.x \in I$ for all $a \in \mathcal{A}$. Suppose$x \notin I, \ x \in J$ implies that $s < x_n > \ inI$ convergent to x. Hence $a.x_n \to a.x \ for \ every \ a \in \mathcal{A}$. Hence $\ a.x \in J$ for every $a \in \ \mathcal{A}$. Similarly J is subspace of $\mathcal{A}$. Hence J is ideal. Next, we claim that J is proper ideal of $\mathcal{A}$.

Clearly J is an ideal. If $a \in I$ $\|1 - a\| < 1$, then a is invertible (because $r(a) < |s|$, then $a - s$ is invertible) implies that I is not proper ideal. Thus $\|1 - a\| \geq 1$ for every a in I and hence for every a in J. Hence $1 \notin J$, so J is proper ideal.

Claim : maximal ideal is closed. Let M is a maximal ideal of algebra $\mathcal{A}$. We know that $M \subset \bar{M} \subset A$ because closure of every proper ideal is proper. We know M is maximal ideal hence $M = \bar{M}$. Hence M is closed. □

Theorem 1.2.8. *Let $\mathcal{A}$ be a real commutative Banach algebra with unit 1.*

1. *If $\phi \in Car(\mathcal{A})$, then the kernel of ϕ, denoted by $ker\phi = \{a \in \mathcal{A} : \phi(a) = 0\}$ is maximal ideal in $\mathcal{A}$.*
2. *If M is a maximal ideal in $\mathcal{A}$, then $M = ker\phi$ for some $\phi \in Car(\mathcal{A})$.*
3. *For ϕ, ψ in $Car(\mathcal{A})$, if $ker\phi = ker\psi$, then $\psi = \phi$ or $\psi = \bar{\phi}$.*

Proof. 1) If $a \in ker\phi, b \in \mathcal{A}$, then $\phi(ab) = \phi(a).\phi(b) = 0 \ hence \ ab \in ker\phi$, hence $ker\phi$ is ideal. Since $\phi(1) = 1$, $ker\phi$ is proper. Suppose that $ker\phi$ is properly contained in an ideal I. We consider two cases to show that $I = \mathcal{A}$.

Case(a): Suppose that $\phi(\mathcal{A}) \subset \mathbb{R}$. Then we can find $b \in I$ such that $\phi(b) = s \neq 0, \ s \in \mathbb{R}$. Let a $\in \mathcal{A}$. If $c = a - \phi(a)b/s, \ then \ \phi(c) = 0$. Hence $c \in ker(\phi) \subset I$. Thus $a = c + \phi(a)b/s \in I$. This shows that $I = \mathcal{A}$.

Case(b) If $\phi(\mathcal{A})$ is not contained in $\mathbb{R}$, we can find $b \in \mathcal{A}$ such that $\phi(b) = s + it \ and \ t \neq 0$ let $c = (b - s)/t$ then $\phi(c) = i$. There exist d in I such that $\phi(d) = p + iq \neq 0, p, q \ in \ \mathbb{R}$. Let $e = d(-pc^2 - qc)/(p^2 + q^2)$. Then $e \in I$and $\phi(e) = 1$. Let $a \in \mathcal{A}$ and $\phi(a) = x + iy$. Then $a - xe - yce \in ker(\phi) \subset I$ Also, $xe + yce \in I$. Hence $a \in I$. This shows that $I = \mathcal{A}$.

Let M be a maximal ideal. Then M is a closed. Show that $\mathcal{A}/M$ is a real commutative Banach algebra with unit. Since M is a maximal, it is easy to prove that $\mathcal{A}/M$ is division algebra. Hence by the Mazur - Gelfand theorem. $\mathcal{A}/M$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Let θ be such isomorphism and $\pi : \mathcal{A} \to \mathcal{A}/M$ be the canonical map, then $\phi = \theta \circ \pi \in Car(\mathcal{A})$.

$$\begin{aligned} ker(\phi) &= \{a : \theta \circ \pi(a) = 0\} \\ &= \{a : \theta(\pi(a)) = 0\} \\ &= \{a : \theta(a + M) = 0\} \ (\theta \ is \ isomorphism \quad = \{a : a + M = M\}) \\ &= M. \end{aligned}$$

(3) Let $\phi, \psi \in Car(\mathcal{A})$ and $ker(\phi) = ker(\psi)$. *Let* $\pi : \mathcal{A} \to \mathcal{A}/M$ be a canonical map and $\tilde{\phi}$ *and* $\tilde{\psi}$ be defined by $\tilde{\phi}((M + a)) = \phi(a)$ and $\tilde{\psi}((M + a)) = \psi(a)$ for a in A. Then $\tilde{\phi}$ and $\tilde{\psi}$ are both isomorphisms from $\mathcal{A}/M$ to $\mathbb{R}$ or $\mathbb{C}$. Hence $\tilde{\phi} \circ (\tilde{\psi})^{-1}$ is isomorphism from $\mathbb{R}$ to $\mathbb{R}$ or $\mathbb{C}$ to $\mathbb{C}$. he only isomorphism from $\mathbb{R}$ to $\mathbb{R}$ is the identity map and the only isomorphism from $\mathbb{C}$ to $\mathbb{C}$, where $\mathbb{C}$ is regarded as real algebra are the identity map the complex conjugation that is, the map $z \mapsto \bar{z}$ this implies that $\tilde{\phi} = \tilde{\psi}$ or $\tilde{\phi} = \bar{\tilde{\psi}}$. Since $\phi = \tilde{\phi} \circ \pi$ and $\psi = \tilde{\psi} \circ \pi$, we have $\phi = \psi$ or $\phi = \bar{\psi}$. $\square$

Remark 1.2.9. *The set of all maximal ideals of real commutative Banach algebra with unit* 1 *is denoted by* $M(\mathcal{A})$ *and is called the* ***maximal ideal space*** *of* $\mathcal{A}$*. The above theorem shows that the map* $ker : Car(\mathcal{A}) \to M(\mathcal{A})$ *(i.e the map* $\phi \mapsto ker\phi$ *) is onto. Thus implies that Car(A) is non empty. Further, for M in* $M(\mathcal{A})$*, the inverse image* $ker^{-1}(M)$ *consists of* $\{\phi, \bar{\phi}\}$ *for some* ϕ *in* $Car(\mathcal{A})$*.*

Remark 1.2.10. *Let* $\mathcal{A}$ *be a complex commutative Banach algebra with unit 1. The carrier space* $\mathcal{A}$*, denoted by* $Car(\mathcal{A})$*, is the set of all non zero homomorphisms from* $\mathcal{A}$ *to* $\mathbb{C}$*. Where c is regarded as a complex algebra. Hence if* $\phi \in Car(A)$*, then* $\phi(i) = i$*. Let* A_R *denote* $\mathcal{A}$*, regarded as a real algebra. It is easy to see that if* $\phi \in Car(\mathcal{A})$*, then* ϕ *as well as* $\bar{\phi}$ *are in* $Car(\mathcal{A}_R)$*, on other hand, if* $\phi \in Car(\mathcal{A}_R)$*, then* $\phi(i)^2 = -1$*,* $\phi(i) = \pm i$ *thus exactly one of* ϕ *and* $\bar{\phi}$ *belongs to Car(A), that is* $Car(\mathcal{A}_R) = Car(\mathcal{A}) \bigcup \{\bar{\phi} : \phi \in Car\mathcal{A}\}$*.*

Theorem 1.2.11. *Let* $\mathcal{A}$ *be a real commutative Banach algebra with unit* 1*. Then*

1. $Car(\mathcal{A})$ *endowed with the Gelfand topology, is compact Hausdorff space.*
2. *The map* $\tau : Car(\mathcal{A}) \to Car(\mathcal{A})$*, defined by* $\tau(\phi) = \bar{\phi}$ *is homomorphism and* $\tau \circ \tau$ *is identity map on* $Car(\mathcal{A})$*.*
3. *The set of all fixed points of* τ *is close in* $Car(\mathcal{A})$*.*
4. *The mapping* $a \mapsto \tilde{a},\ a \in \mathcal{A}$*, is a isomorphism of* $\mathcal{A}$ *into* $C(Car(\mathcal{A}))$*.*
5. *a is singular in* $\mathcal{A}$ *if and only if* $\tilde{a}(\phi) = 0$ *for some* ϕ *in* $Car(\mathcal{A})$*.*
6. *For* $a \in \mathcal{A}$ *,* $\tilde{a}(Car(\mathcal{A})) = Sp(a, \mathcal{A})$*.*
7. $\|\tilde{a}\| = \sup\{|\tilde{a}(\phi)| : \phi \in Car(\mathcal{A})\} = r(a)$*.*

Proof. 1) Let $\phi, \psi \in Car(\mathcal{A})$ and $\psi \neq \psi$. This means that for some $a \in \mathcal{A}$, $\phi(a) \neq \psi(a)$; that is $\tilde{a}(\phi) \neq \tilde{a}(\psi)$. Since $\tilde{a}$ is continuous with respect to Gelfand topology, we enclose ϕ and ψ in disjoint open sets, namely

$$N_\phi = \{\theta \in Car(\mathcal{A}) : |\tilde{\phi}(a) - \tilde{\psi}(a)| < \epsilon\} \textit{ and}$$
$$N_\psi = \{\theta \in Car(\mathcal{A}) : |\tilde{a}(\psi) - \tilde{a}(\theta)| < \epsilon\}$$

with $0 < \epsilon < |\phi(a) - \psi(a)|/2$. $N_\phi \cap N_\psi = \phi$. Thus $Car(\mathcal{A})$ is Hausdorff.

To prove we proceed as follows. For each $a \in \mathcal{A}$, let $K_a = \{z \in C : |z| \leq \|a\|\}$ and $K = \prod_{a \in A} K_a$ be a topological product. Since each K_a is compact, K is compact by Tychonoffs theorem. We have $\phi \in Car(\mathcal{A})$ and $a \in \mathcal{A}$, $|\phi(a)| \leq \|a\|$. Thus $\phi(a) \in K_a$. Hence ϕ can be considered as a point of K. Consequently, $Car(\mathcal{A})$ can be considered as a subset of K. Further, it is clear that the Gelfand topology on $Car(\mathcal{A})$ is same as the (relative) topology of $Car(\mathcal{A})$ as subspace of K. Thus to prove the compactness of $Car(\mathcal{A})$, it is sufficient to show that $Car(\mathcal{A})$ is closed in K. Let $\psi \in K$ be in the closure of $Car(\mathcal{A})$. Since $\psi \in K, \psi$ is a function on $\mathcal{A}$ such that $|\phi(a)| \leq \|a\|$ for each a in $\mathcal{A}$. Since ψ is in the closure of $Car(\mathcal{A})$. There is net $\{\psi_\alpha\}$ in Car(A) converging to ψ this means that $\psi_\alpha(a) \to \psi(a)$ for every $a \in \mathcal{A}$. Hence, for $a, b \in \mathcal{A}$,

$$\begin{aligned}\psi(a+b) &= \lim \psi_\alpha(a+b) \\ &= \lim(\psi_\alpha(a) + \psi_\alpha(b) \\ &= \psi(a) + \psi(b).\end{aligned}$$

$$\begin{aligned}\psi(sa) &= \lim \psi_\alpha(sa) \\ &= s \lim \psi_\alpha(a) \\ &= s\psi_(a) \\ \psi(ab) &= \lim \psi_\alpha(ab) \\ &= \lim \psi_\alpha(a)\psi_\alpha(b) \\ &= \psi(a)\psi(b).\end{aligned}$$

And $\psi(1) = 1$ for $a, b \in \mathcal{A}$ and $s \in \mathbb{R}$. Thus $\psi \in Car(\mathcal{A})$. Hence $Car(\mathcal{A})$ is closed subset of K. Hence $Car(\mathcal{A})$ is compact.

(2) Clearly, $\tau \circ \tau(\phi) = \tau(\bar{\phi}) = \phi$ for all $\phi \; in \; Car(\mathcal{A})$. Now suppose that a net $\{\phi_\alpha\}$ convergent to ϕ in $Car(\mathcal{A})$. Then for each $a \in \mathcal{A}$, $\tilde{a}(\phi_\alpha) \to \tilde{a}(\phi)$ hence $\phi_\alpha(a) \to \phi(a)$ or $\overline{\phi_\alpha(a)} \to \overline{\phi(a)}$; that is, $\tilde{a}(\tau(\phi_\alpha)) \to \tilde{a}(\tilde{(\phi)})$ this means that $\tau(\phi_\alpha) \to \tau(\phi)$. Thus τ is continuous. Since $\tau^{-1} = \tau, \tau$ is a homeomorphism.
3) $S = \{\phi : \tau(\phi) = \phi\}$, S is closed because τ is continuous on $Car(\mathcal{A})$.

4) Let $a, b \in \mathcal{A}$ and $s \in \mathbb{R}$. For each ϕ in $Car(\mathcal{A})$, we have $\phi(a+b) = \phi(a) + \phi(b)$ that is $\widetilde{(a+b)} = \tilde{a} + \tilde{b}$ similarly $\widetilde{(sa)} = s\tilde{a}$ and $\widetilde{(ab)} = \tilde{a}\tilde{b}$.

5) If a is invertible, then $\phi(a)\phi(a^{-1}) = 1$, hence $\phi(a) \neq 0$ for every ϕ in $Car(\mathcal{A})$. If a is singular, then $I = \{ab : b \in A\}$ is proper ideal. I is contained

in a maximal ideal M. Hence$M = ker\phi$ for some ϕ in $Car(\mathcal{A})$. For this $\phi(a) = 0 = \tilde{a}(\phi)$.
(6) $s + it \in \tilde{a}(Car(\mathcal{A}))$ if and only if $s + it = \phi(a)$, for some ϕ in $Car(\mathcal{A})$ if and only if $\phi((a - s)^2 + t^2) = 0$ for some ϕ in $Car(\mathcal{A})$ if and only if $(a - s)^2 + t^2$ is singular in $\mathcal{A}$. If and only if $s + it \in Sp(a, \mathcal{A})$.
(7) We know that $r(a) = \sup\{(s^2 + t^2)^{1/2} : s + it \in p(a, \mathcal{A})\}$
from 6) $\tilde{a}(Car\mathcal{A}) = Sp(a, \mathcal{A})$.

$$\begin{aligned}
\|\tilde{a}\| &= \sup\{|\tilde{a}(\phi)| : \phi \in Car(\mathcal{A})\} \\
&= \sup\{|\tilde{a}(\phi)| : \tilde{a}(\phi) \in \tilde{a}(Car(\mathcal{A}))\} \\
&= \sup\{|\phi(a)| : \phi(a) \in Sp(a, \mathcal{A})\} \\
&= r(a).
\end{aligned}$$

□

Chapter 2

Gleason - Kahane - Zelazko theorem

2.1 Introduction

In theory of Banach algebras, the theorem which yields global conclusions from local hypothesis, are considered quite significant. An example of such theorem is the Gleason- Kahane - Zalazko theorem. Let $\mathcal{A}$ be a complex Banach algebra with unit element 1 and ϕ be function on $\mathcal{A}$ such that $\phi(1) = 1$ *and* $\phi(a) \neq 0$ for every invertible element $a \in \mathcal{A}$. Then ϕ is multiplicative.

This theorem is not true in real Banach algebra. This can be seen by the following example.

Let $\mathcal{A} = C_R[0, 1]$, the algebra of real valued continuous functions on $[0, 1]$ and defined ϕ as $\phi(f) = \int_0^1 f(t)dt$ for f in A. Then to see that $\phi(1) = 1, \phi(f) \neq 0$ for every invertible element f in $\mathcal{A}$, but ϕ is not multiplicative.

2.1.1 Definitions and Notations

Let $\mathcal{A}$ be real algebra with unit element 1 and a be an element of $\mathcal{A}$. Then the spectrum, $Sp_A(a) = \{s + it : (a - s)^2 + t^2 \text{ is singular in } \mathcal{A}\}$.

Lemma 2.1.2. *Let $\mathcal{A}$ be a real algebra with unit element* 1 *and* $\phi : \mathcal{A} \to \mathbb{C}$ *a linear map with satisfying* $\phi(a^2) = \phi(a)^2$ *for every* a *in* $\mathcal{A}$. *Then* $\phi(ab) = \phi(a)\phi(b)$ *for every* $a, b \in \mathcal{A}$.

Proof. Let $a, b \in \mathcal{A}$, then by hypothesis, we have $\phi((a+b)^2) = (\phi(a+b))^2$. After simplifying this we get $\phi(ab+ba) = 2\phi(a)\phi(b)$. This implies $\phi \neq 0$ and $\phi(1) = 1$. If ϕ is not multiplicative, there exist $x, y \in \mathcal{A}$ such that $\phi(xy)-\phi(x)\phi(y) = k \neq 0$ We claim that there exist $a, b \in \mathcal{A}$ such that $\phi(a) = 0$ and $\phi(ab) \neq 0$. Let $\phi(x) = s + it$, $\phi(y) = p + iq$.

Case(i): Assume that either $t = 0$ or $q = 0$. Without loss of generality, we assume that suppose $t = 0$. Let $a = x - s, b = y$. Then $\phi(a) = 0$ and $\phi(ab) = \phi((x-s)y)$ $= \phi(xy) - \phi(x)\phi(y) = k$.

Case(ii) Suppose $t \neq 0$ and $q \neq 0$. Consider $x^{'} = \frac{x-s}{t},\ and\ y^{'} = \frac{y-p}{q}$. Then $\phi(x^{'}) = i = \phi(y^{'})$ and

$$\begin{aligned}\phi(x^{'}y^{'}) &= \frac{1}{tq}\phi((x-s)(y-p)) \\ &= \frac{k}{tq} - 1.\end{aligned}$$

Set $a = x^{'} - y^{'}, b = y^{'}$. Then $\phi(a) = 0$ and

$$\begin{aligned}\phi(ab) &= \phi(x^{'}y^{'} - (y^{'})^2) \\ &= \phi(x^{'}y^{'}) - (\phi(y^{'}))^2 \\ &= \frac{k}{tq} \neq 0.\end{aligned}$$

Let a, b belongs to A such that $\phi(a) = 0$ and $\phi(ab) = r \neq 0$ then $\phi(ba) = -r$. For $c = bab$, we get

$$\begin{aligned}0 = 2\phi(a)\phi(c) &= \phi(ac + ca) \\ &= \phi((ab)^2 + (ba)^2) \\ &= 2r^2 \neq 0.\end{aligned}$$

This is a contradiction. Hence ϕ is multiplicative. □

Theorem 2.1.3. *Let $\mathcal{A}$ be a real commutative Banach algebra with unit element 1 and $\phi : \mathcal{A} \to \mathbb{C}$ be a non zero linear map. Then the following are equivalent.*

1. *$\phi(ab) = \phi(a)\phi(b)$ for all a, b in $\mathcal{A}$.*
2. *$\phi(1) = 1$ and $\phi(a)^2 + \phi(b)^2$ belongs to $Sp_A(a^2 + b^2)$ for all a, b in $\mathcal{A}$.*
3. *$\phi(1) = 1$ and $\phi(a)^2 + \phi(b)^2 \neq 0$ for all a, b in $\mathcal{A}$ such that $(a^2 + b^2)$ is invertible.*

Proof. (1) $\Longrightarrow$ (2) Let ϕ is multiplicative, $\phi(1) = 1$ and $\check{a}(Car\mathcal{A}) = Sp(a)$. This implies $\phi(a) \in Sp(a),\ \forall \phi \in Car\mathcal{A}$, thus $\phi(a^2 + b^2) \in Sp(a^2 + b^2)$.
(2) $\Longrightarrow$ (3) If $a^2 + b^2$ is not invertible, then 0 does not belongs to $Sp(a^2 + b^2)$. Hence $\phi(a)^2 + \phi(b)^2$ is non zero.
(3) $\Longrightarrow$ (2) First we prove that $\phi(a) \leq \|a\|$. Let $a \in \mathcal{A}$ and $s + it \in \mathbb{C}$ such that $s^2 + t^2 > \|a\|^2$. Then $(a-s)^2 + t^2$ is invertible in $\mathcal{A}$. By our hypothesis $\phi(a-s)^2 + \phi(t^2) \neq 0$, this implies that $\phi(a) \neq s + it$. Hence $|\phi(a)| \leq \|a\|$.

For $a \in \mathcal{A}$, define the functions u, v as follows.

$$\begin{aligned}&u(x, y) = \phi([(\exp(a)cos(ya)]) \text{ and} \\ &v(x, y) = \phi[(\exp xa)sin(ya)] \text{ for all real } x, y.\end{aligned}$$

Define $F(z) = F(x+iy) = u(x,y) + iv(x,y)$ for $z = x+iy$ in $\mathbb{C}$. A simple calculation shows that $F(z) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\phi(a^n)z^n$, for all z in $\mathbb{C}$.

$$\begin{aligned} |F(z)| &\leq 1 + \sum_{n=1}^{\infty} \frac{1}{n!}|\phi(a^n)||z|^n \\ &= \exp(||a|||z|)\text{for all } z \; in \; \mathbb{C}. \end{aligned}$$

Thus $F(z)$ is entire function of the exponential type. Since

$$[\exp(xa)\cos(ya)]^2 + [\exp(xa)\sin(ya)^2 = \exp(2xa)$$

is invertible, this implies that $F(z) \neq 0$ for every $z \in \mathbb{C}$. By the Hadamard's factorization theorem, then there exist $\alpha \in \mathbb{C}$ such that $F(z) = \exp(\alpha z)$ i.e $F(z) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\alpha^n z^n$ for all $z \in \mathbb{C}$. We get $\phi(a) = \alpha$ and $\phi(a^2) = \alpha^2 = \phi(a)^2$.

Thus we get $\phi(a^2) = \phi(a)^2$, for every a in $\mathcal{A}$. Hence by the Lemma 1.1.2, ϕ is multiplicative. □

Theorem 2.1.4. *Let $\mathcal{A}$ be a real Banach algebra with unit element 1 and let $\phi : \mathcal{A} \to \mathbb{C}$ be a non zero linear map. Then the following are equivalent.*

1. *$\phi(ab) = \phi(a)\phi(b)$.*
2. *$\phi(1) = 1$ and $\phi(a)^2 + \phi(b)^2$ belongs to $Sp(a^2+b^2)$ for all $a, b \in \mathcal{A}$ such that $ab = ba$.*
3. *$\phi(1) = 1$ and $\phi(a)^2 + \phi(b)^2 \neq 0$, if $ab = ba$ and a^2+b^2 is invertible.*

Proof. (1) $\Longrightarrow$ (2) That $\phi(1) = 1$. It is easy to see that $\phi(1) = 1$. Let $a^2+b^2 = c$ and $\phi(a)^2+\phi(b)^2 = \phi(c) = s+it$. If $s+it$ does not belongs to $Sp(c)$, then there exist d in $\mathcal{A}$ such that $1 = \phi(1) = \phi([(c-s)^2+t^2]d) = ([((c)-s)^2+t^2])\phi(d) = 0$, which is contradiction.
(2) $\Longrightarrow$ (3) Observe that, a^2+b^2 is invertible implies that $0 \notin Sp(a^2+b^2)$. Hence $\phi(a^2+b^2) \neq 0$.
(3) $\Longrightarrow$ (1) Let a be an element of $\mathcal{A}$ and $\mathcal{B}$ a closed commutative sub algebra of $\mathcal{A}$, containing a. By apply theorem(1.1.3), we get $\phi(a^2) = \phi(a)^2$ for every element a in $\mathcal{A}$. The result follows from lemma (1.1.2) that ϕ is commutative. □

Corollary 2.1.5. *Let $\mathcal{A}$ be a real algebra with unit element 1. If ϕ is a linear functional on $\mathcal{A}$ such that every pair of elements a, b in $\mathcal{A}$ with $ab = ba$ and a^2+b^2 invertible, we have $\phi(a) \neq 0$ or $\phi(b) \neq 0$ then $\phi(ab) = \phi(a)\phi(b)$ for all a, b in $\mathcal{A}$.*

Proof. Note that ϕ satisfy the condition (3) of theorem(1.1.4). Hence by this ϕ is multiplicative. □

Corollary 2.1.6. *Let $\mathcal{A}$ be a real Banach algebra with unit 1 and X a linear subspace of $\mathcal{A}$ of codimension 1 with the following property: for every a, b in $\mathcal{A}$ such that $ab = ba$ and $a^2 + b^2$ is invertible, a is not in X or b is not X. Then X is a maximal ideal of $\mathcal{A}$.*

Proof. Since $1 \notin X$, there exist a linear functional ϕ on $\mathcal{A}$ such that $\phi(1) = 1$ and $X = ker\phi = \{a \in \mathcal{A} : \phi(a) = 0\}$. Apply corollary 1.1.5 to ϕ as $a^2 + b^2$ is invertible this implies $\phi(a)^2 + \phi(b)^2 \neq 0$, Hence $a \notin X$ or $b \notin X$. Thus ϕ is multiplicative. Hence $X = ker\phi$ is maximal ideal. □

Theorem 2.1.7. *Let $\mathcal{A}$ be a complex Banach algebra with unit 1 and ϕ a non zero linear functional on $\mathcal{A}$. Then the following condition are equivalent.*

1. *$\phi(ab) = \phi(a)\phi(b)$ for all a, b in $\mathcal{A}$.*
2. *$\phi(a)$ lies in $Sp(a)$ for all a in $\mathcal{A}$.*
3. *$\phi(1) = 1$ and $\phi(a) \neq 0$ if a is invertible.*

Proof. (1) $\implies$ (2) $\implies$ (3) are straight forward. The only non trivial implication is (3) $\implies$ (1).

Let a, b belongs to $\mathcal{A}$ such that $ab = ba$ and $a^2 + b^2$ is invertible. As $a^2 + b^2 = (a + ib)(a - ib)$, so both $a + ib$ *and* $a - ib$ are invertible. Hence $\phi(a + ib) = \phi(a) + i\phi(b) \neq 0$ as well as $\phi(a - ib) = \phi(a) - i\phi(b) \neq 0$ and $\phi(a^2)^2 + \phi(b^2)^2 = [\phi(a) + i\phi(b)][\phi(a) - i\phi(b)] \neq 0$. Thus, ϕ satisfy the conditions (3) of theorem (1.1.3). Hence ϕ is multiplicative. □

Theorem 2.1.8. *Let $\mathcal{A}$ and $\mathcal{B}$ two real Banach algebras with unit 1. Suppose that $\mathcal{B}$ is commutative semi-simple. Let $T : \mathcal{A} \to \mathcal{B}$ be a linear map such that $T(1) = 1$. Then the following are equivalent.*

1. *$T(ab) = T(a)T(b)$ for all a, b in $\mathcal{A}$.*
2. *$Sp(T(a)^2 + T(b)^2) \subset (Sp(a^2 + b^2))$ for all a, b in $\mathcal{A}$ such that $ab = ba$.*
3. *$T(a)^2 + T(b)^2$ is invertible for all a, b in $\mathcal{A}$ such that $ab = ba$ and $a^2 + b^2$ is invertible.*

Proof. (1) implies (2): Let $c = a^2 + b^2$. Suppose $s + it$ is not in $Sp(c)$. Then $(c - s)^2 + t^2$ is invertible, that is, $[(c - s)^2 + t^2]d = 1$ for some d in $\mathcal{A}$. Therefore, $1 = T(1) = T[(c - s)^2 + t^2]T(d) = [(T(c) - s)^2 + t^2]T(d)$. Thus, $(T(c) - s)^2 + t^2$, hence $s + it$ does not belongs to $Sp(T(c)) = Sp(T(a^2 + b^2)) = Sp(T(a)^2 + T(b)^2)$. Hence $Sp(T(a)^2 + T(b)^2) \subset (Sp(a^2 + b^2))$.

(2) implies (3) Let a, b belongs to $\mathcal{A}$ such that $ab = ba$ and $a^2 + b^2$ is invertible. Then 0 is not $Sp(a^2 + b^2)$ and hence not in $Sp(T(a)^2 + T(b)^2)$. In other words, $T(a)^2 + T(b)^2$ is invertible.

(3) implies (1): The Carrier space of $\mathcal{B}$, $Car(\mathcal{B})$ is the space of all non-zero real linear homomorphism of $\mathcal{B}$ to the complex plane $\mathbb{C}$. Let ϕ be any element of $Car\mathcal{B}$. Define $\psi : \mathcal{A} \to \mathbb{C}$ as $\psi = \phi \circ T$. Then $\psi(1) = \phi \circ T(1) = \phi(1) = 1$.

Also, if a, b belongs to $\mathcal{A}$ such that $ab = ba$ and $a^2 + b^2$ is invertible, then $\phi(T(a)^2 + T(b)^2)$ is non-zero, that is, $\psi(a)^2 + \psi(b)^2 \neq 0$, by Theorem 1.1.4. $\psi(ab) = \psi(a)\psi(b)$ for all $a, b \in \mathcal{A}$. That is,

$$\begin{aligned}\phi(T(ab)) &= \phi(T(a))\phi(T(b)) \\ &= \phi(T(a)T(b) \ for\ all\ a, b\ in\ \mathcal{A}.\end{aligned}$$

As ϕ arbitrary element of $Car\mathcal{B}$ and $\mathcal{B}$ is semi simple we have $T(ab) = T(a)T(b)$ for all a, b in $\mathcal{A}$. This completes the proof. □

Remark 2.1.9. *Note that the assumption, $\mathcal{B}$ is commutative and semi simple is used only in proving the assertion (3) implies (1) in Theorem 2.1.8. The other two assertions, namely (1) implies (2) and (2) implies (3) are valid for any for real Banach algebra $\mathcal{B}$ with 1. However we cannot altogether drop the assumptions that $\mathcal{B}$ is commutative and semi simple. The following examples illustrate this.*

Example 2.1.10. *Let $\mathcal{A} = \mathcal{B} = B(\mathcal{H})$, where $B(\mathcal{H})$ is algebra of bounded linear operators on real Hilbert space $\mathcal{H}$. (In particulars we may consider $\mathcal{A}$ and $\mathcal{B}$ as the algebra of 3×3 real matrices). Let $T : \mathcal{A} \to \mathcal{A}$ be defined as $T(a) = a^*$, where a^* The adjoint of a in $\mathcal{A}$. Then $T(1) = 1$. Although we can find $a, b \in \mathcal{A}$ such that $ab = ba$ and $a^2 + b^2$ is invertible, but T is not multiplicative.*
Thus the assumption of commutativity cannot be dropped.

Example 2.1.11. *Let $L^1[0,1]$ be the space of all absolutely integrable complex functions on the interval $[0,1]$ with the norm given by*

$$\|f\| = \int_0^1 |f(t)|dt \quad \forall f \in L^1[0,1], \tag{2.1}$$

and product defined as convolution

$$f * g(x) = \int_0^1 f(x-t)g(t)dt, \quad 0 \leq x \leq 1 \quad \forall f, g \in L^1[0,1]$$

This is a Banach algebra without unit. Let $\mathcal{A}$ be the algebra obtained by formally adjoining a unit e to $L^1[0,1]$. Then it is not simple. Also every $a \in \mathcal{A}$ can be written, uniquely as, $a = f + s.e$ where f in $L^1[0,1]$ and s is in $\mathbb{R}$. Moreover, where a is invertible, if and only if, $s \neq 0$. Define $T : \mathcal{A} \to \mathcal{A}$ as $T(a) = -f + s.e$ where $a = f + s.e$ in $\mathcal{A}$. Then $T(e) = e$. Let $a = f + s.e,\ b = g + t.e \in \mathcal{A}$ such that $a^2 + b^2$ is invertible. Now, $a^2 + b^2 = f^2 + g^2 + 2(sf + tg) + (s^2 + t^2)e$. If $s^2 + t^2 \neq 0$ then

$$\begin{aligned}T(a)^2 + T(b)^2 &= (-f + s.e)^2 + (-g + t.e)^2 \\ &= (f^2 + g^2) - 2(sf + tg) + (s^2 + t^2)e.\end{aligned}$$

is invertible. This shows that the assumption of semi simplicity cannot be dropped.

Chapter 3

Real Function Algebras

3.1 Notation

Let X be a compact Hausdorff space. For a nonempty subset K of X and f in $C(X)$, we denote $\|f\|_K = sup\{|f(x)| : x \in K\}$.

Definition 3.1.1. *Let X be compact Hausdorff space and $\mathcal{A}$ be a non empty subset of $C(X)$. For each $x \in X$, the evaluation map at x, denoted by e_x is defined by*

$$e_x(f) = f(x)\ \forall f \in \mathcal{A}.$$

It is easy to seen that if $\mathcal{A}$ is a subspace, then $e_x : \mathcal{A} \to \mathbb{C}$ is a linear map and if $\mathcal{A}$ is a subalgebra, then e_x is a homomorphism. If $\mathcal{A}$ contains 1, then $e_x(1) = 1$ and $e_x \neq 0$. Next theorem shows that all the non zero homomorphism from $C(X)$ to $\mathbb{C}$ are of this form.

Let $\mathbf{F}$ be a collection of sets. Then $\mathbf{F}$ is said to have the finite intersection property if for every finite collection of sets $\{F_1, F_2, F_3, ...F_n\} \subset \mathbf{F}$ we have that that $\cap_1^n F_n \neq \phi$.

Theorem 3.1.2. *Let X be a topological space. Then X is compact if and only if for every collection of closed sets $\mathbf{F}$ in X, that is if $\mathbf{F}$ has a finite intersection property then $\cap_{F\in\mathbf{F}} F \neq \phi$.*

Theorem 3.1.3. *Let X be compact Hausdorff space. Then $Car(C(X)) = \{e_x : x \in X\}$.*

Proof. We know that if $\mathcal{A}$ is a complex commutative Banach algebra with unit 1 and ϕ, $\psi \in Car(\mathcal{A})$, with $ker\phi = ker\psi$, then $\phi = \psi$.

We have observed that $e_x \in Car(\mathcal{A})$ for each $x \in \mathcal{A}$. Let $\phi \in Car(C(X))$. It is enough to prove that $ker\phi = kere_x$ for some $x \in X$. Let $Z(f) = \{y : f(y) = 0\}$, for $f \in C(X)$.

We claim that $K = \cap\{Z(f) : ker\phi\}$ is non empty. Since each $Z(f)$ is closed subset, by the finite intersection property of compact space X, it is enough to prove that for $f_1, f_2, f_3 ... f_n \in ker\phi$, $Z(f_1) \cap Z(f_2) \cap Z(f_3) ... \cap Z(f_n)$ is non empty.

If possible, assume that $Z(f_1) \cap Z(f_2) \cap Z(f_3) ... \cap Z(f_n)$ is empty. Then $g = \sum_{i=1}^{n} f_i \bar{f}_i \in ker\phi$. Since f_i are no common zero, $g(y) > 0$ for all $y \in X$. Hence g is invertible, So that $\phi(g) \neq 0$ this is contradiction to $(g \in ker\phi)$ this proves the claim.

Now $I = \{f \in C(X) : f \equiv 0 \ on \ K\}$ is proper Ideal of $C(X)$. By above claim, $ker\phi \subset I$. Since $ker\phi$ is maximal Ideal, we get $ker\phi = I$ and K is singleton, say $K = \{x\}$. Hence $ker\phi = kere_x$ and as a consequence $\phi = e_x$ for some x in X. □

Corollary 3.1.4. *Let $(C(X))_{\mathbb{R}}$ denote $C(X)$, as a real algebra. Then $Car(C(X))_{\mathbb{R}}$ $=\{e_x : x \in X\} \cup \{\bar{e}_x : x \in X\}$. This proof follows from the following result:*
Let $\mathcal{A}$ be a real commutative Banach algebra with unit 1 and $\phi, \psi \in Car(\mathcal{A})$, if $ker\phi = ker\psi$, then $\psi = \phi$ or $\psi = \bar{\phi}$.

3.2 Involutions

Definition 3.2.1. *(Topological Involutions) Let X be a topological space. A map $\tau : X \to X$ is called a topological involution on X, if τ is a homeomorphism and $\tau(\tau(x)) = x$ for all $x \in X$.*

Definition 3.2.2. *(Linear Involution) Let A be a real or complex vector space. A linear involution on A is a map $a \to a^*$ from A into A satisfying the following axioms: For $a, b \in A$ and a scalar α*

1. $(a + b)^* = a^* + b^*$.
2. $(\alpha a)^* = \bar{\alpha} a^*$.
3. $(a^*)^* = a$.

The scalar α can be real or complex number depending on whether A is a real or complex vector space. Note that if A is a complex vector space, then a linear involution on A is not a linear map, but is conjugate linear.

Definition 3.2.3. *(Algebra Involution) Let $\mathcal{A}$ be an algebra over $\mathbb{R}$ or $\mathbb{C}$. An algebra involution is a linear involution on $\mathcal{A}$ satisfying $(ab)^* = (a)^*(b)^*$ for $a, b \in \mathcal{A}$.*

Let X be a compact Hausdorff space and τ a topological involution on X. Then the algebra $C(X, \tau)$ defined as

$$C(X, \tau) = \{f \in C(X) : f(\tau(x)) = \bar{f}(x) \ for \ all \ a \in X\}.$$

Theorem 3.2.4. *Let X be compact Hausdorff space and τ be a topological involution on X. Define $\sigma : C(X) \to C(X)$ by*

$$\sigma(f)(x) = \bar{f}(\tau(x))$$

for $f \in C(X)$, $x \in X$. Then

1. *σ is an algebra involution on $C(X)$ and*
$$C(X,\tau) = \{f \in C(X) : \sigma(f) = f\}.$$
2. *$C(X) = C(X,\sigma) \oplus iC(X,\tau)$.*
3. *σ is isometry.*
4. *For f in $C(X)$, define $P(f) = [f + \sigma(f)]/2]$. Then P is continuous linear projection.*
5. *Every algebra involution on $C(X)$ arise from a topological involution on X in the manner described above.*

Proof. (1) It is obvious.
(2) Since σ is an algebra involution, for $f \in C(X), \sigma(f+\sigma(f)) = f+\sigma(f)$ *and* $\sigma((f-\sigma(f))/i) = (f-\sigma(f))/i$.
Thus f = $f+\sigma(f)/2$ + i $(f-\sigma(f))/2i$ in $C(X,\tau)$. Further, if h = f+ig with f and g in $C(X)$ then $\sigma(h) = f - ig$. Hence $f = (h+\sigma(h))/2i$, $g = (h-\sigma(h))/2i$. This prove the uniqueness of f and g.
(3) Let $f \in C(X)$. Then
$$\begin{aligned}\|\sigma(f)\| &= \sup\{|\bar{f}(\tau(x))| : x \in X\}\\ &= \{|f(y)| : y \in X\}\\ &= \|f\|.\end{aligned}$$

The statement (4) follows from (1) and (3).
(5) Let σ be an algebra involution on $C(X)$. We prove that σ is induced by a topological involution τ on X. For $x \in X$. define $\phi_x : C(X) \to C$ by $\phi_x(f) = \sigma(f)(x)$, $f \in C(X)$. ϕ_x is Homomorphism on $C(X)$. Hence $\phi_x \in Car(C(X))$. Hence $\phi_x = e_y$ for a unique y in X. We define $y = \tau(x)$. Thus we have $f(\tau(x)) = \overline{\sigma f(x)}$ for all $f \in C(X), x \in X$. Replacing x by $\tau(x)$, $f(\tau(\tau(x))) = \overline{\sigma(f)}(\tau(x)) = \sigma(\overline{\overline{\sigma(f))(x)}}) = f(x)$ for all $f \in C(X)$. Hence $\tau(\tau(x)) = x$ for all x in X. Thus it remains only show that τ is continuous. First note that since $\overline{\sigma(f)}(x) = f(\tau(x))$ for f in $C(X)$, x in X, and $\tau \circ \tau$ is identity map, we have $\|\sigma(f)\| = \|f\|$. Hence σ is continuous. Net suppose that a net x_α convergence to the point in X. Then for every $f \in C(X)$, $f(x_\alpha) \to f(x)$. *Since* σ is continuous. This implies that $\sigma(f)(x_\alpha) \to \sigma(f)(x)$. Hence $f(\tau(x_\alpha)) = \overline{\sigma(f)}(x_\alpha) \to \overline{\sigma(f)}(x) = f(\tau(x))$. This means that $\tau(x_\alpha) \to \tau(x)$; that τ is continuous on X.
Hence τ is one- one onto, continuous, its inverse also continuous hence τ is homeomorphism. □

Definition 3.2.5. *Let A be a set of function on a set X. We say that A separates the points of X if for every every $x, y \in X$ and $x \neq y$, there exist $f \in A$ such that $f(x) \neq f(y)$.*

Let τ be n involution on X and E be set of functions on X such that $u \circ \tau = u$ for every $u \in E$. We say that E separates the points of X/τ, if for every $x, y \in X$ and $x \neq y$, $\tau(x) \neq y$, there exist $u \in E$ such that $u(x) \neq u(y)$.

Lemma 3.2.6. *Let X be a compact Hausdorff space and τ a topological involution on X. Let $x, y \in X$ and $x \neq y$.*

1. *If $y = \tau(x)$, then there exist f in $C(X, \tau)$ such that $f(x) = i$ and $f(y) = -i$.*

2. *If $y \neq \tau(x)$, then there exist f in $C(X, \tau)$ such that $f(x) = 1$ and $f(y) = 0$.*

Proof. We know that if $h \in C(X)$, define $\sigma(h) = \bar{h}(\tau(x))$. Then, by the theorem 2.0.11, $\sigma(h + \sigma(h)) = h + \sigma(h)$ and $\sigma(h\sigma(h)) = h\sigma(h)$. Hence $h + \sigma h$, and $h\sigma(h) \in C(X)$.
(1) We can find h in $C(X)$ such that $h(x) = i$ and $h(y) = 0$. It is possible because of Urysohn's lemma. Let $f = h + \sigma(h)$. Then $f \in C(X, \tau)$, $f(x) = h(x)+\sigma(h)(x) = h(x)+h(\tau(x)) = h(x)+\bar{h}(y) = i$, and $f(y) = h(y)+h(\tau(y)) = h(y) + \bar{h}(x) = -i$
(2) We can find h in $C(X)$ such that $h(x) = i$ and $h(y) = 0$. Let f $= h\sigma(h)$. Then $f \in C(X, \tau)$ f(x)$= h(x)\sigma(h)(x) = 1$, and $f(y) = h(y)\sigma(h)(y) = 0$. □

Definition 3.2.7. *Let X be a compact Hausdorff space and τ is topological involution on X. A real functional algebra on (X, τ) is a real sub algebra $\mathcal{A}$ of $C(X, \tau)$ such that :*

1. *$\mathcal{A}$ is uniformly closed.*

2. $1 \in \mathcal{A}$.

3. *$\mathcal{A}$ separates the point of X.*

A complex functional algebra $\mathcal{A}$ on compact Hausdorff space X is a complex sub algebra of $C(X)$ satisfying following conditions

1. *$\mathcal{A}$ is uniformly closed.*

2. $1 \in \mathcal{A}$.

3. *$\mathcal{A}$ separates the point of X.*

Lemma 3.2.8. *Let X be a compact Hausdorff space and τ is topological involution on X. Let $\mathcal{A}$ be a sub algebra of $C(X, \tau)$ that separates the points of X and contains 1. Let $x, y \in X$ with $x \neq y$. Then:*

1. *If $y = \tau(x)$, then there exist f in $\mathcal{A}$ such that $f(x) = i$ and $f(y) = -i$.*

2. *If $y \neq \tau(x)$, then there exists f in $\mathcal{A}$ such that $f(x) = 1$ and $f(y) = 0$.*

Proof. We know that $\mathcal{A}$ separates points of X, so there exist f_1 in $\mathcal{A}$ such that $\alpha_1 = f_1(x) \neq f_1(y) = \beta_1$.
(1) Let $y = \tau(x)$ and $\alpha_1 = a + ib$. Then $\beta_1 = f_1(y) = f_1(\tau(x)) = \overline{f_1(x)} = a - ib$. Hence $\beta_1 = a - ib$, So that $b \neq 0$ Now $f = (f_1 - a)/b \in A$, $f(x) = i$ and $f(y) = -i$.

(2) Let $y = \tau(x)$. Then there is f_2 in $\mathcal{A}$ such that $\alpha_2 = f_2(\tau(x)) \neq f_2(y) = \beta_2$. Consider

$$f = \Big[1 - \frac{(f_1 - \alpha_1)(f_2 - \alpha_2)}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)}\Big]\Big[1 - \frac{(f_1 - \bar{\alpha}_1)(f_2 - \bar{\alpha}_2)}{(\bar{\beta}_1 - \bar{\alpha}_1)(\bar{\beta}_2 - \bar{\alpha}_2)}\Big]$$

for this function $f(x) = 1$ and $f(y) = 0$. □

Example 3.2.9. *The disk algebra $A(\mathcal{D})$ is set of holomorphic functions $f : \mathcal{D} \to \mathbb{C}$ where $\mathcal{D}$ is open unit disk in complex plane $\mathbb{C}$, f is extended continuous function on closer of $\mathcal{D}$ That is*

$$A(\mathcal{D}) = H^{\infty}(\mathcal{D}) \cap C(\bar{\mathcal{D}}),$$

where H^{∞} is the set of all bounded analytic functions on unit disk $\mathcal{D}$.

$$\|f\| = sup\{|f(z)| : z \in \mathcal{D}\} = max\{|f(z)| : z \in \bar{\mathcal{D}}\}$$

Example 3.2.10. *Let X is a compact subset of $\mathbb{C}$ and $\mathcal{B}$ is a algebra of all functions that are uniform limits of sequences of rational numbers p/q, where p and q are polynomials and q has no zero on X. Then $\mathcal{B}$ is complex functional algebra. Further, if X is symmetric about real axis, we define $\tau : X \to X$ by $\tau(z) = \bar{z},\ z \in X$. Let*

$$A = \{f \in B : f(\tau(z)) = \bar{f}(z)\ for\ all\ z\ in\ X\}.$$

Then $\mathcal{A}$ is real function algebra consisting of all uniform limits of sequences of rational functions p/q, where p and q are polynomials with real coefficients and q does not have any zero on X.

Example 3.2.11. *Let $0 < c < d$ and X be a annular region $X = \{z \in \mathbb{C} : c \leq |z| \leq d\}$. Let*

$$\mathcal{B} = \{f \in C(X) : f\ is\ analytic\ in\ interior\ of\ X\ \}.$$

Then $\mathcal{B}$ is complex function algebra on X. Let $0 < c < 1$ and $d = \frac{1}{c}$. Then define a map $\tau(z) = \frac{1}{\bar{z}}$ map is topological involution on X. Define

$$\mathcal{A} = \{f \in B : f(\tau(z)) = \bar{f}(z)\ for\ all\ z \in X\}.$$

Then $\mathcal{A}$ is real function algebra on (X, τ).

Definition 3.2.12. *A real uniform algebra is a real commutative Banach algebra $\mathcal{A}$ with unit 1 such that $\|a\|^2 = \|a^2\|$ for all $a \in\ \mathcal{A}$.*

Example 3.2.13. *Let $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$ be a open unit disk and $\mathcal{A}$ be a set of all complex valued bounded analytic functions f on $\mathcal{D}$ satisfying $f(\bar{z}) = \bar{f}(z)$ for all $z \in \mathcal{D}$ with supremum norm, $\mathcal{A}$ is obviously a real uniform algebra.*

Example 3.2.14. *Let μ be a positive measure function on a σ algebra on a measure space X, and $\tau : X \to X$ be a map such that $\tau(\tau(x)) = x$ for all x in X, $\tau(E)$ is measurable when ever E is measurable subset of X. Let*

$$\mathcal{A} = \{f \in L^\infty : f(\tau(x)) = \bar{f}(x) \text{ for all } x \in \ X\}.$$

Then $\mathcal{A}$ is clearly real uniform algebra.

Theorem 3.2.15. *Every uniform algebra isometrically isomorphic to a real function algebra.*

Proof. Let $\mathcal{A}$ be a real uniform algebra, $X = Car(\mathcal{A})$ and $\tau : X \to X$ be a map define $\tau(\phi) = \bar{\phi}$. We know that $Car(\mathcal{A})$ is compact Hausdorff space and τ is a topological involution on X. Let $\tilde{\mathcal{A}} = \{\tilde{a} : a \in \mathcal{A}\}$ where $\tilde{a}$ denote Gelfand transform of a. We know that Gelfand mapping is Isometric isomorphism from $\mathcal{A}$ onto $\tilde{\mathcal{A}}$ Thus it suffices to prove that $\tilde{\mathcal{A}}$ is real functional algebra on (X, τ). For each $a \in \mathcal{A}$ and ϕ in X, $\tilde{a}(\tau\phi) = \bar{\phi}(a) = \overline{\tilde{a}(\phi)}$. Hence $\tilde{a} \in C(X, \tau)$. and $\tilde{\mathcal{A}}$ is real sub algebra of $C(X, \tau)$. It contains constant function 1 and separates points of X. Since $\|\tilde{a}\| = \|a\|$ for each $a \in \mathcal{A}$, thus $\tilde{\mathcal{A}}$ is uniformly closed. $\square$

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