WORKING PAPERS IN ECONOMICS

DEPARTMENT OF ECONOMICS
UNIVERSITY OF ZIMBABWE

OPTION PRICING MODEL WITH TIME-VARYING VOLATILITY

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October 1991

Abstract

The paper extends the option pricing model of Merton (1973) with time-varying volatility of the underlying security. We develop the theoretical option model. Time-varying volatility is constructed by fitting a time-polynomial to implied volatility values where the order of the polynomial is approximated by the number of options considered. We then predict the option price one day forward and compare the results with the standard Black and Scholes model. When applied to FT-SE 100 index European options the new model was found to be more accurate than the Black and Scholes.

Key words: Option, Time-Varying, Volatility, Black and Scholes.

JEL Classification: G13

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I am grateful to S.E.Satchell for his comments and suggestions on this paper. I am also grateful to all members of the Cambridge Finance Workshop for the stimulating discussions on the topic. All errors are mine.

1. INTRODUCTION

The only unknown parameter in pricing options using the Black and Scholes (1973) model is the variance of the underlying security. As Merton (1980) observed, for the efficient estimation and management of risky investment in the market, we require an efficient estimation method for market volatility. An accurate estimate would render hedging strategies more effective in dealing with risky investments. But the estimated variance of the stock returns has been found to change over time, a phenomenon that is discussed widely in the literature.[1] Black and Scholes discovered the seemingly non-stationary characteristics of the variance when they were testing their model. Again, the literature on volatility estimation is quite wide spread.[2]

Cognisant of the empirical evidence demonstrating the stochastic character of volatility, it has become apparent that a more general option pricing model than the Black and Scholes model applies in reality. Such a broader model would have to rid itself of the assumption of constant volatility employed in the standard Black and Scholes model. Subsequently, option pricing models have been developed on the basis that both the stock price and its variance are driven by Geometric Brownian motions and other processes. These stochastic volatility models were put forward by Merton (1973), Cox (1975), Cox and Ross (1976), Merton (1976a), Rubistein (1983), Jones (1984), Ball and Torous (1985), Butler and Schachter (1986), Hull and White (1987), Johnson and Shanno (1987), Scott (1987), Wiggins (1987), and Madan and Seneta (1990). However, the models of Merton (1976a), Jones (1984), and Ball and Torous (1985) are different from the rest even though they still take into account the fact that volatility changes over time. They present Diffusion-Jump models, where the stochastic differential equation of the stock price has two parts, a continuous Geometric Brownian motion, and a part comprising a discontinuous Poisson process that captures the discrete arrival of new information in the market. Cox and Ross (1976) present a Pure-jump process for the stock returns as opposed to a diffusion-jump process.

Madan and Seneta use a Variance Gamma model of random volatility, where the volatility is a product of a constant variance and a random variable that has a gamma distribution. Their approach is related to Praetz's (1972) approach where he considers the reciprocal of the random variable as having a gamma distribution that results in a t distribution. Rubistein's (1983) derives a more general model than the Black and Scholes model where he considers the stochastic process driving the stock price as arising from fundamental characteristics of the firm. The models includes debt as in the case of Geske (1979a)

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[1] See Rosenberg (1972), Merton (1980), French and Roll (1986), Porteba and Summers (1986), Taylor (1986), French, Schwert and Stambaugh (1987), Schwert (1989), Skinner (1989), Bailie and DeGenaro (1990) and Schwert and Seguin (1990).

[2] Parkinson (1976, 1980), Boyle and Ananthanarayanan (1977), Garman and Klass (1980), Beckers (1983), Ball and Torous (1984), Rogers and Satchell (1990).

and further decomposes the firms assets into risky and riskless assets. This results in stochastic volatility of the stock.

Hull and White, and Wiggins' models are special cases of Johnson and Shanno's model. In this model both the stock price returns and the variance follow Geometric Brownian motion and its variations, and thus can be characterised by two stochastic differential equations that are correlated. Cox's model is a constant elasticity of variance (CEV) model. Again, his model can be derived as a special case of that of Johnson and Shanno. Scott's model is slightly different. He assumes that the stock returns follow a Geometric Brownian process but the instantaneous variance of the stock price follows an Ornstein-Uhlenbeck process that is mean-reverting. Butler and Schachter (1986) developed an unbiased Black and Scholes model by taking a Taylor series expansion of the formula and the density of the estimated variance. Their model also performs better than the standard Black and Scholes model.

The main aim of this presentation is to propose a new time-varying volatility option pricing model where the volatility of stock returns is a deterministic function of time-to-expiration of the option, and that functional form is known or the method of determining it is known. The motivation for this model comes from Merton's (1973) model. Taylor (1986) also argues for time-varying volatility. Time-varying volatility can be estimated from the volatility implied by the option at any time before the option expires. The implied volatility is quite important in the literature because it is perceived to be an information preserving estimator of volatility. We propose that the time-varying volatility can thus be characterised and approximated by a time polynomial with a known functional form, whose order is determined by, and may be equal to, the number of options written on the underlying stock price. We emphasize the use of several options written on the security in order to make use of all the information about the security. Hence the new estimator is both time-varying in character and is based on market information making it information-preserving. In this analysis we use data on the FT-SE 100 Index European options traded in the London Traded Options Market which is part of the London International Stock Exchange.

In the next section we develop the time-varying volatility option pricing model using the continuous hedging method. Section 3 will examine the time-varying volatility more closely, bringing out how the coefficients of the volatility polynomial are estimated. The empirical tests on the new model are discussed in section 4 and a comparison is made with the standard Black and Scholes model that uses the historical variance of logarithmic returns. Section 5 contains the results and a discussion on them. The conclusion is in section 6 and the appendices contain mathematical derivations and tables of results.

2. OPTION PRICING MODEL WITH TIME-VARYING VOLATILITY

On the onset let us state the assumptions under which our model obtains, which are:

i. no transactions costs and taxes,

ii. no penalties for short sales,

iii. the market operates continuously,

iv. the riskless rate of interest is constant

v. the stock pays no dividends during the life of the option, and

vi. the option can only be exercised on the expiration date.

The model will basically be for pricing European type options which can only be exercised on the date of expiration (assumption vi) and the underlying stock is dividend protected. As in the standard Black and Scholes model only five variables determine the price of the option namely the underlying stock price (S), exercise price (K), time-to-expiration (τ), stock volatility (V), and the riskless rate of interest (r). The only unknown parameter is the volatility (V), otherwise the rest of the parameters are known.

We perceive investors as economic agents who revise their perception of volatility and risk in the market all the time. The revision is determined by the time-to-expiration of the option. The stochastic volatility option models described above are consistent with a random revision of volatility, while our proposition hinges on a deterministic revision of volatility and perception of market risk. The motivation for this model is Merton's approach, which although does not provide a functional expression for volatility, it expresses volatility as a "time variable"(Merton, 1973, p.166).

In conformity with this approach, we define volatility as a "time variable" of the form

$$V(\tau) = \int_0^{\tau} \sigma^2(t)\, dt, \qquad (1)$$

where t is time, τ is the time-to-expiration of the option, and $\sigma^2(t)$ is the instantaneous variance of the stock returns at any time t. Expression (1) implies that we obtain the volatility of the stock returns, at any time, by integrating the instantaneous volatility with respect to time, and value the result over the remaining life of the option written on the stock. This bestows the time-varying characteristic on the volatility estimator, determined by the remaining life of the derivative security, the option.

It is standard to assume that the stock price is generated by a Geometric Brownian motion, the corollary being that the stock returns follow the stochastic *differential equation*[3]

$$dS/S = \mu dt + \sigma dW(t), \qquad (2)$$

[3] For a description of Itô type stochastic differential equations refer to Schuss (1980). The Itô processes derive from the assumption of a continuous time stochastic process which leads to continuous price changes with independent increments.

where S is the stock price, μ the expected return on the stock price per unit time, σ the local standard deviation of returns, and dW(t) the derivative of a Gauss-Wiener process.

We now create a riskless hedge portfolio continuously, comprising a stock and a European call option, whose mixture is chosen in a particular way. The changes in the call price arise from changes in the stock price, and so are the changes in the hedge portfolio. Because changes in the hedge portfolio are only due to the stock price, the hedge portfolio is a self-financing portfolio. The quantities of the stock and the option should be continuously adjusted appropriately as the stock price changes, such that the rate of return on the hedge portfolio is riskless.

Denote the number of shares of stock with ω_s, and the number of options purchased as ω_c, and the call price as C. Then, the value of the hedge portfolio, H, can be expressed as the stock price times the number of shares plus the call price times the number of options purchased. That is[4],

$$H = S\omega_s + C\omega_c, \tag{3}$$

Next, we have to find the value of the change in the hedge portfolio. This we do by taking the total derivative of H in equation (3), yielding

$$dH = \omega_s dS + \omega_c dC, \tag{4}$$

Next we need an expression for dC. The call option price function can be expressed as $C(S,K,r,\tau,V(\tau))$. Since the stock price follows a Geometric Brownian motion culminating in stochastic differential equation (2), *Itô's Lemma* is employed to obtain the total differential equation of the call option price given by[5]

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[4] Merton's (1973) hedge portfolio also contains a riskless bond as third asset. One can include it explicitly as a separate asset such that the rate of return of the portfolio is zero or as in our case, exclude it but make sure that in equilibrium the rate of return of the hedge portfolio is equal to that of a riskless asset. Also see Smith (1976) for this approach.

[5] *Itô's Lemma* for a function f(x(t),t) is given by

$$df(x(t),t) = (\partial f(x,t)/\partial x)dx + (\partial f(x,t)/\partial t)dt + (1/2)\sigma^2(\partial^2 f(x,t)/\partial x^2)dt.$$

The formula corrects the classical chain rule

$$df(x(t),t) = (\partial f(x,t)/\partial x)dx + (\partial f(x,t)/\partial t)dt,$$

by the additional term

$$(1/2)\sigma^2(\partial^2 f(x,t)/\partial x^2)dt.$$

$$dC = (\partial C/\partial S)dS + (\partial C/\partial t)dt + (1/2)(\partial^2 C/\partial S^2)\sigma^2 S^2 dt. \quad (5)$$

All the terms in the expression for dC are deterministic except the stochastic term dS. Substituting for dC in equation (4) yields

$$dH = \omega_s dS + \omega_c((\partial C/\partial S)dS + (\partial C/\partial t)dt + (1/2)(\partial^2 C/\partial S^2)\sigma^2 S^2 dt). \quad (6)$$

To ensure that the return on the hedge portfolio is riskless, we choose the portfolio such that we have a long position on one share of stock and a short position of $(\partial C/\partial S)^{-1}$ calls.[6]
Then, the ratio of the number of shares per unit number of calls must equal the number of calls held short. Then, we have

$$\omega_s/\omega_c = -\partial C/\partial S. \quad (7)$$

We can also derive expression (7) from saying that the change in the value of the hedge in equation (4) should be equal to zero. That is,

$$\omega_s dS + \omega_c(\partial C/\partial S)dS = 0, \quad (8)$$

from which condition (7) follows directly.

Since the investor holds one share only, meaning that $\omega_s = 1$, the number of options held are, $\omega_c = -1/(\partial C/\partial S)$. Under these conditions the change in the value of the hedge in equation (6) becomes

$$dH = -(\partial C/\partial S)^{-1}((\partial C/\partial t) + (1/2)(\partial^2 C/\partial S^2)\sigma^2 S^2)dt. \quad (9)$$

In equilibrium the return on the hedge portfolio must equal that of the riskless asset, such that

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Notice that dx is defined by the stochastic differential equation

$$dx/x = \mu dt + \sigma dW(t)$$

where W is a Gauss-Weiner process. Refer to Schuss (1980) for a general discussion.

[6] This method is similar to that proposed by Smith (1976). Merton's (1973) approach on p.164-165 is rather long winded.

$$dH/H = rdt, \tag{10}$$

where r is the riskless rate and dH/H is the return on the hedge portfolio. Substituting equations (3) and (9) into equation (10) and rearranging we obtain the partial differential equation for the call price given by

$$(1/2)(\partial^2 C/\partial S^2)\sigma^2 S^2 + rS\partial C/\partial S + \partial C/\partial t - rC = 0, \tag{11}$$

subject to the boundary condition that the terminal value of the option price must be equal to the maximum of either the difference between the stock price and the exercise price or zero. That is,

$$(12) \quad C(S(\tau),K,r,0,V(\tau)) = \text{Max}(0, S(\tau) - K),$$

where $S(\tau)$ denotes the stock price at the expiration date.

To solve equation (11) subject to (12) we use the *Feynman-Kac Formula* as suggested by Duffie (1988a, 1988b). For completeness we shall state the formula and here we are guided by Oksendal (1985).

Theorem 1 (Feynman-Kac Formula): *Consider a function*

$$Y(x,t) = E_{x,t}[exp(-\int_0^T r(X_s)ds\ g(X_T)], \tag{13}$$

where E is an expectation operator, r(X) is bounded and continuous, and X solves the equation, $dX/X = \mu dt + \sigma dW$. *Then*

$$\partial Y/\partial t - LY + rY = 0, 0 \le t \le T, \tag{14}$$

where L is the operator

$$\mu\partial(.)/\partial x + (1/2)\sigma^2\partial(.)^2/\partial x^2. \tag{15}$$

Proof: For the proof see Oksendal (1985), pages 95-96.

The *Feynman-Kac Formula* is a generalisation of the *Kolmogorov backward equation* and is closely

related to *Dynkin's Formula* (see Oksendal (1985), pages 91-96).

In equation (13) the term $\int_0^\tau r(X_s)ds$ is the discount factor which in our case is $r\tau$ and r is a constant. Also, in our case X = S, the security price, and $g(X_\tau) = \max(0, (S(\tau) - K)$, the value of the call option at expiration. The variable $Y = C(S,K,r,\tau,V(\tau))$, the value of the option at any time before expiration. Therefore, the value of the call option, which solves equation (11) subject to (12) is given by

$$C(S,K,r,\tau,V(\tau)) = E\{\exp(-r\tau)\max(0, S(\tau) - K), V(\tau)\}, \tag{16}$$

where E is the risk-adjusted expectation.[7] Equation (16) gives the option price as a martingale with respect to some risk-adjusted probability. This is consistent with Harrison and Kreps' (1979) pricing of securities by the no-arbitrage argument and is also consistent with the efficient market hypothesis. To solve equation (16) we shall draw from the Cox and Ross (1976) risk-neutral approach, and the derivation of the solution is given in appendix 2A.B.1. The solution is the option model with time-varying volatility of the form[8]

$$C(S,K,r,\tau,V(\tau)) = S\Phi(d_1) - K\exp(-r\tau)\Phi(d_2), \tag{17}$$

where

$$d_1 = (\ln(S/K) + r\tau + (1/2)V(\tau))/\sqrt{V(\tau)},$$

$$d_2 = d_1 - \sqrt{V(\tau)},$$

$$V(\tau) = \int_0^\tau \sigma^2(t)dt,$$

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[7] This solution to the option price is in conformity with the Cox, Ingersoll, and Ross (1985b) general intertemporal equilibrium model of asset prices in Lemma 4. The solution is also similar to that of Merton (1973), equation (38), page 166.

[8] Merton (1973) employs the Fourier transformation approach to solve the model and he expresses it in terms of the error complement function. However, he solves the model for the case where $r = 0$, $\sigma^2 = 1$, and $K = 1$. Under normal conditions his model for the option price could be rewritten as, $C(S,K,r,\tau,V(\tau)) = S\text{erfc}(h_1) - K\exp(-r\tau)\text{erf}(h_2)$, where $h_1 = -(\ln(S/K) + r\tau + 0.5V(\tau))/\sqrt{V(\tau)}$, and $h_2 = -(\ln(S/K) + r\tau - 0.5V(\tau))/\sqrt{V(\tau)}$, and erfc(.) is the error complement function of the form, $\text{erfc}(h) = 1 - (2/\sqrt{\pi})\int_0^h \exp(-u^2)du$, and $V(\tau) = \int_0^\tau \sigma^2(t)\,dt$. (See Merton (1973), page 167). The model is similar to our time-varying volatility option pricing model.

and $\Phi(.)$ *is a standard normal distribution of the form* $\Phi(d_i) = \int_0^{d_i}(1/\sqrt{2\pi})\exp(-u^2/2)du$, *with mean zero and* unit variance. Assuming that $\sigma^2(t) = \sigma^2$ for all t, meaning that $V(\tau) = \sigma^2\tau$, we obtain the standard Black and Scholes model, which is in fact a special case of the new model (17).

Alternatively, the solution for the option pricing model could be derived using the *Cameron-Martin-Girsanov Theorem* (see Oksendal (1985), pages 115-119). This theorem involves an explicit transformation of diffusions with a drift (μ in equation (2)) to diffusions without a drift. Subsequently, we can transform the probability measures. The derivation of the option model using the *Cameron-Martin-Girsanov Theorem* is in appendix B.2.

The time-varying volatility option pricing model still has to be integrated over the functional form of the volatility, $V(\tau)$. Now, suppose there are N options written on the stock, each with a different expiry date but with the same strike price, and k strike prices, yielding Nxk options. Since the volatility changes with time-to-expiration, the implied volatility of the underlying stock, implied by each observed option price, is different for each of the Nxk options. We can thus estimate the average implied volatility for each time-to-expiration by summing the volatilities accross different strike prices and dividing by the number of strike prices. Then, the average implied volatility (σ_j^2) for each time-to-expiration is given by

$$\sigma_j^2 = (1/k)\sum_{i=1}^{k}\sigma_i^2, \tag{18}$$

where σ_i is the implied volatility for each time-to-expiration (τ) and each strike price (K), and j denotes each average implied volatility. Then, $V(\tau)$ can be thought of as the expression of average implied volatility, implied by different options with varying expiry dates.

Now, the question is what functional form does the polynomial $\sigma^2(t)$ take? We suggest that the volatility $\sigma^2(t)$, which here is thought of as an implied volatility, that can be approximated by an (N-1)th-order polynomial function of time (t). That is,

$$\sigma^2(t) = \alpha_1 + \alpha_2 t + \ldots\ldots + \alpha_{N-1}t^{N-1}, \quad \alpha_i \neq 0, \tag{19}$$

and more generally expressed as

$$\sigma^2(t) = \sum_{i=1}^{N-1}\alpha_i t^{i-1} \qquad \alpha_i \neq 0, \tag{20}$$

where t is time, α_i is the coefficient, and i denotes each option. To derive $V(\tau)$ we integrate the polynomial (20) with respect to time, t, and value the result over the remaining life of the option, τ. That is,

$$V(\tau) = \int_0^{\tau} \left(\sum_{i=1}^{N-1} \alpha_i t^{(i-1)} \right) dt, \quad \alpha_i \neq 0,$$

$$= \alpha_0 + \alpha_1 \tau + (\alpha_2/2)\tau^2 + (\alpha_3/3)\tau^3 + \ldots\ldots + (\alpha_N/N)\tau^N, \tag{21}$$

which in general can be expressed as

$$V(\tau) = \sum_{i=0}^{N} \beta_i \tau^i, \tag{22}$$

where $\beta_i = \alpha_i/i$, which gives us the functional form of the time-varying volatility estimator. In simple terms, one has to estimate the coefficients, α_i, and then for each option substitute its time-to-expiration in equation (22) and evaluate the polynomial. Having obtained the expression for the volatility the time-varying volatility option pricing model can be rewritten as,

$$C(S,K,r,\tau,V(\tau)) = S\Phi(d_1) - K\exp(-r\tau)\Phi(d_2), \tag{23}$$

where

$$d_1 = (\ln(S/K) + r\tau + (1/2)V(\tau))/\sqrt{V(\tau)},$$

$$d_2 = d_1 - \sqrt{V(\tau)},$$

$$V(\tau) = \sum_{i=0}^{N} \beta_i \tau^i,$$

and $\Phi(.)$ is a standard normal distribution defined as before in equation (17). In the polynomial $V(\tau)$ we shall assume that the constant, $\beta_0 = 0$. This is because at expiration, $\tau = 0$, and the implied volatility is *indeterminate*. Since $V(\tau)$ can be viewed as an area under the curve of the function, $\sigma^2(t)$, from 0 to point τ, the area at expiration should be zero. Hence the assumption that, $\beta_0 = 0$. Now that the model is completely solved we need to discuss how to *estimate the coefficients*, β_i, of the volatility polynomial, $V(\tau)$, and the next section is devoted to that.

3. ESTIMATING THE TIME-VARYING VOLATILITY

In order to calculate option prices using the time-varying volatility option model (23) we need to calculate the coefficients of the volatility polynomial, $V(\tau)$. If the data generating process of security prices is governed by a Geometric Brownian motion and investors price options according to the Black and Scholes model, we can obtain the volatility of the returns of the stock price, implied by the observed actual option price. In the literature it has been argued that the at-the-money options yield the most accurate implied volatility estimates, because they are perceived to contain accurate market information of each stock price.

We stated earlier that we want our volatility estimator to be both time-varying and information-preserving. In this respect we shall make use of the implied volatilities to estimate, $V(\tau)$. Jarrow and Wiggins (1989) have argued that the implied volatility is quite robust under conditions where the Black and Scholes assumptions have been violated. For instance, if stock prices follow an arbitrary process, Rudd and Jarrow (1982) using Edgeworth series expansion, demonstrated that a Black and Scholes model could still be obtained for certain classes of stock price distributions, but the volatility needs to be adjusted. The implicit volatility could be used as an estimate of this adjusted volatility.

When interest rates are stochastic rather than constant, estimating volatility becomes difficult, because one has to estimate the volatility of the stock, the correlation coefficient between the Gauss-Wiener processes of the stock price and interest rate, and the volatility of the interest rate.[9] But we could employ the implied volatility under this scenario to absorb the effects of the stochastic interest rate.

In the presence of market friction, which includes margin requirements, transactions costs, and tax payments, the implied volatility is demonstrated to be the most accurate estimator of volatility. Jarrow and Wiggins' argument successfully demonstrates the robustness of the implied volatility approach under different conditions of market behaviour, which seems to suggest that a more general model,than the Black and Scholes model applies in reality. In this respect we employ the implied volatility in estimating the parameters of $V(\tau)$, because of its accuracy, information content, and general robustness under different market conditions.

To estimate the implied volatilities of the stock price from the actual option prices, we shall employ the Newton-Raphson method.[10] Let us denote the observed market call price by C_m, and the Black and Scholes model value by $C(\sigma^2)$. In this method, the aim is to find the root of the equation,

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[9] Merton (1973) first obtained a stochastic interest rate option model where the interest rate is governed by geometric Brownian motion. Rabinovitch (1989) later derived a stochastic interest rate option model under conditions where the interest rate follows an Ornstein-Uhlenbeck process, as suggested by Vasicek (1977). Rabinovitch's option model is similar to Merton's, except that it contains an expression that explicitly reflects the effect of a mean-reverting term structure of interest rates.

[10] For a discussion of the Newton-Raphson method and other numerical optimisation procedures refer to Harvey (1990), pages 123-145.

$(C(\sigma^2) - C_m)$. Since $C(\sigma^2)$ is monotonic with respect to the volatility, σ, there is only one root. This method uses successive substitutions of σ^2, such that,

$$\sigma_*^2 = \hat{\sigma}^2 - [\partial C(\hat{\sigma}^2)/\partial(\hat{\sigma}^2)]/[\partial^2 C(\hat{\sigma}^2)/\partial(\hat{\sigma}^2)^2] \tag{24}$$

where (*) denotes the current estimate, (^) denotes the previous estimate, and $[\partial^2 C(\hat{\sigma}^2)/\partial(\hat{\sigma}^2)^2]$ is the Hessian. The value of the implied volatility is accepted if it yields a pricing error, C(.) - Cm, that is within the preset tolerance limit. The method requires that we choose an initial value, σ_0^2, for the iterative process. Manaster and Koehler (1982), have suggested that for faster convergence we should choose the initial absolute value, σ_0^2, in the following way,

$$\sigma_0^2 = 2(\,|\ln(S/K) + r\tau\,|)/\tau. \tag{25}$$

Notice that the expression (25) uses all the information contained in the other four known variables, namely, S, K, r, and τ, for choosing the initial value of σ^2. In the Black and Scholes model we need to estimate the standard normal distribution, $\Phi(d_1)$, by numerical integration since it cannot be evaluated directly.[11] A numerical approximation procedure of $\Phi(d_1)$ is discussed in Appendix 2AC.

Whaley (1982) suggested an iterative Ordinary Least Squares procedure for calculating the implied volatility of the underlying stock.[12] Day and Lewis (1988) use a generalized least squares (GLS) procedure in which actively traded options are weighted more heavily than thinly traded ones. MacBeth and Merville (1979) have suggested ways of estimating the at-the-money implied volatility from the implied volatilities by regressing the implied volatilities on the degree to which the option is in- or out-of-the money.[13] The at-the-money implied volatility is taken to be the constant term. But one would still

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[11] See Benninga (1989) for a discussion of numerical integration of a normal distribution.

[12] Whaley (1982) put forward an iterative OLS regression procedure where we minimise the sum of squares of the error in the econometric model, $C_m = C(\sigma) + \varepsilon$, where ε is the random error term with zero mean and constant variance, and σ is unknown. Through a Taylor series expansion and rearranging the terms we obtain, $C - C(\sigma) + \sigma_0\partial C/\partial\sigma = \sigma\partial C/\partial\sigma + \varepsilon$, where σ_0 is given. We apply OLS regression and check the value of σ_i against a designed acceptance tolerance limit, $(\sigma_i - \sigma_{i-1}) < z$, where for small $z > 0$, σ_i is the estimate of σ. We keep doing the OLS regression and obtaining the estimates, σ_i, until we get a value that is acceptable within the tolerance limit.

[13] MacBeth and Merville (1979) regress the implied standard deviation, ISD, on the degree to which the option is in- or out-of-the money, $M = (S\text{-}Kexp(-r\tau))/Kexp(-r\tau)$. They estimate the regression, $ISD_i = \alpha + \beta M_i + \varepsilon_i$, where ε is the random error term with zero mean and constant variance. When the option is at-the-money, M = 0. Under this condition, ISD = α, implying that α is the at-the-money implied volatility.

have to estimate the implied volatilities before doing such a refinement to obtain the at-the-money implied volatility.

Having discussed how to obtain the implied volatilities, we have to discuss how to obtain the coefficients of the polynomial for estimating time-varying volatility given above. As before, assume that the stock price has N options written on it differentiated by their times-to-expiration. At any time the options have the same strike price, K, the same stock price, S, and the riskless interest rate, r, is constant and is the same for all options. What differentiates one option from another is the time-to-expiration. The times-to-expiration of the option can be represented as, τ_1, τ_2, ..., τ_N, where $\tau_1 < \tau_2 < \tau_3 <$.....$< \tau_N$. The options values can thus be denoted by their times-to-expiration as, $C(\tau_1)$, $C(\tau_2)$, ..., $C(\tau_N)$, with their respective implied volatilities, $\sigma^2(\tau_1)$, $\sigma^2(\tau_2)$, ..., $\sigma^2(\tau_N)$. For any given intrinsic value, the value of an option decreases, as the option moves closer to maturity, a phenomenon known as time-value decay. This is true if the stock price upon which the option is written does not pay dividends. Therefore, $C(\tau_1) < C(\tau_2) <$ ..... $< C(\tau_N)$.

For each of the N instantaneous volatilities, there is some polynomial of time explaining it. In all we have N polynomial equations. The problem could be represented as a system of N polynomial equations,

$$
\begin{aligned}
V(\tau_1) &= \beta_1\tau_1 + \beta_2\tau_1^2 + \beta_3\tau_1^3 + \ldots\ldots + \beta_N\tau_1^N \\
V(\tau_2) &= \beta_1\tau_2 + \beta_2\tau_2^2 + \beta_3\tau_2^3 + \ldots\ldots + \beta_N\tau_2^N \\
&\;\;\vdots \\
V(\tau_N) &= \beta_1\tau_N + \beta_2\tau_N^2 + \beta_3\tau_N^3 + \ldots\ldots + \beta_N\tau_N^N
\end{aligned}
\qquad (26)
$$

where $V(\tau_i)$ are the implied volatilities. The system can be solved since there are N equations with N unknown coefficients. In matrix notation we can represent the problem as

$$\underset{\sim}{V}(\tau) = \tau\,\beta \qquad (27)$$

where $\underset{\sim}{V}(\tau)$ is an N x 1 vector of the average implied volatilities, τ is an N x N matrix of the variable, time-to-expiration, and β is the N x 1 vector of coefficients, β_i. To solve for the vector of coefficients, β, we invert equation (27) and obtain

$$\hat{\beta} = \tau^{-1} \underset{\sim}{V}(\tau) \qquad (28)$$

We substitute the known times-to-expiration in the matrix, τ. Then, the vector of coefficients, β, can be solved for, since the vector, $\underset{\sim}{V}(\tau)$, and inverse of the matrix, τ, are known.

This completes the estimation procedure for the time-varying volatility polynomial, $V(\tau)$. Note that in estimating the parameters for the volatility polynomial we have used all the options, with different strike prices, written on each stock price. This is to make use of all available market information about the underlying stock price and the coefficients should preserve this information. As we stated earlier, $V(\tau)$ is perceived to be closely related to $\sigma^2\tau$, the volatility of the stock when it follows a logarithmic random walk. Now, let us turn to the empirical analysis.

4. EMPIRICAL ANALYSIS

In this section, we shall discuss how we use our new option pricing model to value the FT-SE 100 index European call options. The results will be compared with those obtained from the standard Black and Scholes model that uses the historical variance of logarithmic returns. The accuracy of the models will be measured by percentage estimation errors, sum of squared errors and mean square error.

4.1 DATA BASE

Daily data was collected from the Financial Times and the London Traded Options Market (LTOM) of the London International Stock Exchange for the period 1 February 1990 to 31 March 1990. The FT-SE 100 index options are relatively new, having been introduced in the market on 1 February 1990. The options have the expiry cycle March - June - September - December, and the minimum period for trading is three months and the maximum is twelve months. For the year 1990 the expiry dates of the options are 29 March, 29 June, 28 September, and 31 December. The strike prices are set at 25 and/or 75 index point levels.

The normal size of each contract is £10 x index value, unlike equity options where the volume of each contract is 1000 shares, except for Vaal Reef shares with contracts of 100 shares each. Since the options are of the European type they can only be exercised on the expiry day, at 11.20 am, as contrasted to equity options which expire normally two days before the last day of dealings for the last complete Stock Exchange account of the expiry month. Also, the index does not pay dividends to shareholders during the life of the option. The American type index option can be exercised on any business day before the expiry day of the option.

The FT-SE 100 index was introduced on 3 January 1984 at a value of 1000, as an indicator of market movement. As the name suggests it is based on 100 leading British companies, which account for 70 percent of all equities in United Kingdom. Like the American Standard & Poor index, the 100 Index used for Chicago traded options, the FT-SE 100 index is a Weighted Arithmetic Index, weighted by the size of each company's equity in the stock market. Membership of the SE 100 is revised quarterly requiring the revision of the index as per membership revision. However, the FT 30 index, which was introduced in 1935 and based on 30 leading companies, provides an up-to-the-minute indication of the temper of the stock market. The FT 30 index is a geometric mean of each share price, divided by the price at the base date. It follows that the calculation is performed by adding the logarithms of the share prices, subtracting the logarithm of the base value, dividing by 30, and then taking the antilog of the result.

Regarding the riskless interest rate, we shall use the Treasury Bill rate which was 15 percent during the period defined in the sample. This rate also coincides with the one-year London interbank offer rate (Libor) of 15 percent, for the defined period.

In spite of the fact that daily data seems to have distributional problems, we shall have to use daily data. Fama (1965) first suggested that the distribution of daily stock returns seems to depart from normality more than monthly returns. Evidence reveals that the distribution of daily returns has fatter tails than the normal distribution, but the distribution converges to a normal distribution for cross-sectional daily stock returns.[14] But in this study the use of daily data is unavoidable since the time-to-expiration of an option is measured in days and the quotation of the option prices is per day. Now let us look at the empirical tests we can employ to measure the accuracy of our new model.

4.2 EMPIRICAL TESTS

To test the accuracy of the time-varying volatility option pricing model we shall compare it with the standard Black and Scholes model that uses the historical variance of the logarithmic index returns. The returns on the index are measured by the change in the natural logarithm of the index between successive periods. At any time t, the logarithmic return on the index is therefore given by

[14] For a discussion on this point, on the distributional anomalies of daily stock returns, see Hagerman (1978). Scholes and Williams (1977) suggest a further problem, that of non-synchronous trading, arising from the fact that the return on the market index and that of the security may be measured over different time intervals. This introduces a problem of errors of measurement in econometrics, producing biased and inconsistent OLS estimators. Brown and Warner (1985) mention a further problem, mainly that non-synchronicity in trading which may produce serially correlated daily stock returns. The variance of returns becomes non-stationary and it increases around events such as earnings announcement, as Patell and Wolfson (1979) argue. The rest of the literature on stock returns distribution and event studies is discussed in Part I (Introduction).

$$x_n = \ln(S_t/S_{t-1}), \tag{29}$$

where S is the index. Then, the historical estimate of the variance of the logarithmic returns on the index, x_n, is given by

$$\hat{\sigma}^2 = (1/T-1) \sum_{t=1}^{T} (x_n - \bar{x})^2, \tag{30}$$

where T is the sample size of the time series, and $\bar{x}$ is the mean of x_n. The estimator (30) of the variance is an unbiased estimator of the true variance σ^2.

We shall use the estimator (30) of the variance in the standard Black and Scholes model, which we shall compare with the time-varying volatility option model (23). Because the distribution of the estimated call prices from the two models is unknown, it is difficult to rely on parametric tests. The estimation errors that will be employed are discussed in the sections below.

4.2.1 PERCENTAGE ESTIMATION ERROR

This simple approach merely compares the percentage errors of estimation in both models. Define the percentage error of estimation of each option model by, $D_i = |(C(.) - C_m/C_m)|100\%$, where i stands for each option model, C_m is the actual option price, and C(.) is the estimated model option price. D_i could be either positive or negative. We merely compare the magnitude of the percentage errors, D_i, if we use both models to forecast the option price one day forward.

The presence of mispricing errors provides opportunities for making riskless arbitrage profit. In this respect one could consider the monetary value of the errors defined by, $|C(.) - C_m)|$. The monetary value of the error is $£10|C(.) - C_m|$. If the error, is positive then an arbitrage profit can be made by buying the stock at the strike of K index points, and selling it in the market at a higher price S. The profit you realise is £10C(.) which is larger than, $£10C_m$, the monetary value of the difference between S and K. The converse argument applies when the model underprices the option. The investor realises a lower profit, which is reduced by, $£10|C_m - C(.)|$. The investor could even make more sophisticated combination of calls and puts and realise more profit, on the same stock and/or accross different stocks.

4.2.2 SUM OF SQUARED ERRORS AND MEAN SQUARE ERROR

Apart from comparing the magnitude of errors we could also compare squares of the errors and the mean square errors. This approach disregards the sign of the error but considers the degree of

dispersion of the estimated value from the actual option value. The sum of squared errors is given by $\Sigma_i(C(.)_i - C_{mi})^2$. The mean square error is basically the mean of the squares of several estimation errors. Then the mean square error is the sum of squared errors divided by the number of options considered.

5. RESULTS AND DISCUSSION

In this section we shall discuss the results. We calculated the average implied volatilities of logarithmic returns on the index, for the following 14 consecutive trading dates: 16/02/90, 19/02/90, 20/02/90, 21/02/90, 22/02/90, 23/02/90, 26/02/90, 27/02/90, 28/02/90, 01/03/90, 02/03/90, 05/03/90, 06/03/90, and 07/03/90[15]. From 16/02/90 to 26/02/90, the eight strike prices across which the average implied volatility was calculated were, 2225, 2275, 2325, 2375, 2425, 2475, 2525, and 2625. But for the remaining seven days the strike prices were 2125, 2175, 2225, 2275, 2325, 2375, 2425, and 2475. For each stock price there are four expiry months, namely March, June, September, and December. For reasons of accuracy we only used the options for March, June, and September expiry months, because the December implied volatilities are less accurate. The Newton-Raphson method, based on the Black and Scholes model and discussed above, was used for estimating the implied volatilities. We also estimated the historical variance of index returns for each day from 01/11/89 to 07/03/90. Table A1 in appendix A, shows the implied volatilities and the historical standard deviations of the FT-SE 100 index, for the 14 trading days.

Next, we estimated the coefficients of the time-varying volatility polynomial, for the consecutive trading dates referred to above, using the method described in section 3. Since we considered only three expiry months, the polynomial is a cubic one, or third order. Therefore, only three coefficients namely, β_1, β_2, and β_3, were estimated. Table 1 below shows the estimated coefficients for each trading date.

<<<<Table 1 Here>>>>

From table 1 above, we notice that each coefficient of the volatility polynomial has a consistent sign, in the sense that $\hat{\beta}_1$ and $\hat{\beta}_2$ are positive while $\hat{\beta}_3$ is consistently negative. Having obtained the coefficients one can now proceed to estimate future volatility by substituting for time-to-expiration in the polynomial (19).

In the next stage of the analysis we wish to assess the accuracy of each volatility polynomial in estimating option prices one day forward for 14 consecutive trading days (fortnight). Next, we wanted

[15] The weekends, namely saturday and sunday, and holidays are excluded because the market is closed at these times.

to find out how accurate the time-varying volatility model is in predicting options that are in-the-money, near the money, and out-of-the-money. Options with strike prices 2225, 2325, and 2425, which are in-the-money, near-the-money, and out-of-the-money, respectively, were chosen. We therefore estimated option prices one day forward, under the three strike prices, for the three expiry months. We only need to calculate the option price one day forward because the following day we have a new security price and new information for another one day forecast. Option price estimates from each model were tested against the actual option price observed in the market to assess the accuracy of the time-varying volatility option model against the standard Black and Scholes model. Subsequently, the average percentage errors of estimation, sum of squared errors, and mean square errors, were calculated.

Table A2 appendix A shows the results for the options that are in-the-money with the strike price of 2225. The average percentage error of estimation is generally positive for both models, for the 14 dates, except for only one case for the option model with time-varying volatility. The implication is that both models tend to overprice the option prices, with the potential of allowing for the realisation of riskless arbitrage profits by investors. But taking a closer look at the percentage data again, we notice that the time-varying volatility option model, has lower percentage errors than Black-Scholes model in all 14 cases. The sums of squared errors and mean square errors also confirm these results. The sum of squared errors and the mean squared errors of the time-varying volatility model are lower than those of the standard Black and Scholes model in all the cases. This implies that the new model is more accurate for estimating option prices.

In table A3 in appendix A, we have the results for options that are near the money, with a strike price of 2325. In all but two cases, the Black and Scholes model shows positive average percentage errors. This implies that the model tends to overestimate the true option price. The picture of the time-varying volatility option model is somewhat different. Under the new model, we have seven cases of negative average percentage errors, implying that the remaining seven are positive. In is not clear which direction the new model misprices the option. Looking at the sum of squared errors and the mean squared errors we notice that those of the time-varying volatility option model are lower than those of the Black and Scholes model in all but one cases. The exceptional case is the prediction of 07/03/90. Again, the results reveal that the new model is more accurate than the standard Black and Scholes model for options that are near the money.

The case of the out-of-the-money options, with strike price 2425, is depicted in table A4 and in appendix A. For the Black and Scholes model, 9 out of the 14 cases show positive average percentage errors. The implication is that the model weakly overprices the option. The case of the time-varying volatility model is more consistent. This model shows negative average percentage errors in all but two cases for dates 19/02/90 and 28/02/90. We are therefore led to conclude that the new model tends to

underestimate out-of-the-money options. Looking at the sum of squared errors and mean squared errors, we notice that the time-varying volatility model has in ten cases errors that are lower than those of the standard Black and Scholes model. In this respect, we would conclude that the new model is more accurate than the standard Black and Scholes model for options that are out-of-the-money as well.

6. CONCLUSION

In the foregoing analysis we have developed an option pricing model with time-varying volatility that can be solved analytically. The motivation for this approach is from Merton's (1973) model. The new model is compared with the standard Black and Scholes model, in pricing the FT-SE 100 index call options of the European type which were introduced in the London Traded Options Market on 1 February 1990. The time-varying volatility option model has on the overall proved to be more accurate than the standard Black and Scholes model.

The results seem to support the assertion that investors may revise their perception of risk, measured by the volatility of stock returns, as the options written on the index move closer to maturity. This renders the volatility dependent on the time-to-expiration, and its functional form could be approximated by a polynomial whose order is determined by the number of options written on the index. The results also put pointers to the fact that a more general model than the standard Black and Scholes model can be applied. The results, although applied on a limited sample, support this assertion.

This approach could be extended to the pricing of other types of options such as individual stock options, currency options, and commodity options, where again we could assume time-varying volatility. However, this is a subject for future research.

Table 1.

TIME-VARYING VOLATILITY POLYNOMIAL ($V(\tau)$)

No. of Trading Days = 14

DATE	β_1	β_2	β_3
16/02/90	2.045048	- 7.939370	7.814084
19/02/90	1.734859	- 6.384520	6.314906
20/02/90	2.086579	- 7.778200	7.540097
21/02/90	2.254998	- 8.332740	7.957987
22/02/90	2.209441	- 8.034980	7.649384
23/02/90	2.480910	- 9.095430	8.742783
26/02/90	2.841785	-11.120800	11.151240
27/02/90	3.819039	-16.125000	16.919300
28/02/90	2.366383	- 9.514540	9.827108
01/03/90	2.640698	-11.149800	11.618970
02/03/90	3.255821	-13.747300	14.460600
05/03/90	4.023202	-17.189000	18.474000
06/03/90	3.487456	-15.359300	16.635030
07/03/90	3.493746	-15.557600	17.206700

Notes: In estimating coefficients, β_i, time-to-expiration is defined as a fraction of a year.

APPENDIX A

Table A1
IMPLIED VOLATILITIES AND HISTORICAL STANDARD DEVIATIONS OF FT-SE 100 INDEX

16 February 1990 to 7 March 1990
No. of Trading Days = 14

DATE	τ	S	MONTH	σ_i	$\hat{\sigma}$
16/02/90	41 133 224	2325	March June Sept	0.1406 0.0691 0.0710	0.1522
19/02/90	38 130 221	2299	March June Sept	0.1185 0.0933 0.1116	0.1538
20/02/90	37 129 220	2282	March June Sept	0.1394 0.0987 0.0830	0.1537
21/02/90	36 128 219	2259	March June Sept	0.1490 0.1092 0.0721	0.1546
22/02/90	35 127 218	2267	March June Sept	0.1447 0.1182 0.0831	0.1534
23/02/90	34 126 217	2238	March June Sept	0.1592 0.1322 0.0973	0.1557
26/02/90	31 123 214	2248	March June Sept	0.1680 0.1215 0.0908	0.1547
27/02/90	30 122 213	2251	March June Sept	0.2101 0.1103 0.0936	0.1548
28/02/90	29 121 212	2237	March June Sept	0.1300 0.0988 0.0868	0.1530
01/03/90	28 120 211	2259	March June Sept	0.1390 0.0784 0.0414	0.1534

02/03/90	27 119 210	2254	March June Sept	0.1588 0.1105 0.0632	0.1522
05/03/90	24 116 207	2239	March June Sept	0.1899 0.1391 0.1166	0.1520
06/03/90	23 115 206	2216	March June Sept	0.1580 0.0978 0.0609	0.1530
07/03/90	22 115 205	2233	March June Sept	0.1528 0.1011 0.0970	0.1527

Notes: τ is the number of days to maturity; S is the FT-SE 100 index; MONTH is the expiry month; σ_j is the square root of the average annualised implied volatility, averaged accross strike prices; $\hat{\sigma}$ is the historical annualised standard deviation calculated from 01/11/89.

Table A2

ERRORS OF ESTIMATION PREDICTING ONE DAY FORWARD

19 February 1990 to 8 March 1990
No. of option prices predicted with each volatility estimate = 3
Total number of options = 52
K = 2225 r = 0.15

DATE	PE	SSE	MSE	PE*	SSE*	MSE*
19/02/90	49.67	19095.28	6365.09	43.23	13783.79	4594.60
20/02/90	40.33	13049.84	4349.95	32.26	8705.60	2901.87
21/02/90	40.73	11146.19	3715.40	31.73	6327.04	2109.01
22/02/90	20.95	5066.01	1688.67	12.03	1712.45	570.82
23/02/90	48.19	11758.57	3919.53	39.25	6968.19	2322.73
26/02/90	12.44	3063.06	1021.02	5.33	864.56	288.19
27/02/90	12.06	3183.28	1061.09	5.89	868.12	289.37
28/02/90	17.86	3345.67	1115.22	16.24	1047.66	349.22
01/03/90	24.30	5068.92	1689.64	13.07	1813.66	604.55
02/03/90	30.48	6414.87	2138.29	16.68	2153.47	717.82
05/03/90	28.80	4889.73	1629.91	19.02	1468.11	489.00
06/03/90	5.36	1323.81	441.27	5.20	278.37	92.79
07/03/90	5.40	1614.03	538.01	- 5.13	179.39	59.79
08/03/90	41.97	5979.40	1993.13	31.78	2582.60	860.87

Notes: PE, SSE and MSE are the average percentage error, sum of squared errors, and mean square error of the Black and Scholes model, respectively; PE*, SSE*, and MSE* are the average percentage error, sum of squared errors, and mean squared error of the new option model with time-varying volatility.

Table A3

ERRORS OF ESTIMATION PREDICTING ONE DAY FORWARD

19 February 1990 to 8 March 1990
No. of option prices predicted with each volatility estimate = 3
Total number of options = 52
K = 2325 r = 0.15

DATE	PE	SSE	MSE	PE*	SSE*	MSE*
19/02/90	57.66	11349.86	3783.29	36.87	4144.18	1381.39
20/02/90	44.07	7597.27	2532.42	19.65	3013.12	1004.37
21/02/90	46.06	6397.69	2132.56	21.81	1350.03	450.01
22/02/90	18.12	2729.20	909.73	- 2.97	38.17	12.72
23/02/90	51.95	6276.82	2092.27	29.49	1507.24	502.41
26/02/90	7.46	1641.21	547.07	- 7.63	34.41	11.47
27/02/90	4.13	1703.47	567.82	- 8.50	56.17	18.72
28/02/90	10.57	1585.76	528.59	13.10	112.83	37.61
01/03/90	16.00	2088.06	696.02	- 12.14	136.58	45.53
02/03/90	26.77	3256.59	1085.53	- 12.21	75.39	25.13
05/03/90	24.44	2357.55	785.85	1.81	27.81	9.27
06/03/90	- 9.22	525.30	175.10	- 7.35	25.73	8.58
07/03/90	- 4.94	630.15	210.05	- 29.04	1260.44	420.15
08/03/90	- 4.94	2899.67	966.56	10.48	226.76	75.59

Notes: PE, SSE and MSE are the average percentage error, sum of squared errors, and mean square error of the Black and Scholes model, respectively; PE*, SSE*, and MSE* are the average percentage error, sum of squared errors, and mean squared error of the new option model with time-varying volatility.

Table A4

ERRORS OF ESTIMATION PREDICTING ONE DAY FORWARD

19 February 1990 to 8 March 1990
No. of option prices predicted with each volatility estimate = 3
Total number of options = 52
K = 2425 r = 0.15

DATE	PE	SSE	MSE	PE*	SSE*	MSE*
19/02/90	57.37	6119.99	2039.99	7.75	302.00	100.67
20/02/90	41.01	3967.22	1322.41	- 8.18	860.62	286.87
21/02/90	43.90	3168.12	1056.04	- 4.54	28.77	9.59
22/02/90	12.54	1320.26	440.09	- 26.47	644.80	214.93
23/02/90	40.99	2862.19	954.06	- 0.40	16.41	5.47
26/02/90	- 5.01	750.88	250.29	- 30.17	353.26	117.76
27/02/90	- 6.82	708.02	236.01	- 28.58	535.72	178.57
28/02/90	- 0.21	712.80	237.60	9.80	501.85	167.28
01/03/90	7.31	1074.99	358.33	- 41.22	505.39	168.46
02/03/90	14.54	1474.78	491.59	- 58.09	2230.96	743.65
05/03/90	6.24	864.47	288.16	- 34.70	1280.87	426.96
06/03/90	- 21.72	138.67	46.22	- 19.72	322.55	107.52
07/03/90	- 19.98	188.78	62.93	- 61.96	3130.31	1043.44
08/03/90	6.83	1594.83	531.61	- 34.66	99.01	33.00

Notes: PE, SSE and MSE are the average percentage error, sum of squared errors, and mean square error of the Black and Scholes model, respectively; PE*, SSE*, and MSE* are the average percentage error, sum of squared errors, and mean squared error of the new option model with time-varying volatility.

APPENDIX B

In this section we solve for the option model with time-varying volatility by two alternative ways, namely the risk-neutral valuation on the *Feynman-Kac Formula* results, and by the *Cameron-Martin-Girsanov Theorem*.

B.1 SOLUTION FOR THE OPTION MODEL WITH TIME-VARYING VOLATILITY BY RISK-NEUTRAL VALUATION

Using the *Feynman-Kac Formula*, the general solution to valuing the option as a state-contingent payoff is given by

$$C(S,K,r,\tau,V(\tau)) = E\{\exp(-r\tau)\max[0, (S_t - K)]\}, \tag{B1}$$

where t denotes the date at expiration of the option. This solution also follows from the *Feynman-Kac Formula* for solving partial differential equations, demonstrated by Duffie (1988a, 1988b). For the case $(S_t - K) > 0$, we can rewrite equation (B1) as

$$C(S,K,r,\tau,V(\tau)) = \exp(-r\tau)E[S_t - K | S_t > K]. \tag{B2}$$

Now, define the return on the stock as the change in the logarithm of the stock price. Then, the return on the stock is given by

$$x = \ln(S_t/S_{t-1}), \tag{B3}$$

and

$$S_t = S_{t-1}\exp(x). \tag{B4}$$

Substituting equation (B4) into equation (B2), we obtain

$$C(S,K,r,\tau,V(\tau)) = \exp(-r\tau)E[S_{t-1}\exp(x) - K | S_{t-1}\exp(x) > K]$$

$$= \exp(-r\tau)E[S_{t-1}\exp(x) - K | x > \ln(K/S_{t-1})]. \tag{B5}$$

The variable, x, is normally distributed, with a density function f(x). To solve equation (B5) we integrate it with respect to x. With no loss of generality, let us replace S_{t-1} with S. Integrating with respect to x, we have

$$C(S,K,r,\tau,V(\tau)) = \exp(-r\tau) \int_{\ln(K/S)}^{\infty} (S\exp(x) - K)f(x)\,dx. \qquad (B6)$$

Firstly, let us solve the first term on the right hand side of equation (B6). We can rewrite it as,

$$S\exp(-r\tau) \int_{\ln(K/S)}^{\infty} \exp(x)f(x)\,dx = S\exp(-r\tau) \int_{\ln(K/S)}^{\infty} (1/\sigma\sqrt{2\pi})\exp(-((x-\mu)^2 + 2\sigma^2 x))/2\sigma^2\,dx. \qquad (B7)$$

where μ is the mean of x, and σ^2 is the instantaneous variance of x also given by $\sigma^2 = (1/\tau)V(\tau)$. Let the exponent within the integral be y. Then, by expanding it we obtain

$$y = -(x^2 - 2(\mu+\sigma^2)x + \mu^2), \qquad (B8)$$

By completing the square we obtain

$$y = -((x^2 - 2(\mu+\sigma^2)x + (\mu+\sigma^2)^2 - (\mu+\sigma^2)^2 + \mu^2), \qquad (B9)$$

which becomes

$$y = (2\mu\sigma^2 + \sigma^4) - (x - (\mu + \sigma^2))^2. \qquad (B10)$$

Dividing (B10) by $2\sigma^2$, we obtain

$$y/2\sigma^2 = \exp(\mu+\sigma^2/2) + (1/2\sigma^2)(x - (\mu + \sigma^2))^2 \qquad (B11)$$

Substituting (B11) in (B7) we obtain

$$S\exp(-r\tau+\mu+\sigma^2/2) \int_{\ln(K/S)}^{\infty} (1/\sigma\sqrt{2\pi})\exp(-(1/2)(((x-(\mu+\sigma^2))^2/\sigma)^2)\,dx. \qquad (B12)$$

Secondly, we write the normal density function of x in the second term on the right hand side of (B7) Equation (B6) can now be rewritten as

$$C(S,K,r,\tau,V(\tau)) = Sexp(-r\tau+\mu+\sigma^2/2) \int_{\ln(K/S)}^{\infty} (1/\sigma\sqrt{2\pi})exp(-(1/2)(((x-(\mu+\sigma^2))^2/\sigma)^2)\, dx$$

$$- Kexp(-r\tau) \int_{\ln(K/S)}^{\infty} (1/\sigma\sqrt{2\pi})exp(-(1/2\sigma^2)(x-\mu)^2)\, dx. \quad (B13)$$

Let $z = (\ln(x) - \mu)/\sigma$, and $w = (\ln(K/S) - \mu)/\sigma$. Then, equation (B13) becomes

$$C(S,K,r,\tau,V(\tau)) = Sexp(-r\tau+\mu+\sigma^2/2) \int_{w}^{\infty} (1/2\sqrt{\pi})exp(-(z-\sigma)^2/2)\, dz$$

$$- Kexp(-r\tau) \int_{w}^{\infty} (1/2\sqrt{\pi})exp(-z^2/2)\, dz, \quad (B14)$$

which becomes

$$C(S,K,r,\tau,V(\tau)) = Sexp(-r\tau+\mu+\sigma^2/2) \int_{-w}^{\infty} (1/2\sqrt{\pi})exp(-(z-\sigma)^2/2)\, dz$$

$$- Kexp(-r\tau) \int_{-w}^{\infty} (1/2\sqrt{\pi})exp(-z^2/2)\, dz, \quad (B15)$$

which simplifies to

$$C(S,K,r,\tau,V(\tau)) = Sexp(-r\tau+\mu+\sigma^2/2) \int_{-w+\sigma}^{\infty} (1/2\sqrt{\pi})exp(-z^2/2)\, dz$$

$$- Kexp(-r\tau) \int_{-w}^{\infty} (1/2\sqrt{\pi})exp(-z^2/2)\, dz. \quad (B16)$$

We can now rewrite (B16) as

$$C(S,K,r,\tau,V(\tau)) = Sexp(-r\tau+\mu+\sigma^2/2)\Phi(-w+\sigma) - exp(-r\tau)K\Phi(-w), \quad (B17)$$

where, $w = (\ln(K/S) - \mu)/\sigma$ and $\Phi(.)$ is a standard normal distribution.

Since we have risk neutrality[16]

$$\exp(\mu+\sigma^2/2) = \exp(r\tau), \tag{B18}$$

which means that

$$\mu_x + \sigma_x^2/2 = r\tau. \tag{B19}$$

Since the stock price follows a stationary random walk and we are considering a one period change in the stock price, we have

$$\sigma^2 = V(\tau), \tag{B20}$$

where $V(\tau)$ is the volatility polynomial in equation (19), which implies that

$$\mu = r\tau - V(\tau)/2. \tag{B21}$$

Substituting equation (B21) in the expression for w, and then in equation (B18) we obtain

$$C(S,K,r,\tau,V(\tau)) = S\Phi(d_1) - K\exp(-r\tau)\Phi(d_2), \tag{B22}$$

where,

$$d_1 = -w + \sqrt{V(\tau)}$$

$$= (\ln(S/K) + r\tau + (1/2)V(\tau))/\sqrt{V(\tau)}, \tag{B23}$$

and

$$d_2 = d_1 - \sqrt{V(\tau)}.$$

This gives us the option pricing model with time-varying volatility. Substituting $\sigma^2\tau$ for $V(\tau)$ in equation (B23) yields the standard Black and Scholes model.

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[16]. See Cox and Ross (1976).

B.2 SOLUTION FOR OPTION MODEL USING CAMERON-MARTIN-GIRSANOV THEOREM

We could also solve for the option model with time-varying volatility using the *Cameron-Martin-Girsanov Theorem* (see Oksendal (1985), pages 115-119). The theorem characterises the behaviour of semimartingales under a change of probability measure. The theorem transforms the stochastic diffusion equation with a drift to one without a drift. In option pricing the theorem has been employed by Harrison and Kreps (1979), Duffie (1988b), and Cheng (1991).

Consider a security price whose diffusion is represented by the stochastic differential equation with a drift

$$dS/S = \mu dt + \sigma dW(t) \tag{B24}$$

where W(t) is a Gauss-Wiener process, and also consider a riskless bond whose stochastic equation is

$$dB = rBdt. \tag{B25}$$

The security price, S, is a semimartingale relative to some probability space $(\Omega,\{F_t\},P)$. Now, let

$$\alpha(t,T) = -\mu/\sigma, \tag{B26}$$

and a Gauss-Wiener process,

$$W^*(t) = W(t) - \int_0^t \alpha(s,T)ds,\ 0 \le t \le T, \tag{B27}$$

where T is the maturity time of the bond.

From equation (B27) we have

$$\alpha(t,T) = dW(t)/dt - dW^*(t,T)/dt. \tag{B28}$$

Equating (B26) to (B28) we obtain

$$dW(t) = dW^*(t,T) - (\mu/\sigma)dt. \tag{B29}$$

Substituting (B29) in (B24) we obtain a stochastic differential equation

$$dS/S = \sigma dW^*(t,T). \tag{B33}$$

Equation (B30) now has no drift parameter, μ. We have thus transformed equation (B24) with a drift to equation (B30) without a drift. But for (B30) to hold the following conditions should be satisfied (see Harrison and Kreps (1979), pages 396-398):

$$\int_0^T \alpha(t,T)^2 dt < \infty \text{ a.s.}, \tag{B31}$$

that is, $\alpha(t,T)$ must be square integrable. The *Radon-Nikodym* derivative which defines change of measure must be defined. The *Radon-Nikodym* derivative is given by

$$\rho(T) = \exp(\int_0^T \alpha(t,T)\, dW(t) - (1/2)\int_0^T \alpha(t,T)^2\, dt, \tag{B32}$$

and $E(\rho(T)^2) < \infty$, and $E(\rho(T)) = 1$. Also $\rho(T) = dQ/dP$ where both Q and P are defined on $\{F_t\}$. These conditions are sufficient for the application of the *Cameron-Martin-Girsanov Theorem*.

Hence,

$$E(S\rho(T)) = E^Q(S), \tag{B33}$$

where E^Q is the expectation with respect to the new measure Q. At expiration the option price is given by $C(S,K) = (S(T) - K)^+$, otherwise it is

$$C(S,K,r,\tau,V(\tau)) = E^Q\{\exp(-r\tau)(S - K)^+\}, \tag{B34}$$

where S is obtained from equation (B30). Equation (B34) reveals that the option price is a martingale with respect to the new measure Q. This is consistent with Harrison and Kreps' pricing of securities by the no-arbitrage argument. By substituting for S, given by equation (B30), in (B34), and solving the expectation yields the option model with time-varying volatility, which is given by

$$C(S,K,r,\tau,V(\tau)) = S\Phi(d_1) - K\exp(-r\tau)\Phi(d_2), \tag{B35}$$

where,

$$d_1 = (\ln(S/K) + r\tau + (1/2)V(\tau))/\sqrt{V(\tau)},$$

$$d_2 = d_1 - \sqrt{V(\tau)}.$$

This completes the derivation of the model.

Also, refer to Cheng (1991) who checks the suitability of various stochastic processes of bond prices for the pricing of options by whether they satisfy the *Cameron-Martin-Girsanov Theorem*. She considers Brownian motion, Ornstein-Uhlenbeck process, Brownian Bridge process and Exponential Brownian Bridge process.

APPENDIX C

NUMERICAL APPROXIMATION OF A STANDARD NORMAL DISTRIBUTION

To approximate a Standard Normal Distribution, $\Phi(d)$, for the random variable, d, numerically, we use the polynomial

$$\Phi(d) = 1 - [g(d)y(\alpha_0 + \alpha_1 y + \alpha_2 y^2 + \alpha_3 y^3 + \alpha_4 y^4)] + \varepsilon, \qquad \text{(C1)}$$

where,

$$d > 0,$$
$$g(d) = (1/\sqrt{2\pi})\exp(-d^2/2),$$
$$y = 1/(1 + pd),$$
$$p = +0.2316419,$$
$$\alpha_0 = +0.319381530,$$
$$\alpha_1 = -0.356563782,$$
$$\alpha_2 = +1.781477937,$$

$\alpha_3 = -1.821255978,$

$\alpha_4 = +1.330274429,$

and ε, is an error term. From this numerical approximation procedure, the value of the error is less than 7.5×10^{-8}. For the case where, $d < 0$, set $\Phi(d) = 1 - \Phi(-d)$.

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