IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received December 1, 2019, accepted December 12, 2019, date of publication December 20, 2019,
date of current version December 31, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2961165

Probabilistic Message Passing for Decentralized Control of Stochastic Complex Systems

RANDA HERZALLAH

Institute for Urban Technology and the Environment (ASTUTE), Aston University, Birmingham B4 7ET, U.K.

e-mail: r.herzallah@aston.ac.uk

This work was supported by the Leverhulme Trust under Grant RPG-2017-337.

ABSTRACT This paper proposes a novel probabilistic framework for the design of probabilistic message passing mechanism for complex and large dynamical systems that are operating and governing under a decentralized way. The proposed framework considers the evaluation of probabilistic messages that can be passed between mutually interacting quasi-independent subsystems that will not be restricted by the assumption of homogeneity or conformability of the subsystems components. The proposed message passing scheme is based on the evaluation of the marginal density functions of the states that need to be passed from one subsystem to another. An additional contribution is the development of stochastic controllability analysis of the controlled subsystems that constitute a complex system. To facilitate the understanding and the analytical analysis of the proposed message passing mechanism and the controllability analysis, theoretical developments are demonstrated on linear stochastic Gaussian systems.

INDEX TERMS Probabilistic control, probabilistic message passing, stochastic systems.

I. INTRODUCTION

Complex systems are ubiquitous in nature and man-made systems. They appear in a wide range of domains including neuronal [1], intracellular, ecological [2], and engineering and infrastructure [3]. They are composed of a large number of interacting parts and exhibit collective dynamical behaviour that cannot be predicted from the properties of the individual parts themselves. Advances in communication, network science, and computing technologies have over the last decades created a burst of research activity, aiming to uncover new efficient and cost-effective approaches to model and control a complex system. As a result, several promising competing approaches appeared to address the decentralization of the modelling and control of a complex system. Here we mention some of the more promising methods.

Current advances include, multiagent systems [4], distributed control [5], [6], pinning control [7]–[9] and decentralized control [10], [11] to name a few. These advances however suffer from either over-representing single-agent architectures as far as the controller design is concerned, which are centralized and so complete observation of the global state must be known, or are decentralized and decisions are based only on incomplete and disconnected knowledge, or based on an approximate solution considering only a single simplifying averaged effect. In addition, these advances tend to avoid the dynamical characteristic of a control process, thus cannot effectively handle many of the dynamical properties of a complex network and do not allow intelligent and adaptive decisions to be taken. Similarly, recent developments on distributed and decentralized control still suffer from the lack of a reliable message passing framework that can operate efficiently under the presence of heterogenous or uncertain complex systems components [12]–[15]. Although some recent studies have considered the distributed synchronization of multi-agent systems with heterogeneous agents [15], [16], the agents are assumed to be represented by linear equations and they are assumed to have identical dimensions in most of these studies. Other properties and challenges in controlling a complex system have aslo been addressed and discussed in the recent literature. Examples include, the development of adaptive control methods which considered uncertain switched systems for networks that change typology overtime [17], [18], the development of synchronisation methods for complex networks with time delay [19], and the development of synchronisation methods for uncertain multiagent systems [20].

A fundamental property often overlooked conventionally is that the control process of a subsystem of these typical networks-of-networks needs to consider constraints imposed

The associate editor coordinating the review of this manuscript and approving it for publication was Zheng H. Zhu.

from the external environment and neighboring subsystems. This has been addressed in our recent work [21], [22] where we postulated a decentralized architecture that incorporates higher interaction across the network through decoupling the effect of a subsystem into the subsystem's own state and external inputs from neighboring subsystem states estimated via probabilistic message passing. However, the decentralized architecture in [21], [22] has focused on the problem of determining optimal control inputs that make the complex system behave in a pre-specified way and have only developed incipient message passing technique that pass the parameters from one subsystem to another.

Nonetheless, message passing has significant impact on the design of optimal control inputs as it provides the collaborative element and offers informative feedback on the states of the neighbouring subsystems which is an enabler for closing the loop and optimising design and operation. Therefore, to address the aforementioned challenges, we provide in this paper one such approach to the design of a probabilistic message passing mechanism that is applicable to networked systems that are operating and governing in a decentralized way. The proposed message passing scheme is based on the evaluation of the marginal density functions of the states that need to be passed from one subsystem to another. Our solution follows a fully probabilistic framework where a local subsystem is controlled purely based on local information and driven only by local coupling to neighbouring subsystems by probabilistic message passing. The proposed framework can achieve the control objective of the decentrally controlled complex system even in the presence of heterogeneous and dimensionally nonidentical subsystems components that also operate under uncertainty and are affected by noise and randomness. An important consequence of the local control of complex systems which is underappreciated in the study of complex systems is that following a small perturbation these systems may undergo undesirable faulty states when in fact there are other accessible desirable states in which those undesirable states could be avoided. We will show that this drawback will not arise in our probabilistic control and message passing framework. Therefore, we will prove the stochastic controllability of the decentrally controlled system that exchanges messages using our proposed probabilistic message passing method. To facilitate the analytical analysis of the proposed method, the theoretical development will be demonstrated on a class of linear stochastic systems that can be described by Gaussian probability density function.

To summarize, the main contribution of this paper is the development of the probabilistic message passing method and the analysis of the stochastic controllability of the controlled systems. Compared with the existing results on the topic, this article has the following distinct features that have not been reported in the literature. Firstly, a fully probabilistic framework for the design of decentralized controllers and probabilistic message passing is developed where local controllers, systems models, and communications between the subsystems of a complex system are characterized by probability density functions. It will be demonstrated that this probabilistic framework guarantees synchronization in the presence of noise and systems uncertainties. Secondly, the subsystems components are not restricted by the homogeneity assumption and they are not required to have identical lengths. This makes the proposed framework more appropriate for application to real world problems, such as national power grid systems, water and gas supply networks, a city's communication infrastructure and vehicle transport network, which usually have non-homogeneous and non-identical components. Thirdly, the subsystems pass only partial information about the states of their dynamics. This partial information is received by neighbouring nodes as probabilistic messages and treated as external signals. Finally, the stochastic controllability of the subsystems is analyzed and the required result is obtained.

The rest of the paper is organized as follows. The problem formulation is given in Section II. Here the subsystems representation with the notion of external signals is discussed and the subsystems local randomized controllers are introduced and their optimized pdfs are given. The main results are given in Section III. In particular, this section develops the proposed probabilistic message passing scheme. The analysis of the stochastic controllability of the subsystems constituting a complex system is given in Section IV. Section V, provides the simulation results where the proposed decentralized probabilistic control and message passing framework is tested and compared to the centralized control approach. Finally, Section VI concludes the paper by providing a brief summary of the proposed framework and the obtained results.

II. PROBLEM FORMULATION

As discussed in the introduction section, this paper considers a fully probabilistic decentralized control framework where each subsystem in the complex system is controlled based on its local information and uncertain information provided by the external signals from the neighbouring subsystems states estimated via probabilistic message passing. Within the proposed framework, system complexity, variability and uncertainty will be dealt with by using probabilistic design methods to design local controllers. Once the closed loop behaviour from local controllers is obtained, local controllers will be required to diffuse information to neighbouring subsystems. Information diffusion will be achieved through probabilistic message passing in order to update the knowledge of the subsystems about their external inputs which will be achieved by using probabilistic inference methods. The mathematical representation of the system dynamics with external signals from neighbouring subsystem states is given in the next section.

A. SUBSYSTEMS REPRESENTATION

This paper considers the decentralized control of a complex stochastic system which consists of a collection of N mutually interacting quasi-independent subsystems that evolve under local constraints driven only by local coupling to neighbouring subsystems by probabilistic message passing. Each subsystem is locally controlled by a randomized controller, $c(u_{t;i}|z_{t-1;i})$. Here $u_{t;i}$ represents a sequence of multivariate inputs that governs subsystem i, $t \in \{1, \ldots, H\}$ is the time index, H is the control horizon, and $z_{t;i} = [y_{t;i}, x_{t;i}]^T$ is the subsystem state vector with $y_{t;i}$ being the multivariate output of the local subsystem and $x_{t;i}$ being the multivariate observed external signals received from neighbouring subsystems. The interaction of these multivariate random variables is assumed to be modelled by a Markov type pdf as follows,

$$\begin{aligned} s(y_{t;i}, u_{t;i}, x_{t;i}|u_{t-1;i}, \ldots u_{0;i}, y_{t-1;i}, \ldots, y_{0;i}, x_{t-1;i}, \ldots, x_{0;i}) \\ = s(y_{t;i}|u_{t;i}, z_{t-1;i})s(x_{t;i}|x_{t-1;i})c(u_{t;i}|z_{t-1;i}) \end{aligned} \quad (1)$$

where $s(y_{t;i}|u_{t;i}, z_{t-1;i})$ and $c(u_{t;i}|z_{t-1;i})$ represent the pdf of the multivariate output and randomized controller of the local subsystem i respectively. Also, $s(x_{t;i}|x_{t-1;i})$ represents the pdf of the external random signals to subsystem i. The conditioning of this pdf of external signals on the previous external signals only stems from our assumed legitimate fact that the inherent dynamics of these external variables cannot be influenced by the inputs $u_{t;i}$ or outputs $y_{t;i}$ of the ith local subsystem.

Remark 1: In most of the existing literature, the subsystems dynamics are described by dynamical equations which do not provide a complete characterization for stochastic systems that operate under high levels of uncertainty and noise. In this paper the system dynamics are completely characterized by their pdfs as given in Equation (1)*. Furthermore, These pdfs are not assumed to be known apriori, therefore they are estimated online using the method proposed in [23].*

Remark 2: In many real world complex systems, the subsystems components constituting the complex system are heterogeneous. Therefore, the formulation in this paper will not be restricted by the assumption of homogeneity of the subsystems components. In particular, we consider N non-identical mutually interacting quasi-independent subsystems that can be characterized by nonidentical pdfs and that can have different lengths. Furthermore, the mutual interaction of the independent subsystems is not necessarily assumed to be symmetric, which implies that the message passing of the corresponding typological network is allowed to be either directed or undirected.

Each local subsystem i, is controlled by a local controller that is optimized to achieve its control objectives. In this paper local controllers are designed using the fully probabilistic design (FPD) control method [23], [24]. This method specifies the control objective of subsystem i by an ideal pdf that determines the steady state behaviour of the joint distribution of the closed loop system dynamics,

$$\begin{aligned} {}^I s(y_{t;i}, u_{t;i}, x_{t;i}|u_{t-1;i}, \ldots u_{0;i}, y_{t-1;i}, \ldots, y_{0;i}, x_{t-1;i}, \ldots, x_{0;i}) \\ = {}^I s(y_{t;i}|u_{t;i}, z_{t-1;i})s(x_{t;i}|x_{t-1;i})\,{}^I c(u_{t;i}|z_{t-1;i}), \end{aligned} \quad (2)$$

where here the superscript I is used to denote the ideal pdf of the corresponding factor of pdf in Equation (1). The pdf factor, $s(x_{t;i}|x_{t-1;i})$ in Equation (2) is taken to be equal to its corresponding factor in Equation (1) to reflect our assumed legitimate fact that x_t are external multivariate signals, thus they cannot be influenced or changed in node i. The randomized controller is then optimized such that the Kullback–Leibler divergence between the actual joint distribution (1) and ideal joint distribution (2) is minimized,

$$\begin{aligned} -\ln\left(\gamma\left(z_{t-1;i}\right)\right) = \min_{c(u_{t;i}|z_{t-1;i})} \mathcal{D}(s(y_{t;i}, u_{t;i}, x_{t;i}|z_{t-1;i})|| \\ \times {}^I s(y_{t;i}, u_{t;i}, x_{t;i}|z_{t-1;i})), \end{aligned} \quad (3)$$

where, $-\ln\left(\gamma\left(z_{t;i}\right)\right)$ is the value function, and $\mathcal{D}(.)$ represents the Kullback-Leibler divergence.

Since the focus of this paper is on the challenging problem of probabilistic message passing, the next section will only briefly give the results of the solution of the optimized subsystems local randomized controllers. No details on the FPD method, its procedure, or the optimisation methodology of the randomized controllers will be provided here, however, they can be found in [23]–[25].

B. SUBSYSTEMS LOCAL CONTROLLERS

Given the probabilsitic description of the joint distribution of the controlled system dynamics given in Equation (1) and its ideal joint pdf given in Equation (2), the optimal randomized controller that minimizes the Kullback-Leibler divergence specified in Equation (3) is given in the following proposition.

Proposition 1: The optimal randomized controller that minimizes the Kullback–Leibler divergence defined in Equation (3) *subject to the joint distribution of the stochastic system given in Equation* (1) *and its ideal distribution given in Equation* (2) *is given by,*

$$\begin{aligned} c(u_{t;i}|z_{t-1}) &= \frac{{}^I c(u_{t;i}|z_{t-1;i})\exp[-\beta(u_{t;i}, z_{t-1;i})]}{\gamma(z_{t-1;i})}, \\ \gamma(z_{t-1;i}) &= \int {}^I c(u_{t;i}|z_{t-1;i})\exp[-\beta(u_{t;i}, z_{t-1;i})]\mathrm{d}u_{t;i}, \\ \beta(u_{t;i}, z_{t-1;i}) &= \int s(y_{t;i}|u_{t;i}, z_{t-1;i}) \\ &\quad \times \ln\left(\frac{s(y_{t;i}|u_{t;i}, z_{t-1;i})}{{}^I s(y_{t;i}|u_{t;i}, z_{t-1;i})\tilde{\gamma}(y_{t;i}, x_{t-1;i})}\right)\mathrm{d}y_{t;i}, \\ \ln(\tilde{\gamma}&(y_{t;i}, x_{t-1;i})) \\ &= \int s(x_{t;i}|x_{t-1;i})\ln(\gamma(z_{t;i}))\mathrm{d}x_{t;i}. \end{aligned} \quad (4)$$

Proof: The derivation of the above result can be found in [21].

To emphasize, the randomized control solution given in Equation (4) is not restricted by the pdf of the system dynamics or its ideal distribution. It provides the general solution without constraints on the required pdfs. However, the evaluation of the analytic solution for this randomized controller is not possible except for the special case of linear and Gaussian pdfs. Therefore, to facilitate the understanding and the analytical solution of the proposed probabilistic message passing method, the rest of the paper will focus on the development of the required solutions for the optimal randomized controllers and the probabilistic message passing for the case where the pdfs given in Equations (1) and (2) are assumed to be Gaussian.

Thereupon, consider subsystems that are characterized by linear Gaussian pdfs and that receive messages from neighbouring subsystems as external multivariate signals. To be more specific, the pdf of the mutivariate output of local subsystem i is given by,

$$\begin{aligned} s(y_{t;i}|u_{t;i}, z_{t-1;i}) &= \mathcal{N}(\bar{y}_{t;i}, Q_i), \\ s(x_{t;i}|x_{t-1;i}) &= \mathcal{N}(\bar{x}_{t;i}, R_i), \end{aligned} \tag{5}$$

where Q_i and R_i are the covariances of the subsystem output and external signals respectively, and where,

$$\begin{aligned} \bar{y}_{t;i} &= A_i z_{t-1;i} + B_i u_{t;i}, \\ \bar{x}_{t;i} &= C_i x_{t-1;i}. \end{aligned} \tag{6}$$

Within the FPD control framework, the control objective can be achieved through the specification of the appropriate parameters of the ideal distribution that will realise the desired objective. Therefore, the solution in this section will, without any loss of generality, be given for the regulation problem where it is required to bring all the system states from their initial values back to zero. Thus, given this control objective the ideal distribution is assumed to be given by,

$$\begin{aligned} {}^I s(y_{t;i}|u_{t;i}, z_{t-1;i}) &= \mathcal{N}(0, \Sigma_i), \\ {}^I s(x_{t;i}|x_{t-1;i}) &= \mathcal{N}(\bar{x}_{t;i}, R_i), \end{aligned} \tag{7}$$

where Σ_i specifies the desired fluctuations of the system output around zero that need to be achieved. Also, note that ${}^I s(x_{t;i}|x_{t-1;i})$ is taken to be the same as the subsystem distribution of the external signals, emphasising that the external signals should not be governed or even affected by the subsystem output. Similarly, the ideal distribution of the controller is assumed to be given by,

$${}^I c(u_{t;i}|z_{t-1;i}) = \mathcal{N}(0, \Gamma_i), \tag{8}$$

where Γ_i determines the allowed range of the optimal control inputs. The next proposition specifies the solution to the optimized randomized controller based on (3) for subsystems with observed external signals.

Proposition 2: The optimal randomized controller for the subsystem described by (5) *and ideal distributions of system dynamics and control inputs described by* (7) *and* (8) *respectively is given by,*

$$c(u_{t;i}|z_{t-1;i}) = \mathcal{N}(\bar{u}_{t;i}, \Gamma_{t;i}) \tag{9}$$

where,

$$\begin{aligned} \bar{u}_{t;i} &= -K_{t;i} z_{t-1;i}, \\ \Gamma_{t;i} &= (\Gamma_i^{-1} + B_i^T M_{t;i} B_i), \\ K_{t;i} &= \Gamma_{t;i}^{-1} B_i^T \left[M_{t;i} A_{y_{t-1};i} \quad M_{t;i} A_{x_{t-1};i} + S_{t,2;i} C_i \right], \\ M_{t;i} &= \Sigma_i^{-1} + S_{t,1;i}, \end{aligned} \tag{10}$$

and where,

$$-\ln\left(\gamma\left(z_{t;i}\right)\right) = 0.5 z_{t;i}^T S_{t;i} z_{t;i} + 0.5\omega_{t;i}, \tag{11}$$

with

$$\begin{aligned} S_{t-1,1;i} &= -A^T_{y_{t-1};i} M_{t;i} B_i \Gamma_{t;i}^{-1} B_i^T M_{t;i}^T A_{y_{t-1};i} \\ &\quad + A^T_{y_{t-1};i} M_{t;i} A_{y_{t-1};i}, \end{aligned} \tag{12}$$

$$\begin{aligned} S_{t-1,2;i} &= -2A^T_{y_{t-1};i} M_{t;i} B_i \Gamma_{t;i}^{-1} B_i^T M_{t;i}^T A_{x_{t-1};i} \\ &\quad + 2A^T_{y_{t-1};i} M_{t;i} A_{x_{t-1};i} + 2A^T_{y_{t-1};i} M_{t;i} C_i \\ &\quad - 2A^T_{y_{t-1};i} M_{t;i} B_i \Gamma_{t;i}^{-1} B_i^T S_{t,2;i} C_i, \end{aligned} \tag{13}$$

$$\begin{aligned} S_{t-1,3;i} &= C_i^T S_{t,3;i} C_i + A^T_{x_{t-1};i} M_{t;i} A_{x_{t-1};i} \\ &\quad + 2A^T_{x_{t-1};i} S_{t,2;i} C_i - A^T_{x_{t-1};i} M_{t;i} B_i \Gamma_t^{-1} B^T M_{t;i} A_{x_{t-1};i} \\ &\quad - C_i^T S_{t,2;i} B_i \Gamma_t^{-1} B_i^T S_{t,2;i} C_i \\ &\quad - 2A^T_{x_{t-1};i} M_{t;i} B_i \Gamma_{t;i}^{-1} B_i^T S_{t,2;i} C_i, \\ \omega_{t-1;i} &= \omega_{t;i} + tr(S_{t,1;i}\Sigma_i) + tr(S_{t,3;i} R_i) \\ &\quad + \ln|I + (B_i \Gamma_i^{0.5})^T M_{t;i} (B_i \Gamma_i^{0.5})|, \end{aligned} \tag{14}$$

is the quadratic cost function. We have also introduced the following partitioning of the matrices, $S_{t;i} = \begin{bmatrix} S_{t,1;i} & S_{t,2;i} \\ S_{t,2;i} & S_{t,3;i} \end{bmatrix}$, *and* $A_i = \left[A_{y_{t-1};i} \ A_{x_{t-1};i} \right]$.

Proof: The proof of this proposition can be obtained by evaluating Equation (4) using the corresponding pdfs specified in Equations (5), (7), and (8). Its detailed derivation can be found in [21]. As can be seen from Equation (10), only the two blocks defined in Equations (12) and (13) of the full Riccati matrix $S_{t;i}$ need to be solved. The third block defined in (14) of the Riccati equation does not need to be solved. This decreases the computational efforts in obtaining the optimal randomized control law compared to the global solution.

The following sections will focus on the development of the proposed probabilistic message passing algorithm for the non-identical mutually interacting quasi-independent subsystems and the controllability analysis of these subsystems constituting a complex system, which are the main contributions of this paper.

III. MAIN RESULTS

Our decentralized framework is based on the decomposition of a complex system into smaller subsystems that can be controlled individually to achieve their local control objectives. Messages can then be passed between the subsystems to keep them informed about each others objectives and the whole system objectives thus ensuring the achievement of these objectives without the need to centrally control each subsystem in the complex system. When passed from one subsystem to another, messages enter the receiving subsystem as external multivariate signals. External here emphasizes our hypothesis that the receiving subsystem can only receive these signals from its neighbouring subsystems without being allowed to influence these signals or change their values or dynamics. In another word, the message passing in our decentralized framework is done with the objective of sending information about the state of the passing subsystem to the receiving neighbour subsystem keeping it informed about its surrounding environment. This allows the local controllers to control their local environments, and at the same time harmonize their actions with the surrounding environment by making use of uncertain information provided by the external signals. Once the closed loop behaviour from local controllers is obtained, local controllers will be required to diffuse information to neighbouring subsystems. Information diffusion will be achieved through probabilistic message passing in order to update the knowledge of the subsystems about their external inputs which will be achieved by using probabilistic inference methods as will be detailed in this section.

A. PROBABILISTIC MESSAGE PASSING

As discussed in previous sections, the state of each subsystem is decoupled into its own state and external inputs from neighbouring subsystem states estimated via probabilistic message passing. In the probabilistic framework proposed in this paper, information of the subsystem about its external states that are received from neighbouring subsystems can be obtained using probabilistic inference methods. Since the subsystems constituting the complex system are assumed in this paper to be inherently stochastic, complete description of the closed loop behaviour of each subsystem can be described by the joint probability density function of its interacting variables including its internal and external states and its control input. This can be expressed as follows,

$$s(y_{t;i}, x_{t;i}, u_{t;i} \,|z_{t-1;i}). \tag{15}$$

Subsystem i will then be required to pass information about a subset of its internal state variables, $y_{t;i}$ to its neighbouring subsystems. This means that the marginal distribution of the subset of the states that will be required to be passed from one subsystem to another need to be evaluated. To achieve this objective we introduce the following definition for the messages to be passed from subsystem i to subsystem j

Definition 1: Let $s(y_{t;i}, x_{t;i}, u_{t;i} \,|z_{t-1;i})$ be the complete description of the interacting variables of subsystem i, and assume that the probabilistic message to be passed from subsystem i to subsystem j constitutes information about a subset q of its internal states, $y_{t;i}$, then the probabilistic message to be passed from subsystem i to subsystem j is defined as,

$$\begin{aligned} &M_{j\leftarrow i}(y^q_{t;i} \,|z_{t-1;i}) \\ &= \int s(y_{t;i}, x_{t;i}, u_{t;i} \,|z_{t-1;i}) \mathrm{d}y^{q+1}_{t;i} \ldots \mathrm{d}y^n_{t;i} \mathrm{d}x_{t;i} \mathrm{d}u_{t;i}. \end{aligned} \tag{16}$$

Following Definition 1, the probability density function of the states to be passed from subsystem i to subsystem j can be shown to be given by the following theorem.

Theorem 1: Given the probability density function of the multivariate output of local subsystem i defined in Equation (5)*, the probability density function of the randomized controller of subsystem i defined in Equation* (9)*, and Definition 1, the probabilistic message to be passed from subsystem i about a subset q of its internal states to subsystem j is given by,*

$$M_{j\leftarrow i}(y^q_{t;i} \,|z_{t-1;i}) = \mathcal{N}(\mu^q_{y_{t;i}}, \Sigma^q_{y_{t;i}}), \tag{17}$$

where,

$$\begin{aligned} \mu_{y_{t;i}} &= A_i z_{t-1;i} + B_i \bar{u}_{t;i}, \\ \Sigma_{y_{t;i}} &= [B_i^T Q_i^{-1} B_i + \Gamma^{-1}_{t;i}]. \end{aligned} \tag{18}$$

and where we introduced the following notation and partitioning of the matrices,

$$\begin{aligned} &y^q_{t;i} = \begin{bmatrix} y_{1,t;i} \\ \ldots \\ y_{q,t;i} \end{bmatrix}, \quad y^{n-q}_{t;i} = \begin{bmatrix} y_{q+1,t;i} \\ \ldots \\ y_{n,t;i} \end{bmatrix}, \quad y_{t;i} = \begin{bmatrix} y^q_{t;i} \\ y^{n-q}_{t;i} \end{bmatrix}, \\ &\mu^q_{y_{t;i}} = \begin{bmatrix} \mu_{y_{1,t;i}} \\ \ldots \\ \mu_{y_{q,t;i}} \end{bmatrix}, \quad \mu^{n-q}_{y_{t;i}} = \begin{bmatrix} \mu_{y_{q+1t;i}} \\ \ldots \\ \mu_{y_{n,t;i}} \end{bmatrix}, \\ &\mu_{y_{t;i}} = \begin{bmatrix} \mu^q_{y_{t;i}} \\ \mu^{n-q}_{y_{t;i}} \end{bmatrix}, \quad \Sigma_{y_{t;i}} = \begin{bmatrix} \Sigma^q_{y_{t;i}} & \Sigma^{q,n-q}_{y_{t;i}} \\ \Sigma^{n-q,q}_{y_{t;i}} & \Sigma^{n-q}_{y_{t;i}} \end{bmatrix}, \\ &\Sigma^{-1}_{y_{t;i}} = \Omega = \begin{bmatrix} \Omega^q & \Omega^{q,n-q} \\ \Omega^{n-q,q} & \Omega^{n-q} \end{bmatrix}. \end{aligned} \tag{19}$$

Proof: The evaluation of the probabilistic message to be passed from subsystem i to subsystem j can be obtained by applying the chain rule to the integral on the right hand side of Equation (16). This yields,

$$\begin{aligned} &M_{j\leftarrow i}(y^q_{t;i} \,|z_{t-1;i}) \\ &\quad = \int s(y_{t;i} \,|u_{t;i}, z_{t-1;i}) s(x_{t;i} \,|x_{t-1;i}) s(u_{t;i} \,|z_{t-1;i}) \\ &\qquad \times \mathrm{d}y^{q+1}_{t;i} \ldots \mathrm{d}y^n_{t;i} \mathrm{d}x_{t;i} \mathrm{d}u_{t;i}. \end{aligned} \tag{20}$$

Integrating over $x_{t;i}$ in Equation (20), we obtain,

$$\begin{aligned} M_{j\leftarrow i}(y^q_{t;i} \,|z_{t-1;i}) &= \int s(y_{t;i} \,|u_{t;i}, z_{t-1;i}) s(u_{t;i} \,|z_{t-1;i}) \\ &\qquad \times \mathrm{d}y^{q+1}_{t;i} \ldots \mathrm{d}y^n_{t;i} \mathrm{d}u_{t;i}. \end{aligned} \tag{21}$$

Using Equations (5), (6) and (9) in Equation (21) we get,

$$\begin{aligned} &M_{j\leftarrow i}(y^q_{t;i} \,|z_{t-1;i}) \\ &= \int \Bigg(\int \exp\Bigg\{ -0.5 \Big[y_{t;i} - (A_i z_{t-1;i} + B_i u_{t;i}) \Big]^T Q_i^{-1} \\ &\quad \times \Big[y_{t;i} - (A_i z_{t-1;i} + B_i u_{t;i}) \Big] \\ &\quad - 0.5 \Big[u_{t;i} - \bar{u}_{t;i} \Big]^T \Gamma^{-1}_{t;i} [u_{t;i} - \bar{u}_{t;i}] \Bigg\} \mathrm{d}u_{t;i} \Bigg) \mathrm{d}y^{q+1}_{t;i} \ldots \mathrm{d}y^n_{t;i}. \end{aligned} \tag{22}$$

Integrating over $u_{t;i}$ in Equation (22) we get,

$$
\begin{aligned}
&M_{j\leftarrow i}(y^q_{t;i}\,|z_{t-1;i})\\
&= \int \exp\Bigg\{ -0.5\Big[y^T_{t;i}Q_i^{-1}y_{t;i}\\
&\quad + z^T_{t-1;i}A_i^T Q_i^{-1}A_i z_{t-1;i} - 2\, y^T_{t;i}Q_i^{-1}A_i z_{t-1;i} + \bar{u}^T_{t;i}\Gamma^{-1}_{t;i}\bar{u}_{t;i}\\
&\quad - \Big(\Gamma^{-1}_{t;i}\bar{u}_{t;i} + B_i^T Q_i^{-1}y_{t;i} - B_i^T Q_i^{-1}A_i z_{t-1;i}\Big)^T\\
&\quad \times [B_i^T Q_i^{-1}B_i + \Gamma^{-1}_{t;i}]^{-1}\Big(\Gamma^{-1}_{t;i}\bar{u}_{t;i} + B_i^T Q_i^{-1}y_{t;i}\\
&\quad - B_i^T Q_i^{-1}A_i z_{t-1;i}\Big)\Big]\Bigg\} dy^{q+1}_{t;i}\ldots dy^n_{t;i}. \qquad (23)
\end{aligned}
$$

The integral in Equation (23) can then be rewritten as,

$$
\begin{aligned}
&M_{j\leftarrow i}(y^q_{t;i}\,|z_{t-1;i})\\
&\quad = \int \exp\Bigg\{ -0.5\Big[y_{t;i} - (A_i z_{t-1;i}\\
&\qquad + B_i\bar{u}_{t;i})\Big]^T [B_i^T Q_i^{-1}B_i + \Gamma^{-1}_{t;i}]^{-1}\Big[y_{t;i} - (A_i z_{t-1;i}\\
&\qquad + B_i\bar{u}_{t;i})\Big]\Bigg\} dy^{q+1}_{t;i}\ldots dy^n_{t;i}. \qquad (24)
\end{aligned}
$$

Using Equations (18) and (19) in Equation (24), yields

$$
\begin{aligned}
&M_{j\leftarrow i}(y^q_{t;i}\,|z_{t-1;i})\\
&= \int \exp\Bigg\{ -0.5\Big[y_{t;i} - \mu_{y_{t;i}}\Big]^T \Sigma^{-1}_{y_{t;i}}\\
&\quad \Big[y_{t;i} - \mu_{y_{t;i}}\Big]\Bigg\} dy^{q+1}_{t;i}\ldots dy^n_{t;i},\\
&= \int \exp\Bigg\{ -0.5 \begin{bmatrix} y^q_{t;i} - \mu^q_{y_{t;i}} \\ y^{n-q}_{t;i} - \mu^{n-q}_{y_{t;i}} \end{bmatrix}^T \begin{bmatrix} \Omega^q & \Omega^{q,n-q} \\ \Omega^{n-q,q} & \Omega^{n-q} \end{bmatrix}\\
&\quad \times \begin{bmatrix} y^q_{t;i} - \mu^q_{y_{t;i}} \\ y^{n-q} - \mu^{n-q}_{y_{t;i}} \end{bmatrix}\Bigg\} dy^{q+1}_{t;i}\ldots dy^n_{t;i}. \qquad (25)
\end{aligned}
$$

The evaluation of the above integral can be achieved by rewriting the exponent in Equation (25) in the following form,

$$
\begin{aligned}
&-0.5 \begin{bmatrix} y^q_{t;i} - \mu^q_{y_{t;i}} \\ y^{n-q}_{t;i} - \mu^{n-q}_{y_{t;i}} \end{bmatrix}^T \begin{bmatrix} \Omega^q & \Omega^{q,n-q} \\ \Omega^{n-q,q} & \Omega^{n-q} \end{bmatrix}\\
&\quad \times \begin{bmatrix} y^q_{t;i} - \mu^q_{y_{t;i}} \\ y^{n-q} - \mu^{n-q}_{y_{t;i}} \end{bmatrix}\\
&= -0.5\Bigg\{\Big[(y^q_{t;i} - \mu^q_{y_{t;i}})^T(\Omega^q - \Omega^{q,n-q}\Omega^{n-q^{-1}}\Omega^{n-q,q})(y^q_{t;i}\\
&\quad - \mu^q_{y_{t;i}})\Big] + \Big[(y^{n-q}_{t;i} - \mu^{n-q}_{y_{t;i}}) + \Omega^{n-q^{-1}}\Omega^{n-q,q}(y^q_{t;i} - \mu^q_{y_{t;i}})\Big]^T\\
&\quad \times \Omega^{n-q}\Big[(y^{n-q}_{t;i} - \mu^{n-q}_{y_{t;i}}) + \Omega^{n-q^{-1}}\Omega^{n-q,q}(y^q_{t;i} - \mu^q_{y_{t;i}})\Big]\Bigg\}\\
&= -0.5\Big\{(y^q_{t;i} - \mu^q_{y_{t;i}})^T \Sigma^{q^{-1}}_{y_{t;i}}(y^q_{t;i} - \mu^q_{y_{t;i}}) + (y^{n-q}_{t;i} - h_{t;i})^T\\
&\quad \Omega^{n-q}(y^{n-q}_{t;i} - h_{t;i})\Big\}, \qquad (26)
\end{aligned}
$$

where we have introduced the definition, $h_{t;i} = \mu^{n-q}_{y_{t;i}} - \Omega^{n-q^{-1}}\Omega^{n-q,q}(y^q_{t;i} - \mu^q_{y_{t;i}})$, and where based on Theorem 8.2.1 in [26] we replaced $(\Omega^q - \Omega^{q,n-q}\Omega^{n-q^{-1}}\Omega^{n-q,q})$ with $\Sigma^{q^{-1}}_{y_{t;i}}$. Noting that since the elements in $y^q_{t;i}$ are constants with respect to the variables of integration in Equation (25), the evaluation of the integral in (25) gives,

$$
M_{j\leftarrow i}(y^q_{t;i}\,|z_{t-1;i}) = \frac{\exp\{-0.5[y^q_{t;i} - \mu^q_{y_{t;i}}]^T \Sigma^{q^{-1}}_{y_{t;i}} [y^q_{t;i} - \mu^q_{y_{t;i}}]\}}{(2\pi)^{q/2}|\Sigma^q_{y_{t;i}}|^{1/2}}, \qquad (27)
$$

which proves the theorem. □

According to the proposed fully probabilistic decentralized control and message passing framework, subsystem j will then use the message passed from subsystem i about the subset q of the internal state values of subsystem i as defined in Equation (17), to update its knowledge about its external state variables. In particular, the message received at node j about the subset q of the internal states of subsystem i represents the observation information on the external states of subsystem j,

$$
M_{j\leftarrow i}(y^q_{t;i}\,|z_{t-1;i}) = \mathcal{N}(\mu^q_{y_{t;i}}, \Sigma^q_{y_{t;i}}), \quad \text{where, } x_{t;j} \leftarrow y^q_{t;i}. \qquad (28)
$$

Therefore, the prior information that node j retains about its external signals, $x_{t;j}$ can be fused using Bayes' rule with the new observed information received through the passed message from node i about the subset q of its internal states, thus updating the knowledge of node j about its external signals. This is stated in the following theorem.

Theorem 2: The information provided by the message passed from node i to node j as given in Equation (28) *and the prior information retained by node j about its external states,* $s(x_{t;j}|x_{t-1;j}) = \mathcal{N}(\bar{x}_{t;j}, R_j)$ *can be fused using Bayes' rule, thus yielding the following message passing update of the external states of node j,*

$$
s(x_{t,j,fused}) = \mathcal{N}(\bar{x}_{t,j,fused}, \Sigma_{t;j,fused}) \qquad (29)
$$

where,

$$
\begin{aligned}
\bar{x}_{t,j,fused} &= \bar{x}_{t;j} + K_{t;j}(\mu^q_{y_{t;i}} - \bar{x}_{t;j}), &(30)\\
\Sigma_{t;j,fused} &= R_j - K_{t;j}R_j, &(31)
\end{aligned}
$$

and where,

$$
K_{t;j} = R_j(R_j + \Sigma^q_{y_{t;i}})^{-1}. \qquad (32)
$$

Proof: The new pdf of the external signals in node j that represents the fusion of the information from their prior distribution in node j, $s(x_{t;j}|x_{t-1;j}) = \mathcal{N}(\bar{x}_{t;j}, R_j)$ and the passed probabilistic message from node i, $M_{j\leftarrow i}(y^q_{t;i}\,|z_{t-1;i})$ defined in Equation (28) where $y^q_{t;i}$ is mapped to $x_{t;j}$, can be obtained using Bayes' rule by multiplying the two together,

$$\begin{aligned}
&s(x_{t,j,fused}) \\
&= \exp\{-0.5[x_{t;j}-\mu^q_{y_{t;i}}]^T \Sigma^{q^{-1}}_{y_{t;i}} [x_{t;j}-\mu^q_{y_{t;i}}] \\
&\quad -0.5[x_{t;j}-\bar{x}_{t;j}]^T R_j^{-1}[x_{t;j}-\bar{x}_{t;j}]\}, \\
&= \exp\{-0.5x^T_{t;j}[\Sigma^{q^{-1}}_{y_{t;i}}+R_j^{-1}]x_{t;j}+x^T_{t;j}[\Sigma^{q^{-1}}_{y_{t;i}}\mu^q_{y_{t;i}} \\
&\quad +R_j^{-1}\bar{x}_{t;j}]-0.5\mu^{q^T}_{y_{t;i}}\Sigma^{q^{-1}}_{y_{t;i}}\mu^q_{y_{t;i}}-0.5\bar{x}^T_{t;j}R_j^{-1}\bar{x}_{t;j}\}, \\
&= \exp\{-0.5(x_{t;j}-\bar{x}_{t,j,fused})^T[\Sigma^{q^{-1}}_{y_{t;i}}+R_j^{-1}]^{-1} \\
&\quad (x_{t;j}-\bar{x}_{t,j,fused})+[\Sigma^{q^{-1}}_{y_{t;i}}\mu^q_{y_{t;i}}+R_j^{-1}\bar{x}_{t;j}]^T \\
&\quad \times[\Sigma^{q^{-1}}_{y_{t;i}}+R_j^{-1}]^{-1}[\Sigma^{q^{-1}}_{y_{t;i}}\mu^q_{y_{t;i}}+R_j^{-1}\bar{x}_{t;j}] \\
&\quad -0.5\mu^{q^T}_{y_{t;i}}\Sigma^{q^{-1}}_{y_{t;i}}\mu^q_{y_{t;i}}-0.5\bar{x}^T_{t;j}R_j^{-1}\bar{x}_{t;j}\}, \qquad (33)
\end{aligned}$$

where we have used the following definitions,

$$\bar{x}_{t,j,fused} = [\Sigma^{q^{-1}}_{y_{t;i}}+R_j^{-1}]^{-1}[\Sigma^{q^{-1}}_{y_{t;i}}\mu^q_{y_{t;i}}+R_j^{-1}\bar{x}_{t;j}], \qquad (34)$$

$$\Sigma_{t;j,fused} = [\Sigma^{q^{-1}}_{y_{t;i}}+R_j^{-1}]^{-1} \qquad (35)$$

Applying the Woodbury identity to Equations (34) and (35) and introducing the definition $K_{t;j} = R_j(R_j+\Sigma^q_{y_{t;i}})^{-1}$ yields the results given in Equations (30) and (31). This proves the theorem. □

IV. STOCHASTIC CONTROLLABILITY

This section is concerned with the analysis of the stochastic controllability of the subsystems that are controlled using the decentralized probabilistic message passing and control framework. Here, we will show that most of the developed results in the study of controllability [27], [28] of complex systems are not valid under the proposed framework and better controllability can be achieved through the developed collaborative and cooperative control as proposed in this paper. As will be seen from further analysis, the probabilistic passing of messages from one subsystem to another will facilitate the controllability of the controlled complex system even if it has inaccessible states. To show this, we consider the stochastic representation of the subsystems given in Equations (5) and (6). In particular, consider the stochastic representation of subsystem j,

$$y_{t;j} = A_j z_{t-1;j} + B_j u_{t;j} + \epsilon_{t;j}, \qquad (36)$$

$$x_{t;j} = C_j x_{t-1;j} + \nu_{t;j}, \qquad (37)$$

where $\epsilon_{t;j}$ and $\nu_{t;j}$ are Gaussian noises with zero means and Q_j and R_j covariances respectively as can be inferred from Equation (5). The optimized randomized controller for the subsystems in the proposed decentralized control framework is defined in Equation (9). Based on (9), the stochastic representation of the randomized controller of subsystem j is given by,

$$u_{t;j} = \bar{u}_{t;j} + \eta_{t;j}, \qquad (38)$$

where $\eta_{t;j}$ is Gaussian noise with zero mean and $\Gamma_{t,j}$ covariance. Using Equation (38) in Equation (36) and partitioning the matrix A_j in Equation (36) into the part that is multiplying the internal states, $A_{y_{t-1};j}$ and the part that is multiplying the external signals, $A_{x_{t-1};j}$, Equations (36) and (37) can be rewritten in the following compact form,

$$\underbrace{\begin{bmatrix} y_{t;j} \\ x_{t;j} \end{bmatrix}}_{z_{t;j}} = \underbrace{\begin{bmatrix} A_{y_{t-1};j} & A_{x_{t-1};j} \\ 0 & C_j \end{bmatrix}}_{\tilde{A}_j} \underbrace{\begin{bmatrix} y_{t-1;j} \\ x_{t-1;j} \end{bmatrix}}_{z_{t-1;j}} + \underbrace{\begin{bmatrix} B_j \\ 0 \end{bmatrix}}_{\tilde{B}_j} \bar{u}_{t;j} + \underbrace{\begin{bmatrix} \epsilon_{t;j}+\eta_{t;j} \\ \nu_{t;j} \end{bmatrix}}_{\tilde{\kappa}_{t;j}}. \qquad (39)$$

The above equation represents the stochastic representation of subsystem j before it communicates with its neighbours through the proposed probabilistic message passing. From this equation it is clear that $x_{t;j}$, being treated as external signals, are inaccessible to subsystem j and their values cannot be changed or affected by the local controller designed for subsystem j. Therefore this subsystem cannot be controlled by controlling its internal states only. However, within the proposed decentralized control framework subsystem j, updates its information about its external signals through probabilistic message passing. With this update we will show here that the external states are in fact controllable as their values are controlled in neighbouring subsystems before being passed to the corresponding subsystem.

To proceed with the controllability analysis of the subsystems in the complex network, the external signals to subsystem j are assumed to be received from subsystem i only. This provides no restriction of any kind and subsystem j can still be allowed to be connected to other neighbouring subsystems in the complex network. Using the message that is passed from subsystem i to subsystem j as defined in Equations (29) and (30), the stochastic representation of the dynamics of the external signals of node j as defined in Equation (37) can be rewritten as,

$$x_{t;j} = C_j x_{t-1;j} + K_{t;j}(A^q_i z^q_{t-1;i} + B^q_i \bar{u}^q_{t;i} - C_j x_{t-1;j}) + \tilde{\nu}_{t;j}, \qquad (40)$$

where $\tilde{\nu}_{t;j}$ is a Gaussian noise with zero mean and $\Sigma_{t;j,fused}$ covariance matrix. Using Equations (36), (38) and (40), the stochastic description of node j can be re-expressed as,

$$\begin{aligned}
\underbrace{\begin{bmatrix} y_{t;j} \\ x_{t;j} \end{bmatrix}}_{z_{t;j}} &= \underbrace{\begin{bmatrix} A_{y_{t-1};j} & A_{x_{t-1};j} \\ 0 & C_j - K_{t;j}C_j \end{bmatrix}}_{\bar{A}_{t;j}} \underbrace{\begin{bmatrix} y_{t-1;j} \\ x_{t-1;j} \end{bmatrix}}_{z_{t-1;j}} \\
&\quad + \underbrace{\begin{bmatrix} B_j & 0 \\ 0 & K_{t;j}B^q_i \end{bmatrix}}_{\bar{B}_{t;j}} \underbrace{\begin{bmatrix} \bar{u}_{t;j} \\ \bar{u}_{t;i} \end{bmatrix}}_{\bar{u}_{t;j,aug}} + \underbrace{K_{t;j}\begin{bmatrix} 0 & 0 \\ A^q_{y_{t-1};i} & A^q_{x_{t-1};i} \end{bmatrix}}_{\bar{A}^q_{t;i}} \\
&\quad \times \underbrace{\begin{bmatrix} y^q_{t-1;i} \\ x^q_{t-1;i} \end{bmatrix}}_{z_{t-1;j}} + \underbrace{\begin{bmatrix} \epsilon_{t;j}+\eta_{t;j} \\ \tilde{\nu}_{t;j} \end{bmatrix}}_{\kappa_{t;j}},
\end{aligned}$$

$$z_{t;j} = \bar{A}_{t;j} z_{t-1;j} + \bar{B}_{t;j}\bar{u}_{t;j,aug} + \bar{A}^q_{t;i} z^q_{t-1;i} + \kappa_{t;j}. \qquad (41)$$

A quick inspection to the above equation, shows that the external signals, $x_{t,j}$ to subsystem j are in fact controllable through the control signal of subsystem i, $\bar{u}_{t;i}$ where the values of these external signals to subsystem j can be changed as desired in subsystem i. In addition, the solution of Equation (41) can be easily verified to be given by,

$$\begin{aligned} z_{t+L;j} &= \phi(t+L, t+1)z_{t;j} \\ &\quad + \sum_{n=1}^{L} \phi(t+L, n+t+1)\bar{B}_{t+n;j}\bar{u}_{t+n;j,aug} \\ &\quad + \sum_{n=1}^{L} \phi(t+L, n+t+1)\bar{A}^{q}_{t+n;i}z^{q}_{t+n-1;i} \\ &\quad + \sum_{n=1}^{L} \phi(t+L, n+t+1)\kappa_{t+n;j}, \end{aligned} \tag{42}$$

where,

$$\phi(t+L, t+1) = \begin{cases} A_{t+L;j}A_{t+L-1;j}\dots A_{t+1;j} & \text{if } t+1 < t+L \\ I & \text{if } t+1 > t+L \end{cases} \tag{43}$$

It can be seen from Equation (42), that the mean and covariance of the state of subsystem j at time $t+L$ are given by,

$$\begin{aligned} \bar{z}_{t+L;j} &= \phi(t+L, t+1)z_{t;j} \\ &\quad + \sum_{n=1}^{L} \phi(t+L, n+t+1)\bar{B}_{t+n;j}\bar{u}_{t+n;j,aug} \\ &\quad + \sum_{n=1}^{L} \phi(t+L, n+t+1)\bar{A}^{q}_{t+n;i}z^{q}_{t+n-1;i}, \end{aligned} \tag{44}$$

$$\mathrm{cov}(z_{t+L;j}) = \sum_{n=t+2}^{L} \phi(t+L, n)\mathrm{cov}(\kappa_{t+n;j})\phi^{T}(t+L, n). \tag{45}$$

Therefore, obviously, for complete controllability of the subsystem j defined in Equation (41), the covariance of the stochastic subsystem distribution, $\mathrm{cov}(z_{t+L;j})$ defined in Equation (45) should remain bounded [25]. This condition guarantees that the residual error of subsystem j remains bounded.

V. NUMERICAL SIMULATION

The proposed probabilistic decentralized control and message passing framework is validated in this section on the following stochastic discrete time dynamical system,

$$\begin{aligned} \begin{bmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \\ X_{4,t} \\ X_{5,t} \end{bmatrix} &= \begin{bmatrix} -0.6 & 0.87 & 0 & 0 & 0.50 \\ -0.31 & -0.178 & 0 & 0 & 1.2 \\ 0 & 1.0 & 0 & 0 & 0 \\ 1 & 0.064 & 0 & 0 & 0 \\ 0.062 & -0.1 & 0 & 0.1 & -0.236 \end{bmatrix} \\ &\quad \times \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \\ X_{3,t-1} \\ X_{4,t-1} \\ X_{5,t-1} \end{bmatrix} + \begin{bmatrix} -0.13 & 0.035 \\ -0.012 & -0.025 \\ 0 & 0 \\ 0 & 0 \\ 0.002 & 0.008 \end{bmatrix} \begin{bmatrix} U_{1,t} \\ U_{2,t} \end{bmatrix} + \kappa_t, \end{aligned} \tag{46}$$

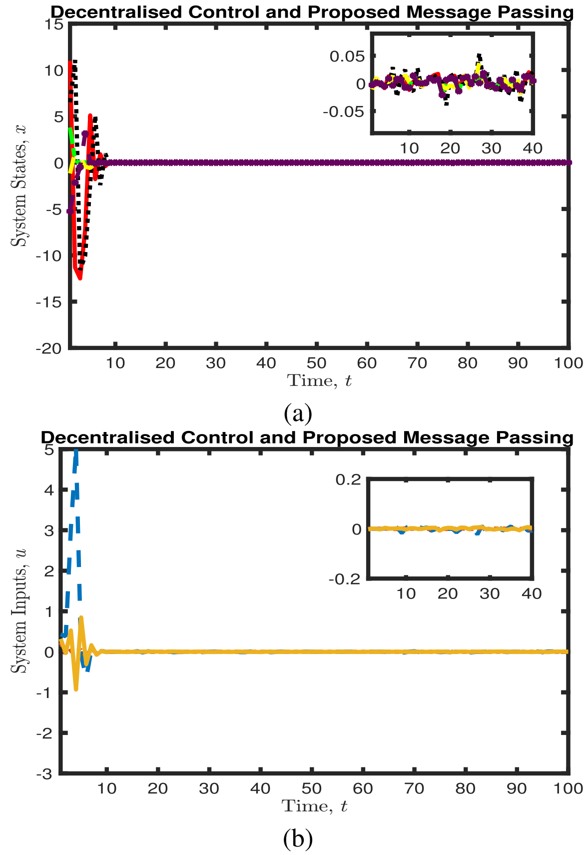

FIGURE 1. **The state and control inputs of the controlled system as a result of using decentralized control and the proposed message passing: (a) the states of the system derived using decentralized control and the proposed message passing. Red solid line is system state 1, green dash-dot line is state 2, purple with asterisk line is state 3, black dotted line is state 4, and dashed yellow line is state 5 (b) the control inputs of the system using decentralized control and the proposed message passing. Blue dashed line is the control input of subsystem *i*, red dotted line is the control input of subsystem *j*, and Yellow solid line is the control input of subsystem *k*.**

where κ_t is Gaussian noise with zero mean and covariance matrix equal to $0.01 I_{5\times5}$, I is the identity matrix, and where X and U refer to the state and control input respectively of the global complex system. Three sets of experiments were then conducted for comparison. The first set considers the globally centralized FPD randomized control [24] of the dynamical system (46), the second considers the decentralized FPD where the subsystems communicate by passing information about the parameters of their models [22], while the third considers the decentralized control of the system (46) according to the proposed probabilistic message passing. In these experiments, the high level control aim is to return the whole system state of 5 nodes from its initial value $x_0 = \begin{bmatrix} 10.989 & 5.2551 & 3.7985 & 6.5140 & -1.1645 \end{bmatrix}^T$ to the origin or a state close to the origin. In addition all pdfs of the systems/subsystems dynamics are assumed to be unknown apriori, therefore they are estimated online as discussed in [23].

In the decentralized control experiments, the control task is designated by three separated subsystems to be controlled

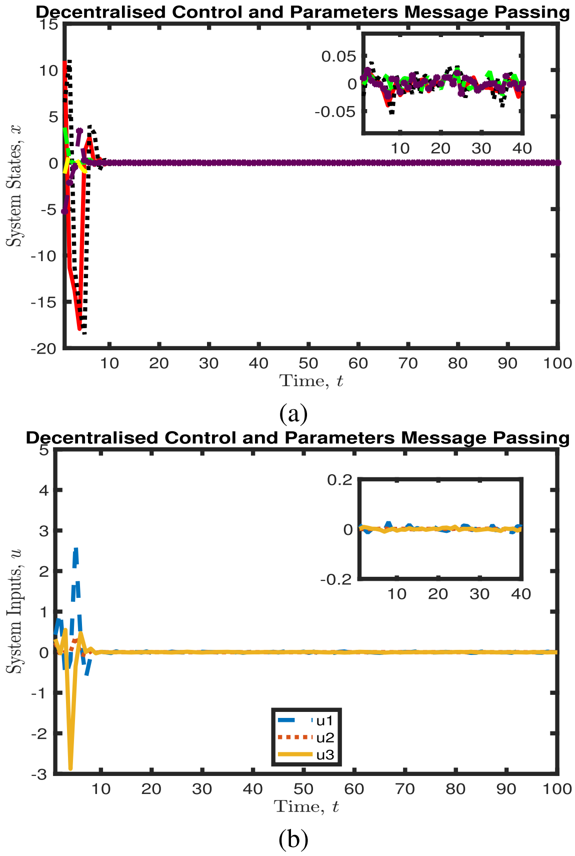

FIGURE 2. **The state and control inputs of the controlled system as a result of using decentralized control and parameters message passing: (a) the states of the system derived using decentralized control and parameters message passing. Red solid line is system state 1, green dash-dot line is state 2, purple with asterisk line is state 3, black dotted line is state 4, and dashed yellow line is state 5 (b) the control inputs of the system using decentralized control and parameters message passing. Blue dashed line is the control input of subsystem *i*, red dotted line is the control input of subsystem *j*, and Yellow solid line is the control input of subsystem *k*.**

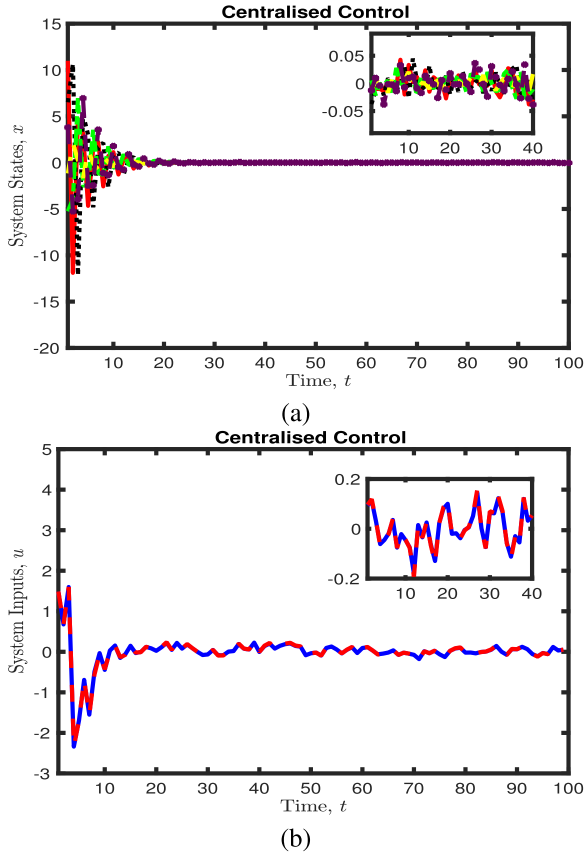

FIGURE 3. **The state and control inputs of the controlled system as a result of using centralized control: (a) the states of the system derived using centralized control. Red solid line is system state 1, green dash-dot line is state 2, purple with asterisk line is state 3, black dotted line is state 4, and dashed yellow line is state 5 (b) the control inputs of the system using centralized control. Blue solid line is control input 1, and red dashed line is control input 2.**

by local knowledge where each subsystem is responsible for controlling a different set of states of the global system (46). To clarify, node i takes $X_{1,t} = y_{1,t;i}$, $X_{4,t} = y_{2,t;i}$ as internal states, and $X_{2,t} = x_{1,t;i}$, $X_{5,t} = x_{2,t;i}$ as external states. Hence, the system model of node i is described by,

$$\begin{aligned} s(y_{t;i}|u_{t;i}, z_{t-1;i}) &= N(A_i z_{t-1;i}, Q_i), \\ \text{where } A_i &= \begin{bmatrix} -0.6000 & 0 & 0.8700 & 0.5000 \\ 1.0000 & 0 & 0.0640 & 0 \end{bmatrix}, \\ B_i &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ s(x_{t;i}|x_{t-1;i}) &= N(C_i x_{t-1;i}, R_i), \\ \text{where } C_i &= \begin{bmatrix} 0 & 0 & c_{1;i} & c_{2;i} \\ 0 & 0 & c_{3;i} & c_{4;i} \end{bmatrix}. \end{aligned} \tag{47}$$

Node j takes $X_{t,3} = y_{1,t;j}$, and $X_{t,5} = y_{2,t;j}$ as internal states, and $X_{t,4} = x_{1,t;j}$ as external state. Hence the system model of node j is described by,

$$\begin{aligned} s(y_{t;j}|u_{t;j}, z_{t-1;j}) &= N(A_j z_{t-1;j}, Q_j), \\ \text{where } A_j &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -0.2360 & 0.100 \end{bmatrix}, \\ B_j &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ s(x_{t;j}|x_{t-1;j}) &= N(C_j x_{t-1;j}, R_j), \\ \text{where } C_j &= \begin{bmatrix} 0 & 0 & c_{1;j} \end{bmatrix}. \end{aligned} \tag{48}$$

Node k takes $X_{t,2} = y_{1,t;k}$ as internal state, and $X_{t,1} = x_{1,t;k}$ as external state. Hence the system model of node k is described by,

$$\begin{aligned} s(y_{t;k}|u_{t;k}, z_{t-1;k}) &= N(A_k z_{t-1;k}, Q_k), \\ \text{where } A_k &= \begin{bmatrix} -0.178 & -0.31 \end{bmatrix}, \\ B_k &= \begin{bmatrix} 1 \end{bmatrix}. \\ s(x_{t;k}|x_{t-1;k}) &= N(C_k x_{t-1;k}, R_k), \\ C_k &= \begin{bmatrix} 0 & c_{1;k} \end{bmatrix}. \end{aligned} \tag{49}$$

These three subsystems are nonidentical as can be seen from their parameters and they also have different lengths and different number of external signals.

The states of the system and the control signals that result from the decentralized control approach and the proposed probabilistic message passing are shown in Figures 1, (a) and (b) respectively. From these figures, it can be clearly seen that the controlled systems are globally synchronised and that the designed probabilistic control and message passing approach has been effective in reconstructing the global desired state using only decentralized local knowledge.

The states of the system and the control signals as a result of the decentralized control approach and the message passing of the parameters of the subsystems models are shown in Figures 2, (a) and (b) respectively. The figures show that the local controllers exhibit larger transient overshoot compared to the decentralized controllers with the proposed probabilistic message passing. This is expected because in this method, the subsystems communicate information about the parameters of their models which will not have converged in this transient period.

The third experiment considers the control of system (46) using the fully probabilistic control design method [24] where the 5 states of the system are controlled using two control inputs as specified in Equation (46). The resulting optimized states of the system and control input are shown in Figure 3, (a) and (b) respectively. As can be concluded from these figures, the centralized controller is again capable of bringing all the states of the system to the required zero value, but it shows higher fluctuations in the transient and steady state periods compared to the decentralized controller.

VI. CONCLUSION

This paper developed a new probabilistic message passing framework for a class of complex and large dynamical systems that are controlled decentrally by controlling their individual subsystems components. The proposed probabilistic message passing scheme for the important decentralized control problems is the main contribution of this paper. It uses the probabilistic inference method to evaluate the marginal distributions of the states to be passed from one subsystem to another keeping the receiving subsystems informed about their surrounding environment. Following the successful development of this message passing scheme, the stochastic controllability of the subsystems constituting a complex system is analyzed. It is shown here that because of the message passing between the subsystems, the subsystems states remain controllable even if they are inaccessible in that subsystem. The developed message passing method is not constrained by the assumption of the homogeneity of the individual subsystems and they do not require them to have identical lengths thus, extending the results of many of the existing methods. Finally, the theoretical development of the proposed message passing framework is demonstrated on a stochastic dynamical system consisting of five nodes and its effectiveness is proved.

The proposed framework is readily applicable to a wide range of application areas including biological networks, autonomous unmanned vehicles, animal cooperative aggregation and flocking, and societal networks. It is also suitable for industry 4.0 [29] and can be applied to a broad range of production processes and complex cyber-physical systems. Future work will consider the extension of the decentralized randomized control solution to take into considerations delays in the control input and state of the stochastic system. Although it is desirable but hard, the application of the proposed solution to practical real world problems will also be sought.

REFERENCES

[1] A. J. Whalen, S. N. Brennan, T. D. Sauer, and S. J. Schiff, "Observability and controllability of nonlinear networks: The role of symmetry," *Phys. Rev. X*, vol. 5, no. 1, 2015, Art. no. 011005.

[2] M. Arcak, "Diagonal stability on cactus graphs and application to network stability analysis," *IEEE Trans. Autom. Control*, vol. 56, no. 12, pp. 2766–2777, Dec. 2011.

[3] A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou, "Synchronization in complex networks," *Phys. Rep.*, vol. 469, no. 3, pp. 93–153, Dec. 2008.

[4] B. van den Broek, W. Wiegerinck, and B. Kappen, "Graphical model inference in optimal control of stochastic multi-agent systems," *J. Artif. Intell. Res.*, vol. 32, no. 1, pp. 95–122, May 2008.

[5] W. Yu, P. DeLellis, G. Chen, M. di Bernardo, and J. Kurths, "Distributed adaptive control of synchronization in complex networks," *IEEE Trans. Autom. Control*, vol. 57, no. 8, pp. 2153–2158, Aug. 2012.

[6] W. Yu, J. Lü, X. Yu, and G. Chen, "Distributed adaptive control for synchronization in directed complex networks," *SIAM J. Control Optim.*, vol. 53, no. 5, pp. 2980–3005, 2015.

[7] F. Sorrentino, M. D. Bernardo, F. Garofalo, and G. Chen, "Controllability of complex networks via pinning," *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top.*, vol. 75, no. 4, 2007, Art. no. 046103.

[8] A. Das and F. L. Lewis, "Distributed adaptive control for synchronization of unknown nonlinear networked systems," *Automatica*, vol. 46, no. 12, pp. 2014–2021, 2010.

[9] R. Yu, H. Zhang, Z. Wang, and J. Wang, "Synchronization of complex dynamical networks via pinning scheme design under hybrid topologies," *Neurocomputing*, vol. 214, pp. 210–217, Nov. 2016.

[10] T. Keviczky, F. Borrelli, and G. J. Balas, "A study on decentralized receding horizon control for decoupled systems," in *Proc. Amer. Control Conf.*, vol. 1. Boston, MA, USA, USA, Jun./Jul. 2004, pp. 4921–4926.

[11] L. Magni and R. Scattolini, "Stabilizing decentralized model predictive control of nonlinear systems," *Automatica*, vol. 42, no. 7, pp. 1231–1236, 2006.

[12] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, no. 1, pp. 215–233, Jan. 2007.

[13] G. S. Seyboth, W. Ren, and F. Allgower, "Cooperative control of linear multi-agent systems via distributed output regulation and transient synchronization," *Automatica*, vol. 68, no. 1, pp. 132–139, 2016.

[14] Z. Ding and Z. Li, "Distributed adaptive consensus control of nonlinear output feedback systems on directed graphs," *Automatica*, vol. 72, no. 1, pp. 46–52, 2016.

[15] M. Mazouchi, M.-B. N. Sistani, S. K. H. Sani, F. Tatari, and H. Modares, "Observer-based adaptive optimal output containment control problem of linear heterogeneous multi-agent systems with relative output measurements," *Adapt. Control Signal Process.*, vol. 33, no. 2, pp. 262–284, 2018.

[16] H. Haghshenas, M. A. Badamchizadeh, and M. Baradarannia, "Containment control of heterogeneous linear multi-agent systems," *Automatica*, vol. 54, pp. 210–216, Apr. 2015.

[17] S. Baldi and P. Frasca, "Adaptive synchronization of unknown heterogeneous agents: An adaptive virtual model reference approach," *J. Franklin Inst.*, vol. 356, no. 2, pp. 935–955, 2018.

[18] M. R. Rosa, "Adaptive synchronization of unknown heterogeneous agents: An adaptive virtual model reference approach," *Machines*, vol. 6, no. 1, p. 7, 2018.
[19] R. Yu, H. Zhang, Z. Wang, and Y. Liu, "Synchronization criterion of complex networks with time-delay under mixed topologies," *Neurocomputing*, vol. 295, no. 21, pp. 8–16, 2018.
[20] H. L. Trentelman, K. Takaba, and N. Monshizadeh, "Robust synchronization of uncertain linear multi-agent systems," *IEEE Trans. Autom. Control*, vol. 58, no. 6, pp. 1511–1523, Jun. 2013.
[21] R. Herzallah and M. Kárný, "Towards probabilistic synchronisation of local controllers," *Int. J. Syst. Sci.*, vol. 48, no. 3, pp. 604–615, 2017.
[22] R. Herzallah, "A fully probabilistic decentralised control design for complex stochastic systems," in *Proc. IEEE 15th Int. Conf. Control Automat. (ICCA)*, vol. 1. Edinburgh, Scotland, Jul. 2019, pp. 806–811.
[23] R. Herzallah and M. Kárný, "Fully probabilistic control design in an adaptive critic framework," *Neural Netw.*, vol. 24, no. 11, pp. 1128–1135, 2011.
[24] M. Karny, "Towards fully probabilistic control design," *Automatica*, vol. 32, no. 12, pp. 1719–1722, 1996.
[25] R. Herzallah, "Probabilistic synchronisation of pinning control," *Int. J. Control*, vol. 85, no. 11, pp. 1708–1716, 2012.
[26] F. A. Graybill, *Matrices With Applications in Statistics*, vol. 1. Massachusetts, CA, USA: Cengage Learning, 2001.
[27] Y. Y. Liu and A.-L. Barabási, "Control principles of complex systems," *Rev. Mod. Phys.*, vol. 88, no. 3, 2016, Art. no. 035006.
[28] Y. Y. Liu, J. J. Slotine, and A. L. Barabási, "Controllability of complex networks," *Nature*, vol. 473, no. 3, pp. 167–173, 2011.
[29] M. Brettel, N. Friederichsen, M. Keller, and M. Rosenberg, "How virtualization, decentralization and network building change the manufacturing landscape: An industry 4.0 perspective," *Int. J. Mech. Ind. Sci. Eng.*, vol. 8, no. 1, pp. 37–44, 2014.

RANDA HERZALLAH received the B.Sc. degree in industrial engineering and the M.Sc. degree in industrial engineering/manufacturing and design from Jordan University, Jordan, and the Ph.D. degree in control theory from Aston University, Birmingham, U.K. She was an Associate Professor with Al-Ahliyya Amman University and a Professor with Al-Balqa Applied University, Jordan. She is currently a Senior Lecturer with the Faculty of Engineering and Applied Science, Aston University. Her general research focus is on the general topic of systems modeling, optimization, design and control. Her current research interests include nonlinear control theory, probabilistic and stochastic control, intelligent control, pinning control, adaptive control, distributed and decentralized control, optimal control, signal processing, time-delay systems, and networked and complex systems control with applications to power grids, manufacturing systems, and biomedical signals.