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Mersaid Aripov
National University of Uzbekistan, mirsaidaripov@mail.ru

Askar Mukimov
National University of Uzbekistan, mukimov_askar@mail.ru

ASYMPTOTICS OF SOLUTIONS AND NUMERICAL SIMULATION OF THE NONLINEAR HEAT CONDUCTIVITY PROBLEM WITH ABSORPTION AND VARIABLE DENSITY

Aripov M., Mukimov A.
National University of Uzbekistan, Tashkent, Uzbekistan
e-mail: mirsaidaripov@mail.ru, mukimov_askar@mail.ru

Abstract

In the present work, the asymptotic behavior of the solutions of the nonlinear variable-density thermal conductivity problem with absorption is obtained. The critical value parameter is considered. The resulting asymptotics was used as an initial approximation, numerical calculations were performed. As a difference scheme, a three-layer difference scheme was used, which, unlike a two-layer scheme, has greater accuracy.

Keywords: nonlinear heat equation, variable density, asymptotics of solutions, critical value of the parameter, upper solution, lower solution, principle of comparison of solutions.

Mathematics Subject Classification (2010): 35K61, 65N06.

Introduction

As is well known for the numerical computation of a nonlinear problem, the choice of the initial approximation is essential, which preserves the properties of the final speed of propagation, spatial localization, bounded and blow-up solutions, which guarantees convergence with a given accuracy to the solution of the problem with minimum number of iterations.

1 Formulation of the problem

Consider the following Cauchy problem in the region $Q = [0, \infty) \times R^N$

$$\begin{array}{l} \frac{\partial u}{\partial t} = \nabla\left(|x|^n u^{m-1} \left|\nabla u^k\right|^{p-2} \nabla u\right) - u^\beta, \\ |x|^n u^{m-1} \left|\nabla u^k\right|^{p-2} \nabla u|_{x \to +\infty} = 0, \end{array} \tag{1}$$

$$u(0, x) = u_0(x),\ x \in R^N. \tag{2}$$

t and x are, respectively, the temporal and spatial coordinates where $m \geq 1, k \geq 1, p \geq 2, n \geq 1$ given numerical parameters, characterizing the nonlinear medium, $\nabla(\cdot) = \underset{x}{grad}(\cdot)$.

This problem describes the processes of nonlinear filtering of the diffusion thermal conductivity, when the thermal conductivity coefficient is a power function of the derivative in the presence of absorption.

Under some suitable assumptions, the existence, uniqueness and regularity of a weak solution to the Cauchy problem (1) - (2) and their variants have been extensively investigated by many authors (see [1]-[3] and the references therein).

If the initial value $u(x, 0) = u_0(x)$ is respectively smooth, there are many papers on the solvability of the Cauchy problem (1)-(2), we can refer to Wu-Zhao [4], Gmira [5], Yang-Zhao [6], Zhao [7]-[9], Zhao-Yuan [10], Dibenedetto-Friedman [11], LiHia [12], Dibenedetto-Herrero [13] and links to them for details.

As is known for obtaining approximate solutions by numerical methods, the main thing is a correctly selected initial approximation. Namely, it guarantees convergence to the solution of the problem with minimal iterations.

The authors of work [14] investigated the properties of spatial localization, existence and non-existence of global solutions for problem (1) - (2) for $k = 1$. Where the density function has the form

$$|x|^{-n}, \frac{1}{(1+|x|)^n}.$$

In [15] given an asymptotic analysis of the behavior of blow-up solutions of the following equation

$$\rho(x)u_t = (u^m)_{xx} \ \ in \ \ Q = R \times R_+,$$

where

$$\rho(x) = |x|^{-\alpha}, e^{-x^2}, e^{-x}.$$

In [16] authors considered a nonlinear parabolic equation with a source and an inhomogeneous density of the following form:

$$\rho(x)\frac{\partial u}{\partial t} = div\left(u^{m-1}|Du|^{p-2}Du\right) + u^{\beta},$$

where

$$\rho(x) = |x|^{-n} \ \ or \ \ \rho(x) = (1+|x|)^{-n}, \ n \geq 0.$$

They found conditions which the solution of the Cauchy problem explodes in a finite time at critical parameters $\beta_* = m + p - 2 + p/N$,

$$n^* = \begin{cases} \frac{N(m+p-3)+p}{m+p-2} & where \ \ p < N, \\ p & where \ \ p \geq N. \end{cases}$$

In [17] authors gave estimates of the blow-up speed of the solution of the following nonlinear parabolic equation

$$u_t - div(u^{m-1}|Du|^{\lambda-1}Du) = f(x)u^p \ \ in \ \ Q_T = \Omega x(0,T), 0 < T < \infty,$$

where $f(x)$ the radial function and for simplicity has the following form $f(x) = |x|^{-\alpha}$, $-\infty < \alpha < \min(N, \lambda + 1)$.

Here are some of them

$$u(x,t) \leq \gamma(T-t)^{-B},$$

where $\gamma > 0$, $|x| < (T-t)^{1/H}/2$, $0 < m+\lambda-1 < p < m+\lambda-1+(\lambda+1-\alpha)/N$,

$$H = \frac{(p-1)(\lambda+1)-\alpha(m+\lambda-2)}{p-m-\lambda+1},$$
$$B = \frac{\lambda+1-\alpha}{(p-1)(\lambda+1)-\alpha(m+\lambda-2)}$$

(H gives the correct space-time scaling near the blow-up time, B gives the blow-up rate).

In [18] established conditions of norm of solution with critical exponent $q^* = \frac{K+N}{Nv+1}$ for following Cauchy problem

$$u_t - div(u^{m-1}|Du|^{\lambda-1}Du) = -\varepsilon|Du^v|^{q_*} + \delta u^p \;\; in \;\; R^N \times (0,T),$$

$$u(x,0) = u_0(x) \geq 0.$$

Here are some of them

$$if\ q < q^*\ then\ \ \|u(t)\| \leq \gamma t^{-A},$$
$$if\ q = q^*\ then\ \ \|u(t)\| \leq \gamma[\ln t]^{-\frac{1}{vq-1}}.$$

$H = (\lambda+1)(vq-1) - q(m+\lambda-2),\ K = N(m+\lambda-2)+\lambda+1,\ \delta = 0, \varepsilon = 1, m+\lambda-2 > 0.$

In [19] the long time asymptotic of the solution were established for the following problem with critical parameter $p_c = 1 + 2m/N$

$$u_t = -(-\Delta)^m - |u|^{p-1}u \;\; in \;\; area \;\; R^N \times R_+,$$

$$u(x,0) = u_0(x).$$

Following asymptotic

$$u(x,t) = \pm C_0 t^{-N/2m}(\ln t)^{-N/(2m+Q)}\left[f(\frac{x}{t^{1/2m}}) + o(1)\right],$$

where f is the rescaled kernel of the fundamental solution of the linear parabolic operator.

In [20] were obtained an estimate and the asymptotics of the solutions of self-similar equations for the problem (1) - (2) for $n = 0$ and without the absorption term.

Authors of the work [21] established asymptotics of the solutions and gave global solvability for the following problem

$$\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial u^m}{\partial x}\right|^{p-2}\frac{\partial u^m}{\partial x}\right), \qquad (x,t) \in R_+ \times (0, +\infty),$$

$$u(0,x) = u_0(x),\ x \in R^N.$$

The authors of the work [22] obtained the asymptotic behavior of problem (1) - (2) without variable density with an additional source term instead of absorption for the second critical exponent of the Fujita type. The asymptotics of self-similar solutions has the following form

$$\overline{f}(\xi) = c(a + \xi^{\frac{p}{p-1}})^{-\frac{p-1}{\beta-(k(p-2)+l+m-1)}}.$$

Where the value of constant c was found for different conditions relative to β and numerical calculations were performed based on the above qualitative studies.

In [23] authors were established the long time asymptotic of solutions for the critical value of parameter for problem (1) - (2) in case $m = 1$, $p = 2$, $n = 0$. They considered following semi-linear parabolic equation

$$u_t - \Delta u + u^\beta = 0, t > 0, \tag{3}$$

$$u(0, x) = u_0(x) \geq 0. \tag{4}$$

$$\Delta = \sum\nolimits_{i=1}^{N} \partial^2/\partial x_i^2, \beta = 1 + \frac{2}{N}.$$

The solution of problem (3) - (4) is "infinity" energy. The initial data is

$$u_0(x) = o\{\exp(-\gamma|x|^2)\}, x \to +\infty, \gamma > 0.$$

They proved that for problem (3) - (4) the long time asymptotic of the solutions is the following approximate self-similar solution

$$u(t, x) = [(T + t)\ln(T + t)]^{-\frac{N}{2}} g_* \left(\frac{x}{(T + t)^{\frac{1}{2}}} \right). \tag{5}$$

For g_* function upper and lower bounds were obtained $A \exp\left(-\frac{|\xi|^2}{4}\right) \leq g_*(\xi) \leq H \exp\left(-\frac{|\xi|^2}{4}\right)$,

$$|\xi|^2 = \frac{|x|^2}{T + t},$$

where A, H are constants.

For $\beta \neq 1 + \frac{2}{N}$, the approximate self-similar differs from (5), which means that for critical values the asymptotic of the solutions will change.

In [22] was considered following nonlinear heat equation with absorption

$$u_t = \Delta(u^{\sigma+1}) - u^\beta \quad in \;\; area \;\; Q = R^N \times (0, \infty), \tag{6}$$

$$u(x, 0) = u_0(x) \quad for \;\; x \in R^N. \tag{7}$$

Authors established the long time asymptotic of the solution for the critical exponent $\beta = \beta_* = \sigma + 1 + 2/N$.

The following asymptotic

$$u(x, t) = ((T + t)\ln(T + t))^{-k} F(\xi; a). \tag{8}$$

$$k = N/(N\sigma + 2), F(\xi; a) = C_0(a^2 - |\xi|^2)_+^{1/\sigma},$$
$$C_0 = [k\sigma/2N(\sigma+1)]^{1/\sigma}, T > 1,$$
$$\xi = x(T+t)^{-k/N}\ln(T+t)^{k\sigma/2},$$

where the value of the numerical parameter a is determined from the law of energy conservation

$$\int w_M(x,t)dx = \int F(\xi; a)d\xi = M,$$

$M = C_1 a^{N/k\sigma}$, where $C_1 = \pi^{N/2} C_0 B(N/2, 1 + 1/\sigma)/(N/2)$.

B and Γ is beta and gamma of Euler function.

They proved that solution (8) is the long time asymptotic of the solution to problem (6) - (7) by constructing lower and upper solutions. The following lower and upper solution with variable a

$$((T+t)\log(T+t))^{-k}F(\xi; a_-) \leq u(x,t) \leq ((T+t)\log(T+t))^{-k}F(\xi; a_-),$$

$$0 < a_- < a_+.$$

The main target of this paper is to obtain the main part of the asymptotic behavior of the solutions of problem (1) - (2), which can be calculated after bringing equation (1) - (2) into a self-similar form convenient for research. Based on the asymptotics of the solutions, suitable initial approximations for the iterative process are proposed depending on the value of the numerical parameters.

2 Reducing the equation to a self-similar form

Consider the solution of equation (1) of the following form

$$u(t,x) = \overline{u}(t)\varpi(\tau(t), x), \tag{9}$$

where

$$\overline{u}(t) = (T + (\beta-1)t)^{\frac{1}{1-\beta}}, \varpi(\tau, x) = f(\xi), \xi = \frac{\varphi(|x|)}{\tau^{1/p}}.$$

Put (9) in (1) and select $\tau(t)$

$$\tau(t) = \frac{[T + (\beta-1)t]^{\frac{\beta-(m+k(p-2))}{\beta-1}}}{\beta - (m + k(p-2))}.$$

For $\omega(\tau(t), x)$ we get following equation

$$\frac{\partial\varpi}{\partial\tau} = \nabla\left(|x|^n \varpi^{m-1}\left|\nabla\varpi^k\right|^{p-2}\nabla\varpi\right) - \frac{(\varpi^\beta - \varpi)}{\tau(m + k(p-2) - \beta)}. \tag{10}$$

Now put

$$\varpi(\tau, x) = f(\xi), \xi = \frac{\varphi(|x|)}{\tau^{1/p}},$$

$$\varphi(x) = \frac{p}{p-n}|x|^{\frac{p-n}{p}} \quad if\ p > n,$$

equation (10) transforms to following self-similar equation

$$\begin{array}{l} \xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}f^{m-1}\left|\frac{df^k}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right) + \frac{\xi}{p}\frac{df}{d\xi} + \frac{(f^\beta+f)}{\beta-(m+k(p-2)} = 0, \\ s = pN/(p-n),\ p > n. \end{array} \tag{11}$$

A very interesting point is the behavior of the function $\varphi(x)$. In the case p = n, the function has the form:

$$\varphi(x) = \ln|x|\,.$$

For $\omega(\tau(t), x)$ equation has the following form

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial\varphi}\left(w^{m-1}\left|\frac{\partial w}{\partial\varphi}^k\right|^{p-2}\frac{\partial w}{\partial\varphi}\right) + \left(w^{m-1}\left|\frac{\partial w}{\partial\varphi}^k\right|^{p-2}\frac{\partial w}{\partial\varphi}\right) - w^\beta.$$

3 Asymptotics of solutions of self-similar problems

The next stage of the research is to study the asymptotics of self-similar solutions to problem (11), which allowed to obtain numerical results.

We show that the function $\overline{f}(\xi) = \left(a - b\xi^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{m+k(p-2)-1}}$ which obtained on the based of the standard equation method [24] is the asymptotic behavior of the self-similar problem (11).

Theorem 1. *A finite solution of the problem (1) - (2) has an the following asymptotic behavior*

$$f(\xi)\sim\bar{f}(\xi)\,.$$

Proof. We will seek a solution to equation (11) in the following form

$$f = \bar{f}(\xi)\,w(\eta), \tag{12}$$

where

$$\eta = -\ln\left(a - b\xi^{\frac{p}{p-1}}\right),$$

then taking into account (12) we obtain

$$f'(\xi) = -bp\xi^{\frac{1}{p-1}}\left(a - b\xi^{\frac{p}{p-1}}\right)^{\frac{p-1}{m+k(p-2)-1}-1}\left(\frac{w}{m+k(p-2)-1} - \frac{w'}{p-1}\right),$$

$$(f^m)'(\xi) = -bmp\xi^{\frac{1}{p-1}}\left(a - b\xi^{\frac{p}{p-1}}\right)^{\frac{m(p-1)}{m+k(p-2)-1}-1}w^{m-1}\left(\frac{w}{m+k(p-2)-1} - \frac{w'}{p-1}\right),$$

$$\left(f^k\right)'(\xi) = -bkp\xi^{\frac{1}{p-1}}\left(a - b\xi^{\frac{p}{p-1}}\right)^{\frac{k(p-1)}{m+k(p-2)-1}-1} w^{k-1}\left(\frac{w}{m+k(p-2)-1} - \frac{w'}{p-1}\right),$$

substituting (12) into (11) for $w(\eta)$ we obtain the following equation

$$\left(\frac{z_1(\eta)}{z_2(\eta)}\frac{(p-1)}{bp} - \frac{p-1}{m+k(p-2)-1}\right) w^{(k-1)(p-2)+m-1}\left|\frac{w}{m+k(p-2)-1} - \frac{w'}{p-1}\right|^{p-2}\left(\frac{w}{r} - \frac{w'}{p-1}\right) +$$
$$+\frac{d}{d\eta}w^{(k-1)(p-2)+m-1}\left|\frac{w}{m+k(p-2)-1} - \frac{w'}{p-1}\right|^{p-2}\left(\frac{w}{r} - \frac{w'}{p-1}\right) + \frac{p-1}{b^{p-1}p^p k^{p-2}}\left(\frac{w}{m+k(p-2)-1} - \frac{w'}{p-1}\right) +$$
$$+\frac{(p-1)\omega}{b^p p^p k^{p-2}}\frac{z_1(\eta)}{z_2(\eta)}\frac{\omega^{\beta-1} e^{\frac{\beta(p-1)-p+m+kp-2k}{m+k(p-2)-1}+1}}{2k-m-k+\beta}, \tag{13}$$

where

$$z_1(\eta) = e^{-\eta}, z_2(\eta) = \left(a - e^{-\eta}\right)/b.$$

Note that the study of the solutions of the last equation is equivalent to the study of those solutions of equation (1), each of which in some interval $[\eta_0, +\infty)$ satisfies inequalities:

$$w(\eta) > 0, \frac{w(\eta)}{m+k(p-2)-1} - \frac{w'(\eta)}{p-1} \neq 0.$$

First we show that the solutions of $w(\eta)$ of equation (13) have a finite limit of w_0 at $\eta \to +\infty$. Introduce the notation

$$x(\eta) = w^{(k-1)(p-2)+m-1}\left|\frac{w}{m+k(p-2)-1} - \frac{w'}{p-1}\right|^{p-2}\left(\frac{w}{r} - \frac{w'}{p-1}\right).$$

Then equation (13) has the following form

$$x' = -\left(\frac{z_1(\eta)}{z_2(\eta)}\frac{(p-1)}{bp} - \frac{p-1}{m+k(p-2)-1}\right)x - \frac{p-1}{b^{p-1}p^p k^{p-2}}\left(\frac{w}{m+k(p-2)-1} - \frac{w'}{p-1}\right) -$$
$$-\frac{(p-1)\omega}{b^p p^p k^{p-2}}\frac{z_1(\eta)}{z_2(\eta)}\frac{\omega^{\beta-1} e^{\frac{\beta(p-1)-p+m+kp-2k}{m+k(p-2)-1}+1}}{2k-m-k+\beta}.$$

To analyze the solutions of the last equation we introduce an auxiliary function

$$\phi(\tau,\eta) = -\left(\frac{z_1(\eta)}{z_2(\eta)}\frac{(p-1)}{bp} - \frac{p-1}{m+k(p-2)-1}\right)\tau - \frac{p-1}{b^{p-1}p^p k^{p-2}}\left(\frac{w}{m+k(p-2)-1} - \frac{w'}{p-1}\right) -$$
$$-\frac{(p-1)\omega}{b^p p^p k^{p-2}}\frac{z_1(\eta)}{z_2(\eta)}\frac{\omega^{\beta-1} e^{\frac{\beta(p-1)-p+m+kp-2k}{m+k(p-2)-1}+1}}{2k-m-k+\beta}.$$

Where τ is a real number. From here it is easy to see that for each value τ the function $\phi(\tau,\eta)$ retains the sign on some interval $[\eta_1, +\infty) \subset [\eta_0, +\infty)$ and for all $\eta \in [\eta_1, +\infty)$ one of the inequalities is satisfied

$$x'(\eta) > 0, x'(\eta) < 0.$$

And so for function $x(\eta)$ exist is a limit at $\eta \in [\eta_1, +\infty)$. The expression for $x(\eta)$ follows that

$$\lim_{\eta\to\infty} x'(\eta) = -\lim_{\eta\to\infty}\left\{\left(\frac{z_1(\eta)}{z_2(\eta)}\frac{(p-1)}{bp} - \frac{p-1}{m+k(p-2)-1}\right)x - \frac{p-1}{b^{p-1}p^p k^{p-2}}\left(\frac{w}{m+k(p-2)-1} - \frac{w'}{p-1}\right) -\right.$$
$$\left.-\frac{(p-1)\omega}{b^p p^p k^{p-2}}\frac{z_1(\eta)}{z_2(\eta)}\frac{\omega^{\beta-1} e^{\frac{\beta(p-1)-p+m+kp-2k}{m+k(p-2)-1}+1}}{2k-m-k+\beta}\right\} = 0. \tag{14}$$

Let us now make the limit transition.

$$\lim_{\eta\to+\infty} z_1(\eta) \to 0,\ \lim_{\eta\to+\infty} \phi_2(\eta) \to \frac{a}{b},\ w' = 0.$$

Then from (14) for we obtain the following algebraic equation

$$w^{(k-1)(p-2)+m-1+p-2}\frac{1}{(m+k(p-2)-1)^{p-1}} - \frac{1}{b^{p-1}p^p k^{p-2}} = 0,$$

where

$$b = (m+k(p-2)-1)\left(\frac{1}{p^p k^{p-2}}\right)^{\frac{1}{p-1}},$$

the given expression for b is $w = 1$ and by virtue of (12)

$$f(\xi) \sim \bar{f}(\xi).$$

The theorem is proved.

4 Numerical analysis of solutions

It is shown in [25]-[28] that conservative difference schemes of through-counting obtained by the balance method can be successfully used for approximate calculation of generalized solutions of nonlinear heat equation. As noted in [28], three-layer implicit schemes are more effective in nonstationary nonlinear heat conduction problems with discontinuities at the temperature wave front or sharply varying initial data, since the presence of temperature waves is usually a source of monotonicity, and two-layer schemes are nonmonotonic [27].

The authors of [29] solved numerically the problem (1)-(2) at $m = \sigma + 1, k = 0, p = 2$ using a three-layer difference scheme.

For the numerical solution it is proposed to use a three-layer difference scheme, which has a higher accuracy.

In $\bar{\Omega}$ we construct the spatial grid $\bar{\omega}_h$ with steps h:

$$x_i = ih,\ h > 0,\ i = 0, 1, ..., n,\ hn = b$$

and temporary grid with τ

$$t_j = j\tau,\ \tau > 0,\ j = 0, 1, ..., m,\ \tau m = T.$$

Replace problem (1)-(2) implicit three-layer difference scheme and obtain the difference task with error $O(h^2 + \tau^2)$:

$$\frac{3}{2\tau}(y_i^{j+1} - y_i^j) - \frac{1}{2\tau}(y_i^j - y_i^{j-1}) = \frac{1}{h^2}\left[\begin{array}{l} \left|\frac{x_i+x_{i+1}}{2}\right|^n a_{i+1}(y^{j+1})(y_{i+1}^{j+1} - y_i^{j+1}) - \\ -\left|\frac{x_{i-1}+x_i}{2}\right|^n a_i(y^{j+1})(y_i^{j+1} - y_{i-1}^{j+1}) \end{array}\right]$$
$$-(y_i^{j+1})^\beta \ i = 1, 2, .., n-1; j = 1, ..m-1,$$

$$y_i^0 = u_0\left(x_i\right)\ i = 0, 1, ..n,$$
$$y_0^j = \varphi_1\left(t_j\right)\ j = 1, 2, ..m,$$
$$y_n^j = \varphi_2\left(t_j\right)\ j = 1, 2, ..m,$$

where

$$a_{i+1} = \frac{1}{2}\left[\left(y_i^{j+1}\right)^{m-1}\left|\frac{\left(y_{i+1}^{j+1}\right)^k - \left(y_i^{j+1}\right)^k}{h}\right|^{p-2} + \left(y_{i+1}^{j+1}\right)^{m-1}\left|\frac{\left(y_i^{j+1}\right)^k - \left(y_{i-1}^{j+1}\right)^k}{h}\right|^{p-2}\right],$$

$$a_i = \frac{1}{2}\left[\left(y_{i-1}^{j+1}\right)^{m-1}\left|\frac{\left(y_i^{j+1}\right)^k - \left(y_{i-1}^{j+1}\right)^k}{h}\right|^{p-2} + \left(y_i^{j+1}\right)^{m-1}\left|\frac{\left(y_{i-1}^{j+1}\right)^k - \left(y_{i-2}^{j+1}\right)^k}{h}\right|^{p-2}\right].$$

This nonlinear system of equations was solved by the iteration method in combination with the Thomas method and for its linearization the representation was used in particular

$$(u_i^{j+1})^\beta \approx (\overline{u}_i^{j+1})^\beta + \beta(\overline{u}_i^{j+1})^{\beta-1}(u_i^{j+1} - \overline{u}_i^{j+1}).$$

Then the linearized system of equations has the following form

$$\overset{s}{\underset{i}{A}}\,\overset{s+1}{\bar{y}_{i-1}} - \overset{s}{\underset{i}{C}}\,\overset{s+1}{\bar{y}_i} + \overset{s}{\underset{i}{B}}\,\overset{s+1}{\bar{y}_{i+1}} = -\overset{s}{\underset{i}{F}},$$

where

$$\underset{i}{A} = \tfrac{2\tau}{h^2}\left|\tfrac{x_{i-1}+x_i}{2}\right|^n a_i(y^{j+1}),\ B_i = \tfrac{2\tau}{h^2}\left|\tfrac{x_i+x_{i+1}}{2}\right|^n a_{i+1}(y^{j+1}),$$
$$C_i = A_i + B_i + 3 + 2\tau\beta(\overline{u}_i^{j+1})^{\beta-1}, F_i = 4u_i^j - u_i^{j-1} - 2\tau(\overline{u}_i^{j+1})^\beta(1-\beta).$$

The values from the upper solution were taken as a zero approximation. The values of the function u on the first layer were determined by an explicit difference scheme with a time step (for $t = T$).

The ending condition of the iteration:

$$|u_k - \overline{u}_k| \le \varepsilon.$$

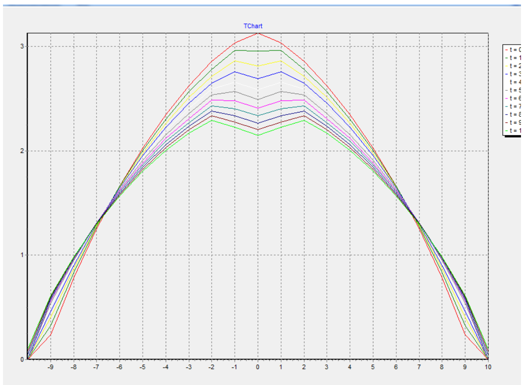
Note. In all numerical calculations we assumed $\varepsilon = 10^{-3}$.

The results of numerical experiments show fast convergence of the iterative process due to the successful choice of the initial approximation. Below are some results of numerical experiments for different values of numerical parameters. The following solutions were used as an initial approximation

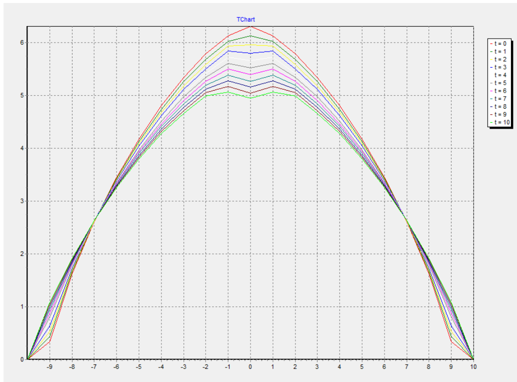
Case $p > n$

$$u_0(t,x) = \bar{u}(t)\overline{f}(\xi),\ \overline{u}(t) = (T + (\beta-1)t)^{\frac{1}{1-\beta}},\ \overline{f}(\xi) = \left(a - b\xi^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{m+k(p-2)-1}},$$

$$\xi = \tfrac{\varphi(x)}{\tau^{1/p}},\ \varphi\left(x\right) = \tfrac{p}{p-n}|x|^{\frac{p-n}{p}}, \tau\left(t\right) = \frac{[T+(\beta-1)t]^{\frac{\beta-(m+k(p-2))}{\beta-1}}}{\beta-(m+k(p-2))}.$$

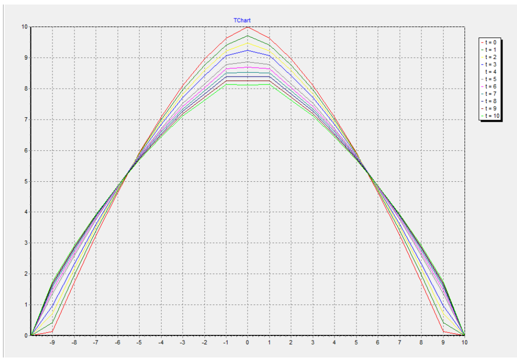


$\beta = 5, p = 3, m = 2, n = 2, k = 2.$

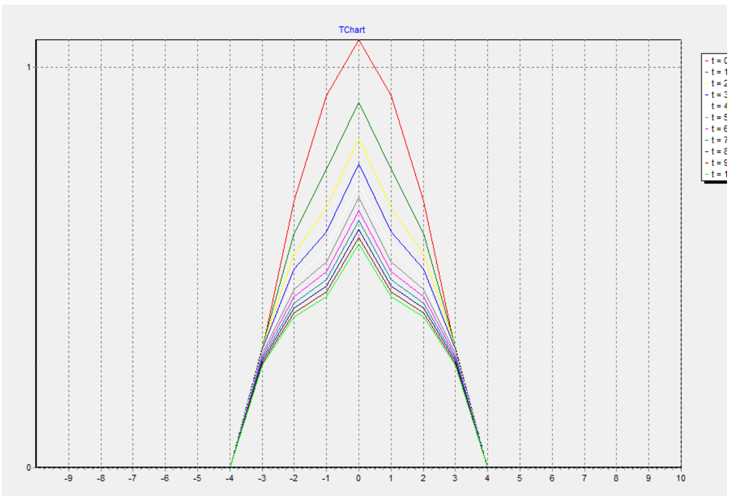


$\beta = 5, p = 3, m = 2, n = 2, k = 3.$

$\beta = 5, p = 4, m = 2, n = 2, k = 2.$



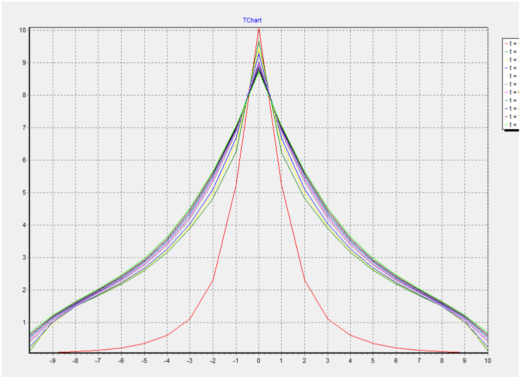
$\beta = 5, p = 3, m = 2, n = 2, k = 3.$



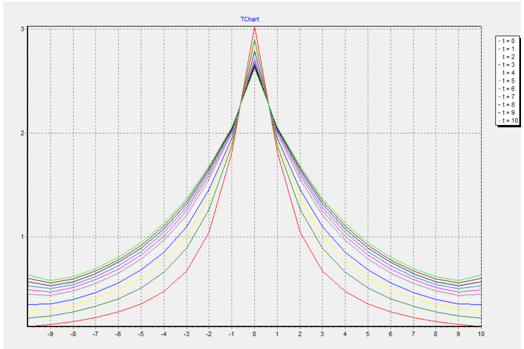
$\beta = 5, p = 3, m = 2, n = 2, k = 2.$

Case $p = n$

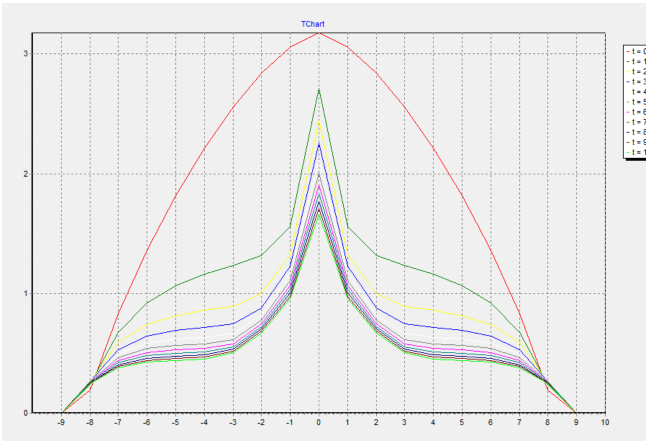
As mentioned above, in the case of $p = n$ the function $\varphi(x)$ will change . The asymptotics will be the same as in the case of p>n except for the function $\varphi(x)$. In this case it will be equal to $\varphi(x) = \ln|x|$.



$\beta = 5, p = 3, m = 2, n = 3, k = 2.$



$\beta = 5, p = 4, m = 4, n = 4, k = 2.$



$\beta = 6, p = 3, m = 2, n = 3, k = 4.$

Conclusions

The search for new effects of the variable density problem is a very important and interesting study. This extends the application of this research. The variable density function may be different in certain cases. We found this function at a critical value, when the denominator vanishes. Using the founded asymptotics of the solution we obtained more accurate solutions using a three-layer difference scheme.In the future, we are faced with the task of finding the boundaries(conditions) of a variable density function in which the solution of the problem (1)-(2) explodes in a finite time.

References

[1] Cianci P., Martynenko A.V., Tedeev A.F. The blow-up phenomenon for degenerate parabolic equations with variable coefficients and nonlinear source. Nonlinear Analysis: Theory, Methods and Applications A, Vol. 73, Issue 7, 2310–2323 (2010).

[2] Benedetto E.Di. Degenerate Parabolic Equations. Universitext, Springer, New York, (1993).

[3] Zhao J.N. On the Cauchy problem and initial traces for the evolution p-Laplacian equations with strongly nonlinear sources. Journal of Differential Equations, Vol. 121, No. 2, 329–383 (1995).

[4] Wu Z., Zhao J., Yun J., Li F. Nonlinear Diffusion Equations. World Scientific Publishing, New York, (2001).

[5] Gmira A. On quasilinear parabolic equations involving measure date. Asymptotic Analysis, Vol. 3, Issue 3, 43–56 (1990).

[6] Yang J., Zhao J. A note to the evolutional P-Laplace equation with absorption. Acta. Sci. Nat, Vol. 2, Issue 2, 35–38 (1995).

[7] Zhao J. Source-type solutions of quasilinear degenerate parabolic equation with absorption. Chin. Ann. of Math, Vol. 3, 89–104 (1994).

[8] Zhao J. Existence and nonexistence of solution for $u_t = div(|\nabla u|^{p-2}\nabla u) + f(\nabla u, u, x, t)$. J. Math. Anal. Appl, Vol. 172, Issue 1, 130–146 (1993).

[9] Zhao J. The Cauchy problem for $u_t = div(|\nabla u|^{p-2}\nabla u)$ when $2N/(N+1) < p < 2$. Nonlinear Analysis, Vol. 24, Issue 5, 615–630 (1995).

[10] Zhao J., Yuan H. The Cauchy problem of a class of doubly degenerate parabolic equation. Chinese Ann. Of Math, Vol. 16, 181–196 (1995).

[11] Dibenedetto E., Friedman A. Hölder estimates for nonlinear degenerate parabolic systems. J. Reine Angew. Math., Vol. 1985, Issue 357, 1–22 (1985).

[12] Li Y., Xie Ch. Blow-up for p-Laplace parabolic equations. Electron. J. Diff. Eqns., Vol. 2003, No. 20, 1–12 (2003).

[13] Dibenedetto E., Herrero M.A. On Cauchy problem and initial traces for a degenerate parabolic equations. Trans. Amer. Math. Soc., Vol. 314, No. 1, 187–224 (1989).

[14] Xiang Zh., Mu Ch., Hu X. Support properties of solutions to a degenerate equation with absorption and variable density. Nonlinear Analysis: Theory, Methods & Applications, Vol. 68, Issue 7, 1940–1953 (2008).

[15] Galaktionov V., King J.R. On the behaviour of blow-up interfaces for an inhomogeneous filtration equation. Journal of Applied Mathematics, Vol. 57, Issue 1, 53–77 (1996).

[16] Martynenko A., Tedeev A. Cauchy Problem for a quasilinear parabolic equation with source and inhomogeneous density. Computational Mathematics and Mathematical Physics, Vol. 47, No. 2, 245–255 (2007).

[17] Andreucci D., Tedeev A. Universal bounds at the blow-up time for nonlinear parabolic equations. Adv. Differential Equations, Vol. 10, No. 1, 89–120 (2005).

[18] Andreucci D., Tedeev A., Ughi M. The Cauchy problem for degenerate parabolic equations with source and damping. Ukr. Mat. Visn, Vol. 1, No. 1, 1–19 (2004).

[19] Galaktionov V. Critical Global Asymptotics In Higher-Order Semilinear Parabolic Equations. International Journal of Mathematics and Mathematical Sciences, Vol. 2003, Issue 60, 3809-–3825 (2003).

[20] Aripov M., Rakhmonov Z. Numerical simulation of a nonlinear problem of a fast diffusive filtration with a variable density and nonlocal boundary conditions. Mathematical Models and Simulation in Science and Engineering, Vol. 23, 72–77 (2014).

[21] Aripov M., Rakhmonov Z. Asymptotics of self-similar solutions of a nonlinear polytropic filtration problem with a nonlinear boundary condition. Vychislitel'nye tekhnologii, Vol. 18, No. 4, 50–55 (2013).

[22] Aripov M., Mukimov A. An asymptotic solution radially symmetric self-similar solution of nonlinear parabolic equation with source in the second critical exponent case. Acta NUUz, No. 2/2, 21–30 (2017).

[23] Galaktionov V.A., Kurdyumov S.P., Samorarskiy A.A. On asymptotic "eigenfunctions" of the Cauchy problem for a nonlinear parabolic equation. Mathematics of the USSR-Sbornik, Vol. 54, No. 2, 421–455 (1986). DOI: 10.1070/SM1986v054n02ABEH002979

[24] Aripov M. Methods of standart equations for solving nonlinear boundary value problems. Fan, Tashkent, (1988).

[25] Samarskiy A.A., Sobol I.M. Examples of numerical calculation of temperature waves. Computational Mathematics and Mathematical Physics, Vol. 3, No. 4, 702–719 (1963).

[26] Tikhonov A., Samarskiy A. Equations of mathematical physics. Nauka, Moscow, (1966).

[27] Richtmayer R. Difference methods for solving problems of mathematical physics. Ed. lit, Moscow, (1960).

[28] Zuev A. On a three-layer scheme for the numerical integration of the gas dynamics equations and the nonlinear heat equation. Numerical methods for solving problems math. Fiz, Vol. 3, 230–236 (1966).

[29] Golaido S., Martinson L., Pavlov K. Unsteady nonlinear heat conduction problems with bulk heat absorption. Computational Mathematics and Mathematical Physics, Vol. 13, No. 5, 1351–1356 (1973).