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SYMMETRIC PRODUCTS, DUALITY AND HOMOLOGICAL DIMENSION OF CONFIGURATION SPACES

SADOK KALLEL

To Fred Cohen on his 60th Birthday

ABSTRACT. We discuss various aspects of "braid spaces" or configuration spaces of unordered points on manifolds. First we describe how the homology of these spaces is affected by puncturing the underlying manifold, hence extending some results of Fred Cohen, Goryunov and Napolitano. Next we obtain a precise bound for the cohomological dimension of braid spaces. This is related to some sharp and useful connectivity bounds that we establish for the reduced symmetric products of any simplicial complex. Our methods are geometric and exploit a dual version of configuration spaces given in terms of truncated symmetric products. We finally refine and then apply a theorem of McDuff on the homological connectivity of a map from braid spaces to some spaces of "vector fields".

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1. INTRODUCTION

Braid spaces or configuration spaces of *unordered pairwise distinct* points on manifolds have important applications to a number of areas of mathematics and physics. They were of crucial use in the seventies in the work of Arnold on singularities and then later in the eighties in work of Atiyah and Jones on instanton spaces in gauge theory. In the nineties they entered in many works on the homological stability of holomorphic mapping spaces. No more important perhaps had been their use than in stable homotopy theory in the sixties and early seventies through the work of Milgram, May, Segal and Fred Cohen who worked out the precise connection with loop space theory. This work has led in particular to the proof of Nishida's nilpotence theorem and to Mahowald's infinite family in the stable homotopy groups of spheres to name a few.

Given a space M, define $B(M,n)$ to be the space of finite subsets of M of cardinality n. This is usually referred to as the n-th "braid space" of M and in the literature it is often denoted by $C_n(M)$ [3, 7, 8]. Its fundamental group written $Br_n(M)$ is the "braid group" of M. The object of this paper is to study the homology of braid spaces and the main approach we adopt is that of duality with the symmetric products. In so doing we take the opportunity to refine and elaborate on some classical material. Next is a brief content summary.

Section 2 describes the homotopy type of braid spaces of some familiar spaces and discusses orientation issues. Section 3 introduces truncated products as in [6, 22], states the duality with braid spaces and then proves our first main result on the cohomological dimension of braid spaces. Section 4 uses truncated product constructions to split in an elementary fashion the homology of braid spaces for punctured manifolds. In section 5 we prove our sharp connectivity result for *reduced* symmetric products of CW complexes which seems to be new and a significant improvement on work of Nakaoka and Welcher [42]. In section 5.2 we make the link between the homology of symmetric and truncated products by discussing a spectral sequence introduced by C.F. Bodigheimer, Fred Cohen and R.J. Milgram and exploited by them to study "braid homology" $H_*(B(M,n))$. Finally Section 6 completes a left out piece from McDuff and Segal's work on configuration spaces [23]. In that paper, $H_*(B(M,n))$, for closed manifolds M, is compared to the homology of some spaces of "compactly supported vector fields" on M and the main theorem there states that these homologies are isomorphic up to a range that increases with n. We make this range more explicit and use it for example to determine the abelianization of the braid groups of a closed Riemann surface. A final appendix collects some homotopy theoretic properties of section spaces that we use throughout.

Below are precise statements of our main results which we have divided up into three main parts. Unless explicitly stated, all spaces are assumed to be connected. The n-th symmetric group is written $\mathfrak{S}_n$.

1.1. **Connectivity and Cohomological Dimension.** For M a manifold, write $H^*(M,\pm\mathbb{Z})$ the cohomology of M with coefficients in the orientation sheaf $\pm\mathbb{Z}$; that is $H^*(M,\pm\mathbb{Z})$ is the homology of $Hom_{\mathbb{Z}[\pi_1(X)]}(C_*(\tilde{M}),\mathbb{Z})$, where $C_*(\tilde{M})$ is the singular chain complex of the universal cover $\tilde{M}$ of M, and where the action of (the class of) a loop on the integers $\mathbb{Z}$ is multiplication by ± 1 according to whether this loop preserves or reverses orientation. Similarly one defines $H_*(M,\pm\mathbb{Z}) := H_*(C_*(\tilde{M})\otimes_{\mathbb{Z}[\pi_1(x)]}\mathbb{Z})$.

Remark 1.1. (see Lemma 2.6) When M is simply connected and $\dim M := d > 2$, $\pi_1(B(M,k)) = \mathfrak{S}_k$ and $\tilde{B}(M,k) = F(M,k)\subset M^k$ is the subspace of k *ordered* pairwise distinct points in M (§2). It follows that $H^*(B(M,k);\pm\mathbb{Z})$ is the homology of the chain complex $Hom_{\mathbb{Z}[\mathfrak{S}_k]}(C_*(F(M,k),\mathbb{Z})$ where $\mathfrak{S}_k$ acts on $\mathbb{Z}$ via $\sigma(1) = (-1)^{sg(\sigma)\cdot d}$ and $sg(\sigma)$ is the sign of the permutation $\sigma\in\mathfrak{S}_k$.

We denote by $\mathrm{cohdim}_{\pm\mathbb{Z}}(M)$ (cohomological dimension) the smallest integer with the property that

$$H^i(M;\pm\mathbb{Z}) = 0 \quad , \quad \forall i > \mathrm{cohdim}_{\pm\mathbb{Z}}(M)$$

If M is orientable, then $H^*(M,\pm\mathbb{Z}) = H^*(M,\mathbb{Z})$ and $\mathrm{cohdim}_{\pm\mathbb{Z}}(M) = \mathrm{cohdim}(M)$; the cohomological dimension of M.

A space X is r-connected if $\pi_i(X) = 0$ for $0\le i\le r$. The connectivity of X; $conn(X)$, is the largest integer with such a property. This connectivity is infinite if X is contractible. The following is our first main result

Theorem 1.2. *Let M be a compact manifold of dimension $d\ge 1$, with boundary ∂M, and let $U\subset M$ be a closed subset such that $U\cap\partial M = \emptyset$ and $M - U$ connected. We denote by r the connectivity of M if $U\cup\partial M = \emptyset$, or the connectivity of the quotient $M/U\cup\partial M$ if $U\cup\partial M\neq\emptyset$. We assume $0\le r<\infty$ and $k\ge 2$. Then*

$$cohdim_{\pm\mathbb{Z}}(B(M-U,k)) \le \begin{cases} (d-1)k - r + 1, & \text{if } U\cup\partial M = \emptyset \\ (d-1)k - r, & \text{if } U\cup\partial M \neq \emptyset \end{cases}$$

When M is even dimensional orientable, then replace $cohdim_{\pm\mathbb{Z}}$ by $cohdim$.

Remark 1.3. We check this theorem against some known examples:

(1) $B(S^d - \{p\},2) = B(\mathbb{R}^d,2)\simeq\mathbb{R}P^{d-1}$ (see Section 3) and $\mathrm{cohdim}_{\pm\mathbb{Z}}(B(\mathbb{R}^d,2)) = 2(d-1) - r = d-1 = \mathrm{cohdim}_{\pm\mathbb{Z}}(\mathbb{R}P^{d-1})$ indeed, where $r = d-1 = conn(S^d)$.
(2) $B(S^d,2)\simeq\mathbb{R}P^d$ (see Section 3) and $\mathrm{cohdim}_{\pm\mathbb{Z}}(B(S^d,2)) = d$ in agreement with our formula.
(3) It is known that for odd primes p and $d\ge 2$, $H^{(d-1)(p-1)}(B(\mathbb{R}^d,p);\mathbb{F}_p)$ is non-trivial and an isomorphic image of $H^{(d-1)(p-1)}(\mathfrak{S}_p;\mathbb{F}_p)$ [31, 38]. Our result states that, at least for even d, no higher homology can occur. The cohomological dimension of $B(\mathbb{R}^d,k)$ when using $\mathbb{F}_2$ coefficients is known to be $(k-\alpha(k))\cdot(d-1)$ where $\alpha(k)$ is the number of 1's in the dyadic decomposition of k (see [32]). In the case $d=2$, $B(\mathbb{R}^2,k)$ is the classifying space of Artin braid group $B_k := Br_k(\mathbb{R}^2)$ and is homotopy equivalent to a $(k-1)$-dimensional CW complex so that $\mathrm{cohdim}(B(\mathbb{R}^2,k))\le k-1$ in agreement with our calculation.

Remark 1.4. The theorem applies to when $M = S^1$ and U is either empty or a single point. In that case $M - U = S^1,\mathbb{R}$. But one knows that for $k\ge 1$, $B(S^1,k)\simeq S^1$ (Proposition 2.5) and $B(\mathbb{R},k)$ is contractible.

Corollary 1.5. *Let S be a Riemann surface and $Q\subset S$ a finite subset. Then $H^i(B(S-Q,k)) = 0$ if $i\ge k+1$ and $Q\cup\partial S\neq\emptyset$; or if $i > k+1$ and $Q\cup\partial S = \emptyset$.*

This corollary gives an extension of the "finiteness" result of [29]. When S is an open surface, then $B(S,k)$ is a Stein variety and hence its homology vanishes above the complex dimension; i.e. $H_i(B(S,k)) = 0$ for $i > k$. This also agrees with the above computed bounds.

The proof of Theorem 1.2 relies on a useful connectivity result of Nakaoka (Theorem 3.8). We also use this result to produce sharp connectivity bounds for the *reduced* symmetric products §5. Recall that $SP^n(X)$, the n-th symmetric product of X, is the quotient of X^n by the permutation action of the symmetric group $\mathfrak{S}_n$ so that $B(X,n) \subset SP^n(X)$ is the subset of configurations of distinct points. We always assume X is based so there is an embedding $SP^{n-1}(X) \hookrightarrow SP^n(X)$ given by adjoining the basepoint, with cofiber $\overline{SP}^n(X)$ the "n-th reduced symmetric" product of X. The following result expresses the connectivity of $\overline{SP}^n X$ in terms of the connectivity of X.

Theorem 1.6. *Suppose X is a based r-connected simplicial complex with $r \geq 1$. Then $\overline{SP}^n(X)$ is $2n+r-2$- connected.*

In particular the embedding $SP^{n-1}(X) \longrightarrow SP^n(X)$ induces homology isomorphisms in degrees up to $(2n+r-3)$. The proof of this theorem is totally inspired from [19] where similar connectivity results are stated, and it uses the fact that the homology of symmetric products only depends on the homology of the underlying complex [11]. Note that the bound $2n+r-2$ is sharp as is illustrated by the case $X = S^2$, $r = 1$ and $\overline{SP}^n(S^2) = S^{2n}$. A slightly weaker connectivity bound than ours can be found in [42], corollary 4.9.

Note that Theorem 1.6 is stated for simply connected spaces. To get connectivity results for reduced symmetric products of a compact Riemann surface for example we use geometric input from [20]. This applies to any two dimensional complex.

Proposition 1.7. *Let $X = \bigvee^w S^1 \cup (D^2_1 \cup \cdots \cup D^2_r)$ be a two dimensional CW complex with one skeleton a bouquet of w circles. Then $\overline{SP}^n X$ is $(2n - min(w,n) - 1)$-connected.*

1.2. **Puncturing Manifolds.** We give generalizations and a proof simplification of results of Napolitano [29, 30]. For S a two dimensional topological surface, p and the p_i points in S, it was shown in [29] that for field coefficients $\mathbb{F}$,

$$H^j(B(S-\{p_1,p_2\},n);\mathbb{F}) \cong \bigoplus_{t=0}^{n} H^{j-t}(B(S-\{p\},n-t);\mathbb{F}) \tag{1}$$

Here and throughout $H^* = 0$ when $* < 0$ and $B(X,0)$ is basepoint. When S is a closed orientable surface and $\mathbb{F} = \mathbb{F}_2$, [29] establishes furthermore a splitting

$$H^j(B(S,n);\mathbb{F}_2) \cong H^j(B(S-\{p\},n);\mathbb{F}_2) \oplus H^{j-2}(B(S-\{p\},n-1);\mathbb{F}_2) \tag{2}$$

Similar splittings occur in [9, 16]. These splittings as we show extend to any closed topological manifold M and to any number of punctures. If V is a vector space, write $V^{\oplus k} := V \oplus \cdots \oplus V$ (k-times). Given positive integers r and s, we write $p(r,s)$ the number of ways we can partition s into a sum of r *ordered* positive (or null) integers. For instance $p(1,s) = 1$, $p(2,s) = s+1$ and $p(r,1) = r$.

Theorem 1.8. *Let M be a closed connected manifold of dimension d and $p \in M$. Then*

$$H^j(B(M,n);\mathbb{F}_2) \cong H^j(B(M-\{p\},n);\mathbb{F}_2) \oplus H^{j-d}(B(M-\{p\},n-1);\mathbb{F}_2) \tag{3}$$

If moreover M is oriented and even dimensional, then

$$H^j(B(M-\{p_1,\cdots,p_k\},n);\mathbb{F}) \cong \bigoplus_{0\leq r\leq n} H^{j-(n-r)(d-1)}(B(M-\{p\},r);\mathbb{F})^{\oplus p(k-1,n-r)} \tag{4}$$

For an arbitrary closed manifold, (4) is still true with $\mathbb{F}_2$-coefficients.

Remark 1.9. As an example we can set $M = S^2, k = 2 = d$ and obtain the additive splitting $H^j(B(\mathbb{C}^*,n);\mathbb{F}) \cong \bigoplus_{0\leq r\leq n} H^{j-(n-r)}(B(\mathbb{C},r);\mathbb{F})$ as in (1), where $\mathbb{C}^*$ is the punctured disk (this isomorphism holds integrally according to [16]). Note that the left hand side is the homology of the hyperplane arrangement of 'Coxeter type' B_n; that is $B(\mathbb{C}^*,n)$ is an Eilenberg-MacLane space $K(Br_n(\mathbb{C}^*),1)$ with fundamental group isomorphic to the subgroup of Artin's braids $Br_{n+1}(\mathbb{C})$ consisting of those braids which leave the last strand fixed. It can be checked that the abelianization of this group for $n \geq 2$ is $\mathbb{Z}^2$ which is consistent with the calculation of H^1 obtained from the above splitting.

Napolitano's approach to (1) is through spectral sequence arguments and "resolution of singularities" as in Vassiliev theory. Our approach relies on a simple geometric manipulation of the truncated symmetric products as discussed earlier (see Section 4). Theorem 1.8 is a consequence of combining a Poincaré-Lefshetz duality statement, the identification of truncated products of the circle with real projective space [26] and a homological splitting result due to Steenrod (Section 3). Note that the splitting in (3) is no longer true with coefficients other than $\mathbb{F}_2$ and is replaced in general by a long exact sequence (lemma 4.1).

1.3. **Homological Stability.** This is the third and last part of the paper. For M a closed smooth manifold of dimension $\dim M = d$, let τ^+M be the fiberwise one-point compactification of the tangent bundle τM of M with fiber S^d. We write $\Gamma(\tau^+M)$ the space of sections of τ^+M. Note that this space has a preferred section (given by the points at infinity). There are now so called "scanning" maps for any $k \in \mathbb{N}$ [23, 7, 18]

$$S_k : B(M,k)\longrightarrow\Gamma_k(\tau^+M) \tag{5}$$

where $\Gamma_k(\tau^+M)$ is the component of degree k sections (see §6.2). In important work, McDuff shows that S_k induces a homology isomorphism through a range that increases with k. In many special cases, this range needs to be made explicit and this is what we do next.

We say that a map $f : X \to Y$ is homologically k-connected (or a homology equivalence up to degree k) if f_* in homology is an isomorphism up to and including degree k.

Proposition 1.10. *Let M be a closed manifold of dimension $d \geq 2$. Assume the map $+ : B(M-p,k)\longrightarrow B(M-p,k+1)$ which consists of adding a point near $p \in M$ (see Section 6) is homologically $s(k)$-connected. Then scanning S_k is homologically $s(k-1)$-connected. Moreover $s(k) \geq [k/2]$ (Arnold).*

When $k = 1$, we give some information about $S_1 : M\longrightarrow\Gamma_1(\tau^+M)$ in lemma 6.5. Note that $s(k)$ is an increasing function of k. Arnold's inequality $s(k) \geq [k/2]$ is proven in [34]. This bound is far from being optimal in some cases since for instance, for M a compact Riemann surface, $s(k) = k - 1$ [21]. Note that the actual connectivity of the map $+ : B(M-p,k)\longrightarrow B(M-p,k+1)$ is often 0 since if $\dim M > 2$, this map is never trivial on π_1 (see Lemma 2.6).

The utility of Proposition 1.10 is that in some particular cases, knowledge of the homology of braid spaces in a certain range informs on the homology of some mapping spaces. Here's an interesting application to computing the abelianization of the braid group of a surface (this was an open problem for some time).

Corollary 1.11. *For S a compact Riemann surface of genus $g \geq 1$, and $k \geq 3$, we have the isomorphism: $H_1(B(S,k);\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}^{2g}$.*

Proof. τ^+S is trivial since S is stably parallelizable and $\Gamma(\tau^+S) \simeq Map(S,S^2)$. Suppose S has odd genus, then $S_k : H_1(B(S,k))\longrightarrow H_1(Map_k(S,S^2))$ is degree preserving (where degree is k) and according to Proposition 1.10 it is an isomorphism when $k \geq 3$ using the bound provided by Arnold. But $\pi := \pi_1(Map_k(S,S^2))$ was computed in [17] and it is some extension

$$0\longrightarrow\mathbb{Z}_{2|k|}\longrightarrow\pi\longrightarrow\mathbb{Z}^{2g}\longrightarrow 0$$

with a generator τ and torsion free generators $e_1,\dots,e_{2g}$ with non-zero commutators $[e_i,e_{g+i}] = \tau^2$ and with $\tau^{2|k|} = 1$. Its abelianization H_1 is $\mathbb{Z}^{2g}\oplus\mathbb{Z}_2$ as desired. When g is even, $S_k : B(S,k)\longrightarrow Map_{k-1}(S,S^2)$ decreases degree by one (see Section 6.1) but the argument and the conclusion are still the same. □

Remark 1.12. The above corollary is also a recent calculation of [4] which is more algebraic in nature and relies on the full presentation of the braid group $\pi_1(B(S,k))$ for a positive genus Riemann surface S.

Example 1.13. We can also apply Proposition 1.10 to the case when M is a sphere S^n. Write $\mathrm{Map}(S^n,S^n) = \coprod_{k\in\mathbb{Z}} \mathrm{Map}_k(S^n,S^n)$ for the space of self-maps of S^n; $\mathrm{Map}_k(S^n,S^n)$ being the component of degree k maps. Since τ^+S^n is trivial there is a homeomorphism $\Gamma(\tau^+S^n) \cong \mathrm{Map}(S^n,S^n)$. However and as pointed out in [33], one has to pay extra care about components : $\Gamma_k(\tau^+S^n) \cong \mathrm{Map}_k(S^n,S^n)$

if n is odd and $\Gamma_k(\tau^+S^n)\cong \mathrm{Map}_{k-1}(S^n,S^n)$ if n is even (see section 6.1). Let $p(n)=1$ if n is even and 0 if n is odd. Vassiliev [38] checks that $H_*(B(\mathbb{R}^n,k);\mathbb{F}_2)\longrightarrow H_*(B(\mathbb{R}^n,k+1);\mathbb{F}_2)$ is an isomorphism up to degree k and so we get that the map of the k-th braid space of the sphere into the higher free loop space

$$B(S^n,k)\longrightarrow \mathrm{Map}_{k-p(n)}(S^n,S^n)$$

is a mod-2 homology equivalence up to degree $k-1$. The homology of $\mathrm{Map}(S^n,S^n)$ is worked out for all field coefficients in [33].

Remark 1.14. The braid spaces fit into a filtered construction

$$B(M,n)=:B^1(M,n)\hookrightarrow B^2(M,n)\hookrightarrow\cdots\hookrightarrow B^n(M,n):=SP^n(M)$$

where $B^p(M,n)$ for $1\leq p\leq n$ is defined to be the subspace

$$\{[x_1,\ldots,x_n]\in SP^n(M)\mid \text{no more than } p \text{ of the } x_i\text{'s are equal}\}\tag{6}$$

Many of our results can be shown to extend with straightforward changes to $B^p(M,n)$ and $p\geq 1$ when M is a compact Riemann surface. Some detailed statements and calculations can be found in [21].

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2. Basic Examples and Properties

As before we write an element of $SP^n(X)$ as an unordered n-tuple of points $[x_1,\ldots,x_n]$ or sometimes also as an abelian finite sum $\sum x_i$ with $x_i\in X$. For a closed manifold M, $SP^n(M)$ is again a manifold for $n>1$ if and only if M is of dimension less or equal to two [40]. We define

$$B(M,n)=\{[x_1,\ldots,x_n]\in SP^n(M), x_i\neq x_j, i\neq j\}$$

It is convenient as well to define the "ordered" n-fold configuration space $F(M,n)=M^n-\Delta_{fat}$ where

$$\Delta_{fat}:=\{(x_1,\ldots,x_n)\in M^n\mid x_i=x_j \text{ for some } i=j\}\tag{7}$$

is the *fat diagonal* in M^n. The configuration space $B(M,n)$ is obtained as the quotient $F(M,n)/\mathfrak{S}_n$ under the free permutation action of $\mathfrak{S}_n$ [1]. Both $F(M,n)$ and $B(M,n)$ are (open) manifolds of dimension nd, $d=\dim M$.

Next are some of the simplest non-trivial braid spaces one can describe.

Lemma 2.1. *$B(S^n,2)$ is an open n-disc bundle over $\mathbb{R}P^n$. When $n=1$, this is the open Möbius band (see Proposition 2.5).*

Proof. There is a surjection $\pi: B(S^n,2)\longrightarrow\mathbb{R}P^n$ sending $[x,y]$ to the unique line $L_{[x,y]}$ passing through the origin and parallel to the non-zero vector $x-y$. The preimage $\pi^{-1}(L_{[x,y]})$ consists of all pairs $[a,b]$ such that $a-b$ is a multiple of $x-y$. This can be identified with an "open" hemisphere determined by the hyperplane orthogonal to $L_{[x,y]}$ (i.e. $B(S^n,2)$ can be identified with the dual tautological bundle over $\mathbb{R}P^n$). □

Example 2.2. Similarly we can see that $B(\mathbb{R}^{n+1},2)\simeq\mathbb{R}P^n$ and that $B(S^n,2)\hookrightarrow B(\mathbb{R}^{n+1},2)$ is a deformation retract. Alternatively one can see directly that $B(S^n,2)\simeq\mathbb{R}P^n$ for there are an inclusion i and a retract r

$$\begin{array}{rlcrl} i: S^n & \hookrightarrow F(S^n,2) & , & r: F(S^n,2) & \longrightarrow S^n\\ x & \longmapsto (x,-x) & & (x,y) & \mapsto \dfrac{x-y}{|x-y|}\end{array}$$

[1] In the early literature on embedding theory [15], $B(M,2)$ was referred to as the "reduced symmetric square".

Identify S^n with $i(S^n)$ as a subset of $F(S^n,2)$. Then $F(S^n,2)$ deformation retracts onto this subset via

$$f_t(x,y) = \left(\frac{x-ty}{|x-ty|}, \frac{y-tx}{|y-tx|}\right)$$

(which one checks is well-defined). We have that f_t is $\mathbb{Z}_2$-equivariant with respect to the involution $(x,y)\mapsto(y,x)$, that $f_0 = id$ and that $f_1 : F(S^n,2)\longrightarrow S^n$ is $\mathbb{Z}_2$-equivariant with respect to the antipodal action on S^n. That is S^n is a $\mathbb{Z}_2$-equivariant deformation retraction of $F(S^n,2)$ which yields the claim.

Example 2.3. $B(\mathbb{R}^2,3)$ is up to homotopy the complement of the trefoil knot in S^3.

Example 2.4. There is a projection $B(\mathbb{R}P^2,2)\longrightarrow\mathbb{R}P^2$ which, to any two distinct lines through the origin in $\mathbb{R}^3$, associates the plane they generate and this is an element of the Grassmann manifold $Gr_2(\mathbb{R}^3)\cong Gr_1(\mathbb{R}^3) = \mathbb{R}P^2$. The fiber over a given plane parameterizes various choices of two distinct lines in that plane and that is $B(\mathbb{R}P^1,2) = B(S^1,2)$. As we just discussed, this is an open Möbius band M and $B(\mathbb{R}P^2,2)$ fibers over $\mathbb{R}P^2$ with fiber M (see [15]). Interestingly $\pi_1(B(\mathbb{R}P^2,2))$ is a quaternion group of order 16 ([41]).

To describe the braid spaces of the circle we can consider the multiplication map

$$m : SP^n(S^1)\longrightarrow S^1 \quad , \quad [x_1,\ldots,x_n]\mapsto x_1x_2\cdots x_n$$

Morton [25] shows that m is a locally trivial bundle with fiber the closed $(n-1)$- dimensional disc and this bundle is trivial if n is odd and non-orientable if n is even. In particular $SP^2(S^1)$ is the closed Möbius band. In fact one can identify $m^{-1}(1)$ with a closed simplex Δ^{n-1} so that the configuration space component $m^{-1}(1)\cap B(S^1,n)$ corresponds to the open part. This is a non-trivial construction that can be found in [25, 24]. Since $B(S^1,n)$ fits in $SP^n(S^1)$ as the open disk bundle one gets that

Proposition 2.5. *$B(S^1,n)$ is a bundle over S^1 with fiber the open unit disc D^{n-1}. This bundle is trivial if and only if n is odd.*

Examples 2.2 and 2.4 show that when $\dim M$ is odd $\neq 1$ or M is not orientable, then $B(M,k)$ fails to be orientable. The following explains why this needs to be the case.

Lemma 2.6. *(folklore) Suppose M is a manifold of dimension $d\geq 2$ and pick $n\geq 2$. Then $B(M,n)$ is orientable if and only if M is orientable of even dimension.*

Proof. We consider the $\mathfrak{S}_n$-covering $\pi : F(M,n)\xrightarrow{\mathfrak{S}_n} B(M,n)$. If M is not orientable, then so is M^n. Now $i : F(M,n)\hookrightarrow M^n$ is the inclusion of the complement of codimension at least two strata so that $\pi_1(F(M,n))\longrightarrow\pi_1(M)^n$ is surjective and hence so is the map on H_1. The dual map in cohomology is an injection mod 2 and hence $w_1(F(M,n)) = i^*(w_1(M^n))\neq 0$ since $w_1(M^n)\neq 0$. This implies that $F(M,n)$ is not orientable if M isn't. It follows that the quotient $B(M,n)$ is not orientable as well.

Suppose then that M is orientable. If $d:=\dim M = 2$, then M is a Riemann surface, $B(M,n)$ is open in $SP^n(M)$ which is a complex manifold and hence is orientable. Suppose now that $d:=\dim M > 2$ so that $\pi_1F(M,n) = \pi_1(M^n)$ (since the fat diagonal has codimension > 2). Notice that we have an embedding $\iota : B(\mathbb{R}^d,n)\hookrightarrow B(M,n)$ coming from the embedding of an open disc $\mathbb{R}^d\hookrightarrow M$. Now $\pi_1(B(\mathbb{R}^d,n)) = \mathfrak{S}_n$ when $d>2$, and ι induces a section of the short exact sequence of fundamental groups for the $\mathfrak{S}_n$-covering π so we have a semi-direct product decomposition

$$\pi_1(B(M,n)) = \pi_1(M^n)\ltimes\mathfrak{S}_n \quad , \quad d>2$$

Let's argue then that $B(\mathbb{R}^d,n)$ is orientable if and only if d is even. Denote by τ_x the tangent space at $x\in\mathbb{R}^d$ and write $\pi : F(\mathbb{R}^d,n)\longrightarrow B(\mathbb{R}^d,n)$ the quotient map. A transposition $\sigma\in\mathfrak{S}_n$ acts on the tangent space to $B(\mathbb{R}^d,n)$ at some chosen basepoint say $[x_1,\ldots,x_n]$ which is identified with the tangent space $\tau_{x_1}\times\cdots\times\tau_{x_n}$ at say $(x_1,\ldots,x_n)\in\pi^{-1}([x_1,\ldots,x_n])\subset F(\mathbb{R}^d,n)\subset(\mathbb{R}^d)^n$. The action of $\sigma=(ij)$ interchanges both copies $\tau_{x_i}M$ and $\tau_{x_j}M\cong\mathbb{R}^d$ and thus has determinant $(-1)^d$. Orientation is preserved only when d is even and the claim follows (for the relation between orientation and fundamental group see [13], Chapter 4). ∎

Note that the lemma above is no longer true in the one-dimensional case according to proposition 2.5.

3. Truncated Symmetric Products and Duality

The heroes here are the truncated symmetric product functors TP^n which were first put to good use in [6, 22]. For $n \geq 2$, define the identification space

$$TP^n(X) := SP^n(X)/_\sim \ , \quad [x, x, y_1 \ldots, y_{n-2}] \sim [*, *, y_1, \cdots, y_{n-2}]$$

where as always $* \in X$ is the basepoint. Clearly $TP^1X = X$ and we set $TP^0(X) = *$. Note that by adjunction of basepoint $[x_1, \ldots, x_n] \mapsto [*, x_1, \ldots, x_n]$, we obtain topological embeddings $SP^n(X) \longrightarrow SP^{n+1}X$ and $TP^n(X) \longrightarrow TP^{n+1}X$ of which limits are $SP^\infty(X)$ and $TP^\infty(X)$ respectively. We identify $SP^{n-1}(X)$ and $TP^{n-1}(X)$ with their images in $SP^n(X)$ and TP^nX under these embeddings and we write

$$\overline{TP}^n(X) := TP^n(X)/TP^{n-1}(X) \tag{8}$$

for the *reduced* truncated product. These are based spaces by construction. We will set $\overline{TP}^0(X) := S^0$. The following two properties are crucial.

Theorem 3.1.

(1) [12] $\pi_i(TP^\infty(X)) \cong \tilde{H}_i(X; \mathbb{F}_2)$

(2) [22] *There is a splitting* $H_*(TP^n(X); \mathbb{F}_2) \cong H_*(TP^{n-1}(X); \mathbb{F}_2) \oplus \tilde{H}_*(\overline{TP}^n X; \mathbb{F}_2)$

The splitting in (2) is obtained from the long exact sequence for the pair $(TP^n(X), TP^{n-1}(X))$ and the existence of a retract $H_*(TP^n(X); \mathbb{F}_2) \longrightarrow H_*(TP^{n-1}(X); \mathbb{F}_2)$ constructed using a transfer argument. In fact this splitting can be viewed as a consequence of the following homotopy equivalence discussed in [22, 43].

Lemma 3.2. $TP^\infty(TP^n(X)) \simeq TP^\infty(\overline{TP}^n(X)) \times TP^\infty(TP^{n-1}(X))$.

Further interesting splittings of the sort for a variety of other functors are investigated in [43]. The prototypical and basic example of course is Steenrod's original splitting of the homology of symmetric products (which holds with integral coefficients).

Theorem 3.3. *(Steenrod, Nakaoka) The induced basepoint adjunction map on homology* $H_*(SP^{n-1}(X); \mathbb{Z}) \longrightarrow H_*(SP^n(X); \mathbb{Z})$ *is a split monomorphism.*

3.1. **Duality and Homological Dimension.** The point of view we adopt here is that $B(M, n) = TP^n(M) - TP^{n-2}(M)$ as spaces. A version of Poincaré-Lefshetz duality (Lemma 3.5) can then be used to relate the cohomology of $B(M, k)$ to the homology of reduced truncated products. This idea is of course not so new (see [7, 27]).

If $U \subset X$ is a closed cofibrant subset of X, define in $SP^n(X)$ the "ideal"

$$\underline{U} := \{[x_1, \ldots, x_n] \in SP^n(X), x_i \in U \text{ for some } i\} \tag{9}$$

For example and if $* \in X$ is the basepoint, then $\underline{*} = SP^{n-1}(X) \subset SP^n(X)$. Let S be the "singular set" in $SP^n(X)$ consisting of unordered tuples with at least two repeated entries. This is a closed subspace.

Lemma 3.4. *With* $U \neq \emptyset$, $SP^n(X)/(\underline{U} \cup S) = \overline{TP}^n(X/U)$.

Proof. Denote by $*$ the basepoint of X/U which is the image of U under the quotient $X \longrightarrow X/U$. Then by inspection

$$SP^n(X)/(\underline{U} \cup S) = SP^n(X/U)/(\underline{*} \cup S)$$

Moding out $SP^n(X/U)$ by S we obtain $TP^n(X/U)/TP^{n-2}(X/U)$. Moding out further by $\underline{*}$ we obtain the desired quotient. $\square$

The next lemma is the fundamental observation which states that for M a compact manifold with boundary and $U \hookrightarrow M$ a closed cofibration, $B(M-U,n) \cong SP^n(M) - \underline{U \cup \partial M} \cup S$ is Poincaré-Lefshetz dual to the quotient $SP^n(M)/(\underline{U\cup\partial M}\cup S)$. More precisely, set

$$\overline{M} = M/(U\cup\partial M) \tag{10}$$

with the understanding that $\bar{M} = M$ if $U\cup\partial M = \emptyset, \{\text{point}\}$. The following elaborates on [6], Theorem 3.2.

Lemma 3.5. *If M is a compact manifold of dimension $d\geq 1$, $U\subset M$ a closed subset with $M-U$ connected, $U\cap\partial M=\emptyset$ and $\overline{M}$ as in (10), then*

$$H^i(B(M-U,k);\pm\mathbb{Z}) \cong \begin{cases} H_{kd-i}(TP^k(\overline{M}), TP^{k-1}(\overline{M});\mathbb{Z}), & \text{if } U\cup\partial M\neq\emptyset\\ H_{kd-i}(TP^k(M), TP^{k-2}(M);\mathbb{Z}), & \text{if } U\cup\partial M=\emptyset\end{cases}$$

The isomorphism holds with coefficients $\mathbb{F}_2$. When M is even dimensional and orientable, we can replace $\pm\mathbb{Z}$ by the trivial module $\mathbb{Z}$.

Proof. Suppose X is a compact oriented d-manifold with boundary ∂X. Then Poincaré-Lefshetz duality gives an isomorphism $H^{d-q}(X;\mathbb{Z})\cong H_q(X,\partial X;\mathbb{Z})$. Apply this to the following situation: X is a finite d-dimensional CW-complex, $V\subset X$ is a closed subset of X, and N is a tubular neighborhood of V which deformation retracts onto it;

$$V\subset N\subset X$$

$\bar{N}$ its closure and $\partial\bar{N} = \partial(X-N) = \bar{N}-N$. Assume that $X-N$ is an orientable d-dimensional manifold with boundary $\partial\bar{N}$. Then we have a series of isomorphisms

$$H^{d-q}(X-V;\mathbb{Z})\cong H^{d-q}(X-N;\mathbb{Z})\cong H_q(X-N,\partial\bar{N};\mathbb{Z})\cong H_q(X,V;\mathbb{Z}) \tag{11}$$

Let's now apply (11) to the case when $X = SP^k(\overline{M})$ with M as in the lemma and with V the closed subspace consisting of configurations $[x_1,\ldots,x_k]$ such that (i) $x_i = x_j$ for some $i\neq j$ or (ii) for some i, $x_i = *$ the point at which $U\cup\partial M$ is collapsed out. As discussed in Lemma 3.4, $SP^k(\overline{M})/\underline{*} = SP^k(M)/(\underline{U\cup\partial M})$ so that $SP^k(\overline{M})/V = SP^k(M)/(\underline{U\cup\partial M}\cup S)$ with S again being the image of the fat diagonal in $SP^k(M)$. Then, according to Lemma 3.4 and to its proof we see that

$$SP^k(\overline{M})/V = \begin{cases} TP^k(\overline{M})/TP^{k-1}(\overline{M}), & \text{if } \partial M\neq\emptyset \text{ or } U\neq\emptyset\\ TP^k(M)/TP^{k-2}(M), & \text{if } M \text{ closed and } U=\emptyset\end{cases}$$

Now $B(M-U,k)\cong SP^k(M) - \underline{U\cup\partial M}\cup S = SP^k(\overline{M}) - V$ is connected (since $M-U$ is), it is kd dimensional and is orientable if M is even dimensional orientable (Lemma 2.6). Applying (11) yields the result in the orientable case. When $B(M-U,k)$ is non orientable, Poincaré-Lefshetz duality holds with twisted coefficients. $\square$

A version of this lemma has been greatly exploited in [6, 21] to determine the homology of braid spaces and analogs. The following is immediate.

Corollary 3.6. *With M, $U\subset M$ as in Lemma 3.5, let*

$$R_k = \begin{cases} conn(TP^k(\overline{M})/TP^{k-1}(\overline{M})), & \text{if } U\cup\partial M\neq\emptyset,\\ conn(TP^k(M)/TP^{k-2}(M)), & \text{if } U\cup\partial M=\emptyset\end{cases}$$

Then $cohdim_{\pm\mathbb{Z}}(B(M-U,k)) = dk - R_k - 1$.

Theorem 1.2 is now a direct consequence of the following result.

Lemma 3.7. *Let M, U and $\overline{M}$ as above, $r = conn(\overline{M})$ with $r\geq 1$. Then*

$$R_k \geq \begin{cases} k+r-1, & \text{if } U\cup\partial M\neq\emptyset,\\ k+r-2, & \text{if } U\cup\partial M=\emptyset\end{cases}$$

The proof of this key lemma is based on a computation of Nakaoka ([28], proposition 4.3). We write $Y^{(k)}$ for the k-fold smash product of a based space Y and $X_{\mathfrak{S}_k}$ the orbit space of a $\mathfrak{S}_k$-space X.

Theorem 3.8. *(Nakaoka) If Y is r-connected, then $(Y^{(k)}/\Delta_{fat})_{\mathfrak{S}_k}$ is $r+k-1$ connected.*

Remark 3.9. In fact nakaoka only proves the homology version of this result and also assumes $r \geq 1$. An inspection of his proof shows that $r \geq 0$ works as well. Also his homology statement can be upgraded to a genuine connectivity statement. To see this, we can assume that $k \geq 2$ (the case $k = 1$ being trivial). One needs to show in that case that $\pi_1((Y^{(k)}/\Delta_{fat})_{\mathfrak{S}_k}) = 0$. This follows by an immediate application of Van Kampen and the fact that $\pi_1(Y^{(k)}/\mathfrak{S}_k) = \pi_1(\overline{SP}^k Y) = 0$ for $k \geq 2$. To see this last statement, recall that the natural map $\pi_1(Y)\longrightarrow\pi_1(SP^kY)$ factors through $H_1(Y;\mathbb{Z})$ and then induces an isomorphism $H_1(Y;\mathbb{Z}) \cong \pi_1(SP^kY)$ when $k \geq 2$ [36]. But if $SP^{k-1}(Y) \hookrightarrow SP^k(Y)$ induces a surjection on fundamental groups, then the cofiber is simply connected (Van-Kampen).

Proof. (of Lemma 3.7 and Theorem 1.2) By construction we have the equality $\overline{TP^k}(Y) = (Y^{(k)}/\Delta_{fat})_{\mathfrak{S}_k}$. The connectivity of $TP^k(M)/TP^{k-1}(M)$ is (at least) $k+r-1$ according to Theorem 3.8, while that of $TP^{k-1}(M)/TP^{k-2}(M)$ is at least $k+r-2$ which means that $conn(TP^k(M)/TP^{k-2}(M)) \geq k+r-2$ (by the long exact sequence of the triple $(TP^{k-2}(M), TP^{k-1}(M), TP^k(M))$). This produces the lower bounds on R_k in Lemma 3.7. Since the cohomology of $B(M-U,k)$ starts to vanish at $dk-R_k$ (Corollary 3.6), Theorem 1.2 follows. $\square$

4. Braid Spaces of Punctured Manifolds

We start with a simple proof of Theorem 1.8, (3); $\dim M = d \geq 2$ throughout.

Proof. (Theorem 1.8-(3)) This is a direct computation (with M closed)

$$\begin{aligned} H^j(B(M,n);\mathbb{F}_2) &\cong H_{nd-j}(TP^nM, TP^{n-2}M;\mathbb{F}_2) && \text{(Lemma 3.5)} \\ &\cong \tilde H_{nd-j}(\overline{TP}^nM;\mathbb{F}_2) \oplus \tilde H_{nd-j}(\overline{TP}^{n-1}M;\mathbb{F}_2) && (\textit{by } 3.1, (2)) \\ &\cong H^j(B(M-\{p\},n);\mathbb{F}_2) \oplus H^{j-d}(B(M-\{p\},n-1);\mathbb{F}_2) \end{aligned}$$

In this last step we have rewritten H_{nd-j} as $H_{(n-1)d-(j-d)}$ and reapplied Lemma 3.5. $\square$

Example: When $M = S^d$ and $n = 2$, then $B(S^d,2) \simeq \mathbb{R}P^d$ and $B(S^d - p, 2) = B(\mathbb{R}^d, 2) = \mathbb{R}P^{d-1}$ in full agreement with the splitting. This shows more importantly that the splitting is not valid for coefficients other than $\mathbb{F}_2$. The general case is covered by the following observation of Segal and McDuff.

Lemma 4.1. [23] *There is a long exact sequence*

$$\to H_{*-d+1}(B(M-*,n-1)) \to H_*(B(M-*,n)) \to H_*(B(M,n)) \to H_{*-d}(B(M-*,n-1))\cdots$$

Proof. Let U be an open disc in M of radius $< \epsilon$ and let $N = M - U$. We have that $B(M-*,n) \simeq B(N,n)$. There is an obvious inclusion $B(N,n)\longrightarrow B(M,n)$ and so we are done if we can show that the cofiber of this map is $\Sigma^d B(N,n-1)_+$. To that end using a trick as in [23] (proof of theorem 1.1) we replace $B(M,n)$ by the homotopy equivalent model $B'(M,n)$ of configurations $[x_1,\ldots,x_n] \in B(M,n)$ such that at most one of the x_i's is in U. The cofiber of $B(N,n) \hookrightarrow B'(M,n)$ is a based space at $*$ and consists of pairs $(x,D) \in \bar U \times B(N,n-1)$ such that if $x \in \partial \bar U$ then everything is collapsed out to $*$. But $U \cong D^d$ and $\bar U/\partial \bar U = S^d$ so that the cofiber is the half-smash product $S^d \rtimes B(N,n-1) = \Sigma^d B(N,n-1)_+$ as asserted. $\square$

In order to prove Theorem 1.8 we need the following result of Mostovoy.

Lemma 4.2. [26] *There is a homeomorphism $TP^n(S^1) \cong \mathbb{R}P^n$.*

Remark 4.3. We only need that the spaces be homotopy equivalent. It is actually not hard to see that $\mathrm{TP}^n(S^1)$ has the same homology as $\mathbb{R}P^n$ since it can be decomposed into cells one for each dimension less than n and with the right boundary maps. The k-th skeleton is $\mathrm{TP}^k(S^1)$. Indeed identify S^1 with

$[0,1]/\sim$. A point in $\mathrm{TP}^k(S^1)$ can be written as a tuple $0\le t_1\le\cdots\le t_k\le 1$ with identifications at $t_1=0, t_k=1$ and $t_i=t_{i+1}$. The set of all such points is therefore the image σ^k of a k-simplex $\Delta^k\longrightarrow\mathrm{TP}^k(S^1)$ with identifications along the faces $F_i\Delta^k$. Since all faces corresponding to $t_i=t_{i+1}$ map to the lower skeleton ($\mathrm{TP}^{k-2}(S^1)$) and since the last face $F_k\Delta^k$ (when $t_k=1$) is identified with the zeroth face ($t_1=0$) in $\mathrm{TP}^k(S^1)$, the corresponding *chain* map sends the boundary chain $\partial\sigma_k$ to the image of $\partial\Delta^k=\sum_{i=0}^k(-1)^iF_i\Delta^k$; that is to the image of $F_0\Delta^k+(-1)^kF_k\Delta^k$ which is $(1+(-1)^k)\sigma^{k-1}$.

We need one more lemma.

Lemma 4.4. *Set* $\overline{TP}^0(X)=S^0$. *Then* $\overline{TP}^n(X\vee Y)=\bigvee_{r+s=n}\overline{TP}^r(X)\wedge\overline{TP}^s(Y)$.

Proof. Here the smash products are taken with respect to the canonical basepoints of the various $\overline{TP}$'s. A configuration $[z_1,\dots,z_n]$ in $TP^n(X\vee Y)$ can be decomposed into a pair of the form $[x_1,\dots,x_r]\times[y_1,\dots,y_s]$ in $TP^r(X)\times TP^s(Y)$ for some $r+s=n$. This decomposition is unique if we demand that the basepoint (chosen to be the wedgepoint $*$) is not contained in the configuration. The ambiguity coming from this basepoint is removed when we quotient out $TP^n(X\vee Y)$ by $\underline{*}=TP^{n-1}(X\vee Y)$, and when we quotient out $\bigcup_{r+s=n}TP^r(X)\times TP^s(Y)$ by those pairs of configurations with the basepoint in either one of them. The proof follows. □

We are now in a position to prove the second splitting (4)

Proof. (of Theorem 1.8-(4)) Let $Q_k=\{p_1,\dots,p_k\}$ be a finite subset of M of cardinality k. We note that the quotient M/Q_k is of the homotopy type of the bouquet $M\vee\underbrace{S^1\vee\cdots\vee S^1}_{k-1}$, and that $\overline{TP}^l(S^1)=\mathbb{R}P^l/\mathbb{R}P^{l-1}=S^l$. Using field coefficients we then have (whenever we quote Lemma 3.5 below we assume that either M is even dimensional orientable or that $\mathbb{F}=\mathbb{F}_2$)

$$\begin{aligned}
&H^j(B(M-Q_k,n);\mathbb{F})\\
\cong\ &\tilde H_{nd-j}(\overline{TP}^n(M/Q_k)) &&\text{(Lemma 3.5 with } U\cup\partial M=Q_k)\\
\cong\ &\tilde H_{nd-j}(\overline{TP}^n(M\vee\bigvee_{k-1}S^1))\\
\cong\ &\tilde H_{nd-j}\left(\bigvee_{r+s_1+\cdots+s_{k-1}=n}\overline{TP}^r(M)\wedge\overline{TP}^{s_1}(S^1)\wedge\cdots\wedge\overline{TP}^{s_{k-1}}(S^1)\right)\\
\cong\ &\tilde H_{nd-j}\left(\bigvee_{r+s_1+\cdots+s_{k-1}=n}S^{n-r}\wedge\overline{TP}^rM\right)\\
\cong\ &\bigoplus_{r+s_1+\cdots+s_{k-1}=n}\tilde H_{nd-j-n+r}(\overline{TP}^rM)\\
\cong\ &\bigoplus_r\tilde H_{nd-j-n+r}(\overline{TP}^rM)^{\oplus p(k-1,n-r)}\\
\cong\ &\bigoplus_{r=0}^nH^{j-(n-r)(d-1)}(B(M-\{p\},r);\mathbb{F})^{\oplus p(k-1,n-r)} &&\text{(Lemma 3.5)}
\end{aligned}$$

This is what we wanted to prove. □

5. Connectivity of Symmetric Products

In this section we prove Theorem 1.6 and Proposition 1.7 of the introduction.

Theorem 5.1. *Suppose X is a based r-connected simplicial complex with $r\ge 1$ and let $n\ge 1$. Then $\overline{SP}^n(X)$ is $2n+r-2$- connected.*

Proof. The claim is tautological for $n=1$ and so we assume throughout that $n>1$. We use some key ideas from [2, 19]. Start with X simply connected and choose a CW complex Y such that $H_*(\Sigma Y) = H_*(X)$. If X is based and r-connected, then Y is based and $(r-1)$-connected. A crucial theorem of Dold [11] now asserts that $H_*(SP^nX)$, and hence $H_*(\overline{SP}^nX)$, only depends on $H_*(X)$ so that in our case $H_*(\overline{SP}^nX) = H_*(\overline{SP}^n\Sigma Y)$. As before we write $X^{(n)}$ the n-fold smash product of X so that we can identify $\overline{SP}^nX$ with the quotient $X^{(n)}/\mathfrak{S}_n$ by the action of $\mathfrak{S}_n$. It will also be convenient to write $X^{(n)}_{\mathfrak{S}_n} := X^{(n)}/\mathfrak{S}_n$. Note that $X^{(n)}$ has a preferred basepoint which is fixed by the action of $\mathfrak{S}_n$ (i.e the action is *based*). By construction we have equivalences

$$\overline{SP}^n(\Sigma Y) = (\Sigma Y)^{(n)}_{\mathfrak{S}_n} = (S^1\wedge Y)^{(n)}_{\mathfrak{S}_n} = (S^1)^{(n)}\wedge_{\mathfrak{S}_n} Y^{(n)} \tag{12}$$

where here $A\wedge_{\mathfrak{S}_n} B$ is the notation for the quotient by the diagonal action of $\mathfrak{S}_n$ on $A\wedge B$ where A admits a based right action of $\mathfrak{S}_n$ and B a based left action.

We next observe that the quotient $(S^1)^{(n)}/K$ is contractible for any non-trivial subgroup $K\subset\mathfrak{S}_n$. This can be deduced from the fact ([19], section 6) that the permutation action desuspends in the sense that

$$(S^1)^{(n)}/K \simeq \Sigma\left(S^{n-1}/K\right)$$

where S^{n-1} is viewed as the unit sphere in $\mathbb{R}^n$ on which $K\subset\mathfrak{S}_n$ acts by permutation of coordinates that is by reflections across the hyperplanes $x_i=x_j$ in $\mathbb{R}^n$. Since S^{n-1}/K is obviously contractible for non-trivial K, then so is $(S^1)^{(n)}/K$. We can then use proposition 7.11 of [2] to conclude that $(S^1)^{(n)}\wedge_{\mathfrak{S}_n}\Delta_{fat}$ is contractible with Δ_{fat} as in (7). This subspace can then be collapsed out in the expression of $\overline{SP}^n(\Sigma Y)$ of (12) without changing the homotopy type and one obtains

$$\overline{SP}^n(\Sigma Y) \simeq (S^1)^{(n)}\wedge_{\mathfrak{S}_n}\left(Y^{(n)}/\Delta_{fat}\right) \tag{13}$$

The point of expressing $\overline{SP}^n(X)$ in this form is to take advantage of the fact that the action of $\mathfrak{S}_n$ on $Y^{(n)}/\Delta_{fat}$ is based free (i.e. free everywhere but at a single fixed point say x_0 to which the entire Δ_{fat} is collapsed out).

Consider the projection $W_n := S^n\times_{\mathfrak{S}_n}(Y^{(n)}/\Delta_{fat})\to (Y^{(n)}/\Delta_{fat})_{\mathfrak{S}_n}$. This map is a fibration on the complement of the point x_0 with fiber S^n there, and over x_0 the fiber is $F_0 = S^n/\mathfrak{S}_n$ (which is contractible). The space $\overline{SP}^n(\Sigma Y)$ in (13) is obtained from W_n by collapsing out F_0 (being contractible this won't matter) and $X_n := *\times_{\mathfrak{S}_n}(Y^{(n)}/\Delta_{fat}) = (Y^{(n)}/\Delta_{fat})_{\mathfrak{S}_n}$. Consider the sequence of maps $(S^n,*)\longrightarrow(W_n,X_n)\longrightarrow(X_n,X_n)$. This is a fibration away from the point $x_0\in X$ as we pointed out. One can then construct a relative serre spectral sequence (as in [19], §4) with E^2-term

$$E^2 = \tilde H_*(X_n;\tilde H_*(S^n)) \implies H_*(W_n,X_n)\cong H_*(\overline{SP}^n(\Sigma Y))$$

But X_n is $r+n-2$-connected (Theorem 3.8), $r+n-2\geq 1$, so that the E^2-term is made out of terms of homological dimension $r+n-1+n = 2n+r-1$ or higher which implies that $\overline{SP}^n(\Sigma Y) = \overline{SP}^n(X)$ has trivial homology up to $2n+r-2$. But $\overline{SP}^n(X)$ is simply connected if $n\geq 2$ (see remark after Theorem 3.8) and the proof follows. $\square$

Example 5.2. There is a homotopy equivalence $\overline{SP}^2(S^k)\simeq\Sigma^{k+1}\mathbb{R}P^{k-1}$ (see Hatcher, chapter 4 , example 4K.5). This space is $k+1 = 4+(k-1)-2$-connected as predicted and this is sharp.

5.1. **Two dimensional complexes.** To prove Proposition 1.7 we use a minimal and explicit complex constructed in [20]. The existence of this complex is due to the simple but exceptional property in dimension two that SP^nD, where $D\subset\mathbb{R}^2$ is a disc, is again a disc of dimension $2n$. Write $X = \bigvee^w S^1\cup(D^2_1\cup\cdots\cup D^2_r)$ and denote by $\star$ the symmetric product at the chain level. In [20] we constructed a space $\widetilde{SP}^nX$ homotopy equivalent to $SP^n(X)$ and such that $\widetilde{SP}X\simeq\coprod_{n\geq 0}\widetilde{SP}^nX$ has a multiplicative cellular chain complex generated under $\star$ by a zero dimensional class v_0, degree one classes $e_1,\ldots,e_w$

and degree $2s$ classes SP^sD_i, $1\leq i\leq r$, $1\leq s$, under the relations

$$e_i\star e_j = -e_j\star e_i\ \ (i\neq j)\ \ ,\ \ e_i\star e_i = 0$$
$$SP^sD_i\star SP^tD_i = \binom{s+t}{t}SP^{s+t}D_i$$

The cellular boundaries on these cells were also explicitly computed (but we don't need them here). The point however is that a cellular chain complex for $\widetilde{SP}^n(X)$ consists of the subcomplex generated by cells

$$v_0^r\star e_{i_1}\star\cdots\star e_{i_t}\star SP^{s_1}(D_{j_1})\star\cdots\star SP^{s_l}(D_{j_l})$$

with $r+t+s_1+\cdots+s_l=n$ and $t\leq w$ where w again is the number of leaves in the bouquet of circles. The dimension of such a cell is $t+2(s_1+\cdots+s_l)$ for pairwise distinct indices among the e_i's.

A cellular complex for $\overline{SP}^nX$ can then be taken to be the quotient of $C_*(\widetilde{SP}^nX)$ by the summand $C_*(\widetilde{SP}^{n-1}X)$ and this has cells of the form

$$e_{i_1}\star\cdots\star e_{i_t}\star SP^{s_1}(D_{j_1})\star\cdots\star SP^{s_l}(D_{j_l})$$

with $t+s_1+\cdots+s_l=n$. The dimension of such a cell is $t+2(s_1+\cdots+s_l)=2n-t$. The smallest such dimension is $2n-min(w,n)$. This means that $conn(\widetilde{SP}^nX/\widetilde{SP}^{n-1}X)=conn(\overline{SP}^nX)\geq 2n-min(w,n)-1$ and Proposition 1.7 follows.

Example 5.3. A good example to illustrate Proposition 1.7 is when S is a closed Riemann surface of genus g. It is well-known that for $n\geq 2g-1$, $SP^n(S)$ is an analytic fiber bundle over the Jacobian (by a result of Mattuck)

$$\mathbb{P}^{n-g}\longrightarrow SP^n(S)\xrightarrow{\mu}J(S)$$

where μ is the Abel-Jacobi map. In fact this is the projectivisation of an $n-g+1$ complex vector bundle over $J(S)$. Collapsing out fiberwise the hyperplanes $\mathbb{P}^{n-g-1}\subset\mathbb{P}^{n-g}$ we get a fibration ζ_n : $S^{2n-2g}\longrightarrow E_n\longrightarrow J(S)$ with a preferred section, so that for $n\geq 2g$, $\overline{SP}^n(S)$ is the cofiber of this section. This is $2n-2g-1$-connected as predicted, and in fact $\tilde{H}_*(\overline{SP}^n(S))=\sigma^{2n-2g}H_*(J(S))$ where σ is a formal suspension operator which raises degree by one.

5.2. **Connectivity and Truncated Products.** The homology of truncated products, and hence of braid spaces, is related to the homology of symmetric products via a very useful spectral sequence introduced in [6]. This spectral sequence has been used and adapted with relative success to other situations; eg. [21]. The starting point is the duality in lemma 3.5. The problem of computing $H^*(B(M,n);\mathbb{F})$ becomes then one of computing the homology of the relative groups $H_*(TP^n\overline{M},TP^{n-2}\overline{M};\mathbb{F})$. The key tool is the following *Eilenberg-Moore type* spectral sequence with field coefficients $\mathbb{F}$.

Theorem 5.4. [6] *Let X be a connected space with a non-degenerate basepoint. Then there is a spectral sequence converging to $H_*(TP^n(X),TP^{n-1}(X);\mathbb{F})$, with E^1-term*

$$\bigoplus_{i+2j=n}H_*(SP^iX,SP^{i-1}X)\otimes H_*(SP^j(\Sigma X),SP^{j-1}(\Sigma X))\tag{14}$$

and explicit d^1 differentials.

Field coefficients are used here because this spectral sequence uses the Kunneth formula to express E^1 as in (14). Here $SP^{-1}(X)=\emptyset$ and $SP^0(X)$ is the basepoint.

Example 5.5. When $X=S^1$, then $H_*(TP^n(S^1),TP^{n-1}(S^1))=\tilde{H}_*(S^n)$. Since $SP^iS^1\simeq S^1$ for all $i\geq 1$, the spectral sequence in this case has E^1-term of the form

$$H_*(S^1,*)\otimes H_*(\mathbb{P}^{\frac{n-1}{2}},\mathbb{P}^{\frac{n-1}{2}-1})=\sigma\tilde{H}_*(S^{n-1})=\tilde{H}_*(S^n)$$

if n is odd (where σ is the suspension operator), or $E^1_{*,*}=H_*(\mathbb{P}^{(n/2)},\mathbb{P}^{(n/2)-1})=\tilde{H}_*(S^n)$ if n is even. In all cases the spectral sequence collapses at E^1.

Now Lemma 3.5 combined with Theorem 5.4 gives an easy method to produce upper bounds for the non-vanishing degrees of $H^*(B(M,n))$. The least connectivity of the terms $\overline{SP}^i X \times \overline{SP}^j(\Sigma X)$ for $i+2j=n$ translates by duality to such an upper bound. This was in fact originally our approach to the cohomological dimension of braid spaces. We illustrate how we can apply this spectral sequence by deriving Corollary 1.5 from Proposition 1.7.

Proof. (Corollary 1.5) Suppose $Q \cup \partial S \neq \emptyset$. The spectral sequence of Theorem 5.4 converging to the homology of $(TP^k(\overline{S}), TP^{k-1}(\overline{S}))$ takes the form

$$E^1 = \tilde{H}_*(\overline{SP}^k\overline{S}) \bigoplus \oplus_{i+2j=k}(H_*(\overline{SP}^i\overline{S}) \otimes H_*(\overline{SP}^j(\Sigma\overline{S})) \bigoplus \tilde{H}_*(\overline{SP}^{k/2}(\Sigma\overline{S})) \tag{15}$$

(if k odd, the far right term is not there). We have that R_k (as in Corollary 3.6) is at least the connectivity of this E^1-term. Since $\overline{S}$ is a two dimensional complex, the connectivity of $\overline{SP}^i(\overline{S})$ is at least $2i - min(w,i) - 1$ (for some $w \geq 0$). The connectivity of $\overline{SP}^j(\Sigma\overline{S})$ is at least $2j + r - 2 \geq 2j - 1$ since $\Sigma\overline{S}$ is now simply connected (Theorem 5.1). The connectivity of $\overline{SP}^i(\overline{S}) \wedge \overline{SP}^j(\Sigma\overline{S})$ for non-zero i and j is then at least

$$(2i - min(w,i) - 1) + (2j - 1) + 1 = i + k - min(w,i) - 1$$

When $i = 0$, then $j = \frac{k}{2}$ (k even) and $conn(\overline{SP}^{k/2}(\Sigma\overline{S})) \geq k - 1$. The connectivity of the E^1-term (15) is at least the minimum of

$$\begin{cases} i + k - min(w,i) - 1 & 1 \leq i \leq k-1 \\ 2k - min(w,k) - 1 & i = k \\ k - 1 & i = 0 \end{cases}$$

which is $k - 1$. By duality $H^*(B(S - Q, k)) = 0$ for $* \geq 2k - k + 1 = k + 1$. If S is closed, then the same argument shows that this bound needs to be raised by one. □

6. Stability and Section Spaces

In this final section, we extrapolate on standard material and make slightly more precise a well-known relationship between configuration spaces and section spaces [23, 7, 34, 18].

When manifolds have a boundary or an end (eg. a puncture), one can construct embeddings

$$+ : B(M,k) \longrightarrow B(M,k+1) \tag{16}$$

by "addition of points" near the boundary, near "infinity" or near the puncture. In the case when $\partial M \neq \emptyset$ for example, one can pick a component A of the boundary and construct a nested sequence of collared neighborhoods $V_1 \supset V_2 \supset \cdots \supset A$ together with sequences of points $x_k \in V_k - V_{k+1}$. There are then embeddings $B(M - V_k, k) \longrightarrow B(M - V_{k+1}, k+1)$ sending $\sum z_i$ to $\sum z_i + x_k$. Now we can replace $B(M - V_k, k)$ by $B(M - A, k)$ and then by $B(M,k)$ up to small homotopy. In the direct limit of these embeddings we obtain a space denoted by $B(M, \infty)$. Note that an easy analog of Steenrod's splitting [6] gives the splitting

$$H_*(B(M,\infty)) \cong \bigoplus_{k=0} H_*(B(M,k+1), B(M,k)) \tag{17}$$

(here $B(M,0) = \emptyset$). In fact (17) is a special case of a trademark *stable splitting* result for configuration spaces of open manifolds or manifolds with boundary. Denote by $D_k(M)$ the cofiber of (16). For example $D_1(M) = B(M,1) = M$.

Theorem 6.1. [5, 8] *For M a manifold with non-empty boundary, there is a stable splitting (i.e. after sufficiently many suspensions)*

$$B(M,k) \simeq_s \bigvee_{i=0}^{k} D_i(M)$$

The classical case of $M = D^n$ (closed n-ball) is due to Victor Snaith. A short and clever argument of proof for this sort of splittings is due to Fred [8]. The next stability bound is due to Arnold and a detailed proof is in an appendix of [34].

Theorem 6.2. *(Arnold) The embedding $B(M,k) \hookrightarrow B(M,k+1)$ induces a homology monomorphism and a homology equivalence up to degree $[k/2]$.*

The monomorphism statement is in fact a consequence of (17). Arnold's range is not optimal. For instance

Theorem 6.3. [21] *If S is a compact Riemann surface and $S^* = S - \{p\}$, then $B(S^*,k) \hookrightarrow B(S^*,k+1)$ is a homology equivalence up to degree $k-1$.*

We define $s(k)$ to be the homological connectivity of $+ : B(M,k) \longrightarrow B(M,k+1)$ (see §1.3) . By Arnold, $s(k) \geq [k/2]$.

6.1. **Section Spaces.** If $\zeta : E \longrightarrow B$ is a fiber bundle over a base space B, we write $\Gamma(\zeta)$ for its space of sections. If ζ is trivial then evidently $\Gamma(\zeta)$ is the same as maps into the fiber. Let M be a closed smooth manifold of dimension d, $U \subset M$ a closed subspace and τ^+M the fiberwise one-point compactification of the tangent bundle over M with fiber $S^d = \mathbb{R}^d \cup \{\infty\}$. Then $\tau^+M \longrightarrow M$ has a preferred section s_∞ which is the section at ∞ and we let $\Gamma(\tau^+M;U)$ be those sections which coincide with s_∞ on U. Note that $\Gamma(\tau^+M)$ splits into components indexed by the integers as in

$$\Gamma(\tau^+M) := \coprod_{k\in\mathbb{Z}} \Gamma_k(\tau^+M)$$

This degree arises as follows. Let $s : M \longrightarrow \tau^+M$ be a section. By general position argument it intersects s_∞ at a finite number of points and there is a sign associated to each point. This sign is defined whether the manifold is oriented or not (as in the definition of the Euler number). The degree is then the signed sum. Similarly we can define a (relative) degree of sections in $\Gamma(\tau^+M;U)$.

Observe that if τ^+M is trivial, then $\Phi : \Gamma(\tau^+M) \xrightarrow{\simeq} \mathrm{Map}(M,S^d)$, where $d = \dim M$. The components of $\mathrm{Map}(M,S^d)$ are indexed by the degree of maps (Hopf), but at the level of components we have the equivalence

$$\Gamma_k(\tau^+M) \simeq \mathrm{Map}_{k+\ell}(M,S^d)$$

where ℓ is such that $\Phi(s_\infty) \in \mathrm{Map}_\ell$. In the case when $M = S^{even}$, then $\Phi(s_\infty)$ is the antipodal map which has degree $\ell = -1$ [33]. When $M = S$ is a compact Riemann surface, $\ell = -1$ when the genus is even and $\ell = 0$ when the genus is odd [21]. Further relevant homotopy theoretic properties of section spaces are summarized in the appendix.

6.2. **Scanning and Stability.** A beautiful and important connection between braid spaces and section spaces can be found for example in [35, 23, 18] (see [10] for the fiberwise version). This connection is embodied in the "scanning" map

$$S_k : B(M-U,k) \longrightarrow \Gamma_k(\tau^+M; U \cup \partial M) \tag{18}$$

where U is a closed subspace of M. Here and throughout we assume that removing a subspace as in $M-U$ doesn't disconnect the space. The scanning map has very useful homological properties. A sketch of the construction of S_k for closed Riemaniann M goes as follows (for a construction that works for topological manifolds see for example [14]). First construct $S_1 : M-U \longrightarrow \Gamma_1$. We can suppose that M has a Riemannian metric and use the existence of an exponential map for τM which is a continuous family of embeddings $exp_x : \tau_x M \longrightarrow M$ for $x \in M$ such that $x \in im(exp_x)$ and $im(exp_x)^+ \cong \tau_x^+ M$ (the fiber at x of τ^+M). By collapsing out for each x the complement of $im(exp_x)$ we get a map $c_x : M \longrightarrow im(exp_x)^+ \cong \tau_x^+M$ Let V be an open neighborhood of U, $M-V \longrightarrow M-U$ being a deformation retract. Then we have the map

$$S_1 : M - V \longrightarrow \Gamma(\tau^+M)\ ,\ y \mapsto (x \mapsto c_x(y)) \in \tau_x^+M$$

Observe that for x near U, the section $S_1(y)$ agrees with the section at infinity (i.e. we say it is *null*). In fact and more precisely, S_1 maps into $\Gamma^c(\tau^+M,U)$ the space of sections which are null outside a compact subspace of $M-U$. A deformation argument shows that $\Gamma^c\simeq\Gamma$. It will be convenient to say that a section $s\in\Gamma$ is *supported* in a subset $N\subset M$ if $s=s_\infty$ outside of N. A useful observation is that if s_1,s_2 are two sections supported in closed A and B and $A\cap B=\emptyset$, then we can define a new section which is supported in $A\cup B$, restricting to s_1 on A and to s_2 on B.

Extending S_1 to S_k is now easy. We first choose $\epsilon>0$ so that $B^\epsilon(M,k)$ the closed subset of $B(M,k)$ where particles have pairwise separation $\geq 2\epsilon$ is homotopic to $B(M,k)$ (this is verified in [23], lemma 2.3). We next choose the exponential maps to be supported in neighborhoods of radius ϵ. Given a finite subset $Q:=\{y_1,\dots,y_k\}\in B^\epsilon(M-U,k)$, each point y_i determines a section supported in $V_i:=im(exp_{y_i})$. Since the V_i's are pairwise disjoint, these sections fit together to give a section s_Q supported in $\bigcup V_i$ so that $S_k(Q):=s_Q$.

When M is compact with boundary, then we get the map in (18) by replacing $B(M-U,k)$ by $B(M-U\cup\partial M,k)$ and $\Gamma^c(\tau^+M,U)$ by $\Gamma(\tau^+M,U\cup\partial M)$ the space of sections that are null outside a compact subspace of $M-U\cup\partial M$. We let $s(k)$ be the stability range of the map $B(M-U,k)\longrightarrow B(M-U,k+1)$ (as in §6.1)

The next proposition is a follow up on a main result of [23] (see also [18]).

Proposition 6.4. *Suppose M is a closed manifold and $U\subset M$ a non-empty closed subset, $M-U$ connected. Then the map $S_{k*}:H_*(B(M-U,k))\longrightarrow H_*(\Gamma_k(\tau^+M,U))$ is a monomorphism in all dimensions and an isomorphism up to dimension $s(k)$.*

Proof. It is easy to see that the maps S_k for various k are compatible up to homotopy with stabilization so we obtain a map $S:B(M,\infty)\longrightarrow\Gamma_\infty(\tau^+M,U):=\lim_k\Gamma_k(\tau^+M,U)$ which according to the main theorem of McDuff is a homology equivalence (in fact all components of $\Gamma(\tau^+M,U)$ are equivalent and Γ_∞ can be chosen to be the component containing s_∞). But according to (17) $H_*(B(M-U,k))\to H_*(B(M-U,\infty))$ is a monomorphism, and then an isomorphism up to dimension $s(k)$. The claim follows. ∎

This now also implies our last main result from the introduction.

Proof. (of Proposition 1.10) Suppose that M is a closed manifold of dimension d, U a small open neighborhood of the basepoint $*$ and consider the fibration (see appendix)

$$\Gamma_k(\tau^+M;\bar U)\longrightarrow\Gamma_k(\tau^+M)\longrightarrow S^d$$

The main point is to use the fact as in ([23], proof of theorem 1.1) that scanning sends the exact sequence in Lemma 4.1 to the Wang sequence of this fibration. Let $N=M-U$ so that we can identify $\Gamma_k(\tau^+M;\bar U)$ with $\Gamma_k(\tau^+N;\partial N)$ which we write for simplicity $\Gamma^c_k(\tau^+N)$ as before. Under these identifications and by a routine check we see that scanning induces commutative diagrams

$$\begin{array}{ccccccccc}
\to & H_{q-d+1}(B(N,k-1)) & \to & H_q(B(N,k)) & \to & H_q(B(M,k)) & \to & H_{q-d}(B(N,k-1)) & \to\\
 & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S} & \\
\to & H_{q-d+1}(\Gamma^c_k(\tau^+N)) & \to & H_q(\Gamma^c_k(\tau^+N)) & \to & H_q(\Gamma_k(\tau^+M)) & \to & H_{q-d}(\Gamma^c_k(\tau^+N)) & \to
\end{array}$$

where the top sequence is the homology exact sequence for the pair $(B(M,k),B(N,k))$ as discussed in Lemma 4.1 and the lower exact sequence is the Wang sequence of the fibration $\Gamma_k(\tau^+M)\longrightarrow S^d$. According to Proposition 6.4, the map $S_{k*}:H_q(B(N,k))\longrightarrow H_q(\Gamma^c_k(\tau^+N))$ is an isomorphism up to degree $q=s(k)$. It follows that all vertical maps in the diagram above involving the subspace N together with the next map on the right (which doesn't appear in the diagram) are isomorphisms whenever $q\leq s(k-1)\leq s(k)$. By the 5-lemma the middle map is then an isomorphism within that range as well. This proves the proposition. ∎

We can say a little more when $k=1$, M closed always.

Lemma 6.5. *The map $S_1 : M \longrightarrow \Gamma_1(\tau^+M)$ induces a monomorphism in homology in degrees $r+1, r+2$, where $r = conn(M)$, $r \geq 1$.*

Proof. Consider $\Gamma(s\tau^+M)$ the space of sections of the fibration $s\tau^+M \longrightarrow M$ obtained from τ^+M by applying fiberwise the functor SP^∞. It is easy to see that scanning has a stable analog $st : SP^\infty(M_+) \longrightarrow \Gamma(s\tau^+M)$ but harder to verify that st is a (weak) homotopy equivalence [14, 18]. Note that $SP^\infty(M_+) \simeq SP^\infty M \times \mathbb{Z}$ and $SP^\infty(M)$ is equivalent to a connected component (any of them) say $\Gamma_0(s\tau^+M)$. By construction the following diagram homotopy commutes

$$\begin{array}{ccc} M & \xrightarrow{S_1} & \Gamma_1(\tau^+M) \\ \big\downarrow & & \big\downarrow \alpha \\ SP^\infty(M) & \xrightarrow{st} & \Gamma_0(s\tau^+M) \end{array}$$

where the right vertical map α is induced from the natural fiber inclusion $\alpha : S^d \hookrightarrow SP^\infty(S^d)$. When M is r-connected, the map $M \longrightarrow SP^\infty(M)$ induces an isomorphism in homology in dimensions $r+1$ and $r+2$ ([28], Corollary 4.7). This means that the composite $M \to \Gamma_1(\tau^+M) \to \Gamma_1(s\tau^+M)$ is a homology isomorphism in those dimensions and the claim follows. $\square$

Remark 6.6. If M has boundary, then by scanning $M_0 := M - \partial M$ we obtain a map into the compactly supported sections $\Gamma(\tau^+M)$. This map extends to a map $S : M/\partial M \longrightarrow \Gamma(\tau^+M)$ which is according to [1] $(d-r+1)$-connected if M is r-connected of dimension $d \geq 2$.

7. Appendix: Some Homotopy Properties of Section Spaces

All spaces below are assumed connected. We discuss some pertinent statements from [37]. Let $p : E \longrightarrow B$ be a Serre fibration, $i : A \hookrightarrow X$ a cofibration (A can be empty) and $u : X \longrightarrow E$ a given map. Slightly changing the notation in that paper, we define

$$\Gamma_u(X, A; E, B) = \{f : X \longrightarrow E \mid f \circ i = u \circ i, p \circ f = p \circ u\}$$

This is a closed subspace of the space of all maps $\mathrm{Map}(X, E)$ and is in other words the solution space for the extension problem

$$\begin{array}{ccc} A & \xrightarrow{ui} & E \\ {\scriptstyle i}\big\downarrow & \overset{u}{\nearrow} & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow[pu]{} & B \end{array}$$

with data $u_{|A} : A \longrightarrow E$ and $pu : X \longrightarrow B$. When $A = \{x_0\}$ and $B = \{y_0\}$ then $\Gamma(X, x_0; E, y_0) = \mathrm{Map}^*(X, E)$ is the space of based maps from X to Y sending x_0 to y_0. On the other hand and when $X = B$ and $A = \emptyset$, then $\Gamma_u(B, \emptyset; E, B) = \Gamma(E)$ is the section space of the fibration $\zeta = (E \xrightarrow{p} B)$.

Proposition 7.1. [37]

- *If $A \subset X' \subset X$ is a nested sequence of NDR pairs, and $j : X' \hookrightarrow X$ the inclusion, then the induced map $\Gamma_u(X, A; E, B) \longrightarrow \Gamma_{uj}(X', A; E, B)$ yields a fibration with $\Gamma_u(X, X'; E, B)$ as fibre.*
- *If $E \longrightarrow E' \longrightarrow B$ are two fibrations and $q : E \longrightarrow E'$ the projection, then the induced map $\Gamma_u(X, A; E, B) \longrightarrow \Gamma_{qu}(X, A; E', B)$ is a fibration with $\Gamma_u(X, A; E, E')$ as fibre.*

The first part of Switzer's result implies that restriction of the bundle $\zeta : E \longrightarrow B$ to $X \subset B$ is a fibration $\Gamma(\zeta) \longrightarrow \Gamma(\zeta_{|X})$ with fiber the section space $\Gamma(\zeta, X)$ i.e. those sections of ζ which are "stationary" over X (compare [10], chapter 1, §8). An example of relevance is when $\zeta = \tau^+M$ is the fiberwise one-point compactification and s_∞ is the section mapping at infinity. Denote by S^d the fiber over $x_0 \in M$. If U is a small open neighborhood of x_0, then $\Gamma(\zeta_{|\bar{U}}) \simeq S^d$ and we have a fibration

$$\Gamma(\tau^+M, \bar{U}) \longrightarrow \Gamma(\tau^+M) \xrightarrow{res} S^d \tag{19}$$

where the fiber consists of those sections which coincide with s_∞ on U. So for instance if $M = S^d$, $\Gamma(\tau^+M, \bar{U}) \simeq \Omega^d S^d$ and the fibration reduces to the evaluation fibration $\Omega^d S^d \to \mathrm{Map}(S^d, S^d) \to S^d$.

Finally and according to [10] (p. 29), if $E \longrightarrow B$ is a Hurewicz fibration and s, t are two sections, then s and t are homotopic if and only if they are section homotopic. We use this to deduce the following lemma.

Lemma 7.2. *Let $\pi : E \longrightarrow B$ be a fibration with a preferred section s_∞ (which we choose as basepoint). Then the inclusion $\Gamma(E) \longrightarrow Map(B, E)$ induces a monomorphism on homotopy groups.*

Proof. We give $\Gamma(E) \subset \mathrm{Map}(B, E)$ the common basepoint s_∞. An element of $\pi_i\Gamma(E)$ is the homotopy class of a (based) map $\phi : S^i \longrightarrow \Gamma(E)$ or equivalently a map $\phi : S^i \times B \longrightarrow E$ (where $\phi(-, b) \in \pi^{-1}(b)$ and $\phi(N, -) = s_\infty(-)$, N the north pole of S^i) and the homotopy is through similar maps. Write Φ the image of ϕ via the composite $S^i \longrightarrow \Gamma(E) \longrightarrow \mathrm{Map}(B, E)$. Now Φ can be viewed as a section of $S^i \times E \longrightarrow S^i \times B$ and a null-homotopy of Φ is a homotopy to $id \times s_\infty$. Since this null-homotopy can be done fiberwise it is a null-homotopy in $\Gamma(E)$ from ϕ to s_∞. □

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Université des Sciences et Technologies de Lille, Laboratoire Painlevé, U.F.R de Mathématiques, 59655 Villeneuve d'Ascq, France

E-mail address: `sadok.kallel@math.univ-lille1.fr`