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Quasiclassical Geometry and Integrability of AdS/CFT Correspondence^{*}

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We discuss the quasiclassical geometry and integrable systems related to the gauge/string duality. The analysis of quasiclassical solutions to the Bethe ansatz equations arising in the context of the AdS/CFT correspondence is performed, compare to stationary phase equations for the matrix integrals. We demonstrate how the underlying geometry is related to the integrable sigma-models of dual string theory, and investigate some details of this correspondence.

1 Introduction

Gauge/string duality is an old fascinating subject [1], based, in particular, on relationship between the string worldsheets and Yang-Mills Feynman diagrams at strong coupling. A well-known example of such a duality is the matrix model description of two-dimensional quantum gravity and low-dimensional noncritical strings [2, 3].

A more complicated case of this duality is the so called AdS/CFT correspondence – an asserted equivalence of a four-dimensional $\mathcal{N} = 4$ supersymmetric gauge theory with a type IIB string theory in the ten-dimensional $AdS_5 \times S^5$ background. Among other predictions, the AdS/CFT conjecture relates the dimensions of gauge-invariant operators with the energies of particular closed string states propagating in the ten-dimensional $AdS_5 \times S^5$ spacetime background [4, 5, 6].

These anomalous dimensions in supersymmetric Yang-Mills theory (SYM) or energies of string states appear to be the quantities of particular interest, since they can be sometimes evaluated on different sides of the gauge/string duality, providing in this sense a quantitative test of the AdS/CFT correspondence. At present, there is a various number of methods and approaches for such tests, as well as vast literature on the subject, but below we are going to concentrate only on a particular way of testing the equality of the string energies to anomalous dimensions of the gauge operators.

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This way is based on appearance of integrable systems on both sides of the AdS/CFT correspondence. On gauge side this is *quantum* integrable models, observed even in the context of non-supersymmetric QCD and, in particular, responsible for diagonalization of the mixing matrix of renormalization of the constituent operators [7, 8]. A particular simple form [9] this matrix acquires in the sector of scalar operators, which are absent in non-supersymmetric gauge theories (generalized set of the famous BMN operators [10], see e.g. [11] on gauge-theory calculations of their renormalization). When restricted to the subsector of operators, consisting of two (among three) complex holomorphic scalars, it becomes literally equivalent to the Hamiltonian of the Heisenberg spin chain, solved long ago via the Bethe ansatz [12, 13].

On string side this is a classically integrable string sigma-model, satisfying the world-sheet Virasoro constraints (unfortunately nothing is really known yet about the quantum theory on this side, see however [14]). Only in particular so-called "pp-wave" limit of the $AdS_5 \times S^5$ background, the world-sheet Green-Schwarz theory can be quantized in the light-cone gauge [15]. However, this formulation is based on use of the Fock space of two-dimensional *massive* oscillators, which is hardly consistent with two-dimensional conformal invariance and, is rather a technically convenient way to describe a collection of effective *classical* oscillator-like modes of an integrable string model (see [16]).

Based on quite a number of achievements in computations of particular examples of anomalous dimensions [17] and energies of classical string trajectories [18], as well as matching between the higher charges of the entire integrable structures for the elliptic solutions [19], in [20] a general approach to comparing the gauge and string integrable systems was proposed. Up to now this approach was developed only for restricted sector of particular scalar operators and classical motions of string in the (subsector of) compact sphere S^5 of the $AdS_5 \times S^5$ geometry. It directly relates the *quasiclassical* solutions to Bethe equations, where elementary magnon excitations form condensates or "Bethe strings", with the collective classical configurations of spin chains, which can be identified with the particular limit of the the string sigma-model's finite-gap solutions.

In these notes we are going to review the results of [20] and present some of them in more general and invariant geometric way. As a toy model of solving quasiclassically the Bethe ansatz equations we first discuss the more simple quasiclassical solutions of matrix models. Then we present the comparison of gauge and string geometry underlying the correspondence. Finally, we discuss some physical consequences of the derived formulas and open problems.

2 Duality and geometry in matrix models

In contrast to higher-dimensional field theories, the zero-dimensional Yang-Mills theories – matrix models do not have renormalizations of any operators and the main aim there is to compute the (generalized) partition function – a generating function for the correlators ¹. The partition function of a matrix model

$$Z = \int d\Phi e^{-\frac{1}{\hbar} \text{Tr} W(\Phi)} = \frac{1}{N!} \int \prod_{i=1}^N \left(dz_i e^{-\frac{1}{\hbar} W(z_i)} \right) \prod_{i < j} (z_i - z_j)^2 \quad (2.1)$$

with some potential $W(\Phi) = \sum_{k=1}^{L+1} t_k \Phi^k$ is related to free energy of dual string theory

$$\mathcal{F} = -\log Z = \sum_{g=0}^{\infty} \hbar^{2g-2} \mathcal{F}_g \quad (2.2)$$

¹Here and everywhere below we consider only the gauge-invariant single-trace operators in matrix theories.

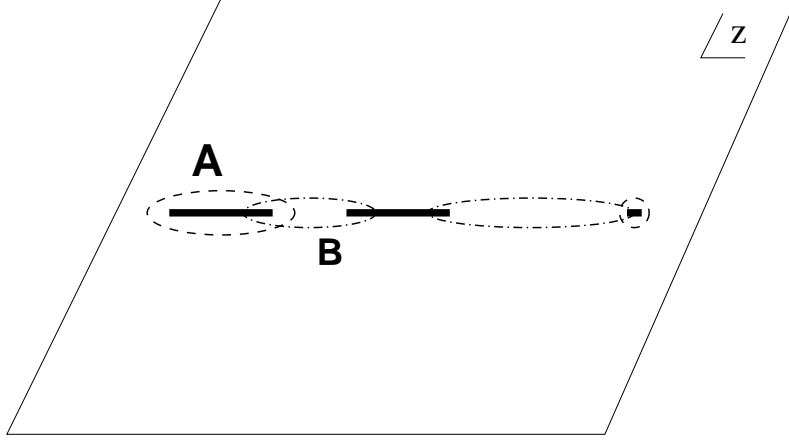


Figure 1: Riemann surface of the one-matrix model (2.1) as a double cover of the z -plane. The eigenvalues are condensed along the cuts, surrounded by A -cycles. The partition function (in leading in $\frac{1}{N}$ or quasiclassical approximation) is defined by its derivatives over the fractions of eigenvalues equal to the integrals along the dual B -cycles.

by the quasiclassical expansion in \hbar , which is equivalent to the 't Hooft's $\frac{1}{N}$ -expansion, provided by fixing $\hbar N = t_0$. The quasiclassical solution at $\hbar \rightarrow 0$, corresponding therefore to the planar limit $N \rightarrow \infty$ in summing over diagrams, where all closed string loops or higher topologies in (2.2) are suppressed, can be found studying the extrema of effective potential in (2.1)

$$W'(z_j) = 2\hbar \sum_{k \neq j} \frac{1}{z_j - z_k} \quad (2.3)$$

If $\hbar = 0$ the interaction is switched off, and all eigenvalues z_j are somehow distributed over the minima where $W'(z) = 0$. Introducing at $N \rightarrow \infty$ the eigenvalue density

$$\rho(z) = \hbar \sum_{j=1}^N \delta(z - z_j), \quad \int_{\mathbf{C}} dz \rho(z) = \hbar N \equiv t_0 \quad (2.4)$$

or resolvent, defined on the double cover of z -plane, cut along some segments $\mathbf{C} = \bigcup_j \mathbf{C}_j$, see fig. 1

$$G(z) = \int_{\mathbf{C}} \frac{\rho(\zeta) d\zeta}{z - \zeta}, \quad \frac{1}{2\pi i} \oint_{\mathbf{C}} dz G(z) = t_0 \quad (2.5)$$

equation (2.3) can be reduced to the integral equation

$$W'(z) = \int_{\mathbf{C}} \frac{\rho(\zeta) d\zeta}{z - \zeta} = G(z_+) + G(z_-), \quad z \in \forall \mathbf{C}_j \subset \mathbf{C} \quad (2.6)$$

where $z_{\pm} = z \pm i0$ are two "close" points on two different sides of the cut – above and below if cut is along the real axis like it can happen for polynomial potentials. This equation holds in any point of the eigenvalue

support \mathbf{C} , consisting of several disjoint pieces, and therefore formula (2.6) can be further rewritten as an algebraic equation on the resolvent G (see e.g. [21])

$$G^2 - W'(z)G = f(z) \quad (2.7)$$

where $f(z)$ in the r.h.s. is a polynomial of the power $L-1$ or one less than that of $W'(z)$. Equation (2.7) defines a hyperelliptic curve and the quasiclassical free energy \mathcal{F}_0 (the first term in (2.2)) can be entirely defined in terms of the curve (2.7) and the generating differential Gdz (2.5). For polynomial potentials the curve (2.7) is algebraic curve of finite genus $L-1$ and equation (2.7) defines resolvent G as *algebraic function* on the double-cover of z -plane. However, if one allows all possible long operators $\text{Tr}\Phi^L$ for $L \rightarrow \infty$ (whose renormalization in four-dimensional theory will be considered below) it becomes a curve of infinite genus and nothing can be said about the resolvent G immediately.

Indeed, introducing $Y = W'(z) - 2G$ and rewriting (2.7) as $Y^2 = W'(z)^2 + 4f(z)$ one immediately finds that this is a genus $g = L-1$ hyperelliptic Riemann surface and the auxiliary $L = g+1$ parameters – the coefficients of the polynomial $f(z)$ can be "eaten" by the fractions of eigenvalues on the cuts

$$S_j = \frac{i}{4\pi} \oint_{A_j} Y dz = \frac{1}{2\pi i} \oint_{A_j} G dz = \int_{\mathbf{C}_j} \rho(z) dz, \quad j = 1, \dots, L-1 \quad (2.8)$$

together with their total number t_0 (see (2.5)), and the dependence of free energy upon these variables is given by

$$\frac{\partial \mathcal{F}_0}{\partial S_j} = \frac{1}{2} \oint_{B_j} Y dz$$

where dual contours B_j are drawn on fig. 1.

Finally in this section let us point out that for non-polynomial potentials, when $L \rightarrow \infty$, to get any reasonable answer one should consider the situation when only finite number of extrema of the potential $W'(x) = 0$ are filled in by eigenvalues. The number of condensates K then becomes an extra parameter of the problem and, generally speaking, one should consider any $K < L$. This is in quite direct analogy with what are going to do below: for the long operators (nontrivial analogs of $\text{Tr}\Phi^L$ for $L \rightarrow \infty$) one has to define the Riemann surface "by hands", and not simply using the quasiclassical Baxter curve of, as in [22], – an analog of the matrix model curve (2.7) for a non-polynomial potential.

3 SYM and geometry of quasiclassical Bethe equations

In contrast to matrix model the AdS/CFT conjecture deals with the *four-dimensional* $\mathcal{N} = 4$ SYM with the $SU(N)$ gauge group. Again, the main contribution in $\frac{1}{N}$ -expansion, where closed string loops are suppressed, comes from the planar diagrams when $N \rightarrow \infty$ at fixed 't Hooft coupling $\lambda = g_{YM}^2 N = g_s N$, analogous to t_0 of the previous section, while string coupling $g_s = g_{YM}^2$ is an analog of quasiclassical parameter \hbar . At $\lambda \gg 1$ the $\mathcal{N} = 4$ SYM theory is believed to be dual to string theory in $AdS_5 \times S^5$ with the equal radii of curvature $\frac{R}{\sqrt{\alpha'}} = \lambda^{1/4}$. Therefore any test of the AdS/CFT conjecture implies comparing analytic series at $\lambda = 0$ (SYM perturbation theory) with analytic in $\alpha' \propto \frac{1}{\sqrt{\lambda}}$ worldsheet expansion ².

²There were, however, some attempts (not very promising from our point of view) to consider the world-sheet theory around so called "null-string" limit with $\alpha' \rightarrow \infty$, see [23].

A possible way-out from this discrepancy in parameters of expansion can be to consider the *classical* string solutions with large values of integrals of motion (usually referred as "spins" J) on $AdS_5 \times S^5$ side [16, 18], whose energies should correspond to anomalous dimensions of "long" operators on gauge side. In this case the classical string energy of the form $\Delta = \sqrt{\lambda} \mathcal{E} \left(\frac{J}{\sqrt{\lambda}} \right)$ may have an expansion of the form $\Delta = J + \sum_{l=1}^{\infty} E_l \left(\frac{\lambda}{J^2} \right)^l$ over the integer powers of 't Hooft coupling, which can be treated as series at $\lambda = 0$ even at $\lambda \gg 1$ provided large λ is suppressed by large value of the integrals of motion J . If it happens (this is not, of course, guaranteed) the classical string energy can be tested by direct comparison with perturbative series for gauge theory.

The four-dimensional $\mathcal{N} = 4$ SYM is conformal theory, i.e. $\beta(g_{YM}) = 0$, but the anomalous dimensions γ of the composite operators, e.g. $\text{Tr}(\Phi_{i_1} \dots \Phi_{i_L})$ are still renormalized nontrivially. In [20] and below we consider the particular scalar operators from this set, though the proposed approach can be applied in much more general situation. On string side such operators correspond to the string motion in the compact S^5 -part of ten-dimensional target-space, due to standard Kaluza-Klein argument. To simplify the situation maximally, choose two complex $\Phi_1 = \Phi_1 + i\Phi_2$ and $\Phi_2 = \Phi_3 + i\Phi_4$ fields among six real Φ_i and consider the *holomorphic* operators

$$\text{Tr}(\Phi_1 \Phi_1 \Phi_1 \Phi_2 \Phi_2 \Phi_1 \Phi_2 \Phi_2 \Phi_1 \Phi_1 \Phi_1 \Phi_2 \dots) \quad (3.9)$$

which can be conveniently labeled by arrows as $|\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\uparrow\downarrow\dots\rangle \in (\mathbf{C}^2)^{\otimes L}$. The holomorphic subsector is "closed" under renormalization and anomalous dimensions are eigenvalues of the $2^L \times 2^L$ mixing matrix

$$H = \frac{\lambda}{16\pi^2} \sum_{l=1}^L (1 - \sigma_l \cdot \sigma_{l+1}) + O(\lambda^2) \quad (3.10)$$

which is, up to addition of a constant, the permutation operator in $(\mathbf{C}^2) \otimes (\mathbf{C}^2)$, whose appearance is determined by structure of the Φ^4 -vertex in SYM Lagrangian, or the Hamiltonian for Heisenberg magnetic [9].

It is well-known, that this matrix can be diagonalized using the Bethe ansatz [12], (see e.g. [13] for present status of this technique and comprehensive list of references). Eigenvectors of H are parameterized by Bethe roots $\{u_1, \dots, u_J\}$, ($J \leq \frac{1}{2}L$ due to an obvious \mathbf{Z}_2 -symmetry) satisfying the Bethe ansatz equations for the Heisenberg spin chain

$$L \log \left(\frac{u_j + i/2}{u_j - i/2} \right) = 2\pi i n_j + \sum_{k \neq j}^J \log \frac{u_j - u_k + i}{u_j - u_k - i} \quad (3.11)$$

(with mode numbers $n_j \in \mathbf{Z}$ being some integers, never vanishing for a nontrivial solution). For the operators (3.9) equations (3.11) are supplied by the "trace condition"

$$e^{iP} = \prod_{j=1}^J \frac{u_j + i/2}{u_j - i/2} = 1, \quad (3.12)$$

i.e. integrality of the total momentum: $\frac{P}{2\pi} \in \mathbf{Z}$. The energy of the solution, is given by sum over magnons, i.e. in the leading order

$$\gamma = \frac{\lambda}{8\pi^2} \sum_{j=1}^J \frac{1}{u_j^2 + 1/4} + O(\lambda^2) \quad (3.13)$$

and equals to the (one-loop) Yang-Mills scaling dimension γ up to a factor.

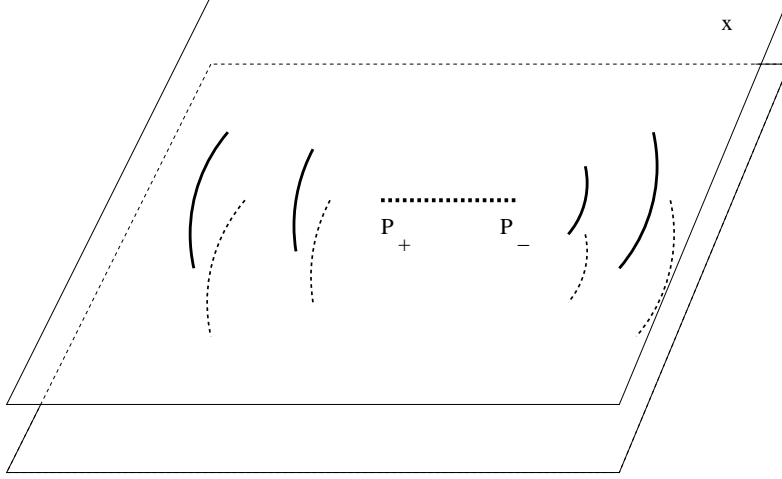


Figure 2: Riemann surface Σ , which is a double-cover of the x -plane cut along the Bethe strings (four slightly curved lines on each sheet in this example), which cross the real axis at $x = \frac{1}{2\pi n_l}$. The upper sheet is physical, while on the lower sheet the resolvent may have extra singularities.

For comparison with dual string theory we are interested in the long operators with $L \rightarrow \infty$, for which the Bethe roots are typically of the order of $u_j \sim L$. Rescaling $u_j = Lx_j$, and neglecting the higher in $\frac{1}{L}$ terms, one gets from (3.11)

$$\frac{1}{x_j} = 2\pi n_j + \frac{2}{L} \sum_{k \neq j}^J \frac{1}{x_j - x_k} \quad (3.14)$$

Like in the matrix model case (2.3), in the absence of the second in the r.h.s. of (3.14) "interaction term", $x_j = \frac{1}{2\pi n_j}$ for each n_j , and when we switch on the interaction the roots corresponding to n_j will "concentrate" around $\frac{1}{2\pi n_j}$ and form the so called "Bethe strings", shown at fig. 2. Introduce at $L \rightarrow \infty$, as in (2.4), (2.5), the density

$$\rho(x) = \frac{1}{L} \sum_{j=1}^J \delta(x - x_j), \quad \int_{\mathbf{C}} dx \rho(x) = \frac{J}{L} \quad (3.15)$$

or resolvent

$$G(x) = \frac{1}{L} \sum_{j=1}^J \frac{1}{x - x_j} = \int_{\mathbf{C}} \frac{d\xi \rho(\xi)}{x - \xi}, \quad \frac{1}{2\pi i} \oint_{\mathbf{C}} dx G(x) = \frac{J}{L} \quad (3.16)$$

Suppose we have *finite* number of different $n_l \neq n_{l'}$, with $l, l' = 1, \dots, K$, then in the scaling limit the total eigenvalue support is again $\mathbf{C} = \mathbf{C}_1 \cup \dots \cup \mathbf{C}_K$, where on each component one gets from (3.14)

$$2 \int_{\mathbf{C}} \frac{d\xi \rho(\xi)}{x - \xi} = G(x_+) + G(x_-) = \frac{1}{x} - 2\pi n_l, \quad x \in \mathbf{C}_l \quad (3.17)$$

where $G(x_{\pm})$ are values of the resolvent on two different sides of the cut. The integrality of total momentum condition, using the Bethe equations (3.14), acquires the form

$$\frac{1}{L} \sum_{j=1}^J \frac{1}{x_j} = 2\pi \sum_{l=1}^K n_l \int_{\mathbf{C}_l} \rho(x) dx = 2\pi m, \quad m \in \mathbf{Z} \quad (3.18)$$

or

$$\frac{1}{2\pi i} \oint_{\mathbf{C}} \frac{G(x) dx}{x} = 2\pi m, \quad n_l, m \in \mathbf{Z}. \quad (3.19)$$

Different $n_l \neq n_{l'}$ on different parts of support $\mathbf{C}_l \cap \mathbf{C}_{l'} = \emptyset$ mean that, in contrast to the matrix model case $G(x) = \int^x dG$ is not already a single-valued function, but an Abelian integral on some hyperelliptic curve Σ

$$y^2 = R_{2K}(x) = x^{2K} + r_1 x^{2K-1} + \dots + r_{2K} = \prod_{j=1}^{2K} (x - x_j) \quad (3.20)$$

where x_j are roots of the polynomial $R_{2K}(x)$. Equations (3.16), (3.17) and (3.19) can be solved after reformulating them as a set of properties of the meromorphic differential dG :

- dG is the second-kind Abelian differential with the only second-order pole at the point P_0 , ($x(P_0) = 0$ on unphysical sheet of the Riemann surface Σ);
- dG has integral B -periods

$$\oint_{B_i} dG = 2\pi(n_i - n_K) \quad (3.21)$$

More exactly one can write [20]

$$\int_{B'_j} dG = 2\pi n_j, \quad j = 1, \dots, K+1 \quad (3.22)$$

where B'_j is the contour from ∞_- on the lower sheet to ∞_+ on the upper sheet, passing through the j -th cut, so that $B_j = B'_j - B'_K$, for $j = 1, \dots, K$;

- dG has the following behavior at infinity

$$dG \underset{x \rightarrow \infty}{=} \frac{J dx}{L x^2} + \dots \quad (3.23)$$

and the Abelian integral $G(x)$ itself is fixed by

$$G(x) = 2\pi m + \int_0^x dG, \quad \text{or} \quad G(0) = 2\pi m \quad (3.24)$$

The general solution for the differential dG on hyperelliptic curve (3.20), satisfying the above requirements (3.21), (3.22), (3.23) and (3.24) reads [20]

$$dG = -\frac{dx}{2x^2} \left(1 - \frac{\sqrt{r_{2K}}}{y} \right) + \frac{r_{2K-1}}{4\sqrt{r_{2K}}} \frac{dx}{xy} + \sum_{k=1}^{K-1} a_k \frac{x^{k-1} dx}{y} \quad (3.25)$$

together with the extra conditions, ensuring single-valuedness of the resolvent on "upper" physical sheet

$$\oint_{A_i} dG = 0, \quad i = 1, 2, \dots, K - 1 \quad (3.26)$$

to be easily solved for the coefficients $\{a_k\}$. The rest of parameters is "eaten by" fractions of roots on particular pieces of support

$$S_j = -\frac{1}{2\pi i} \oint_{A_j} x dG = \int_{C_j} \rho(x) dx, \quad (3.27)$$

$$j = 1, \dots, g = K - 1$$

the total amount of Bethe roots (3.15), and the total momentum (3.19).

The energy or one-loop anomalous dimension for generic finite-gap solution [20] can be read from (3.13), (3.25)

$$\gamma = \frac{\lambda}{8\pi^2 L} \oint_{\mathbf{C}} \frac{dx}{2\pi i x^2} G(x) = \frac{\lambda}{8\pi^2 L} \left(\frac{r_{2K-2}}{4r_{2K}} - \frac{r_{2K-1}^2}{16r_{2K}^2} - \frac{a_1}{\sqrt{r_{2K}}} \right) \quad (3.28)$$

The anomalous dimensions defined by (3.28) are functions of the coefficients of the embedding equation (3.20) and a_1 which again is expressed through these coefficients by means of (3.26). The moduli of the curve (3.20) are themselves (implicitly) expressed through the mode numbers n_j and root fractions S_j via (3.21) or (3.22) and (3.27) (together with the total momentum (3.19) and the total number of Bethe roots (3.16)).

4 Geometry of classical string solutions

The general solution for anomalous dimension (3.28) is expressed through the integrals of motion on some *classical* configurations of the Heisenberg magnet [20]³. In the dual string picture one has the classical trajectories of string, moving in (subspace of) $AdS_5 \times S^5$ and the finite gap solutions to string sigma-model in AdS-like spaces were first constructed in [25]. In the Appendix to [20] it was demonstrated that this construction, slightly modified, can be easily applied to the case of compact S^d sigma-models. In this section we are going to show how these classical solutions can be compared with the quasiclassical solutions on the gauge side.

In particular subsector of only two holomorphic fields one gets the $S^3 \subset S^5$ sigma-model (in the AdS_5 -sector the only nontrivial string co-ordinate on the solution is "time" $X_0 = \frac{\Delta}{\sqrt{\lambda}} \tau$), which is equivalent [26] (since S^3 is the group-manifold of $SU(2)$) to the $SU(2)$ principal chiral field with the Lax pair, (see [20] for more details)

$$J_{\pm}(x) = \frac{\Delta}{\sqrt{\lambda}} \frac{i\mathbf{S}_{\pm} \cdot \boldsymbol{\sigma}}{1 \mp X} \quad (4.29)$$

$$\partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0$$

$$\partial_+ J_- + \partial_- J_+ = 0$$

which has two simple poles at values of string spectral parameter $X = X(P_{\pm}) = \pm \frac{\sqrt{\lambda}}{4\pi\Delta}$. In different words,

³Such correspondence with classical solutions was first noticed in [24] for the non-linear Schrödinger equation.

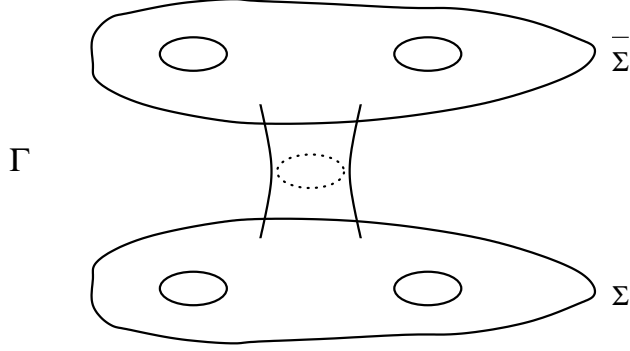


Figure 3: Riemann surface Γ , which is a double cover of Σ with a single cut.

such sigma-model is equivalent to a system of *two* interacting relativistic spins \mathbf{S}_+ and \mathbf{S}_- :

$$\begin{aligned}\partial_+ \mathbf{S}_- + \frac{2\Delta}{\sqrt{\lambda}} \mathbf{S}_- \times \mathbf{S}_+ &= 0, \\ \partial_- \mathbf{S}_+ - \frac{2\Delta}{\sqrt{\lambda}} \mathbf{S}_- \times \mathbf{S}_+ &= 0.\end{aligned}\tag{4.30}$$

which in some "non-relativistic limit" degenerates into the Heisenberg magnet [20], which is similar to a limit, studied in the papers [33].

However, this method can be used only for the group-manifolds. Nevertheless, in general situation the string sigma-model solution for the complex co-ordinates $Z_I(\tau, \sigma)$ and $\bar{Z}_I(\tau, \sigma)$ on S^{2D-1} , (constraint by $\sum_I |Z_I|^2 = 1$)

$$Z_I(\sigma_{\pm}) = r_I \Upsilon(q_I, \sigma_{\pm}) \quad \bar{Z}_I(\sigma_{\pm}) = r_I \Upsilon(\bar{q}_I, \sigma_{\pm}) = r_I \overline{\Upsilon(q_I, \sigma_{\pm})}, \quad I = 1, \dots, D\tag{4.31}$$

can be found [25] in terms of the Baker-Akhiezer (BA) functions

$$\Upsilon(P, \sigma_{\pm}) \underset{P \rightarrow P_{\pm}}{=} e^{k_{\pm} \sigma_{\pm}} \left(1 + \sum_{j=1}^{\infty} \frac{\xi_j(\sigma_{\pm})}{k_{\pm}^j} \right) \propto e^{\Omega_+(P)\sigma_+ + \Omega_-(P)\sigma_- - \theta(\mathcal{A}(P) + \mathbf{U}_+ \sigma_+ + \mathbf{U}_- \sigma_-)}\tag{4.32}$$

defined on double cover Γ (branched at P_+ and P_-) of a Riemann surface Σ (see fig. 3). For only two complex co-ordinates Z_I (like in the S^3 case) the curve Σ is hyperelliptic and directly related with the curve (3.20) of the Heisenberg chain. For the S^5 case Σ can be presented as some three-sheet cover of an X -plane, and is presumably related to the three-sheet covers arising in solving of the Bethe ansatz equations for the operators beyond the $SU(2)$ sector [27, 28, 29].

The BA function (4.32) (and hence the solution to sigma-model) is constructed in terms of *two* second-kind Abelian differentials $d\Omega_{\pm}$ on Riemann surface Γ

$$d\Omega_{\pm} \underset{P \rightarrow P_{\pm}}{=} \pm dk_{\pm} (1 + O(k_{\pm}^{-2})), \quad \oint_{\mathbf{A}} d\Omega_{\pm} = 0 \quad (4.33)$$

with the only second-order pole at P_{\pm} respectively; $\mathbf{U}_{\pm} = \oint_{\mathbf{B}} d\Omega_{\pm}$ are the vectors of their \mathbf{B} -periods.

The proof of the fact that formulas (4.31) are solutions to the sigma-model, satisfying classical Virasoro constraints, is based on existence of the third-kind Abelian differential $d\Omega$ on Σ with the simple poles at P_{\pm} and zeroes in the poles of the BA functions Υ and conjugated $\bar{\Upsilon}$. Then one may define the string resolvent or quasimomentum by the following formula

$$d\mathcal{G} = \frac{1}{2} (d\Omega_{+} - d\Omega_{-}) \quad (4.34)$$

$$d\mathcal{G} \underset{P \rightarrow P_{\pm}}{=} \frac{1}{2} dk_{\pm} (1 + O(k_{\pm}^{-2})), \quad \oint_{\mathbf{A}} d\mathcal{G} = 0$$

For periodic in σ solution, as follows from (4.32), the B -periods of the resolvent $\frac{1}{2\pi} \oint_{\mathbf{B}} d\mathcal{G} \in \mathbb{Z}$ are integer-valued, and for the periodic solutions one can write

$$d\Omega = \frac{\bar{\Upsilon}\Upsilon}{\langle \bar{\Upsilon}\Upsilon \rangle} d\mathcal{G} \quad (4.35)$$

where brackets mean the average over the period in σ -variable.

The BA function (4.32) satisfies the second-order differential equation

$$(\partial_{+}\partial_{-} + u)\Upsilon(P, \sigma_{\pm}) = 0, \quad P \in \Gamma \quad (4.36)$$

where $u \propto \sum_i (\partial_{+}Z_i\partial_{-}\bar{Z}_i + \partial_{-}Z_i\partial_{+}\bar{Z}_i)$. This fact and the Virasoro constraints $\lambda \sum_I |\partial_{\pm}Z_I|^2 = \Delta$ are guaranteed by the properties of the differential (4.35) and existence of the function E on Σ with D simple poles $\{q_I\}$ and the following behavior at the vicinities of the points P_{\pm} : $E = E_{\pm} \pm \frac{4\pi\Delta}{\sqrt{\lambda}} \frac{1}{k_{\pm}^2} + \dots$, (cf. with [25]).

The normalization factors in the expressions for the sigma-model co-ordinates (4.31) are determined by the formulas

$$r_I^2 = \frac{\text{res}_{q_I} E d\Omega}{E_{-} - E_{+}}, \quad I = 1, \dots, D \quad (4.37)$$

where $E_{\pm} = E(P_{\pm})$ and normalizations (4.37) satisfy $\sum_{I=1}^D r_I^2 = 1$ due to vanishing of the total sum over the residues $\sum \text{res}(E d\Omega) = 0$.

Rescaling $\partial_{\pm} \rightarrow \frac{\sqrt{\lambda}}{4\Delta} \partial_{\tau} \pm \partial_{\sigma}$, $\Upsilon \rightarrow e^{\frac{2\Delta}{\sqrt{\lambda}}\tau} \Psi$, $\bar{\Upsilon} \rightarrow e^{-\frac{2\Delta}{\sqrt{\lambda}}\tau} \bar{\Psi}$, $u \rightarrow u - \frac{4\Delta^2}{\lambda}$, in the limit $\lambda/\Delta^2 \ll 1$, one gets from (4.36) the non-stationary Schrödinger equation

$$\begin{aligned} (\partial_{\tau} - \partial_{\sigma}^2 + u) \Psi &= 0 \\ (-\partial_{\tau} - \partial_{\sigma}^2 + u) \bar{\Psi} &= 0 \end{aligned} \quad (4.38)$$

where now both BA functions Ψ and $\bar{\Psi}$ can be defined on Riemann surface Σ (see fig. 2), with the ends of the extra cut P_{\pm} are shrunk to a single point P_0 , with the expansion at the vicinity of this point (with new local

parameter $k(P_0) = \infty$)

$$\begin{aligned}\Psi_{P \rightarrow P_0} &= e^{k\sigma + k^2\tau} \left(1 + \frac{\psi_1}{k} + \frac{\psi_2}{k^2} + \dots \right) \\ \bar{\Psi}_{P \rightarrow P_0} &= e^{-k\sigma - k^2\tau} \left(1 + \frac{\bar{\psi}_1}{k} + \frac{\bar{\psi}_2}{k^2} + \dots \right)\end{aligned}\tag{4.39}$$

Substituting expansions (4.39) into (4.38) one gets

$$\begin{aligned}u &= 2\frac{\partial\psi_1}{\partial\sigma} = -2\frac{\partial\bar{\psi}_1}{\partial\sigma} \\ \frac{\partial\psi_1}{\partial\tau} - 2\frac{\partial\psi_2}{\partial\sigma} - \frac{\partial^2\psi_1}{\partial\sigma^2} + u\psi_1 &= 0 \\ -\frac{\partial\bar{\psi}_1}{\partial\tau} + 2\frac{\partial\bar{\psi}_2}{\partial\sigma} - \frac{\partial^2\bar{\psi}_1}{\partial\sigma^2} + u\bar{\psi}_1 &= 0\end{aligned}\tag{4.40}$$

For the conjugated functions one has $\psi_1 + \bar{\psi}_1 = 0$ and then $\frac{\partial}{\partial\sigma}(\psi_2 + \bar{\psi}_2 + \psi_1\bar{\psi}_1) = -\frac{\partial u}{\partial\sigma}$. Therefore, one gets an expansion

$$\bar{\Psi}\Psi_{P \rightarrow P_0} = 1 + \frac{u}{k^2} + \dots\tag{4.41}$$

The differential (4.35) in this limit turns into $d\Omega = \frac{\bar{\Psi}\Psi}{\langle\bar{\Psi}\Psi\rangle}dG$ and function E acquires a simple pole at P_0 , i.e. $E_{P \rightarrow P_0} = k + \dots$, if written in terms of new local parameter k . From vanishing of the sum of the residues of differential $E d\Omega$ one gets now

$$u = \text{res}_{P_0} F dQ = - \sum_{q_I} F \frac{\bar{\Psi}\Psi}{\langle\bar{\Psi}\Psi\rangle} dG \propto \sum_I \bar{\Psi}(q_I)\Psi(q_I)\tag{4.42}$$

It means, that $\Psi_I(\tau, \sigma) \propto \Psi(q_I)$ satisfy some vector non-linear Schrödinger equation [30]

$$\left(\partial_\tau - \partial_\sigma^2 + \sum_J |\Psi_J|^2 \right) \Psi_I = 0\tag{4.43}$$

In the case of $D = 2$ the curve Σ is hyperelliptic and one can take the function E with the only two poles, see below. Then (4.43) turns into the ordinary non-linear Schrödinger equation, which can be transformed to the Heisenberg magnetic chain [31, 32]:

$$\begin{aligned}|\Psi|^2 &\propto \mathbf{S}_\sigma^2 \\ \bar{\Psi}\partial\Psi - \Psi\partial\bar{\Psi} &\propto (\mathbf{S}_\sigma \cdot \mathbf{S} \times \mathbf{S}_{\sigma\sigma})\end{aligned}\tag{4.44}$$

and so on, which is a gauge transformation for the Lax operators.

Another way to describe classical string geometry was proposed in [20] and was based on reformulating of geometric data of the principal chiral field (4.29) in terms of some Riemann-Hilbert problem. The spectral problem on string side (a direct analog of the formulas (3.16), (3.17), (3.19) and (3.28)) can be formulated in the following way. Let X and $\mathcal{G}(X)$ be string spectral parameter and resolvent, equal to the quasimomentum of

the classical solution (maybe up to an exact one-form). The spectral Riemann-Hilbert problem on string side can be written as [20]

$$\begin{aligned}\frac{1}{2\pi i} \oint_{\mathbf{C}} \mathcal{G}(X) dX &= \frac{J}{\Delta} + \frac{\Delta - L}{2\Delta} \\ \frac{1}{2\pi i} \oint_{\mathbf{C}} \frac{dX \mathcal{G}(X)}{X} &= 2\pi m \\ \oint_{\mathbf{C}} \frac{2t dX}{X^2} \frac{\mathcal{G}(X)}{2\pi i} &= \Delta - L\end{aligned}\tag{4.45}$$

and

$$\mathcal{G}(X_+) + \mathcal{G}(X_-) - 2\pi n_l = \frac{X}{X^2 - t}\tag{4.46}$$

where we introduced the notation $t = \frac{\lambda}{16\pi^2 \Delta^2}$.

Consider now $x = X + \frac{t}{X}$ as *exact* change of spectral parameter, together with $G(x) = \mathcal{G}(X)$ ⁴. This is literally an exact change of the local co-ordinate in the vicinity of an extra cut in the general construction discussed above. Indeed, in terms of k_{\pm} one has $X \underset{P \rightarrow P_{\pm}}{=} \pm\sqrt{t} + \frac{1}{k_{\pm}} + \dots$, then $x \underset{P \rightarrow P_{\pm}}{=} \pm 2\sqrt{t} \pm \frac{1}{\sqrt{t} k_{\pm}} + \dots$, i.e. the function $E = \frac{1}{x}$ (up to an overall constant) satisfies all desired properties, e.g. when the cut between P_+ and P_- on fig. 2 shrinks to a point P_0 with $x(P_0) = 0$ the function E acquires a simple pole at this point.

Proceeding further

$$dx = dX \left(1 - \frac{t}{X^2}\right) = \frac{dX}{X} \left(X - \frac{t}{X}\right) = \frac{dX}{X} \sqrt{x^2 - 4t}\tag{4.47}$$

and combination of the first and third lines in (4.45) gives

$$\frac{1}{2\pi i} \oint \mathcal{G}(X) dX = \frac{J}{\Delta} + \frac{\Delta - L}{2\Delta} = \frac{J}{\Delta} + t \oint \frac{\mathcal{G}(X) dX}{2\pi i X^2}\tag{4.48}$$

or

$$\frac{1}{2\pi i} \oint dx G(x) = \frac{J}{\Delta}\tag{4.49}$$

The second line of (4.45) is then

$$\frac{1}{2\pi i} \oint \frac{dx G(x)}{\sqrt{x^2 - 4t}} = 2\pi m\tag{4.50}$$

where the integral is taken around the cut between the points $-2\sqrt{t}$ and $2\sqrt{t}$ in the x -plane, and the third line of (4.45) gives

$$\oint \frac{dx G(x)}{2\pi i} \left(\frac{x}{\sqrt{x^2 - 4t}} - 1\right) = \Delta - L\tag{4.51}$$

The "string Bethe" equation on the cuts (4.46) turns now into

$$G(x_+) + G(x_-) - 2\pi n_l = \frac{1}{\sqrt{x^2 - 4t}}\tag{4.52}$$

⁴In [20] it has been introduced in the leading order in λ . A similar change of variables has been proposed in [37], with Δ replaced with its "bare value" L .

We now see from (4.49), (4.50), (4.51) and (4.52) that the classical string theory spectral problem looks identically to the quasiclassical Bethe equations on gauge side upon replacements

$$\begin{aligned} \frac{1}{x} &\rightarrow \frac{1}{\sqrt{x^2 - 4t}} = \frac{1}{x} + \frac{2t}{x^3} + \dots \\ L &\rightarrow \Delta \\ \gamma &\rightarrow \Delta - L \end{aligned} \tag{4.53}$$

In other words, this leads to a nonlinear relation

$$\Delta - L = \Gamma(\lambda, \Delta) \tag{4.54}$$

where $\Gamma(\lambda, L) = \gamma + O(\lambda^2)$ should be compared with the multi-loop anomalous dimension of the supersymmetric gauge theory.

A simplest non-trivial example of such relation is the solitonic limit of small number of Bethe roots, leading to the "modified" BMN formula [20]

$$\Delta - L = \sum_k N_k \left(\sqrt{1 + \frac{\lambda n_k^2}{\Delta^2}} - 1 \right) \tag{4.55}$$

for $J = \sum_k N_k$ expressed as a total amount of "positive" $n_k > 0$ and "negative" $n_k < 0$ massive oscillators [15]. Formulas (4.54) and (4.55) show, that the solution for Δ of classical string theory is given in terms of the highly non-linear formulas, and the oscillator language of [15, 10] is rather an effective tool for description of certain quasiclassical modes of an integrable string model in pp-wave geometry, than an exact world-sheet quantization.

5 Discussion

In these notes we have discussed the recent attempts of quantitative verification of the AdS/CFT correspondence based on appearance of integrable structures on both sides of the gauge/string duality. It turns out that the quasiclassical solution to the Bethe ansatz equations arising in the process of diagonalization of the mixing matrix for constituent operators can be formulated in terms of (discrete) families of complex curves endowed with a generating one-form, quite similar to the quasiclassical solutions of matrix models and Seiberg-Witten gauge theories.

In contrast to the matrix model case, the Bethe roots for the "compact" chains form strings never lying along the real axis. Moreover, the resolvent for the infinite number of Bethe roots cannot be expressed through an algebraic function on a curve of finite genus due to nontrivial mode numbers n_j , whose total number is fixed to be finite for the class of finite-gap or algebro-geometric solutions.

The condensation of Bethe roots on the cuts of Riemann surfaces leads to the fact that for long $L \rightarrow \infty$ operators the corresponding anomalous dimensions are expressed through the integrals of motion of some *classical* spin waves of the corresponding magnetic. This is a very nontrivial "continuum limit", since many quantum spins condense into collective classical mode; this nontriviality was also discussed in [33], though the approach presented above is much more simple. Hence, considering long operators and solving Bethe equations for them we finally come to some finite-gap solutions determined by particular complex curves.

On the other side of duality one has classically integrable string sigma-model. Despite there are no arguments why this approximation on string side maybe valid for comparison with the gauge theory (and this is not true, e.g. for the operators corresponding to motion of string in the non-compact part of $AdS_5 \times S^5$ [16], see also discussion of this issue e.g. in [34]), for certain solutions nevertheless the large λ is suppressed by large values of the integrals of motion and field-theoretic perturbation theory can be reproduced from the string calculations.

Generally the string sigma-model (even restricted to bosonic part of compact S^5 or its subspace) is a system with infinitely many degrees of freedom. However, following [25] it is possible to construct its finite-gap solutions, satisfying the world-sheet Virasoro constraints. We have investigated the properties of such solutions and demonstrated that underlying geometry can be naturally sewed with the quasiclassical geometry of the Bethe ansatz solutions.

In more exact terms the generic sigma-model solution is formulated using the complex curve Γ , being a simple "one-cut" double cover of some curve Σ , where exists a function E with D simple poles. It means that Σ can be considered as a D -sheet cover of Riemann sphere, being for $D = 2$ (the S^3 case) a hyperelliptic curve. The solution is constructed using the standard technique of the finite-gap potentials for non-stationary one-dimensional and two-dimensional the Schrödinger operators. For $D > 2$ the appearance of vector non-linear Schrödinger provides an intuition for what kind of solutions to the nested Bethe ansatz should be looked for beyond the $SU(2)$ subsector.

However, the "weak-coupling" limit to the gauge side is rather nontrivial. The geometry of Γ changes so that it becomes just two copies of Σ with the extra cut on fig. 2 shrunk to a single point P_0 . The sigma-model solution turns into solution of $(D - 1)$ -dimensional vector non-linear Schrödinger equation, equivalent for $D = 2$ to the Heisenberg magnetic chain.

The subtleties of this limit were widely discussed in the literature. In [35] it was proposed to describe higher loops on the SYM side in terms of the integrable chain [36], whose transfer matrix is defined on elliptic curve with one of the periods proportional to the length of the chain. Such reformulation of perturbative SYM leads to difference with the predictions of classical string theory starting from three loops, the most comprehensive discussion of these discrepancies can be found in [37].

It would be interesting to understand how the integrable models, associated to the elliptic curve, can appear in the framework of approach discussed in these notes. The periodicity in the length of the chain L naturally arises when one computes more than L loops on the gauge side, but this is certainly inconsistent with taking the $L \rightarrow \infty$ limit first. It means that one has to study the finite-size $\frac{1}{L}$ -corrections to the quasiclassical Bethe equations, see e.g. [38]. From the point of view of generic sigma-model solutions the periodicity in σ comes automatically, when one considers the curve Σ as cover of some elliptic curve, instead of the x -plane or Riemann sphere. This suggests a possible way of development of the considered problems along the lines of [39].

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