

# THE BIANCHI IX (MIXMASTER) COSMOLOGICAL MODEL IS NOT INTEGRABLE

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## Abstract

The perturbation of an exact solution exhibits a movable transcendental essential singularity, thus proving the nonintegrability. Then, all possible exact particular solutions which may be written in closed form are isolated with the perturbative Painlevé test; this proves the inexistence of any vacuum solution other than the three known ones.

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# 1 Introduction

The Bianchi IX cosmological model [32, 33, 4, 34] [8, 9, 10, 7, 19] is governed by a system of three coupled second order differential equations

$$\sigma^2(\text{Log } A)'' = A^2 - (B - C)^2 \text{ and cyclically,} \quad (1)$$

where a prime means a derivation with respect to the “logarithmic time”  $\tau$ , and  $\sigma^2$  is 1 or  $-1$  according as the metric is asymptotically Euclidean or Minskowskian. An equivalent definition is

$$\sigma^2(\text{Log } \omega_1)'' = \omega_2^2 + \omega_3^2 - \omega_2^2 \omega_3^2 / \omega_1^2 \text{ and cyclically,} \quad (2)$$

with

$$A = \omega_2 \omega_3 / \omega_1, \quad \omega_1^2 = BC \text{ and cyclically.} \quad (3)$$

This system possesses the first integral

$$I = \sigma^2 \{ (\text{Log } B)' (\text{Log } C)' + (\text{Log } C)' (\text{Log } A)' + (\text{Log } A)' (\text{Log } B)' \} \\ + A^2 + B^2 + C^2 - 2(BC + CA + AB) \quad (4)$$

$$= 2\sigma^2 \{ (\text{Log } \omega_2)' (\text{Log } \omega_3)' + (\text{Log } \omega_3)' (\text{Log } \omega_1)' + (\text{Log } \omega_1)' (\text{Log } \omega_2)' \} \\ - 2\sigma^2 \{ (\text{Log } \omega_1)'^2 + (\text{Log } \omega_2)'^2 + (\text{Log } \omega_3)'^2 \} \\ + \omega_1^{-2} \omega_2^2 \omega_3^2 + \omega_2^{-2} \omega_3^2 \omega_1^2 + \omega_3^{-2} \omega_1^2 \omega_2^2 - 2(\omega_1^2 + \omega_2^2 + \omega_3^2), \quad (5)$$

equal to the Ricci tensor component  $R_0^0$  which must vanish in the absence of matter (vacuum). Nevertheless, we will also discuss cases with  $I \neq 0$  which represents the case of noninteracting matter.

The question to be settled is the generic behaviour of the system: is it chaotic or not? A necessary condition for chaos is the existence of at least one positive Lyapunov exponent, but numerical simulations in this direction [2, 21, 11, 29, 27, 37] are very difficult, a typical result for the largest exponent being a slightly positive value not distinguishable from zero in the vacuum case  $I = 0$ . More recent computations of the correlation dimension [18] indicate, without definite proof, a probable chaotic behaviour.

The approach adopted here is analytic. The main relevant feature for a global knowledge of the behaviour of the system is the singularity structure of the solutions [35, 26]. This study, independent of the choice of the metric, must be done in the complex plane of  $\tau$  and the issue is: has the system the Painlevé property (PP) or not? This property is defined as the absence of movable critical singularities in the general solution of (1), where a singularity is said to be movable if its location in the complex plane of  $\tau$  depends on the initial conditions, and critical if branching takes place around it (for a tutorial introduction to these questions, see [6]).

The violation of the PP is not enough to decide about chaos. It is in addition necessary, but not sufficient, that the general solution takes an infinity of values

around the movable critical points [35] (which happens for instance with  $\text{Log}(\tau - \tau_0)$ ). On the other hand, the presence of an infinity of movable logarithms is generally admitted to be sufficient to ensure the existence of some chaotic régimes.

Throughout the paper, the term “integrable” means “which has the PP”.

In section 2, we recall the exact solutions known in closed form. Section 3 proves the nonintegrability with a perturbation à la Poincaré. In sections 4 and 5, the Painlevé test isolates all possible particular solutions which can be written as single valued expressions in closed form: this proves the inexistence of any vacuum solution other than the three known ones.

## 2 The three known exact solutions

The first integral  $I$  is identically zero under the condition that the curvature be self-dual [23]

$$\sigma(\text{Log } A)' = B + C - A - 2\lambda_1\sqrt{BC}, \quad \lambda_1 = \lambda_2\lambda_3, \quad \text{and cyclically,} \quad (6)$$

$$\sigma\omega_1' = \omega_2\omega_3 - \omega_1(\lambda_2\omega_2 + \lambda_3\omega_3), \quad \text{and cyclically,} \quad (7)$$

where  $(\lambda_1, \lambda_2, \lambda_3)$  are constants. The equations (6)–(7) are real only in the Euclidean case. The equations for  $(\lambda_1, \lambda_2, \lambda_3)$  have three solutions  $(0, 0, 0)$ ,  $(1, 1, 1)$ ,  $(1, -1, -1)$ , but the third one is not distinct from the second one due to the invariance  $(\lambda_1, \lambda_2, \lambda_3, \omega_1, \omega_2, \omega_3) \rightarrow (\lambda_1, -\lambda_2, -\lambda_3, \omega_1, -\omega_2, -\omega_3)$ . The two remaining solutions define respectively the *Euler system* (1750) [3] describing the motion of a rigid body around its center of gravity

$$\sigma\omega_1' = \omega_2\omega_3, \quad \text{and cyclically,} \quad (8)$$

and the *Darboux system* ([17] eq. (124) p. 149) describing a problem of geometry of second degree surfaces

$$\sigma\omega_1' = \omega_2\omega_3 - \omega_1\omega_2 - \omega_1\omega_3, \quad \text{and cyclically.} \quad (9)$$

The Euler system has been integrated by Abel and Jacobi (and in the Bianchi IX model by Belinskii *et al.* [3]) with elliptic functions,

$$\begin{aligned} A &= \sigma(\text{Log } \text{pq}(\lambda(\tau - \tau_0), k))', \\ B &= \sigma(\text{Log } \text{rq}(\lambda(\tau - \tau_0), k))', \\ C &= \sigma(\text{Log } \text{xq}(\lambda(\tau - \tau_0), k))', \end{aligned} \quad (10)$$

depending on the three arbitrary parameters  $(\tau_0, \lambda, k)$ , where  $(\text{q,p,r,x})$  is any permutation of the letters  $(\text{s,c,d,n})$  used in the notation for the Jacobi functions (the choice of ref. [3] is  $\text{q=s}$ ). The Darboux system has been integrated by Halphen

([24], [25] tome I chap. IX) and Bureau [12] with Hermite modular elliptic functions. The general solution of the Euler system has for only singularities movable simple poles. The general solution of the Darboux-Halphen system is only defined inside or outside a movable circle, it is holomorphic in its domain of definition and its only singularity is a movable natural boundary defined by the circle. Therefore both three-dimensional systems have the PP. These two systems also occur as reductions of the self-dual Yang-Mills equations [13, 38].

A third exact solution was found more than forty years ago by Taub [39] who noticed that the constraint  $B = C, \omega_2 = \omega_3$  is a consistent reduction of the order from six to four

$$\begin{aligned}
A &= \delta \frac{k_1}{\cosh k_1(\tau - \tau_1)}, \quad \delta^2 = -\sigma^2, \\
\omega_2 = \omega_3 &= \delta \frac{k_2}{\cosh k_2(\tau - \tau_2)}, \\
B = C = \omega_1 &= \delta \frac{k_2^2 \cosh k_1(\tau - \tau_1)}{k_1 \cosh^2 k_2(\tau - \tau_2)}, \\
I &= (4k_2^2 - k_1^2)\sigma^2
\end{aligned} \tag{11}$$

(the above real writing is adapted to a Minkowskian metric  $\sigma^2 = -1$ ). It depends on the four arbitrary parameters  $(\tau_1, \tau_2, k_1, k_2)$ , which reduce to three if one satisfies the vacuum constraint that  $I$  be zero. Its only singularities are movable poles, located either at  $\tau'_1 = \tau_1 + i(n + 1/2)\pi/k_1$  or at  $\tau'_2 = \tau_2 + i(n + 1/2)\pi/k_2$ , with  $n$  integer, and the four-dimensional dynamical system also has the PP.

### 3 Proof of nonintegrability

The existence of the above three subsystems of order less than six tells nothing about the presence or absence of movable branching in the general solution of Bianchi IX.

Following the method initiated by Poincaré for the three-body problem [36], let us look for solutions close to the one  $(A_0, B_0 = C_0)$  of Taub in the vacuum case  $\sigma^2 = -1, k_1 = 2k_2 = 2k$  ( $k$  is only defined by its square), in order to introduce the three missing arbitrary parameters. One sets

$$A = A_0(1 + \varepsilon A_1), \quad B = B_0(1 + \varepsilon B_1), \quad C = C_0(1 + \varepsilon C_1), \tag{12}$$

to obtain, at first order in the perturbation variable  $\varepsilon$ , the linear differential system

$$A_1'' - 2\sigma^{-2}A_0^2A_1 = 0, \tag{13}$$

$$P_1'' - 2\sigma^{-2}A_0B_0P_1 = 4\sigma^{-2}(A_0B_0 - A_0^2)A_1, \tag{14}$$

$$M_1'' + 2\sigma^{-2}(A_0B_0 - 2B_0^2)M_1 = 0, \tag{15}$$

with the notation  $P_1 = B_1 + C_1$ ,  $M_1 = B_1 - C_1$ . The homogeneous parts of (13) and (14) only differ by the permutation of  $(k_1, \tau_1)$  and  $(k_2, \tau_2)$ , and the equation (13) integrates as

$$A_1 = k_3 \tanh 2k(\tau - \tau_1) + k_4 \left( \tau \tanh 2k(\tau - \tau_1) - \frac{1}{2k} \right), \quad (k_3, k_4) \text{ arbitrary}, \quad (16)$$

which only changes the order of the poles of  $A$ . The equation (15) possesses singular irregular points of rank two since the coefficient  $B_0^2$  has fourth order poles located at the points  $\tau_2'$ . Therefore [40, 28] its solution contains essential singular points, and a sufficient condition that they be critical for (15) is that either the rank of the singular irregular points be odd (which is not the case here), or the formal solution in the neighborhood of  $\tau_2'$

$$M_1 = k_5 e^{\alpha_+ / (\tau - \tau_2')} \sum_{k=0}^{+\infty} \lambda_k^+ (\tau - \tau_2')^{k+s_+} + k_6 e^{\alpha_- / (\tau - \tau_2')} \sum_{k=0}^{+\infty} \lambda_k^- (\tau - \tau_2')^{k+s_-}, \quad \lambda_0^+ \lambda_0^- \neq 0, \quad (17)$$

where  $(\alpha_{\pm}, s_{\pm}, \lambda_k^{\pm})$  are two sets of constants, have one of its two Thomé indices  $s_{\pm}$  irrational. The two values  $(\alpha_{\pm}, s_{\pm})$  are given by the first two most singular terms in (15):

$$\alpha_{\pm} = \pm k^{-1} \sinh 2k(\tau_2' - \tau_1'), \quad s_{\pm} = 1 \mp 2 \cosh 2k(\tau_2' - \tau_1'). \quad (18)$$

The generically complex values, with irrational real and imaginary parts, for the Thomé indices  $s_{\pm}$  introduce a nonalgebraic (i.e. transcendental) critical branching in  $M_1$  and therefore a movable critical transcendental essential singularity in the general solution of the Bianchi IX model. It may be useful to insist on the impossibility to remove this critical singularity by considering either Stokes sectors or some algebraic transform of  $(A, B, C)$ , as would be the case with  $u^2$  for the equation whose general solution is

$$u = K_1 e^{\frac{1}{\tau - \tau_0}} \sqrt{\tau - \tau_0} + K_2 e^{-\frac{1}{\tau - \tau_0}} / \sqrt{\tau - \tau_0}, \quad (K_1, K_2) \text{ arbitrary}. \quad (19)$$

The reason for this impossibility is the irrational nature of the two Thomé indices.

Thus, the Bianchi IX model has not the Painlevé property and its general solution contains an infinity of movable logarithms. As explained in the introduction, it is therefore quite probably chaotic, a fact supported by some recent numerical simulations [18].

In the limit  $k_1 \rightarrow 0, k_2 \rightarrow 0$  where the solution of Taub degenerates into the two-parameter solution ([17] p. 150) of the Darboux-Halphen system

$$\frac{A}{\sigma} = \frac{1}{\tau - \tau_1'}, \quad \frac{B}{\sigma} = \frac{C}{\sigma} = \frac{\tau - \tau_1'}{(\tau - \tau_2')^2}, \quad k_1(\tau_1 - \tau_1') = k_2(\tau_2 - \tau_2') = i\frac{\pi}{2}, \quad (20)$$

the movable essential singularity becomes noncritical ( $\alpha_{\pm} = \pm 2(\tau_2' - \tau_1')$ ,  $s_{\pm} = 1 \mp 2$ ) and the formal series for  $M_1$  can be summed with the Euler function  $E(x)$

$$M_1 = k_5 e^{\frac{1}{2x}} x^{-1} + k_6 e^{-\frac{1}{2x}} (1 - x + 2x^2 - E(x)), \quad x = \frac{\tau - \tau_2'}{4(\tau_2' - \tau_1')}, \quad (21)$$

$$E(x) = \int_0^{+\infty} \frac{e^{-u}}{1+xu} du. \quad (22)$$

It could be interesting to consider an equation equivalent to (15) with polynomial coefficients, by the change of variables  $T = \tanh c(\tau - \tau_2)$ ,  $M_1 = m_1 \cosh c(\tau - \tau_2)$ :

$$\begin{aligned} & 4a^2(1-T^2)^2 \frac{d^2 m_1}{dT^2} \\ & + [a^4(1+T)^4 + (1-T)^4 + 2a^2(1-T^2)^2 + 8a^2T^2 - 4a^2]m_1 = 0, \\ & a = e^{2c(\tau_2 - \tau_1)}, \end{aligned} \quad (23)$$

which sends the singular irregular point to  $T = \infty$ .

*Remark.* A perturbation of the Taub solution has already been performed [5] up to and including second order, without finding critical singularities. The effect is due to the neglect of terms involving  $A$  in the three right hand sides of (1) (see system (4.4)–(4.6) in ref. [5]), and the present results *a posteriori* prove that these terms are crucial.

## 4 Local singularity analysis

Let us now try to obtain the list of *all* the exact solutions which can be written *in closed form*. Such solutions can only be particular and depend on at most four arbitrary parameters, since they cannot [35] contain any of the two infinities of movable logarithms found in section 3. Three of them are already known, see section 2. To achieve this goal, let us express necessary conditions for the absence of any infinity of movable logarithms in local representations of the general solution. The *local, necessary* information thus obtained will then have to be transformed into a *global, sufficient* one in the shape of a closed form solution. The method involved is called singularity analysis [6].

The singularity analysis of this dynamical system has recently been undertaken [15] and there exist two different local representations of  $(A, B, C)$  by Laurent series bounded from below, i.e. describing a meromorphic-like behaviour.

The first family of movable singularities, about a point which we denote  $\tau_1$ , is

$$\begin{aligned} A/\sigma &= \chi^{-1} + a_2\chi + O(\chi^3), \quad \chi = \tau - \tau_1, \\ B/\sigma &= b_0\chi + b_1\chi^2 + O(\chi^3), \\ C/\sigma &= c_0\chi + c_1\chi^2 + O(\chi^3), \\ I/\sigma^2 &= 6a_2 - 2(b_0 + c_0) + b_1c_1/(b_0c_0), \end{aligned} \quad (24)$$

and its Fuchs indices are  $(-1, 0, 0, 1, 1, 2)$ , corresponding to the orders at which the six arbitrary coefficients enter the expansion, respectively  $(\tau_1, b_0, c_0, b_1, c_1, a_2)$ .

The second family, about a movable singularity which we denote  $\tau_2$ , is

$$\begin{aligned} A/\sigma &= \chi^{-1} + a_2\chi + O(\chi^3), \quad \chi = \tau - \tau_2, \\ B/\sigma &= \chi^{-1} + b_2\chi + O(\chi^3), \\ C/\sigma &= \chi^{-1} + c_2\chi + O(\chi^3), \\ I/\sigma^2 &= -6(a_2 + b_2 + c_2), \end{aligned} \tag{25}$$

its Fuchs indices are  $(-1, -1, -1, 2, 2, 2)$ , but the arbitrary coefficients  $(\tau_2, a_2, b_2, c_2)$  correspond to only four Fuchs indices, respectively  $(-1, 2, 2, 2)$ .

The first series represents a locally meromorphic behaviour of the general solution. On the contrary, the second series only represents a four-parameter locally single valued *particular* solution, and it tells nothing about some possible multivaluedness about  $\tau_2$  in the *general* solution. The reason is the presence of two negative indices in addition to the ever present  $-1$ , counted once even if it is multiple (contrary to the erroneous argument presented in [15]). In such a situation, the method of pole-like expansions initiated by Sonia Kowalevskaya [30, 31, 22, 1] is unable to represent the general solution and to check the absence of movable branching about  $\tau_2$ .

Therefore, one must perturb the second Laurent series in order to extract the information contained in the two remaining negative indices.

The similar local representations of the particular solutions are the following. For the Belinskii *et al.* solution (10) of the subsystem (8), the family (24) with  $b_1 = c_1 = 0, 3a_2 = b_0 + c_0$  and indices  $(-1, 0, 0)$ , and the family (25) with  $a_2 + b_2 + c_2 = 0$  and indices  $(-1, 2, 2)$ . For the Halphen solution of the Darboux subsystem (9), the family (24) with  $b_1 = 2b_0\sqrt{c_0}, c_1 = 2c_0\sqrt{b_0}, a_2 = (\sqrt{b_0} - \sqrt{c_0})^2/3$  and indices  $(-1, 0, 0)$ , and the family

$$\begin{aligned} A/\sigma &= \chi^{-1} + a_{-1}\chi^{-2} + \dots, \quad \chi = \tau - \tau_2, \\ B/\sigma &= \chi^{-1} + b_{-1}\chi^{-2} + \dots, \\ C/\sigma &= \chi^{-1} + c_{-1}\chi^{-2} + \dots, \end{aligned} \tag{26}$$

with indices  $(-1, -1, -1)$ . This “descending” Laurent series shares its first term with (25) and it evidently describes a solution missing in (25).

As to the Taub particular solution of the full system (1), near  $\tau'_1$  it is represented by (24) with  $b_0 = c_0, b_1 = c_1$  but, near  $\tau'_2$ , it is impossible to fit its Laurent series into one of the above ones. This is due to the presence of the movable essential singularity detected in perturbation, which makes meaningless the computation of Fuchs indices ([28] chap. XVII). Its Laurent series near  $\tau'_2$  will be obtained in next section.

## 5 All possible solutions in closed form

An algorithmic method to perform the perturbation of the second family (25) has

recently been proposed [20, 14], and later shown [6] to be a natural application of the theorem of perturbations of Poincaré and Lyapunov. At each perturbative order, this method checks the absence of movable logarithms likely to occur at all indices, then adds to the previous Laurent series another Laurent series, also bounded from below but starting with a strictly lower singularity power. The resulting “doubly infinite” Laurent series, which is the well known standard local representation of a single valued function, is the union of the two pieces (25) and (26), with some “interference terms” of the highest interest.

Our interest here is not to prove multivaluedness, which has been done in section 3, but to exploit the necessary conditions for the absence of movable logarithms in order to detect particular locally single valued solutions, which will then have to be written in closed form if the conditions are sufficient.

In our case where the smallest index is  $-1$  and the highest one  $2$ , to perform this perturbative test up to some maximal order  $N$ , one sets [14]

$$\frac{A}{\sigma} = \chi^{-1} \sum_{n=0}^N \varepsilon^n \sum_{j=-n}^{2+N-n} a_j^{(n)} \chi^j, \quad \chi = \tau - \tau_2, \quad \text{and cyclically,} \quad (27)$$

and puts these polynomials (with negative and positive powers of  $\chi$ ) into a polynomial definition of the dynamical system

$$\sigma^4 E_A \equiv \sigma^2(AA'' - A'^2) - A^4 + A^2(B - C)^2 = 0 \quad \text{and cyclically.} \quad (28)$$

Writing the LHS  $E_A$  of (28) as a polynomial similar to (27)

$$E_A = \chi^{-4} \sum_{n=0}^N \varepsilon^n \sum_{j=-n}^{2+N-n} E_{A,j}^{(n)} \chi^j + \dots \quad \text{and cyclically,} \quad (29)$$

one then solves the set of equations  $\mathbf{E}_j^{(n)} = 0$  by respecting the ordering that  $\mathbf{E}_{j'}^{(n')} = 0$  must be solved before  $\mathbf{E}_j^{(n)} = 0$ , with  $n' \leq n, j' + n' \leq j + n, (n', j') \neq (n, j)$ . The first equation  $(n, j) = (0, 0)$  is the only nonlinear one

$$E_{A,0} = a_0^{(0)^2} [1 - a_0^{(0)^2} + (b_0^{(0)} - c_0^{(0)})^2] = 0 \quad \text{and cyclically,} \quad (30)$$

and it has already been solved, see eq. (25) or (26), as  $a_0^{(0)} = b_0^{(0)} = c_0^{(0)} = 1$  since  $\sigma$  is only defined by its square. As usual for a perturbative method, all other equations are linear

$$\forall (n, j) \neq (0, 0) : \mathbf{P}(j) \begin{pmatrix} a_j^{(n)} \\ b_j^{(n)} \\ c_j^{(n)} \end{pmatrix} + \mathbf{Q}_j^{(n)} = 0, \quad \mathbf{P}(j) = -(j+1)(j-2)\mathcal{I}, \quad (31)$$

where  $\mathbf{P}$  is a multiple of the identity matrix  $\mathcal{I}$  independent of  $n$  and the column vector  $\mathbf{Q}_j^{(n)}$  only depends on previously computed coefficients.



At every perturbation order  $n$  and every triple Fuchs index  $j \in \{-1, 2\}$ , the linear system for  $a_j^{(n)}, b_j^{(n)}, c_j^{(n)}$  is singular and has a zero rank; if the orthogonality condition  $\mathbf{Q}_j^{(n)} = 0$  is satisfied, three more arbitrary coefficients enter the expansion, but if it does not an impossibility occurs and logarithms must be introduced. In the former case, the arbitrary coefficients need only be introduced the first time the Fuchs index is encountered, i.e.  $(n, j) = (1, -1), (0, 2)$ , because next times they would only perturbate these ones. Since the formal solution evidently does not depend on the seven parameters  $(\tau_2, a_2^{(0)}, b_2^{(0)}, c_2^{(0)}, \varepsilon a_{-1}^{(1)}, \varepsilon b_{-1}^{(1)}, \varepsilon c_{-1}^{(1)})$ , one can freely choose one of the values of the four quantities  $(\tau_2, a_{-1}^{(1)}, b_{-1}^{(1)}, c_{-1}^{(1)})$  related to the triple Fuchs index  $-1$ . This choice of gauge will be used below.

Due to the nice symmetry of the second family, only one of the three scalar equations of (31) needs to be considered, the results for the two others being derived by cyclic permutations. Parity makes  $\mathbf{Q}_j^{(n)}$  hence  $a_j^{(n)}$  vanish for  $n + j$  odd.

The costless violation occurs at third order for the Fuchs index  $-1$

$$a_0^{(0)} = 1, a_{-1}^{(1)} = \text{arbitrary}, a_2^{(0)} = \text{arbitrary}, \quad (32)$$

$$4a_{-2}^{(2)} = 4a_{-1}^{(1)2} - (b_{-1}^{(1)} - c_{-1}^{(1)})^2, \quad (33)$$

$$4a_{-3}^{(3)} = 4a_{-1}^{(1)3} - (a_{-1}^{(1)} + b_{-1}^{(1)} + c_{-1}^{(1)})(b_{-1}^{(1)} - c_{-1}^{(1)})^2, \quad (34)$$

$$4a_0^{(2)} = (b_{-1}^{(1)} - c_{-1}^{(1)})\{ (5a_2^{(0)} - 4b_2^{(0)} - 4c_2^{(0)})(b_{-1}^{(1)} - c_{-1}^{(1)})^2 - (2a_{-1}^{(1)} - b_{-1}^{(1)} - c_{-1}^{(1)})(b_2^{(0)} - c_2^{(0)}) \} \quad (35)$$

$$a_1^{(1)} = -a_2^{(0)}a_{-1}^{(1)} + (b_2^{(0)} - c_2^{(0)})(b_{-1}^{(1)} - c_{-1}^{(1)}), \quad (36)$$

$$(n, j) = (3, -1) : \text{violation.} \quad (37)$$

Unless the three conditions  $\mathbf{Q}_{-1}^{(3)} \equiv \mathbf{E}_{-1}^{(3)} = 0$  are satisfied, movable logarithms start entering the expansion and the Painlevé test fails. The formal solution will thus contain an infinity of logarithmic terms. The three conditions  $\mathbf{Q}_{-1}^{(3)} = 0$  are not independent and, together with the first integral  $I$ , they are best expressed in coordinates adapted to the ternary symmetry:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = R \begin{pmatrix} a_2^{(0)} \\ b_2^{(0)} \\ c_2^{(0)} \end{pmatrix}, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R \begin{pmatrix} a_{-1}^{(1)} \\ b_{-1}^{(1)} \\ c_{-1}^{(1)} \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -\sqrt{2}/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$$

$$Q_{A,-1}^{(3)} \equiv 9C_1, \quad Q_{B,-1}^{(3)} \equiv -\frac{9}{2}C_1 + \frac{9\sqrt{3}}{2}C_2, \quad Q_{C,-1}^{(3)} \equiv -\frac{9}{2}C_1 - \frac{9\sqrt{3}}{2}C_2, \quad (38)$$

$$C_1 \equiv -x(ux^2 - 4vxy + uy^2) = 0 \quad (39)$$

$$C_2 \equiv vx^3 - 2ux^2y + vxy^2 + 2uy^3 = 0, \quad (40)$$

$$I/\sigma^2 = -6\sqrt{3}w + (18 \text{ terms})\varepsilon^2 + O(\varepsilon^4). \quad (41)$$

Equations  $C_1 = 0, C_2 = 0$  are independent of  $(w, z)$  and have five solutions

$$u = x = 0 : \quad b_2^{(0)} = c_2^{(0)}, \quad b_{-1}^{(1)} = c_{-1}^{(1)}, \quad (42)$$

$$\begin{aligned}
\frac{v}{u} = \frac{y}{x} = \frac{1}{\sqrt{3}} : \quad & c_2^{(0)} = a_2^{(0)}, \quad c_{-1}^{(1)} = a_{-1}^{(1)}, \\
\frac{v}{u} = \frac{y}{x} = -\frac{1}{\sqrt{3}} : \quad & a_2^{(0)} = b_2^{(0)}, \quad a_{-1}^{(1)} = b_{-1}^{(1)}, \\
u = v = 0 : \quad & a_2^{(0)} = b_2^{(0)} = c_2^{(0)}, \\
x = y = 0 : \quad & a_{-1}^{(1)} = b_{-1}^{(1)} = c_{-1}^{(1)},
\end{aligned} \tag{43}$$

$$\tag{44}$$

corresponding to only three distinct cases.

The first constraint (42) implies the equality of two of the components  $(A, B, C)$  at every order and thus represents the four-parameter solution of Taub (11).

The second constraint (43) is not refined at next condition  $(n, j) = (2, 2)$  but it is restricted at  $(n, j) = (5, -1)$  by the relation  $wx(x^2 - 3y^2) = 0$ , which splits into either  $x(x^2 - 3y^2) = 0$ , i.e. a subset of case (42), or  $w = 0$ , so the second constraint (43) becomes  $u = v = w = 0$ . This represents the three-parameter solution of the Darboux-Halphen system (9).

For the third and last constraint (44), the doubly infinite Laurent series (27) has the same sum than the semi-infinite Laurent series (25) because one can assign the gauge  $a_{-1}^{(1)} = b_{-1}^{(1)} = c_{-1}^{(1)}$  to zero, exactly like the series of negative powers  $\sum_{j=-\infty}^{-1} (\tau_2 - \tau_0)^{-j-1} (\tau - \tau_0)^j$ , convergent outside a disk centered at  $\tau_0$  containing  $\tau_2$ , and the series of positive powers  $-\sum_{j=0}^{+\infty} (\tau_2 - \tau_1)^{j-1} (\tau - \tau_1)^j$ , convergent inside a disk centered at  $\tau_1$  not containing  $\tau_2$ , both represent the same function  $(\tau - \tau_2)^{-1}$ . The perturbative Painlevé test reduces in this case to the test of Kowalevskaya and one cannot rule out the possibility of a four-parameter, global, closed form, single valued exact solution extending the three-parameter solution (10) of Belinskii *et al.*, which it contains for  $a_2^{(0)} + b_2^{(0)} + c_2^{(0)} = \sqrt{3}w = 0$ . We have not yet succeeded in finding this closed form.

Since the local singularity analysis shows that the only movable branching is logarithmic (no algebraic branching such as  $\chi^q$ ,  $q$  rational), this solves entirely the question of closed form global solutions in the vacuum case  $I = 0$ : there exists no solution to the Bianchi IX model in vacuum other than the three known ones. Let us recall once more that the local, not in closed form solutions are illusory [35].

*Remarks.*

1. The lowest perturbation order (two) at which the failure of the Painlevé test occurs is one more than the perturbation order of the solution of Taub, because the present perturbation starts from a formal solution with a simple, not double, pole.
2. This dynamical system is one more example [14] of a family of movable singularities with negative Fuchs indices which contains *all the information* on the integrability, while the family with positive indices, in this particular case, contains no useful information at all.

## 6 Conclusion

With two different proofs using perturbative methods, we have shown the non-integrability in the Painlevé sense of the Bianchi IX model, by exhibiting movable essential transcendental critical singularities which quite probably imply chaos. Moreover, we have proven the inexistence of any vacuum solution other than the three known ones.

One important open problem is the understanding of the apparently zero value for the highest Lyapunov exponent. Has this something to do with the existence of a local singlevalued *formal* representation of the general solution by the meromorphic series (24), whose radius of convergence could then be worth studied?

Our result should probably not affect the overall understanding of the Kasner epochs, for which a detailed description has been given [8, 32, 7, 19].

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*Note added in proof.* After the submission of this paper, we obtained a copy of a paper [16] correcting a previous one [15]: the authors use the perturbative Painlevé test for testing the negative indices and find movable logarithms.

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