Non-Anticommutative Deformations of N=(1,1) Supersymmetric Theories

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Abstract

We discuss chirality-preserving nilpotent deformations of four-dimensional $N{=}(1,1)$ Euclidean harmonic superspace and their implications in $N{=}(1,1)$ supersymmetric gauge and hypermultiplet theories, basically following [hep-th/0308012] and [hep-th/0405049]. For the SO(4)×SU(2) invariant deformation, we present non-anticommutative Euclidean analogs of the $N{=}2$ gauge multiplet and hypermultiplet off-shell actions. As a new result, we consider a specific non-anticommutative hypermultiplet model with $N{=}(1,0)$ supersymmetry. It involves free scalar fields and interacting right-handed spinor fields.

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1 Introduction

In recent years, non-(anti)commutative deformations of supersymmetric field theories received a great deal of attention.

The simplest type of non-commutativity affects the space-time coordinates

$$x^m \star x^n - x^n \star x^m = i\Theta^{mn} \tag{1.1}$$

where Θ^{mn} is some constant tensor specifying the deformation. Such non-commutative coordinates arise in the field-theory limit of string theory in a constant B-field background [1, 2]. For local fields $f(x)$ and $g(x)$, this non-commutativity implies the use of the Moyal-Weyl star-product which can be defined via the bi-differential operator P (Poisson structure)

$$f \star g = f e^P g, \quad P = \frac{i}{2}\Theta^{mn}\overleftarrow{\partial}_m \overrightarrow{\partial}_n\,. \tag{1.2}$$

Moyal-Weyl type deformations of supersymmetric theories in superspace are characterized by a generic Poisson bracket APB where A and B are some superfields and the Poisson operator P is in general some quadratic form in derivatives with respect to both the even and odd superspace coordinates [3, 4]. Symmetry properties of the operator P determine unbroken symmetries of the deformed superfield theory: these symmetries are those generators of which commute with P. [1]

The specific deformed superfield field theories studied so far correspond to some particular degenerate choices of the generic superdifferential Poisson operator P. E.g., the authors of [5] considered the deformations of some theories in harmonic $N = 2$ superspace [6, 7] corresponding to the standard pure bosonic Poisson structure (1.2).

Deformations of a different kind are the nilpotent or non-anticommutative ones for which the operator P is bilinear in the proper derivations with respect to Grassmann coordinates. As such one can choose either generators of supersymmetry (Q-deformations), or spinor covariant derivatives (D-deformations). A surge of interest in superfield theories deformed in such a way was triggered by a recent paper [8] where a minimal deformation of the Euclidean $N{=}(\frac{1}{2}, \frac{1}{2})$ superspace was considered. For the chiral $N{=}(\frac{1}{2}, \frac{1}{2})$ coordinates $(x^m_L, \theta^\alpha, \bar\theta^{\dot\alpha})$ the operator P defining the relevant star product is given by the simple bracket

$$APB \;=\; -\tfrac{1}{2}(-1)^{p(A)} C^{\alpha\beta}\partial_\alpha A \partial_\beta B\,, \quad P = -\tfrac{1}{2} C^{\alpha\beta}\overleftarrow{\partial}_\alpha \overrightarrow{\partial}_\beta\,, \tag{1.3}$$

with $C^{\alpha\beta}$ being some constant symmetric matrix, $\partial_\alpha = \partial/\partial\theta^\alpha$, and $p(A)$ the Z_2-grading. The operator P defined by (1.3) acts on the θ^α coordinates only and retains the $N{=}(\frac{1}{2}, 0)$ fraction of the original $N{=}(\frac{1}{2}, \frac{1}{2})$ supersymmetry. It is very important that the corresponding noncommutative product of superfields preserves the chiral and antichiral representations of the $N = (\frac{1}{2}, \frac{1}{2})$ supersymmetry. Like the bosonic deformation (1.1), (1.2),

[1] In general, this criterion should be applied in a weak sense, i.e. for the commutator sandwiched between the superfields A and B.

this purely fermionic deformation also originates from string theory, as discussed in [8] and [9]-[12].

Deformations of the N=2 superfield theories along similar lines were discussed in [13]. In this contribution we shall focus on the harmonic-superspace formalism of the nilpotently deformed Euclidean N=$(1,1)$ theories, basically following Refs. [14, 15, 16] (see also [17, 18]).

The Grassmann harmonic analyticity is the key notion of the off-shell superfield description of N=2 supersymmetric field theories in four dimensions [6, 7] where it plays the role analogous to chirality in $N = 1$ superfield theories. In particular, the analytic gauge and hypermultiplet superfields are the building-blocks of off-shell interactions, and the harmonic analytic superspace formalism is indispensable for quantum supergraph calculations. By construction, the nilpotent Q-deformations (and some special D-deformations) of N=$(1,1)$ Euclidean superspace preserve this harmonic G-analyticity [14, 15]. Yet, the chirality also plays the important role in N=2 and N=$(1,1)$ supersymmetric gauge theories, so the deformations which we shall consider preserve as well both chiralities.

In Section 2 we review the nilpotent Q-deformations of the Euclidean chiral N=$(1,1)$ superspace and analyze the role of the standard conjugation or an alternative pseudoconjugation in Euclidean N=$(1,1)$ supersymmetric theories. The corresponding bi-differential operator P preserves chirality and anti-chirality, and half of the original N=$(1,1)$ supersymmetry (N=$(1,0)$ supersymmetry). For special choices, however, N=$(1,\frac{1}{2})$ supersymmetry or the whole automorphism group SO(4)$\times$ SU(2) can be retained.

Section 3 is devoted to the chirality-preserving SO(4)$\times$ SU(2) invariant deformation of the gauge N=$(1,1)$ theories in the harmonic superspace. This singlet deformation breaks half of supersymmetries and gives rise to some additional interactions of the scalar field $\bar{\phi}$ of the $N = (1,1)$ gauge multiplet with the remaining components of the latter [16]. [2]

Non-anticommutative interactions of the Grassmann-analytic hypermultiplets are considered in Section 4. Formally these interactions resemble those considered in the bose-deformed harmonic superspace of [5], however, the component contents of these two theories are entirely different. As a new explicit example, we analyze in some detail the simplest hypermultiplet self-interaction which vanishes in the anticommutative-superspace limit. In the component action of this model, the scalar fields do not interact with fermions, and only some specific fermionic self-interaction is present, with two derivatives on fermions. The solvable equation for the right-handed fermions contains the nonlinear source constructed from the left-handed ones which are free.

[2] The singlet Q-deformation of U(1) gauge theory was independently considered in [18].

2 Deformations of N=(1,1) Euclidean chiral superspace

The Euclidean N=$(1,1)$ superspace has as its automorphisms the Euclidean space spinor group $Spin(4) \sim$ SU$(2)_L\times$ SU$(2)_R$ and the R-symmetry group SU(2)× O(1,1) properly acting on the coordinates $x^m, \theta^\alpha_k, \bar\theta^{\dot\alpha k}$. We prefer to use the chiral coordinates $z_L \equiv (x^m_L, \theta^\alpha_k, \bar\theta^{\dot\alpha k})$ to parametrize this superspace. These Euclidean coordinates z_L are real with respect to the standard conjugation [19]

$$\widetilde{\theta^\alpha_k} \;=\; \varepsilon^{kj}\varepsilon_{\alpha\beta}\theta^\beta_j\;, \qquad \widetilde{\bar\theta^{\dot\alpha k}} \;=\; -\varepsilon_{kj}\varepsilon_{\dot\alpha\dot\beta}\bar\theta^{\dot\beta j}\;, \qquad \widetilde{AB} \;=\; \widetilde{B}\widetilde{A}\,. \tag{2.1}$$

This conjugation squares to identity on any object, and with respect to it the N=$(1,1)$ superspace has the real dimension $(4|8)$. However, if we wish to treat the N=$(\frac12,\frac12)$ superspace as a real subspace of the N=$(1,1)$ superspace (like $N=1$ supersubspace in the standard Minkowski $N=2,4D$ superspace), e.g. in order to be able to make reductions to the theories considered in [8], we cannot limit ourselves merely to this standard conjugation. Indeed, the Euclidean N=$(\frac12,\frac12)$ superspace *cannot* be real with respect to the complex conjugation: two independent SU(2) spinor coordinates have the real dimension 8 which coincides with the Grassmann dimension of the whole N=$(1,1)$ superspace.

The alternative SU(2)-breaking pseudoconjugation in the same Euclidean N=$(1,1)$ superspace was considered in [14]:

$$(\theta^\alpha_k)^* \;=\; \varepsilon_{\alpha\beta}\theta^\beta_k\;, \qquad (\bar\theta^{\dot\alpha k})^* \;=\; \varepsilon_{\dot\alpha\dot\beta}\bar\theta^{\dot\beta k}\;, \qquad (x^m_L)^* \;=\; x^m_L\;, \qquad (AB)^* \;=\; B^*A^*. \tag{2.2}$$

The existence of this pseudoconjugation does not impose any further restriction on the N=$(1,1)$ superspace which has the same dimension $(4|8)$ as with respect to the complex conjugation. Clearly, with respect to this pseudoconjugation, θ^α_1 and $\bar\theta^{\dot\alpha 1}$ are 'real', so they form an N=$(\frac12,\frac12)$ subspace of the 'real' dimension $(4|4)$ in N=$(1,1)$ superspace (such subspaces can be singled out in a few different ways). The standard conjugation (2.1) and the pseudoconjugation (2.2) act differently on the objects transforming by non-trivial representations of the R-symmetry SU(2).[3] The map * squares to -1 on the Grassmann coordinates and the associated spinor fields, and to $+1$ on any bosonic monomial or field. On the singlets of SU(2), both maps act as the standard complex conjugation. In particular, the invariant actions are real with respect to both * and $\tilde{}$, despite the fact that the component fields may have different properties under these (pseudo)conjugations.

After this digression, let us come back to our main subject, Q-deformations of N=$(1,1)$ theories. In chiral coordinates, the simplest Poisson structure operator is

$$P = -\frac12 C^{\alpha\beta}_{ik}\overleftarrow{Q}{}^i_\alpha\overrightarrow{Q}{}^k_\beta = -\frac12 C^{\alpha\beta}_{ik}\overleftarrow{\partial}{}^i_\alpha\overrightarrow{\partial}{}^k_\beta \tag{2.3}$$

[3]Some ambiguities of generalized conjugations in Grassmann algebras (C-antilinear maps with squares equal to ±1) were discussed in [20].

and the Poisson bracket for two superfields A and B is defined as

$$APB = -\tfrac{1}{2}(-1)^{p(A)}(\partial^k_\alpha A)C^{\alpha\beta}_{kj}(\partial^j_\beta B) = -(-1)^{p(A)p(B)}BPA\,. \tag{2.4}$$

Here, $C^{\alpha\beta}_{kj} = C^{\beta\alpha}_{jk}$ are some constants, $p(A)$ is the Z_2-grading, and the partial spinor derivatives act as

$$\partial^k_\alpha\theta^\beta_i = \delta^k_i\delta^\beta_\alpha \qquad \text{and} \qquad \bar\partial_{\dot\alpha i}\bar\theta^{\dot\beta k} = \delta^k_i\delta^{\dot\beta}_{\dot\alpha}\,. \tag{2.5}$$

By definition, the bracket (2.4) preserves both chirality and anti-chirality and does not touch $\mathrm{SU}(2)_R$ acting on dotted indices. Generically, it breaks half of the original $N{=}(1,1)$ supersymmetry since the generators $\bar Q_{\dot\alpha k}$ do not commute with the operator P. We demand P to be real, i.e. invariant under some antilinear map in the algebra of superfields. The two possible (pseudo)conjugations lead to different conditions on the constants $C^{\alpha\beta}_{kj}$. The constant deformation matrix can be split into two irreducible parts,

$$C^{\alpha\beta}_{kj} \;=\; C^{(\alpha\beta)}_{(kj)} + 2\varepsilon^{\alpha\beta}\varepsilon_{kj}I\,, \tag{2.6}$$

where I is a real parameter. The second, singlet part preserves the full $\mathrm{SO}(4)\times\mathrm{SU}(2)$ symmetry:

$$P_s \;=\; -I\,\overleftarrow{Q}{}^k_\alpha\overrightarrow{Q}{}^\alpha_k\,, \quad AP_sB \;=\; -I(-1)^{p(A)}Q^k_\alpha AQ^\alpha_kB\,. \tag{2.7}$$

Given the operator (2.4), the Moyal product of two superfields reads

$$A\star B \;=\; A\,\mathrm{e}^PB \;=\; A\,B + A\,P\,B + \tfrac{1}{2}A\,P^2B + \tfrac{1}{6}A\,P^3B + \tfrac{1}{24}A\,P^4B \tag{2.8}$$

where the identity $P^5 = 0$ was used. This star product preserves both chirality and antichirality and breaks $N{=}(0,1)$ supersymmetry. In the approach with the star product only free actions preserve all supersymmetries while interactions get deformed and they are not invariant under the $N{=}(0,1)$ supersymmetry transformations.

The (3,3) part $C^{(\alpha\beta)}_{(kl)}$ of the deformation matrix breaks the R-symmetry SU(2), so we should choose one of the alternative reality conditions to define the minimal form of the matrix $C^{(\alpha\beta)}_{(kl)}$. The minimal representation of this (3,3) part has the following form:

$$\begin{aligned} C^{(\alpha\beta)}_{(12)} \;&=\; C^{(\alpha\beta)}, \quad C^{(\alpha\beta)}_{(11)} = C^{(\alpha\beta)}_{(22)} = 0\,,\\ AP_CB &= -\frac{1}{2}(-1)^{p(A)}C^{(\alpha\beta)}(Q^1_\alpha AQ^2_\beta B + Q^2_\alpha AQ^1_\beta B)\,, \end{aligned} \tag{2.9}$$

if we assume that $C^{(\alpha\beta)}_{(ik)}$ is real with respect to the $\widetilde{\ }$ conjugation, $\widetilde{C^{(\alpha\beta)}_{(ik)}} = C^{(ik)}_{(\alpha\beta)}$.

The choice of the * pseudoconjugation (2.2) is compatible with the decomposition of $N{=}(1,1)$ into two $N{=}(\frac{1}{2},\frac{1}{2})$ superalgebras. Therefore, it allows one to choose a degenerate deformation

$$P(Q^2) \;=\; -\tfrac{1}{2}C(\overleftarrow{Q}{}^2_1\overrightarrow{Q}{}^2_2 + \overleftarrow{Q}{}^2_2\overrightarrow{Q}{}^2_1)\,, \tag{2.10}$$

which does not involve Q^1_α and contains the real parameter C. In this case, only $\bar Q_{\dot\alpha 2}$ are broken, but not the supercharges $\bar Q_{\dot\alpha 1}$. Hence, the deformation $P(Q^2)$ preserves the larger fraction $N{=}(1,\frac{1}{2})$ of the original $N{=}(1,1)$ supersymmetry.

It is of course possible to consider more general deformations affecting both the chiral and anti-chiral sectors. E.g. one can take the anticommuting set of pseudoreal generators $Q^2_\alpha, \bar{Q}_{\dot\alpha 1}$ and construct the real deformation operator $\hat{P}$ and the corresponding bracket for even superfields A and B as

$$A\hat{P}B = -C^{\alpha\beta}Q^2_\alpha A Q^2_\beta B - B^{\alpha\dot\alpha}(Q^2_\alpha A\bar{Q}_{\dot\alpha 1}B + \bar{Q}_{\dot\alpha 1}AQ^2_\alpha B) - \bar{C}^{\dot\alpha\dot\beta}\bar{Q}_{\dot\alpha 1}A\bar{Q}_{\dot\beta 1}B. \qquad (2.11)$$

It is evident that this deformation operator defines an associative star-product and it commutes with all spinor derivatives $D^k_\alpha, \bar{D}_{\dot\alpha k}$, as well as with 4 generators of supersymmetry $Q^2_\alpha, \bar{Q}_{\dot\alpha 1}$. Hence it breaks half of supersymmetry and preserves both chiralities.

3 Chirality-preserving singlet deformations of N=(1,1) harmonic superspace

Harmonic superspace with noncommutative bosonic coordinates x^m_A has been discussed in [5]. This deformation yields nonlocal theories but preserves the whole N=2 supersymmetry. The nilpotent D-deformations of Euclidean N=(1,1) superspace also preserving the full amount of supersymmetry were considered in [13]. Within the harmonic superspace formalism, a special case of such deformations, the singlet one preserving the SO(4)×SU(2) symmetry, one of two chiralities and harmonic analyticity, was addressed in [14, 15]. In particular, in [15] N=(1,1) gauge theory with such D-deformation was studied (see also a recent preprint [22]). Further in this contribution we shall not discuss this type of nilpotent deformations. Instead, we shall concentrate on the supersymmetry-breaking singlet nilpotent Q-deformation associated with the operator P_s (2.7). We shall essentially use the Euclidean version of the harmonic superspace approach, following refs. [14, 16].

The basic concepts of the harmonic superspace approach in its Euclidean variant coincide, up to a few minor distinctions, with those of the standard (Minkowski) N=2, D=4 harmonic superspace as collected in the book [7]. In both versions, the key ingredient is the SU(2)/U(1) harmonics $u^\pm_i, \quad u^{+i}u^-_i = 1$, where SU(2) is the R-symmetry group. The chiral-analytic coordinates $Z_C = (x^m_L, \theta^{\pm\alpha}, \bar\theta^{\pm\dot\alpha}, u^\pm_i)$ in the N=(1,1) harmonic superspace are related to the analytic coordinates via the shift of the bosonic coordinate

$$x^m_A = x^m_L - 2i(\sigma^m)_{\alpha\dot\alpha}\theta^{-\alpha}\bar\theta^{+\dot\alpha}\,, \quad \theta^{\pm\alpha} = \theta^{\alpha i}u^\pm_i\,,\ \bar\theta^{\pm\dot\alpha} = \bar\theta^{\dot\alpha i}u^\pm_i\,. \qquad (3.1)$$

The (pseudo)conjugations (2.1) and (2.2) can be extended to the harmonics and the coordinates of the harmonic superspace [14]. These two (pseudo)conjugations act identically on invariants and harmonic superfields, e.g. $(A^kB_k)^* = \widetilde{(A^kB_k)}$ or $(q^+)^* = \widetilde{q^+}$, but they differ when acting on harmonics or R-spinor component fields, e.g. $(A_k)^* \neq \widetilde{A_k}$. An important invariant pseudoreal subspace is the analytic Euclidean harmonic superspace, parametrized by the coordinates

$$(x^m_A\ ,\ \theta^{+\alpha}\ ,\ \bar\theta^{+\dot\alpha}, u^\pm_k)\ \equiv\ (\zeta, u)\,. \qquad (3.2)$$

The supersymmetry-preserving spinor and harmonic derivatives in different coordinate bases are defined in [14, 15, 16]. A Grassmann-analytic (G-analytic) superfield $\Phi = \Phi(\zeta, u)$ is defined by the constraints

$$D^+_\alpha \Phi(\zeta, \theta^-, \bar\theta^-, u) \;=\; \bar D^+_{\dot\alpha} \Phi(\zeta, \theta^-, \bar\theta^-, u) \;=\; 0. \tag{3.3}$$

It is important that the chirality-preserving operator P (2.4) also preserves Grassmann analyticity:

$$[P, (D^+_\alpha, \bar D^+_{\dot\alpha})] \;=\; 0\,. \tag{3.4}$$

In what follows it will be convenient to deal with harmonic projections of the $N{=}(1,1)$ supersymmetry generators

$$Q^k_\alpha = u^{+k} Q^-_\alpha - u^{-k} Q^+_\alpha, \quad \bar Q_{\dot\alpha k} = u^+_k \bar Q^- - u^-_k \bar Q^+. \tag{3.5}$$

For instance, in the chiral-analytic coordinates we have

$$Q^+_\alpha = \partial_{-\alpha}, \quad Q^-_\alpha = -\partial_{+\alpha} \tag{3.6}$$

where $\partial_{\pm\alpha} = \partial/\partial\theta^{\pm\alpha}$. In these coordinates, different terms in the product (2.8) with the singlet Q-deformation operator P_s are explicitly expressed as

$$\begin{aligned}
&AP_sB = I(-1)^{p(A)}\left(\partial_{-\alpha}A\partial^\alpha_+B + \partial^\alpha_+A\partial_{-\alpha}B\right),\\
&\frac{1}{2}AP_s^2B = -\frac{I^2}{4}(\partial_+)^2A(\partial_-)^2B - \frac{I^2}{4}(\partial_-)^2A(\partial_+)^2B + I^2\partial_{+\beta}\partial^\alpha_-A\partial_{+\alpha}\partial^\beta_-B\,,\\
&\frac{1}{6}AP_s^3B = \frac{I^3}{4}(-1)^{p(A)}\partial^\alpha_-(\partial_+)^2A\partial_{+\alpha}(\partial_-)^2B + \frac{I^3}{4}(-1)^{p(A)}\partial_{+\alpha}(\partial_-)^2A\partial^\alpha_-(\partial_+)^2B\,,\\
&\frac{1}{24}AP_s^4B = \frac{I^4}{16}(\partial_+)^2(\partial_-)^2A(\partial_-)^2(\partial_+)^2B\,.
\end{aligned} \tag{3.7}$$

Note that the last two terms vanish for the analytic superfields.

Now we turn to some details of the deformed $N{=}(1,1)$ gauge theory in harmonic superspace. It largely mimics the harmonic superspace formulation of non-abelian $N = 2$ gauge theory in 4D Minkowski space [7].

The basic superfield of the $N = (1,1)$ gauge theory is the analytic anti-Hermitian potential V^{++} with the values in the algebra of the gauge group which we choose to be U(n). The gauge transformation of the U(n) gauge potential V^{++} reads

$$\delta_\Lambda V^{++} \;=\; D^{++}\Lambda + [V^{++}, \Lambda]_\star \tag{3.8}$$

where Λ is an anti-Hermitian analytic gauge parameter and D^{++}, in the chiral-analytic basis, is

$$D^{++} = \partial^{++} + \theta^{+\alpha}\partial_{-\alpha} + \bar\theta^{+\dot\alpha}\partial_{-\dot\alpha}\,, \quad \partial^{++} = u^{+i}\frac{\partial}{\partial u^{-i}}\,. \tag{3.9}$$

In the Wess-Zumino (WZ) gauge we shall use the expansion of the potential in $\bar\theta^{+\dot\alpha}$

$$\begin{aligned}
&V^{++}_{WZ} = \bar\phi^{++} + \bar\theta^+_{\dot\alpha} V^{+\dot\alpha} + (\bar\theta^+)^2 V\,,\\
&\bar\phi^{++}(x_A,\theta^+,u) = (\theta^+)^2\bar\phi, \quad V^{+\dot\alpha}(x_A,\theta^+,u) = 2\theta^{+\alpha}A^{\dot\alpha}_\alpha + 4(\theta^+)^2\bar\Psi^{-\dot\alpha}\,,\\
&V(x_A,\theta^+,u) = \phi + 4\theta^{+\alpha}\Psi^-_\alpha \; + 3(\theta^+)^2\mathcal{D}^{--}
\end{aligned} \tag{3.10}$$

where $\Psi^-_\alpha = u^-_k\Psi^k_\alpha, \bar\Psi^-_{\dot\alpha} = u^-_k\bar\Psi^k_{\dot\alpha}, \mathcal{D}^{--} = u^-_k u^-_l\mathcal{D}^{kl}$ and all component fields are functions of x^m_A.

For what follows it will be convenient to rewrite the expression for the WZ-potential in the chiral-analytic basis, using the relation (3.2)

$$V^{++}_{WZ}(Z_C,u) = v^{++}(z_C,u) + \bar\theta^+_{\dot\alpha}v^{+\dot\alpha}(z_C,u) + (\bar\theta^+)^2 v(z_C,u) \tag{3.11}$$

where the chiral superfunctions depend on the coordinates $x^m_L, \theta^{+\alpha}, \theta^{-\alpha}$ and $u^\pm_i$ only

$$\begin{aligned}
&v^{++}(z_C,u) = (\theta^+)^2\bar\phi(x_L)\,,\\
&v^{+\dot\alpha}(z_C,u) = V^{+\dot\alpha}(x_L,\theta^+,u) - 2i\theta^{-\alpha}\partial^{\dot\alpha}_\alpha\bar\phi^{++}(x_L,\theta^+,u)\\
&= -2\theta^+_\alpha A^{\alpha\dot\alpha} + 4(\theta^+)^2u^-_k\bar\Psi^{\dot\alpha k} + 2i\theta^-_\alpha(\theta^+)^2\partial^{\alpha\dot\alpha}\bar\phi\,,\\
&v(z_C,u) = V(x_L,\theta^+,u) + i\theta^{-\alpha}\partial_{\alpha\dot\alpha}V^{+\dot\alpha}(x_L,\theta^+,u) - (\theta^-)^2\Box\bar\phi^{++}(x_L,\theta^+,u)\\
&= \phi + 4\theta^{+\alpha}\Psi^-_\alpha + 3(\theta^+)^2\mathcal{D}^{--} - 2i(\theta^+\theta^-)\partial_m A_m + \theta^+\sigma_{mn}\theta^- F_{mn}\\
&+4i\theta^{-\alpha}(\theta^+)^2\partial_{\alpha\dot\alpha}\bar\Psi^{-\dot\alpha} - (\theta^-)^2(\theta^+)^2\Box\bar\phi\,.
\end{aligned} \tag{3.12}$$

Here all component fields (after separating the harmonic dependence) are functions of x^m_L.

Now we specialize to the simplest case of the U(1) gauge group. The corresponding P_s-deformed gauge and $N{=}(1,0)$ supersymmetry transformations of the component fields can be readily found [16]. They are given, respectively, by

$$\begin{aligned}
&\delta_a\phi = -8IA_m\partial_m a\,, \quad \delta_a\bar\phi = 0\,, \quad \delta_a A_m = (1+4I\bar\phi)\partial_m a\,,\\
&\delta_a\Psi^k_\alpha = -4I\bar\Psi^{\dot\alpha k}\partial_{\alpha\dot\alpha}a\,, \quad \delta_a\bar\Psi^k_{\dot\alpha} = 0\,, \quad \delta_a\mathcal{D}^{kl} = 0
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
&\delta_\epsilon\phi = 2\epsilon^{\alpha k}\Psi_{\alpha k}\,, \quad \delta_\epsilon\bar\phi = 0\,, \quad \delta_\epsilon A_m = \epsilon^{\alpha k}(\sigma_m)_{\alpha\dot\alpha}\bar\Psi^{\dot\alpha}_k\,,\\
&\delta_\epsilon\Psi^k_\alpha = -\epsilon_{\alpha l}\mathcal{D}^{kl} + \frac{1}{2}(1+4I\bar\phi)(\sigma_{mn}\epsilon^k)_\alpha F_{mn} - 4iI\epsilon^k_\alpha A_m\partial_m\bar\phi\,,\\
&\delta_\epsilon\bar\Psi^k_{\dot\alpha} = -i\epsilon^{\alpha k}(1+4I\bar\phi)\partial_{\alpha\dot\alpha}\bar\phi\,,\\
&\delta_\epsilon\mathcal{D}^{kl} = i\partial_m[(\epsilon^k\sigma_m\bar\Psi^l + \epsilon^l\sigma_m\bar\Psi^k)(1+4I\bar\phi)]
\end{aligned} \tag{3.14}$$

where $F_{mn} = \partial_m A_n - \partial_n A_m$.

The nonpolynomial superfield action of the Q-deformed gauge theory has been given in [14] as an integral over the full superspace in the chiral coordinates, by analogy with the undeformed $N{=}2$ superfield action [21]. It was shown in [16] that the P_s-deformed U(1) gauge action can be conveniently rewritten as the integral over the chiral superspace

$$S^{(I)} = \frac{1}{4}\int d^4x_L d^4\theta\mathcal{A}^2 \tag{3.15}$$

where $\mathcal{A}(x_L, \theta^+, \theta^-, u)$ is the deformed chiral superfield strength. The latter appears as the lowest component in the $\bar\theta^{+\dot\alpha}$ expansion of the covariantly chiral superfield strength $\mathcal{W}$:

$$\mathcal{W} \equiv -\tfrac{1}{4}(\bar D^+)^2 V^{--} = \mathcal{A} + \bar\theta^+_{\dot\alpha}\tau^{-\dot\alpha} + (\bar\theta^+)^2\tau^{-2} \tag{3.16}$$

and the action (3.15) can be rewritten as

$$S^{(I)} = \frac{1}{4}\int d^4x_L d^4\theta \mathcal{W}^2 \,. \tag{3.17}$$

It can be shown that the remaining two components in (3.16) do not contribute to (3.17).

The composite harmonic connection V^{--} is connected with the basic potential V^{++} via the deformed harmonic zero curvature equation [14]

$$D^{++}V^{--} - D^{--}V^{++} + [V^{++}, V^{--}]_\star = 0 \tag{3.18}$$

where, in the chiral-analytic basis,

$$D^{--} = \partial^{--} + \theta^{-\alpha}\partial_{+\alpha} + \bar\theta^{-\dot\alpha}\partial_{+\dot\alpha}\,, \quad \partial^{--} = u^{-i}\frac{\partial}{\partial u^{+i}}\,.$$

As a consequence of (3.18), the chiral superfield $\mathcal{A}$ satisfies the homogeneous harmonic equation

$$[\partial^{++} + (1 + 4I\bar\phi)\theta^{+\alpha}\partial_{-\alpha}]\mathcal{A} = 0 \tag{3.19}$$

and some additional nonlinear inhomogeneous equation [16]:

$$\begin{aligned}&[\partial^{++} + (1 + 4I\bar\phi)\theta^{+\alpha}\partial_{-\alpha}]\varphi^{--} + 2(\mathcal{A} - v) - I\left(\partial^\alpha_- v^+_{\dot\alpha}\partial_{+\alpha}v^{-\dot\alpha} - \partial^\alpha_+ v^+_{\dot\alpha}\partial_{-\alpha}v^{-\dot\alpha}\right)\\ &+ \frac{I^3}{4}\,\partial^\alpha_-(\partial_+)^2 v^+_{\dot\alpha}\partial_{+\alpha}(\partial_-)^2 v^{-\dot\alpha} = 0\end{aligned} \tag{3.20}$$

where $v^{-\dot\alpha}$ and φ^{--} are the proper chiral coefficients of the expansion of V^{--} in $\bar\theta^{\pm\dot\alpha}$. They can be calculated in terms of the component fields.

The undeformed chiral U(1) superfield strength has the following component field content

$$\begin{aligned}&W_0(x_L, \theta^+, \theta^-, u) = \varphi + 2\theta^+\psi^- - 2\theta^-\psi^+ + (\theta^+)^2 d^{--}\\ &-2(\theta^+\theta^-)d^{+-} + (\theta^-)^2 d^{++} + (\theta^-\sigma_{mn}\theta^+)f_{mn}\\ &+2i[(\theta^-)^2\theta^+\sigma_m\partial_m\bar\psi^+ + (\theta^+)^2\theta^-\sigma_m\partial_m\bar\psi^-] - (\theta^+)^2(\theta^-)^2\Box\bar\phi\end{aligned} \tag{3.21}$$

where $f_{mn} = \partial_m a_n - a_m a_n$, $\psi^\pm_\alpha = \psi^i_\alpha(x_L)u^\pm_i$, $d^{+-} = u^+_k u^-_l d^{kl}(x_L)$, etc. This superfield obeys the free harmonic equation $D^{++}W_0 = 0$ and transforms under $N = (1,0)$ supersymmetry as

$$\delta_\epsilon W_0 = (\epsilon^{-\alpha}\partial_{-\alpha} + \epsilon^{+\alpha}\partial_{+\alpha})W_0\,. \tag{3.22}$$

It is rather straightforward to show that $\mathcal{A}$ can be constructed as a nonlinear transformation of the undeformed U(1) superfield strength W_0

$$\mathcal{A}(x_L, \theta^+, \theta^-, u) = (1 + 4I\bar\phi)^2 W_0(x_L, \theta^+, (1 + 4I\bar\phi)^{-1}\theta^-, u)\,. \tag{3.23}$$

The nonlinear relations between the undeformed and deformed U(1) component fields following from (3.23) are

$$\begin{aligned}
&\varphi=(1+4I\bar\phi)^{-2}[\phi+4I(1+4I\bar\phi)^{-1}(A_m^2+4I^2(\partial_m\bar\phi)^2)]\,,\\
&a_m=(1+4I\bar\phi)^{-1}A_m,\quad \bar\psi^k_{\dot\alpha}=(1+4I\bar\phi)^{-1}\bar\Psi^k_{\dot\alpha}\,,\\
&\psi^k_\alpha=(1+4I\bar\phi)^{-2}[\Psi^k_\alpha+4I(1+4I\bar\phi)^{-1}A_{\alpha\dot\alpha}\bar\Psi^{\dot\alpha k}]\,,\\
&d^{kl}=(1+4I\bar\phi)^{-2}[\mathcal{D}^{kl}+8I\bar\Psi^k_{\dot\alpha}\bar\Psi^{\dot\alpha l}]\,.
\end{aligned}\tag{3.24}$$

The $N{=}(1,0)$ supersymmetry transformation of the deformed chiral superfield is given by

$$\delta_\epsilon\mathcal{A}=[(1+4I\bar\phi)\epsilon^{-\alpha}\partial_{-\alpha}+\epsilon^{+\alpha}\partial_{+\alpha}]\mathcal{A}\,.\tag{3.25}$$

The deformed U(1) gauge superfield action can be expressed in terms of the abelian undeformed objects up to a total spinor derivative in the integrand

$$S^{(I)}=\frac14\int d^4x_Ld^4\theta\,\mathcal{A}^2=\frac14\int d^4x_Ld^4\theta\,(1+4I\bar\phi)^2W_0^2\,.\tag{3.26}$$

Using the redefinitions of the deformed fields (3.24), one can obtain the component Lagrangian of the deformed U(1) gauge theory as $L^{(I)}=(1+4I\bar\phi)^2L_0$ where L_0 is the free undeformed Lagrangian

$$L_0=-\frac12\varphi\Box\bar\phi+\frac14(f_{mn}^2+\frac12\varepsilon_{mnrs}f_{mn}f_{rs})-i\psi^\alpha_k\partial_{\alpha\dot\alpha}\bar\psi^{\dot\alpha k}+\frac14(d^{kl})^2\,.\tag{3.27}$$

It is obvious that the scalar, fermionic and auxiliary terms in the action can be given the form of the free kinetic terms by properly rescaling the fields φ,ψ^k_α and d^{kl}. However, the nonlinear interaction of the fields $\bar\phi$ and f_{mn} ,

$$\frac14(1+4I\bar\phi)^2(f_{mn}^2+\frac12\varepsilon_{mnrs}f_{mn}f_{rs})\,,\tag{3.28}$$

cannot be removed by any field redefinition.

Now let us shortly discuss how the above generalizes to the nonabelian U(n) case ($n\geq 2$). We use the WZ-gauge for the U(n) potential (3.10), and the corresponding deformed component gauge transformations are

$$\begin{aligned}
&\delta_a\bar\phi=-i[a,\bar\phi],\quad \delta_r\bar\Psi^k_{\dot\alpha}=-i[a,\bar\Psi^k_{\dot\alpha}],\quad \delta_r\mathcal{D}^{kl}=-i[a,\mathcal{D}^{kl}]\,,\\
&\delta_aA_m=\partial_ma+i[A_m,a]+2I\{\bar\phi,\partial_ma\}\,,\\
&\delta_a\phi=-i[a,\phi]-4I\{A_m,\partial_ma\}-4iI^2[\Box a,\bar\phi]\,,\\
&\delta_a\Psi^k_\alpha=-i[a,\Psi^k_\alpha]-2I(\sigma_m)_{\alpha\dot\alpha}\{\bar\Psi^{\dot\alpha k},\partial_ma\}.
\end{aligned}\tag{3.29}$$

The P_s-deformed U(n) chiral gauge superfield $\mathcal{A}$ satisfies the following equation:

$$D^{++}\mathcal{A}+I\theta^{+\alpha}\{\bar\phi,\partial_{-\alpha}\mathcal{A}\}+(\theta^+)^2[\bar\phi,\mathcal{A}]+I^2[\bar\phi,(\partial_-)^2\mathcal{A}]=0\tag{3.30}$$

where $\bar\phi$ is the Hermitian matrix scalar field. It is convenient to define the following matrix operator:

$$L = 1 + 2I\{\bar\phi, \quad\}\,, \tag{3.31}$$

then the first two terms in eq.(3.30) can be rewritten as $(\partial^{++} + L\theta^{+\alpha}\partial_{-\alpha})\mathcal{A}$. The undeformed harmonic chiral U(n) superfield A has the following component expansion

$$\begin{aligned} A \;=\;& \varphi + 2\theta^+\psi^- - 2\theta^-\psi^+ + (\theta^+)^2 d^{--} + (\theta^+\theta^-)([\varphi,\bar\phi] - 2d^{+-}) + (\theta^-)^2 d^{++} \\ & + (\theta^+\sigma_{mn}\theta^-) f_{mn} + 2(\theta^-)^2\theta^+\left(i\xi^+ - [\bar\phi,\psi^+]\right) + 2i(\theta^+)^2\theta^-\xi^- \\ & - (\theta^+)^2(\theta^-)^2\left(p + [\bar\phi, d^{+-}]\right) \end{aligned} \tag{3.32}$$

where all the component fields are $n\times n$ matrices and the following short-hand notation is used:

$$\begin{aligned} &\nabla_m = \partial_m + i[a_m, \quad]\,, \quad f_{mn} = \partial_m a_n - \partial_n a_m + i[a_m, a_n]\,, \\ &\xi^k_\alpha = (\sigma_m)_{\alpha\dot\alpha}\nabla_m\bar\psi^{\dot\alpha k}, \quad p = \nabla^2_m\bar\phi + \{\bar\psi^{\dot\alpha k}, \bar\psi_{\dot\alpha k}\} + \frac{1}{2}[\bar\phi,[\bar\phi,\varphi]]\,. \end{aligned} \tag{3.33}$$

The deformed chiral U(n) superfield can be written as a sum of two $N{=}(1,0)$ covariant objects

$$\begin{aligned} &\mathcal{A}(x_L,\theta^+,\theta^-,u) = [L^2 + L(1-L)(\theta^-\partial_-) - \frac{1}{4}(1-L)^2(\theta^-)^2(\partial_-)^2]A(x_L,\theta^+,\theta^-,u) \\ &-4I^2\hat A(x_L,\theta^+,u) \end{aligned} \tag{3.34}$$

where A is the undeformed U(n) superfield (3.32), and the $\bar\phi$-dependent matrix operator L (3.31) commutes with $\theta^{\pm\alpha}$ and $\partial_{-\alpha}$ and acts on all matrix quantities standing to the right. The second part $\hat A$ is a traceless chiral-analytic $N{=}(1,0)$ superfield

$$\begin{aligned} &\hat A(x_L,\theta^+,u) = \hat p - [\bar\phi, d^{+-}] + 2\theta^{+\alpha}(i[\bar\phi,\xi^-_\alpha] - [\bar\phi,[\bar\phi,\psi^-_\alpha]]) \\ &+(\theta^+)^2[\bar\phi,[\bar\phi,d^{--}]]\,, \quad \hat p = p - \tfrac{1}{n}\mathrm{Tr}\,p\,. \end{aligned} \tag{3.35}$$

Both parts of $\mathcal{A}$ are thus expressed in terms of the undeformed field components of the superfield A (3.32).

The $N{=}(1,0)$ supersymmetry transformation of $\mathcal{A}$ has the following form:

$$\delta_\epsilon\mathcal{A} = 2(\epsilon^-\theta^+)[\bar\phi,\mathcal{A}] + L\epsilon^{-\alpha}\partial_{-\alpha}\mathcal{A} + \epsilon^{+\alpha}\partial_{+\alpha}\mathcal{A}\,. \tag{3.36}$$

It is worth noting that the undeformed anti-self-duality equation in the $N{=}(1,1)$ supersymmetric U(n) gauge theory [23, 24] can be written in the pure chiral superfield form as

$$A = 0\,, \tag{3.37}$$

which, as follows from (3.32), amounts to the following set of matrix component equations

$$\begin{aligned} &f_{mn}(\sigma_{mn})^\beta_\alpha = 0\,, \quad \varphi = \psi^k_\alpha = d^{kl} = 0\,, \\ &(\sigma_m)_{\alpha\dot\alpha}(\partial_m\bar\psi^{\dot\alpha k} + i[a_m,\bar\psi^{\dot\alpha k}]) = 0\,, \quad (\nabla_m)^2\bar\phi + \{\bar\psi^{\dot\alpha k}, \bar\psi_{\dot\alpha k}\} = 0\,. \end{aligned} \tag{3.38}$$

These anti-self-dual U(n) solutions preserve only the $N{=}(1,0)$ supersymmetry, so it is natural that the same undeformed solutions survive in the I-deformed U(n) gauge theory

$$A = 0 \;\Leftrightarrow\; \mathcal{A} = 0\,. \tag{3.39}$$

The I-deformed U(n) gauge theory component action can be directly obtained from the superfield chiral action

$$\mathcal{S}_n = \frac{1}{4}\int d^4x_L d^4\theta\, \mathrm{Tr}\, \mathcal{A}^2 = \int d^4x_L d^4\theta\, \mathrm{Tr}\, \{\frac{1}{4}(LA)^2 - 2I^2\hat{A}A\}\,, \tag{3.40}$$

using relations (3.32) and (3.35). In the limit $I \to 0$ the first term yields the action of the undeformed U(n) gauge theory. The non-standard second term contains higher derivative terms, in particular $I^2(\Box\bar{\phi})^2$, which can hopefully be removed by a redefinition of the scalar field φ (so far we have checked this only for the bilinear free part of the total action).

4 Interactions of hypermultiplets in deformed harmonic superspace

The free q^+ hypermultiplet actions of ordinary harmonic theory [7] are not deformed in the non-anticommutative superspace:

$$S_0(q^+) \;=\; \frac{1}{2}\int \mathrm{d}u\, \mathrm{d}\zeta^{-4}\; q^+_a \star D^{++}q^{+a} = \frac{1}{2}\int \mathrm{d}u\, \mathrm{d}\zeta^{-4} q^+_a D^{++}q^{+a}\,. \tag{4.1}$$

Here $\mathrm{d}\zeta^{-4} = \mathrm{d}^4x_{\scriptscriptstyle A}(D^-)^4$ and the additional 'Pauli-Gürsey' SU(2)$_P$ indices $a, b = 1, 2$ were introduced: $q^{+a} = \varepsilon^{ab}q^+_b = (\tilde{q}^+, q^+)$. Let us consider the $\bar{\theta}^{+\dot{\alpha}}$-expansion of the superfield doublet q^{+a} in the analytic basis

$$\begin{aligned} q^{+a} \;&=\; c^{+a} + \bar{\theta}^+_{\dot{\alpha}}\kappa^{\dot{\alpha}a} + (\bar{\theta}^+)^2 b^{-a}, \\ D^{++}q^{+a} \;&=\; \partial^{++}c^{+a} + \bar{\theta}^+_{\dot{\alpha}}(\partial^{++}\kappa^{\dot{\alpha}a} + 2i\theta^+_\alpha\partial^{\alpha\dot{\alpha}}c^{+a}) \\ &\quad + (\bar{\theta}^+)^2(\partial^{++}b^{-a} + i\theta^{+\alpha}\partial_{\alpha\dot{\alpha}}\kappa^{\dot{\alpha}a}) \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} &c^{+a} = f^a + \theta^{+\alpha}\rho^a_\alpha + (\theta^+)^2 g^a, \quad\; \kappa^{\dot{\alpha}a} = \chi^{a\dot{\alpha}} + \theta^{+\alpha}r^{\dot{\alpha}a}_\alpha + (\theta^+)^2\bar{\Sigma}^{a\dot{\alpha}}, \\ &b^{-a} = h^a + \theta^{+\alpha}\Sigma^a_\alpha + (\theta^+)^2 X^a \end{aligned} \tag{4.3}$$

and, for brevity, the U(1) charges of the component fields $f^a, g^a, h^a, \ldots$ are suppressed. The component fields are functions of $x^m_{\scriptscriptstyle A}$ and harmonics. The chiral representation of the free action (i.e., with the integration over $\bar{\theta}^{+\dot{\alpha}}$ manifestly performed) reads

$$\begin{aligned} S_0(q^+) \;=\; &-\int du\, d^4x_{\scriptscriptstyle A}\, d^2\theta^+[\tfrac{1}{2}b^{-a}\partial^{++}c^+_a + \tfrac{1}{2}c^{+a}\partial^{++}b^-_a + \tfrac{1}{4}\kappa^{\dot{\alpha}a}\partial^{++}\kappa_{\dot{\alpha}a} \\ &+ \tfrac{i}{2}\theta^{+\alpha}(c^{+a}\partial_{\alpha\dot{\alpha}}\kappa^{\dot{\alpha}}_a - \kappa^{\dot{\alpha}}_a\partial_{\alpha\dot{\alpha}}c^{+a})]\,. \end{aligned} \tag{4.4}$$

The non-anticommutativity shows up in the hypermultiplet self-interactions. If we prefer to work in the manifestly $\mathrm{SU}(2)_P$ covariant formalism, it is convenient to define two independent combinations:

$$\{q^{+a},q^{+b}\}_\star, \quad \left[q^{+a},q^{+b}\right]_\star = 2q^{+a}P_s q^{+b} = \varepsilon^{ab}\mathcal{C}^{++}\,. \tag{4.5}$$

The square of the first superfield contracted with some $\mathrm{SU}(2)_P$-breaking constant parameter $C_{(ab)}$ gives a non-anticommutative generalization of the self-interaction $[q^{+a}q^{+b}C_{(ab)}]^2$ which yields the familiar Taub-NUT hyper-Kähler metric on the bosonic target space [7]. Leaving this generalization for the future study, we shall consider a simpler example of the deformed self-interaction constructed out of the second combination in (4.5) and vanishing in the anticommutative limit $I\to 0$

$$S_\nu(q^+) \;=\; -\frac{\nu}{4}\int \mathrm{d}u\,\mathrm{d}\zeta^{-4}\,\mathcal{C}^{++}\star\mathcal{C}^{++} = -\frac{\nu}{4}\int \mathrm{d}u\,\mathrm{d}\zeta^{-4}\,\mathcal{C}^{++}\mathcal{C}^{++} \tag{4.6}$$

where ν is a coupling constant and the overall sign was chosen for further convenience. Note that this superfield interaction is nilpotent, $(\mathcal{C}^{++})^2 \sim (\bar\theta^+)^2$, and preserves both $\mathrm{SU}(2)_P$ and the R-symmetry SU(2) which acts on harmonics.

One can easily calculate the chiral components of the composite superfields

$$\begin{aligned}
\mathcal{C}^{++} \;&=\; q^+_a P_s q^{+a} = -4iI\bar\theta^{+\dot\alpha}\partial_{\alpha\dot\alpha}q^+_a\partial^\alpha_+ q^{+a} = -4iI\bar\theta^{+\dot\alpha}\partial_{\alpha\dot\alpha}c^+_a\partial^\alpha_+ c^{+a}\\
&\quad + 2iI(\bar\theta^+)^2(\partial_{\alpha\dot\alpha}\kappa^{\dot\alpha}_a\partial^\alpha_+ c^{+a} - \partial_{\alpha\dot\alpha}c^+_a\partial^\alpha_+\kappa^{\dot\alpha a})\,,\\
(\mathcal{C}^{++})^2 \;&=\; -8I^2(\bar\theta^+)^2 B^+_{\dot\alpha}B^{+\dot\alpha}, \quad B^{+\dot\alpha}(c^+) = \partial^{\alpha\dot\alpha}c^+_a\partial_{+\alpha}c^{+a}\,.
\end{aligned} \tag{4.7}$$

The deformed interaction (4.6) contains superfields c^{+a} only

$$\begin{aligned}
S^s_\nu(q^+) \;&=\; 2\nu I^2\int du\, d^4x_{A}\, d^2\theta^+ B^+_{\dot\alpha}B^{+\dot\alpha}\\
&=\; -2\nu I^2\int du\, d^4x_{A}\, d^2\theta^+\partial_{\alpha\dot\alpha}c^+_a\partial^\alpha_+ c^{+a}\partial^{\beta\dot\alpha}c^+_b\partial_{+\beta}c^{+b}\,.
\end{aligned} \tag{4.8}$$

The total superfield action $S_0(q^+) + S_\nu(q^+)$ yields the hypermultiplet equation of motion

$$D^{++}q^{+a} = \nu q^{+a}P_s(q^+_b P_s q^{+b}) \equiv J^{(+3)a}(q^+) \tag{4.9}$$

where $J^{(+3)a}(q^+)$ is the nonlinear nilpotent source. After performing the θ-integration, the total action contains an infinite number of auxiliary fields coming from the harmonic expansions of the components in (4.3). These auxiliary fields can be eliminated using the appropriate non-dynamical equations collected in the $\theta^+,\bar\theta^+$ expansion of (4.9).

The non-dynamical equations of motion for c^{+a} and $\kappa^{\dot\alpha a}$ have the following form:

$$\partial^{++}c^{+a} = 0\,, \quad \partial^{++}\kappa^a_{\dot\alpha} - 2i\theta^{+\alpha}\partial_{\alpha\dot\alpha}c^{+a} = 0\,. \tag{4.10}$$

In components, the solution to these equations is given by

$$c^{+a} = u^+_k f^{ak}(x) + \theta^{+\alpha}\rho^a_\alpha(x)\,, \quad \kappa^a_{\dot\alpha} = \chi^a_{\dot\alpha}(x) + 2iu^-_k\theta^{+\alpha}\partial_{\alpha\dot\alpha}f^{ak}(x)\,. \tag{4.11}$$

The last equation, for the chiral component b^{-a} , also follows from eq. (4.9)

$$\begin{aligned} \partial^{++}b^{-a} + i\theta^{+\alpha}\partial_{\alpha\dot\alpha}\kappa^{\dot\alpha a} \;&=\; -4\nu I^2[\partial_{\alpha\dot\alpha}c^{+a}\partial^{\beta\dot\alpha}\partial^\alpha_+c^{+b}\partial_{+\beta}c^+_b - \partial_m c^{+a}\partial_m c^+_b(\partial_+)^2c^{+b} \\ &+\partial^\alpha_+c^{+a}\Box c^+_b\partial_{+\alpha}c^{+b} + \partial^\alpha_+c^{+a}\partial^{\beta\dot\alpha}c^+_b\partial_{\alpha\dot\alpha}\partial_{+\beta}c^{+b}] \end{aligned} \tag{4.12}$$

and is solved by

$$b^{-a} = -4\nu I^2 u^-_k[\partial_{\alpha\dot\alpha}f^{ak}(\partial^{\beta\dot\alpha}\rho^{\alpha b})\rho_{\beta b} + \rho^{\alpha a}\partial^{\beta\dot\alpha}f^k_b\partial_{\alpha\dot\alpha}\rho^b_\beta + \rho^{\alpha a}\rho^b_\alpha\Box f^k_b] \tag{4.13}$$

(eq.(4.12) involves also the set of dynamical equations for the physical fields $f^{ak}(x), \rho^a_\alpha(x)$ and $\chi^a_{\dot\alpha}(x)$; these equations can be re-derived from the on-shell action written in terms of the physical fields). Actually, b^{-a} does not contribute to the total physical on-shell action $S_0 + S_\nu$: the only place where it appears is the first two terms in (4.4), and these terms vanish after employing first of eqs.(4.10) and integrating by parts with respect to ∂^{++}.

Eliminating the auxiliary component fields from the action $S_0 + S_\nu$ by the substitution (4.11), one obtains the physical action of this model. It contains the standard free kinetic terms for the physical bosonic and fermionic fields, as well as some fermionic self-interaction with two derivatives:

$$S = \int d^4x[\tfrac{1}{2}\partial_m f^{ak}\partial_m f_{ak} + \tfrac{i}{2}\rho^{\alpha a}\partial_{\alpha\dot\alpha}\chi^{\dot\alpha}_a - \nu I^2(\partial_{\alpha\dot\alpha}\rho_{\gamma a})\rho^{\alpha a}(\partial^{\beta\dot\alpha}\rho^{\gamma b})\rho_{\beta b}]\,. \tag{4.14}$$

The scalar field f^{ak} and the left-handed spinor field ρ^a_α satisfy the free massless equations in this model

$$\Box f^{ak} = 0, \quad \partial_{\alpha\dot\alpha}\rho^{\alpha a} = 0\,. \tag{4.15}$$

At the same time, the equation for the right-handed spinor field $\chi^a_{\dot\alpha}$ contains the nonlinear spinor source depending on the left-handed spinor field

$$\begin{aligned} &i\partial_{\alpha\dot\alpha}\chi^{\dot\alpha}_a = -4\nu I^2[\rho_{\beta b}(\partial_{\alpha\dot\alpha}\rho_{\gamma a})(\partial^{\beta\dot\alpha}\rho^{\gamma b}) + \rho_{\beta a}(\partial_{\gamma\dot\alpha}\rho_{\alpha b})(\partial^{\beta\dot\alpha}\rho^{\gamma b}) \\ &+\rho_{\beta b}(\partial_{\gamma\dot\alpha}\rho^\gamma_a)(\partial^{\beta\dot\alpha}\rho^b_\alpha) + \rho^{\beta b}\rho_{\beta a}\Box\rho_{\alpha b}] = -\nu I^2 J_{\alpha a}[\rho(x)]\,. \end{aligned} \tag{4.16}$$

Note that the last two terms in $J_{\alpha a}$ are vanishing on the mass-shell of the free fields ρ^a_α. The exact classical solution for $\chi^{\dot\alpha a}$ is a sum of the free right-handed fermion $\chi^{\dot\alpha a}_0$ and the inhomogeneous solution with the above nilpotent spinor source:

$$\begin{aligned} &\chi^{\dot\alpha a} = \chi^{\dot\alpha a}_0 + i\nu I^2\int d^4y\partial^{\alpha\dot\alpha}_x D^0(x-y)J^a_\alpha[\rho(y)]\,, \\ &\partial_{\alpha\dot\alpha}\chi^{\dot\alpha a}_0 = 0, \quad \Box_x D^0(x-y) = \delta^4(x-y). \end{aligned} \tag{4.17}$$

Thus the considered model is exactly solvable at the classical level.

The component form of some other nilpotently deformed q^+ self-interactions and the deformed hypermultiplet interactions with the analytic gauge superfield V^{++} will be studied elsewhere.

5 Conclusions

In this contribution, basically following refs. [14, 15, 16], we briefly reviewed recent results on the nilpotent non-anticommutative deformations of Euclidean N=$(1,1)$ superspace, with the main emphasis on the structure of the singlet Q-deformation of $N=(1,1)$ gauge theories. This deformation breaks half of $N=(1,1)$ supersymmetry, but preserves O(4) and SU(2) automorphism symmetries, as well as both chiralities and harmonic Grassmann analyticity. We also considered a simple new example of the Q-deformed hypermultiplet action, with the self-interaction vanishing in the anticommutative limit. This model is exactly solvable at the classical level.

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