IRREDUCIBLE SUBGROUPS OF ALGEBRAIC GROUPS

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Abstract

A closed subgroup of a semisimple algebraic group G is said to be G-irreducible if it lies in no proper parabolic subgroup of G. We prove a number of results concerning such subgroups. Firstly they have only finitely many overgroups in G; secondly, with some specified exceptions, there exist G-irreducible subgroups of type A_1; and thirdly, we prove an embedding theorem for G-irreducible subgroups.

1. Introduction

Let G be a semisimple algebraic group over an algebraically closed field K of characteristic $p \geqslant 0$. Following Serre, we define a subgroup Γ of G to be *G-irreducible* if Γ is closed, and lies in no proper parabolic subgroup of G. When $G = SL(V)$, this definition coincides with the usual notion of irreducibility on V. The definition follows the philosophy, developed over the years by Serre, Tits and others, of generalizing standard notions of representation theory (morphisms $\Gamma \to SL(V)$) to situations where the target group is an arbitrary semisimple algebraic group. For an exposition, see for example [**8**, Part II].

In this paper we study the collection of connected G-irreducible subgroups of semisimple algebraic groups G. Our first theorem is a finiteness result, showing that connected G-irreducible subgroups are 'nearly maximal'.

THEOREM 1 *Let G be a connected semisimple algebraic group, and let A be a connected G-irreducible subgroup of G. Then A is contained in only finitely many subgroups of G.*

Since connected G-irreducible subgroups are necessarily semisimple (see Lemma 2.1), the smallest possibility for such a subgroup is A_1. The next result shows that G-irreducible A_1 subgroups usually exist. In large characteristic this is hardly surprising, as maximal A_1 subgroups usually exist; but in low characteristic maximal A_1 subgroups do not exist (see [**5**]), and the result provides a supply of nearly maximal A_1 subgroups.

THEOREM 2 *Let G be a simple algebraic group over K. If $G = A_n$, assume that $p > n$ or $p = 0$. Then G has a G-irreducible subgroup of type A_1.*

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In the excluded case $G = A_n,\ 0 < p \leqslant n$, it is easy to see that an irreducible subgroup A_1 exists if and only if all prime factors of $n + 1$ are at most p.

In a subsequent paper [**6**] we shall use the G-irreducible A_1s constructed in the proof of Theorem 2 to exhibit examples of *epimorphic* subgroups of minimal dimension in simple algebraic groups, as defined in [**2**]. (A closed subgroup H of the connected algebraic group G is said to be *epimorphic* if any morphism of G into an algebraic group is determined by its restriction to H. [**2**, Theorem 1] has a number of equivalent formulations of this definition: for example, H is *epimorphic* if and only if, whenever V is a rational G-module and $V \downarrow H = X \oplus Y$, then X, Y are G-invariant.)

Our final theorem concerns the description of conjugacy classes of connected G-irreducible subgroups of semisimple algebraic groups G. When G is simple, it has only finitely many classes of maximal connected subgroups (see [**5**, Corollary 3]). This is in general not the case for connected G-irreducible subgroups (see for example Corollary 4.5 below). However, Theorem 3 below shows that there is a finite collection of conjugacy classes of closed connected subgroups such that every G-irreducible subgroup is embedded in a specified way in a member of one of these classes. For the precise statement we require the following definition.

DEFINITION Let X, Y be connected linear algebraic groups over K.

(i) Suppose X is simple. We say X is a *twisted diagonal* subgroup of Y if $Y = Y_1 \dots Y_t$, a commuting product of simple groups Y_i of the same type as X, and if each projection $X \to Y_i/Z(Y_i)$ is non-trivial and involves a different Frobenius twist.

(ii) More generally, if X is semisimple, say $X = X_1 \dots X_r$ with each X_i simple, we say X is a *twisted diagonal* subgroup of Y if $Y = Z_1 \dots Z_r$, a commuting product of semisimple subgroups Z_i, and, writing $\bar{X} = X/Z(X) = \bar{X}_1 \dots \bar{X}_r$ and $\bar{Y} = Y/Z(Y) = \bar{Z}_1 \dots \bar{Z}_r$, each $\bar{X}_i$ is a twisted diagonal subgroup of $\bar{Z}_i$.

THEOREM 3 *Let G be a connected semisimple algebraic group of rank l. Then there is a finite set $\mathcal{C}$ of conjugacy classes of connected semisimple subgroups of G, of size depending only on l, with the following property. If X is any connected G-irreducible subgroup of G, then there is a subgroup $Y \in \bigcup \mathcal{C}$ such that X is a twisted diagonal subgroup of Y.*

The above results concern connected G-irreducible subgroups. Examples of non-connected G-irreducible subgroups X such that X^0 is not G-irreducible are easy to come by: for instance, $X = N_G(T)$, the normalizer of a maximal torus T is such an example, and there are many others for which $C_G(X^0)$ contains a non-trivial torus. However, we have not found any examples for which $C_G(X^0)$ contains no non-trivial torus. It may be the case that if X is a non-connected G-irreducible subgroup such that X^0 is not G-irreducible, then $C_G(X^0)$ necessarily contains a non-trivial torus; this is easily seen to be true when $G = A_n$.

NOTATION For G a simple algebraic group over K and λ a dominant weight, we denote by $V_G(\lambda)$ (or just λ) the rational irreducible KG-module of high weight λ. When $p > 0$, the irreducible module λ twisted by a p^r-power field morphism of G is denoted by $\lambda^{(p^r)}$. Finally, if $V_1, \dots, V_k$ are X-modules then $V_1/\dots/V_k$ denotes a G-module having the same composition factors as $V_1 \oplus \dots \oplus V_k$.

2. Preliminaries

As above, let G be a semisimple connected algebraic group over the algebraically closed field K of characteristic p. We begin with two elementary results concerning G-irreducible subgroups.

LEMMA 2.1 *If X is a connected G-irreducible subgroup of G, then X is semisimple, and $C_G(X)$ is finite.*

Proof. Suppose $C = C_G(X)^0 \neq 1$. If C contains a non-trivial torus T, then $X \leqslant C_G(T)$, which lies in a parabolic; otherwise C is unipotent, so $X \leqslant N_G(C)$ which lies in a parabolic by [**3**]. In either case we have a contradiction, and so $C_G(X)^0 = 1$, giving the result.

LEMMA 2.2 *Suppose G is classical, with natural module $V = V_G(\lambda_1)$. Let X be a semisimple connected closed subgroup of G. If X is G-irreducible then one of the following holds:*

(i) *$G = A_n$ and X is irreducible on V;*

(ii) *$G = B_n, C_n$ or D_n and $V \downarrow X = V_1 \perp \ldots \perp V_k$ with the V_i all non-degenerate, irreducible and inequivalent as X-modules;*

(iii) *$G = D_n$, $p = 2$, X fixes a non-singular vector $v \in V$, and X is a G_v-irreducible subgroup of $G_v = B_{n-1}$.*

Proof. Part (i) is clear, so assume $G = Sp(V)$ or $SO(V)$. Let W be a minimal non-zero X-invariant subspace of V. Then W is either non-degenerate or totally isotropic. In the first case induction gives a non-degenerate decomposition as in (ii); note that no two of the V_i are equivalent as X-modules since otherwise, if say $V_1 \downarrow X \cong V_2 \downarrow X$ via an isometry $\phi : V_1 \to V_2$, then X fixes the diagonal totally singular subspace $\{v + i\phi(v) : v \in V_1\}$ of $V_1 + V_2$ (where $i^2 = -1$), hence lies in a parabolic. Finally, if W is totally isotropic it can have no non-zero singular vectors (as X does not lie in a parabolic), so we must have $G = SO(V)$ with $p = 2$ and $W = \langle v \rangle$ non-singular, yielding (iii).

The next result is fairly elementary for classical groups G, but rests on the full weight of the memoirs [**5**, **7**] for exceptional groups.

PROPOSITION 2.3 [**5**, Corollary 3] *If G is a simple algebraic group then G has only finitely many conjugacy classes of maximal closed subgroups of positive dimension. The number of conjugacy classes is bounded in terms of the rank of G.*

We shall also require a description of the maximal closed connected subgroups of semisimple algebraic groups. Let G be a semisimple algebraic group, and write $G = G_1 \cdots G_r$, a commuting product of simple factors G_i. Define $\mathcal{M}(G)$ to be the following set of connected subgroups of G:

(1) for $j \in \{1, \ldots, r\}$, subgroups $(\Pi_{i \neq j} G_i) \cdot M_j$, with M_j a maximal connected proper subgroup of G_j, and

(2) for $r \geqslant 2$ and distinct $j, k \in \{1, \ldots, r\}$ such that there is a surjective morphism $\phi : G_j \to G_k$, subgroups of the form

$$G_{j,k}(\phi) = (\Pi_{i \neq j,k} G_i) \cdot D_{j,k},$$

where $D_{j,k} = \{(g, \phi(g)) : g \in G_j\}$, a closed connected diagonal subgroup of $G_j G_k$.

LEMMA 2.4 *The collection $\mathcal{M}(G)$ comprises all the maximal closed connected subgroups of the semisimple group G.*

Proof. It is clear that the members of $\mathcal{M}(G)$ are maximal closed connected subgroups of G. Conversely, suppose that M is a maximal closed connected subgroup of G. Factoring out $Z(G)$, we may assume that $Z(G) = 1$. Let π_i be the projection map $M \to G_i$. If some π_i is not surjective, then M lies in $(\prod_{j \neq i} G_j) \cdot \pi_i(M)$, which is contained in a member of $\mathcal{M}(G)$ under (1) of the definition above. Otherwise, all π_i are surjective and we easily see that M lies in a member of $\mathcal{M}(G)$ under (2) above.

By Proposition 2.3, there are only finitely many G-classes of subgroups in $\mathcal{M}(G)$ under (1) in the definition above. If the collection of subgroups under (2) is non-empty, then it consists of finitely many G-classes if $p = 0$, and infinitely many classes if $p > 0$, since in this case we can adjust the morphism ϕ by an arbitrary field twist.

Write $\mathcal{M}_1(G)$ for the collection of subgroups of G under (1), so that $\mathcal{M}_1(G)$ consists of finitely many G-classes of subgroups.

If H is a proper connected G-irreducible subgroup of G, then there is a sequence of subgroups

$$H = H_0 < H_1 < \cdots < H_s = G$$

such that for each i, H_i is semisimple and $H_i \in \mathcal{M}(H_{i+1})$. Write $\mathcal{M}_0(G)$ for the collection of G-irreducible subgroups H for which there is such a sequence with $H_i \in \mathcal{M}_1(H_{i+1})$ for all i. By Proposition 2.3 again, there are only finitely many G-classes of subgroups in $\mathcal{M}_0(G)$.

3. Proof of Theorem 1

Let G be a connected semisimple algebraic group, and let A be a connected G-irreducible subgroup of G. We prove that A is contained in only finitely many subgroups of G.

The proof proceeds by induction on $\dim G$. The base case $\dim G = 3$ is obvious. Clearly we may assume without loss that $Z(G) = 1$. Write $G = G_1 \cdots G_r$, a direct product of simple groups G_i, and let $\pi_i : G \to G_i$ be the ith projection map.

LEMMA 3.1 *If H is a subgroup of G containing A, then H is closed and H^0 is semisimple.*

Proof. Observe that $A^H = \langle A^h : h \in H \rangle$ is closed and connected, and hence $N_{\bar{H}}(A^H)$ is also closed. This normalizer contains H, hence contains $\bar{H}$. Thus $A^H \triangleleft \bar{H}^0$. By Lemma 2.1, $\bar{H}^0$ is semisimple and $C_G(A)^0 = 1$. It follows that $A^H = \bar{H}^0$. Thus $\bar{H}^0 \leqslant H \leqslant \bar{H}$. This means that H is a union of finitely many cosets of $\bar{H}^0$, hence is closed, as required.

In view of this lemma, it suffices to show that the number of closed connected overgroups of A in G is finite. Suppose this is false, so that A is contained in infinitely many connected subgroups of G. We shall obtain a contradiction in a series of lemmas.

By Lemma 2.1, $C_G(A)$ and $N_G(A)/A$ are finite. Recall the definitions in section 2 of the collections $\mathcal{M}(G)$ and $\mathcal{M}_1(G)$ of maximal connected subgroups of G.

LEMMA 3.2 *There exists $M \in \mathcal{M}(G)$ such that A lies in infinitely many G-conjugates of M.*

Proof. First, if $A \leqslant M \in \mathcal{M}(G)$, then M is semisimple by Lemma 2.1, and by induction A has only finitely many overgroups in M. It follows that A lies in infinitely many members of $\mathcal{M}(G)$.

We next claim that the overgroups of A in $\mathcal{M}(G)$ represent only finitely many G-conjugacy classes of subgroups. For if not, there must exist j, l such that A lies in subgroups $G_{j,l}(\phi)$ for morphisms ϕ involving infinitely many different field twists. Since the high weights of composition factors of $L(G_l) \downarrow A$ are ϕ-twists of those of $L(G_j) \downarrow A$ this implies that the highest weight of A on $L(G)$ is arbitrarily large, a contradiction. This proves the claim, and the lemma follows.

From now on, let M be the subgroup provided by Lemma 3.2.

LEMMA 3.3 *M contains infinitely many G-conjugates of A, no two of which are M-conjugate.*

Proof. By the previous lemma, A lies in infinitely many conjugates of M; say A lies in distinct conjugates M^g for $g \in C$, where C is an infinite subset of G. Let $g, h \in C$, so $A^{g^{-1}}$ and $A^{h^{-1}}$ lie in M; if these subgroups are M-conjugate, say $A^{g^{-1}} = A^{h^{-1}m}$ with $m \in M$, then $h^{-1}mg \in N_G(A)$. Letting $n_1, \ldots, n_t$ be coset representatives for A in $N_G(A)$, we have $h^{-1}mg = an_i$ for some $a \in A$ and some i. Thus $M^g = M^{han_i}$, so as $a \in M^h$, we have $M^g = M^{hn_i}$.

To summarize: fix $g \in C$; then if $h \in C$ is such that $A^{g^{-1}}$ and $A^{h^{-1}}$ are M-conjugate, we have $M^h = M^{gn_i^{-1}}$ for some i, so there are only finitely many such h. The lemma follows.

LEMMA 3.4 $M \in \mathcal{M}_1(G)$.

Proof. Suppose not. Then there exist distinct $j, k \in \{1, \ldots, r\}$ and a surjective morphism $\phi : G_j \to G_k$, such that

$$M = G_{j,k}(\phi) = G_0 \cdot D_{j,k},$$

where $G_0 = \Pi_{i \neq j,k} G_i$ and $D_{j,k} = \{g \cdot \phi(g) : g \in G_j\}$.

We may take it that $A \leqslant M$, so that each element of A is of the form $a = a_0 \cdot a_j \cdot \phi(a_j)$, where $a_0 \in G_0, a_j \in G_j$. Since M contains infinitely many G-conjugates of A, no two of them M-conjugate, it follows that M contains infinitely many conjugates of the form A^{g_k} ($g_k \in G_k$). If $a \in A$ is as above, then $a^{g_k} = a_0 \cdot a_j \cdot \phi(a_j)^{g_k}$, so it follows that $\phi(a_j)^{g_k} = \phi(a_j)$ for all $a_j \in \pi_j(A)$. But this means that $g_k \in C_{G_k}(\pi_k(A))$, which is finite; a contradiction.

LEMMA 3.5 *There exists $M_1 \in \mathcal{M}_1(M)$ such that M_1 contains infinitely many G-conjugates of A, no two of which are M-conjugate.*

Proof. By Lemma 3.3, M contains infinitely many G-conjugates of A, no two of which are M-conjugate. Call these conjugates A^{g_λ} ($\lambda \in \Lambda$), where Λ is an infinite index set. For each $\lambda \in \Lambda$, there exists $M_\lambda \in \mathcal{M}(M)$ containing A^{g_λ}. Then infinitely many M_λ are in $\mathcal{M}_1(M)$, since otherwise there exist j, k such that $A^{g_\lambda} \leqslant M_{j,k}(\phi)$ for morphisms ϕ involving infinitely many different field twists, which is impossible as in the proof of Lemma 3.2.

Since there are only finitely many M-classes of subgroups in $\mathcal{M}_1(M)$, infinitely many of the M_λ lie in a single M-class of subgroups, with representative say M_1. Then M_1 contains infinitely many G-conjugates $A^{g_\lambda m_\lambda}$ ($m_\lambda \in M$), no two of which are M-conjugate.

Recall the definition of $\mathcal{M}_0(G)$ from section 2. Choose $N \in \mathcal{M}_0(G)$, minimal subject to containing infinitely many G-conjugates of A, no two of which are N-conjugate.

LEMMA 3.6 *There are infinitely many distinct G-conjugates of A lying in $\mathcal{M}(N)$, no two of which are N-conjugate.*

Proof. Say A^{g_λ} $(\lambda \in \Lambda)$ are infinitely many conjugates of A lying in N, no two of them N-conjugate. If the conclusion of the lemma is false, then for infinitely many λ, there is a subgroup $N_\lambda \in \mathcal{M}(N)$ such that $A^{g_\lambda} \leqslant N_\lambda$. As in the previous proof, infinitely many of these N_λ are in $\mathcal{M}_1(N)$, of which there are only finitely many N-classes, so infinitely many N_λ are N-conjugate to some $N_1 \in \mathcal{M}_1(N)$. But then N_1 contains infinitely many G-conjugates of A (namely $A^{g_\lambda n_\lambda}$ for some $n_\lambda \in N$), no two of which are N-conjugate, contradicting the minimal choice of N.

At this point we can obtain a contradiction. Write $N = N_1 \cdots N_k$, a commuting product of simple factors N_i. By Lemma 3.6, there are infinitely many distinct G-conjugates A^{g_λ} lying in $\mathcal{M}(N)$, no two of which are N-conjugate. As $\mathcal{M}_1(N)$ consists of only finitely many N-classes of subgoups, infinitely many of the A^{g_λ} are in $\mathcal{M}(N)\backslash\mathcal{M}_1(N)$. Hence there exist j, l such that infinitely many A^{g_λ} are of the form $N_{j,l}(\phi_\lambda)$, where ϕ_λ is a surjective morphism $N_j \to N_l$, and no two of these subgroups are N-conjugate. Then the morphisms ϕ_λ must involve infinitely many different field twists, which is a contradiction as usual, as it implies that the highest weight of A on $L(G)$ (which is of course the highest weight of each conjugate A^{g_λ}) is arbitrarily large.

This completes the proof of Theorem 1.

4. Proof of Theorem 2

Let G be a simple algebraic group over K in characteristic p, as in Theorem 2 (so that if $G = A_n$ then $p > n$ or $p = 0$). We aim to construct a G-irreducible subgroup $A \cong A_1$.

LEMMA 4.1 *The conclusion of Theorem* 2 *holds if* $p = 0$.

Proof. Suppose $p = 0$. First consider the case where G is classical. The irreducible representation of A_1 of high weight r embeds A_1 in Sp_{r+1} if r is odd, and in SO_{r+1} if r is even. Hence SL_n, Sp_{2n} and SO_{2n+1} all have irreducible subgroups A_1. As for the remaining case $G = SO_{2n}$, an A_1 embedded irreducibly in a subgroup SO_{2n-1} is G-irreducible.

When G is of exceptional type, but not E_6, it has a maximal subgroup A_1 (see [**7**]), and this is obviously G-irreducible; and for $G = E_6$, a maximal A_1 in a subgroup F_4 is G-irreducible (its connected centralizer in G is trivial, so it cannot lie in any Levi subgroup).

In view of Lemma 4.1, we assume from now on that $p > 0$.

LEMMA 4.2 *The conclusion of Theorem* 2 *holds if G is classical.*

Proof. Assume G is classical. If $G = A_n = SL_{n+1}$ then $p > n$ by hypothesis, so G has a subgroup A_1 acting irreducibly on the natural $n+1$-dimensional G-module (with high weight n); clearly this subgroup does not lie in a parabolic of G.

Next, if $G = C_n = Sp_{2n}$, then G has a subgroup $(Sp_2)^n = (A_1)^n$, and we choose a subgroup $A \cong A_1$ of this via the embedding $1, 1^{(p)}, 1^{(p^2)}, \ldots, 1^{(p^{n-1})}$; then A fixes no non-zero totally isotropic subspace of the natural module, hence lies in no parabolic of G. Similarly, if $G = D_{2n} = SO_{4n}$, then G has a subgroup $(SO_4)^n = (A_1)^{2n}$, and we choose $A \cong A_1$ in this via the embedding $1, 1^{(p)}, \ldots, 1^{(p^{2n-1})}$.

Now let $G = D_{2n+1} = SO_{4n+2}$. Then G has a subgroup $SO_6 \times (SO_4)^{n-1} \cong A_3 \times (A_1)^{2(n-1)}$, which contains a subgroup $(A_1)^{2n}$ lying in no parabolic of G; choose $A \cong A_1$ in this $(A_1)^{2n}$ via the embedding $1, 1^{(p)}, \ldots, 1^{(p^{2n-1})}$ again.

Finally, for $G = B_{2n} = SO_{4n+1}$, choose $A \cong A_1$ in a subgroup $(SO_4)^n = (A_1)^{2n}$ via the above embedding, while for $G = B_{2n+1} = SO_{4n+3}$ choose A in a subgroup $SO_3 \times (SO_4)^n \cong (A_1)^{2n+1}$. This completes the proof.

Assume from now on that G is of exceptional type. We choose our subgroup $A \cong A_1$ as follows. For $G = E_8, E_7, F_4$ or G_2, there is a maximal rank subgroup $(A_1)^l$ (where $l = 8, 7, 4$ or 2 respectively), and we choose

$$A < (A_1)^l, \text{ via embedding } 1, 1^{(p^2)}, 1^{(p^4)}, \ldots, 1^{(p^{2(l-1)})}.$$

For $G = E_6$ with $p > 2$, there is a maximal rank subgroup $(A_2)^3$, and we choose

$$A < (A_2)^3, \text{ via embedding } 2, 2^{(p^2)}, 2^{(p^4)}.$$

Finally, for $G = E_6$ with $p = 2$, take a subgroup F_4 of G, and a subgroup C_4 of that, generated by short root groups in F_4; now take $A < C_4$, embedded via the irreducible symplectic 8-dimensional representation $1 \otimes 1^{(2)} \otimes 1^{(4)}$.

LEMMA 4.3 (i) *For $G \neq E_6$, $L(G)/L(A_1^l)$ restricts to A as follows:*
$G = E_8$: 14 distinct 4-fold tensor factors,
$G = E_7$: seven distinct 4-fold tensor factors,
$G = F_4$: one 4-fold factor and six distinct 2-fold factors,
$G = G_2$: $1 \otimes 3^{(p^2)}$ $(p \neq 2, 3)$; $1 \otimes 1^{(9)}/1 \otimes 1^{(27)}$ $(p = 3)$; $1 \otimes 1^{(4)} \otimes 1^{(8)}$ $(p = 2)$.
Moreover, $L(A_1^l)$ restricts to A as $2/2^{(p^2)}/\ldots/2^{(p^{2(l-1)})}$ if $p \neq 2$, and as $1^{(2)}/1^{(8)}/\ldots/1^{(2^{2l-1})}/0^l$ if $p = 2$.
In particular, the non-trivial composition factors of $L(G) \downarrow A$ are all distinct.
(ii) *For $G = E_6$ $(p \neq 2)$, $L(G)/L(A_2^3)$ restricts to A as $(2 \otimes 2^{(p^2)} \otimes 2^{(p^4)})^2$; and $L(A_2^3)$ restricts to A as $2/2^{(p^2)}/2^{(p^4)}/4/4^{(p^2)}/4^{(p^4)}$ if $p \neq 3$, and as $2/2^{(3^2)}/2^{(3^4)}/1 \otimes 1^{(3)}/1^{(3^2)} \otimes 1^{(3^3)}/1^{(3^4)} \otimes 1^{(3^5)}/0^3$ if $p = 3$.*
(iii) *For $G = E_6$ $(p = 2)$, letting $V_{27} = V_G(\lambda_1)$, we have*

$$V_{27} \downarrow A = 1^{(2)} \otimes 1^{(4)}/1^{(2)} \otimes 1^{(8)}/1^{(4)} \otimes 1^{(8)}/1^{(2)}/1^{(2)}/1^{(4)}/1^{(4)}/1^{(8)}/1^{(8)}/0^3.$$

Proof. (i) For $G = E_8$, the restriction of $L(G)$ to a subsystem D_4D_4 is given by [**4**, 2.1]: it is $L(D_4D_4)/\lambda_1 \otimes \lambda_1/\lambda_3 \otimes \lambda_3/\lambda_4 \otimes \lambda_4$. Now consider the restriction further to A_1^8. This is embedded as $SO_4 \cdot SO_4$ in each D_4 factor, so the factor $\lambda_1 \otimes \lambda_1$ of $L(G) \downarrow D_4D_4$ restricts to A_1^8 as a sum of 4-fold tensor factors, each of dimension 16. The normalizer $N_G(A_1^8)$ acts as the 3-transitive permutation group $AGL_3(2)$ on the eight factors, and the smallest orbit of this on 4-sets has size 14. It follows that $L(G) \downarrow A_1^8$ has at least 14 distinct 4-fold tensor factors. Since $14 \cdot 16 + \dim A_1^8 = \dim G$, these 14 modules comprise all the composition factors of $L(G)/L(A_1^8)$ restricted to A_1^8. Part (i) follows for $G = E_8$. The other types are handled similarly.

(ii) The restriction $L(E_6) \downarrow (A_2)^3$ is given by [**4**, 2.1], and (ii) follows easily.

(iii) We have $V_{27} \downarrow F_4 = V_{F_4}(\lambda_4)/0$, and $V_{F_4}(\lambda_4) \downarrow C_4 = V_{C_4}(\lambda_2)$. Hence $V_{27} \downarrow C_4$ has the same composition factors as the wedge-square of the natural 8-dimensional C_4-module, minus one trivial composition factor. Now, to get the conclusion, calculate the composition factors of the A_1-module $\wedge^2(1 \otimes 1^{(2)} \otimes 1^{(4)})$.

LEMMA 4.4 *The subgroup A is G-irreducible.*

Proof. First assume $G \neq E_6$. If $A < P = QL$, a parabolic subgroup with unipotent radical Q and Levi subgroup L, then the composition factors of A on $L(Q)$ are the same as those on $L(Q^{\mathrm{opp}})$, the Lie algebra of the opposite unipotent radical. By the last sentence of Lemma 4.3(i), it follows that all composition factors of A on $L(Q)$ must be trivial, whence from Lemma 4.3(i) we see that $\dim Q \leqslant l/2$, which is impossible.

Now assume $G = E_6$ with $p \neq 2$. If $p \neq 3$ then $L(G) \downarrow A$ has no trivial composition factors, so A cannot lie in a parabolic. Now suppose $p = 3$. By Lemma 4.3(ii), $L(G) \downarrow A$ has two isomorphic 27-dimensional composition factors. If $A < QL$ as above, then these factors must occur in $L(Q) + L(Q^{\mathrm{opp}})$, and the only other possible composition factors in $L(Q) + L(Q^{\mathrm{opp}})$ are trivial. Hence $\dim Q$ must be 27 or 28. There is no such unipotent radical in E_6.

Finally, assume $G = E_6$ with $p = 2$. Suppose $A < P = QL$, with the parabolic P chosen minimally. By minimality, A must project irreducibly to any A_r factor of L'; since the irreducible representations of A have dimension a power of 2, it follows that the only possible such factors are A_3 and A_1. Consequently either $L' = A_3A_1$, or L' lies in a subsystem D_5. If $L' = A_3A_1$, then A acts on the natural modules for A_3, A_1 as $1 \otimes 1^{(q)}, 1^{(q')}$ respectively, for some powers q, q' of 2. The restriction $V_{27} \downarrow A_3A_1$ is given by [**4**, 2.3], and it follows that $V_{27} \downarrow A$ has a composition factor $1 \otimes 1^{(q)} \otimes 1^{(q')}$ if $q \neq q'$, and has two composition factors $1 \otimes 1^{(q)}$ if $q = q'$. This conflicts with Lemma 4.3(iii). Therefore $L' \neq A_3A_1$. The remaining possibilities for L' lie in a subsystem D_5. The irreducible orthogonal A_1-modules of dimension 10 or less have dimensions 4 and 8, and do not extend the trivial module (see [**1**, 3.9]). It follows that $L' \leqslant D_4$. Observe that $V_{27} \downarrow D_4 = \lambda_1/\lambda_3/\lambda_4/0^3$. Hence it is readily checked that no possible embedding of A in D_4 gives composition factors for $V_{27} \downarrow A$ consistent with Lemma 4.3(iii).

This completes the proof of Theorem 2.

By varying the field twists involved in the definitions of A above, we obtain the following.

COROLLARY 4.5 *Let G be a simple algebraic group in characteristic $p > 0$, and assume that $G \neq A_n$. Then G has infinitely many conjugacy classes of G-irreducible subgroups of type A_1.*

5. Proof of Theorem 3

Let G be a connected semisimple algebraic group of rank l. The proof proceeds by induction on $\dim G$. The base case $\dim G = 3$ is trivial. Let X be a connected G-irreducible subgroup of G. By Lemma 2.1, X is semisimple. Write $G = G_1 \dots G_r$ and $X = X_1 \dots X_s$, commuting products of simple factors G_i and X_i. Without loss we can factor out the finite group $Z(G)$, and hence assume that $Z(G) = 1$.

Suppose first that X projects onto every simple factor G_i of G. Say X_1 projects onto the factors $G_1, \dots, G_t$. Identifying the direct product $G_1 \dots G_t$ with $G_1 \times \dots \times G_1$ (t factors), and replacing X by a suitable G-conjugate, we can take

$$X_1 = \{(x^{\tau_1}, \dots, x^{\tau_t}) : x \in G_1\},$$

where each $\tau_i = \gamma_i q_i$ with γ_i a graph automorphism or 1, and q_i a Frobenius morphism or 1. For each k let $S_k = \{i : q_i = q_k\}$, and define a corresponding subgroup $G_{S_k} \leqslant \prod_{i \in S_k} G_i$ by

$$G_{S_k} = \left\{ \prod_{i \in S_k} x^{\gamma_i} : x \in G_1 \right\}.$$

Then X_1 is a twisted diagonal subgroup of $G_1^+ := \prod_{S_k} G_{S_k}$. Repeating this construction for each simple factor X_i of X, we obtain a subgroup $G_1^+ \dots G_s^+$ of G containing X as a twisted diagonal subgroup. There are only finitely many such subgroups $G_1^+ \dots G_s^+$ in G. Hence if we include the conjugacy classes of these subgroups in our collection $\mathcal{C}$, we have the conclusion of Theorem 3 in this case.

Now suppose X does not project onto some factor, say G_1, of G. Then there exists a maximal connected subgroup M_1 of G_1 such that $X \leqslant M_1 G_2 \cdots G_r$. By Proposition 2.3, up to G_1-conjugacy there are only finitely many possibilities for M_1. Since $M_1 G_2 \dots G_r$ is a semisimple group of dimension less than dim G, the result now follows by induction.

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References

1. H. H. Andersen, J. Jorgensen, and P. Landrock, The projective indecomposable modules of $SL(2, p^n)$, *Proc. London Math. Soc.* **46** (1983), 495–528.
2. F. Bien and A. Borel, Sous-groupes epimorphiques de groupes linéaires algébriques I, *C.R. Acad. Sci. Paris* **315** (1992), 649–653.
3. A. Borel and J. Tits, Éléments unipotents et sousgroupes paraboliques de groupes réductifs, *Invent. Math.* **12** (1971), 95–104.
4. M. W. Liebeck and G. M. Seitz, Reductive subgroups of exceptional algebraic groups, *Mem. Amer. Math. Soc.* **121** (1996).
5. M. W. Liebeck and G. M. Seitz, The maximal subgroups of positive dimension in exceptional algebraic groups, *Mem. Amer. Math. Soc.* to appear.
6. M. W. Liebeck and D. M. Testerman, Some epimorphic subgroups of simple algebraic groups, to appear.
7. G. M. Seitz, Maximal subgroups of exceptional algebraic groups, *Mem. Amer. Math. Soc.* **90** (1991).
8. J.-P. Serre, *Moursund Lecture Notes*, University of Oregon Mathematics Department, 1998.