Two-fluid hydrodynamics of a Bose gas including damping from normal fluid transport coefficients

T. Nikuni
Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro, Tokyo 152, Japan

A. Griffin* and E. Zaremba†
JILA, University of Colorado, Boulder, Colorado 80309-0440

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Abstract

We extend our recent work on the two-fluid hydrodynamics of the condensate and non-condensate in a trapped Bose gas by including the dissipation associated with viscosity and thermal conduction. For purposes of illustration, we consider the hydrodynamic modes in the case of a uniform Bose gas. A finite thermal conductivity and shear viscosity give rise to a damping of the first and second sound modes in addition to that found previously due to the lack of diffusive equilibrium between the condensate and non-condensate. The relaxational mode associated with this equilibration process is strongly coupled to thermal fluctuations and reduces to the usual thermal diffusion mode above the Bose-Einstein transition. In contrast to the standard Landau two-fluid hydrodynamics, we predict a damped mode centered at zero frequency, in addition to the usual second sound doublet.

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I. INTRODUCTION

Since the original discovery in 1995, the subject of Bose-Einstein condensation in trapped atomic gases has become a major field of research [1,2]. Even though these systems consist of a very dilute vapor, they exhibit robust collective oscillations which are strongly influenced by mean-field interactions and collisions between the atoms. There has been considerable theoretical work devoted to describing the collective modes of a trapped gas at very low temperatures in terms of the solution of the time-dependent Gross-Pitaevskii (GP) equation for the macroscopic wavefunction of the condensate. As discussed in several recent reviews [1,2], there is excellent agreement between experimental observations for $T \ll T_{\mathrm{BEC}}$ and theoretical calculations based on the $T = 0$ GP equation.

At elevated temperatures where the condensate is appreciably depleted by thermal excitations, one enters a more complex regime in which collisions between the atoms must be considered. Two limiting cases for the dynamics of the gas correspond to the collisionless and hydrodynamic regimes [2,3]. In the collisionless regime, the main effect of the non-condensate component appears to be a shift in the collective mode frequencies as a result of the change in the condensate *number*, and to the appearance of Landau damping. There have been several studies of this collisionless region in trapped gases where dynamic mean fields dominate the physics (we refer to the review articles in Ref. [2]). The second regime arises when collisions between atoms are rapid enough to establish a state of dynamic local equilibrium in the non-condensate gas. This collision-dominated hydrodynamic regime is the subject of this paper. To be in this regime the collective modes of frequency ω must satisfy the condition $\omega\tau \ll 1$, where τ is some appropriate relaxation time for reaching local equilibrium. As a rough estimate, we can take τ to be the collision time τ_{cl} for a classical gas. Here $1/\tau_{\mathrm{cl}} = \sqrt{2}n\sigma\bar{v}$, where $\sigma = 8\pi a^2$ is the low energy quantum mechanical cross-section for bosonic atoms and $\bar{v}$ is the average thermal velocity [4]. It is clear that a high density and/or a large atomic scattering cross-section are favorable for reaching the hydrodynamic region. Experiments which can probe this region in trapped Bose gases are now feasible and promise to provide much new physics.

One finds in the hydrodynamic regime that the dynamics of the condensate and non-condensate components can *both* be described in terms of a few macroscopic (coarse-grained) variables (such as the local densities and velocities of the two components). The coupled equations of motion for these local quantities will be referred to as the two-fluid hydrodynamic equations. The microscopic basis of these two-fluid equations rests on a generalized Gross-Pitaevskii equation for the condensate atoms and a quantum kinetic equation for the non-condensate atoms. These two components are coupled through mean-field interactions as well as collisions between the atoms. The authors have recently given a detailed derivation and discussion of such a two-fluid hydrodynamics for trapped atomic gases at finite temperatures [5,4,6].

These equations were derived for temperatures where the dominant thermal excitations in the trap can be treated as atoms moving in a self-consistent Hartree-Fock field. In particular, the original ZGN hydrodynamic equations derived in Ref. [5] were generalized in Refs. [4,6] to include collisions between condensate and non-condensate atoms. This allows for the possibility of treating the situation in which atoms of the non-condensate are in local thermodynamic equilibrium among themselves but are *not* in diffusive equilibrium with the condensate atoms. The resulting ZGN′ hydrodynamic equations [4,6] involve a characteristic relaxation time τ_μ which is the time scale on which local diffusive equilibrium is established. This equilibration process leads to a novel damping mechanism which is associated with the collisional exchange of atoms between the two components. The ZGN′ equations are briefly reviewed in Section II.

In Section III, we further generalize the ZGN′ equations by considering the effects of deviations from local equilibrium within the non-condensate. At the finite temperatures of interest, this deviation from local equilibrium gives rise to damping associated with the thermal conductivity and the shear viscosity. This generalization has already been discussed in Section V of Ref. [11] starting from the ZGN hydrodynamic equations.

In the limit that the two components are in complete local equilibrium with each other, our two-fluid hydrodynamic equations reduce to those first derived by Landau in 1941 [7]. The Landau two-fluid equations give an excellent description of the low frequency response of superfluid ^{4}He [8]. As noted by several authors, the Landau theory is also valid for Bose-condensed gases. The thermal conductivity and shear viscosity were first derived for a uniform Bose-condensed gas at finite temperatures in a pioneering paper by Kirkpatrick and Dorfman [9]. Their results were used by Gay and Griffin [10] to evaluate the temperature-dependent damping of first and second sound as predicted by the Landau two-fluid hydrodynamic equations. Our present results are consistent with both of these early papers in the appropriate Landau limit, namely when $\omega\tau_\mu \ll 1$ [4,6]. However it is important to point out that our generalized two-fluid hydrodynamic equations provide a more complete description than the original Landau version since they can be used in situations in which the superfluid and normal fluid are *not* in local diffusive equilibrium with each other.

To illustrate the physics, we use our ZGN′ equations in Section IV to study the hydrodynamic normal mode spectrum of a uniform Bose gas in the presence of hydrodynamic dissipation. In particular, we show how first and second sound modes are affected by viscosity and thermal conduction, and also discuss how the new relaxational mode exhibited in Refs. [4,6] is modified. In another paper, we apply these same equations to a discussion of the damping of the out-of-phase dipole mode recently observed [5,12] in a trapped Bose gas.

II. A REVIEW OF THE ZGN′ EQUATIONS

In this section, we first briefly review the finite temperature ZGN′ equations [5,6] based on the assumption that the non-condensate is in local equilibrium. In the next section, we calculate the corrections to these equations which arise from a small deviation from local equilibrium. The non-condensate atoms are described by the distribution function $f(\mathbf{r},\mathbf{p},t)$, which obeys the quantum kinetic equation (we set $\hbar = 1$ throughout this paper):

$$\frac{\partial f(\mathbf{r},\mathbf{p},t)}{\partial t} + \frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla} f(\mathbf{r},\mathbf{p},t) - \boldsymbol{\nabla} U\cdot\boldsymbol{\nabla}_{\mathbf{p}} f(\mathbf{r},\mathbf{p},t) = C_{12}[f] + C_{22}[f]. \tag{1}$$

Here the effective potential $U(\mathbf{r},t) \equiv U_{\rm ext}(\mathbf{r}) + 2g[n_c(\mathbf{r},t) + \tilde{n}(\mathbf{r},t)]$ includes the self-consistent Hartree-Fock (HF) mean field, and as usual, we treat the inter-atomic interaction in the s-

wave approximation with $g = 4\pi a/m$. The condensate density is $n_c(\mathbf{r},t) \equiv |\Phi(\mathbf{r},t)|^2$ and the non-condensate density $\tilde{n}(\mathbf{r},t)$ is given by

$$\tilde{n}(\mathbf{r},t) = \int \frac{d\mathbf{p}}{(2\pi)^3} f(\mathbf{r},\mathbf{p},t). \tag{2}$$

The two collision terms in (1) are given by

$$\begin{aligned} C_{22}[f] &\equiv 4\pi g^2 \int \frac{d\mathbf{p}_2}{(2\pi)^3} \int \frac{d\mathbf{p}_3}{(2\pi)^3} \int d\mathbf{p}_4 \\ &\times \delta(\mathbf{p}+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\delta(\tilde{\varepsilon}_p+\tilde{\varepsilon}_{p_2}-\tilde{\varepsilon}_{p_3}-\tilde{\varepsilon}_{p_4}) \\ &\times \left[(1+f)(1+f_2)f_3f_4 - ff_2(1+f_3)(1+f_4)\right], \end{aligned} \tag{3}$$

$$\begin{aligned} C_{12}[f] &\equiv 4\pi g^2 n_c \int \frac{d\mathbf{p}_1}{(2\pi)^3} \int d\mathbf{p}_2 \int d\mathbf{p}_3 \\ &\times \delta(m\mathbf{v}_c+\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3)\delta(\varepsilon_c+\tilde{\varepsilon}_{p_1}-\tilde{\varepsilon}_{p_2}-\tilde{\varepsilon}_{p_3}) \\ &\times[\delta(\mathbf{p}-\mathbf{p}_1)-\delta(\mathbf{p}-\mathbf{p}_2)-\delta(\mathbf{p}-\mathbf{p}_3)] \\ &\times[(1+f_1)f_2f_3 - f_1(1+f_2)(1+f_3)], \end{aligned} \tag{4}$$

with $f \equiv f(\mathbf{r},\mathbf{p},t)$, $f_i \equiv f(\mathbf{r},\mathbf{p}_i,t)$. The expression in (4) takes into account the fact that a condensate atom locally has energy $\varepsilon_c(\mathbf{r},t) = \mu_c(\mathbf{r},t) + \frac{1}{2}mv_c^2(\mathbf{r},t)$ and momentum $m\mathbf{v}_c$, where the condensate chemical potential μ_c and velocity $\mathbf{v}_c$ will be defined shortly. On the other hand, a non-condensate atom locally has the HF energy $\tilde{\varepsilon}_p(\mathbf{r},t) = \frac{p^2}{2m} + U(\mathbf{r},t)$. This particle-like dispersion relation limits our analysis to finite temperatures.

The equation of motion for the condensate was derived in Ref. [6] (see also Ref. [13]) and is given by a generalized Gross-Pitaevskii equation for the macroscopic wavefunction $\Phi(\mathbf{r},t)$

$$i\frac{\partial\Phi(\mathbf{r},t)}{\partial t} = \left[-\frac{\nabla^2}{2m} + U_{\rm ext}(\mathbf{r}) + gn_c(\mathbf{r},t) + 2g\tilde{n}(\mathbf{r},t) - iR(\mathbf{r},t)\right]\Phi(\mathbf{r},t), \tag{5}$$

where

$$R(\mathbf{r},t) = \frac{\Gamma_{12}(\mathbf{r},t)}{2n_c(\mathbf{r},t)}, \tag{6}$$

with

$$\Gamma_{12} \equiv \int \frac{d\mathbf{p}}{(2\pi)^3} C_{12}[f(\mathbf{r},\mathbf{p},t)]. \tag{7}$$

The dissipative term R in (5) is associated with the exchange of atoms between the condensate and non-condensate as described by the collision integral $C_{12}[f]$ in (4). We see that (1) and (5) must be solved self-consistently. It is customary to rewrite the GP equation (5) in terms of the amplitude and phase of $\Phi(\mathbf{r},t) = \sqrt{n_c(\mathbf{r},t)}e^{i\theta(\mathbf{r},t)}$, which leads to ($\mathbf{v}_c = \boldsymbol{\nabla}\theta(\mathbf{r},t)/m$)

$$\frac{\partial n_c}{\partial t} + \boldsymbol{\nabla}\cdot(n_c\mathbf{v}_c) = -\Gamma_{12}[f],$$

$$m\left(\frac{\partial}{\partial t}+\mathbf{v}_c\cdot\boldsymbol{\nabla}\right)\mathbf{v}_c=-\boldsymbol{\nabla}\mu_c\ , \tag{8}$$

where the condensate chemical potential is given by

$$\mu_c(\mathbf{r},t)=-\frac{\nabla^2\sqrt{n_c(\mathbf{r},t)}}{2m\sqrt{n_c(\mathbf{r},t)}}+U_{\text{ext}}(\mathbf{r})+gn_c(\mathbf{r},t)+2g\tilde{n}(\mathbf{r},t)\,. \tag{9}$$

One sees that Γ_{12} in (8) plays the role of a "source function" in the continuity equation for the condensate, arising from the fact that C_{12} collisions do not conserve the number of condensate atoms [6]. Because of the structure of the equations in (8), they are often referred to as "hydrodynamic equations", even though they are completely equivalent to the generalized GP equation in (5).

Following the standard procedure in the classical kinetic theory of gases [14], we take moments of (1) to derive the most general form of hydrodynamic equations for the non-condensate. These moment equations take the form (μ and ν are Cartesian components):

$$\begin{aligned}
&\frac{\partial\tilde{n}}{\partial t}+\boldsymbol{\nabla}\cdot(\tilde{n}\mathbf{v}_n)=\Gamma_{12}[f]\,,\\
&m\tilde{n}\left(\frac{\partial}{\partial t}+\mathbf{v}_n\cdot\boldsymbol{\nabla}\right)v_{n\mu}=-\frac{\partial P_{\mu\nu}}{\partial x_\nu}-\tilde{n}\frac{\partial U}{\partial x_\mu}-m(v_{n\mu}-v_{c\mu})\Gamma_{12}[f]\,,\\
&\frac{\partial\tilde{\epsilon}}{\partial t}+\nabla\cdot(\tilde{\epsilon}\mathbf{v}_n)=-\boldsymbol{\nabla}\cdot\mathbf{Q}-D_{\mu\nu}P_{\mu\nu}+\left[\frac{1}{2}m(\mathbf{v}_n-\mathbf{v}_c)^2+\mu_c-U\right]\Gamma_{12}[f].
\end{aligned} \tag{10}$$

The non-condensate density was defined earlier in (2) while the non-condensate local velocity is defined by

$$\tilde{n}(\mathbf{r},t)\mathbf{v}_n(\mathbf{r},t)\equiv\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{\mathbf{p}}{m}f(\mathbf{r},\mathbf{p},t)\,. \tag{11}$$

In addition, we have

$$\begin{aligned}
P_{\mu\nu}(\mathbf{r},t)&\equiv m\int\frac{d\mathbf{p}}{(2\pi)^3}\left(\frac{p_\mu}{m}-v_{n\mu}\right)\left(\frac{p_\nu}{m}-v_{n\nu}\right)f(\mathbf{r},\mathbf{p},t),\\
\mathbf{Q}(\mathbf{r},t)&\equiv\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{2m}(\mathbf{p}-m\mathbf{v}_n)^2\left(\frac{\mathbf{p}}{m}-\mathbf{v}_n\right)f(\mathbf{r},\mathbf{p},t),\\
\tilde{\epsilon}(\mathbf{r},t)&\equiv\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{2m}(\mathbf{p}-m\mathbf{v}_n)^2f(\mathbf{r},\mathbf{p},t)\,.
\end{aligned} \tag{12}$$

Finally, the symmetric rate-of-strain tensor appearing in (10) is defined as

$$D_{\mu\nu}(\mathbf{r},t)\equiv\frac{1}{2}\left(\frac{\partial v_{n\mu}}{\partial x_\nu}+\frac{\partial v_{n\nu}}{\partial x_\mu}\right). \tag{13}$$

Formally, these results are *exact* consequences of the kinetic equation (1).

The lowest order approximate solution of (1) is based on the assumption that C_{22} collisions are sufficiently rapid to force the distribution function to have the form of the local equilibrium Bose distribution

$$\tilde{f}(\mathbf{r},\mathbf{p},t) = \frac{1}{e^{\beta[\frac{1}{2m}(\mathbf{p}-m\mathbf{v}_n)^2+U-\tilde{\mu}]}-1}\,. \tag{14}$$

Here, the temperature parameter β, normal fluid velocity $\mathbf{v}_n$, chemical potential $\tilde{\mu}$, and mean field U are all functions of $\mathbf{r}$ and t. One may immediately verify that $\tilde{f}$ satisfies $C_{22}[\tilde{f}]=0$ independent of the value of $\tilde{\mu}$. In contrast, one finds that $C_{12}[\tilde{f}]$ is in general finite, namely

$$\begin{aligned}
C_{12}[\tilde{f}] = 4\pi g^2 n_c[1 - e^{-\beta(\tilde{\mu}-\frac{1}{2}m(\mathbf{v}_n-\mathbf{v}_c)^2-\mu_c)}]& \\
\times\int\frac{d\mathbf{p}_1}{(2\pi)^3}\int d\mathbf{p}_2\int d\mathbf{p}_3\delta(m\mathbf{v}_c+\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3)\delta(\tilde{\varepsilon}_1+\varepsilon_c-\tilde{\varepsilon}_2-\tilde{\varepsilon}_3)& \\
\times[\delta(\mathbf{p}-\mathbf{p}_1)-\delta(\mathbf{p}-\mathbf{p}_2)-\delta(\mathbf{p}-\mathbf{p}_3)](1+\tilde{f}_1)\tilde{f}_2\tilde{f}_3.&
\end{aligned} \tag{15}$$

Using the local distribution function (14) to evaluate the moments in (2) and (12), we find that the heat current $\mathbf{Q}(\mathbf{r},t)=0$, and that

$$\tilde{n}(\mathbf{r},t) = \int\frac{d\mathbf{p}}{(2\pi)^3}\,\tilde{f}(\mathbf{r},\mathbf{p},t)\Big|_{\mathbf{v}_n=0} = \frac{1}{\Lambda^3}g_{3/2}(z)\,, \tag{16}$$

$$P_{\mu\nu}(\mathbf{r},t) = \delta_{\mu\nu}\tilde{P}(\mathbf{r},t) \equiv \delta_{\mu\nu}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{p^2}{3m}\,\tilde{f}(\mathbf{r},\mathbf{p},t)\Big|_{\mathbf{v}_n=0} = \delta_{\mu\nu}\frac{1}{\beta\Lambda^3}g_{5/2}(z). \tag{17}$$

Here $z(\mathbf{r},t)\equiv e^{\beta[\tilde{\mu}-U(\mathbf{r},t)]}$ is the local fugacity, $\Lambda(\mathbf{r},t)\equiv[2\pi/mk_BT(\mathbf{r},t)]^{1/2}$ is the local thermal de Broglie wavelength and $g_n(z)=\sum_{l=1}^{\infty}z^l/l^n$ are the Bose-Einstein functions. The kinetic energy density is given by $\tilde{\epsilon}(\mathbf{r},t)=\frac{3}{2}\tilde{P}(\mathbf{r},t)$ which is the same relation as found for a uniform ideal gas.

To summarize, using $f\simeq\tilde{f}$, we obtain the ZGN′ lowest-order hydrodynamic equations for the non-condensate given in Refs. [4,6]

$$\begin{aligned}
&\frac{\partial\tilde{n}}{\partial t}+\boldsymbol{\nabla}\cdot(\tilde{n}\mathbf{v}_n) = \Gamma_{12}[\tilde{f}]\,,\\
&m\tilde{n}\left(\frac{\partial}{\partial t}+\mathbf{v}_n\cdot\boldsymbol{\nabla}\right)\mathbf{v}_n = -\boldsymbol{\nabla}\tilde{P}-\tilde{n}\boldsymbol{\nabla}U-m(v_{n\mu}-v_{c\mu})\Gamma_{12}[\tilde{f}]\,,\\
&\frac{\partial\tilde{P}}{\partial t}+\nabla\cdot(\tilde{P}\mathbf{v}_n) = -\frac{2}{3}\tilde{P}\boldsymbol{\nabla}\cdot\mathbf{v}_n+\frac{2}{3}\left[\frac{1}{2}m(\mathbf{v}_n-\mathbf{v}_c)^2+\mu_c-U\right]\Gamma_{12}[\tilde{f}].
\end{aligned} \tag{18}$$

where $\Gamma_{12}[\tilde{f}]$ is obtained from (7) with $C_{12}[\tilde{f}]$ given by (15).

III. ZGN′ EQUATIONS WITH HYDRODYNAMIC DISSIPATION

We next derive the additional terms which arise from the equations in (10) due to a deviation of the distribution function from local equilibrium, $f\simeq\tilde{f}+\delta f$ [14]. Following Refs. [9,11], we write this deviation in the form

$$\delta f = \tilde{f}(\mathbf{r},\mathbf{p},t)[1+\tilde{f}(\mathbf{r},\mathbf{p},t)]\psi(\mathbf{r},\mathbf{p},t). \tag{19}$$

To first order in ψ, the C_{22} collision integral in (3) reduces to

$$C_{22}[\tilde{f}+\delta f] \simeq 4\pi g^2 \int \frac{d\mathbf{p}_2}{(2\pi)^3} \int \frac{d\mathbf{p}_3}{(2\pi)^3} \int d\mathbf{p}_4 \delta(\mathbf{p}+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\delta(\tilde{\varepsilon}_1+\tilde{\varepsilon}_2-\tilde{\varepsilon}_3-\tilde{\varepsilon}_4)$$
$$\times \tilde{f}\tilde{f}_2(1+\tilde{f}_3)(1+\tilde{f}_4)(\psi_3+\psi_4-\psi_2-\psi) \equiv \hat{L}_{22}[\psi]. \tag{20}$$

In the left hand side of (1) and in the C_{12} collision integral, we approximate f by the local Bose distribution $\tilde{f}$. The various derivatives of $\mathbf{v}_n$, $\tilde{\mu}$, T and U with respect to $\mathbf{r}$ and t can be written using the lowest-order hydrodynamic equations given in (18). The resulting linearized equation which determines ψ is (for details, see the Appendix)

$$\left\{\frac{\mathbf{u}\cdot\boldsymbol{\nabla}T}{T}\left[\frac{mu^2}{2k_{\mathrm{B}}T} - \frac{5g_{5/2}(z)}{2g_{3/2}(z)}\right] + \frac{m}{k_{\mathrm{B}}T}D_{\mu\nu}\left(u_\mu u_\nu - \frac{1}{3}\delta_{\mu\nu}u^2\right)\right.$$
$$\left. + \left(\sigma_2 + \frac{mu^2}{3k_{\mathrm{B}}T}\sigma_1 + \frac{m}{k_{\mathrm{B}}T}\mathbf{u}\cdot\mathbf{w}\right)\frac{\Gamma_{12}[\tilde{f}]}{\tilde{n}}\right\}\tilde{f}(1+\tilde{f}) - C_{12}[\tilde{f}] = \hat{L}_{22}[\psi]. \tag{21}$$

Here the thermal velocity $\mathbf{u}$ is defined by $m\mathbf{u} \equiv \mathbf{p} - m\mathbf{v}_n$ and $\mathbf{w} \equiv \mathbf{v}_c - \mathbf{v}_n$. The dimensionless thermodynamic functions σ_1, σ_2 are defined by

$$\sigma_1(\mathbf{r},t) \equiv \frac{\gamma\tilde{n}\left[\frac{1}{2}mw^2 + \mu_c - U\right] - \frac{3}{2}\tilde{n}^2}{\frac{5}{2}\tilde{P}\gamma - \frac{3}{2}\tilde{n}^2},$$
$$\sigma_2(\mathbf{r},t) \equiv \beta\frac{\frac{5}{2}\tilde{P}\tilde{n} - \tilde{n}^2\left[\frac{1}{2}mw^2 + \mu_c - U\right]}{\frac{5}{2}\tilde{P}\gamma - \frac{3}{2}\tilde{n}^2}, \tag{22}$$

where $\gamma(\mathbf{r},t) \equiv \frac{\beta}{\Lambda^3}g_{1/2}(z(\mathbf{r},t))$. We note that Refs. [4,6] introduce a related dimensionless quantity $\tilde{\gamma} \equiv g\gamma$.

Since (21) is a linear equation for ψ, one may write the solution as $\psi = \psi^{(1)} + \psi^{(2)}$, where $\psi^{(1)}$ is the solution of

$$\left\{\frac{\mathbf{u}\cdot\boldsymbol{\nabla}T}{T}\left[\frac{mu^2}{2k_{\mathrm{B}}T} - \frac{5g_{5/2}(z)}{2g_{3/2}(z)}\right] + \frac{m}{k_{\mathrm{B}}T}D_{\mu\nu}\left(u_\mu u_\nu - \frac{1}{3}\delta_{\mu\nu}u^2\right)\right\}\tilde{f}(1+\tilde{f}) = \hat{L}_{22}[\psi^{(1)}], \tag{23}$$

and $\psi^{(2)}$ is the solution of

$$\left(\sigma_2 + \frac{mu^2}{3k_{\mathrm{B}}T}\sigma_1 + \frac{m}{k_{\mathrm{B}}T}\mathbf{u}\cdot\mathbf{w}\right)\frac{\Gamma_{12}[\tilde{f}]}{\tilde{n}}\tilde{f}(1+\tilde{f}) - C_{12}[\tilde{f}] = \hat{L}_{22}[\psi^{(2)}]. \tag{24}$$

We note from its definition that $\tilde{f}(\mathbf{r},\mathbf{p},t)$ is a function of the variable u^2 while the linearized operator $\hat{L}_{22}$ defined in (20) is a function of $\mathbf{u}$. The expression (15) for $C_{12}[\tilde{f}]$ can be written in the form

$$C_{12}[\tilde{f}] = \frac{2g^2n_cm^3}{(2\pi)^2}[1 - e^{-\beta(\mu_{\mathrm{diff}} - \frac{1}{2}mw^2)}] \int d\mathbf{u}_1 \int d\mathbf{u}_2 \int d\mathbf{u}_3$$
$$\times \delta(\mathbf{w}+\mathbf{u}_1-\mathbf{u}_2-\mathbf{u}_3)\delta(\mu_c - U + \frac{m}{2}(w^2+u_1^2-u_2^2-u_3^2))$$
$$\times [\delta(\mathbf{u}-\mathbf{u}_1) - \delta(\mathbf{u}-\mathbf{u}_2) - \delta(\mathbf{u}-\mathbf{u}_3)](1+\tilde{f}_1)\tilde{f}_2\tilde{f}_3, \tag{25}$$

where $\mu_{\mathrm{diff}} \equiv \tilde{\mu} - \mu_c$. This expression shows that $C_{12}[\tilde{f}]$ depends on $\mathbf{u}$ and $\mathbf{w}$, and in particular, obeys the relation $C_{12}[\mathbf{u},\mathbf{w}] = C_{12}[-\mathbf{u},-\mathbf{w}]$. Finally, we note that $\Gamma_{12}[\tilde{f}]$ in (24) is independent of $\mathbf{u}$, but depends on $\mathbf{w}$.

The equations (23) and (24) can be shown to have unique solutions if we impose the following constraints on $\psi^{(1)}$ and $\psi^{(2)}$:

$$\int d\mathbf{p}\ \tilde{f}(1+\tilde{f})\psi^{(i)} = \int d\mathbf{p}\ p_\mu \tilde{f}(1+\tilde{f})\psi^{(i)} = \int d\mathbf{p}\ p^2 \tilde{f}(1+\tilde{f})\psi^{(i)} = 0. \tag{26}$$

Physically these constraints mean that the deviations from local equilibrium make no contribution to $\tilde{n}$, $\mathbf{v}_n$ and the diagonal component of $P_{\mu\nu}$. The solution for $\psi^{(1)}$ of (23) has been given already in Ref. [11], namely

$$\psi^{(1)} = \left[\frac{\boldsymbol{\nabla} T\cdot\mathbf{u}}{T}A(u) + 2D_{\mu\nu}\left(u_\mu u_\nu - \frac{1}{3}u^2\delta_{\mu\nu}\right)B(u)\right]. \tag{27}$$

Using this solution for $\psi^{(1)}$, one finds that the heat current density $\mathbf{Q}$ and the pressure tensor $P_{\mu\nu}$ are given by

$$\begin{aligned} P_{\mu\nu} &= \delta_{\mu\nu}\tilde{P} - 2\eta\left[D_{\mu\nu} - \frac{1}{3}\mathrm{Tr}D\delta_{\mu\nu}\right] + P^{(2)}_{\mu\nu}, \\ \mathbf{Q} &= -\kappa\boldsymbol{\nabla} T + \mathbf{Q}^{(2)}, \end{aligned} \tag{28}$$

where $P^{(2)}_{\mu\nu}$ and $\mathbf{Q}^{(2)}$ are the contribution from $\psi^{(2)}$. The explicit expressions for the transport coefficients η and κ are associated with $\psi^{(1)}$. They are given in Eqs. (38) and (34), respectively, of Ref. [11] for a trapped Bose gas below T_{BEC}. The analogous transport coefficients for a uniform degenerate Bose gas above T_{BEC} were first calculated by Uehling and Uhlenbeck [15].

The additional corrections due to $\psi^{(2)}$ are given by

$$\begin{aligned} P^{(2)}_{\mu\nu} &= m\int\frac{d\mathbf{p}}{(2\pi)^3}u_\mu u_\nu \tilde{f}(1+\tilde{f})\psi^{(2)}, \\ \mathbf{Q}^{(2)} &= \int\frac{d\mathbf{p}}{(2\pi)^3}\frac{m}{2}u^2\mathbf{u}\tilde{f}(1+\tilde{f})\psi^{(2)}. \end{aligned} \tag{29}$$

The most general solution of (24) for $\psi^{(2)}$ is of the form

$$\psi^{(2)} = [1 - e^{-\beta(\mu_{\mathrm{diff}} - \frac{1}{2}mw^2)}]D(\mathbf{u},\mathbf{w}). \tag{30}$$

If $\mathbf{w}\equiv\mathbf{v}_n - \mathbf{v}_c = 0$, the left hand side of (24) and $D(\mathbf{u})$ are isotropic functions of $\mathbf{u}$. Using this fact in conjunction with the relation $C_{12}[\mathbf{u},\mathbf{w}] = C_{12}[-\mathbf{u},-\mathbf{w}]$ as noted below (25), we conclude that D must have the following form

$$D(\mathbf{u},\mathbf{w}) \simeq D_0(u) + D_1(u)\mathbf{w}\cdot\mathbf{u} + D_2(u)w_\mu w_\nu\left(u_\mu u_\nu - \frac{1}{3}u^2\delta_{\mu\nu}\right) + O(w^3). \tag{31}$$

Using this, the additional terms given by (29) have the following form

$$\begin{aligned} P^{(2)}_{\mu\nu} &\propto [1 - e^{-\beta(\mu_{\mathrm{diff}} - \frac{1}{2}mw^2)}]\left(w_\mu w_\nu - \frac{1}{3}w^2\delta_{\mu\nu}\right), \\ \mathbf{Q}^{(2)} &\propto [1 - e^{-\beta(\mu_{\mathrm{diff}} - \frac{1}{2}mw^2)}]\mathbf{w}. \end{aligned} \tag{32}$$

We note that the constraints in (26) imply that

$$\int d\mathbf{u}\ u^2\tilde{f}(1+\tilde{f})\psi^{(i)} = 0, \tag{33}$$

and thus it follows that the isotropic term $D_0(u)$ in (31) makes no contribution to $P^{(2)}_{\mu\nu}$ in (29).

In summary, we have obtained the following hydrodynamic equations for the non-condensate including the normal fluid transport coefficients

$$\begin{aligned}
&\frac{\partial\tilde{n}}{\partial t} + \boldsymbol{\nabla}\cdot(\tilde{n}\mathbf{v}_n) = \Gamma_{12}[\tilde{f}],\\
&mn\left(\frac{\partial}{\partial t} + \mathbf{v}_n\cdot\boldsymbol{\nabla}\right)v_{n\mu} + \frac{\partial\tilde{P}}{\partial x_\mu} + \tilde{n}\frac{\partial U}{\partial x_\mu} = -m(v_{n\mu}-v_{c\mu})\Gamma_{12}[\tilde{f}]\\
&\qquad\qquad + \frac{\partial}{\partial x_\nu}\left\{2\eta\left[D_{\mu\nu} - \frac{1}{3}(\mathrm{Tr}D)\delta_{\mu\nu}\right]\right\} - \frac{\partial P^{(2)}_{\mu\nu}}{\partial x_\nu},\\
&\frac{\partial\tilde{\epsilon}}{\partial t} + \boldsymbol{\nabla}\cdot(\tilde{\epsilon}\mathbf{v}_n) + (\boldsymbol{\nabla}\cdot\mathbf{v}_n)\tilde{P} = \left[\frac{1}{2}m(\mathbf{v}_n-\mathbf{v}_c)^2\mu_c - U\right]\Gamma_{12}[\tilde{f}]\\
&\qquad\qquad + \boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T) + 2\eta\left[D_{\mu\nu} - \frac{1}{3}(\mathrm{Tr}D)\delta_{\mu\nu}\right]^2 - \boldsymbol{\nabla}\mathbf{Q}^{(2)}.
\end{aligned} \tag{34}$$

Since $P^{(2)}_{\mu\nu}$ and $\mathbf{Q}^{(2)}$ in (32) are at least of second order in the fluctuations in $\delta\mu_{\mathrm{diff}}$ and $\delta\mathbf{w}$ around static equilibrium, these terms can be neglected when discussing the *linearized* form of these hydrodynamic equations. The equivalent "quantum" hydrodynamic equations for the condensate are given in (8).

In closing this Section, it is useful to summarize the logical structure of our analysis. The kinetic equation (1) leads to the exact set of equations involving the variables $\tilde{n}$, $\mathbf{v}_n$, $\tilde{\varepsilon}$, $P_{\mu\nu}$ and $\mathbf{Q}$ which are defined in terms of various moments of the distribution function $f(\mathbf{r},\mathbf{p},t)$. In addition, the equations in (10) contain the function Γ_{12} in (7) associated with collisions between condensate and non-condensate atoms. In Refs. [4,6], the closed set of hydrodynamic equations displayed in (18) were derived by making use of the local equilibrium distribution function. In the present paper, we have extended this analysis to include a small deviation (19) from local equilibrium, following the Chapman-Enskog approach. However, we have only included contributions to the function $\psi(\mathbf{r},\mathbf{p},t)$ [see (19)] associated with the linearized C_{22} collision integral ($\hat{L}_{22}$ in (20)), which leads to the linear integral equations for ψ in (23) and (24). We have omitted any contribution to $\psi(\mathbf{r},\mathbf{p},t)$ associated with the linearized C_{12} collision integral.

Our neglect of the latter contribution can be justified by noting that C_{12} in (4) is proportional to the condensate density $n_c(\mathbf{r},t)$. Thus the deviations from local equilibrium due to the C_{12} collision integral are relatively unimportant at temperatures close to T_{BEC}. However at lower temperatures, the deviations from local equilibrium due to C_{12} collisions will become increasingly important and one can expect corrections to the values of κ, η and τ_μ obtained in the present paper. Such corrections were evaluated in Ref. [9] for a uniform Bose-condensed gas, although these authors made the further assumption that the condensate and non-condensate were in diffusive equilibrium with each other (i.e., $\tau_\mu \to 0$). As expected, these corrections to the values of κ and η are of order $n_c/\tilde{n}$ at temperatures close to T_{BEC} (see Eq. (26) of the second reference in [9]) and hence can be neglected.

IV. NORMAL MODES FOR A UNIFORM BOSE GAS

In this section, we discuss the normal mode solutions of the linearized hydrodynamic equations, as given by

$$\frac{\partial\delta n_c}{\partial t} = -\boldsymbol{\nabla}\cdot(n_{c0}\delta\mathbf{v}_c) - \delta\Gamma_{12},$$
$$\frac{\partial\delta\mathbf{v}_c}{\partial t} = -\boldsymbol{\nabla}\delta\mu_c, \tag{35}$$

and

$$\frac{\partial\delta\tilde{n}}{\partial t} + \boldsymbol{\nabla}\cdot(\tilde{n}_0\delta\mathbf{v}_n) = \delta\Gamma_{12},$$
$$m\tilde{n}_0\frac{\partial\delta v_{n\mu}}{\partial t} = -\frac{\partial\delta\tilde{P}}{\partial x_\mu} - \delta\tilde{n}\frac{\partial U}{\partial x_\mu} - 2g\tilde{n}_0\frac{\partial\delta n}{\partial x_\mu} + \frac{\partial}{\partial x_\nu}\left\{2\eta\left[D_{\mu\nu} - \frac{1}{3}(\mathrm{Tr}D)\delta_{\mu\nu}\right]\right\},$$
$$\frac{\partial\delta\tilde{P}}{\partial t} = -\frac{5}{3}\boldsymbol{\nabla}\cdot(\tilde{P}_0\delta\mathbf{v}_n) + \frac{2}{3}\delta\mathbf{v}_n\cdot\boldsymbol{\nabla}\tilde{P}_0 + (\mu_{c0} - U_0)\delta\Gamma_{12} + \frac{2}{3}\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}\delta T). \tag{36}$$

As discussed in Refs. [4,6], the source term $\delta\Gamma_{12}$ is conveniently expressed in terms of the fluctuation of the local chemical potential difference $\mu_{\rm diff} \equiv \tilde{\mu} - \mu_c$,

$$\delta\Gamma_{12} = -\frac{\beta_0 n_{c0}}{\tau_{12}}\delta\mu_{\rm diff}. \tag{37}$$

Here we have introduced an equilibrium relaxation time involving collisions between condensate and non-condensate atoms,

$$\frac{1}{\tau_{12}} \equiv 4\pi g^2\int\frac{d\mathbf{p}_1}{(2\pi)^3}\int\frac{d\mathbf{p}_2}{(2\pi)^3}\int d\mathbf{p}_3(1+\tilde{f}_1)\tilde{f}_2\tilde{f}_3$$
$$\times\delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)\delta(\mu_c + \tilde{\varepsilon}_{p_1} - \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3}), \tag{38}$$

where it is understood that all quantities now pertain to static thermal equilibrium.

One can now eliminate δT and $\delta\mu_{\rm diff}$ from the above linearized equations using the relations (see (A6) and (A4) of Appendix A)

$$\delta\tilde{\mu} = \frac{\sigma_{10}}{\tilde{n}_0}\delta\tilde{P} + \frac{\sigma_{20}}{\beta_0\tilde{n}_0}\delta\tilde{n} + 2g(\delta\tilde{n} + \delta n_c),$$
$$\delta T = \frac{T_0\sigma_{30}}{\tilde{P}_0}\delta\tilde{P} - \frac{T_0\sigma_{40}}{\tilde{n}_0}\delta\tilde{n}. \tag{39}$$

The various coefficients appearing are given by

$$\sigma_{10} = -\frac{\frac{3}{2}\tilde{n}_0^2 - \gamma_0\tilde{n}_0(\mu_{c0} - U_0)}{\frac{5}{2}\tilde{P}_0\gamma_0 - \frac{3}{2}\tilde{n}_0^2}, \quad \sigma_{20} = \beta_0\tilde{n}_0\frac{\frac{5}{2}\tilde{P}_0 - \tilde{n}_0(\mu_{c0} - U_0)}{\frac{5}{2}\tilde{P}_0\gamma_0 - \frac{3}{2}\tilde{n}_0^2},$$
$$\sigma_{30} = \frac{\tilde{P}_0\gamma_0}{\frac{5}{2}\tilde{P}_0\gamma_0 - \frac{3}{2}\tilde{n}_0^2}, \quad \sigma_{40} = \frac{\tilde{n}_0^2}{\frac{5}{2}\tilde{P}_0\gamma_0 - \frac{3}{2}\tilde{n}_0^2}, \tag{40}$$

where $\mu_{c0} - U_0 = -gn_{c0}$ in the Thomas-Fermi approximation [1]. Using (39) and (40), we see that our two-fluid hydrodynamic equations in (35) and (36) reduce to five coupled equations (three for the non-condensate and two for the condensate) in the five variables $\delta\tilde{n}$, δn_c, $\delta\mathbf{v}_n$, $\delta\mathbf{v}_c$, and $\delta\tilde{P}$. Thus one expects a total of five normal modes from the extended ZGN$'$ equations.

We now consider the special case of a *uniform* gas ($U_{\rm ext} = 0$) and look for plane-wave solutions $\sim e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$. It is convenient to introduce dimensionless variables

$$\begin{aligned} \bar{n}_c \equiv n_c/n, \quad & \bar{n} \equiv \tilde{n}/n, \\ \delta\bar{v}_c \equiv i\hat{\mathbf{k}}\cdot\delta\mathbf{v}_c/v_{\rm cl}, \quad & \delta\bar{v}_n \equiv i\hat{\mathbf{k}}\cdot\delta\mathbf{v}_n/v_{\rm cl}, \\ \bar{P} \equiv \tilde{P}/nk_{\rm B}T_0, \qquad & t \equiv T/T_{\rm BEC}, \qquad \lambda \equiv gn/k_{\rm B}T_{\rm BEC}, \end{aligned} \tag{41}$$

where $v_{\rm cl} \equiv (5k_{\rm B}T_{\rm BEC}/3m)^{1/2}$ is the sound velocity of a classical gas at $T = T_{\rm BEC}$. We also introduce dimensionless frequency and wavenumber variables

$$\bar{\omega} \equiv \omega\tau_0, \ \bar{k} \equiv kv_{\rm cl}\tau_0, \tag{42}$$

where $\tau_0^{-1} \equiv \sigma n(16k_{\rm B}T_{\rm BEC}/\pi m)^{1/2}$ is the *classical* gas collision time [3], evaluated at $T = T_{\rm BEC}$ (but with the quantum collision cross-section for bosons, $\sigma = 8\pi a^2$.) Finally we define dimensionless transport coefficients and a dimensionless collision time

$$\bar{\kappa} \equiv \kappa/nv_{\rm cl}^2\tau_0k_{\rm B}, \ \bar{\eta} \equiv \eta/nv_{\rm cl}^2m\tau_0, \ \bar{\tau}_{12} \equiv \tau_{12}/\tau_0. \tag{43}$$

In terms of the above dimensionless quantities, we obtain the following closed set of equations

$$i\bar{\omega}\delta\bar{n}_c = -\frac{\lambda\bar{n}_{c0}}{\bar{\tau}_{12}t}\delta\bar{n}_c + \bar{k}\bar{n}_{c0}\delta\bar{v}_c - \frac{\sigma_{10}\bar{n}_{c0}}{\bar{\tau}_{12}\bar{n}_0}\delta\bar{P} - \frac{\sigma_{20}\bar{n}_{c0}}{\bar{\tau}_{12}\bar{n}_0}\delta\bar{n}, \tag{44a}$$

$$i\bar{\omega}\delta\bar{v}_c = -\frac{3}{5}\lambda\bar{k}(\delta\bar{n}_c + 2\delta\bar{n}), \tag{44b}$$

$$i\bar{\omega}\delta\bar{n} = \frac{\lambda\bar{n}_{c0}}{\bar{\tau}_{12}t}\delta\bar{n}_c + \bar{k}\bar{n}_0\delta\bar{v}_n + \frac{\sigma_{10}\bar{n}_{c0}}{\bar{\tau}_{12}\bar{n}_0}\delta\bar{P} + \frac{\sigma_{20}\bar{n}_{c0}}{\bar{\tau}_{12}\bar{n}_0}\delta\bar{n}, \tag{44c}$$

$$i\bar{\omega}\delta\bar{v}_n = -\frac{6}{5}\lambda\bar{k}(\delta\bar{n}_c + \delta\bar{n}) - \frac{3t}{5\bar{n}_0}\bar{k}\delta\bar{P} + \frac{4\bar{\eta}}{3\bar{n}_0}\bar{k}^2\delta\bar{v}_n, \tag{44d}$$

$$\begin{aligned} i\bar{\omega}\delta\bar{P} = {} & \frac{5\bar{P}_0}{3}\bar{k}\delta\bar{v}_n - \frac{2}{3}\left(\frac{\sigma_{20}\lambda\bar{n}_{c0}^2}{\bar{\tau}_{12}t\bar{n}_0} + \bar{k}^2\frac{\sigma_{40}\bar{\kappa}}{\bar{n}_0}\right)\delta\bar{n} - \frac{2\lambda^2\bar{n}_{c0}^2}{3\bar{\tau}_{12}t^2}\delta\bar{n}_c \\ & -\frac{2}{3}\left(\frac{\sigma_{10}\lambda\bar{n}_{c0}^2}{\bar{\tau}_{12}t\bar{n}_0} - \bar{k}^2\frac{\sigma_{30}\bar{\kappa}}{\bar{P}_0}\right)\delta\bar{P}. \end{aligned} \tag{44e}$$

We emphasize that our hydrodynamic equations are restricted by the assumption of local equilibrium and thus are only valid if $\bar{\omega} \ll 1$, $\bar{k} \ll 1$.

It is straightforward to solve the coupled set of equations in (44). Above $T_{\rm BEC}$, they reduce to three equations for three variables, which gives rise to two sound modes ($\pm uk$) and a thermal diffusion mode. Below $T_{\rm BEC}$, these equations give two first sound modes ($\pm u_1k$), two second sound modes ($\pm u_2k$) and one relaxational mode. If the transport coefficients η and κ are set to zero, these equations are identical to the ZGN$'$ equations as discussed in Refs. [4,6]. Using (44), we can discuss the effect of η and κ on the first and second sound modes, as well as on the relaxational mode. In Fig. 1 of Ref. [6], a graph is given of the first and second sound velocities vs. temperature. They are almost identical in both the ZGN limit ($\omega\tau_\mu \gg 1$) and the Landau limit ($\omega\tau_\mu \ll 1$). Calculations based on the generalized ZGN$'$ equations in (44) show essentially no change in the first and second sound velocities when the effects of κ and η are included, for $\bar{k} < 0.5$. In Figs. 1–3, we illustrate the effect of these transport coefficients on the damping Γ of the first and second sound modes (where $\omega_i = uk_i - i\Gamma_i$) and on the relaxational mode, all as a function of the dimensionless wavevector $\bar{k}$. These specific results are for a temperature of $T/T_{\rm BEC} = 0.9$ and $gn/k_{\rm B}T_{\rm BEC} = 0.2$, where one finds $\tau_0/\tau_\mu = 0.91$, $\bar{\kappa} = 2.41$ and $\bar{\eta} = 0.34$. We note that in the absence of η and κ, only second sound is significantly damped through coupling to the relaxational mode. When we include η and κ, Γ_1 is large compared to Γ_2.

Figs. 1–3 only show the damping in a case where $\omega\tau_\mu \ll 1$. It is interesting to see how the relaxational mode depends on the value of $\omega\tau_\mu$. In Fig. 1 of Ref. [4], we have plotted the temperature dependence of τ_{12}, τ_0 and τ_μ for $gn = 0.1k_{\rm B}T_{\rm BEC}$. In Fig. 4, we plot the damping of the relaxational mode for $\bar{k} = 0.4$, as a function of the temperature. We also show the corresponding temperature dependence of τ_μ/τ_0. As discussed in Ref. [4], our use of a particle-like spectrum with a HF mean-field gives rise to a spurious finite value of the condensate density n_{c0} at $T_{\rm BEC}$. As a result, τ_μ is also finite at $T_{\rm BEC}$.

In Appendix B, we derive an analytical expression for the width of the mode centered at zero frequency. Working to first order in κ, η, $1/\tau_{12}$ and second order in $\lambda = gn/k_{\rm B}T_{\rm BEC}$, we obtain

$$\Gamma_{\rm R} \simeq \frac{1}{\tau_\mu} + \frac{2}{5}\frac{\sigma_{40}\kappa}{(\tilde{P}_0/T_0)}k^2. \tag{45}$$

As shown in Fig. 4, this approximate expression is in good agreement with the result of a direct numerical evaluation of $\Gamma_{\rm R}$ from (44) at temperatures close to $T_{\rm BEC}$ (it deviates more and more at lower temperatures because of the non-linear dependence of $\Gamma_{\rm R}$ on κ and $1/\tau_{12}$). The approximate analytic expression in (45) is useful in that it shows clearly that below $T_{\rm BEC}$, the relaxational mode is strongly coupled to thermal diffusion which arises from the deviation from local equilibrium of the non-condensate distribution function. Indeed, one may view this damped mode as a renormalized version of the well-known thermal diffusion mode found above $T_{\rm BEC}$ [14]. In the classical high-temperature limit, we have $\sigma_{40} = 1$ and $\tilde{P}_0 = nk_{\rm B}T_0$, and (45) reduces in this case to $\Gamma_{\rm R} = \kappa k^2/nC_{\rm P}$, where the specific heat $C_{\rm P} = 5k_{\rm B}/2$. This is the well-known classical gas result for the thermal diffusion mode [10,14].

In the ZGN limit ($\tau_\mu \to \infty$), (45) shows that the damping rate of the relaxational mode is mainly due to the finite thermal conductivity (if we set $\kappa = 0$, we obtain the zero frequency mode discussed in Section V of Ref. [6]). In the opposite Landau limit of the ZGN$'$ equations ($\tau_\mu \to 0$), the width of this mode is dominated by $1/\tau_{12}$, associated with the C_{12} collisions which bring about diffusive equilibrium between the condensate and non-condensate components (see Fig. 4). Eq. (45) is in complete agreement with the qualitative picture sketched at the end of Section V of Ref. [6]. The mode spectrum which the coupled ZGN$'$ equations in (44) imply below $T_{\rm BEC}$ is different from the usual Landau two-fluid hydrodynamics [8,9,3]. In the latter theory, the thermal diffusion mode above $T_{\rm BEC}$ does not persist as a relaxational mode below $T_{\rm BEC}$. Rather, the thermal diffusion mode is interpreted as being replaced by two damped second sound modes ($\pm u_2 k - i\Gamma_2$) below $T_{\rm BEC}$.

Fig. 4 also shows that the width of the relaxational mode increases sharply (over and above the spurious jump at $T_{\rm BEC}$ noted above) as the temperature passes from above to below $T_{\rm BEC}$, as a result of the onset of C_{12} collisions. This increased width of a mode peaked at $\omega = 0$ will be a useful experimental signature of the new physics implied by the extended ZGN′ hydrodynamic equations derived in the present paper. Of course, all of the above analysis says nothing about the relative weight of the first sound, second sound and relaxational modes below $T_{\rm BEC}$. This requires the evaluation of a specific response function, such as the dynamic structure factor [17]. Further calculations along these lines are in progress.

V. CONCLUSIONS

In this paper, we have extended our recent derivation [4,6] of two-fluid hydrodynamic equations (referred to as the ZGN′ equations) to include the effects of a small deviation from local equilibrium of the non-condensate atoms. This brings in the usual kind of hydrodynamic damping due to the thermal conductivity and shear viscosity of the thermal cloud. In calculating these transport coefficients we have only taken into account the deviations from local equilibrium due to the C_{22} collision integral but have neglected the contribution coming from the C_{12} collision integral. This limits the validity of the present calculations to the vicinity of the transition temperature. The damping due to hydrodynamic dissipation is in addition to that due to the equilibration of the condensate and non-condensate components already contained in the original ZGN′ equations, as discussed in Refs. [4,6].

For illustration, we presented some numerical results for a *uniform* Bose gas. In this case, we find damped first and second sound modes ($\pm u_1 k - i\Gamma_1, \pm u_2 k - i\Gamma_2$) and a purely relaxational mode $\omega = -i\Gamma_{\rm R}$. The latter mode is not exhibited by the usual two-fluid hydrodynamic equations which assume that the condensate and non-condensate are in local thermodynamic equilibrium. The overall effect of η and κ on the ZGN′ predictions in Ref. [6] is to introduce additional damping of all three modes. Above $T_{\rm BEC}$, the relaxational and second sound modes ($\pm u_2 k$) merge into a single thermal diffusive mode. The relaxational mode below $T_{\rm BEC}$ may therefore be viewed as the superfluid analogue of the normal thermal diffusion mode.

In another paper [16], we shall use these ZGN′ equations with transport coefficients to discuss the damping of hydrodynamic modes in a *trapped* Bose gas. In general this is a more complex situation, since one must contend with the fact that the local equilibrium solution becomes invalid as a starting point in the low density tail of the thermal cloud [11,18]. However, one can derive general expressions for the damping and renormalization of the hydrodynamic modes as given by the ZGN equations. In particular, we shall show that the damping of the out-of-phase dipole oscillation of the condensate and non-condensate components in a trapped Bose-condensed gas is only weakly affected by the non-condensate transport coefficients. As a result, the damping of this mode is entirely due to the collisions between the condensate and non-condensate atoms which are responsible for bringing these two components into diffusive equilibrium. Further experimental studies [12] of this analogue of second sound in superfluid ^{4}He [8] would be of great interest.

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APPENDIX A:

We briefly sketch the derivation of the kinetic equation in (21). Using (14) in the left hand side of (1), one has

$$\begin{aligned}
&\left[\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla} - \boldsymbol{\nabla}U(\mathbf{r},t)\cdot\boldsymbol{\nabla}_{\mathbf{p}}\right]\tilde{f}(\mathbf{r},\mathbf{p},t) \\
&= \left[\frac{1}{z}\left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla}\right)z + \frac{mu^2}{2k_{\mathrm{B}}T^2}\left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla}\right)T\right. \\
&\quad \left. + \frac{m\mathbf{u}}{k_{\mathrm{B}}T}\cdot\left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla}\right)\mathbf{v} + \frac{\boldsymbol{\nabla}U(\mathbf{r},t)}{k_{\mathrm{B}}T}\cdot\mathbf{u}\right]\tilde{f}(1+\tilde{f}).
\end{aligned} \tag{A1}$$

Using the expressions for the density $\tilde{n}$ given by (16) and the pressure $\tilde{P}$ in (17), one finds

$$\begin{aligned}
d\tilde{n} &= \frac{3\tilde{n}}{2T}dT + \frac{\gamma k_{\mathrm{B}}T}{z}dz\,, \\
d\tilde{P} &= \frac{5\tilde{P}}{2T}dT + \frac{\tilde{n}k_{\mathrm{B}}T}{z}dz\,,
\end{aligned} \tag{A2}$$

where γ is the variable introduced after (22). One may combine to these equations to obtain

$$\begin{aligned}
\frac{dT}{T} &= \frac{\sigma_3}{\tilde{P}}d\tilde{P} - \frac{\sigma_4}{\tilde{n}}d\tilde{n}, \\
\frac{dz}{z} &= \frac{1}{k_{\mathrm{B}}T}\left(\frac{5\sigma_3}{2\gamma}d\tilde{n} - \frac{3\sigma_4}{2\tilde{n}}d\tilde{P}\right),
\end{aligned} \tag{A3}$$

where the thermodynamic functions σ_3 and σ_4 are defined by

$$\sigma_3(\mathbf{r},t) = \frac{\tilde{P}\gamma}{\frac{5}{2}\tilde{P}\gamma - \frac{3}{2}\tilde{n}^2}, \qquad \sigma_4(\mathbf{r},t) = \frac{\tilde{n}^2}{\frac{5}{2}\tilde{P}\gamma - \frac{3}{2}\tilde{n}^2}. \tag{A4}$$

Using the lowest-order hydrodynamic equations given in (18), one finds that the equations in (A3) reduce to

$$\begin{aligned}
\frac{\partial T}{\partial t} &= -\frac{2}{3}T(\boldsymbol{\nabla}\cdot\mathbf{v}_n) - \mathbf{v}_n\cdot\boldsymbol{\nabla}T + \frac{2T}{3\tilde{n}}\sigma_1\Gamma_{12}[\tilde{f}], \\
\frac{\partial z}{\partial t} &= -\mathbf{v}_n\cdot\boldsymbol{\nabla}z + \sigma_2 z\frac{\Gamma_{12}[\tilde{f}]}{\tilde{n}},
\end{aligned} \tag{A5}$$

where the local equilibrium thermodynamic functions σ_1 and σ_2 are defined in (22). The analogous equation for $\partial \mathbf{v}_n/\partial t$ is given directly by (18). Using these results in (A1), one finds that it reduces to

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}+\frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla}-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}_{\mathbf{p}}\right)\tilde{f}\\
&=\left\{\frac{1}{T}\mathbf{u}\cdot\boldsymbol{\nabla}T\left(\frac{mu^2}{2k_{\mathrm{B}}T}-\frac{5\tilde{P}}{2\tilde{n}k_{\mathrm{B}}T}\right)+\frac{m}{k_{\mathrm{B}}T}\left[\mathbf{u}\cdot(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v}-\frac{u^2}{3}\boldsymbol{\nabla}\cdot\mathbf{v}\right]\right.\\
&\left.+\left(\sigma_2+\frac{mu^2}{3k_{\mathrm{B}}T}\sigma_1+\frac{m}{k_{\mathrm{B}}T}\mathbf{u}\cdot\mathbf{w}\right)\frac{\Gamma_{12}[\tilde{f}]}{\tilde{n}}\right\}\tilde{f}(1+\tilde{f}),
\end{aligned}\tag{A6}$$

where we recall $\mathbf{u}\equiv\mathbf{p}/m-\mathbf{v}_n$. This can be rewritten in the form shown on the left hand side of (21).

Using $dz/z=\beta[-(\tilde{\mu}-U)dT/T+d\tilde{\mu}-2gdn]$, one obtains from (A3)

$$d\tilde{\mu}=\frac{\sigma_1'}{\tilde{n}}d\tilde{P}+\frac{\sigma_2'}{\beta\tilde{n}}d\tilde{n}+2g(d\tilde{n}+dn_c).\tag{A7}$$

Here σ_1' and σ_2' are obtained from σ_1 and σ_2 in (22) with the replacement of $\frac{1}{2}m(\mathbf{v}_n-\mathbf{v}_c)^2+\mu_c$ by $\tilde{\mu}$. The relations (A2)-(A4) and (A7) are used in Sec. IV to eliminate $\delta\mu_{\mathrm{diff}}$ and δT from the linearized hydrodynamic equations in (35) and (36).

APPENDIX B:

In this Appendix, we sketch the derivation of the approximate expression for the relaxation rate Γ_{R} given in (45). It is convenient to introduce the five-component vector

$$\mathbf{y}^{\mathrm{T}}=(\delta\bar{n},\delta\bar{P},\delta\bar{v}_n,\delta\bar{n}_c,\delta\bar{v}_c).\tag{B1}$$

The linearized hydrodynamic equations in (44) can then be written in the matrix form

$$i\bar{\omega}\mathbf{y}=K\mathbf{y},\tag{B2}$$

where the 5×5 matrix K is given by

$$K=\begin{pmatrix}
\frac{\sigma_{20}\bar{n}_{c0}}{\bar{\tau}_{12}\bar{n}_0} & \frac{\sigma_{10}\bar{n}_{c0}}{\bar{\tau}_{12}\bar{n}_0} & \bar{n}_0\bar{k} & \frac{\lambda\bar{n}_{c0}}{\bar{\tau}_{12}t} & 0\\
-\frac{2}{3}\left(\frac{\sigma_{20}\lambda\bar{n}_{c0}^2}{\bar{\tau}_{12}t\bar{n}_0}+\frac{\sigma_{40}\bar{\kappa}\bar{k}^2}{\bar{n}_0}\right) & -\frac{2}{3}\left(\frac{\sigma_{10}\lambda\bar{n}_{c0}^2}{\bar{\tau}_{12}t\bar{n}_0}-\frac{\sigma_{30}\bar{\kappa}\bar{k}^2}{\bar{P}_0}\right) & \frac{5}{3}\bar{P}_0\bar{k} & -\frac{2\lambda^2\bar{n}_{c0}^2}{3\bar{\tau}_{12}t^2} & 0\\
-\frac{6}{5}\lambda\bar{k} & -\frac{3t}{5\bar{n}_0}\bar{k} & \frac{4\bar{\eta}}{3\bar{n}_0}\bar{k}^2 & -\frac{6}{5}\lambda\bar{k} & 0\\
-\frac{\sigma_{20}\bar{n}_{c0}}{\bar{\tau}_{12}\bar{n}_0} & -\frac{\sigma_{10}\bar{n}_{c0}}{\bar{\tau}_{12}\bar{n}_0} & 0 & -\frac{\lambda\bar{n}_{c0}}{\bar{\tau}_{12}t} & \bar{n}_{c0}\bar{k}\\
-\frac{6}{5}\lambda\bar{k} & 0 & 0 & -\frac{3}{5}\lambda\bar{k} & 0
\end{pmatrix}.\tag{B3}$$

Although it is difficult to obtain general analytical solutions of the matrix equation (B2), we can obtain approximate analytical expression for Γ_{R} by making use of the expression for the determinant of this matrix at zero frequency:

$$\det K=\tau_0^5(\Omega_1^2+\Gamma_1^2)(\Omega_2^2+\Gamma_2^2)\Gamma_{\mathrm{R}}.\tag{B4}$$

Working to first order in $\bar{\kappa}$, $\bar{\eta}$, $1/\bar{\tau}_{12}$ and second order in λ, a direct evaluation of the determinant of the matrix in (B3) gives

$$\det K = \frac{3}{5}\lambda\bar{n}_{c0}\frac{t\bar{P}_0}{\bar{n}_0}\bar{k}^4\left\{\frac{2\sigma_{40}\bar{\kappa}\bar{k}^2}{5\bar{P}_0}\left(1-\frac{2\lambda\bar{n}_0^2\sigma_{30}}{t\bar{P}_0\sigma_{40}}\right)\right.$$
$$\left.\frac{\bar{n}_{c0}}{\bar{\tau}_{12}\bar{n}_0}\left[\sigma_{20}+\frac{\lambda\bar{n}_0}{t\bar{P}_0}\left(\frac{2}{5}\bar{n}_{c0}\sigma_{20}+2\bar{P}_0\sigma_{10}-2\bar{P}_0\right)\right]\right\}. \tag{B5}$$

If we set κ, $\eta = 0$, the first (Ω_1) and second (Ω_2) sound mode frequencies for small wavevectors are given by the solution of Eq. (93) of Ref. [6]. To second order in λ, this gives

$$\tau_0^4\Omega_1^2\Omega_2^2 = \bar{k}^4\frac{3}{5}\lambda\bar{n}_{c0}\frac{t\bar{P}_0}{\bar{n}_0}\left(1-\frac{2\lambda\bar{n}_0^2}{t\bar{P}_0\sigma_{40}}-\frac{6\lambda\bar{n}_0^2}{5t\bar{P}_0}\right). \tag{B6}$$

Using (B6) and (B5) in (B4), we obtain

$$\Gamma_{\rm R}\tau_0 \approx \frac{\det K}{\Omega_1^2\Omega_2^2\tau_0^4}$$
$$= \frac{\tau_0}{\tau_\mu}+\frac{2\bar{\kappa}\bar{k}^2}{5\bar{P}_0}\left[\sigma_{40}+\frac{\lambda\bar{n}_0^2}{t\bar{P}_0}\left(\frac{6}{5}\sigma_{40}+\frac{2}{5}-2\sigma_{30}\right)\right], \tag{B7}$$

where τ_μ is given by

$$\frac{\tau_{12}}{\tau_\mu} = \frac{n_{c0}}{k_{\rm B}T_0}\left[\frac{\frac{5}{2}\tilde{P}_0+2g\tilde{n}_0 n_{c0}+\frac{2}{3}g^2\gamma_0 n_{c0}^2}{\frac{5}{2}\tilde{P}_0\gamma_0-\frac{3}{2}\tilde{n}_0^2}-g\right]$$
$$\simeq \frac{\bar{n}_{c0}}{\bar{n}_0}\left[\frac{5\bar{P}_0}{2\bar{n}_0}+\frac{\lambda}{t}(2\bar{n}_{c0}\sigma_{40}-\bar{n}_0)\right]+O(\lambda^2). \tag{B8}$$

Neglecting the second small term in the square bracket of (B7), we obtain for $\Gamma_{\rm R}$ the simple expression

$$\Gamma_{\rm R}\tau_0 = \frac{\tau_0}{\tau_\mu}+\frac{2\sigma_{40}\bar{\kappa}}{5\bar{P}_0}\bar{k}^2. \tag{B9}$$

Numerical calculations show that (B7) is very well approximated by (B9). Using the definitions of the various dimensionless quantities in (41)-(43), (B9) can be written in the form given in (45).

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* Permanent address: Department of Physics, University of Toronto, Toronto, Ontario, Canada, M5S 1A7.

† Permanent address: Department of Physics, Queen's University, Kingston, Ontario, Canada, K7L 3N6.

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FIGURE CAPTIONS

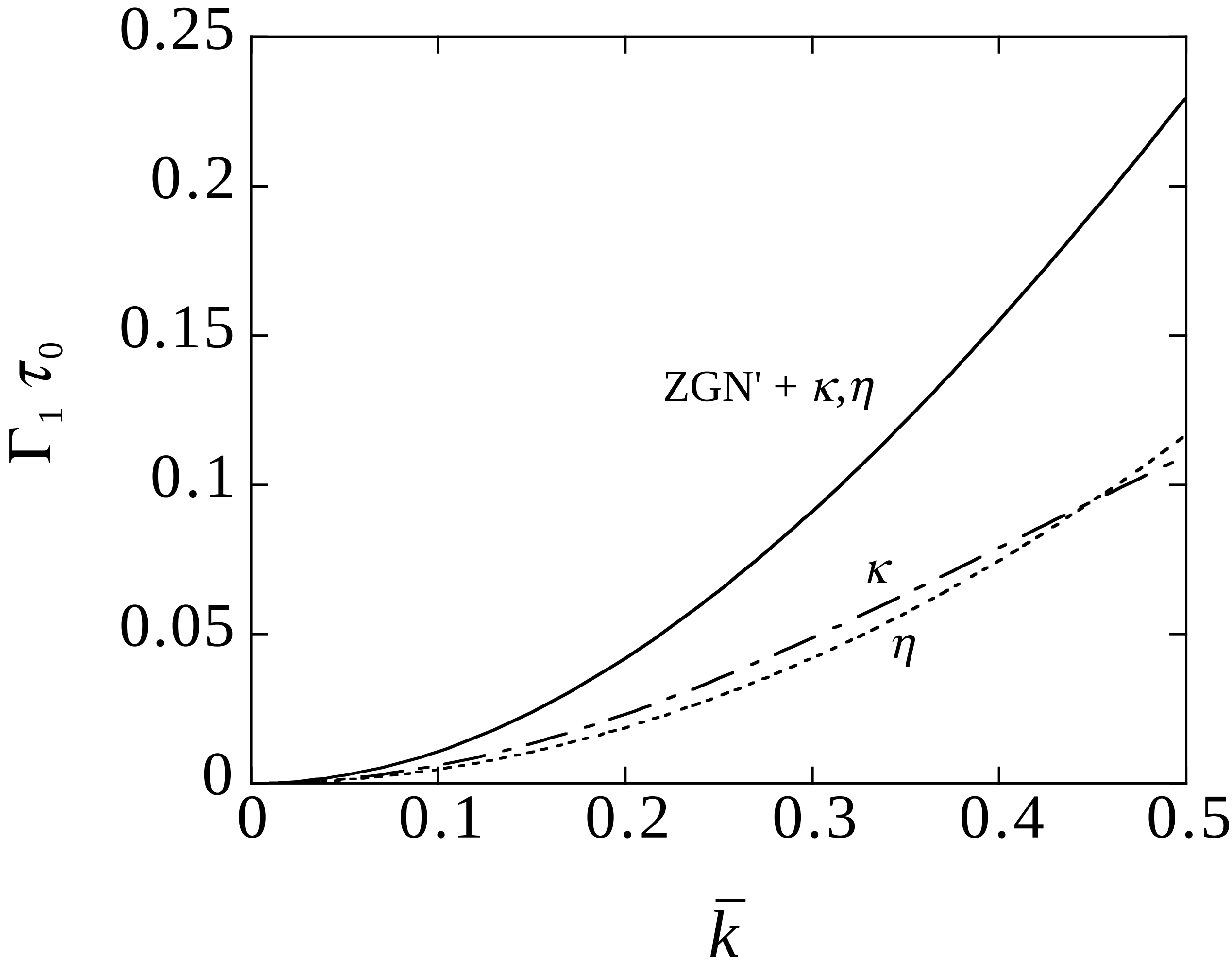
FIG.1: Damping of first sound in a uniform gas for $T/T_{\mathrm{BEC}} = 0.9$ and $gn = 0.2k_{\mathrm{B}}T_{\mathrm{BEC}}$, as a function of the dimensionless wavevector defined in (42). The ZGN$'$ results including only κ and only η are also shown.

FIG.2: Damping of second sound in a uniform gas for the same parameters as in Fig. 1.

FIG.3: Damping of the relaxational mode vs. wavevector (see Fig. 1). The damping due to the thermal conductivity is shown for both the ZGN ($\tau_\mu \to \infty$) and ZGN$'$ theories. The effect of the shear viscosity is negligible.

FIG.4: Damping of the relaxational mode vs. temperature, for $gn = 0.1k_{\mathrm{B}}T_{\mathrm{BEC}}$ and $\bar{k} = 0.4$. The ZGN$'$ relaxation time τ_μ is also shown. All results are normalized to the classical gas collision time τ_0 defined below (42). The approximate analytical expression for Γ_{R} in (45) is compared with a direct numerical solution of the linearized equations in (44). The discontinuity at T_{BEC} is spurious, being a result of the mean-field approximation used for the thermal excitations [4].

Figure 1



0.25
0.2
0.15
0.1
0.05
0
0.1
0.2
0.3
0.4
0.5
$\Gamma_1 \tau_0$
ZGN' + κ,η
κ
η
$\bar{k}$

Figure 2

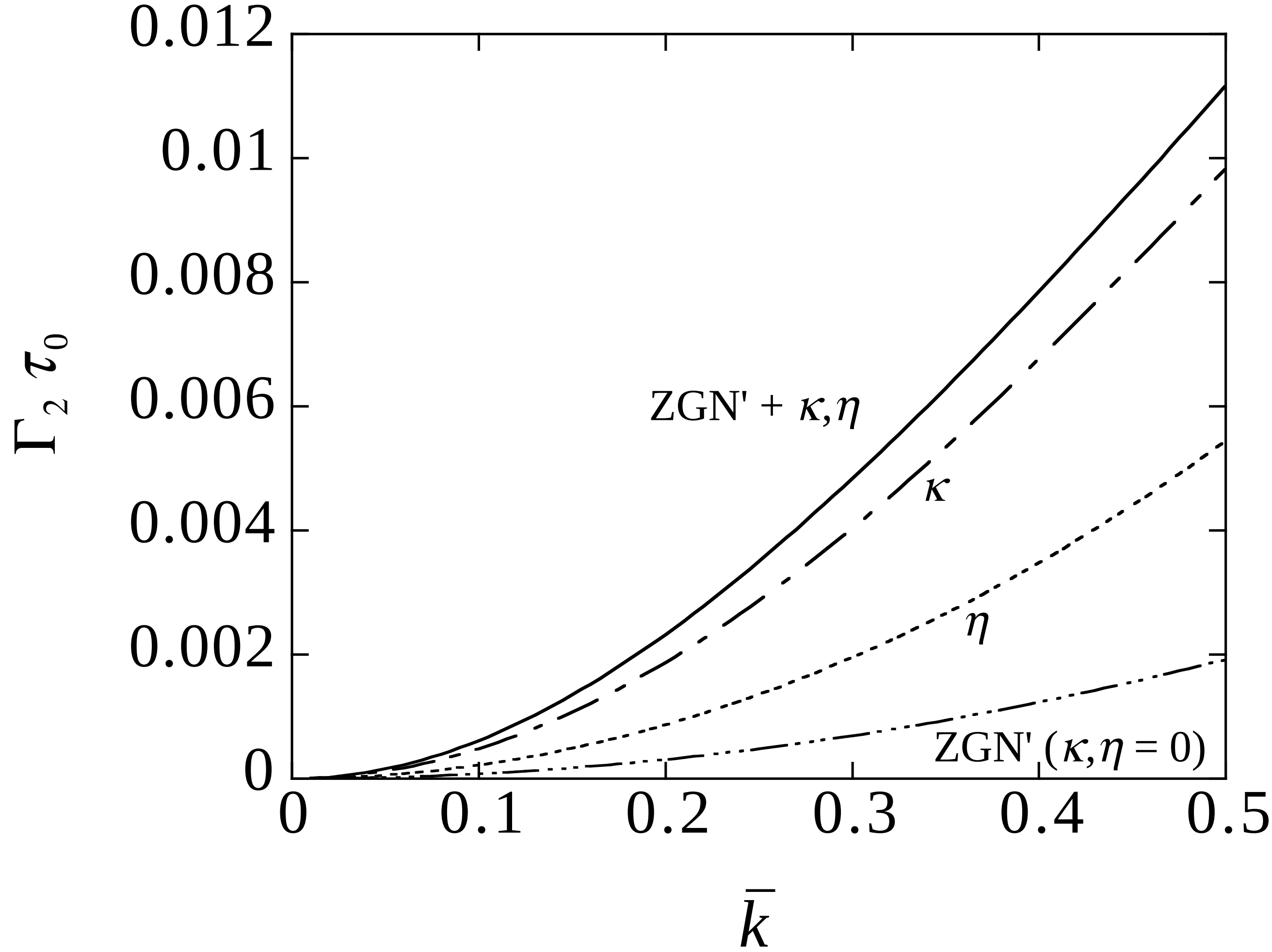

0.012
0.01
0.008
0.006
0.004
0.002
0
$\Gamma_2\,\tau_0$
ZGN' + κ,η
κ
η
ZGN' (κ,η = 0)
0
0.1
0.2
0.3
0.4
0.5
$\bar{k}$

Figure 3

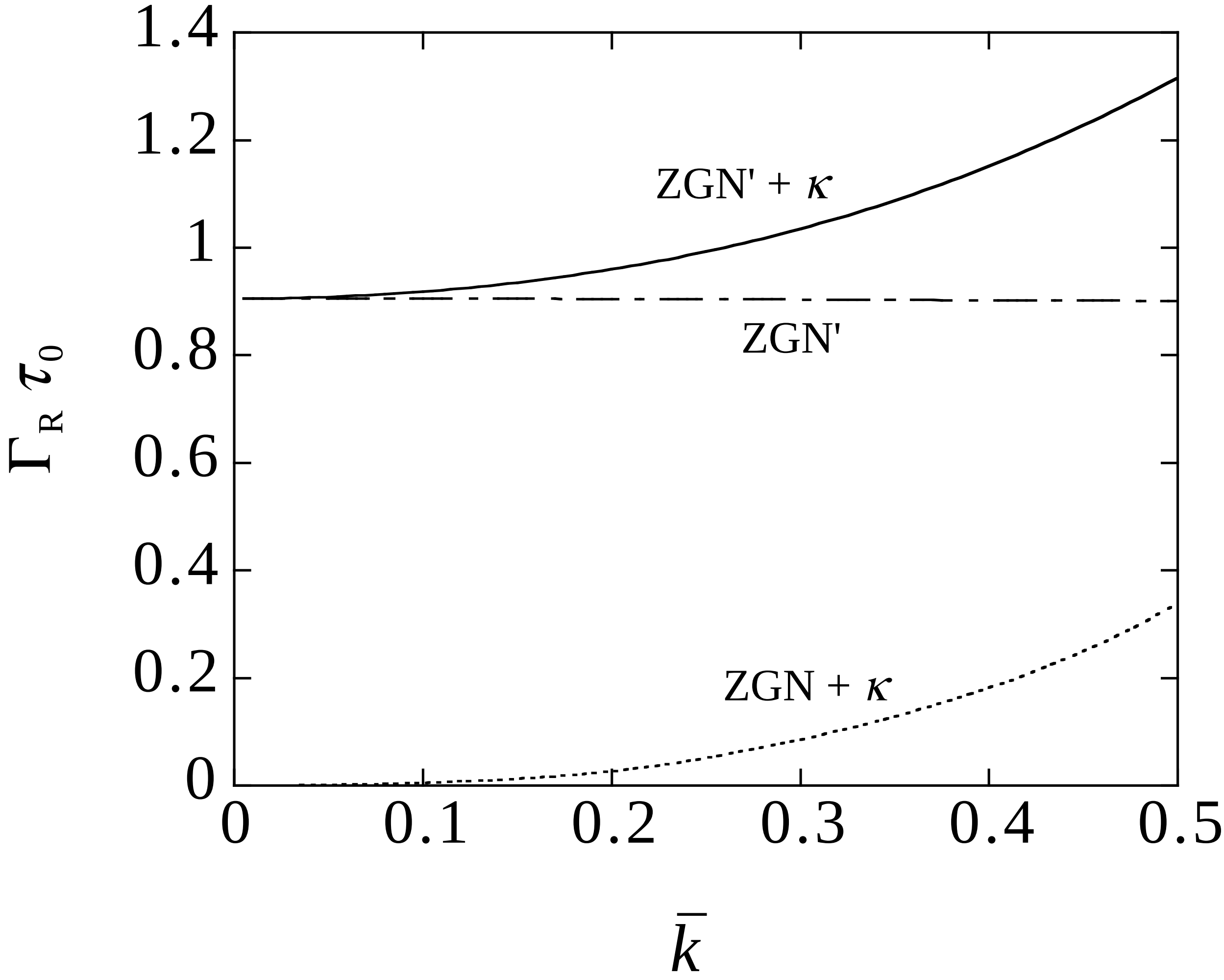
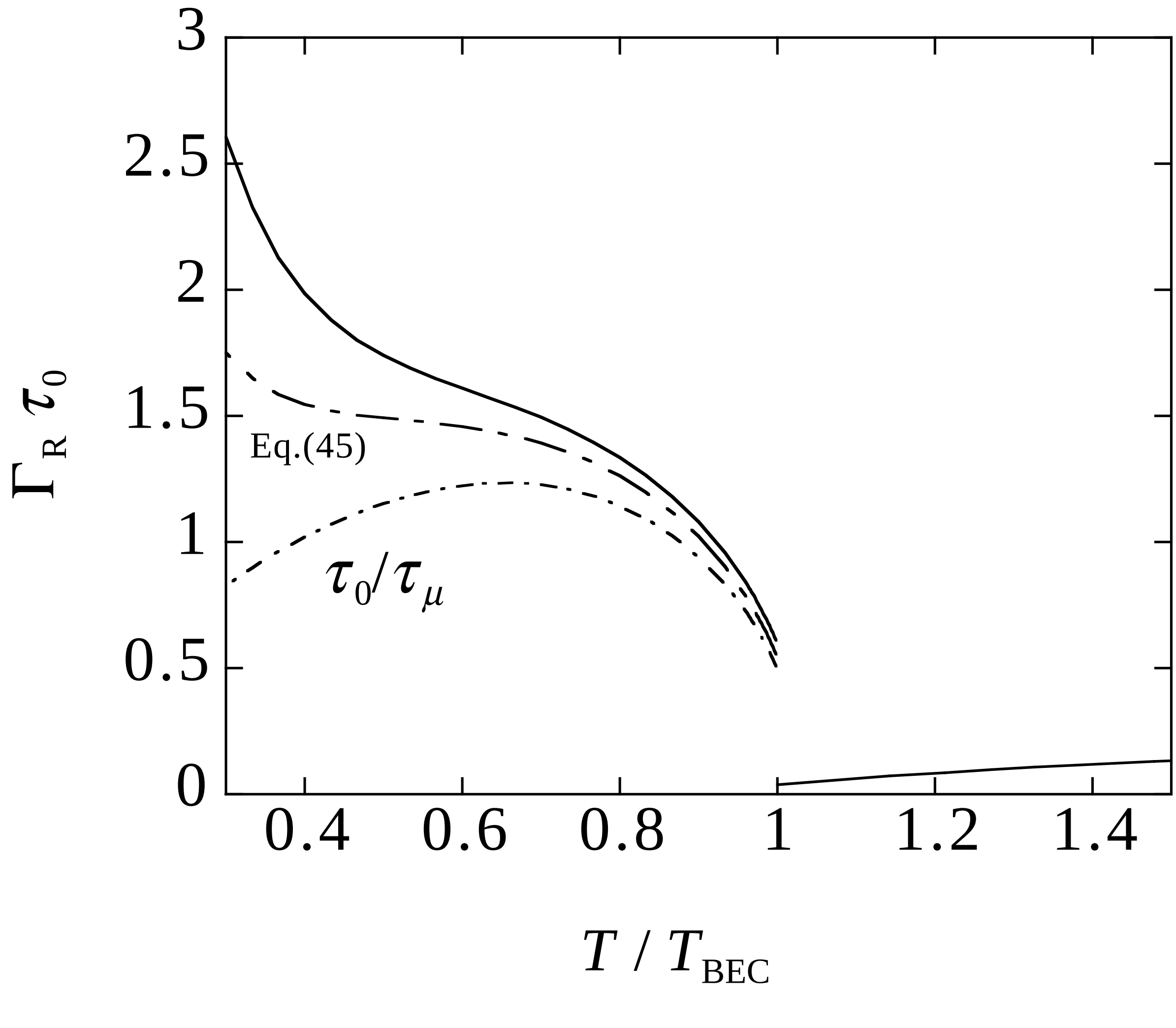


Figure 4


3
2.5
2
1.5
1
0.5
0
0.4
0.6
0.8
1
1.2
1.4
$\Gamma_R \tau_0$
Eq.(45)
τ_0/τ_μ
T / $T_{\rm BEC}$