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Quantum Contextuality as a Measurement Disturbance Effect[1]

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Abstract. The question of quantum contextuality in the Mermin-Peres square is considered. It is shown that a deterministic, noncontextual hidden variable model of this problem is not inconsistent with quantum mechanics, contrary to the Kochen-Specker theorem. The key idea is that measurement outcomes may be viewed as deterministic functions of hidden variable states which are disturbed through the process of measurement. This, in turn, implies that the outcome of measuring the product of two commensurate observables need not be equal to the product of the outcomes that would have been obtained had they been measured individually. A critical analysis of some recent and proposed experimental tests of contextuality is also provided.

Keywords: contextuality, Kochen-Specker, hidden variables, measurement
PACS: 03.65.-w, 03.65.Ta, 03.65.Ud, 03.67.-a

INTRODUCTION

Contextuality is a property ascribed to quantum systems which appears to be at odds with a deterministic, "hidden variable" description. However, due to its implicit counter-factual nature, the notion of contextuality can only be defined in terms of a hypothetical hidden variable description. In the broadest sense, a measurement of an observable is said to be noncontextual if the outcome of the measurement does not depend upon which other compatible observables are measured subsequently, simultaneously, or previously. The Kochen-Specker theorem purports to prove that no noncontextual hidden variable model exists which is consistent with quantum mechanics (for a Hilbert space of dimension three or greater). Therefore, quantum mechanics is said to be contextual.

Spekkens [1] has recently argued that such a definition of noncontextuality is overly restrictive since, as Bell observed much earlier, the particular outcome of a measurement may very well depend implicitly upon which other compatible observables are measured previously or simultaneously [2]. A better definition of a noncontextual measurement would require only that the joint statistics of commuting observables be unchanged by the details of how they are measured. A noncontextual hidden variable model may then be defined as one which associates a single measurable function with each observable yet reproduces the correct joint statistics.

The Kochen-Specker theorem was first introduced by Bell [2], following his refutation of von Neumann's impossibility proof, and was itself based on the mathematical work of Gleason [3]. It was independently proven by Kochen and Specker shortly afterwards [4] and became popular in philosophical circles. The original theorem applied to Hilbert spaces of three dimensions, which may be viewed as describing the angular momentum of a spin-1 particle. Simpler but more restrictive versions of the theorem in higher dimensions have since been published [5, 6, 7].

Unlike Bell's inequality [8], the Kochen-Specker theorem is entirely nonstatistical — in theory a single measurement suffices for empirical confirmation. Thus, it is an example of an "all-versus-nothing" proof of the impossibility of hidden variables. It was not until recently, however, that a empirical test was proposed [9]. Subsequently, experiments using single and correlated photons [10, 11, 12] as well as neutron interferometry [13] have all shown results which are consistent with quantum theoretic predictions. These experimental results appear to corroborate the theoretical prediction that quantum mechanics is inescapably contextual.

The aim of this paper is to demonstrate that such a conclusion is unwarranted and that, in fact, quantum theory is perfectly consistent with a deterministic, noncontextual theory. To this end, it will be argued that the standard proofs of the Kochen-Specker theorem are invalid due to the possible dependence of the *post-measurement* hidden variable probability distribution on the particular set of mutually commensurate observables chosen for measurement.

[1] An expanded version of this work will appear in *Physical Review A*.

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Furthermore, it will be shown how this dependence may arise naturally through the process of measurement and attendant interactions with the measuring devices. Although the discussion is restricted to a four-dimensional Hilbert space, the approach and conclusions are expected to generalize to any Hilbert space.

The organization of the paper is as follows. First, a particular representation of the four-dimensional Hilbert space is introduced. The Kochen-Specker theorem is considered in the context of nine composite Pauli spin operators on this space, and the proof for this case is shown to be invalid. The reasons for this conclusion are then further elaborated upon, whereupon a hidden variable interpretation is offered. A critical analysis of some recent and proposed experimental tests follows, where the concept of operator decomposability is introduced.

PROBLEM STATEMENT

A general, four-dimensional Hilbert space may be mapped to a notional composite system of two spin-1/2 particles. For the single-particle component subspace, any self-adjoint operator may be written as a linear combination of the Pauli spin operators, $\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$, and the identity, $\hat{1}$. The discussion that follows will be cast in terms of this representation.

Let us begin by considering the example of the Mermin-Peres "magic square" [6, 7], which consists of nine operators arranged as follows:

$$\begin{array}{ccc} \hat{\sigma}_x \otimes \hat{1} & \hat{1} \otimes \hat{\sigma}_x & \hat{\sigma}_x \otimes \hat{\sigma}_x \\ \hat{1} \otimes \hat{\sigma}_y & \hat{\sigma}_y \otimes \hat{1} & \hat{\sigma}_y \otimes \hat{\sigma}_y \\ \hat{\sigma}_x \otimes \hat{\sigma}_y & \hat{\sigma}_y \otimes \hat{\sigma}_x & \hat{\sigma}_z \otimes \hat{\sigma}_z \end{array}$$

Let $\hat{V}_{ij}$ denote the operator in row i, column j. From the properties of the Pauli spin operators it is readily verified that the three operators in each row are mutually commuting, as are those in each column. Furthermore, it can be shown that each operator *anticommutes* with the four operators not in its row or column. Finally, we observe that the product of the three operators in each row, as well as those in the first two columns, is $+\hat{1} \otimes \hat{1}$. The product of the operators in Column 3, by contrast, is $-\hat{1} \otimes \hat{1}$.

For a noncontextual model of the Mermin-Peres magic square, we seek a set of hidden variables, Ω, and a collection of nine functions $V_{ij} : \Omega \to \mathbb{R}$ (for $i, j = 1, 2, 3$) such that the outcome of measuring operator $\hat{V}_{ij}$ is $V_{ij}(\omega)$, where $\omega \in \Omega$. Suppose Ω and all nine V_{ij} are given. The aforementioned operator relations for the product of each row and column *suggest* a similar relation in the hidden variable model.

For $i = 1, 2, 3$, let us define R_i to be the set of all $\omega \in \Omega$ such that $V_{i1}(\omega), V_{i2}(\omega), V_{i3}(\omega) \in \{-1, +1\}$ and

$$V_{i1}(\omega)V_{i2}(\omega)V_{i3}(\omega) = +1. \tag{1}$$

Similarly, for $j = 1, 2$, let C_j be the set of all $\omega \in \Omega$ such that $V_{1j}(\omega), V_{2j}(\omega), V_{3j}(\omega) \in \{-1, +1\}$ and

$$V_{1j}(\omega)V_{2j}(\omega)V_{3j}(\omega) = +1. \tag{2}$$

Finally, let $C_3 \subseteq \Omega$ be such that, for all $\omega \in C_3$, $V_{13}(\omega), V_{23}(\omega), V_{33}(\omega) \in \{-1, +1\}$ but, by contrast,

$$V_{13}(\omega)V_{23}(\omega)V_{33}(\omega) = -1. \tag{3}$$

Now, let us suppose that there exists at least one point, ω, that is common to all six row/column sets. From Eqn. (1) it follows that

$$\prod_{i=1}^{3} V_{i1}(\omega)V_{i2}(\omega)V_{i3}(\omega) = (+1)(+1)(+1) = +1. \tag{4}$$

Furthermore, from Eqns. (2) and (3) it follows that

$$\prod_{j=1}^{3} V_{1j}(\omega)V_{2j}(\omega)V_{3j}(\omega) = (+1)(+1)(-1) = -1. \tag{5}$$

But

$$\prod_{i=1}^{3} V_{i1}(\omega)V_{i2}(\omega)V_{i3}(\omega) = \prod_{j=1}^{3} V_{1j}(\omega)V_{2j}(\omega)V_{2j}(\omega), \tag{6}$$

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so we arrive at a contradiction and conclude that

$$(R_1 \cap R_2 \cap R_3) \cap (C_1 \cap C_2 \cap C_3) = \varnothing. \tag{7}$$

Thus far, we have not incorporated quantum theory other than to suggest the form for R_i and C_j. The standard proof of the Kochen-Specker theorem assumes that the functional relations held by the operators imply that

$$R_1 = R_2 = R_3 = C_1 = C_2 = C_3 = \Omega. \tag{8}$$

In other words, that Eqns. (1)–(3) hold for *all* $\omega \in \Omega$. If this is so, then Eqn. (7) implies that $\Omega = \varnothing$ and no (non-vacuous) noncontextual model is possible.

The proof is almost trivial, but it relies on one key assumption: the validity of Eqn. (8). This assumption may,in turn, be seen as a consequence of the so-called Functional Composition Principle [14]. Greenberger *et al.* [5] have noted that such equalities are overly restrictive, as the statistical nature of quantum mechanics requires only that each set have unit probability measure. This is true, but a more subtle observation, which has largely gone unnoticed, is that the relevant probability measure may in fact be different for each of the six sets. How this is possible, and what it implies, are the subject of the following sections.

UNDERSTANDING CONTEXTUALITY

Given an $\omega_0 \in \Omega$, we know that it will *not* be contained in at least one of the six row/column sets $R_1, \ldots, C_3$. If it happens to be the case that $\omega_0 \in R_1$, then $V_{11}(\omega_0)V_{12}(\omega_0)V_{13}(\omega_0) = +1$, as one might expect. If, however, it happens to be the case that $\omega_0 \notin C_1$, then we find, perhaps surprisingly, that $V_{11}(\omega_0)V_{21}(\omega_0)V_{31}(\omega_0) \neq +1$. Since ω_0 was arbitrary, the question arises why this is never observed.

One possible answer lies in a taking a closer look at the measurement process itself. Measuring an observable such as $\hat{V}_{11} = \hat{\sigma}_x \otimes \hat{1}$ requires a particular apparatus designed to interact with the system under interrogation. This process need not be benign. Suppose $\omega_0 \in \Omega$ describes the initial *microstate* of the system. (Here the term "system" may refer not only to the specific object of inquiry but also to the measuring device, surrounding environment, etc.) Interaction with the measuring apparatus may cause it to change its microstate from ω_0 to some $\varphi_{11}(\omega_0) \in \Omega$. Let us call this function the *measurement interaction map* (MIM). An observation then maps this microstate to some *macrostate* $g_{11}(\varphi_{11}(\omega_0)) \in \mathbb{R}$. If we consider an *ensemble* of initial microstates described by the probability measure P_0, then the ensemble after interaction becomes $P_0 \circ \varphi_{11}^{-1}$. The distribution for the macrostate is then $P_0 \circ \varphi_{11}^{-1} \circ g_{11}^{-1}$.

The situation is quite different in classical statistical mechanics, where the process of extracting a macrostate from the system is often ignored or irrelevant. Thus, $g_{11}(\omega_0)$ may be the true macrostate of the system prior to measurement, but we cannot observe it directly. Though we may *hypothesize* its existence, we may only *access* it via measurement. The process of measurement, however, results in our measuring $g_{11}(\varphi_{11}(\omega_0))$ which, depending upon the nature of φ_{11}, may not be the same as $g_{11}(\omega_0)$. (In the positivist philosophical tradition, one may go further and assert that $g_{11}(\omega_0)$, having no operational definition, is simply meaningless.)

If a subsequent measurement of, say, $\hat{V}_{12} = \hat{1} \otimes \hat{\sigma}_x$ is made, a similar process unfolds. The microstate $\varphi_{11}(\omega_0)$ is now transformed into $\varphi_{12}(\varphi_{11}(\omega_0))$, and the observed macrostate is $g_{12}(\varphi_{12}(\varphi_{11}(\omega_0)))$. The ensemble is transformed in a like manner from $P_0 \circ \varphi_{11}^{-1}$ to $P_0 \circ \varphi_{11}^{-1} \circ \varphi_{12}^{-1}$, and the joint distribution of the two measurements is therefore $P_0 \circ (g_{11} \circ \varphi_{11},\, g_{12} \circ \varphi_{12} \circ \varphi_{11})^{-1}$. Had we chosen to measure $\hat{V}_{21} = \hat{1} \otimes \hat{\sigma}_y$ instead of $\hat{1} \otimes \hat{\sigma}_x$, the observed macrostate would have been $g_{21}(\varphi_{21}(\varphi_{11}(\omega_0)))$, and the final ensemble would have been $P_0 \circ \varphi_{11}^{-1} \circ \varphi_{21}^{-1}$.

Of course, simultaneous measurements may also be possible, in which case the MIMs φ_{11}, φ_{12}, and φ_{13}, say, will be replaced by a single MIM, Φ_{Row1}. Similarly, φ_{11}, φ_{21}, and φ_{31}, will be replaced by a single Φ_{Col1}. In this case, the ensemble following a measurement of Row 1 will be $P_0 \circ \Phi_{\mathrm{Row1}}^{-1}$, while that of Column 1 will be $P_0 \circ \Phi_{\mathrm{Col1}}^{-1}$. Letting G_{ij} denote the macroscopic map corresponding to the operator $\hat{V}_{ij}$, a measurement of, say, Row 1 results in the values $G_{1j}(\Phi_{\mathrm{Row1}}(\omega_0))$ for $j = 1,2,3$, while a measurement of, say, Column 1 results in the values $G_{i1}(\Phi_{\mathrm{Col1}}(\omega_0))$ for $i = 1,2,3$. It is an academic matter whether one considers the random variable $V_{11} = G_{11}$ as noncontextual, with the contextual probability measures $P_{\mathrm{Row1}} = P_0 \circ \Phi_{\mathrm{Row1}}^{-1}$ and $P_{\mathrm{Col1}} = P_0 \circ \Phi_{\mathrm{Col1}}^{-1}$, or whether one considers $G_{11} \circ \Phi_{\mathrm{Row1}}$ and $G_{11} \circ \Phi_{\mathrm{Col1}}$ as contextual, with $P = P_0$ now noncontextual.

If such interactions do indeed exist, their effect must be consistent with the statistical predictions of quantum mechanics. It is desirable that they also satisfy our various intuitive notions of physical realism. For example, if a measurement of $\hat{V}_{11}$ is followed by a time-like separated measurement of either $\hat{V}_{12}$ or $\hat{V}_{21}$, we expect the outcome of

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the first measurement to be independent of which observable is chosen for the second measurement. Furthermore, the outcome of measuring $\hat{V}_{11}$ should not depend upon whether $\hat{V}_{12}$ is measured before $\hat{V}_{13}$ or $\hat{V}_{13}$ is measured before $\hat{V}_{12}$. Indeed, this should be true even if a measurement of $\hat{V}_{11}$ is followed by a measurement of an incompatible observable, such as $\hat{V}_{22}$. Since, according to the above description, the outcome of the first measurement is always $g_{11}(\varphi_{11}(\omega_0))$, all these conditions are clearly satisfied.

Now suppose $\hat{V}_{12}$ is measured first, followed by a time-like measurement of $\hat{V}_{11}$. Should we demand that the outcome of the latter, namely $g_{11}(\varphi_{11}(\varphi_{12}(\omega_0)))$, be identical to the outcome that would have resulted if $\hat{V}_{11}$ were measured first, namely $g_{11}(\varphi_{11}(\omega_0))$? This is certainly possible, but it is unreasonable and unwarranted to demand it. Although the joint distributions must be the same, i.e.,

$$P_0 \circ (g_{11} \circ \varphi_{11},\, g_{12} \circ \varphi_{12} \circ \varphi_{11})^{-1} = P_0 \circ (g_{11} \circ \varphi_{11} \circ \varphi_{12},\, g_{12} \circ \varphi_{12})^{-1}, \tag{9}$$

it is not necessary that $g_{11} \circ \varphi_{11} = g_{11} \circ \varphi_{11} \circ \varphi_{12}$, nor $g_{12} \circ \varphi_{12} = g_{12} \circ \varphi_{12} \circ \varphi_{11}$, in order for this to be true [1]. Thus, as long as the correct quantum statistics are reproduced, this constraint is also satisfied.

Finally, if a measurement of $\hat{V}_{11}$ is repeated, even after measurements of other compatible observables have been made, then quantum theory predicts (and observation dictates) that the same outcome must be obtained. Thus, for example, we require that $g_{11} \circ \varphi_{11} = g_{11} \circ \varphi_{11} \circ \varphi_{12} \circ \varphi_{11}$ P_0-almost everywhere. This, and similar relations, do place important constraints on a noncontextual hidden variable model and reflect, in part, the von Neumann postulate regarding wavefunction collapse.

Of course, if the measurements are simultaneous and not colocated (or merely space-like separated), then local realism imposes more severe constraints. If $\hat{\sigma}_x \otimes \hat{1}$ and $\hat{1} \otimes \hat{\sigma}_x$, say, represent spin measurements on two distant spin-1/2 particles, then we certainly would expect that relations such as $g_{11} \circ \varphi_{11} \circ \varphi_{12} = g_{11} \circ \varphi_{11}$ hold exactly and not just in their distributions. This, and similar relations, place severe constraints on the choice of MIMs consistent with a local hidden variable theory. A nonlocal, noncontextual theory of space-like separated measurements may, however, still possible. It remains to be shown whether a suitable set of MIMs, for time-like or space-like separated measurements, does in fact exist.

OPERATOR DECOMPOSABILITY AND SOME RECENT EXPERIMENTS

A frequent assumption made in discussions of contextuality is that, in a noncontextual theory, the outcome of a measurement on the product of two commuting operators, commonly written $v[\hat{A}\hat{B}]$, is equal to the product, $v[\hat{A}]v[\hat{B}]$, of the outcomes that would have been obtained had either of the two operators been measured individually [15]. This section critically examines the basis for this assumptions.

Consider the operators $\hat{\sigma}_x \otimes \hat{1}$ and $\hat{1} \otimes \hat{\sigma}_y$ from the magic square. For a given $\omega \in \Omega$ we may make the following associations:

$$v[\hat{\sigma}_x \otimes \hat{1}] = V_{11}(\omega) \tag{10a}$$

$$v[\hat{1} \otimes \hat{\sigma}_y] = V_{21}(\omega) \tag{10b}$$

$$v[\hat{\sigma}_x \otimes \hat{\sigma}_y] = V_{31}(\omega)\,. \tag{10c}$$

Now, the aforementioned assumption is that $V_{31}(\omega) = V_{11}(\omega)V_{21}(\omega)$. By definition, this equality holds if and only if $\omega \in C_1$, and this, in turn, will almost always hold whenever a measurement of Column 1 is performed. (Recall that, as per the discussion of the previous section, ω is interpreted as the post-measurement microstate.) If, however, a measurement of, say, Row 1 is performed, it may well be that $\omega \notin C_1$, in which case the assumption of equality may be false.

What can explain this dependency? Suppose $\hat{\sigma}_x \otimes \hat{1}$ measures the spin of a neutron in the x-direction, while $\hat{1} \otimes \hat{\sigma}_y$ measures whether it has passed through a certain path in an interferometer. The product $\hat{\sigma}_x \otimes \hat{\sigma}_y$ represents a measurement of the two quantities in a single experimental run. If the neutron passes through the beamsplitter first and later has its spin measured, then it is quite conceivable that the particular outcome of the spin measurement may have been different had the beamsplitter not been present (and hence the path measurement not been made). In other words, $V_{11}(\omega)$ might be different from $V_{31}(\omega)/V_{21}(\omega)$. In this section we will consider conditions under which one may legitimately decompose the noncontextual random variables into a product of constituent random variables. We will then turn to consider the implications of this decomposability property for some recent experimental tests of quantum contextuality.

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Decomposability

Each of the nine operators in the magic square may be written in terms of the four basic operators $\hat{\sigma}_x \otimes \hat{1}$, $\hat{1} \otimes \hat{\sigma}_x$, $\hat{\sigma}_y \otimes \hat{1}$, and $\hat{1} \otimes \hat{\sigma}_y$. This raises the question of whether it is possible to write each of the nine noncontextual random variables in terms of the four basic random variables V_{11}, V_{12}, V_{22}, V_{21}, which we shall denote here by X_1, X_2, Y_1, Y_2, respectively.

From the definitions of the six row/column sets, we note the following:

$$\omega \in R_1 \Rightarrow V_{13}(\omega) = X_1(\omega)X_2(\omega) \tag{11a}$$
$$\omega \in R_2 \Rightarrow V_{23}(\omega) = Y_1(\omega)Y_2(\omega) \tag{11b}$$
$$\omega \in R_3 \Rightarrow V_{33}(\omega) = V_{31}(\omega)V_{32}(\omega) \tag{11c}$$
$$\omega \in C_1 \Rightarrow V_{31}(\omega) = X_1(\omega)Y_2(\omega) \tag{11d}$$
$$\omega \in C_2 \Rightarrow V_{32}(\omega) = X_2(\omega)Y_1(\omega) \tag{11e}$$
$$\omega \in C_3 \Rightarrow V_{33}(\omega) = -V_{13}(\omega)V_{23}(\omega)\,. \tag{11f}$$

Since ω is not contained in at least one of these six sets, we have at most five equations to define the five remaining unknowns. If ω is contained in only four or fewer sets, then a full decomposition may not be possible. If, however, ω is contained in exactly five sets, then we have six possible, and distinct, decompositions, each corresponding to the single set which does not contain ω. These are given as follows.

First, suppose $\omega \notin R_3$ (i.e., $\omega \in R_1 \cap R_2 \cap C_1 \cap C_2 \cap C_3$). We cannot assume that $V_{33}(\omega) = V_{31}(\omega)V_{32}(\omega)$, but, since $\omega \in C_3$, we know that $V_{33}(\omega) = -V_{13}(\omega)V_{23}(\omega)$. Furthermore, since ω is contained in both R_1 and R_2, we may decompose $V_{13}(\omega) = X_1(\omega)X_2(\omega)$ and $V_{23}(\omega) = Y_1(\omega)Y_2(\omega)$. From this we conclude that

$$\omega \notin R_3 \Rightarrow V_{33}(\omega) = -X_1(\omega)X_2(\omega)Y_1(\omega)Y_2(\omega). \tag{12}$$

This provides a full decomposition of all nine random variables in terms of the four basic ones. Note that the above decomposition of $V_{33}(\omega)$ will be valid whenever $\omega \in C_3 \cap R_1 \cap R_2$.

Next, suppose that $\omega \notin C_3$ (but, again, is contained in the other five). Now $V_{33}(\omega)$ is decomposed as follows.

$$\omega \notin C_3 \Rightarrow V_{33}(\omega) = X_1(\omega)Y_2(\omega)X_2(\omega)Y_1(\omega). \tag{13}$$

Of course, the order of the four factors in unimportant. The above decomposition of $V_{33}(\omega)$ will be valid whenever $\omega \in R_3 \cap C_1 \cap C_2$.

If ω is supposed to be in all sets but R_1, then we can no longer decompose $V_{13}(\omega)$ as $X_1(\omega)X_2(\omega)$. Since $\omega \in R_3 \cap C_3$, however, we may deduce that

$$V_{31}(\omega)V_{32}(\omega) = -V_{13}(\omega)V_{23}(\omega), \tag{14}$$

and from this we conclude that

$$\omega \notin R_1 \Rightarrow V_{13}(\omega) = -X_1(\omega)X_2(\omega). \tag{15}$$

Note that $V_{33}(\omega)$ is decomposed according to Eqn. (13).

Proceeding in a similar manner, find

$$\omega \notin R_2 \Rightarrow V_{23}(\omega) = -Y_1(\omega)Y_2(\omega) \tag{16}$$
$$\omega \notin C_1 \Rightarrow V_{31}(\omega) = -X_1(\omega)Y_2(\omega) \tag{17}$$
$$\omega \notin C_2 \Rightarrow V_{32}(\omega) = -X_2(\omega)Y_1(\omega)\,. \tag{18}$$

We conclude that it may be possible to decompose any $V_{ij}(\omega)$ in terms of one or more of $X_1(\omega)$, $Y_1(\omega)$, $X_2(\omega)$, $Y_2(\omega)$, provided that ω is contained in all but one of the six row/column sets. The decomposition is not unique, however, as it depends upon which of the six row/column sets ω is not contained in. As discussed previously, this, in turn, will be determined by which operators one chooses to measure.

Huang Single-Photon Experiment

An early experiment to test noncontextuality was performed by Huang *et al.* [10] using photon path and polarization measurements. The concept of this experiment was based on the theoretical work of Simon *et al.* [9], who suggested

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a possible test of noncontextual hidden variable theories using two degrees of freedom (path and spin) for a single spin-1/2 particle. By measuring polarization in place of spin, the experimenters were able to perform an equivalent test using a single photon.

In the experiment, the photon is initially prepared in the entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|u\rangle \otimes |z+\rangle \; + \; |d\rangle \otimes |z-\rangle\right], \tag{19}$$

where the first component corresponds to the path (u = up, d = down) and the second component corresponds to the polarization ($z+$ = vertical, $z-$ = horizontal). The former correspond to eigenstates of $\hat{\sigma}_z \otimes \hat{1}$, while the latter correspond to eigenstates of $\hat{1} \otimes \hat{\sigma}_z$. Specifically,

$$\hat{\sigma}_z \otimes \hat{1} = \Big(|u\rangle\langle u| - |d\rangle\langle d|\Big) \otimes \hat{1} \tag{20a}$$

$$\hat{1} \otimes \hat{\sigma}_z = \hat{1} \otimes \Big(|z+\rangle\langle z+| - |z-\rangle\langle z-|\Big)\,. \tag{20b}$$

Following [10], these two operators will be denoted $\hat{Z}_1$ and $\hat{Z}_2$, respectively. In addition, the authors consider the operators $\hat{X}_1 = \hat{\sigma}_x \otimes \hat{1}$ and $\hat{X}_2 = \hat{1} \otimes \hat{\sigma}_x$. For the particular quantum state, $|\psi\rangle$, chosen by the experimenters, a measurement of either $\hat{Z}_1\hat{Z}_2 = \hat{\sigma}_z \otimes \hat{\sigma}_z$ or $\hat{X}_1\hat{X}_2 = \hat{\sigma}_x \otimes \hat{\sigma}_x$ always results in the value $+1$.

Based on the theoretical work of Simon *et al.*, the authors assert that, for systems prepared in this way, a noncontextual hidden variable theory would predict that any joint measurement of the commuting observables $\hat{Z}_1\hat{X}_2 = \hat{\sigma}_z \otimes \hat{\sigma}_x$ and $\hat{X}_1\hat{Z}_2 = \hat{\sigma}_x \otimes \hat{\sigma}_z$ must result in the same outcome for both observables. Quantum mechanics predicts that the outcomes are always different. The experimental task was to make such a measurement and ascertain whether the outcomes are indeed equal. The result was that only about 19% of the measurements showed identical outcomes for the two observables, in agreement with quantum mechanics and at variance with their prediction for a noncontextual theory.

The theoretical argument of Simon *et al.* is straightforward but logically flawed. As is common in discussions of noncontextuality, they associate with each operator $\hat{A}$ a predetermined value $v[\hat{A}]$. Thus, for example, $v[\hat{X}_1] = X_1(\omega)$ for some particular $\omega \in \Omega$. The interpretation of $v[\hat{X}_1]$ is, however, subtly different from that of $X_1(\omega)$, as the former is taken to be a preexisting value which remains unchanged by the process of measurement. By contrast, and in accordance with the present measurement disturbance interpretation, $X_1(\omega)$ is viewed here as the post-measurement outcome. The difference in the two interpretations lies in whether ω is viewed as the pre- or post-measurement hidden variable state. It is only in the former interpretation that a contradiction with quantum mechanics arises.

With this notation in mind, Simon *et al.* observe that, for the particular choice of $|\psi\rangle$ in Eqn. (19), the outcomes $v[\hat{Z}_1\hat{Z}_2] = +1$ and $v[\hat{X}_1\hat{X}_2] = +1$ always occur. They then make the following decomposability assumptions:

$$v[\hat{X}_1\hat{X}_2] = v[\hat{X}_1]v[\hat{X}_2] \tag{21a}$$

$$v[\hat{Z}_1\hat{Z}_2] = v[\hat{Z}_1]v[\hat{Z}_2] \tag{21b}$$

$$v[\hat{X}_1\hat{Z}_2] = v[\hat{X}_1]v[\hat{Z}_2] \tag{21c}$$

$$v[\hat{Z}_1\hat{X}_2] = v[\hat{Z}_1]v[\hat{X}_2]\,, \tag{21d}$$

from which one readily deduces that $v[\hat{Z}_1\hat{X}_2] = v[\hat{X}_1\hat{Z}_2]$.

The problem may be mapped to the Mermin-Peres magic square by interchanging $\hat{\sigma}_y$ and $\hat{\sigma}_z$. We may then define six analogous row/column sets, $R'_1, \ldots, C'_3$, and nine noncontextual random variables V'_{ij}. We then see that the decomposition is valid only if $\omega \in R'_1 \cap R'_2 \cap C'_1 \cap C'_2$. Since the actual experiment measures Row 3 (i.e., $\hat{X}_1\hat{Z}_2$ and $\hat{Z}_1\hat{X}_2$), we are guaranteed only that $\omega \in R'_3$. Therefore, if the measurement outcomes for the two observables are not equal, we merely conclude that $\omega \notin R'_1 \cap R'_2 \cap C'_1 \cap C'_2$ and the decomposition was invalid. Thus, the experimental results of Huang *et al.* do not rule out a noncontextual hidden variable interpretation.

Hasegawa Neutron Interferometry Experiment

In a recent experiment using neutron interferometry, Hasegawa *et al.* [13] claim to have obtained empirical confirmation of the Kochen-Specker result by showing violations of a certain Bell-like inequality. The authors consider a

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single-particle system for which two observables are measured: the spin (in a particular direction) and the path taken in the interferometer. In the experiment, the system is prepared in the Bell state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\downarrow\rangle \otimes |I\rangle \; - \; |\uparrow\rangle \otimes |II\rangle\right], \tag{22}$$

where the first component corresponds to the spin (in the z direction) and the second represents the interferometer path.

In each run of the experiment, exactly one of three observables is measured, represented here by the operators $\hat{\sigma}_x \otimes \hat{\sigma}_x$, $\hat{\sigma}_y \otimes \hat{\sigma}_y$, $\hat{\sigma}_z \otimes \hat{\sigma}_z$, where

$$\hat{\sigma}_z \otimes \hat{1} = \Big(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\Big) \otimes \hat{1} \tag{23a}$$

$$\hat{1} \otimes \hat{\sigma}_z = \hat{1} \otimes \Big(|I\rangle\langle I| - |II\rangle\langle II|\Big) \,. \tag{23b}$$

Note that, for this particular choice of $|\psi\rangle$, each such measurement will, theoretically, always result in an outcome of -1. For the experiment, multiple independent runs were performed to get statistical averages of each of these observables.

The resulting measured averages, denoted E_x, E_y, and E_z, are compared against the corresponding quantum predictions. The empirical test consists of comparing the empirical quantity

$$C' := 1 - E_x - E_y - E_z \tag{24}$$

against the quantum prediction

$$C_{\text{QM}} := 1 - \langle\psi|\left[\hat{\sigma}_x \otimes \hat{\sigma}_x + \hat{\sigma}_y \otimes \hat{\sigma}_y + \hat{\sigma}_z \otimes \hat{\sigma}_z\right]|\psi\rangle = 4, \tag{25}$$

and a value, $\overline{C}_{\text{NC}}$, predicted for a noncontextual hidden variable theory. In Eqn. (6) of reference [13], the authors predict that $|\overline{C}_{\text{NC}}| \leq 2$ based on a set of assumptions in Eqn. (2) of the same reference. The experiment yielded a measured value of $C' = 3.138 \pm 0.015$, which clearly violates their noncontextual prediction.

In fact, the noncontextual prediction is based on an invalid assumption regarding the decomposability of the measured observables. To see this, first note that the noncontextual prediction is

$$\overline{C}_{\text{NC}} = 1 - \sum_{i=1}^{3} \int V_{i3}(\omega)\, dP_{\text{Row}i}(\omega) = \int C_{\text{NC}}(\omega)\, dP_{\text{Col3}}(\omega), \tag{26}$$

where

$$C_{\text{NC}}(\omega) = 1 - V_{13}(\omega) - V_{23}(\omega) - V_{33}(\omega) \tag{27}$$

and, since the marginal distributions are noncontextual, for $i = 1,2,3$

$$\int V_{i3}(\omega)\, dP_{\text{Row}i}(\omega) = \int V_{i3}(\omega)\, dP_{\text{Col3}}(\omega). \tag{28}$$

Now, in [13] the authors assume the following decomposition.

$$C_{\text{NC}}(\omega) = 1 - X_1(\omega)X_2(\omega) - Y_1(\omega)Y_2(\omega) - X_1(\omega)X_2(\omega)Y_1(\omega)Y_2(\omega). \tag{29}$$

Such a decomposition holds if and only if $\omega \in R_1 \cap R_2 \cap R_3 \cap C_1 \cap C_2$ — i.e., $\omega \notin C_3$ and is contained in the other five sets. One readily verifies that $C_{\text{NC}}(\omega) \in \{-2, +2\}$ for every such ω. If only such values of ω are possible, the prediction $|\overline{C}_{\text{NC}}| \leq 2$ is obtained. (In fact, for the particular choice of $|\psi\rangle$ used, only $C_{\text{NC}}(\omega) = 2$ will be realized; hence, this decomposition implies $\overline{C}_{\text{NC}} = 2$.)

This assumption regarding ω is, however, unwarranted. Following the previous discussion, a measurement of V_{i3} would entail only that $\omega \in R_i \cup C_3$. It is certainly possible that every such ω is not contained in C_3, and contained in the other five sets, but this need not be so. It may be, for example, that $\omega \in R_1 \cap R_2 \cap C_3$, in which case we have the following, alternative decomposition.

$$C_{\text{NC}}(\omega) = 1 - X_1(\omega)X_2(\omega) - Y_1(\omega)Y_2(\omega) + X_1(\omega)X_2(\omega)Y_1(\omega)Y_2(\omega). \tag{30}$$

In this case, we find that $C_{\text{NC}}(\omega) = \{0, 4\}$, with 4 the only possible value given the choice of $|\psi\rangle$ used in the experiment. If the measurement process results only in such values of ω, then the noncontextual prediction agrees precisely with that of quantum mechanics. Thus, the experimental results of Hasegawa *et al.* do not rule out a noncontextual hidden variable interpretation.

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Proposed Experiment of Cabello *et al.*

In a related and more recent article, Cabello *et al.* [16] suggest an alternative method of testing quantum contextuality, again, using single-neutron interferometry. Using an experimental setup similar to that described in [13] and the same initial entangled state as Eqn. (22), they propose to perform a series five separate measurements of the following sets of observables: (1) $\hat{X}_1$, $\hat{X}_2$, (2) $\hat{Y}_1$, $\hat{Y}_2$, (3) $\hat{V}_{31}$, $\hat{X}_1$, $\hat{Y}_2$, (4) $\hat{V}_{32}$, $\hat{Y}_1$, $\hat{X}_2$, and, finally, (5) $\hat{V}_{31}$, $\hat{V}_{32}$. In each of the five experiments, the product of the observations is taken, and the results are averaged over multiple runs. Quantum mechanics predicts the following:

$$\langle\psi|\hat{X}_1\hat{X}_2|\psi\rangle = \langle\psi|\hat{V}_{13}|\psi\rangle = -1 \tag{31a}$$
$$\langle\psi|\hat{Y}_1\hat{Y}_2|\psi\rangle = \langle\psi|\hat{V}_{23}|\psi\rangle = -1 \tag{31b}$$
$$\langle\psi|\hat{V}_{31}\hat{X}_1\hat{Y}_2|\psi\rangle = 1 \tag{31c}$$
$$\langle\psi|\hat{V}_{32}\hat{Y}_1\hat{X}_2|\psi\rangle = 1 \tag{31d}$$
$$\langle\psi|\hat{V}_{31}\hat{V}_{32}|\psi\rangle = \langle\psi|\hat{V}_{33}|\psi\rangle = -1. \tag{31e}$$

In fact, quantum mechanics predicts that these results hold, not only on average, but for each individual (and ideal) measurement. Based on this observation, the authors assert that a noncontextual hidden variable theory should satisfy the following relations:

$$X_1(\omega)X_2(\omega) = -1 \tag{32a}$$
$$Y_1(\omega)Y_2(\omega) = -1 \tag{32b}$$
$$V_{31}(\omega)X_1(\omega)Y_2(\omega) = 1 \tag{32c}$$
$$V_{32}(\omega)Y_1(\omega)X_2(\omega) = 1 \tag{32d}$$
$$V_{31}(\omega)V_{32}(\omega) = -1, \tag{32e}$$

where the authors assume (implicitly) that these relations hold for all $\omega \in \Omega$. (See Eqns. (3a)–(3e) in [16].) They then note (correctly) that no single ω can possibly satisfy all five relations, since the product of the left-hand side is $+1$, while the product of the right-hand side is -1.

To understand this better, let us define the sets $B_i := \{\omega \in \Omega : V_{i1}(\omega)V_{i2}(\omega) = -1\}$ for $i = 1,2,3$. By definition, Eqn. (32a) is satisfied iff $\omega \in B_1$, Eqn. (32b) is satisfied iff $\omega \in B_2$, and Eqn. (32e) is satisfied iff $\omega \in B_3$. Furthermore, Eqns. (32c) and (32d) are satisfied iff $\omega \in C_1$ and $\omega \in C_2$, respectively. The impossibility of satisfying all five equations simultaneously implies that

$$B_1 \cap B_2 \cap B_3 \cap C_1 \cap C_2 = \varnothing. \tag{33}$$

This result is simular to that for the six row/column sets, which were found to have no common intersection point. It is the probabilities, however, that make this situation appear paradoxical.

For any quantum state, $P_{\text{Col1}}[C_1] = P_{\text{Col2}}[C_2] = 1$. Furthermore, for the particular form of $|\psi\rangle$ chosen, $P_{\text{Row}i}[B_i] = 1$ for $i = 1,2,3$. As has been argued previously, this does not, however, imply that any of these sets is identical to Ω. It is for this reason that the inequalities expressed in Eqns. (4) and (5) of Ref. [16] are invalid. Now, it is also the case that, quite generally, $P_{\text{Col3}}[C_3] = 1$ and $P_{\text{Row}i}[R_i] = 1$ for $i = 1,2,3$. Thus, $P_{\text{Row}i}[R_i \cap B_i] = 1$ and $P_{\text{Col3}}[C_3 \cap B_i] = 1$. In other words, a measurement of, say, Row 1 will result in a post-measurement microstate, ω, such that $X_1(\omega)X_2(\omega) = V_{13}(\omega) = -1$, while a measurement of Column 3 will result in a (possibily different) post-measurement microstate, ω', such that $V_{13}(\omega') = -1$, $V_{23}(\omega') = -1$, and $V_{33}(\omega') = -1$. The mere fact that, say, $V_{13}(\omega) = V_{13}(\omega')$ does not imply, for example, that $X_1(\omega) = X_1(\omega')$ or $X_2(\omega) = X_2(\omega')$.

Since a noncontextual hidden variable theory does not predict that all five equations are ever satisfied, a violation of the proposed inequalities will not rule out the possibility of a noncontextual hidden variable interpretation. Indeed, the Sequential Measurements model presented here will exactly reproduce the predicted quantum results.

SUMMARY AND CONCLUSIONS

In this paper, the question of quantum contextuality in the Mermin-Peres square has been considered. It was shown that a deterministic, noncontextual hidden variable model of this problem is not inconsistent with quantum mechanics,

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contrary to the Kochen-Specker theorem. The key idea proposed is that measurement outcomes are deterministic functions of hidden variable states which may be modified through the process of measurement. Thus, the (inaccessible) pre-measurement value of an observable may be different from its post-measurement outcome.

The flaw in the Kochen-Specker theorem was found to lie in the assumption of the Functional Composition Principle. Contrary to this assumption, quantum mechanics demands only that the set of hidden variable states over which a given functional relation among commuting operators holds must have a probability of 1 with respect to a distribution corresponding to the particular set of commuting operators. This alone merely shifts the question of contextuality from the random variables to the probability measures. One way to understand how such an apparent contextual dependency may arise is to suppose that the hidden variable states are modified through interaction between the measuring device and the system under interrogation.

Finally, empirical tests of quantum contextuality in two recent experiments, Huang *et al.* [10] and Hasegawa *et al.* [13], and one proposed experiment by Cabello *et al.* [16] were considered. In all cases, it was found that the authors' predictions for a noncontextual theory, based implicitly on the Functional Composition Principle, were invalid due to an improper decomposition of the random variables corresponding to each operator. Neither experiment was found capable of ruling out a noncontextual hidden variable interpretation. As both experiments used measurements of path and polarization/spin on a single photon/neutron, neither was capable of ruling out a local hidden variable interpretation either.

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