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THE LOGICALLY SIMPLEST FORM OF THE INFINITY AXIOM

FRANCO PARLAMENTO AND ALBERTO POLICRITI

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ABSTRACT. We provide a way of expressing the existence of infinite sets in the first order set theoretic language, which is of the lowest possible logical complexity.

Let $\mathcal{L}_\in$ be the first order language with identity, based on the membership relation $\in$, and assume the axioms of Zermelo-Fraenkel (ZF). A formula of $\mathcal{L}_\in$ is restricted if it does not contain quantifiers except for the restricted quantifiers $\forall x \in y$ and $\exists x \in y$. Restricted formulas which, under ZF, are satisfiable, but are not satisfied by finite sets, are provided by the usual ways of formulating the axiom of infinity. In fact that a set a is inductive, i.e. contains the empty set and is closed under the successor operation taking a set b into $b \cup \{b\}$, can be expressed by the restricted formula:

$$\exists x \in a \forall u \in x (u \notin x) \wedge \forall x \in a \exists y \in a \forall u \in x$$
$$\forall v \in y (u \in y \wedge x \in y \wedge (v \in x \vee v = x)).$$

Note that this formula contains alternations of quantifiers of the form $\forall\exists\forall$. A logically simpler example is obtained by expressing that a is a limit ordinal, i.e. a nonzero nonsuccessor ordinal, through the following restricted formula, which only involves an alternation of quantifiers of the form $\forall\exists$:

$$\forall x \in a \forall u \in x (u \in a) \wedge \forall x \in a \forall y \in a (x \in y \vee x = y \vee y \in x)$$
$$\wedge \exists x \in a (x = x) \wedge \forall x \in a \exists y \in a (x \in y).$$

Restricted formulas of the above kind with two free variables are provided by the notion of finiteness due to Dedekind, Russell-Whitehead and Tarski (see [**2** and **3**]). In fact "b is a 1-1 but nononto function from a into a", "b is an inductive family of subset of a and $a \notin b$", and "b is a family of subset of a, which does not contain a maximal element with respect to inclusion" are notions that are readily expressed in ZF by restricted formulas involving alternations of quantifiers of the forms $\forall\exists\forall$, $\forall\exists$, and $\forall\exists\forall$ respectively.

A lower bound on the complexity of the satisfiable but not finitely satisfiable restricted formulas follows from [**1**], which gives a decision procedure to test for any given restricted formula involving only universal quantifiers without nestings of bound variables (i.e. no subformula of the form $\forall x_1 \in y_1 \cdots \forall x_n \in y_n \varphi$, where some x_i is a y_j, is allowed), whether there are sets satisfying it or not. As a by-product of such a procedure, one has that if a nonnested universal restricted formula is satisfiable at all, then it is already satisfied by suitable hereditarily finite sets.

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That this reflection property, which immediately extends to the restricted formulas involving only existential quantifiers, held also for the full class of the restricted formulas involving only universal quantifiers, was then commonly expected. On the contrary we have the following example, which shows that the purely universal restricted formulas having two free variables, with nestings of bound variables of the lowest possible level, suffice to express the existence of infinite sets.

PROPOSITION (ZF). *The following formula $\varphi(a,b)$ is satisfiable, but it is not finitely satisfiable; more precisely if either a or b is finite, then $\varphi(a,b)$ does not hold*:

$$\begin{aligned}&a \neq b \wedge a \notin b \wedge b \notin a \wedge \forall x \in a \forall u \in x (u \in b) \wedge \forall x \in b \forall u \in x (u \in a)\\ &\wedge \forall x \in a (x \notin b) \wedge \forall x, y \in a \forall z, w \in b (z \in x \wedge x \in w \wedge w \in y \to z \in y)\\ &\wedge \forall x, y \in b \forall z, w \in a (z \in x \wedge x \in w \wedge w \in y \to z \in y).\end{aligned}$$

PROOF. Let f_n and g_n be the sequences of sets defined by recursion on ω, such that $f_0 = \varnothing$, $g_n = \{f_0, \dots, f_n\}$, $f_{n+1} = \{g_0, \dots, g_n\}$, and let $\omega' = \{f_0, f_1, \dots\}$ and $\omega'' = \{g_0, g_1, \dots\}$. It is straightforward to check that ω' and ω'' satisfy φ. We prove the rest of our claim by showing that

$$\varphi(a,b) \to (\omega' \subseteq a \wedge \omega'' \subseteq b) \vee (\omega'' \subseteq a \wedge \omega' \subseteq b).$$

Assume that a and b satisfy φ. First note that a and b are both nonempty. For, assume $a = \varnothing$. Since $a \neq b$, $b \neq \varnothing$. Since $a \notin b$, $\varnothing \notin b$. Therefore b contains a nonempty set c, but then $\forall x \in b \forall u \in x (u \in a)$ implies that every element of c is a member of a, contrary to the assumption $a = \varnothing$. Thus $a \neq \varnothing$. Symmetrically $b \neq \varnothing$. Furthermore either $\varnothing \in a$ and $\{\varnothing\} \in b$, or $\{\varnothing\} \in a$ and $\varnothing \in b$. For, let c be an element of minimal rank in a. If $c \neq \varnothing$, let d be any element of c. If d were different from $\varnothing$, then every element of d would be a member of a, against the minimality of c. Therefore $c = \{\varnothing\}$, from which it follows that $\varnothing \in b$. On the other hand if $c = \varnothing$, as above, one verifies that the only element of minimal rank in b is $\{\varnothing\}$, since by the fact that a and b are disjoint, one cannot have $\varnothing \in b$. Let us assume, for example, that $\varnothing \in a$ and $\{\varnothing\} \in b$. By induction on n we now prove that:

(i) the set of elements of a of rank$\leq 2n$ is $\{f_0, f_1, \dots, f_n\}$,

(ii) the set of elements of b of rank$\leq 2n+1$ is $\{g_0, g_1, \dots, g_n\}$.

The case $n = 0$ follows immediately from our assumption that $\varnothing \in a$ and $\{\varnothing\} \in b$. Assume (i) and (ii) hold for n. a must contain some element beside $f_0, \dots, f_n$; otherwise a would be equal to g_n and thus a member of b, contrary to $\varphi(a,b)$. Let c be an element of a of minimal rank different from $f_0, \dots, f_n$. Thus $\operatorname{rank}(c) > 2n$. Every element of an element of c belongs to a and has rank less than rank(c), and is therefore in $\{f_0, \dots, f_n\}$; hence it has rank at most equal to $2n$. Therefore every element of c has at most rank $2n+1$, and, in turn, c has at most rank $2n+2$. Now we note that rank(c) cannot be $2n+1$; otherwise in b there would be an element of rank $2n$, against the fact that $g_0, \dots, g_n$ are the only elements of b with rank $\leq 2n+1$, and they all have odd rank. Therefore the rank of c is precisely $2n+2$. This means that c has an element of rank $2n+1$. Since such an element has to belong to b, it has to coincide with g_n. Since for every $i < n$, $g_i \in f_n$, $f_n \in g_n$ and $g_n \in c$, the penultimate clause in $\varphi(a,b)$ implies that $\{g_0, \dots, g_n\} \subseteq c$. Finally c cannot contain any element different from $g_0, \dots, g_n$, because all of its elements belong to b and have rank $\leq 2n+1$ and (ii) applies. Therefore $c = f_{n+1}$ and (i) holds for $n+1$. Assuming this, a similar argument shows, on the basis of the induction hypothesis, that also (ii) holds for $n+1$. Clearly from (i) and (ii) it follows that $\omega' \subseteq a$, and $\omega'' \subseteq b$. A symmetric argument shows that if $\varnothing \in b$ and $\{\varnothing\} \in a$, then $\omega'' \subseteq a$ and $\omega' \subseteq b$. $\square$

It is rather easy to see that if any of the conjuncts of φ is dropped, then there are hereditarily finite sets satisfying the resulting formula. Furthermore, the number of free variables cannot be reduced to one. For, suppose a is a set which satisfies a purely universal restricted formula ψ with a single free variable. By the axiom of foundation there is a finite set $b = \{a_0, a_1, \ldots, a_n\}$ such that $a_0 = \varnothing$, for every $i < n$, $a_i \in a_{i+1}$, and $a_n = a$. The membership relation is clearly extensional on b; thus the Mostowski collapse of $(b, \in)$ is an isomorphism between b and a set which is transitive and hereditarily finite. The image of a under such an isomorphism is readily seen to satisfy ψ. Finally let us note that $\exists a \exists b \varphi(a,b)$ is equivalent to the usual formulation of the axiom of infinity, stating the existence of an inductive set, over the remaining axioms of ZF. ZF proves the existence of the two sets ω' and ω'' which satisfy φ. Conversely, working in ZF-infinity, let us assume that there are sets a and b such that $\varphi(a,b)$ holds. In ZF-infinity one can define the well-founded "class" N of the natural numbers and, as in the proof of the Proposition, introduce by recursion on N two operations F and G, and show that the ranges of F and G on N are included in a and b (or b and a) respectively. Since F and G are 1-1 on N, by the axiom of replacement it follows that N is in fact a set; obviously such a set is inductive. By the above discussion we can therefore claim that $\exists a \exists b \varphi(a,b)$ expresses the axiom of infinity in the logically simplest possible way.

REMARK. The formula φ above is specified by sets of rank ω. At the price of using a greater number of free variables, it is easy to build, for every natural number k, a purely universal restricted formula which is satisfiable, but not by sets of rank less than $\omega \cdot k$. For example, one can use φ to obtain first a set, say a, of rank at least ω, then an additional free variable to characterize $\{a\}$, and finally two more free variables a' and b' to describe sets which are related to a and $\{a\}$ as ω' and ω'' are related to $\varnothing$ and $\{\varnothing\}$. The resulting formula is satisfiable but not by sets of rank less than $\omega \cdot 2$. It would be interesting to know if formulas of this kind can be built with a bounded number of free variables, as well as whether there are universal restricted formulas which are satisfiable but not by sets of rank less than ω^2.

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DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI TORINO, VIA CARLO ALBERTO 10, 10100 TORINO, ITALY

Current address (Franco Parlamento): Dipartimento di Matematica e Informatica, Via Zanon 6, 31100 Udine, Italy

Current address (Alberto Policriti): Department of Computer Science, Courant Institute of Mathematical Sciences, New York University, New York, New York 10012