# HARMONIC ANALYSIS OF FRACTAL MEASURES 

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#### Abstract

We consider affine systems in $\mathbb{R}^{n}$ constructed from a given integral invertible and expansive matrix $R$, and a finite set $B$ of translates, $\sigma_{b} x:=R^{-1} x+b$; the corresponding measure $\mu$ on $\mathbb{R}^{n}$ is a probability measure and fixed by the selfsimilarity $\mu=|B|^{-1} \sum_{b \in B} \mu \circ \sigma_{b}^{-1}$. There are two a priori candidates for an associated orthogonal harmonic analysis : (i) the existence of some subset $\Lambda$ in $\mathbb{R}^{n}$ such that the exponentials $\left\{e^{i \lambda \cdot x}\right\}_{\lambda \in \Lambda}$ form an orthogonal basis for $L^{2}(\mu)$; and (ii) the existence of a certain dual pair of representations of the $C^{*}$-algebra $\mathcal{O}_{N}$ where $N$ is the cardinality of the set $B$. (For each $N$, the $C^{*}$-algebra $\mathcal{O}_{N}$ is known to be simple; it is also called the Cuntz-algebra.) We show that, in the "typical" fractal case, the naive version (i) must be rejected; typically the orthogonal exponentials in $L^{2}(\mu)$ fail to span a dense subspace. Instead we show that the $C^{*}$-algebraic version of an orthogonal harmonic analysis, viz., (ii), is a natural substitute. It turns out that this version is still based on exponentials $e^{i \lambda \cdot x}$, but in a more indirect way. (See details in Section 5 below.) Our main result concerns the intrinsic geometric features of affine systems, based on $R$ and $B$, such that $\mu$ has the $C^{*}$-algebra property (ii). Specifically, we show that $\mu$ has an orthogonal harmonic analysis (in the sense (ii)) if the system $(R, B)$ satisfies some specific symmetry conditions (which are geometric in nature). Our conditions for (ii) are stated in terms of two pieces of data: (a) a unitary generalized Hadamard-matrix, and (b) a certain system of lattices which must exist and, at the same time, be compatible with the Hadamard-matrix. A partial converse to this result is also given. Several examples are calculated, and a new maximality condition for exponentials is identified.


## 1. Introduction

The present paper continues work by the coauthors in [JP3-6], and it also provides detailed proofs of results announced in [JP4]. In addition we have new results going beyond those of the announcement [JP4]. We consider a new class of selfsimilarity fractals $\bar{X}$, each $\bar{X}$ with associated fractal selfsimilar measure $\mu$, such that $L^{2}(\mu)$ has an orthogonal harmonic analysis in the sense of $C^{*}$-algebras (see (ii) below). This possibility is characterized with geometric axioms on the pair

[^0]$(\bar{X}, \mu)$. It is known since $[\mathrm{St} 3-4]$ that $\mu$ is typically singular (in the fractal case), and that in general only an asymptotic Plancherel type formula can be expected in the sense of [Bes]. Our present approach is based instead on $C^{*}$-algebra theory. In particular, we use the $C^{*}$-algebras $\mathcal{O}_{N}$ of Cuntz $[\mathrm{Cu}]$, and we give the orthogonal decompositions in terms of a dual pair of representations of $\mathcal{O}_{N}$ where $N$ denotes the number of translations in the affine system which determines $\mu$.

For an orthogonal harmonic analysis, the following three possibilities appear $a$ priori as natural candidates:
(i) the existence of a subset $\Lambda$ in $\mathbb{R}^{n}$ such that the exponentials $e_{\lambda}(x):=e^{i \lambda \cdot x}$ (indexed by $\lambda \in \Lambda$ ) form an orthogonal basis in $L^{2}(\mu)$;
(ii) the existence of a dual system of representations of some $C^{*}$-algebra $\mathcal{O}_{N}$ say, ( $N=$ the cardinality of $B$ ), such that one representation is acting affinely in $x$-space, and the other (dually) in frequency-space (where the frequency variable is represented by $\lambda$ in the above exponentials $e^{i \lambda \cdot x}$ ); and finally
(iii) one might base the harmonic analysis on an orthogonal basis of polynomials in $n$ variables obtained from the monomials $x^{\alpha}:=x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{n}^{\alpha_{n}}$ (where $\alpha=$ $\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ is a multi-index, $\left.\alpha_{i}=0,1,2, \ldots, 1 \leq i \leq n\right)$, by the familiar GramSchmidt algorithm.

But it is immediate that both of the possibilities (i) and (iii) lack symmetry in the variables $x$ and $\lambda$. Moreover, it turns out that (i) must be ruled out also for a more serious reason. We show in Sections 6-7 below that, for the "typical" fractal measures $\mu$, none of the orthogonal sets $\left\{e_{\lambda}\right\}$ in $L^{2}(\mu)$ will in fact span a dense subspace. Specifically, there is a canonical maximally orthogonal $\left\{e_{\lambda}\right\}$ system such that a finite set of "translates" (details in Section 5) of it does give us a dense subspace. It is this extra operation (i.e., "spreading out" the orthogonal
exponentials) which leads to our dual pair of representations of the algebra $\mathcal{O}_{N}$.

It also turns out that case (ii) is a natural extension of our orthogonality condition, studied earlier in [JP2] for $L^{2}(\Omega)$, now with $\Omega$ some subset in $\mathbb{R}^{n}$ with finite positive Lebesgue measure, and $L^{2}(\Omega)$ considered as a Hilbert space with the restricted Lebesgue measure. For the case, when $\Omega$ is further assumed open and connected, we showed, in [JP2] and [Pe], that (i) holds (i.e., there is a set $\Lambda$ such that $\left\{e_{\lambda}\right\}_{\lambda \in \Lambda}$ forms an orthogonal basis in $\left.L^{2}(\Omega)\right)$ iff the corresponding symmetric operators $\left\{\sqrt{-1} \frac{\partial}{\partial x_{j}}\right\}_{j=1}^{n}$, defined on $C_{c}^{\infty}(\Omega)$, have commuting selfadjoint operator extensions acting in $L^{2}(\Omega)$. It is well known that extension theory for symmetric operators is given by von Neumann's deficiency spaces. But, even when individual selfadjoint extensions exist for commuting symmetric operators, such extensions are typically non-commuting. Hence, we expect that, also for our $L^{2}(\mu)$ analysis, there will be distinct symmetry conditions and selfadjointness conditions.

For our present case, the pair $(R, B)$ is specified as above, the affine maps are given by $\sigma_{b} x=R^{-1} x+b$, and indexed by points $b$ in the finite set $B$. We get the measure $\mu$, and the Hilbert space $L^{2}(\mu)$, by a general limit construction which we show must start with some $L^{2}(\Omega)$ example as discussed. But, for $L^{2}(\mu)$, we show that the analogous symmetry condition is related to a certain lattice configuration in $\mathbb{R}^{n}$ (see Lemma 4.1 below), whereas the analogous selfadjointness now corresponds to a spectral pairing between $B$ and a second subset $L$ in $\mathbb{R}^{n}$, of same cardinality, such that the $N$ by $N$ matrix $\left\{e^{i b \cdot \ell}\right\}$, (for $b \in B, \ell \in L$ ), forms a so-called unitary generalized Hadamard matrix, see [SY]. Then this matrix, together with the lattice configuration leads to a dual pair of representations, as sketched above and worked out in detail below. The two representations will act naturally on $L^{2}(\mu)$ and provide
a non- commutative harmonic analysis with a completely new interpretation of the classical time- frequency duality (see e.g., $[\mathrm{HR}]$ ), of multivariable Fourier series.

When our "symmetry" condition is satisfied, we get a dual pair of self-similar measures, $\mu_{B}$ and $\mu_{L}$, and this pair is used in the proof of our structure theorem. Many examples are given illustrating when the "symmetry" holds and when it doesn't. A connection is made to classical spectral duality, see e.g., [JP1-3].

## 2. Basic Assumptions

We consider affine operations in $\mathbb{R}^{n}$ where $n$ is fixed; the case $n=1$ is also included, and the results are non-trivial and interesting also then. A system $s$ in $\mathbb{R}^{n}$ will consist of a quadruple $(R, B, L, K)$ where $R \in G L_{n}(\mathbb{R}), B$ and $L$ are finite subsets in $\mathbb{R}^{n}$, and both of them are assumed to contain the origin $O$ in $\mathbb{R}^{n}$; finally $K$ is a lattice in $\mathbb{R}^{n}$, i.e., a free additive group with $n$ generators. It will be convenient occasionally to identify a fixed lattice with a matrix whose columns are then taken to be a set of generators for the lattice in question. It is known that generators will always form a linear basis for the vector space (see e.g., [CS]); and it follows that the matrix is then in $G L_{n}(\mathbb{R})$.

With the assumptions (to be specified), it turns out that we may apply Hutchinson's theorem $[\mathrm{Hu}]$ to the affine system $\left\{\sigma_{b}\right\}_{b \in B}$ given by

$$
\begin{equation*}
\sigma_{b} x:=R^{-1} x+b, \quad x \in \mathbb{R}^{n} \tag{2.1}
\end{equation*}
$$

There is a unique probability measure $\mu$ on $\mathbb{R}^{n}$ satisfying

$$
\begin{equation*}
\mu=|B|^{-1} \sum_{b \in B} \mu \circ \sigma_{b}^{-1}, \tag{2.2}
\end{equation*}
$$

which amounts to the condition

$$
\begin{equation*}
\int f d \mu=|B|^{-1} \sum_{b} \int f \circ \sigma_{b} d \mu \tag{2.3}
\end{equation*}
$$

for all $\mu$-integrable functions $f$ on $\mathbb{R}^{n}$. For the matrix $R$, we assume that some positive integral power of it has all eigenvalues in $\{\lambda \in \mathbb{C}:|\lambda|>1\}$, and we refer to this as the expansive property for $R$. (It is actually equivalent to the same condition for $R$ itself.) The use of $[\mathrm{Hu}]$ requires the so called open-set- condition which turns out to hold when our system $s$ has a symmetry property which we proceed to describe. We then also have the following compact subset $\bar{X}$, defined as the closure (in $\mathbb{R}^{n}$ ) of the set of vectors $x$ with representation

$$
\begin{equation*}
x=\sum_{i=0}^{\infty} R^{-i} b_{i}, \quad b_{i} \in B . \tag{2.4}
\end{equation*}
$$

If $|B|<|\operatorname{det} R|$, where $|B|$ denotes the cardinality of $B$, then the fractal dimension of $\bar{X}$ will be less than the vector space dimension $n$ of the ambient $\mathbb{R}^{n}$. (See e.g., [Ke] for details on this point.) In general, the measure $\mu$ is supported by $\bar{X}$, and we may identify $L^{2}(\mu)$ with $L^{2}(\bar{X}, \mu)$ as a Hilbert space. We will refer to $\bar{X}$ as the "fractal" even in the cases when its dimension may in fact be integral, and the "fractal" representation will be understood to be (2.4). Occasionally, we will write $\bar{X}(B)$ to stress the digit-set $B$.

## 3. Generalized Hadamard Matrices

The two sets $B$ and $L$ from the system came up in our previous work (see [JP24] and [JP6]) on multivariable spectral theory. The condition we wish to impose on two sets $B, L$ amounts to demanding that the corresponding exponential matrix

$$
\begin{equation*}
\left(e^{i 2 \pi b \cdot \ell}\right) \tag{3.1}
\end{equation*}
$$

is generalized Hadamard, see [SY]. The term $b \cdot \ell$ refers to the usual dot-product in $\mathbb{R}^{n}$. It will be convenient to abbreviate the matrix entries as, $\langle b, \ell\rangle:=e^{i 2 \pi b \cdot \ell}$. Since $0 \in B$ and $0 \in L$ by assumption, one column, and one row, in the matrix $(\langle b, \ell\rangle)_{B L}$ consists of a string of ones. Let the matrix be denoted by $U$ : We say that it is generalized Hadamard if the two sets $B$ and $L$ have the same cardinality, $N$ say, and if

$$
\begin{equation*}
U^{*} U=N I_{N} . \tag{3.2}
\end{equation*}
$$

It follows from this that then also $U U^{*}=N I_{N}$. (This is just saying, of course, that the complex $N$ by $N$ matrix, $N^{-1 / 2} U$ is unitary in the usual sense.)

We noted in [JP6] that the harmonic analysis of type (ii) is based on this kind of Hadamard matrices. (The matrices also have an independent life in combinatorics.) It turns out that the matrices are known for $N$ up to $N=4$. We will show, in Section 7 below, that this then leads to a classification of the simplest affine fractals (as specified) such that the analysis (ii) exists. We say that two matrices $U$ of the form (3.2) are equivalent, if $N$ is the same for the two matrices, and if one arises from the other by multiplication on the left, or right, with a permutation matrix, or with a unitary diagonal matrix. We now list below (without details) the inequivalent cases of type (3.2) for $N \leq 4$. (For higher $N$, such a classification is not known.) After our present preprint was circulated, we learned that the $N \leq 4$ classification had also been found independently, see references [Cr] and [Wer]. The purpose of our examples in Section 7 is to show how the equivalence classes of (3.2) lead to distinct examples of fractal measures $\mu$, and how the different $U$-matrices lead to different dual pairs of representations.

We will postpone to a later paper a rigorous classification of the different systems
$(R, B)$, and of the corresponding type (ii) harmonic analysis of $L^{2}(\mu)$. But we feel that the $N \leq 4$ examples are sufficiently interesting in their own right. They also serve to illustrate the technical points in our (present) two main theorems.

Notice the $2 \pi$ factor in the exponential (3.1) above. It is put in for technical convenience only.

Remark 3.1. If we pick the string of ones as first row and first column, then the possibilities for $U$ when $N=2$ are

$$
\left(\begin{array}{cc}
1 & 1  \tag{3.3}\\
1 & -1
\end{array}\right) ;
$$

for $N=3$,

$$
\left(\begin{array}{lll}
1 & 1 & 1  \tag{3.4}\\
1 & \zeta & \bar{\zeta} \\
1 & \bar{\zeta} & \zeta
\end{array}\right)
$$

where $\zeta$ is a primitive 3 rd root of 1 ; and for $N=4$,

$$
\left(\begin{array}{cccc}
1 & 1 & 1 & 1  \tag{3.5}\\
1 & 1 & -1 & -1 \\
1 & -1 & u & -u \\
1 & -1 & -u & u
\end{array}\right)
$$

where $|u|=1$, up to equivalence for generalized Hadamard matrices, see e.g. [SY].

## 4. Selfadjoint Systems

Corresponding to the affine mappings (2.1) for a given system $s=(R, B, L, K)$ we have

$$
\begin{equation*}
\tau_{\ell}(t):=R^{*} t+\ell, \quad t \in \mathbb{R}^{n} \tag{4.1}
\end{equation*}
$$

and the inverses

$$
\begin{equation*}
\tau_{\ell}^{-1}(t):=R^{*^{-1}}(t-\ell) \tag{4.2}
\end{equation*}
$$

where the translations for (4.2) are given by the vectors

$$
\begin{equation*}
b^{\prime}:=-R^{*^{-1}}(\ell), \quad \text { as } \ell \text { varies over } L . \tag{4.3}
\end{equation*}
$$

Here $R$ is an $n$ by $n$ matrix as specified above, $B$ and $L$ are finite subsets in $\mathbb{R}^{n}$ both containing $O$, and $K$ is a rank $n$ lattice. The invariance $R(K) \subset K$ will be assumed, and we summarize this by the notation $K \in \operatorname{lat}(R)$.

We introduce the dual system $s^{\circ}$ defined by $s^{\circ}=\left(R^{*}, B^{\prime}, L^{\prime}, K^{\circ}\right)$ where $K^{\circ}$ is the dual latttice,

$$
\begin{equation*}
B^{\prime}:=-R^{*^{-1}}(L) \tag{4.4}
\end{equation*}
$$

and

$$
\begin{equation*}
L^{\prime}:=-R(B) \tag{4.5}
\end{equation*}
$$

The system $s$ is said to be symmetric if

$$
\begin{equation*}
R(B) \subset K \tag{4.6}
\end{equation*}
$$

and if $K \in \operatorname{lat}(R)$; and it is said to be selfadjoint if both $s$ and $s^{\circ}$ are symmetric.
(Also notice that, in general, we have $s^{\circ \circ}=s$ when $s$ is an arbitrary system.)
(The definitions are analogous to familiar ones for closed operators $S$ with dense domain in Hilbert space, see e.g., [Fu]: the operator $S$ is said to be symmetric if $S \subset S^{*}$, where $S^{*}$ denotes the adjoint, and the inclusion refers to inclusion of graphs. It follows that $S$ is selfadjoint, i.e., $S=S^{*}$, iff both $S$ and $S^{*}$ are symmetric.)

We shall need the fact that $B$ embeds into the of coset space $R^{-1}(K) / K$ when additional orthogonality is assumed:

Lemma 4.1. Consider a system $s=(R, B, L, K)$ in $\mathbb{R}^{n}$ with the matrix $R$, the two finite subsets $B$ and $L$ in $\mathbb{R}^{n}$, and a lattice $K$ as described above. Assume that
$L \subset K^{\circ}$ and that the two conditions (3.2), i.e., that Hadamard property, and (4.6) hold, where $K^{\circ}$ is the dual lattice in $\mathbb{R}^{n}$. Then it follows that different points in $B$ represent distinct elements in the finite group $R^{-1}(K) / K$.

Proof. Suppose $b \neq b^{\prime}$ in $B$. Then $\sum_{\ell \in L}\left\langle\ell, b-b^{\prime}\right\rangle=0$ using (3.2). But, for all $k \in K$, we also have $\sum_{\ell \in L}\langle\ell, k\rangle=|L|$, and it follows that $b-b^{\prime} \notin K$; i.e., the $R^{-1}(K) / K$ cosets are distinct.

We shall assume in the following that our given system is of Hadamard type, i.e., that $|B|=|L|$ and that the matrix (3.1) formed from $(B, L)$ is generalized Hadamard, see (3.2) above.

The following lemma is also simple but useful.

Lemma 4.2. A given system $s=(R, B, L, K)$ in $\mathbb{R}^{n}$ is selfadjoint if and only if the following three conditions hold:
(i) $K \in \operatorname{lat}(R)$,
(ii) $R(B) \subset K$, and
(iii) $L \subset K^{\circ}$.

Proof. A calculation shows that $K \in \operatorname{lat}(R)$ holds iff $K^{\circ} \in \operatorname{lat}\left(R^{*}\right)$. For the system $s^{\circ}$ to be symmetric, we need $R^{*}\left(B^{\prime}\right) \subset K^{\circ}$, and that is equivalent to (iii) by virtue of formula (4.4). So both $s$ and $s^{\circ}$ are symmetric precisely when (i)-(iii) hold.

Remark 4.3. (Classical Systems) In [JP2], we considered the following spectral problem for measurable subsets $\Omega \subset \mathbb{R}^{n}$ of finite positive Lebesgue measure, i.e., $0<m(\Omega)<\infty$ where $m=m_{n}$ denotes the $\mathbb{R}^{n}$ - Lebesgue-measure: Let $\Omega$ be given, when is there a subset $\Lambda \subset \mathbb{R}^{n}$ s.t. the exponentials

$$
\begin{equation*}
e_{\lambda}(x)=\langle\lambda, x\rangle=e^{i 2 \pi \lambda \cdot x} \tag{4.7}
\end{equation*}
$$

indexed by $\lambda \in \Lambda$, form an orthonormal basis in $L^{2}(\Omega)$ with inner product

$$
\begin{equation*}
m(\Omega)^{-1} \int_{\Omega} \overline{f(x)} g(x) d x ? \tag{4.8}
\end{equation*}
$$

The problem (in its classical form) goes back to [Fu], and it is motivated by a corresponding one for commuting vector fields on manifolds with boundary, see also [Jo1-2], [Pe], and [JP2].

We showed that the general problem may be "reduced" (by elimination of "trivial" systems) to a special case when the pair $(\Omega, \Lambda)$ is such that the polar

$$
\begin{equation*}
\Lambda^{\circ}=\left\{t \in \mathbb{R}^{n}:\langle t, \lambda\rangle=1, \quad \forall \lambda \in \Lambda\right\} \tag{4.9}
\end{equation*}
$$

is a lattice in $\mathbb{R}^{n}$, say $K:=\Lambda^{\circ}$, and the natural torus mapping $\mathbb{R}^{n} \rightarrow \mathbb{R}^{n} / K$ is then $1-1$ on $\Omega$.

In this case, there is a system $s=(R, B, L, K)$ which is self-adjoint and of Hadamard type. Moreover the set $\Lambda$ (called the spectrum) may be taken as

$$
\begin{equation*}
\Lambda=L+R^{*} K^{\circ} . \tag{4.10}
\end{equation*}
$$

Pairs $(\Omega, \Lambda)$ with the basis-property are called spectral pairs; the "reduced" ones where $\Lambda$ may be brought into the form (4.10) (with $L \neq\{0\}$, i.e., $|L|>1$ ) are called simple factors. We showed in [JP6] that more general ones may be built up from the simple factors.

The following easy fact will be used below: Let $K_{1}$ and $K_{2}$ be lattices, and let $\tilde{K}_{1}$ and $\tilde{K}_{2}$ be corresponding matrices. Then we have the lattice inclusion $K_{1} \subset K_{2}$ if and only if the matrices factor: $\tilde{K}_{1}=\tilde{K}_{2} M$ with $M \in \operatorname{Mat}_{n}(\mathbb{Z})$ where $\operatorname{Mat}_{n}(\mathbb{Z})$ denotes the ring of integral $n$ by $n$ matrices, i.e., $M=\left(m_{i j}\right)_{i, j=1}^{n}$ with $m_{i j} \in \mathbb{Z}$.

This observation allows us to take advantage of the Noetherian property of the ring $\operatorname{Mat}_{n}(\mathbb{Z})$. A minimal choice for $K$ subject to conditions is then always well defined.

For a given lattice $K$, the dual lattice is denoted $K^{\circ}$ and given by

$$
K^{\circ}:=\left\{s \in \mathbb{R}^{n}: s \cdot k \in \mathbb{Z}, \quad \forall k \in K\right\}
$$

If $\tilde{K}$ is a matrix for $K$, then the inverse transpose, i.e., $\left(\tilde{K}^{\operatorname{tr}}\right)^{-1}$ will be a matrix for $K^{\circ}$.

When $R \in G L_{n}(\mathbb{R})$ is given, we denote by $\operatorname{lat}(R)$ the set of all lattices $K$ in $\mathbb{R}^{n}$ such that $R(K) \subset K$. For the matrices, that reads $\tilde{K}^{-1} R \tilde{K} \in \operatorname{Mat}_{n}(\mathbb{Z})$. This fact will be used in the paper; it implies for example that $|\operatorname{det} R|$ is the index of $K$ in $R^{-1}(K)$. It is known (see e.g., [CS] or [JP6]) that, if $\operatorname{lat}(R) \neq \emptyset$, then $\operatorname{det} R \in \mathbb{Z}$. (Remark: If $R$ is not in diagonal form, i.e., $\left(\begin{array}{cccc}r & 0 & \ldots & 0 \\ 0 & r & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & r\end{array}\right)=r I_{n}$ for $r \in \mathbb{Z}$, then there are lattices $K$ not in $\operatorname{lat}(R)$.)

The standing assumption which is placed on $R$ is referred to as the expansive property: We assume that, for some $p \in \mathbb{N}$, all the eigenvalues $\left\{\lambda_{j}\right\}$ of $R^{p}$ satisfy $\left|\lambda_{j}\right|>1$. Recall, $R$ has real entries, but the eigenvalues may be complex. For emphasis, we will denote the transpose of $R$ by $R^{*}$, even though it is the same as $R^{\mathrm{tr}}$. (Note that the assumption on the eigenvalues of $R^{p}$ for some positive power $p$ is equivalent to the same condition on $R$ itself, i.e., to the condition for $p=1$.)

## 5. Iteration Systems

In this paper, we shall study fractals (in the sense of (2.4) above) with a high degree of symmetry; and show that these fractals are precisely those which may be built from systems $s=(R, B, L, K)$ which are selfadjoint, of Hadamard-type, and
where the lattice $K$ is chosen as minimal relative to the three conditions (i)-(iii) in Lemma 4.1. In describing our limit systems (typically fractals), we show again that the Hadamard condition (3.2) is the central one.

Motivated by (4.10), we form the set $\mathcal{L}(L)$ consisting of all (finite) sums

$$
\begin{equation*}
\ell_{0}+R^{*} \ell_{1}+R^{*^{2}} \ell_{2}+\cdots+R^{*^{m}} \ell_{m} \tag{5.1}
\end{equation*}
$$

when $m$ varies over $\{0,1,2, \ldots\}$ and $\ell_{i} \in L$. Using (4.1), also notice that $\mathcal{L}(L)$ is made from iterations

$$
\begin{equation*}
\tau_{\ell_{0}}\left(\tau_{\ell_{1}}\left(\cdots\left(\tau_{\ell_{m}}(0)\right) \cdots\right)\right) \tag{5.2}
\end{equation*}
$$

The set $\Lambda$ in (4.10) is $\bigcup\left\{\tau_{\ell}\left(K^{\circ}\right): \ell \in L\right\}$. We shall also need the corresponding iterations,

$$
\begin{equation*}
\bigcup_{m}\left\{\tau_{\ell_{0}} \circ \cdots \circ \tau_{\ell_{m}}\left(K^{\circ}\right): \ell_{i} \in L\right\} . \tag{5.3}
\end{equation*}
$$

For a given string $\left(\ell_{0}, \ldots, \ell_{m}\right)$, the set in (5.3) will be denoted $K^{\circ}\left(\ell_{0}, \ldots, \ell_{m}\right)$.

Definition 5.1. We say that $K^{\circ}$ formed from a given system $s=(R, B, L, K)$ is total if the functions $\left\{e_{s}: s \in K^{\circ}\right\} \subset L^{2}(\mu)$ span a subspace which is dense in the Hilbert space $L^{2}(\mu)$ defined from the Hutchinson measure $\mu$, see (2.3).

Both of our main results will have the total property for $K^{\circ}$ as an assumption. The way to test it in applications is to rely on our earlier paper [JP2] about spectral pairs, i.e., subsets $\Omega$, and $\Lambda$, in $\mathbb{R}^{n}$ such that $\Omega$ has finite positive $n$-dimensional Lebesgue measure, and the exponentials $\left\{e_{\lambda}: \lambda \in \Lambda\right\}$ form an orthogonal basis for $L^{2}(\Omega)$. We show that, for every such pair, the set

$$
K:=\Lambda^{\circ}=\left\{\xi \in \mathbb{R}^{n}: \xi \cdot \lambda \in \mathbb{Z}, \quad \forall \lambda \in \Lambda\right\}
$$

is a lattice. Analogously to the situation in Lemma 4.1 above, we also show in [JP2] that the set $\Omega$ in a spectral pair embeds in the torus $\mathbb{R}^{n} / K$. We identify a
special class of spectral pairs, called simple factors which produce two finite sets $B, L \subset \mathbb{R}^{n}$, and a matrix $R$ with $K \in \operatorname{lat}(R)$ such that the system $s=(R, B, L, K)$ satisfies the conditions from section 4 above. Our present paper is motivated by getting "invariants" for simple factors from iteration of the affine maps (see (2.1) and (4.1) above). In [JP6] we further study the converse problem of reconstructing simple factors from "fractal" iteration limit-objects. In any case, the fractal limit $\bar{X}(B)$ from (2.4) will also be embedded in the torus $\mathbb{R}^{n} / K$. When equipped with Haar-measure $L^{2}\left(\mathbb{R}^{n} / K\right)$ has the exponentials $\left\{e_{\lambda}: \lambda \in K^{\circ}\right\}$ as an orthogonal basis. In testing for our totality condition relative to $L^{2}(\mu)$, we can then use that $\bar{X}(B)$ is the support of $\mu$, and then apply Stone-Weierstrass to $\left\{e_{\lambda}\right\}_{\lambda \in K^{\circ}}$ when viewed as a subset of $C(\bar{X}(B))$.

We shall say that $\mathcal{L}(L)$ is maximal if $\left\{e_{\lambda}: \lambda \in \mathcal{L}(L)\right\}$ is orthogonal in $L^{2}(\mu)$ and (considering $t \in \mathbb{R}^{n}$ ) if

$$
\begin{equation*}
\text { whenever }\left\langle e_{t}, e_{\lambda}\right\rangle_{\mu}=\hat{\mu}(\lambda-t)=0 \text { for all } \lambda \in \mathcal{L}(L) \text {, then } t \in \mathcal{L}(L) \text {. } \tag{5.4}
\end{equation*}
$$

We have used the transform $\hat{\mu}$ given by

$$
\begin{equation*}
\hat{\mu}(s)=\int e_{s} d \mu=\int e^{i 2 \pi s \cdot x} d \mu(x) \quad \text { for } \quad s \in \mathbb{R}^{n} . \tag{5.5}
\end{equation*}
$$

We say that the system $s$ is $\Lambda$-orthogonal, if the functions $\left\{e_{\lambda}: \lambda \in \Lambda\right\}$ are orthogonal in $L^{2}(\mu)$, here $\Lambda$ is given by (4.10), i.e., $\Lambda=L+R^{*} K^{\circ}$. (See also (5.6) below.)

We are now ready for the

Theorem 5.2. Let $s=(R, B, L, K)$ be a selfadjoint system in $\mathbb{R}^{n}$, and assume
(i) $K^{\circ}$ is total;
(ii) $\mathcal{L}(L)$ is maximal in $L^{2}(\mu)$; and
(iii) $s$ is $\Lambda$-orthogonal, i.e., the points in $\Lambda$ from (4.10) are orthogonal for $\ell \neq \ell^{\prime}$ in $\Lambda$.

Then it follows that $s$ is of Hadamard type; i.e., $|B|=|L|$ and the $B / L$-matrix $U$ satisfies (3.2).

Proof. Condition (i) states that the orthogonal complement of $\left\{e_{s}: s \in K^{\circ}\right\}$ in $L^{2}(\mu)$ is zero. Notice that condition (iii) is equivalent to:

$$
\begin{equation*}
\hat{\mu}\left(\ell-\ell^{\prime}+R^{*} s\right)=0, \quad \forall \ell \neq \ell^{\prime} \text { in } L, \quad \forall s \in K^{\circ} . \tag{5.6}
\end{equation*}
$$

If we set

$$
\begin{equation*}
\mathcal{B}(t):=|B|^{-1} \sum_{b \in B}\langle b, t\rangle, \quad \forall t \in \mathbb{R}^{n} ; \tag{5.7}
\end{equation*}
$$

then (2.4) implies the factorization:

$$
\begin{equation*}
\hat{\mu}(t)=\mathcal{B}(t) \hat{\mu}\left(R^{*^{-1}} t\right) . \tag{5.8}
\end{equation*}
$$

For distinct points $\ell$ and $\ell^{\prime}$ in $L$ we claim that

$$
R^{*^{-1}}\left(\ell-\ell^{\prime}\right) \notin K^{\circ} .
$$

Assuming the contrary, there would be some $s \in K^{\circ}$ such that $\ell-\ell^{\prime}=R^{*} s$. From the orthogonality property (5.6) (see Definition 5.1), we then get

$$
\hat{\mu}(0)=\hat{\mu}\left(\ell-\ell^{\prime}-R^{*} s\right)=\left\langle e_{\ell}, e_{\ell^{\prime}+R^{*} s}\right\rangle_{\mu}=0
$$

contradicting $\hat{\mu}(0)=1$.
Since $\mathcal{L}(L) \subset K^{\circ}$, the point $t:=R^{*^{-1}}\left(\ell-\ell^{\prime}\right)$ is not in $\mathcal{L}(L)$. From the maximal property (5.4), we conclude that there is some $\lambda \in \mathcal{L}(L)$ such that

$$
\left\langle e_{t}, e_{\lambda}\right\rangle_{\mu} \neq 0
$$

This term works out to

$$
\hat{\mu}\left(R^{*^{-1}}\left(\ell-\ell^{\prime}\right)-\lambda\right)(\neq 0) .
$$

From the orthogonality (Definition 5.1), we also have:

$$
0=\hat{\mu}\left(\ell-\ell^{\prime}-R^{*} \lambda\right)=\mathcal{B}\left(\ell-\ell^{\prime}\right) \hat{\mu}\left(R^{*^{-1}}\left(\ell-\ell^{\prime}\right)-\lambda\right)
$$

where the last factor is non-zero. It then follows that $\mathcal{B}\left(\ell-\ell^{\prime}\right)=0$.
Recall, for $u, v \in \mathbb{R}^{n}$, the notation $\langle u, v\rangle:=e^{i 2 \pi u \cdot v}$. Then the vectors $\{\langle\cdot, \ell\rangle\}$ are indexed by points $\ell \in L$, and we showed that they are orthogonal when viewed as elements in $\ell^{2}(B)$.

It follows that $|L| \leq|B|$ where the symbol $|\cdot|$ denotes cardinality. We claim that they are equal. For suppose the contrary, viz., $|L|<|B|$. Then pick coefficients $k_{b} \in \mathbb{C}$, not-all zero, indexed by $b \in B$, such that

$$
\begin{equation*}
\sum_{b \in B} k_{b}\langle b, \ell\rangle=0, \quad \forall \ell \in L . \tag{5.9}
\end{equation*}
$$

For every $s \in K^{\circ}$ and $\ell \in L$, consider $t:=\ell+R^{*} s$; and define

$$
\begin{equation*}
f:=\sum_{b \in B} \bar{k}_{b} \chi_{\left(b+R^{-1}(\bar{X})\right)} \tag{5.10}
\end{equation*}
$$

where $\chi$ denotes "indicator function", the subscript is a $b$-translate, and finally $\bar{X}$ is the $B$-fractal. (Recall, details below, it is compact, and satisfies $\bar{X}=B+R^{-1}(\bar{X})$, with

$$
\begin{equation*}
\left.\mu\left(\left(b+R^{-1}(\bar{X})\right) \cap\left(b^{\prime}+R^{-1}(\bar{X})\right)\right)=0 \quad \text { for all } b \neq b^{\prime} \text { in } B .\right) \tag{5.11}
\end{equation*}
$$

Note, (5.11) is a consequence of the totality of $K^{\circ}$ and the following observation: if $b$ and $c$ are in $B$ and $b+R^{-1} x=c+R^{-1} y$, then $R(b-c)=y-x$; it now follows
from Lemma 4.1 that $x \in y+K$. But then

$$
\begin{align*}
\left\langle f, e_{t}\right\rangle_{\mu} & =\int_{\mathbb{R}^{n}} \overline{f(x)} e_{t}(x) d \mu(x) \\
& =\sum_{b \in B} k_{b} \int_{R^{-1}(\bar{X})} e_{t}(b+x) d \mu(x) \\
& =\sum_{b \in B} k_{b} \int_{R^{-1}(\bar{X})}\left\langle\ell+R^{*} s, b+x\right\rangle d \mu(x) \\
& =\underbrace{\left(\sum_{b \in B} k_{b}\langle\ell, b\rangle\right)}_{(5.9)} \int_{R^{-1}(\bar{X})} e_{t} d \mu \tag{5.12}
\end{align*}
$$

(where we use (5.11) and Lemma 5.3 below), and

$$
\left\langle R^{*} s, b\right\rangle=\langle s, R b\rangle=1 .
$$

The last fact is from axiom (4.6) which makes $R b \in K$. It follows (from (5.9)) that $f$ is in the orthogonal complement of $\left\{e_{\ell+R^{*} s}\right\}$ as $\ell$ varies over $L$, and $s$ over $K^{\circ}$. But from (i), we know that this is a total set of vectors in $L^{2}(\mu)$, so the function $f$ must vanish identically, $\mu$-a.e. If the coefficients $\left\{k_{b}\right\}$ are not all zero, this would contradict (5.11), (2.3), and the basic properties of the Hutchinson measure $\mu$.

From the contradiction, we conclude that $|L|=|B|$; which is to say, both conditions on the matrix $(\langle b, \ell\rangle)_{b, \ell}$, indexed by $B \times L$, to be of generalized Hadamard type, are satisfied. We have $|L|=|B|=N$. If the matrix is denoted $U$, then

$$
\begin{equation*}
U U^{*}=U^{*} U=N I_{N} \tag{5.13}
\end{equation*}
$$

where $I_{N}$ denotes the identity matrix in $N$ variables, and $U^{*}$ is the transpose conjugate. To define it, it is convenient to use a common index labeling, e.g., $\{1,2, \ldots, N\}$.

Remark. Note that if we assume $B$ is a subset of a set of representations for $R^{-1} K / K$, then (5.11) follows from an application of [Ke, Theorem 10] and a related result in [Ma]. (See also [Ba-Gr] for related work.) The Kenyon-Madych result applies in the present context since the mapping from the set of all finite $B$ strings $\left(b_{1}, \ldots, b_{m}\right)$ with $m$ varying in $\mathbb{N}, b_{i} \in B$, into $\sum_{i} R^{i} b_{i} \in K$ is $1-1$. This follows by induction and use of our orthogonality assumptions. To use [Ke]-[Ma], we then extend $B$ so as to get a full set of residue classes $R^{-1}(K) / K$.

Question. Does either of the following two conditions imply the other: (i) $K^{\circ}$ is total, (ii) $B$ is a subset of a set of representatives for the quotient $\left(R^{-1} K\right) / K$ ?

In the calculation (5.12) above, the following lemma was used. (It is needed because we do not know if, in general, $\mu$ is a Hausdorff-measure.)

Lemma 5.3. Under the assumptions of Theorem 5.2, it follows that

$$
\int_{\sigma_{b} \bar{X}} f(x) d \mu(x)=\int_{R^{-1} \bar{X}} f(x+b) d \mu(x)
$$

for all $b$ in $B$ and $f$ in $L^{2}(\mu)$.

Proof. The claim is equivalent to having

$$
\mu\left(\sigma_{b} \Delta\right)=\mu\left(R^{-1} \Delta\right)
$$

for all $\mu$-measurable sets $\Delta \subset \mathbb{R}^{n}$ and all $b$ in $B$; which in turn is equivalent to

$$
\begin{equation*}
\mu\left(\sigma_{b} \Delta\right)=\mu\left(\sigma_{c} \Delta\right) \tag{5.14}
\end{equation*}
$$

for all $\mu$-measurable $\Delta$, and all $b$ and $c$ in $B$. The last equivalence used the assumption that $0 \in B$. By regularity, it suffices to consider the case where $\Delta$ is a closed set.

Let $\Delta$ be a closed subset of $\bar{X}(B)=\bar{X}$, and choose $B_{k} \subset B^{k}=B \times \cdots \times B(k$ terms) such that

$$
\left(b_{1}, \ldots, b_{k}, b_{k+1}\right) \subset B_{k+1} \Rightarrow\left(b_{2}, \ldots, b_{k+1}\right) \in B_{k},
$$

and such that

$$
\Delta=\bigcap_{k=1}^{\infty} E_{k},
$$

where

$$
E_{k}=\bigcup_{\left(b_{1}, \ldots, b_{k}\right) \in B_{k}} \sigma_{b_{1}} \cdots \sigma_{b_{k}} \bar{X} .
$$

The first condition means that $\left(E_{k}\right)$ is a descreasing sequence of compact sets. It follows (analogously to (5.11)) that the overlaps in the definition of $E_{k}$ are $\mu$ -null-sets. Hence, to prove the lemma, it suffices to show that

$$
\mu\left(\sigma_{b_{1}} \cdots \sigma_{b_{k}} \bar{X}\right)=\mu\left(\sigma_{c_{1}} \cdots \sigma_{c_{k}} \bar{X}\right)
$$

for all $\left(b_{1}, \ldots, b_{k}\right),\left(c_{1}, \ldots, c_{k}\right)$ in $B^{k}$.
First note that, for any Borel set $\Delta$, and any $b$ in $B$, we have

$$
\mu(\Delta)=|B|^{-1} \sum_{c \in B} \mu\left(\sigma_{c}^{-1} \Delta\right) \geq|B|^{-1} \mu\left(\sigma_{b}^{-1} \Delta\right)
$$

and hence $\mu\left(\sigma_{b} \Delta\right) \geq|B|^{-1} \mu(\Delta)$. From this inequality, and (5.11), it follows further that

$$
\mu(\bar{X})=\mu\left(\bigcup_{b \in B} \sigma_{b} \bar{X}\right)=\sum_{b \in B} \mu\left(\sigma_{b} \bar{X}\right) \geq \mu(\bar{X}),
$$

and therefore that $\mu\left(\sigma_{b} \bar{X}\right)=|B|^{-1} \mu(\bar{X})$. Assuming

$$
\mu\left(\sigma_{b_{2}} \cdots \sigma_{b_{k}} \bar{X}\right)=|B|^{-k+1} \mu(\bar{X})
$$

it follows (analogously to the above), that

$$
\begin{equation*}
\mu\left(\sigma_{b_{1}} \sigma_{b_{2}} \cdots \sigma_{b_{k}} \bar{X}\right)=|B|^{-k} \mu(\bar{X}) \tag{5.15}
\end{equation*}
$$

Hence, by induction, (5.15) is true for all positive integers $k$, and all $b_{1}, \ldots, b_{k}$ in $B$. This completes the proof of the lemma.

Note also (heuristically) that (5.14) is a consequence of (5.11), and that

$$
\begin{aligned}
\mu\left(\sigma_{b} \Delta\right) & =\lim _{k \rightarrow \infty} \sum_{b_{1}, \ldots, b_{k}}|B|^{-1} \chi_{\sigma_{b} \Delta}\left(\sigma_{b_{1}} \cdots \sigma_{b_{k}} x\right) \\
& =\lim _{k \rightarrow \infty} \sum_{b_{1}, \ldots, b_{k}}|B|^{-1} \chi_{\Delta}\left(\left(\sigma_{b_{2}} \cdots \sigma_{b_{k}} x\right)+R b_{1}-R b\right)
\end{aligned}
$$

where $x$ in $\bar{X}$ is arbitrary. However, the first equality requires that $\chi_{\sigma_{b} \Delta}$ is continuous. We refer to [Fa, p. 121] for further details on this point.

We conclude with the following lemma which is both basic and general; in fact it holds in a context which is more general than where we need it. Such more general contexts occur, e.g., in [St4], [Mat], [MOW], and [Od], (among other places). But we will still restrict the setting presently to where it is needed below for our proof of Theorem 6.1.

Lemma 5.4. Let $(R, B)$ be an affine system in $\mathbb{R}^{n}$ (see details in Section 2) with $R$ expansive and $B \subset \mathbb{R}^{n}$ a finite subset. Let $\mathcal{B}$ be given by (5.7), and let $\mu$ be the probability measure from (2.2). We are assuming the property (5.11). Let $\mathcal{N}:=\left\{t \in \mathbb{R}^{n}: \mathcal{B}(t)=0\right\}$. Then, for the roots of $\hat{\mu}$, we have

$$
\left\{t \in \mathbb{R}^{n}: \hat{\mu}(t)=0\right\}=\bigcup_{k=0}^{\infty} R^{*^{k}}(\mathcal{N})
$$

Proof. We have (5.8) by virtue of [JP6, Lemma 3.4], and it follows that $\hat{\mu}(t)=0$ when $t \in R^{*^{k}}(\mathcal{N})$ for some $k \in\{0,1, \ldots\}$. From the assumed expansivity of $R$, we
also know that the corresponding infinite product formula is convergent, see [JP6, (3.13)]. In fact $\lim _{k \rightarrow \infty} \hat{\mu}\left({R^{*}}^{-k} t\right)=1$, for all $t \in \mathbb{R}^{n}$. This is from continuity of $\hat{\mu}$, and the limit, $R^{*^{-k}} t \rightarrow 0$. Now consider,

$$
\hat{\mu}(t)=\prod_{j=0}^{k-1} \mathcal{B}\left(R^{*^{-j}} t\right) \hat{\mu}\left(R^{*^{-k}} t\right)
$$

and suppose $\hat{\mu}(t)=0$. Pick $k$ (sufficiently large) s.t., $\hat{\mu}\left(R^{*^{-k}} t\right) \neq 0$. (This is possible by continuity, and the fact that $\hat{\mu}(0)=1$ ). We conclude, then that, for some $j, 0 \leq j<k, R^{*^{-j}} t \in \mathcal{N}$; and this is the assertion of the lemma.

## 6. Orthogonal Exponentials

We keep the standing assumptions on the quadruple $s=(R, B, L, K)$ which determine a system in $\mathbb{R}^{n}$. In particular, the matrix $R$ is assumed expansive (see section 2), the sets $B$ and $L$ in $\mathbb{R}^{n}$ are finite both containing 0 . We will assume now that $s$ is selfadjoint and of Hadamard type. We say that the system $s$ is irreducible if there is no proper linear subspace $V \subset \mathbb{R}^{n}$ (i.e., of smaller dimension) which contains the set $B$, and which is invariant under $R$, i.e., $R v \in V$ for all $v \in V$. If such a proper subspace does exist, we say that $s$ is reducible. In that case, it is immediate that the fractal $\bar{X}(B)$ from (2.4) is then contained in $V$. All the examples in Section 7 below can easily be checked to be irreducible. But the following example in $\mathbb{R}^{2}$ is reducible, and serves to illustrate the last conclusion from our theorem in the present section: Let $R=\left(\begin{array}{ll}2 & 1 \\ 0 & 2\end{array}\right), B=\left\{\binom{0}{0},\binom{\frac{1}{2}}{0}\right\}$, $L=\left\{\binom{0}{0},\binom{1}{0}\right\}, K=\mathbb{Z}^{2}$; and $V=\binom{\mathbb{R}}{0}$, i.e., the $x$-axis in $\mathbb{R}^{2}$. Then a direct calculation shows that $\bar{X}(B)=\binom{I}{0}$, where $I=[0,1]$ is the unit-interval on
the $x$-axis; and the Hutchinson measure $\mu$ is the product measure $\lambda_{1} \otimes \delta_{0}$ where $\lambda_{1}$ is the restriction to $I$ of the one-dimensional Lebesgue measure, and $\delta_{0}$ is the point-0-Dirac measure in the second coordinate. For the set $\mathcal{L}(L)$ from (5.1), we have,

$$
\mathcal{L}(L)=\left\{\binom{n}{N(n)}: n=0,1,2, \ldots\right\}
$$

$n$ represented by finite sums, $n=\sum_{j \geq 0} 2^{j} \epsilon_{j}, \epsilon_{j} \in\{0,1\}$, and $N(n)=\sum_{j>0} j 2^{j-1} \epsilon_{j}$.
We will show below that if $K^{\circ}$ is total then in the "fractal case", i.e., when $N=|B|=|L|$ is smaller than $|\operatorname{det} R|$, and when $s$ is irreducible, then $\mathcal{L}(L)$ is maximally orthogonal, and $s$ is $\Lambda$-orthogonal. Recall $\mu$ is the Hutchinson measure, see (2.3), $\mathcal{L}(L)$ is the set given by (5.1); and finally the properties regarding the two sets $K^{\circ}$ and $\mathcal{L}(L)$ refer to the corresponding exponentials $e_{s}$, when $s$ is in the respective sets, and each

$$
\begin{equation*}
e_{s}(x)=\langle s, x\rangle=e^{i 2 \pi s \cdot x} \tag{6.1}
\end{equation*}
$$

is considered a vector (alias function on $\mathbb{R}^{n}$ ) in the Hilbert space $L^{2}(\mu)=L^{2}(\bar{X}, \mu)$. We refer to Definition 5.1 and Theorem 5.2 for further details. Recall that the total property (iii) in Theorem 5.2 amounts to the $\Lambda$-orthogonality, including the assertion

$$
\begin{equation*}
\hat{\mu}\left(\ell-\ell^{\prime}+R^{*} s\right)=0 \tag{6.2}
\end{equation*}
$$

for all $\ell \neq \ell^{\prime}$ in $L$ and all $s \in K^{\circ}$. But if $s$ is selfadjoint and of Hadamard type, then (6.2) follows immediately from (5.8), which is the functional equation of the transform $\hat{\mu}$, see also (5.5).

The purpose of the present section is twofold. First we show that Theorem 5.2 has a partial converse, and secondly that the technical conditions from our two theorems 5.2 and 6.1 amount to the dual pair condition (see Section 1) for
representations of the $C^{*}$ - algebra $\mathcal{O}_{N}$. This is for systems $s=(R, B, L, K)$ as specified where the two given finite sets $B$ and $L$ in $\mathbb{R}^{n}$ are assumed to have the same cardinality $N$, i.e., $|B|=|L|=N$. Our recent paper [JP6] further details how the representation duality relates to our present assumptions. But we shall summarize the essentials here for the convenience of the reader. The Cuntz-algebra $\mathcal{O}_{N}($ see $[\mathrm{Cu}])$ is known to be given universally on $N$ generators $\left\{s_{i}\right\}$ and subject only to the relations:

$$
\begin{equation*}
s_{i}^{*} s_{j}=\delta_{i j} 1 \quad \text { and } \quad \sum_{i} s_{i} s_{i}^{*}=1 \tag{6.3}
\end{equation*}
$$

This means that, if a finite set of $N$ operators $S_{i}$ say, acting on some Hilbert space $\mathcal{H}$ say, are known to satisfy the relations (6.3), then there is a unique represention $\rho$ of $\mathcal{O}_{N}$, acting by bounded operators on $\mathcal{H}$, such that $\rho\left(s_{i}\right)=S_{i}$ for all $i$; or, equivalently, $\rho(a) f=\hat{a} f$ for all $a \in \mathcal{O}_{N}$ and all $f \in \mathcal{H}$, where the operator $\hat{a}$ is given by the same expression in the $S_{i} \mathrm{~s}$ as $a$ is in the $s_{i}$-generators.

For a given system $s=(R, B, L, K)$, there is then the possibility of making a representation duality based on the exponentials $e^{i 2 \pi t \cdot x}$ in (6.1), and treating the two vector-variables $x$ and $t$ symmetrically: The pair $(R, B)$ gives one affine system $\sigma_{b} x=R^{-1} x+b(b \in B)$ in the $x$-variable; and the dual system $\left(R^{*}, L\right)$ given by $\tau_{\ell} t=R^{*} t+\ell(\ell \in L)$; a second one now acting in the $t$-variable. See (2.1) and (4.1) above. To be able to generate the asserted representation pair we need to specify $\left\{\tau_{\ell}\right\}_{\ell \in L}$ for enough values of $t$ such that the corresponding functions $e_{t}(x):=e^{i 2 \pi t \cdot x}$ span a dense subspace in $L^{2}(\mu)$. But it turns out that other conditions must be met as well: For the two affine systems $\left\{\sigma_{b}\right\}_{b \in B}$ and $\left\{\tau_{\ell}\right\}_{\ell \in L}$, the question is if we can associate operator systems $\left\{S_{b}\right\}$ and $\left\{T_{\ell}\right\}$ of $2 N$ operators acting on $L^{2}(\mu)$, each system satisfying (6.3), and the operators collectively defined from the
exponentials $e^{i 2 \pi t \cdot x}$ as specified. When this is so, we have an (orthogonal) dual pair of representations of $\mathcal{O}_{N}$ acting on $L^{2}(\mu)$, and conversely. Then it turns out that, for each pair $(b, \ell)$, the operator $S_{b}^{*} T_{\ell}$ is a multiplication operator, see (6.6) below.

Theorem 6.1. Let $s=(R, B, L, K)$ be a system in $\mathbb{R}^{n}$ and assume that $s$ is selfadjoint and of Hadamard type. Assume further that $K^{\circ}$ is total (with a minimal choice for $K$ ), and that $|B|<|\operatorname{det} R|$, and let $\mu$ be the corresponding measure (see (2.3)) with support $\bar{X}$. Then $s$ is $\Lambda$-orthogonal and carries a dual pair of Cuntz representations (with $\mathcal{O}_{N}$ acting on $L^{2}(\mu)$ for both representations). If $s$ is also irreducible, then $\mathcal{L}(L)$ is maximally orthogonal.

Proof. Let $N=|B|=|L|$ and note that from [JP6] (Theorem 4.1) we get a dual pair of representations $\left\{S_{b}\right\}_{b \in B}$ and $\left\{T_{\ell}\right\}_{\ell \in L}$ of the Cuntz algebra $\mathcal{O}_{N}$, see also $[\mathrm{Cu}]$ and $[\mathrm{Ar}]$, acting on $L^{2}(\mu)$ and given by the respective formulas:

$$
\begin{align*}
& S_{b}^{*} f=N^{-1 / 2} f \circ \sigma_{b} \quad \text { for } \quad f \in L^{2}(\mu), \quad b \in B  \tag{6.4}\\
& T_{\ell} e_{s}=e_{\tau_{\ell}(s)} \quad \text { for } \quad s \in K^{\circ} \quad \text { and } \quad \ell \in L . \tag{6.5}
\end{align*}
$$

Moreover $S_{b}^{*} T_{\ell}$ is the multiplication operator $M_{b \ell}$ on $L^{2}(\mu)$ given by

$$
\begin{equation*}
M_{b \ell} f=N^{-1 / 2}\left(e_{\ell} \circ \sigma_{b}\right) f \quad \text { for } \quad f \in L^{2}(\mu) . \tag{6.6}
\end{equation*}
$$

It follows from [JP6, Theorem 4.1] that for $\ell$ and $\ell^{\prime}$ in $L$

$$
\begin{equation*}
T_{\ell}^{*} T_{\ell^{\prime}}=\delta_{\ell \ell^{\prime}} I \quad \text { and } \quad \sum_{\ell \in L} T_{\ell} T_{\ell}^{*}=I \tag{6.7}
\end{equation*}
$$

Here we use the Kronecker delta notation

$$
\delta_{\ell \ell^{\prime}}= \begin{cases}1 & \text { if } \ell=\ell^{\prime} \\ 0 & \text { if } \ell \neq \ell^{\prime},\end{cases}
$$

and $I$ denotes the identity operator in the Hilbert space $L^{2}(\mu)$.
It follows then from (6.6) that the vectors $e_{\ell+R^{*} s}, s \in K^{\circ}$, are mutually orthogonal in $L^{2}(\mu)$ for distinct values of $\ell$, i.e., for $\ell \neq \ell^{\prime}$ in $L$. For more details on this point, we refer to sections 3-4 in [JP6]. For

$$
\begin{equation*}
\lambda=\sum_{j=0}^{n} R^{*^{j}} \ell_{j}, \tag{6.8}
\end{equation*}
$$

we have

$$
\begin{equation*}
T_{\ell_{0}} T_{\ell_{1}} \cdots T_{\ell_{n}} e_{0}=e_{\lambda} . \tag{6.9}
\end{equation*}
$$

Let $\lambda=\sum_{j=0}^{n} R^{*^{j}} \ell_{j}$, and $\kappa=\sum_{j=0}^{m} R^{*^{j}} k_{j}$, where the $\ell_{j}$ 's and $k_{j}$ 's are in $L$. Then $e_{\lambda}$ and $e_{\kappa}$ are orthongonal in $L^{2}(\mu)$ except in the cases where $m \leq n$, and $\ell_{j}=0$ for $j>m$, and where $m \geq n$ and $k_{j}=0$ for $j>n$. In the exceptional cases, it follows from (6.9) and $T_{0} e_{0}=e_{0}$ that $e_{\lambda}=e_{\kappa}$. To prove the orthogonality assertion above, note that

$$
\left\langle e_{\lambda}, e_{\kappa}\right\rangle_{\mu}=\left\langle T_{\ell_{0}} T_{\ell_{1}} \cdots T_{\ell_{n}} e_{0}, T_{k_{0}} T_{k_{1}} \cdots T_{k_{m}} e_{0}\right\rangle
$$

is $=0$, unless $\ell_{0}=k_{0}$, because $T_{k_{0}}^{*} T_{\ell_{0}}=0$ if $k_{0} \neq \ell_{0}$. If $k_{0}=\ell_{0}$, then $T_{k_{0}}^{*} T_{\ell_{0}}=I$, and we can repeat the argument on $\ell_{1}$ and $k_{1}$. It remains to consider the case where $n>0$ and $m=0$; in this case, we will use the identity, $T_{0} e_{0}=e_{0}$, to write

$$
\left\langle e_{\lambda}, e_{\kappa}\right\rangle_{\mu}=\left\langle T_{\ell_{0}} T_{\ell_{1}} \cdots T_{\ell_{n}} e_{0}, T_{0} e_{0}\right\rangle_{\mu}=0
$$

and we conclude that $\left\langle e_{\lambda}, e_{\kappa}\right\rangle_{\mu}=0$ unless $\ell_{0}=\ell_{1}=\cdots=\ell_{n}=0$.
It follows that the map,

$$
\begin{equation*}
\left(\ell_{0}, \ldots, \ell_{n}\right) \mapsto \sum_{j=0}^{n} R^{*^{j}} \ell_{j} \in \mathcal{L}(L) \tag{6.10}
\end{equation*}
$$

is $1-1$ on the set of finite sequences $\left(\ell_{0}, \ldots, \ell_{n-1}, \ell_{n}\right)$ with $n$ a nonnegative integer, the $\ell_{j}$ 's in $L$, and $\ell_{n} \neq 0$.

The assumption that $s$ be irreducible is now imposed, and we show that $\mathcal{L}(L)$ has the stated maximality property: We show that, if $t \in \mathbb{R}^{n}$ and $\left\langle e_{\lambda}, e_{t}\right\rangle_{\mu}=0$ for all $\lambda \in \mathcal{L}(L)$, then it follows that $t \in \mathcal{L}(L)$. We shall do this by contradiction, assuming the $t \notin \mathcal{L}(L)$. We shall use the functional equation (5.8) for $\hat{\mu}$, recalling that

$$
\begin{equation*}
\left\langle e_{\lambda}, e_{t}\right\rangle_{\mu}=\hat{\mu}(t-\lambda) \tag{6.11}
\end{equation*}
$$

We shall also use that for every $s \in \mathbb{R}^{n}$ there is some $\ell \in L$ such that $\mathcal{B}(\ell-s) \neq 0$. This follows from the formula (5.7) for $\mathcal{B}(\cdot)$, and from the Hadamard property (3.2) which is now assumed.

As a special case of (5.8), we get

$$
0=\hat{\mu}\left(t-\ell_{0}-R^{*} \ell_{1}\right)=\mathcal{B}\left(t-\ell_{0}\right) \hat{\mu}\left(R^{*^{-1}} t-R^{*^{-1}} \ell_{0}-\ell_{1}\right)
$$

Picking $\ell_{0} \in L$ s.t. $\mathcal{B}\left(t-\ell_{0}\right) \neq 0$, we get

$$
\begin{aligned}
0 & =\hat{\mu}\left(R^{*^{-1}} t-R^{*^{-1}} \ell_{0}-\ell_{1}-R^{*} \ell_{2}\right) \\
& =\mathcal{B}\left(R^{*^{-1}} t-R^{*^{-1}} \ell_{0}-\ell_{1}\right) \hat{\mu}\left(R^{*^{-2}} t-R^{*^{-2}} \ell_{0}-R^{*^{-1}} \ell_{1}-\ell_{2}\right) .
\end{aligned}
$$

Picking $\ell_{1} \in L$ s.t.

$$
\mathcal{B}\left(R^{*^{-1}} t-R^{*^{-1}} \ell_{0}-\ell_{1}\right) \neq 0
$$

we conclude next that

$$
\hat{\mu}\left(R^{*^{-2}} t-R^{*^{-2}} \ell_{0}-R^{*^{-1}} \ell_{1}-\ell_{2}\right)=0
$$

and we continue by induction, determining $\ell_{0}, \ell_{1}, \ldots \in L$ such that the points

$$
s_{p}:=R^{*^{-p}} \ell_{0}+\cdots+R^{*^{-1}} \ell_{p-1}+\ell_{p}
$$

are in the dual fractal set $\bar{X}(L)$, see (6.2) above. When $N<|\operatorname{det} R|$, we may pick, inductively, the "digits" $\ell_{i}$ such that the differences

$$
\begin{equation*}
R^{*^{-p}} t-s_{p} \tag{6.12}
\end{equation*}
$$

are distinct as $p$ varies, but

$$
\hat{\mu}\left(R^{*^{-p}} t-s_{p}\right)=0 \quad \text { and } \quad \mathcal{B}\left(R^{*^{-(p-1)}} t-s_{p-1}\right) \neq 0 .
$$

Notice that the analytically extended transform

$$
\begin{equation*}
\hat{\mu}(z)=\int e^{i 2 \pi z \cdot x} d \mu(x) \tag{6.13}
\end{equation*}
$$

is entire analytic on $\mathbb{C}^{n}$, where for $z=\left(z_{1}, \ldots, z_{n}\right) \in \mathbb{C}^{n}, z \cdot x=z_{1} x_{1}+\cdots+z_{n} x_{n}$ is the usual dot-product. Hence its zeros cannot accumulate. But the "dual attractor" $\bar{X}(L)($ see $(7.2))$ is compact in $\mathbb{R}^{n}$ so there a subsequence $s_{p_{i}}$ with limit $s_{p_{i}} \rightarrow s \in$ $\bar{X}(L)$, and

$$
0=\lim _{p_{i}} \hat{\mu}\left(R^{*^{-p_{i}}}(t)-s_{p_{i}}\right)=\hat{\mu}(-s)=\overline{\hat{\mu}(s)}
$$

contradicting that the roots of $\hat{\mu}(\cdot)$ must be isolated (see (6.12)), even isolated in $\mathbb{C}^{n}$. The contradiction completes the proof, and we conclude that $\mathcal{L}(L)$ is maximal.

If only a finite number of the "digits" $\ell_{j}$ are nonzero, then, using the contractive property of $R^{*^{-1}}$, we see that the sequences $R^{*^{-p}}(t)$, and $s_{p}$, both converge to zero as $p \rightarrow \infty$, contradicting that $\hat{\mu}(0)=1$, since $\lambda \rightarrow \hat{\mu}(\lambda)$ is continuous on $\mathbb{R}^{n}$.

Claim 1. The set $B$ is a subset of a set of representatives for $R^{-1}(K) / K$.

Proof of Claim. From the self-adjointness of $s$ we have $R B \subset K$ (by Lemma 4.2).
Therefore, $B \subset R^{-1} K$. If $b$ and $b^{\prime}$ are distinct and both in $B$, and if $b \in b^{\prime}+K$, then $e_{t}(b+x)=e_{t}\left(b^{\prime}+x\right)$ for all $x \in R^{-1} \bar{X}$ (all $t \in K^{\circ}$ ), contradicting the totality of $K^{\circ}$ in $L^{2}(\mu)$.

Claim 2. The finite set $L$ is a subset of a set of representatives for $K^{\circ} / R^{*} K^{\circ}$.

Proof of Claim. By Lemma 4.2, $L \subset K^{\circ}$. If $\ell$ and $\ell^{\prime}$ are in $L$, and $\ell=\ell^{\prime}+R^{*} \gamma$ for some $\gamma \in K^{\circ}$, then

$$
\begin{aligned}
\langle b, \ell\rangle & =\left\langle b, \ell^{\prime}+R^{*} \gamma\right\rangle \\
& =\left\langle b, \ell^{\prime}\right\rangle\langle R b, \gamma\rangle \\
& =\left\langle b, \ell^{\prime}\right\rangle
\end{aligned}
$$

where the last equality used Lemma 4.2 again. But this contradicts the Hadamardproperty, unless $\gamma=0$. Considering,

$$
\begin{aligned}
& x_{p}=R^{*^{-p}} t-\left(R^{*^{-p}} \ell_{0}+\cdots+R^{*^{-1}} \ell_{p-1}+\ell_{p}\right), \quad \text { and } \\
& y_{p}=R^{*^{p}} x_{p}=t-\left(\ell_{0}+R^{*} \ell_{1}+\cdots+R^{*^{p}} \ell_{p}\right),
\end{aligned}
$$

and letting $P=\left\{p: \ell_{p} \neq 0\right\}$; then we showed above that $P$ is infinite, and that $p \in P \rightarrow y_{p}$ is a $1-1$ map. Hence $\left\{y_{p}: p \in P\right\}$ is infinite.

Remark 6.2. For the reducible example (in $\mathbb{R}^{2}$ ) mentioned in the beginning of the present section, we note that all the conditions of the first part of Theorem 6.1 are satisfied. We also described the set $\mathcal{L}(L)$ of orthogonal exponentials for the example. But the maximality condition is not satisfied relative to $L^{2}(\mu)$. Indeed, for the transform $\hat{\mu}(s)$ from (5.5), we have, with $s=\left(s_{1}, s_{2}\right) \in \mathbb{R}^{2}$,

$$
\hat{\mu}(s)=\left\{\begin{array}{clc}
e^{i s_{1} \pi \frac{\sin \left(s_{1} \pi\right)}{s_{1} \pi}} & \text { if } & \left(s_{1} \neq 0\right) \\
1 & \text { if } & s_{1}=0
\end{array}\right.
$$

It follows that the identity from (5.4) will be satisfied whenever $t=\left(t_{1}, t_{2}\right) \in \mathbb{R}^{2}$ is such that $t_{1} \in \mathbb{Z}_{-}$, i.e., negative and integral. (Specifically, $\hat{\mu}(\lambda-t)=0$ for $\forall \lambda \in \mathcal{L}(L)$.$) From the calculation of \mathcal{L}(L)$, we note that such points $t=\left(t_{1}, t_{2}\right)$ will not be in the set $\mathcal{L}(L)$; and so the maximality condition is not satisfied for the example.

## 7. Examples

7.1 Background Material. We now give examples to illustrate the conditions in Theorems 5.2 and 6.1. Since the generalized Hadamard matrices are known up to $N=4$, the examples we give are "typical" for the possibilities when $N \leq 4$, and it is likely that there is a classification; but as it is unclear what is the "correct" notion of equivalence for systems $s=(R, B, L, K)$ we will postpone the classification issue to a later paper. Note that the examples occur in pairs, one for $s$ and a dual one for $s^{\circ}$. Also note that each $s$ will correspond to a spectral pair $(\Omega, \Lambda)$ as well as a selfsimilar iteration limit, typically a "fractal" $\bar{X}$ with a selfsimilar measure $\mu$. When the given system $s$ is selfadjoint, then there will in fact be a pair of "fractals" occurring as iteration limits, a selfsimilar $\mu$ from the affine system:

$$
\begin{equation*}
\sigma_{b} x=R^{-1} x+b \quad \text { leading to } \bar{X}=\bar{X}(B) \tag{7.1}
\end{equation*}
$$

defined from $s$, and also

$$
\begin{equation*}
\tau_{\ell}^{-1}(t)=R^{*^{-1}}(t-\ell) \quad \text { leading to } \bar{X}(L) \tag{7.2}
\end{equation*}
$$

and defining the corresponding dual selfsimilar measure $\mu^{\prime}$. Recall both $\mu$ and $\mu^{\prime}$ are probability measures on $\mathbb{R}^{n} ; \mu$ is determined by (2.2), and $\mu^{\prime}$ by:

$$
\begin{equation*}
\mu^{\prime}=|L|^{-1} \sum_{\ell \in L} \mu^{\prime} \circ \tau_{\ell} \tag{7.3}
\end{equation*}
$$

see also (7.1)-(7.2) and Lemma 4.1 for more details on the dual pair of affine systems.

Our examples below will be constructed from the matrices (3.3)-(3.5) which we listed in section 3. In fact, we shall supply a group of examples for each of the
generalized Hadamard matrices $N=2, N=3$, and $N=4$, all the examples will be symmetric and of Hadamard type; but some will not be selfadjoint. In fact, when considering $s=(R, B, L, K)$ we shall fix the first three $R, B$, and $L$, but allow variations in the lattice. When we insist on the Hadamard type, we shall see that, in some familiar fractal- examples, it will then not be possible to choose any lattice $K$ such that the corresponding system $s=s(-, K)$ is selfadjoint. We will then say that the system is not self-adjoint; it turns out that the obstruction is a certain integrality condition; and, when it is not possible to find a lattice consistent with both selfadjointness and Hadamard type, then it will typically be a simple, case by case computation, and we shall be very brief with detailed calculations. (It will be immediate that each of the examples in the list is irreducible; see section 6.)
7.2 Group 1 Examples. We take $N=2$; the matrix is (3.3), and the examples are illustrated with subsets of the line, i.e., $n=1$, for $\mathbb{R}^{n}$. First, take $R=4$, i.e., multiplication by the integer 4 ; the sets $B$ and $L$ will be $B=\{0,1 / 2\}, L=\{0,1\}$, and lattice $K=\mathbb{Z}$. The $(\Omega, \Lambda)$ spectral pair will be as follows:
$\Omega=[0,1 / 4] \cup[1 / 2,3 / 4]$, (i.e., the union of two intervals);
$\Lambda=\{0,1\}+4 \mathbb{Z}$, (i.e., two residue systems modulo 4, see (4.10) above for the general case)
$\bar{X}=$ iteration fractal, see Figure A, fractal dimension $D=\frac{\ln 2}{\ln 4}=1 / 2$, see the affine system (2.2), and also more details on $\mu$ in Section 2 of [JP6].

It is easy to check that with this choice for $R, B, L$, and $K$, the corresponding system $s$ is selfadjoint and of the Hadamard type. For this particular example, there are only two choices for $K$ such that the corresponding system $s_{K}=s(-, K)$ is selfadjoint. They are $K=\mathbb{Z}$ and $K=2 \mathbb{Z}$. But the following modification,
corresponding to the classical middle-third-Cantor set, will only be symmetric; not selfadjoint: With

$$
\begin{aligned}
R & =3, \\
B & =\{0,2 / 3\}, \quad \text { and } \\
L & =\{0,3 / 4\},
\end{aligned}
$$

we have the Hadamard type, c.f., (3.2); but there is no lattice $K$ in $\mathbb{R}$ which makes the corresponding system $s_{K}$ selfadjoint (Graphic illustration, Figures A and 1).
7.3 Group 2 Examples. We take $N=3$; the matrix is (3.4), and the examples are illustrated with subsets of the plane $\mathbb{R}^{2}$. Take

$$
\begin{aligned}
R= & \left(\begin{array}{ll}
6 & 0 \\
0 & 6
\end{array}\right), \\
B= & \left\{\binom{0}{0},\binom{\frac{1}{2}}{0},\binom{0}{\frac{1}{2}}\right\}, \\
L= & \left\{\binom{0}{0}, \pm \ell\right\} \quad \text { where } \quad \ell=\frac{2}{3}\binom{1}{-1} \\
K= & 3 \mathbb{Z}^{2}, \text { i.e., multiples of the unit-lattice in } 2 \text { dimensions, equivalently } \\
& \text { points in } \mathbb{R}^{2} \text { of the form }\binom{3 m}{3 n} \text { where } m, n \in \mathbb{Z} .
\end{aligned}
$$

The corresponding system will be selfadjoint of Hadamard type. If $K$ is instead taken to be the lattice generated by the two vectors $\binom{1}{1}$ and $\binom{0}{\frac{3}{2}}$ (which turns out to yield $K \subset L^{\circ}$ ), then there is a corresponding spectral pair $(\Omega, \Lambda)$ where $\Omega$ is a suitable union of scaled squares in the plane, and the corresponding spectrum satisfies $\Lambda^{\circ}=K$. But with this $K$, the iteration system $s_{K}$ will not have $K^{\circ}$ total in $L^{2}(\mu)$. In all, there are only three distinct choices, in this case, for lattices $K$ in $\mathbb{R}^{2}$ such that the corresponding system $s_{K}=s(-, K)$ is selfadjoint: They
are given by the respective matrices $3 I_{2},\left(\begin{array}{ll}3 & 0 \\ 3 & \frac{3}{2}\end{array}\right)$, and $\left(\begin{array}{cc}1 & 0 \\ 1 & \frac{3}{2}\end{array}\right)$ with inclusions $K_{1} \subset K_{2} \subset K_{3}$ for the lattices. The fractal dimension of $\bar{X}$ is

$$
D=\frac{\ln 3}{\ln 6} \simeq .61 .
$$

(Graphic illustrations, Figures 2-9.)
7.4 Group 3 Examples. We take $N=4$; the matrix is (3.5) corresponding to $u=-1$, and the examples are illustrated with solid sets, i.e., pictures in 3 -space $\mathbb{R}^{3}$. Take

$$
\begin{aligned}
& R=\left(\begin{array}{lll}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{array}\right), \\
& B=\left\{\left(\begin{array}{l}
0 \\
0 \\
0
\end{array}\right),\left(\begin{array}{c}
-\frac{1}{2} \\
0 \\
0
\end{array}\right),\left(\begin{array}{c}
0 \\
-\frac{1}{2} \\
0
\end{array}\right),\left(\begin{array}{c}
0 \\
0 \\
-\frac{1}{2}
\end{array}\right)\right\}, \\
& L=\left\{\left(\begin{array}{l}
0 \\
0 \\
0
\end{array}\right),\left(\begin{array}{c}
-1 \\
-1 \\
0
\end{array}\right),\left(\begin{array}{c}
-1 \\
0 \\
-1
\end{array}\right),\left(\begin{array}{c}
0 \\
-1 \\
-1
\end{array}\right)\right\}, \\
& K=\mathbb{Z}^{3}=K^{\circ} \quad \text { (i.e., selfduality). }
\end{aligned}
$$

It is convenient to summarize the choices for $R, B, L$ and $K$ as follows:

$$
\begin{aligned}
R & =2 I_{3}, \\
B & =-\frac{1}{2} I_{3}, \quad \text { and } \\
L & =-\left(\begin{array}{lll}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{array}\right),
\end{aligned}
$$

where

$$
I_{3}=\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)
$$

is the unit-matrix. The choices for the lattice $K$ are subjected to the conditions in Lemma 4.1. It turns out that the choice $K=\mathbb{Z}^{3}$ is the minimal one such that the system $s_{K}=s(-, K)$ is selfadjoint; and there is also a unique maximal choice for $K$ with $s_{K}$ selfadjoint, viz., $K=L^{\circ}$ where $L^{\circ}$ is given by (7.4) below. (Since $L$ is symmetric, the matrix for the lattice $L^{\circ}$ is $L^{-1}$.) The lattice $L^{\circ}$ has matrix represented by the inverse

$$
L^{-1}=-\frac{1}{2}\left(\begin{array}{ccc}
1 & 1 & -1  \tag{7.4}\\
1 & -1 & 1 \\
-1 & 1 & 1
\end{array}\right)
$$

The corresponding system $s=(R, B, L, K)$ is selfadjoint of Hadamard type. If the choice for $K=\mathbb{Z}^{3}$ is replaced by $K=L^{\circ}$, then (the modified) $s$ is still selfadjoint: Notice that $K=\mathbb{Z}^{3}$ is the minimal choice for $K$ (subject to (i)-(iii) in Lemma 4.1); and $K=L^{\circ}$ is the maximal one. This means that $K^{\circ}=\mathbb{Z}^{3}$ is maximal among the possible choices for $K^{\circ}$; and this $K^{\circ}$ is total, see Definition 5.1.
7.5 Dual Pairs. The fractal dimension is $D=\frac{\ln 4}{\ln 2}=2$ which is integral, but less than the dimension (viz., 3) of ambient $\mathbb{R}^{3}$. The fractal for the system $s$ arises from scaling iteration of the set $\Omega=$ union of 4 cubes, see the figure (Figure B). For the dual system, $\Omega^{\circ}$ is instead the union of tetrahedra resulting in a 3-dimensional Sierpinski gasket, same fractal dimension $D=2$, but with angles $60^{\circ}$ rather than $90^{\circ}$. The sketch is Figures $10-17$, see also [Sch] for more details; it is the Eiffel tower construction, (maximal strength with least use of iron.)

The corresponding planar Sierpinski-gasket corresponding to $R=\left(\begin{array}{ll}2 & 0 \\ 0 & 2\end{array}\right), B=$ $\frac{1}{2} I_{2}$, and $L=\frac{2}{3}\left(\begin{array}{cc}1 & -1 \\ -1 & 1\end{array}\right)$, does not have a lattice choice for $K$ which makes the
associated system $s=(R, B, L, K)$ in $\mathbb{R}^{2}$ selfadjoint. The fractal dimension is $D=\frac{\ln 3}{\ln 2} \approx 1.58$.

For the matrix (3.5) with primitive 4th roots of 1 (e.g., $u=i$ ), there is also a realization in $\mathbb{R}^{3}$ : We may take $R=2 I_{3}, B=\frac{1}{2} I_{3}$, and $L=\left(\begin{array}{ccc}\frac{1}{2} & 1 & \frac{3}{2} \\ 1 & 0 & 1 \\ \frac{3}{2} & 1 & \frac{1}{2}\end{array}\right)$ will give a system of Hadamard type in 3-space, but again there is no lattice choice for $K$ in $\mathbb{R}^{3}$ such that the corresponding $s_{K}$ is selfadjoint. (Graphic illustrations, Figures 10-17.)

## 8. Concluding Remarks

The operators $\left\{T_{\ell}\right\}$ from (6.5) and (6.7) may also be used in the definition of an endomorphism $\theta$ on a certain $C^{*}$-algebraic $\mathcal{O}_{N}$-crossed product, $\mathfrak{U}$ say. It is given by,

$$
\theta(A)=\sum_{\ell \in L} T_{\ell} A T_{\ell}^{*}, \quad \text { for } A \in \mathfrak{U},
$$

and clearly, $\theta\left(A^{*}\right)=\theta(A)^{*}$, and $\theta(A B)=\theta(A) \theta(B)$ for all $A, B \in \mathfrak{U}$. Continuous versions, also called endomorphism-semigroups, have been studied recently by Arveson and Powers, see e.g., [Ar]. As spectral-invariants for these, Arveson has proposed (in [Ar]) a Cuntz-algebra construction which is based on WienerHopf techniques, and which is inherently continuous, in fact with $\mathbb{R}_{+}$used as index for the generators in place of the usual finite (or infinite) discrete labeling set $\{1, \ldots, N\}$. For our present $B / L$ duality project with dual fractals, $\bar{X}(B)$ and $\bar{X}(L)$; we plan (in a sequel paper) to study an analogous $C^{*}$ - algebra construction which is generated by $\bar{X}(L)$ in place of $\mathbb{R}_{+}$, but still modelled on Arveson's WienerHopf approach. It appears that such an $\bar{X}(L)$-fractal-based $C^{*}$-algebra will serve
as a spectral-invariant for our $B / L$ Hadamard-systems which are only symmetric, but generally not selfadjoint (relative to some choice of lattice $K$, see Section 7 above).

The spectral-invariant question is an important one, and in our case we produce the dual representation pair (6.4) and (6.5) as a candidate. But representations $\left\{S_{b}\right\}$ of $\mathcal{O}_{N}$ in the form (6.4), without a paired dual representation $\left\{T_{\ell}\right\}$, cf. (6.6), are present for iteration systems which are much more general than the affine fractals studied here. As a case in point we mention Matsumoto's [Mat] recent analysis of (von Neumann type) cellular automata (details in [MOW] and [Od]); it is based on an $S$-representation which is given by a formula similar to our (6.4) above. There is also an associated endomorphism with an entropy that can be computed; but we stress that for these (and many other) iteration systems, there is typically not a dualitly based on exponentials $e^{i \lambda \cdot x}$ and typically not a second $\left\{T_{\ell}\right\}$ - representation such that the two form a dual pair in any natural way.

We have studied the class of spectral systems $s=(R, B, L, K)$ in $\mathbb{R}^{n}$ with special view to the selfadjoint ones which are also of Hadamard type, see Lemma 4.1. (When $s$ is given in this class, the two sets $B$ and $L$ then have the same cardinality; it will be denoted $N$ for convenience in the following comments.) It is important (but elementary) that this class of systems is closed under the tensor- product operation; i.e., if $s_{1}$ and $s_{2}$ are systems in $\mathbb{R}^{n_{1}}$ and $\mathbb{R}^{n_{2}}$ respectively, then the two properties (selfadjointness and Hadamard type) carry over to the system $s_{1} \otimes s_{2}$ in $\mathbb{R}^{n_{1}+n_{2}}=$ $\mathbb{R}^{n_{1}} \times \mathbb{R}^{n_{2}}$.

If the Hilbert spaces for the respective systems are $L^{2}\left(\mu_{i}\right), i=1,2$; then the Hilbert space for $s_{1} \otimes s_{2}$ is $L^{2}\left(\mu_{1} \otimes \mu_{2}\right)$, and the measure $\mu_{1} \otimes \mu_{2}$ is the unique probability measure on $\mathbb{R}^{n_{1}} \times \mathbb{R}^{n_{2}}$ which scales the affine tensor operations of $s_{1} \otimes s_{2}$,
see (2.3) above. The set $B$ for $s_{1} \otimes s_{2}$ is $B_{1} \times B_{2}$, and the matrix-operation is, $\left(b_{1}, b_{2}\right) \mapsto\left(R_{1} b_{1}, R_{2} b_{2}\right)$. In verifying the Hadamard property (3.2) for $s_{1} \otimes s_{2}$, we use the important (known) fact that the class of generalized Hadamard matrices is closed under the tensor-product operation, i.e., that $U_{1} \otimes U_{2}$ satisfies (3.2) with order $N=N_{1} N_{2}$ if the individual factors $U_{i}, i=1,2$, do with respective orders $N_{i}$, $i=1,2$.

We say that a system $s$ is irreducible if it does not factor "non-trivially" $s \simeq$ $s_{1} \otimes s_{2}$; and we note that the examples above from Section 7 are all irreducible in this sense. (In fact this irreducibility notion is different from that of Section 6, but the examples are irreducible in both senses.)

The spectral geometry for regions in $\mathbb{R}^{n}$ has a long history, see e.g., [Bo-Gu], [CV], [Ge], and [Gu-St]. But, so far, the Laplace operator has played a favored role despite the known incompleteness for the correspondence between the geometry of the given domain and the spectrum of the corresponding Laplace operator. The approach in [De] is based instead on a multitude of second order differential operators, but the spectral correspondence is still incomplete there. Our present approach leads to a complete spectral picture and is based instead on a system of first order operators. For the fractal case however, the differential operators have no analogue.

While our simultaneous eigenfunctions are based, at the outset, on a commutative operator system, our spectral invariant derives instead from a dual pair of representations of a certain non-abelian (in fact simple) $C^{*}$-algebra.

Self-similar limit constructions have received much recent attention, starting with [Hu], and then more recently, see e.g., [Ba-Gr], [Ed], [Ma], and [Ke]. These results seem to stress the geometry and the combinatorics of the infinite limits, and not
the spectral theory. Our present emphasis is a direct spectral/geometry- correspondence; and we also do not in [JP6] impose the strict expansivity assumption (which has, so far, been standard almost everywhere in the literature). Furthermore, we wish to stress that the sets $\Omega \subset \mathbb{R}^{n}$ which occur in our present spectral pairs are more general than the self-reproducing tiles (SRT) which were characterized in $[\mathrm{Ke}$, Theorem 10]. However, Kenyon's SRT's can be shown to satisfy our conditions, although our class is properly larger; not only because of the expansivity assumption, but also because of the combinatorics, see [Jo-Pe5] for details. Further work on these interconnections is also in progress.

## Acknowledgments

Both authors were supported in part by grants from the U.S. National Science Foundation. It should finally be mentioned that the portion of the problem which related to commuting operator extensions for the partial derivatives $\frac{\partial}{\partial x_{j}}$ originates with suggestions made first in 1958 by Professor I.E. Segal, see [Fu] for details on this point. Encouragement and several correspondences from Professors B. Fuglede and R. Strichartz are also greatly appreciated. Detailed suggestions from Strichartz led to substantial improvements.

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[^0]:    1991 Mathematics Subject Classification. Primary 28A75, 42B10, 46L55; Secondary 05B45.
    Key words and phrases. Iterated function system, affine maps, fractional measure, harmonic analysis, Hilbert space, operator algebras.

    Research supported by the NSF.

