POINT PROCESSES AND TIME SERIES ANALYSIS: THEORY AND APPLICATIONS TO COMPLEX PHYSIOLOGICAL PROBLEMS.

by

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DEDICATED TO

my parents George and Elisavet.

CONTENTS

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I would also like to express my deepest thanks to Dr. J.R. Rosenberg, not only for the endless discussions which we had about the complexity of the physiological problems, but also for his continual support and encouragement during the completion of this work.

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SUMMARY

This thesis presents, develops and applies statistical methods for treating point processes and time series in the analysis and identification of a complex physiological system.

To provide the necessary background for an understanding of the physiological problems, we give in Chapter 1 a brief discussion of the neuromuscular system and a detailed description of the muscle spindle - the component of the neuromuscular system in which we are interested. In addition, the basic data sets obtained from the muscle spindle under different experimental conditions are described and illustrated.

In Chapter 2 the general theory of time series and point processes is reviewed. Definitions of time- and frequency-domain parameters are given under the assumption of stationarity. In the case of point processes the events are assumed to be isolated.

The response of the muscle spindle, recorded from a single nerve fibre (axon), is assumed to be a univariate stationary point process with isolated points. In Chapter 3, certain parameters of the univariate point process are defined in both time and frequency domains and their estimates are considered. The asymptotic properties of these estimates are developed and asymptotic confidence intervals are

constructed for the time- and frequency-domain parameters of the univariate point process. The comparison of three different methods of estimating the power spectrum of the response of the muscle spindle is presented and illustrated using the data sets given in Chapter 1. The importance of the frequency-domain parameters in comparison to the time-domain ones is discussed, and it is shown that important features of the data appear only in the frequency domain description. Finally, we discuss the estimation of time-domain parameters by using the frequency domain methods which in practice allow a rapid calculation of these estimates. For long records it is in fact faster to estimate the time domain parameters by starting in the frequency domain and transforming to the time domain, than it is to calculate the time domain estimates directly from the data.

In Chapter 4 we consider the problem of identification when both the input to and the output from the muscle spindle are assumed to be realizations of a bivariate point process. Estimates of certain parameters of the bivariate point process in both time and frequency domains are considered and computed by extending the procedures given in Chapter 3. The asymptotic properties of these estimates are developed and in some cases confidence limits are constructed. The model used for the linear identification of the muscle spindle is the relevant part of the Volterra expansion for point processes introduced by Brillinger. The results obtained by using this model clearly indicate that the response of the muscle spindle and the input point process are strongly related for low and moderate frequencies. Some non-linear features of the muscle spindle can be examined by adding an additional term to the linear part of the Volterra expansion leading to a discussion of the quadratic model.

In Chapter 5 we discuss the problem of identification when the input to the muscle spindle is a continuous signal (time series) while the output from the same system is a point process. By considering the Fourier-Stieltjes transform of the time series and the point process respectively, we define the cross-periodogram between hybrid processes involving a time series and a point process. An estimate of the cross-spectrum between hybrid processes is obtained by smoothing the cross-periodogram. Estimates of other parameters of the hybrid processes in the time and frequency domains are discussed and illustrated. The model proposed for the linear identification of the muscle spindle is the relevant part of the Volterra expansion for systems having a point process as output and a time series as input. The results obtained by using this model suggest that the responses of the muscle spindle to a continuous input are strongly related at moderate and higher frequencies. The effect on the output from the muscle spindle of a point process input in the presence of a second continuous input to the muscle spindle is also considered. In addition, the effect on the output from the muscle spindle of a continuous input in presence of a second point process input is examined. The results clearly indicate that when both inputs are present simultaneously the output from the muscle spindle is primarily determined by the point process input at low frequencies and by the continuous input at moderate and higher frequencies.

A number of problems which arise when the muscle spindle is affected by several inputs are given in Chapter 6 as topics for further research.

The development of the statistical methods presented in this thesis and the conclusions obtained by applying these methods to neurophysiological data required the close collaboration of applied statistics and neurophysiology. Real progress in collaboration of this kind can only be obtained when the statistician learns a fair amount about the physiological background of the problem, and conversely when the physiologist familiarises himself with the statistical methodology.

ABBREVIATIONS

ACF	AUTO-COVARIANCE FUNCTION
ACH	AUTO-COVARIANCE HISTOGRAM
AIF	AUTO-INTENSITY FUNCTION
CCF	CROSS-CUMULANT FUNCTION
CF	CUMULANT FUNCTION
CIF	CROSS-INTENSITY FUNCTION
CPD	CROSS-PRODUCT DENSITY
CS	CROSS-SPECTRUM
FFT	FAST FOURIER TRANSFORM
MI	MEAN INTENSITY
PS	POWER SPECTRUM

PREFACE

The data sets used in this thesis were obtained from experiments performed at the University of Southern California during the Summers of 1978 and 1979. The experimental work was jointly supported by grants from the National Institutes of Health (U.S.A.) to G.P. Moore, and from the Wellcome Trust and the S.E.R.C. to J.R. Rosenberg.

The Department of Statistics at the University of Glasgow was asked by J.R. Rosenberg, who provided the data sets, for help in the development of methods for the analysis of systems influenced by both point processes and continuous signals and giving rise to point process outputs.

This work follows on from and extends earlier studies by Professor D.R. Brillinger. His book on time series analysis and papers on point processes clarified for me many ideas related to the frequency domain analysis of point processes.

The development of the problems presented in this thesis was entirely the work of the author. Chapters 3, 4 and 5 are original contributions to the theory and applications of point process.

CHAPTER ONE

A DESCRIPTION OF SOME COMPONENTS OF THE NEUROMUSCULAR SYSTEM

1.1 INTRODUCTION

An important feature of many biological systems is that under normal operating conditions they are acted upon by several inputs simultaneously, and in turn may generate several distinct outputs. The field of neurobiology offers many examples. The muscle spindle is an important component of the neuromuscular system and is thought to provide information to other parts of the nervous system that is important in the control of movement and the regulation of posture. During the course of a movement the muscle spindle is acted upon by a continuous change in length, which occurs as a consequence of the movement, in addition to several independent point process inputs which act simultaneously with the length change to alter the point process output from the muscle spindle. Moreover, it has been recognized by some statisticians that the field of neurophysiology provides a rich source of problems and data relating to point process systems (e.g. Brillinger, 1975d; Brillinger, Bryant & Segundo, 1976; Cox & Isham, 1980; Cox & Lewis, 1968; Feinberg, 1974; and Sampath & Srinivasan, 1977).

The object of the first part of this chapter is to describe some elements of the neuromuscular system and to introduce the minimal technical vocabulary required to appreciate the complexity of this system. Furthermore, this will help us to identify the problem areas, and see the scope for the application of statistical procedures.

We will first present a brief description of the organization of a part of the neuromuscular system and then consider in some detail the structure and operation of one of its components. References to various aspects of the neuromuscular system will be included at appropriate points in the text as a guide to the vast literature on this subject.

In the second part of this chapter we present, describe and discuss the kind of data that can be obtained from the muscle spindle. The development of statistical procedures for the analysis of these data sets is the main interest of this thesis.

1.2 THE PERIPHERAL NERVOUS SYSTEM

The neuromuscular control system may be thought of as all those parts of the nervous and muscular systems concerned with the initiation and control of movement and maintenance of posture. On anatomical and functional grounds this system has been divided into peripheral and central parts. The peripheral nervous system, at the level of the spinal cord, is arranged in a sequence of repeating units. These units are organisationally identical and are referred to as "segmental levels" of the spinal cord. The components of the peripheral neuromuscular system at one segmental level of the spinal cord are outlined in Fig.1.2.1. There are several classes of nerve cells which lie within the spinal cord in groups called nuclei some of which may contain as many as 2000 cells. One of these groups, called "alpha-

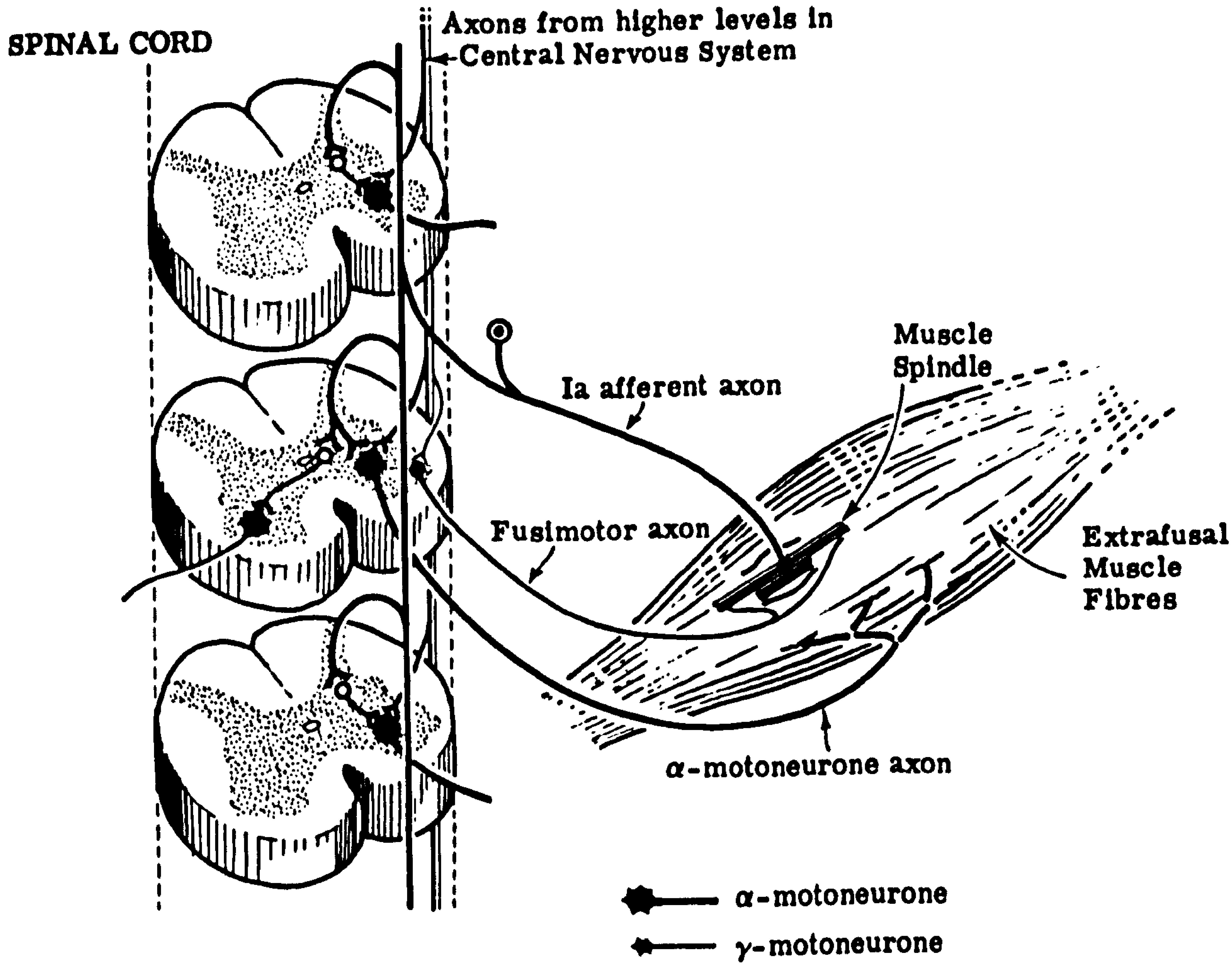


Fig. 1.2.1: Diagram of some of the pathways connecting a muscle spindle and its parent muscle to the spinal cord.

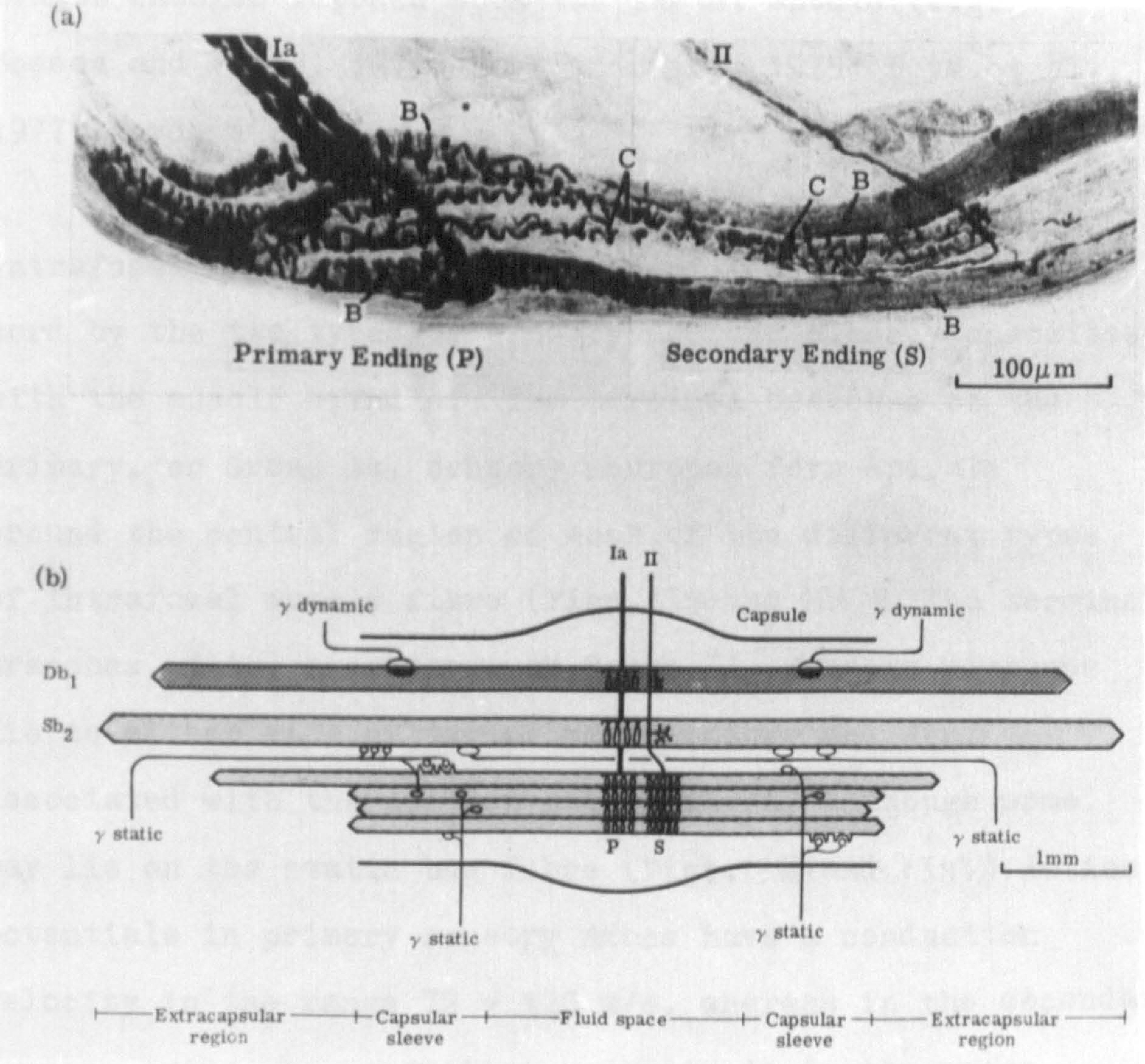
motoneurones", have cell bodies whose diameters range from 25 to 100 μm and have long processes, from 8 to 20 μm in diameter, called "axons", which leave the spinal cord to innervate the load-bearing or "extrafusal" muscle fibres forming the main mass of the muscles responsible for generating forces or changes of length. The axons of the alpha-motoneurones normally conduct nerve impulses from the cell body to the extrafusal muscle fibres. The nerve impulse is a localised voltage change which occurs across the membrane surrounding the nerve cell body and axon. It is propagated along the axon at a velocity which depends, in part, on the diameter of the axon. The nerve impulse is approximately 100 mV in amplitude and 1 ms in duration. In the axon of the alpha-motoneurone this pulse is conducted at velocities in the range 50 - 120 m/s. Nerve impulses are often referred to as "action potentials" or, because of their relatively short duration, as "spikes". Nerve cells can generate propagated action potentials repetitively to produce spike trains having mean frequencies which may vary from one pulse every few seconds to several hundred pulses per second. The fine terminal branches of the alpha-motoneurone axon end on specialised areas of the extrafusal muscle fibre called the "motor endplate". When a nerve impulse reaches the junction between the axon and the muscle fibre a sequence of electro-chemical events occurs which leads to the contraction of the load-bearing muscle fibres. The force of the contraction of the entire muscle may be graded by increasing the number of active alpha-motoneurones associated with a muscle and by altering the frequency of the nerve impulses reaching the muscle over the axons of the alpha-motoneurones. Muscles concerned with posture or movement are called "skeletal" muscles and are made up of a large number of individual muscle fibres innervated by the alpha-motoneurones. Once the axon of an alpha-motoneurone reaches a muscle it divides into a number of fine branches. Each terminal branch innervates a single extrafusal fibre of one muscle and all of the extrafusal fibres innervated by one alpha-motoneurone lie within the same muscle. The alpha-motoneurone and all of the muscle fibres that it innervates is called a "motor-unit". Skeletal muscles are thus, in part, made up of groups of motor-units. The size of a motor-unit (the number of muscle fibres innervated by a particular alpha-motoneurone), and the number of motor units within a muscle, depend on the function of that muscle.

Associated with the extrafusal muscle fibres and the tendons which attach the muscles to bone are a number of physiological transducers, or "receptors", that are sensitive to imposed changes of length or force acting on the parent muscle. The nerves associated with these receptors, which normally conduct action potentials toward the spinal cord, are called "sensory" nerves. These nerves transmit pulse-coded information inherent in some function of the sequence of action potentials transmitted along their axons from the muscle receptors to the groups of nerve cells lying within the spinal cord. The sensory axons make contact with these cells at junctions called "synapses". Each sensory axon divides into a number of branches after entering the spinal cord and these make synaptic contact with a large number of nerve cells over several segmental levels of the spinal cord. Conversely, each cell within the spinal cord receives input from a large number of sensory axons from different receptors in the same muscle and from receptors associated with different muscles. The train of action potentials travelling along the axon releases a sequence of electro-chemical events at the point of contact between the sensory axon and the nerve cell within the spinal cord. These synaptic events then modify the on-going activity of these cells.

In addition to their activity being modified by pulse-coded signals from the muscle receptors, complex interactions occur between cells at one segmental level of the spinal cord as well as between segmental levels within the spinal cord. The segmental and inter-segmental circuits in the spinal cord are further acted upon by axons arising from nerve cells at higher levels of the central nervous system. It has been estimated that a single alpha-motoneurone may receive as many as 10000 inputs. An excellent introductory account of the organisation of the spinal cord may be found in Shepherd (1974), and a detailed review of the organisation and properties of spinal cord neurones and their interconnections is given by Burke and Rudomin (1977).

1.3 THE MUSCLE SPINDLE

Muscle spindles are one particularly important class of muscle receptor which are thought to have an important role in the initiation of movement and the maintenance of posture. The muscle spindle is a transducer which responds primarily to length changes imposed on the parent muscle. Most skeletal muscles contain a number of these receptors which lie in parallel with the extrafusal fibres. They consist of a number of specialised muscle fibres lying parallel to each other and partially contained within a fluid-filled capsule of connective tissue (e.g. Boyd, 1962). The fibres within a muscle spindle, the so-called "intrafusal" fibres, are considerably shorter than the extrafusal muscle fibres. In the cat, for example, the soleus muscle, which is used in extension of the ankle, has extrafusal muscle fibres of average length 3 cm, whereas the longest intrafusal fibres are, on average, only 0.7 cm long. There are three different types of intrafusal muscle fibre. These are the dynamic nuclear bag fibres (Db_1), the static nuclear bag fibres (Sb_2) and the nuclear chain fibres (C). Most muscle spindles contain one of each type of nuclear bag fibre and up to six nuclear chain fibres. The mechanical properties of the three types of intrafusal fibres are different and consequently they respond differently to



<u>Fig. 1.3.1</u>: Primary and secondary sensory endings.
(a) Photograph of the primary and secondary sensory endings in a muscle spindle dissected free of the surrounding extrafusal muscle fibres and stained with gold chloride.
(b) Diagram of the sensory and motor innervation of the muscle spindle. P indicates primary sensory endings. S indicates secondary sensory endings.

length changes imposed upon the parent muscle (e.g. Bessou and Pagès, 1975; Boyd and Ward, 1975; Boyd et al., 1977; Boyd, 1980).

The effects of the imposed length changes on the intrafusal muscle fibres are transmitted to the spinal cord by the two types of sensory neurone closely associated with the muscle spindle. The terminal branches of the primary, or Group Ia, sensory neurones form spirals around the central region of each of the different types of intrafusal muscle fibre (Figs. 1.3.1(a) and 1.3.1(b)). The terminal branches of the secondary, or Group II, sensory neurones lie to either side of the primary endings and are associated with the nuclear chain fibres, although some may lie on the static bag fibre (Figs. 1.3.1(a) and 1.3.1(b)). Action potentials in primary sensory axons have a conduction velocity in the range 72 - 120 m/s, whereas in the secondary sensory axons the conduction velocity is in the range 24 - 72 m/s.

When a muscle is held at a fixed length the sensory axons from the muscle spindle generate action potentials at a constant rate which depends upon the muscle length (Matthews and Stein, 1969). An increase in muscle length will increase the rate of discharge of action potentials in both the primary and secondary sensory endings. The deformation of the intrafusal muscle fibres caused by length changes imposed on the parent muscle distorts the fine terminals of the sensory axons, and, by a process not entirely understood, the rate of pulse-coded activity of these axons is then modified. Each muscle spindle is innervated by a single Ia axon but may have several Group II axons. The changes in activity in the Ia sensory axon in part reflect the responses to imposed length changes in all three kinds of intrafusal fibre, whereas the activity of the secondary axons reflect, mainly, changes in the nuclear chain fibres. It has recently been shown that the primary and secondary sensory axons project largely to different groups of cells within the spinal cord and therefore may be associated with quite different functions (Johansson, 1981).

In addition to the sensory nerves associated with the muscle spindle, the intrafusal muscle fibres are innervated by the axons of a group of cells lying within the spinal cord in the neighbourhood of the alpha-motoneurones. These cells are considerably smaller in diameter that the alpha-motoneurones and conduct nerve impulses with velocities in the range 10 - 50 m/s. These "gamma-motoneurones" innervate only intrafusal muscle fibres and are often referred to as "fusimotor" neurones. Each gamma-motoneurone may innervate intrafusal fibres lying in different muscle spindles within the same muscle. The fusimotor neurones have been divided into two broad categories, gamma-dynamic and gamma-static axons (Matthews, 1962; Crowe and Matthews, 1964; Emonet-Dénand et al., 1977). The gamma-dynamic axons innervate the dynamic nuclear bag fibres whereas the gamma-static axons innervate either the nuclear chain fibres or the static nuclear bag fibres or both (Boyd, 1980; Matthews, 1981). A single muscle spindle may be innervated by as many as six fusimotor neurones.

The main structural features of the muscle spindle are summarised in Fig. 1.3.1 (Boyd, 1962; Boyd, 1980). Fig. 1.3.1(a) is a photograph of an isolated muscle spindle in which the spirals of the primary sensory ending are clearly seen wrapped round all of the intrafusal muscle fibres, whereas the terminals of the secondary ending are largely restricted to chain fibres. Fig. 1.3.1(b) is a simplified diagram of the normal innervation of the muscle spindle. The Ia sensory axon and one gamma-motoneurone of a muscle spindle are also shown in Fig. 1.2.1.

Muscle spindles are extremely sensitive to small dynamic changes of length imposed on the parent muscle. For example, a sinusoidal length variation of 100 μm will produce an appreciable modulation of the pulse-coded response of the primary sensory ending. Under favourable conditions it has been shown that the activity of a primary sensory ending may be modulated by changes of length as small as 1 μm applied to the parent muscle.

The differences in the responses of the primary and secondary sensory axons to a ramp and hold stretch imposed on the parent muscle are illustrated in Fig. 1.3.2. The upper trace of Fig. 1.3.2(a) shows the response of a primary axon. For this case it may be seen that the rate of discharge of nerve impulses, or action potentials,

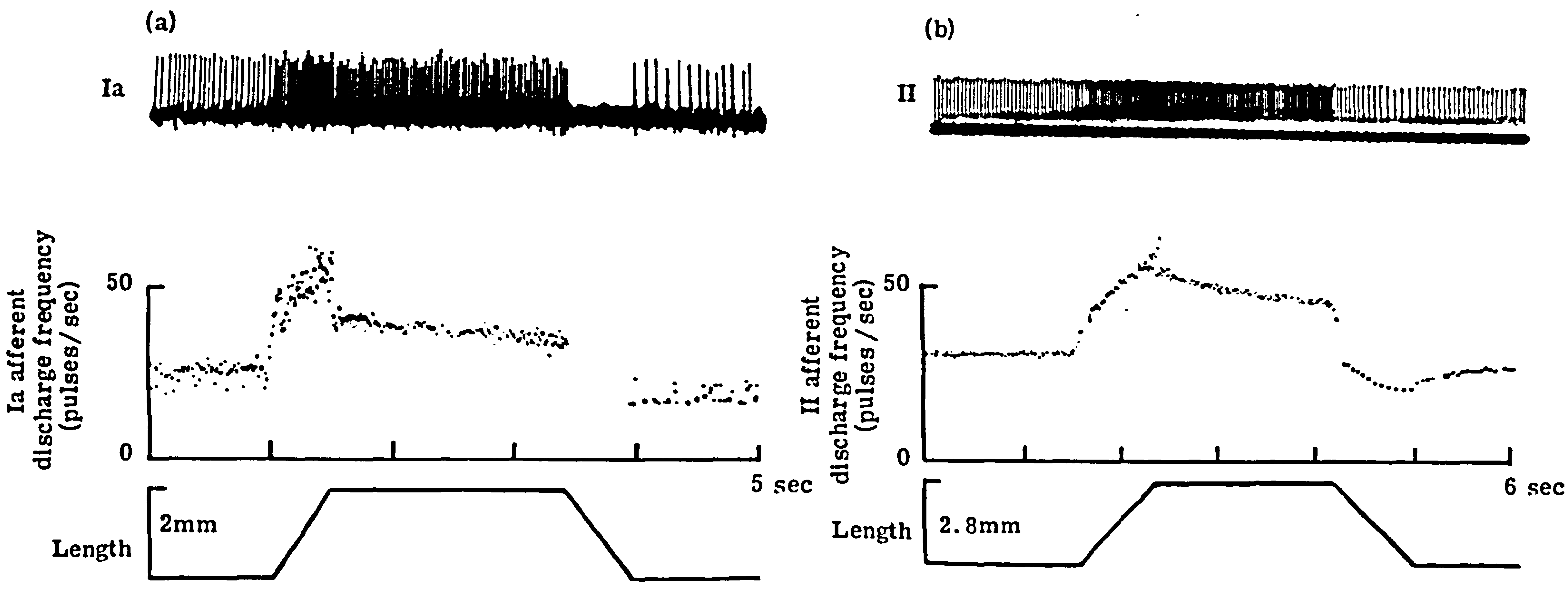


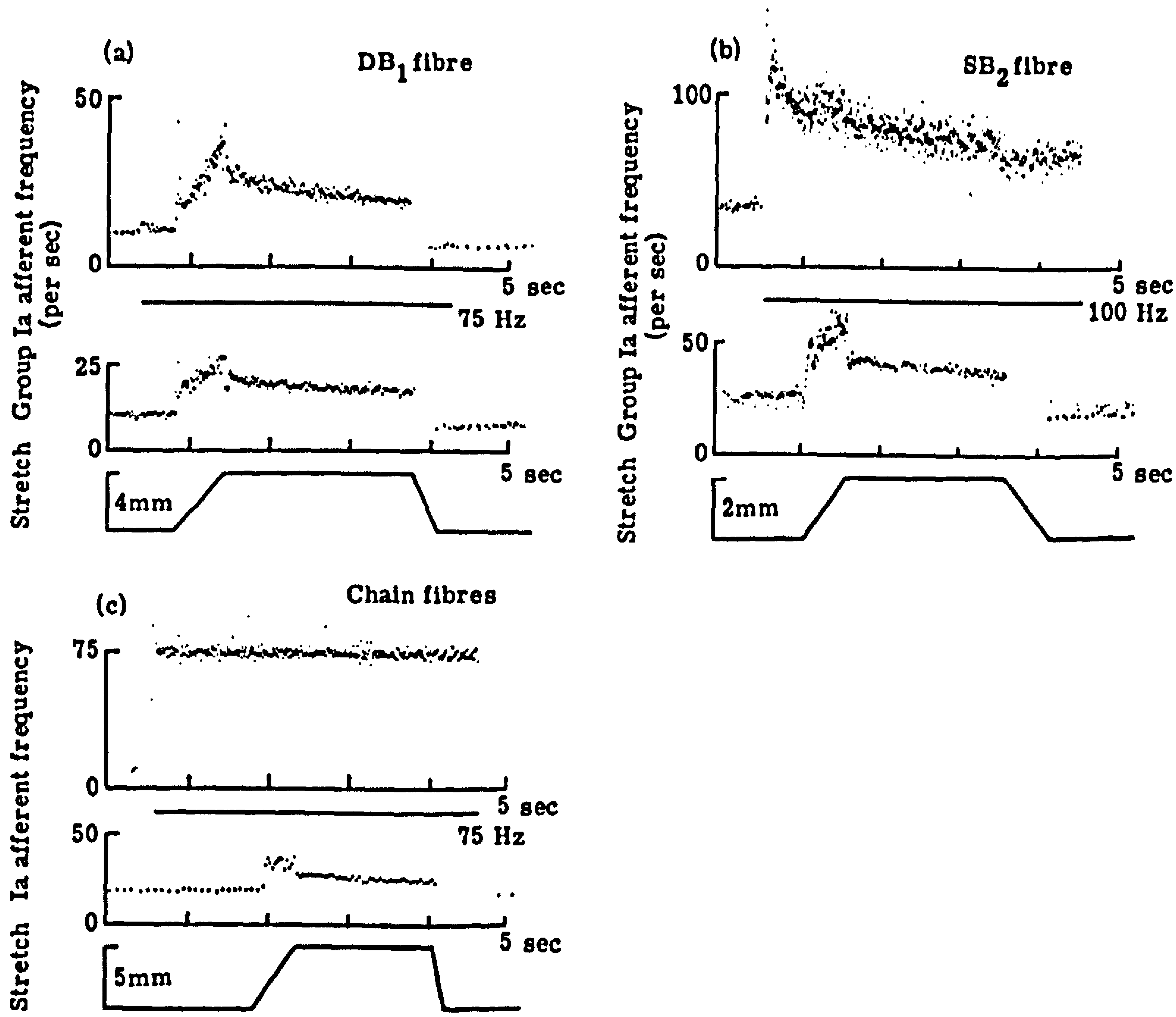
Fig. 1.3.2: The response of (a) muscle-spindle primary sensory axon and (b) secondary sensory axon to a ramp and hold stretch (lower trace) applied to the parent muscle. The upper trace in each figure is a photograph of the actual sequence of action potentials displayed on an oscilloscope screen. The middle trace in each figure represents the instantaneous frequency discharge calculated as the reciprocal of the interval between a spike and the immediately preceeding one.

increases rapidly during the applied stretch. Once the stretching has stopped there is a rapid decline in the discharge rate to a new constant level associated with the increased muscle length. The upper trace of Fig. 1.3.2(b) illustrates the response of a secondary sensory ending for the same type of stimulus.

The main features of the responses of the sensory axons to length changes imposed on the muscle spindle become clearer when the sequence of action potentials is represented by the instantaneous frequency of the discharge calculated as the reciprocal of the time between a given pulse and the immediately preceding one. This pulse frequency representation of the muscle spindle response to stretch is shown in the middle traces of Fig. 1.3.2(a) and Fig. 1.3.2(b). The different shapes of the responses of the primary and secondary sensory axons suggests different electro-mechanical properties of the intrafusal muscle fibres associated with these endings, and perhaps the transmission of different kinds of information to the spinal cord over each of the different kinds of sensory axon.

The gamma-motoneurones which innervate only the intrafusal muscle fibres modify the response of the muscle spindle sensory endings to imposed length changes. The different ways in which the gamma-motoneurones can modify the response of a Ia sensory axon to a ramp and hold stretch imposed on the parent muscle are illustrated in

Fig. 1.3.3. In each part of this figure the response of the primary ending to a ramp and hold stretch is shown in the absence and then in the presence of a constant rate of gamma motor axon stimulation. Fig. 1.3.3(a) illustrates how the dynamics of the response of a Ia axon is modified by the activation of a gamma motor axon selectively innervating a dynamic nuclear bag fibre. The rate of increase of discharge of the Ia ending and the magnitude of the overshoot following the end of the dynamic phase of the stretch are both increased by the presence of gamma motor stimulation. On the other hand, stimulation of a gamma motor axon associated with a static nuclear bag fibre (Fig. 1.3.3(b)) or nuclear chain fibre (Fig. 1.3.3(c)) almost eliminates the Ia response to the ramp and hold stretch. From Fig. 1.3.3(c) it appears that the constant rate of discharge of the gamma axon which leads to the chain fibres is imposed on the discharge of the Ia sensory axon to the extent that changes of muscle length are no longer effective in modulating the Ia discharge. From Fig. 1.3.3(b) it appears that the gamma-motoneurone discharge alone produces a strong dynamic response in the sensory discharge of the Ia axon. The interaction of this response with that of the imposed length change almost obscures the latter. These findings suggest that activity in the gamma motor axons may modify the mechanical properties of the intrafusal fibres or exert a direct effect on the mechanisms



<u>Fig. 1.3.3</u>: The effect of a constant rate of stimulation of a fusimotor axon innervating, (a) dynamic-bag$_1$ intrafusal muscle fibre (Db$_1$), (b) static-bag$_2$ intrafusal muscle fibre (Sb$_2$) and (c) chain fibres (C), on the response of a muscle-spindle primary (Ia) sensory axon to a ramp and hold stretch (lower trace). In (a) - (c) the Ia discharge is shown both in presence (upper trace) and absence (middle trace) of fusimotor stimulation. The duration of fusimotor stimulation is indicated by a straight line in (a) - (c).

involved in the conversion of mechanical strain in the intrafusal fibres to discharge of action potentials in the sensory axon.

1.4 BASIC DATA SETS OBTAINED FROM THE MAMMALIAN MUSCLE SPINDLE

The purpose of this final section is to present and describe the basic statistical characteristics of a sample from each of the data sets which form the experimental material of this thesis. In addition, the basic statistical properties of the two different kinds of input are described.

DATA SET I: Spontaneous discharge recorded from a single Ia axon

In the absence of a fusimotor input and a continuous modulation of the length of the muscle spindle the Ia and II sensory axons generate nerve action potentials at relatively constant rates (Matthews and Stein, 1969). The output which occurs under these conditions is referred to as the spontaneous discharge of the muscle spindle. The rate of the spontaneous discharge depends on the fixed length at which the muscle spindle is held. High rates of discharge correspond to large stretches applied to the muscle spindle and progressive decreases in muscle length lead to a monotonic decrease in the rate of the spontaneous discharge.

Fig. 1.4.1 gives some simple graphical methods of

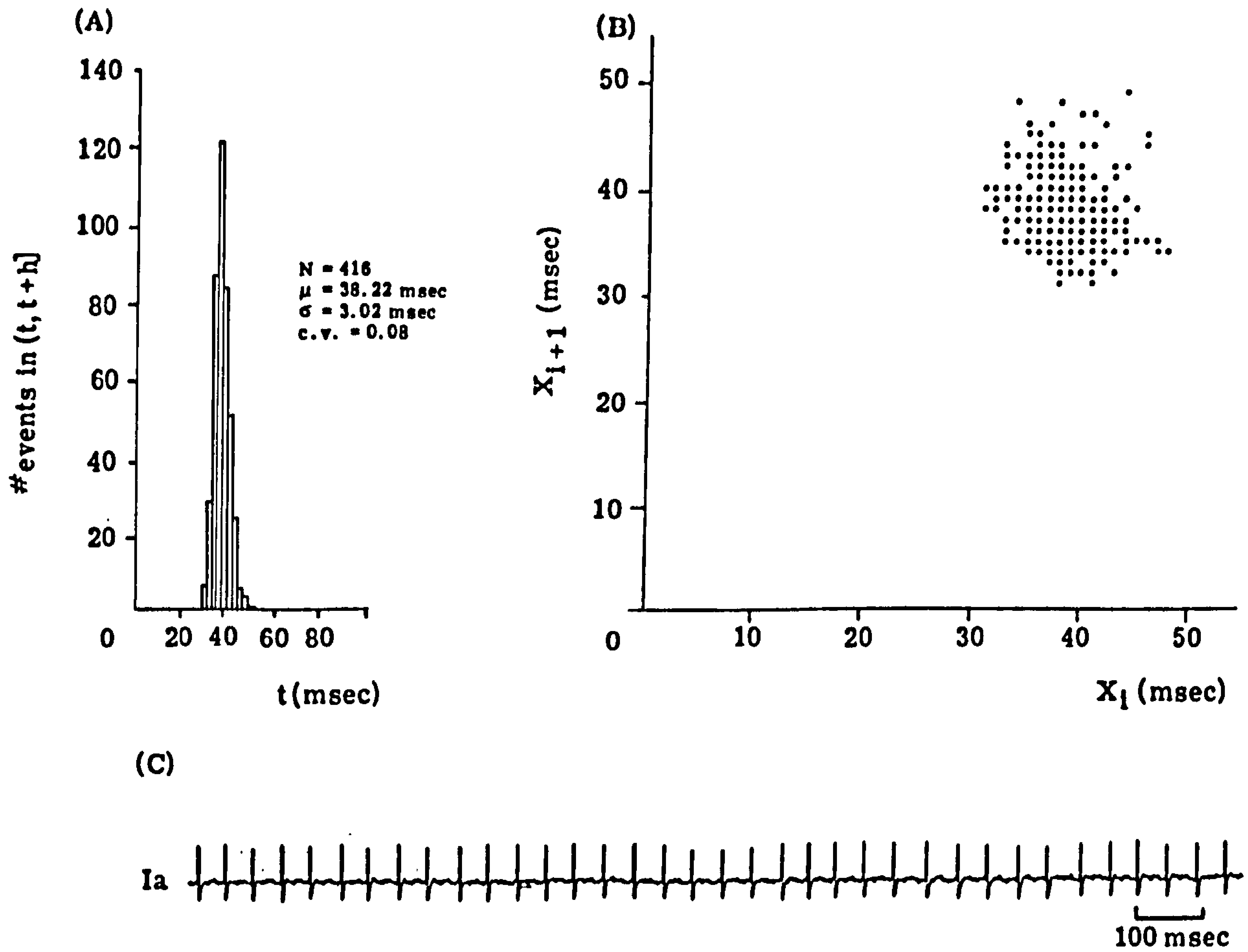


Fig. 1.4.1: Spontaneous Ia discharge. (A) Histogram of the inter-spike intervals of the Ia discharge (h = 2); (B) Scatter diagram of adjacent intervals between spikes of the Ia discharge; (C) Sequence of action potentials (spikes) of the Ia discharge.

representing Data set I. Fig. 1.4.1(A) is the histogram of the intervals between the action potentials of the spike train illustrated in Fig. 1.4.1(C). (h = 2). The estimates of the mean, standard deviation and the coefficient of variation of the intervals between successive events of the spike train are indicated in the upper right corner of Fig. 3.1.4(A). The total number of spikes is also shown for a record length of 15872 msecs.

Fig. 1.4.1(B) is the scatter diagram of the adjacent intervals between events. Graphically this means that we plot $X_{i+1} = \tau_{i+1} - \tau_i$ against $X_i = \tau_i - \tau_{i-1}$ where τ_i is the time of occurrence of the ith event of the spike train. It is obvious that the scatter diagram has a small dispersion and a high concentration of points around the mean value of the inter-spike intervals. This is easily explained because of the regularity of the events of the spontaneous discharge of the Ia sensory axon.

<u>DATA SET II</u>: <u>Ia response recorded in the presence of γs stimulation</u>

This particular set of data describes the Ia response when a fusimotor input (γs) is present. The γs stimulus is derived from the output of a Geiger counter placed close to a radioactive source. The axon of the fusimotor neurone is stimulated at twice threshold with these pulses.

Fig. 1.4.2 gives some simple graphical methods of representing the basic statistical features of the train

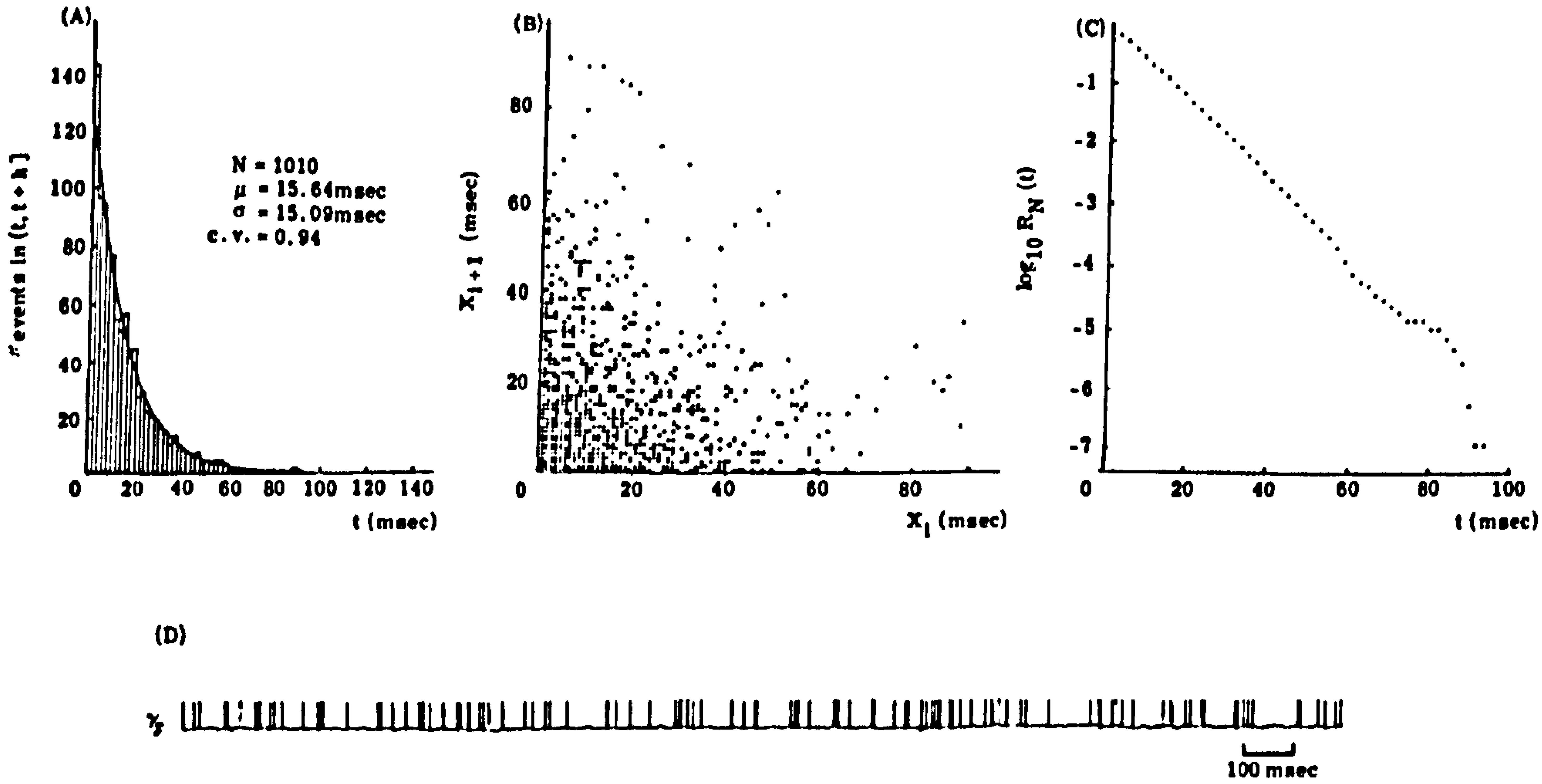


Fig. 1.4.2: Fusimotor input (γs). (A) Histogram of the inter-spike intervals of the fusimotor input fitted with an exponential distribution (h = 2); (B) Scatter diagram of adjacent intervals between pulses of the fusimotor input; (C) Log to base 10 of the empirical survivor function; (D) Sequence of pulses of the fusimotor input.

of pulses in the fusimotor input (γs). Fig 1.4.2(A) is the histogram of the intervals between pulses given in Fig. 1.4.2(D). (h = 2). This histogram has been fitted by an exponential probability density with an estimate of the parameter λ given by

$$\hat{\lambda} = \frac{1}{\bar{X}} = 0.06394 \ .$$

In order to check the goodness of fit a Chi-square test was carried out. The Chi-square statistic found to be

$$\chi^2 = 16.10 \ .$$

The 5% point of the Chi-square distribution with 32 degrees of freedom is approximately equal to 45. This suggests that the fit to data by an exponential distribution is quite good.

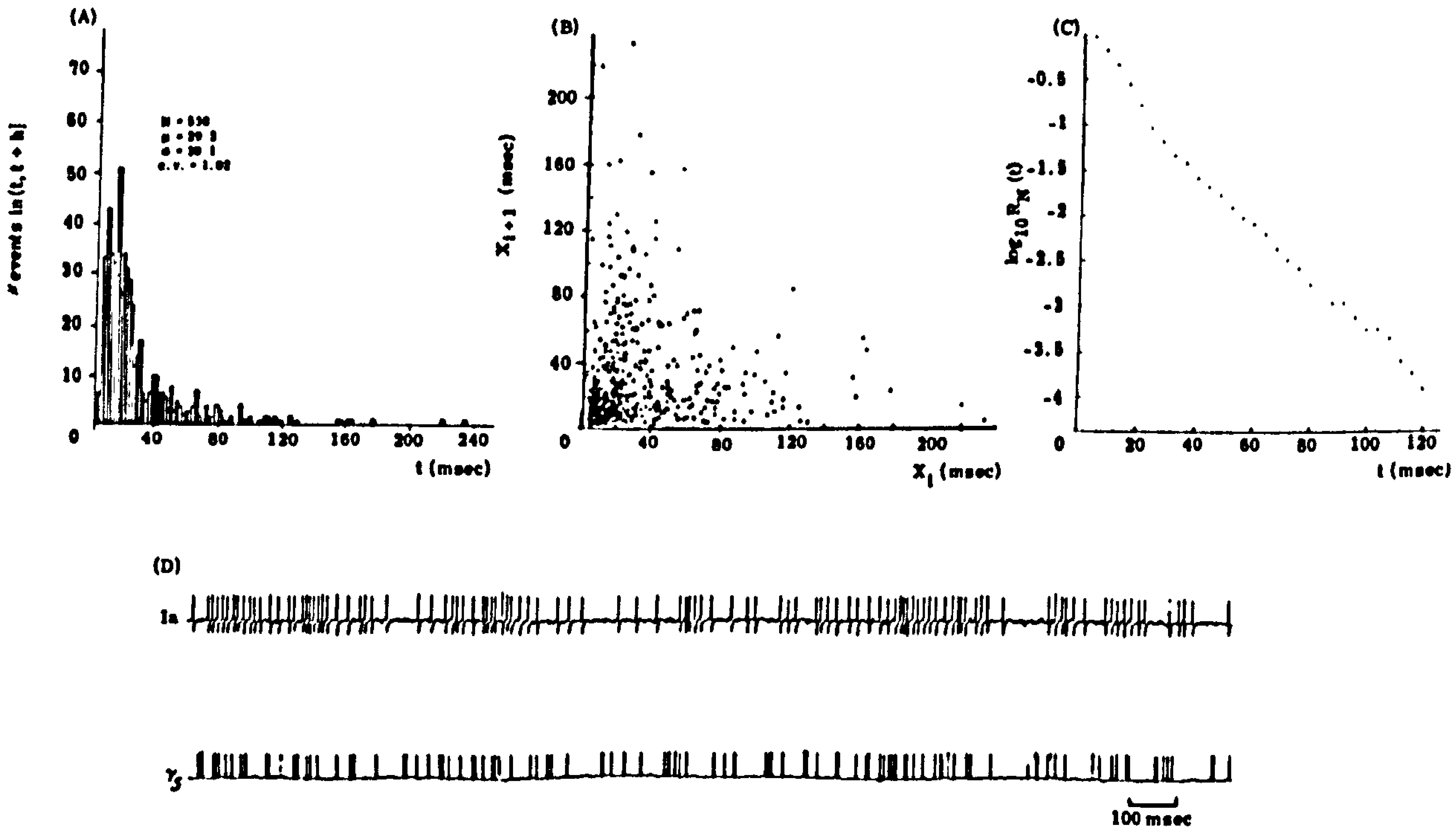
Fig. 1.4.2(B) is the scatter diagram of the adjacent intervals between pulses of the fusimotor input (γs). This figure shows the randomness of the γs pulses.

Fig. 1.4.2(C) is the $\log_{10}$ of the empirical survivor function (Cox and Lewis, 1968; p7) given by

$$\log_{10} R_N(X) = \log_{10} \left\{ \text{proportion of the intervals between pulses longer than } X \right\} .$$

This function also indicates that the exponential distribution is appropriate for this kind of data.

Fig. 1.4.3 gives the simple graphical characteristics of the Ia response in the presence of a fusimotor input.



<u>Fig. 1.4.3:</u> Ia discharge in presence of a fusimotor input. (A) Histogram of the inter-spike intervals of the Ia discharge (h = 2); (B) Scatter diagram of adjacent intervals between spikes of the Ia discharge; (C) Log to base 10 of the empirical survivor function; (D) Sequence of action potentials (spikes) of the Ia discharge in presence of a fusimotor input (γs).

It is clear from Fig. 1.4.3(D) that the presence of the fusimotor input completely destroys the regularity which is a characteristic of the spontaneous activity of Ia axons.

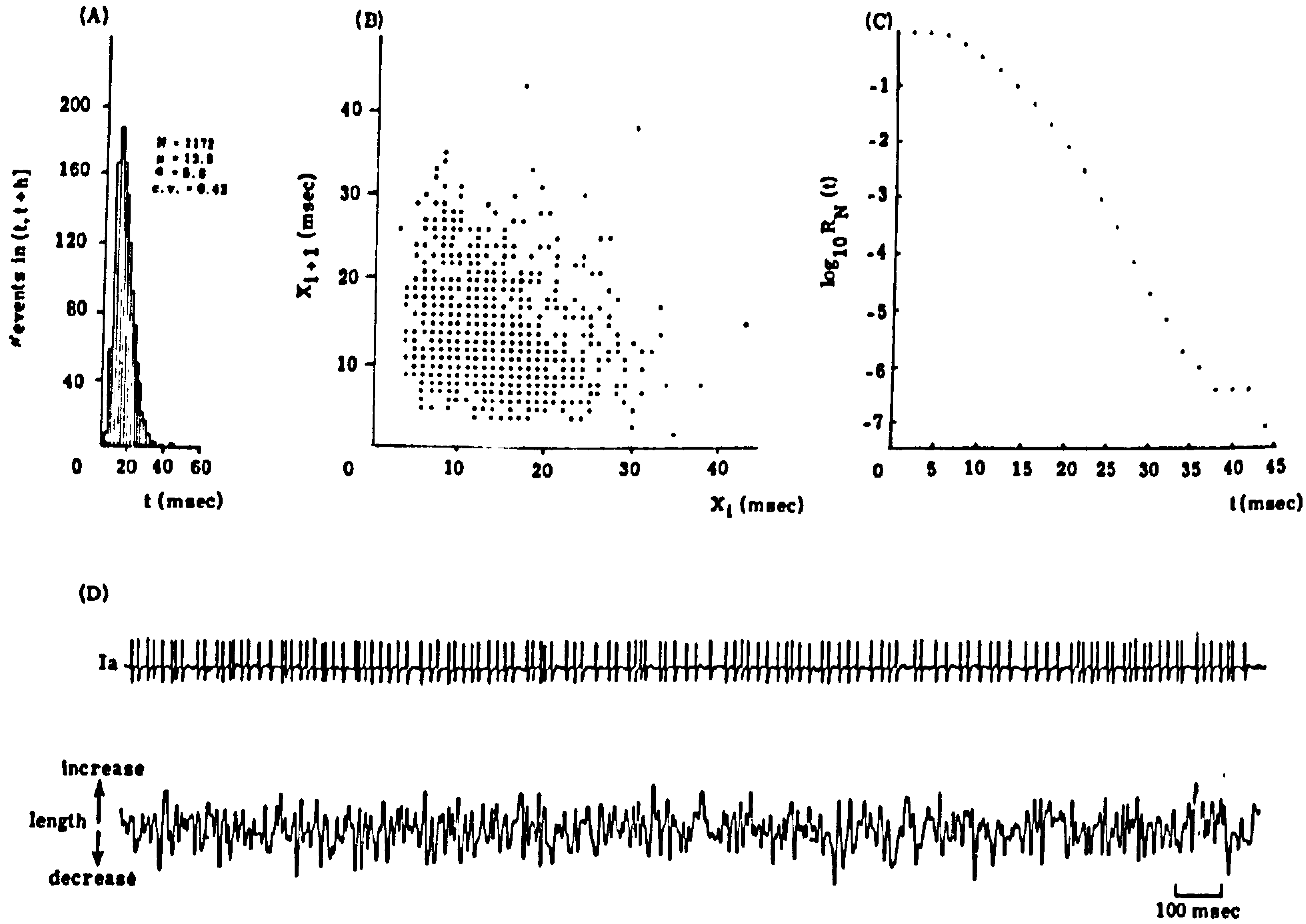
DATA SET III: Ia response recorded in the presence of a length change

The third data set gives the Ia response when a length input is applied to the parent muscle through a servo-controlled muscle puller.

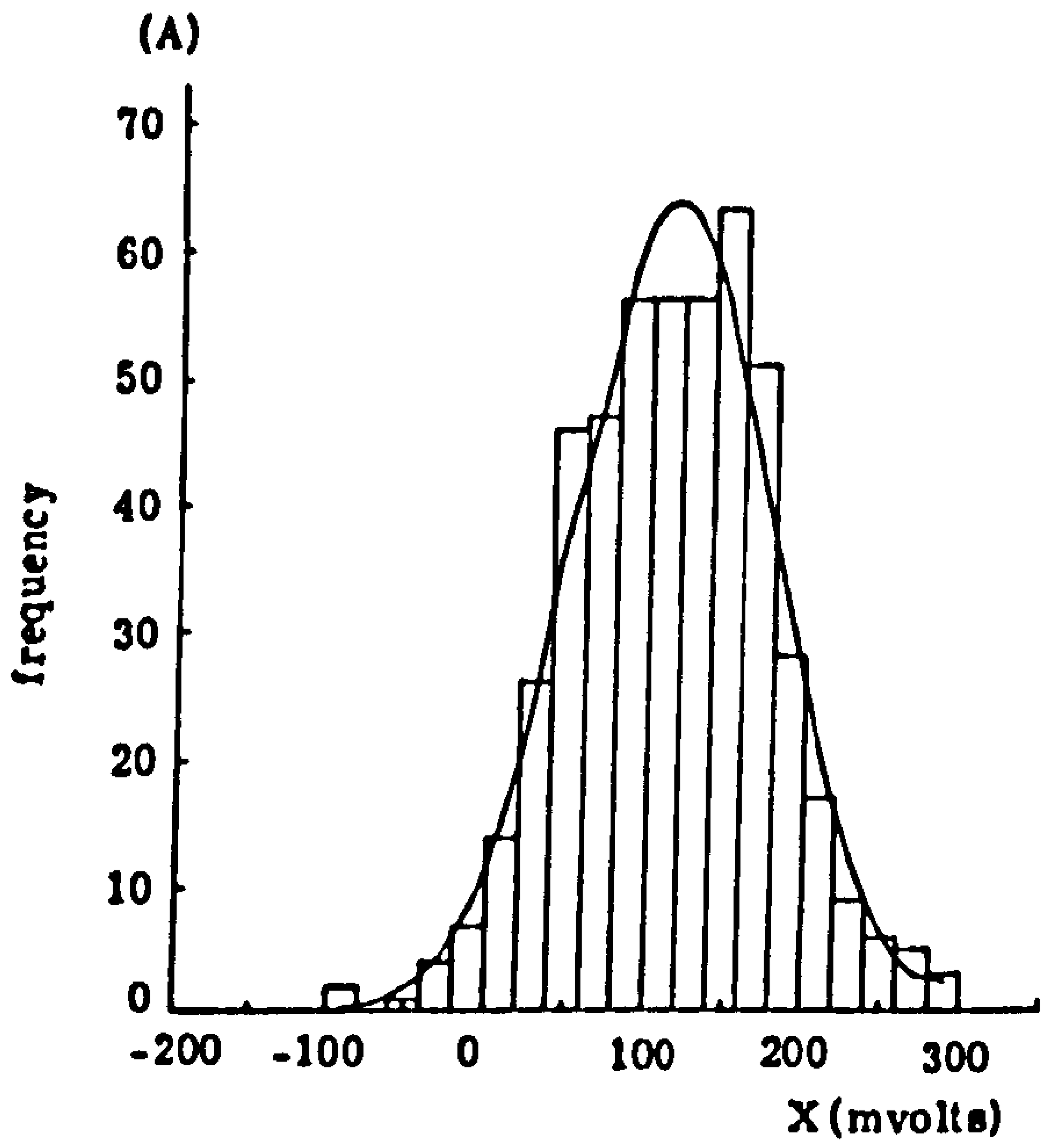
Fig. 1.4.4 presents in simple graphical form the basic statistical characteristics of the Ia response when a length change is present. We note from Fig. 1.4.4(A) that there are no long intervals between events as in the case of a fusimotor input, which presumably suggests that the internal mechanism of the muscle spindle responds differently to a length change and a fusimotor input.

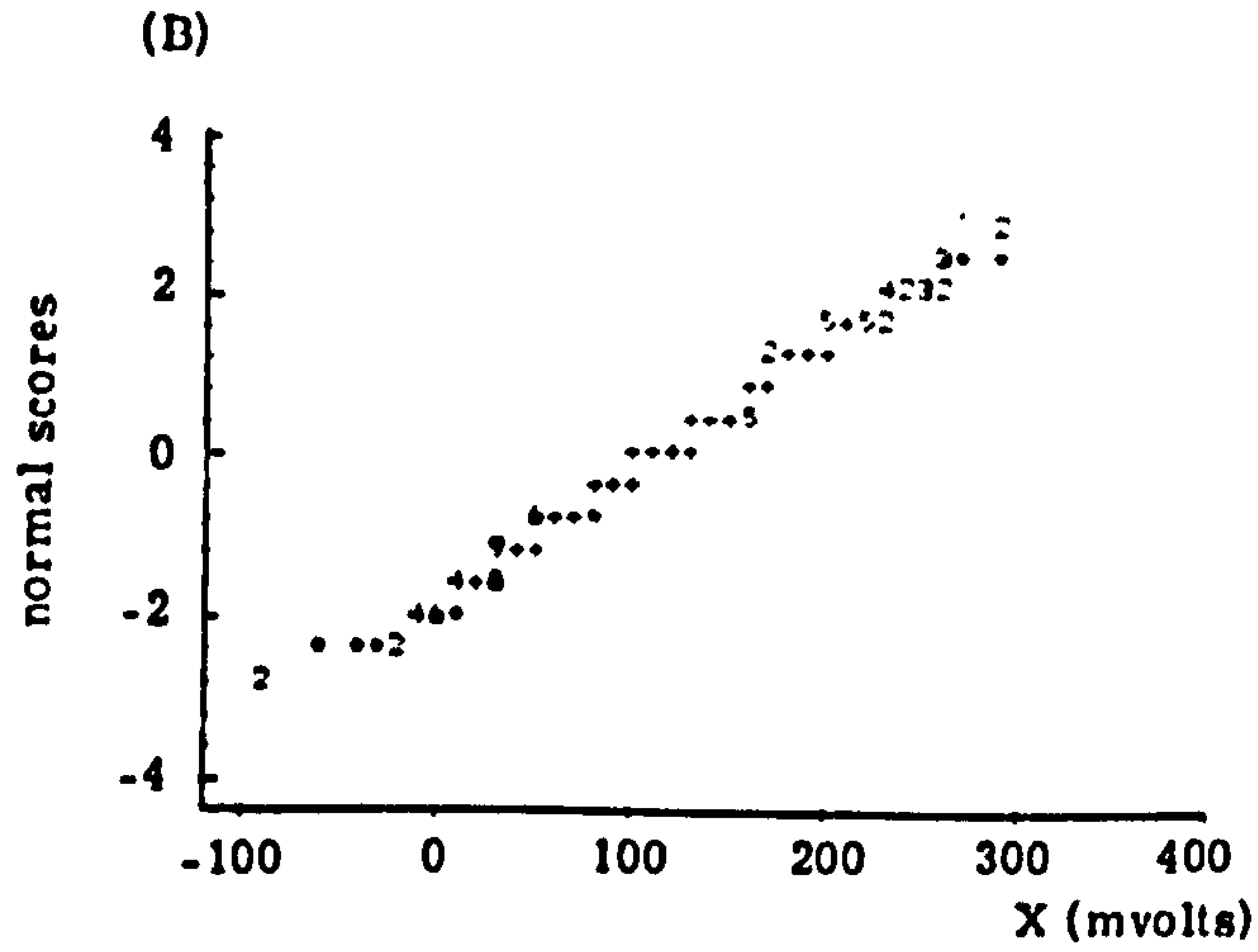
Fig. 1.4.5 shows that the length change has a Gaussian distribution. Fig. 1.4.5(A) gives the Gaussian curve which has been fitted to the histogram. Fig. 1.4.5(B) is the plot of the normal scores against the values of the length change. The fitted line to these points is almost perfect (cf. tables (C) and (D) of Fig. 1.4.5).

Figs. 1.4.6(A) and (B) are the estimates of the autocovariance function and the power spectrum of the length change illustrated in Fig. 1.4.6(C). These two functions are discussed in more detail in Chapter 5.



<u>Fig. 1.4.4:</u> Ia discharge in presence of a length change. (A) Histogram of the inter-spike intervals of the Ia discharge (h = 2); (B) Scatter diagram of adjacent intervals between spikes of the Ia discharge; (C) Log to base 10 of the empirical survivor function; (D) Sequence of action potentials (spikes) of the Ia discharge in presence of a length change.





(C)

THE REGRESSION EQUATION IS
Y = - 1.84 +0.0159 X1

	COLUMN	COEFFICIENT	ST. DEV. OF COEF.	T-RATIO = COEF/S.D.
	--	-1.843117	0.005042	-365.53
X1	C1	0.0159270	0.0000384	415.27

THE ST. DEV. OF Y ABOUT REGRESSION LINE IS
S = 0.05350
WITH (500- 2) = 498 DEGREES OF FREEDOM

R-SQUARED = 99.7 PERCENT

(D)

R-SQUARED = 99.7 PERCENT, ADJUSTED FOR D.F.

ANALYSIS OF VARIANCE

DUE TO	DF	SS	MS=SS/DF
REGRESSION	1	493.6549	493.6549
RESIDUAL	498	1.4256	0.0029
TOTAL	499	495.0801	

Fig. 1.4.5: Length change. (A) Histogram of a series of length changes fitted with a Gaussian curve; (B) Normal scores plotted against the actual values of length change; (C) Table of the regression line fitted to the points of Figure (B); (D) Table of the analysis of variance for the fitted line.

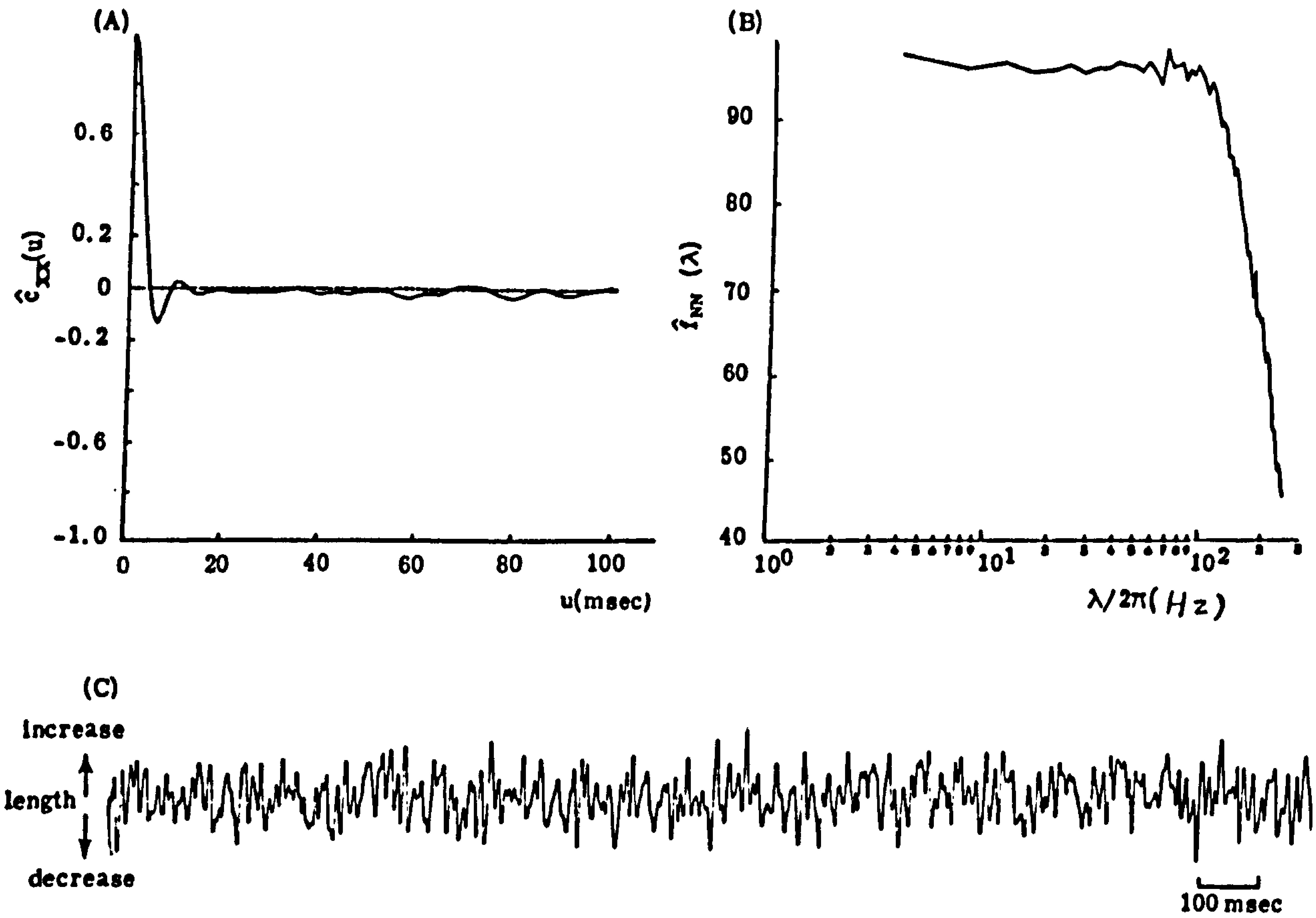


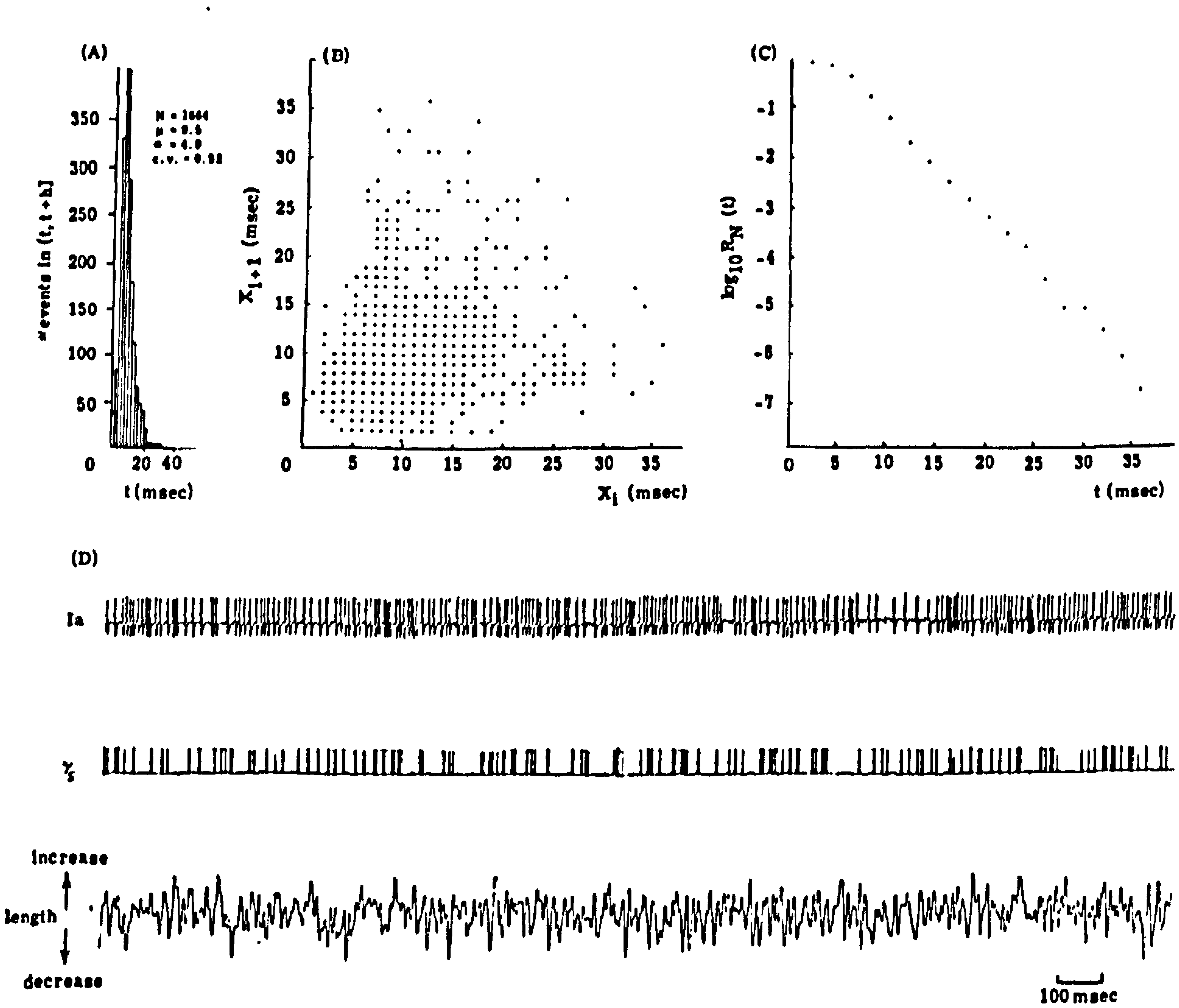
Fig. 1.4.6: Second-order properties of the length change. (A) Estimate of the autocovariance function of the actual values of the length change; (B) Estimate of the power spectrum of the length change; (C) Changes in the length of the parent muscle with time.

DATA SET IV: Ia response recorded in the presence of a combined length change and fusimotor input

This particular data set gives the Ia response to combined length and fusimotor inputs. These inputs are applied to the muscle spindle as we have already described for Data sets II and III.

Fig. 1.4.7 gives some simple graphical methods of representing the basic statistical features of the Ia response when length and fusimotor inputs are simultaneously present. Fig. 1.4.7(A) is the histogram of the intervals between spikes, Fig. 1.4.7(B) is the scatter diagram, Fig. 1.4.7(C) is the $\log_{10}$ of the empirical survivor function and Fig. 1.4.7(D) is a part of the recorded Ia response. These figures suggest that the concurrent effect of a length change and a fusimotor input changes the characteristics of the Ia response, compared with the response observed in Data sets II and III.

The statistical methods which will be developed in the next chapters aim at the further analysis of the four data sets described above, and lead to the extraction of useful information about the internal workings of the muscle spindle.



<u>Fig. 1.4.7:</u> Ia discharge in presence of a fusimotor input (γs) and a length change. (A) Histogram of the intervals between spikes of the Ia discharge (h = 2); (B) Scatter diagram of adjacent intervals between spikes of the Ia discharge; (C) Log to base 10 of the empirical survivor function; (D) Sequence of action potentials (spikes) of the Ia discharge in presence of a fusimotor input (γs) and a length change.

CHAPTER TWO

GENERAL THEORY OF TIME SERIES AND POINT PROCESSES

2.1 INTRODUCTION

The analysis of the data described in Chapter 1 require the use of statistical methods involving time series and point processes. Such processes are particular cases of a more general class of processes called Stochastic Processes which are extensively discussed in the books by Cramer & Leadbetter (1967), Doob (1953) and Yaglom (1962).

To emphasize the importance of the theory of time series and point processes we refer to the large number of applications in different fields ranging from economics to engineering. The main theory and applications of time series can be found in Anderson (1971), Bloomfield (1976), Brillinger (1975a), and Koopmans (1974). Similarly in the case of point processes the main theory and applications are given in Cox & Isham (1980), Cox & Lewis (1968), Lewis (1972) and Srinivasan (1974). It is also worth mentioning the work by Brillinger (Brillinger, 1978) indicating that in certain cases the concepts and procedures of time series have direct analogs in the study of point processes, while in other circumstances unique methods are only applicable to the one kind of process and not to the other.

Our intention here is to introduce the general theory of time series and point processes on which we shall base the statistical analysis of the available physiological data.

We start with the general theory of time series.

2.2 DEFINITION OF TIME SERIES AND HISTORICAL NOTES

A family of real-valued random variables with regard to time is called a time series. The domain of the time parameter is either the set of integer numbers (discrete time series) or the set of real numbers (continuous time series). Examples of time series are:

(a) Export totals in successive months

(b) Air temperature measured in successive days

(c) Signals recorded by a seismometer after a nuclear explosion, and

(d) Length changes imposed on muscle spindle (Fig. 1.4.6(C), Chapter 1).

Historically it may be assumed that the scientific analysis of time series have started in the middle of the 17th century when Issac Newton decomposed a light signal into frequency components by passing it through a glass prism. Although Newton himself did not carry out a quantitative analysis of his series, others developed the mathematical foundations for the analysis of such series in the middle of the 18th century. For example, Goury represented white light as a Fourier series while Rayleigh replaced the series by an integral. In the end of the 19th century Schuster (1898) introduced the periodogram,

$$I_{XX}^{(T)}(\lambda) = \left| \sum_{t=0}^{T-1} X(t) e^{-i\lambda t} \right|^2 \qquad (2.2.1)$$

which is based on the series x(t), t = 0,1,2,.... His intention was to search for hidden periodicities.

M.S. Bartlett, A. Khinchine, A.N. Kolmogorov, J.W. Tukey and N. Wiener were amongst the people who mostly contributed in the theoretical development of the time series during the period 1930-50. More details about their contributions may be found in Wold (1965).

The use of the computers after the 1950s made possible the applications of time series to the analysis of comparatively large sets of data. Moreover, the development of the Fast Fourier Transform a few years later opened new horizons in the analysis of very large data sets because the computation of the periodogram could be done very quickly.

2.3 STOCHASTIC ANALYSIS OF TIME SERIES

Let $(\Omega, \mathcal{A}, P)$ be the basic probability space. We assume that the r vector-valued time series $\underset{\sim}{X}(t) = (X_1(t), \ldots, X_r(t))$ is a member of an ensemble denoted by

$$\left\{ \underset{\sim}{X}(t,\omega) \; ; \; \omega \in \Omega \text{ and } t = 0, \pm 1, \pm 2, \ldots \right\}$$

where ω is a random variable.

If $X(t,\omega)$ is a measurable function of ω, then $X(t,\omega)$ is a random variable and we can define the finite-dimensional probability distrbutions as follows

$$P_{k_1 \ldots k_\ell}(x_1, \ldots, x_\ell \,;\, t_1, \ldots, t_\ell) = Prob\left\{ X_{k_1}(t_1, \omega) \leq x_1, \ldots, X_{k_\ell}(t_\ell, \omega) \leq x_\ell \right\}$$

$$k_1, \ldots, k_\ell = 1, \ldots, r \;;\; \ell = 1, 2, \ldots \quad . \qquad (2.3.1)$$

When ω is fixed, the function $\underset{\sim}{X}(t,\omega)$ is called a realization of the time series. Since there is no need to include ω specifically as an argument in $\underset{\sim}{X}(t,\omega)$ we shall denote $\underset{\sim}{X}(t,\omega)$ by $\underset{\sim}{X}(t)$ from here on.

The general expression (2.3.1) allows us to define the following quantities

$$E\{X_k(t)\} = \int x \, dP_k(x;t) = C_k(t) \ , \quad k = 1,\dots,r \tag{2.3.2}$$

$$Var\{X_k(t)\} = \int [x - E\{X_k(t)\}]^2 dP_k(x;t) = C_{kk}(t,t) \ , \quad k = 1,\dots,r \tag{2.3.3}$$

and

$$cov\{X_{k_1}(t), X_{k_2}(t)\} = \iint [x_1 - E\{X_{k_1}(t_1)\}][x_2 - E\{X_{k_2}(t_2)\}] \, dP_{k_1k_2}(x_1,x_2;t_1,t_2)$$

$$= C_{k_1k_2}(t_1,t_2) \ , \quad k_1, k_2 = 1,\dots,r \ . \tag{2.3.4}$$

The functions $C_k(t)$, $C_{kk}(t)$ and $C_{k_1k_2}(t_1,t_2)$ are called the mean function of the series $X_k(t)$, the auto-covariance function of $X_k(t)$, and the cross-covariance function of $X_{k_1}(t)$ and $X_{k_2}(t)$, respectively. The mean function $C_k(t)$ exists if and only if

$$E|X_k(t)| < \infty \ , \quad k = 1,\dots,r \ .$$

We will say that the series $X_{k_1}(t)$ and $X_{k_2}(t)$ are orthogonal if

$$C_{k_1k_2}(t_1,t_2) = 0 \qquad \text{for all } t_1, t_2.$$

More details about the probabistic theory of the time series can be found in Doob (1953).

2.4 CUMULANT FUNCTIONS OF ORDER 1

Let $(X_1,\ldots,X_r)$ be an r vector-valued random variable with

$$E|X_k|^r < \infty \ , \quad k=1,\ldots,r \ .$$

Definition 2.4.1: The rth order cumulant function, denoted by cum $\{X_1,\ldots,X_r\}$, is defined by the coefficient of $(i)^r t_1 \cdots t_r$ in the Taylor series expansion of log $(E \exp i \sum_{j=1}^{r} X_j t_j)$ about the origin.

Alternatively we have,

Definition 2.4.2: The cumulant of order r is defined as,

$$\mathrm{cum}(X_1,\ldots,X_r) = \sum (-1)^{s-1}(s-1)!\,(E\prod_{j\in\nu_1} X_j)\cdots(E\prod_{j\in\nu_s} X_j) \ . \qquad (2.4.1)$$

where the summation extends over all the partitions $(\nu_1,\ldots,\nu_s)$ of $(1,\ldots,r)$ and $s = 1,\ldots,r$.

Some properties of the cumulant functions are:

(i) $\mathrm{cum}(a_1X_1,\ldots,a_rX_r) = a_1\cdots a_r \ \mathrm{cum}\,(X_1,\ldots,X_r)$, where $a_1,\ldots,a_r$ are constants

(ii) $\mathrm{cum}(X_1,\ldots,X_r)$ is a symmetric function of the variables $X_1,\ldots,X_r$

(iii) If any group of the X's are independent of the remaining X's, then $\mathrm{cum}(X_1,\ldots,X_r) = 0$

(iv) For a random variable $(Y_1, X_1, \ldots, X_r)$

$$\mathrm{cum}(X_1 + Y_1, X_2, \ldots, X_r) = \mathrm{cum}(X_1, X_2, \ldots, X_r) + \mathrm{cum}(Y_1, X_2, \ldots, X_r)$$

(v) For k constant

$$\mathrm{cum}(X_1 + k, X_2, \ldots, X_r) = \mathrm{cum}(X_1, \ldots, X_r), \quad r=2,3,\ldots$$

(vi) If the random variables $(X_1, \ldots, X_r)$ and $(Y_1, \ldots, Y_r)$ are independent, then

$$\mathrm{cum}\,(X_1 + Y_1, \ldots, X_r + Y_r) = \mathrm{cum}\,(X_1, \ldots, X_r) + \mathrm{cum}\,(Y_1, \ldots, Y_r)$$

(vii) $\mathrm{cum}\, X_i = E(X_i)$ for $i = 1, \ldots, r$

(viii) $\mathrm{cum}(X_i, X_i) = \mathrm{Var} X_i$ for $i = 1, \ldots, r$ and X_i real

$\mathrm{cum}(X_i, \bar{X}_i) = \mathrm{Var} X_i$ for X_i complex ($\bar{X}_i$ is the complex conjugate of X_i)

(ix) $\mathrm{cum}(X_i, X_j) = \mathrm{cov}(X_i, X_j)$ for $i, j = 1, \ldots, r$ and X_i, X_j are real, $\mathrm{cum}\,(X_i, \bar{X}_j) = \mathrm{cov}\,(X_i, X_j)$ for X_i, X_j are complex variables

The two definitions and the properties of the cumulant functions given above can be found in Brillinger (1975a).

When $r = 1$ (univariate-case) definition (2.4.1) gives the cumulant function of order r of the random variable X. Cumulant functions of a univariate random variable are extensively discussed in Kendall and Stuart (1966), Vol.1, p69.

Cumulant functions are useful measurements of joint statistical dependence between random variables. They are also called semi-invariants and are discussed by this name in Leonov and Shiryaev (1959).

We proceed now to extend the definition of the mean function and auto-covariance function given in section 2.3. Given the r vector-valued time series $\underset{\sim}{X}(t)$, $t = 0, \pm 1, \pm 2, \ldots$ with components $X_k(t)$, $k = 1,\ldots,r$ and $E\,|X_k(t)|^{\,l} < \infty$ we define

$$C_{k_1\ldots k_l}(t_1,\ldots,t_l) = \mathrm{cum}\{X_{k_1}(t_1),\ldots,X_{k_l}(t_l)\} = C_{X_{k_1}\ldots X_{k_l}}(t_1,\ldots,t_l)$$

$$\text{for } k_1,\ldots,k_l = 1,\ldots,r\,,\ l = 1,2,\ldots \text{ and } t_1,\ldots,t_l = 0,\pm1,\pm2,\pm\ldots \qquad (2.4.2)$$

This function is called a <u>joint cumulant function of order l</u> of the series $\underset{\sim}{X}(t)$, $t = 0, \pm 1, \pm 2, \ldots$, and it is helpful in defining the l-th order spectrum as we shall see later.

2.5 STRICTLY AND WEAKLY STATIONARY TIME SERIES

An r vector-valued time series $\underset{\sim}{X}(t)$, $t = 0, \pm 1, \ldots$ is called <u>strictly stationary</u> when the joint distribution of $X_{k_1}(t_1 + t), \ldots, X_{k_l}(t_l + t)$ does not depend on t for all $t = 0, \pm 1, \pm 2, \ldots$, i.e.

$$P_{k_1\ldots k_l}(X_{k_1}(t_1+t) \le x_1,\ldots, X_{k_l}(t_l+t) \le x_l) = P_{k_1\ldots k_l}(X_{k_1}(t_1) \le x_1,\ldots, X_{k_l}(t_l) \le x_l)$$

$$\text{for all } t_1,\ldots,t_l = 0,\pm1,\ldots \text{ and } k_1,\ldots,k_l = 1,\ldots,r;\ l = 1,2,\ldots \qquad (2.5.1)$$

If the mean and the variance of the random variables exist, it is clear that strict stationarity implies the properties,

$$E\{X_k(t)\} = E\{X_k(0)\} = C_k \ , \ \text{for } k=1,\dots,r \text{ and } t=0,\pm 1,\dots \tag{2.5.2}$$

$$\operatorname{cov}\{X_{k_1}(t+u), X_{k_2}(t)\} = \operatorname{cov}\{X_{k_1}(u), X_{k_2}(0)\} = C_{k_1k_2}(u)$$

$$\text{for } t,u=0,\pm 1,\dots \text{ and } k_1,k_2=1,\dots,r \ . \tag{2.5.3}$$

These relations show that the mean functions are constant in time and the cross-covariance functions depend on the time shift u, but not on t.

Suppose that instead of the expression (2.5.1) we assume that the random variables $X_k(t)$ of the time series have the property

$$Var\{X_k(t)\} = C_{kk}(0) < \infty \ , \ k=1,\dots,r \ ,$$

and satisfy conditions (2.5.2) and (2.5.3). Then the series $\underset{\sim}{X}(t)$ is said to be weakly stationary.

A weakly stationary series is not, in general, strictly stationary. A Gaussian process is an important exception. This follows from the fact that the joint multivariate normal distributions of a weakly stationary Gaussian Process depend only on the vector of the first moments, and the matrix of the second moments (covariance matrix) of the random variables. Since these functions satisfy equations (2.5.2) and (2.5.3), the joint distributions will satisfy equation (2.5.1). Thus, strict stationarity and weak stationarity are equivalent for Gaussian series.

If the vector-valued series $\underset{\sim}{X}(t)$, $t = 0, \pm 1, \pm 2, \ldots$ is strictly stationary with

$$E|X_k(t)|^l < \infty \,, \quad k = 1, \ldots, r \,, \quad \text{then}$$

$$c_{k_1 \ldots k_l}(t_1 + u, \ldots, t_l + u) = c_{k_1 \ldots k_l}(t_1, \ldots, t_l)$$

$$\text{for} \quad t_1, \ldots, t_l, u = 0, \pm 1, \ldots \,; \quad l = 1, 2, \ldots \,. \tag{2.5.4}$$

The assumption of finite moments does not cause any problems, because in practice all series available for analysis are strictly bounded, $X_k(t) < M$, $k = 1, \ldots, r$, for some finite M and so all moments exist.

2.6 THE SPECTRUM OF ORDER 1 OF A STATIONARY TIME SERIES

Suppose that the series $\underset{\sim}{X}(t)$, $t = 0, \pm 1, \pm 2, \ldots$ satisfy

$$\sum_{u_1, \ldots, u_{l-1} = -\infty}^{+\infty} | c_{k_1 \ldots k_l}(u_1, \ldots, u_{l-1}) | < \infty \,, \tag{2.6.1}$$

where

$$c_{k_1 \ldots k_l}(u_1, \ldots, u_{l-1}) = \operatorname{cum}\{ X_{k_1}(t + u_1), \ldots, X_{k_{l-1}}(t + u_{l-1}), X_{k_l}(t) \}$$

$$= c_{k_1 \ldots k_l}(u_1, \ldots, u_{l-1}, 0) \,.$$

Then we define the l-th order spectrum of the series as

$$f_{k_1 \ldots k_l}(\lambda_1, \ldots, \lambda_{l-1}) = (2\pi)^{-l+1} \sum_{u_1, \ldots, u_{l-1} = -\infty}^{+\infty} c_{k_1 \ldots k_l}(u_1, \ldots, u_{l-1}) \exp\Big\{ -\sum_{j=1}^{l-1} u_j \lambda_j \Big\}$$

$$\text{for all} \ -\infty < \lambda_j < \infty \,, \ j = 1, \ldots, l-1 \,; \ k_1, \ldots, k_l = 1, \ldots, r, \ l = 2, 3, \ldots \,. \tag{2.6.2}$$

It is clear that the existence of the l-th order spectrum depends on assumption (2.6.1).

We extend definition (2.6.2) to include the case $l = 1$ by setting

$$f_K = C_K = E\{X_K(t)\} \quad \text{for } K=1,\dots,r \quad . \tag{2.6.3}$$

It is convenient sometimes to add a symbolic argument λ_l to the spectrum $f_{k_1,\dots,k_l}(\lambda_1,\dots,\lambda_{l-1})$, writing $f_{k_1,\dots,k_l}(\lambda_1,\dots,\lambda_l)$, in order to maintain symmetry. The argument λ_l may be taken to be related to the other λ_j by

$$\sum_{j=1}^{l} \lambda_j \equiv 0 (\operatorname{mod} 2\pi) \quad . \tag{2.6.4}$$

We note that $f_{k_1,\dots,k_l}(\lambda_1,\dots,\lambda_l)$ is generally complex-valued. It is also bounded and uniformly continuous in the manifold given by (2.6.4), and measures the statistical dependence of the components of frequency λ_s in $X_{k_s}(t)$, $s = 1,\dots,l$ in the case that $\lambda_l = -(\lambda_1 + \dots + \lambda_{l-1})$. Higher order spectra and their properties are discussed by Brillinger and Rosenblatt (1967a,b) in the case of ordinary time series.

By inverting relation (2.6.2) we get the cumulant function of order l

$$C_{K_1\dots K_l}(u_1,\dots,u_{l-1}) = \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} f_{K_1\dots K_l}(\lambda_1,\dots,\lambda_{l-1}) \exp\Big\{i\sum_{j=1}^{l-1}\lambda_j u_j\Big\} d\lambda_1\cdots d\lambda_{l-1} \tag{2.6.5}$$

In the case of two components $X_{k_1}(t)$ and $X_{k_2}(t)$ of the r vector-valued $\underset{\sim}{X}(t)$ time series and under the assumption

$$\sum_{u=-\infty}^{+\infty} |C_{K_1K_2}(u)| < \infty \quad \text{for } K_1, K_2 = 1,\dots,r \, , \tag{2.6.6}$$

we define the <u>second-order spectrum</u> as

$$f_{k_1k_2}(\lambda) = (2\pi)^{-1} \sum_{u=-\infty}^{+\infty} c_{k_1k_2}(u) e^{-i\lambda u}, \text{ for } -\infty < \lambda < \infty \text{ and } k_1, k_2 = 1, \ldots, r. \tag{2.6.7}$$

It follows from expression (2.6.6) that $f_{k_1k_2}(\lambda)$ is bounded and uniformly continuous. Also, since the components of the $\underset{\sim}{X}(t)$ are assumed to be real-valued functions of t, we have that

$$f_{k_1k_2}(\lambda) = \bar{f}_{k_1k_2}(-\lambda) \tag{2.6.8}$$

where $\bar{f}_{k_1k_2}(\lambda)$ is the conjugate function of $f_{k_1k_2}(\lambda)$.

Finally, we see from (2.6.7) that $f_{k_1k_2}(\lambda)$ is a 2π -periodic function with respect to frequency λ.

By setting $k_1 = k_2 = k$ in (2.6.7) we get the function $f_{kk}(\lambda)$ which is called the power spectrum of the series $X_k(t)$ at frequency λ. If $k_1 \neq k_2$, then $f_{k_1k_2}(\lambda)$ is called the cross-spectrum of the series $X_{k_1}(t)$ and $X_{k_2}(t)$ at frequency λ. The real part of the cross-spectrum is called the co-spectrum and the imaginary part is called the quadrature spectrum. The argument of the cross-spectrum written as

$$\theta_{k_1k_2}(\lambda) = \arg f_{k_1k_2}(\lambda)$$

is called the phase-spectrum while the modulus of $f_{k_1k_2}(\lambda)$ is called the amplitude spectrum.

An expression for the cross-covariance function of the components $X_{k_1}(t)$ and $X_{k_2}(t)$ can be evaluated by inverting (2.6.7), i.e.

$$c_{k_1k_2}(u) = \int_{-\pi}^{\pi} f_{k_1k_2}(\lambda) e^{i\lambda u} d\lambda \ . \tag{2.6.9}$$

In the case that the vector series $\underset{\sim}{X}(t)$ has finite second-order moments, but does not necessarily satisfy a condition of the form (2.6.6), it is still possible to have a spectral representation similar to the one given by (2.6.7). For a discussion of this point we refer to Brillinger (1975a), p25. We will always assume the validity of a condition of the form (2.6.6) in our analysis.

It is clear from (2.6.5) and (2.6.9) that quantities of the time domain can be calculated by using methods involving functions of the frequency domain. The great advantage of the frequency domain becomes apparent when we make use of the FFT which reduces the calculations to a minimum (Bloomfield, 1976; Chapter 4).

The spectral analysis of time series has very interesting applications to the problem of polynomial systems' identification (ref. Brillinger, 1970a). In later parts of this thesis we make use of the Volterra expansion (Volterra, 1959) for processes with stationary increments in order to identify point process systems or hybrid systems of both time series and point processes.

In the rest of this Chapter we deal with the general theory of point processes.

2.7 INTRODUCTION TO POINT PROCESSES AND SOME HISTORICAL NOTES

Point processes refer to isolated events which occur randomly in time. Such events may be emissions from a radioactive source, accidents occurring in time, pulse discharges of a nerve cell, stops of a machine during a time period, times of earthquakes and many others. A wide variety of examples of point processes with applications to different areas of research are discussed in Lewis (1972).

The most important point processes are the Poisson point processes because their properties are simple and well known (Cox and Lewis, 1968). The development of the theory of the Poisson point processes starts with the discovery of the Poisson distribution which is credited to de Moivre and Poisson around the 18th century. In 1868 Boltzman calculated the probability of no events in the interval h, $e^{-\lambda h}$, and in 1910 Bateman determined the counting distributions by solving a set of differential equations (ref. Haight, 1967).

In 1909 Erlang applied the theory of Poisson point processes to traffic problems. He also applied the theory of point processes to the design of communication systems, and in the study of queuing systems involving input and output point processes which correspond to the times of arrival and departure of customers. Such queuing systems are discussed in Haight (1967).

Another group of processes with a long history of study, and many applications to different fields, is known as renewal processes. The intervals of successive events in these processes are independent non-negative variates. The first serious investigation of the renewal processes starts with Herbelot in 1909 (ref. Lotka, 1957).

Recent applications of the point processes to the problems of Physiology and earthquakes are given in Brillinger (1978).

2.8 FORMAL DEFINITION OF A POINT PROCESS

A point process is defined as a Stochastic process specified in relation with events each labelled with the random value of a continuous parameter, which may be the time parameter itself (Bartlett, 1966). In a more mathematical definition, given in Appendix I, point processes are regarded as random, non-negative, integer-valued measures.

A point process may alternatively be described:

(i) As a sequence of the counting variables N(t), where N(t) is the number of events in the interval $(0,t]$, or

(ii) As a sequence of successive intervals between events occurring in time, that is $\{X_j\}$, $j = 1,2,\ldots$.

The two sequences are connected by the fundamental relation

$$Pr\{S_r > t\} = Pr\{N(t) < r\} \tag{2.8.1}$$

since

$$S_r = \sum_{j=1}^{r} X_j > t \quad \text{if and only if } N(t) < r.$$

At this point we must emphasise that the process $\{N(t)\}$ is the basic and most important process. Trends in the series of events will generally occur as a function of the time t and less frequently as a function of the interval parameter j. Moreover stationarity of the series of events is defined in the terms of the process $\{N(t)\}$. This stationarity implies stationarity of the intervals between events $\{X_j\}$ if and only if the point process is a Poisson process. In this case the intervals X_j are independently and exponentially distributed (Lewis, 1970).

In the discrete case (the time parameter t takes values on the integers numbers) stationarity of $\{N_t\}$ implies stationarity of the intervals $\{X_j\}$ if and only if the N_t are binomial random variables.

2.9 STATIONARITY OF POINT PROCESSES

A point process is assumed to be stationary if its probabilistic structure does not change with time. The types of stationarity are three, simple, second-order and complete.

We say that the point process is "simple stationary" if the probability distribution of the number of events $N(t, t+\tau_1]$ is the same as that of the number of events $N(t+\tau, t+\tau_1+\tau]$, for all $t, \tau_1, \tau > 0$.

By extension, we say that the point process is "second-order stationary" (weakly stationary), if the probability distribution of the number of events in two fixed intervals is invariant under translation.

More generally, we say that the point process is "completely stationary" (strong stationary), if the joint probability distribution of the number of events in arbitrary number of intervals is invariant under translation.

Complete stationarity of a point process implies second-order stationarity which further implies simple stationarity. Complete stationarity is very difficult to verify in practice except for a small number of intervals (Cox and Lewis, 1968; p60).

An immediate consequence of simple stationarity is that the distribution of the number of events in an interval depends only on the length of the interval. Further it is easy to show that the expected number of events in an interval is proportional to the length of the interval.

The definition of stationarity for multivariate point processes is given in Cox and Lewis (1972) and in Appendix I.

2.10 ORDERLINESS AND MIXING CONDITION

A point process is said to be orderly if

$$Pr\{N(t,t+\tau] \geqslant 2\} = o(\tau),$$

which implies that the probability of two or more events occurring in the interval $(t, t+\tau]$ tends to 0 as τ becomes smaller.

In practice orderliness is the condition which prevents multiple events to occur in small intervals.

Further we say that a point process satisfies a (strong) mixing condition if events of the same process well-separated in time are independent. This condition can also be applied to the case of multivariate point processes by assuming that events of one point process separated in time by a distance u from the events of the other point process become independent for large values of u. A more formal mathematical definition of (strong) mixing is given in Appendix I.

Orderliness and (strong) mixing are conditions which are satisfied approximately in practice and play a fundamental role in the statistical analysis of point process systems.

2.11 POINT PROCESS PARAMETERS

Let $\underset{\sim}{N}(t) = (N_1(t),\dots,N_r(t)), t \in \mathbb{R}$, be an r vector-valued stationary point process. Then we define the product density of order l as

$$p_{k_1\dots k_l}(u_1,\dots,u_{l-1}) = \lim_{h_1,\dots,h_l \to 0} \text{prob}\{\text{event of } N_{k_1} \text{ in } (u_1,u_1+h_1],\dots,\text{event of } N_{k_l} \text{ in } (0,h_l]\}/h_1\cdots h_l \qquad (2.11.1)$$

for all distinct $u_1,\dots,u_{l-1}, 0$; $k_1,\dots,k_l=1,\dots,r$ and $l=1,2,\dots$. The product densities are assumed to be continuous functions.

Furthermore, when the process $\underset{\sim}{N}$ is orderly, the product densities are defined by

$$p_{k_1\dots k_l}(u_1,\dots,u_{l-1})\,du_1\dots du_{l-1}\,dt = E\{dN_{k_1}(t+u_1)\cdots dN_{k_{l-1}}(t+u_{l-1})\,dN_{k_l}(t)\} \qquad (2.11.2)$$

where $dN_k(t) = N_k(t,t+dt]$ is the differential notation giving the number of events of the process N_k, $k = 1,\ldots,r$, in the interval $(t,t+dt]$.

Special cases of the above definitions are

$$p_k = \lim_{h \to 0} \text{prob}\{\text{event of } N_k \text{ in } (t,t+h]\}/h = E\{dN_k(t)\}/dt, \; k=1,\ldots,r \tag{2.11.3}$$

$$p_{k_1k_2}(u) = \lim_{h_1,h_2 \to 0} \text{prob}\{\text{event of } N_{k_1} \text{ in } (t+u,t+u+h_1] \text{ and event of } N_{k_2} \text{ in } (t,t+h_2]\}/h_1h_2$$

$$= E\{dN_{k_1}(t+u_1)dN_{k_2}(t)\}/dtdu \quad \text{for } k_1,k_2 = 1,\ldots,r \; . \tag{2.11.4}$$

The parameter p_k is called the mean intensity of the process N_k and the function $p_{k_1k_2}(u)$ is called the second-order product density. In addition to the product density function of order 1 we can now define the cumulant function of order 1 of the stationary process $\underset{\sim}{N}$ as

$$q_{k_1\ldots k_\ell}(u_1,\ldots,u_{\ell-1})\,du_1\cdots du_{\ell-1}dt = \text{cum}\{dN_{k_1}(t+u_1),\ldots,dN_{k_{\ell-1}}(t+u_{\ell-1}),dN_{k_\ell}(t)\} \tag{2.11.5}$$

for all distinct $u_1,\ldots,u_{1-1}$ different than 0.

Cumulant functions for interval functions are defined in Brillinger (1972), who also extends the definition to include non-stationary processes (ref. Appendix I).

The cumulant functions are connected directly with the product densities through relations such as

$$q_k = p_k \quad , \; k = 1,\ldots,r \tag{2.11.6}$$

$$q_{k_1k_2}(u) = p_{k_1k_2}(u) - p_{k_1}p_{k_2} \quad , \; k_1,k_2 = 1,\ldots,r \tag{2.11.7}$$

$$q_{K_1K_2K_3}(u,v) = p_{K_1K_2K_3}(u,v) - p_{K_1K_2}(u-v)p_{K_3} - p_{K_1K_3}(u)p_{K_2} - p_{K_2K_3}(v)p_{K_1} + 2p_{K_1}p_{K_2}p_{K_3},$$

$$\text{for} \quad K_1, K_2, K_3 = 1,\dots,r \; . \tag{2.11.8}$$

More general relations are given in Appendix I.

Cumulant functions are useful measures of joint statistical dependence and together with product densities are discussed in Ramakrishnan (1950), Kuznetsov and Stratonovich (1965) and Brillinger (1972).

2.12 THE SPECTRUM OF ORDER 1 OF A POINT PROCESS

We assume that the lth order cumulant function exists and satisfies the following assumption

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} |q_{K_1\dots K_\ell}(u_1,\dots,u_{\ell-1})|\,du_1\dots du_{\ell-1} < \infty \;, \text{ for } K_1,\dots,K_{\ell-1}=1,\dots,r;\ \ell=2,3,\dots \tag{2.12.1}$$

In terms of the condition (2.12.1) we define the spectrum of order l as

$$f_{K_1\dots K_\ell}(\lambda_1,\dots,\lambda_{\ell-1}) = (2\pi)^{-\ell+1}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\exp\Big\{-i\sum_{j=1}^{\ell-1}\lambda_j u_j\Big\}\, q_{K_1\dots K_\ell}(u_1,\dots,u_{\ell-1})\,du_1\dots du_{\ell-1},$$

$$\text{for } -\infty < \lambda_j < +\infty \;,\ j=1,\dots,\ell-1\;;\quad \ell=2,3,\dots \,. \tag{2.12.2}$$

Special cases of the definition (2.12.2) are

$$f_K = q_K \;,\quad K=1,\dots,r \tag{2.12.3}$$

$$f_{K_1K_2}(\lambda) = (2\pi)^{-1}\int_{-\infty}^{+\infty}\exp\{-i\lambda u\}\, q_{K_1K_2}(u)\,du \;,\quad K_1 \neq K_2 \tag{2.12.4}$$

$$f_{K_1K_2K_3}(\lambda,\mu) = (2\pi)^{-2}\iint_{-\infty}^{+\infty}\exp\{-i(\lambda u+\mu v)\}\, q_{K_1K_2K_3}(u,v)\,du\,dv,\quad K_1\neq K_2\neq K_3 \;. \tag{2.12.5}$$

Corresponding to these are the inverse relations

$$g_{K_1K_2}(u) = \int_{-\infty}^{+\infty} \exp\{iu\lambda\} f_{K_1K_2}(\lambda)d\lambda \ , \quad K_1 \neq K_2 \ , \quad K_1, K_2 = 1,\ldots,r \tag{2.12.6}$$

$$g_{K_1K_2K_3}(u,v) = \iint_{-\infty}^{+\infty} \exp\{i(u\lambda + v\mu)\} f_{K_1K_2K_3}(\lambda,\mu)d\lambda d\mu \ , \quad K_1 \neq K_2 \neq K_3 \ , \quad K_1,K_2,K_3 = 1,\ldots,r \ . \tag{2.12.7}$$

Spectra of point processes are discussed in Brillinger (1972).

2.13 EXAMPLES OF POINT PROCESSES

(a) Poisson point processes

Let N be a univariate point process and N(I) denote the number of events in the interval $I = (t, t + \tau]$. Suppose that, for some positive constant ρ as $\tau \longmapsto 0$

$$\mathrm{Prob}\{N(I) = 0\} = 1 - \rho\tau + o(\tau) \tag{2.13.1}$$

$$\mathrm{Prob}\{N(I) = 1\} = \rho\tau + o(\tau) \tag{2.13.2}$$

so that

$$\mathrm{Prob}\{N(I) \geqslant 2\} = o(\tau) \ . \tag{2.13.3}$$

Further, we assume that N(I) is completely independent of occurrences in $(0, t]$. Under these assumptions we say that the process N is a Poisson point process of rate ρ.

The independence condition specifies the randomness of the series, and (2.13.3) is the condition which prevents the occurrence of multiple events. If $N(t, t+\tau]$ does not depend on t we say that the Poisson process is simple stationary.

The second-order properties of a Poisson point process are determined by the constant value ρ of the mean rate of this process (Cox, 1965). This implies that the events of a Poisson point process occur in a random way. In practice the second-order properties of non-Poisson point processes will be compared with the constant properties of the corresponding Poisson point process.

(b) Renewal processes

In the Poisson point process the intervals between successive events are independent and exponentially distributed. If the intervals between events are independent and identically distributed with a common probability density function, then the resulting process is called an (ordinary) renewal process.

An important function related to the renewal processes is the renewal density which is defined as the limit of the probability of a renewal (event) in $(t,t+h]$ divided by h as $h \longmapsto 0$. For a Poisson process the renewal density is a constant equal to the mean rate ρ.

It can be proved that the renewal density tends to a limit as $t \longmapsto \infty$. This limit is equal to the inverse of the mean of the independent intervals between successive events (renewals). (Cox, 1962). The limiting result implies in practice that the renewal density of a non-Poisson process will fluctuate around the mean rate of the same process for large t.

Renewal processes and their applications to physiological problems are discussed in Perkel, Gerstein and Moore (1967a,b) while the statistical analysis of such processes is thoroughly examined by Cox (1962).

CHAPTER THREE

TIME AND FREQUENCY DOMAIN ANALYSIS OF THE UNIVARIATE POINT PROCESS

3.1 INTRODUCTION

The univariate point process is introduced and certain parameters of this process are defined in the time and frequency domain. Estimates of these parameters are discussed and their asymptotic distributions are examined in both domains. The construction of asymptotic confidence limits is also discussed and illustrated.

The importance of these estimates and their properties becomes apparent in the analysis of the output of the Ia sensory axon of the muscle spindle in the presence of different inputs.

However, we must stress that the main theme of this chapter is the usefulness of the frequency-domain analysis of point processes. The reasons for this are:
(1) The estimated parameters of the frequency domain in the case of a univariate point process may reveal more about the structure of the system under investigation (e.g. muscle spindle) than the equivalent time domain parameters and
(2) Estimates of time domain parameters can often be obtained more quickly by first calculating the corresponding parameters in the frequency domain directly from the data, and then inverting to the time domain.

3.2 TIME-DOMAIN PARAMETERS OF THE UNIVARIATE POINT PROCESS

The univariate point process, as we have already seen (ref. Ch. 2), is defined as a Stochastic process whose realizations are sequences of points occurring in time and/or space. Here we only deal with point processes which depend on time.

Suppose, now, that N is a univariate point process with differential increments $dN(t)$, where $dN(t)$ is the number of events in $(t,t+dt]$. The differential process $\{dN(t)\}$ generated from the increments $dN(t)$ provides a convenient way of describing a point process since it takes the value zero almost everywhere, except for events (delta functions) at the random times t, $t+t_1$, $t+t_1+t_2 \ldots$, where t is the time from the origin to the first event and t_i, $i = 1,2,\ldots$, are the times between successive events (Cox and Lewis, 1968, p73).

It is assumed that the univariate point process N is stationary, orderly, and satisfies a (strong) mixing condition of the sort discussed in Chapter 2.

Under these assumptions we have that the mean intensity (MI), denoted by p_N, is a constant while the second-order product density is defined by

$$p_{NN}(u)\,du\,dt = E\{dN(t+u)dN(t)\} \qquad u \neq 0 \tag{3.2.1}$$

and it may be interpreted as

$$Prob\{N \text{ event in } (t+u, t+u+du] \text{ and } (t, t+dt]\}\ .$$

Another very important function which describes the second-order properties of the univariate process is the auto-intensity function (AIF) defined by

$$m_{NN}(u)\,du = E\{dN(t+u)/N\} \qquad u \neq 0 \tag{3.2.2}$$

and it may be interpreted as

$$\text{Prob}\{N \text{ event in } (t+u, t+u+du] / N \text{ event at } t\} \ .$$

It follows from the properties of conditional probability that

$$m_{NN}(u) = \frac{p_{NN}(u)}{p_N} \qquad u \neq 0 \quad . \tag{3.2.3}$$

Furthermore, the (strong) mixing condition ensures that

$$\lim_{u \to \infty} p_{NN}(u) = p_N^2 \tag{3.2.4}$$

and

$$\lim_{u \to \infty} m_{NN}(u) = p_N \quad . \tag{3.2.5}$$

Expression (3.2.5) suggests that, in practice, the AIF should fluctuate around the MI of the point process for large values of u.

In order to take account of the singularity at u = 0 (Bartlett, 1963a) we define the auto-covariance function (ACF) as

$$\text{cov}\{dN(t+u), dN(t)\} = [p_N \delta(u) + q_{NN}(u)]\, du\, dt \quad , \tag{3.2.6}$$

where $\delta(u)$ is the Dirac delta function and $q_{NN}(u)$ is the cumulant function of the process N given by

$$q_{NN}(u) = p_{NN}(u) - p_N^2 \quad , \quad u \neq 0 \qquad (q_{NN}(u) \text{ is continuous at } u = 0).$$

This function will tend to zero for large values of u suggesting that increments of N become independent when they are well-separated.

We now turn to the problem of estimating the time-domain parameters defined above.

In practice, the univariate point process is defined on a finite interval $(0,T]$. Then an obvious estimate of p_N is

$$\hat{p}_N = \frac{1}{T}\int_0^T dN(t) = \frac{N(T)}{T} , \tag{3.2.7}$$

where $N(T)$ is the number of events in the interval $(0,T]$.

The value of this estimate is equal to the inverse of the estimate of the mean of the successive intervals between events of the point process.

Let $t_1, \ldots, t_n$ be the times of occurrence of the events in the interval $(0,T]$.

Then the estimate of the AIF is given by

$$\hat{m}_{NN}(u) = \frac{J_{NN}^{(T)}(u)}{bN(T)} , \tag{3.2.8}$$

where

$$J_{NN}^{(T)}(u) = \#\left\{u - \frac{b}{2} < \tau_k - \tau_j < u + \frac{b}{2} \;,\; j,k = 1,\ldots,n\right\}, \tag{3.2.9}$$

for some bin width b of the small non-overlapping intervals in which the whole record has been divided. The symbol "#" denotes the number of counts.

A simple way of explaining the function $J_{NN}^{(T)}(u)$ is to consider an interval of length T divided into small intervals, $(lb,(l+1)b)$, $l = 0,1,2,\ldots, L(=\frac{T}{b}-1)$ and then count the number of differences, $t_k - t_j$, which fall in each of these intervals. Also, since $J_{NN}^{(T)}(u)$ is an even function, we have only to calculate this function for values of u greater than zero. This can be done by calculating expression (3.2.9) for $k > j$.

The asymptotic distributions of the estimates $J_{NN}^{(T)}(u)$, $\hat{p}_{NN}(u)$ and $\hat{m}_{NN}(u)$ are now examined.

Theorem 3.2.1: Let N be a stationary point process on $(0,T]$ which satisfies a (strong) mixing condition and such that $p_{NN}(u)$, $p_{NNN}(u,v)$, $p_{NNNN}(u,v,w)$ are finite, where $p_{NNN}(u,v)$ and $p_{NNNN}(u,v,w)$ are the third and fourth-order product densities of N. Then the variate $J_{NN}^{(T)}(u)$, as $T \mapsto \infty$ is asymptotically Poisson with mean $bTp_{NN}(u)$. Furthermore, if $b \mapsto 0$ but $bT \mapsto \infty$ the variate $J_{NN}^{(T)}(u)$ is asymptotically normally distributed with mean and variance $bTp_{NN}(u)$.

Proof: The proof can be found in Brillinger (1976a).

Corollary 3.2.1: The estimate, $\hat{p}_{NN}(u) = \frac{J_{NN}^{(T)}(u)}{bT}$, has an asymptotic Poisson distribution $(bT)^{-1}Po(bTp_{NN}(u))$ or an asymptotic normal distribution $N(p_{NN}(u),(bT)^{-1}p_{NN}(u))$ as $bT \mapsto \infty$ (Po(a) denotes a Poisson distribution with mean a).

Proof: The proof follows from Th. 3.2.1.

Corollary 3.2.2: The estimate of the AIF, $\hat{m}_{NN}(u)$, is asymptotically distributed as $(bT)^{-1}p_N^{-1}\, Po(bTp_{NN}(u))$.

Proof: The proof follows from Th. 3.2.1. and Cor. 3.2.1. However, in applying Th. 3.2.1 in practical problems, we must have in mind the following approximation.

Rigorously the function $J_{NN}^{(T)}(u)$ is defined by

$$J_{NN}^{(T)}(u) = \int_0^{T-u-\frac{b}{2}} \int_{u-\frac{b}{2}}^{u+\frac{b}{2}} dN(t+s)\,dN(t) \qquad s \neq 0 \quad \text{(Cox \& Lewis, 1968; p122).}$$

The expected value of $J_{NN}^{(T)}(u)$ can be written as

$$E\left\{ J_{NN}^{(T)}(u) \right\} = \int_0^{T-u-\frac{b}{2}} \int_{u-\frac{b}{2}}^{u+\frac{b}{2}} E\left\{ dN(t+s)\,dN(t) \right\} = \int_0^{T-u-\frac{b}{2}} \int_{u-\frac{b}{2}}^{u+\frac{b}{2}} p_{NN}(s)\,ds\,dt \;, \tag{3.2.10}$$

and by expanding the quantity $p_{NN}(s)$ in a Taylor series around u for small b we get

$$\int_{u-\frac{b}{2}}^{u+\frac{b}{2}} p_{NN}(s)\,ds = b\,p_{NN}(u) + \frac{b^3}{6}\,p_{NN}'(u) + \cdots \simeq b\,p_{NN}(u) \;.$$

On using this result in (3.2.10) we have for small values of b

$$E\left\{ J_{NN}^{(T)}(u) \right\} \simeq \int_0^{T-u-\frac{b}{2}} b\,p_{NN}(u)\,dt = b\left(T-u-\frac{b}{2}\right) p_{NN}(u) \simeq b(T-u)\,p_{NN}(u) \;. \tag{3.2.11}$$

When the argument u is small in comparison with T, then $E\left\{ J_{NN}^{(T)}(u) \right\}$ can be approximated by

$$E\left\{ J_{NN}^{(T)}(u) \right\} \simeq b\,T\,p_{NN}(u) \;. \tag{3.2.12}$$

We now discuss ways of calculating asymptotic confidence intervals for the AIF. We know from Cor. 3.2.2 that the distribution of the estimate $\hat{m}_{NN}(u)$ can be approximated by a normal distribution $N(m_{NN}(u),\ (bTp_N)^{-1}\ m_{NN}(u))$ for large bT.

This normal distribution of $\hat{m}_{NN}(u)$, under the assumption that the events of the process N occur randomly, becomes $N(p_N,\ (bT)^{-1})$.

Then a simple approximate 95% confidence interval for $\hat{m}_{NN}(u)$ is given by

$$\hat{p}_N \pm 1.96\left\{(bT)^{-1}\right\}^{1/2}. \tag{3.2.13}$$

A second way of finding an asymptotic confidence interval for $\hat{m}_{NN}(u)$ is presented in the following proposition.

<u>Proposition 3.2.1</u>: The square root of the estimate of the AIF, $(\hat{m}_{NN}(u))^{\frac{1}{2}}$, is approximately distributed as $N(\{m_{NN}(u)\}^{1/2}; (4bTp_N)^{-1})$ The variance $(4bTp_N)^{-1}$ is stable for all u. (Kendall and Stuart, 1966, Vol.1, p88).

It follows from Prop. 3.2.1 that the 95% approximate confidence interval for $(\hat{m}_{NN}(u))^{\frac{1}{2}}$ under the null hypothesis of independence will be

$$\{\hat{p}_N\}^{1/2} \pm 1.96\left\{(4bTp_N)^{-1}\right\}^{1/2} \simeq \{\hat{p}_N\}^{1/2} \pm \left\{(bTp_N)^{-1}\right\}^{1/2}. \text{(Brillinger, 1976a)} \tag{3.2.14}$$

Finally, we refer to a suggestion of Cox (Cox, 1965) in which he gives another method of estimating the AIF by using weighting functions. In this case the estimate of the AIF will be based on

$$\sum_{i=-I}^{I} W_i J_{NN}^{(T)}(u-ib) \quad , \tag{3.2.15}$$

where the weights W_i satisfy the relation $\sum_i W_i = 1$.

By combining Th. 3.2.1 and Cor. 3.2.2 we find that the distribution of the new estimate of the AIF, denoted by $\hat{\hat{m}}_{NN}(u)$, is asymptotically normal with mean $m_{NN}(u)$ and variance $m_{NN}(u)\sum W_i^2(bTp_N)^{-1}$. Hence the 95% asymptotic confidence interval for $(\hat{m}_{NN}(u))^{\frac{1}{2}}$ will be

$$(m_{NN}(u))^{1/2} \pm \quad \left(\Sigma W_i^2/bTp_N\right)^{1/2}. \qquad (3.2.16)$$

Under the null hypothesis of independence $m_{NN}(u)$ becomes p_N and the confidence intervals in (3.2.16) can be further simplified by substituting p_N for $m_{NN}(u)$.

Brillinger (1975b) gives a more general class of 'window functions', as well as some useful theorems related to the asymptotic properties of the product density functions of second and higher orders.

We now apply the above-mentioned procedures for estimating $m_{NN}(u)$ and its approximate confidence intervals to the four sets of data presented in Chapter 1 . Figs. 3.2.1 (A-D) refer to the data set I and Figs. 3.2.2 (A-D) are a comparison of the four auto-intensities calculated for each data set.

Fig. 3.2.1 (A) is the function $J_{NN}^{(T)}(u)$ which has been calculated for a bin-width of 1msec (b=1) and u ranging from 0-512 msec. This function is basically a histogram, called the autocorrelation histogram (ACH), and is used very commonly by physiologists (e.g. ref. Bryant et al., 1973).

The peaks in the histogram occur at equidistant points because of the periodic response of the Ia sensory axon in the muscle spindle.

Fig. 3.2.1 (B) is the normalised ACH, i.e., the function $J_{NN}^{(T)}(u)$ divided by the product b.N(T), which is the estimate of the AIF. The estimate of the mean intensity (MI) $\hat{p}_N$ is 26.2 pulses/sec. The dotted line in the centre corresponds to the $\hat{p}_N$ and the two solid lines above and below this line

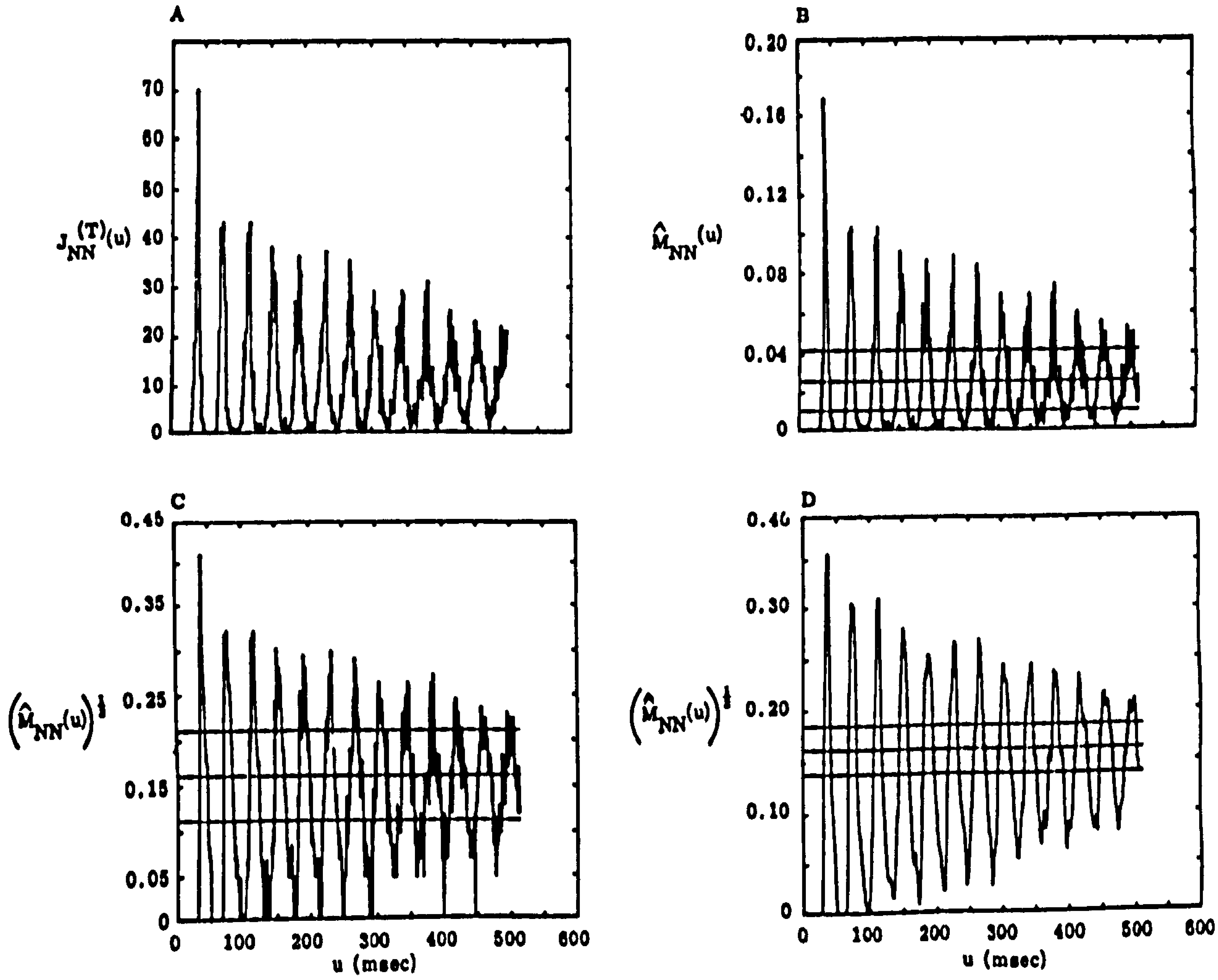
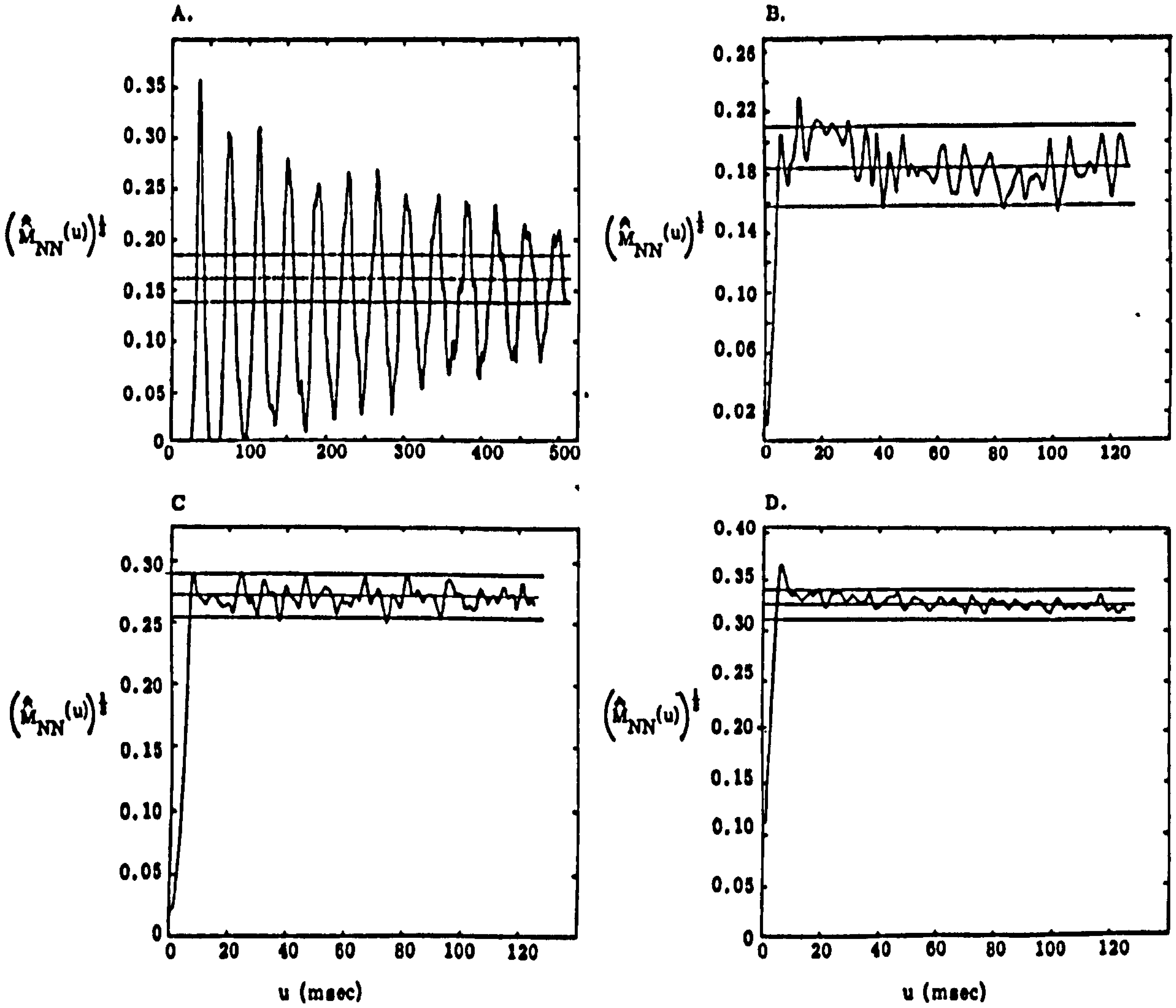


Fig. 3.2.1: Estimates of the AIF of the spontaneous Ia discharge. (A) Histogram-like estimate $J_{NN}^{(T)}(u)$ computed with a bin width b = 1 msec, (B) Estimate of the AIF obtained from the histogram-like estimate by normalising it, (C) Square root of the estimate of the AIF, and (D) Smoothed estimate of the AIF computed with a "hanning window". The dotted line in Figures (B), (C) and (D) corresponds to the estimated value of the mean rate (square root of the mean rate) of the Ia discharge. The horizontal lines are the asymptotic 95% confidence limits.



<u>Fig. 3.2.2:</u> Smoothed estimate of the square root of the AIF computed with a "hanning window". (A) Spontaneous Ia discharge, (B) Ia discharge in presence of a fusimotor input (γs), (C) Ia discharge in presence of a length change (l), and (D) Ia discharge in presence of γs and l. The dotted line in these figures corresponds to the square root of the estimated value of the mean rate. The horizontal lines are the asymptotic 95% confidence limits.

indicate the limits of the approximate 95% confidence interval. It is clear that increments of the point process N are approximately independent more than 0.5 sec apart. The estimate is zero for the first 32 msec because no pulses occurred closer together than that interval. The estimate of the mean interval between pulses is $\bar{X} = 1/\hat{p}_N = 0.038$ sec and is exactly the point at which the first maximum occurs. The other maxima occur at multiples of this value.

Fig. 3.2.1 (C) gives the square root transformation of the estimate of the AIF. The dotted line in the centre corresponds to the $(\hat{p}_N)^{\frac{1}{2}}$ and the solid horizontal lines are the 95% confidence limits, set at plus and minus two standard deviations about the dotted line. This figure improves the symmetry of the estimate given in Fig. 3.2.1 (B), especially for the area below the estimate of the MI.

Fig. 3.2.1 (D) presents the square root of the estimate of the AIF after having been smoothed. The approach employed here makes use of weights of the form $(\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$ which clearly gives less emphasis to values around the middle point. By using weights of this form we can get better estimates of the square root of the AIF, but we must be careful not to oversmooth the estimate, since important features may be obscured.

The next Fig. 3.2.2 is a comparison of the four autointensities for the corresponding sets of data. Each data set gives information about the behaviour of the system in the different conditions described in Chapter 1. Our purpose here is to see what, if any, useful conclusions can be drawn from the AIF for each particular set of data.

Fig. 3.2.2 (A) is the smoothed estimate of the square root of the AIF in the absence of any stimulus. This estimate has been examined thoroughly in Fig. 3.2.1. The other three estimates of the square root of the AIF have been smoothed by using a "hanning window" (Tukey, 1977), i.e. applying a scheme of weights of the form $(\frac{1}{4},\frac{1}{2},\frac{1}{4})$. The dotted line in the centre is the square root of the estimate of MI and the two solid horizontal lines are the 95% confidence limits given by expression (3.2.16).

We now examine the characteristics of each transformed estimate of the AIF for the Data Sets II, III and IV.

Fig. 3.2.2 (B) presents the transformed estimate of the AIF in the case that the response of the Ia sensory axon when gamma stimulation is present. It is obvious that the presence of the stimulus destroys completely the regularity of the Ia response. Other characteristics are the depression near the origin, and possibly the excitation which lasts until 30 msec. After that the estimate becomes a constant and behaves as the mean intensity of a Poisson process.

Fig. 3.2.2 (C) and Fig. 3.2.2 (D) are the estimates of the square root of the AIFs in the cases when the Ia sensory axon is responding to a length change imposed on the muscle spindle and to concurrent static gamma stimulation and an imposed length change, respectively. Again we note the absence of any periodicity. In the first case we have a depression of about 8-10 msec and after that a Poisson-like behaviour, while in the second case the depression is shorter followed by an excitation between 10-15 msec and after that the value of the estimate again becomes a constant.

It is difficult considering Figs. 3.2.2 (B-D) to see any distinguishable effects from the three different stimuli by examining the estimates of the AIF, aside from the loss of periodicity under each input condition. Therefore, we turn our attention to the frequency domain in order to develop estimates of certain parameters which might be able to provide us with more information about the effect of each input condition on the Ia response of the sensory axon.

3.3 FREQUENCY-DOMAIN PARAMETERS OF THE UNIVARIATE POINT PROCESS

The analysis of point process data may also be carried out in the frequency domain. This analysis can be achieved in two ways.

(a) By taking the Fourier transform of time-domain parameters, and

(b) By going straight to the frequency domain from the original data.

The second method, as we shall see, is based on the periodogram of a point process.

At this point we must emphasise that the periodogram is the basis for much of our work on the spectral analysis of point processes. Theorems about the asymptotic properties of the periodogram are developed. Smoothed estimates of the power spectrum of a point process, and asymptotic confidence intervals for these estimates are constructed. Estimates of certain parameters in the time domain are calculated by using the periodogram. Worked examples of several methods of estimating the spectrum and its confidence intervals are given. Finally, examples of the auto-intensity function calculated from the periodogram are compared to the estimates of the AIF calculated from the time domain.

We now proceed to consider certain parameters of the frequency domain. Let N be a univariate point process which is stationary, orderly and satisfies a (strong) mixing condition. Let also p_N and $q_{NN}(u)$ be the mean-intensity (MI) and cumulant function (CF) of the process N. Suppose that the CF exists and satisfies the following condition

$$\int_{-\infty}^{\infty} |q_{NN}(u)| du < \infty \quad . \tag{3.3.1}$$

Assumption (3.3.1) is a form of asymptotic independence condition on the increments of the point process (Brillinger, 1972).

Then the power spectrum of N is defined by Bartlett (1963a) as

$$f_{NN}(\lambda) = (2\pi)^{-1} p_N + (2\pi)^{-1} \int_{-\infty}^{\infty} q_{NN}(u) e^{-i\lambda u} du \, , \; -\infty < \lambda < \infty \quad . \tag{3.3.2}$$

The power spectrum (PS) of a point process N is a non-negative and even function of the parameter λ. In addition, it follows from the Riemann-Lebesque Lemma (Katznelson, 1968, p13) that the PS tends to a constant for large values of λ, that is

$$\lim_{|\lambda| \to \infty} f_{NN}(\lambda) = \frac{p_N}{2\pi} \quad . \tag{3.3.3}$$

This constant value corresponds to the value of the PS of a Poisson point process with the same mean rate.

It is also clear from condition (3.3.1) that the PS is bounded and uniformly continuous (Katznelson, 1968, p121). These properties suggest that (3.3.2) may be inverted to give

$$q_{NN}(u) = \int_{-\infty}^{\infty} \left(f_{NN}(\lambda) - P_N/2\pi \right) e^{i\lambda u} d\lambda \quad . \tag{3.3.4}$$

In practice, however, since $q_{NN}(u)$ is sampled at intervals of length b, we define the power spectrum of the point process as

$$g_{NN}(\lambda) = (2\pi)^{-1} P_N + (2\pi)^{-1} b \sum_j q_{NN}(u_j) e^{-i\lambda u_j} \quad , -\infty < \lambda < \infty \tag{3.3.5}$$

for $u_j = bj$; $j = 0, \pm 1, \pm 2, \pm \cdots$

The PS $g_{NN}(\lambda)$ has period $2\pi/b$, and as $g_{NN}(\lambda) = g_{NN}(-\lambda)$, we may define the interval $[0, \pi/b]$ as the fundamental domain for $g_{NN}(\lambda)$. The frequency π/b is known as Nyquist frequency. It is also called the folding frequency, since effectively higher frequencies are folded down into the interval $[0, \pi/b]$

Now, it easily follows from (3.3.2) and (3.3.5) that the two power spectra $g_{NN}(\lambda)$ and $f_{NN}(\lambda)$ are related through the following expression

$$g_{NN}(\lambda) - \frac{P_N}{2\pi} = \sum_{j=-\infty}^{\infty} \left\{ f_{NN}\left(\lambda + \frac{2\pi j}{b}\right) - \frac{P_N}{2\pi} \right\} \quad . \tag{3.3.6}$$

We see from (3.3.6) that the frequency λ of the spectrum $g_{NN}(\lambda)$ relates to the frequencies λ, $\lambda \pm \frac{2\pi}{b}, \ldots$ of the spectrum $f_{NN}(\lambda)$. It is also related to the frequencies $-\lambda$, $-\lambda \pm \frac{2\pi}{b}, \ldots$ since $g_{NN}(\lambda)$ is an even function. For this reason the frequencies

$$\lambda + \frac{2\pi j}{b}, \; -\lambda + \frac{2\pi j}{b} \; ; \; j = 0, \pm 1, \pm 2, \cdots$$

have been called aliases by Tukey (Tukey, 1959b).

If the power spectrum $f_{NN}(\lambda)$ behaves as a Poisson process with mean rate p_N for λ greater than π/b (Nyquist frequency), then

$$g_{NN}(\lambda) = f_{NN}(\lambda) \; . \tag{3.3.7}$$

Alternatively, as stated in the beginning of this section, the estimation of the PS may be based on the periodogram of the point process. Bartlett (1963a) defines the periodogram of a point process by analogy with the periodogram of a stationary time series, that is

$$I_{NN}^{(T)}(\lambda) = \frac{1}{2\pi} \, J_N^{(T)}(\lambda) \, \overline{J_N^{(T)}(\lambda)} \; , \; -\infty < \lambda < \infty \tag{3.3.8}$$

where $J_N^{(T)}(\lambda)$ is given by

$$J_N^{(T)}(\lambda) = \frac{d_N^{(T)}(\lambda)}{T^{1/2}} \; , \tag{3.3.9}$$

and $d_N^{(T)}(\lambda)$ is the finite Fourier-Stieltjes transform of the counting process $N(t)$ defined as

$$d_N^{(T)}(\lambda) = \int_0^T e^{-i\lambda t} dN(t) \; . \tag{3.3.10}$$

The properties of the statistic (3.3.9) are developed in the next section.

3.4 THE FINITE FOURIER-STIELTJES TRANSFORM OF THE COUNTING PROCESS

Let $N(t)$ be an integer-valued stationary point process defined on the interval $(0,T]$ with differential increments $dN(t) = N(t,t+dt]$. Let $J_N^{(T)}(\lambda)$ be the normalized finite Fourier-Stieltjes transform of the counting process $N(t)$ given by (3.3.9). Then we have the following property. The Fourier transform $J_N^{(T)}(\lambda)$ is a complex function with respect to λ satisfying the relation

$$J_N^{(T)}(-\lambda) = \overline{J_N^{(T)}(\lambda)} \quad , \tag{3.4.1}$$

where $\overline{J_N^{(T)}(\lambda)}$ is the complex conjugate function of $J_N^{(T)}(\lambda)$.

We now turn to the problem of finding the asymptotic distribution of $J_N^{(T)}(\lambda)$. If we denote by $N_1^C(0,2\pi f_{NN}(\lambda))$ the complex univariate normal distribution with mean 0 and variance $2\pi f_{NN}(\lambda)$ (ref. Goodman, 1963), then we have

Theorem 3.4.1: Let $N(t)$ be a stationary point process on $(0,T]$ with differential increments $dN(t) = N(t,t+dt]$, satisfying Lemma 1.2 of Appendix I. For $\lambda \neq 0$, the function $J_N^{(T)}(\lambda)$ is asymptotically $N_1^C(0,2\pi f_{NN}(\lambda))$ variate. Also $J_N^{(T)}(0) = N(0,T]/T^{\frac{1}{2}}$ is asymptotically $N_1(p_N T^{\frac{1}{2}}, 2\pi f_{NN}(0))$ variate.

Proof: We will show that the cumulants of $J_N^{(T)}(\lambda)$ tend to the cumulants of the normal distribution $N_1^C(0, 2\pi f_{NN}(\lambda))$. We have for the first cumulant

$$E\left\{ J_N^{(T)}(\lambda)\right\} = p_N e^{-i\lambda T/2} \frac{\sin \lambda T/2}{T^{1/2}\, \lambda/2} \quad , \quad \lambda \neq 0$$

where p_N is the MI. It is clear that the first cumulant tends to zero as $T \longmapsto \infty$.

Next

$$\operatorname{cov}\left(J_N^{(T)}(\lambda), J_N^{(T)}(\mu)\right) = T^{-1} \operatorname{cum}\left(d_N^{(T)}(\lambda), d_N^{(T)}(-\mu)\right)$$

$$= T^{-1} \int_{-\infty}^{\infty} \frac{\sin(\lambda-\alpha)T/2}{(\lambda-\alpha)/2} \, \frac{\sin(\mu-\alpha)T/2}{(\mu-\alpha)/2} \, f_{NN}(\alpha) d\alpha \longmapsto 0 \quad \text{as } T \longmapsto \infty \, .$$

For $\lambda = \mu$ we have

$$\operatorname{cov}\left(J_N^{(T)}(\lambda), J_N^{(T)}(\lambda)\right) = T^{-1} \int_{-\infty}^{\infty} \left(\frac{\sin(\lambda-\alpha)T/2}{(\lambda-\alpha)/2}\right)^2 f_{NN}(\alpha) d\alpha$$

The function $\frac{1}{2\pi T}\left(\frac{\sin(\lambda-\alpha)T/2}{(\lambda-\alpha)/2}\right)^2$ becomes a delta function as $T \longmapsto \infty$ (Papoulis, 1962) and

$$\operatorname{cov}\left(J_N^{(T)}(\lambda), J_N^{(T)}(\lambda)\right) \longmapsto 2\pi f_{NN}(\lambda) \, .$$

Finally

$$\operatorname{cum}\left\{J_N^{(T)}(\lambda_1), \ldots, J_N^{(T)}(\lambda_k)\right\} = (2\pi)^{k-1} T^{-k/2} \Delta^{(T)}\left(\sum_{j=1}^{k} \lambda_j\right) f_{N \ldots N}(\lambda_1, \ldots, \lambda_{k-1}) + o\left(T^{1-k/2}\right) ,$$

where $\Delta^{(T)}(\alpha) = \int_0^T e^{-i\alpha t} dt$. This result follows from Lemma 2.1 given in Appendix I.

The kth order cumulant of $J_N^{(T)}(\lambda)$ will tend to zero for $k > 2$. The above form of argument was used by Brillinger (1970b, 1972) for continuous time series and interval functions, respectively.

In practice, the Fourier-Stieltjes transform (3.3.9) can be approximated by

$$J_N^{(T)}(\lambda) = \frac{d_N^{(T)}(\lambda)}{T^{1/2}} \quad , \tag{3.4.2}$$

where now

$$d_N^{(T)}(\lambda) \simeq \sum_{t=0}^{T-1} e^{-i\lambda t}\{N(t+1) - N(t)\} \quad . \tag{3.4.3}$$

$J_N^{(T)}(\lambda)$ then becomes the standardized discrete Fourier transform of the number of events of N(t) in the interval (t,t+1], t=0,..., T-1 (ref. Brillinger, 1972 for a similar procedure applied in the analysis of time series).

The difference $\{N(t+1)-N(t)\}$ takes on the value 0 when no events have occurred, and the value 1 when an event has occurred in the interval (t,t+1].

3.5 THE ASYMPTOTIC PROPERTIES OF THE PERIODOGRAM OF A POINT PROCESS

Lemma 2.1 of Appendix I suggests that one could consider estimates of the form $d_N^{(T)}(\lambda)\,\overline{d_N^{(T)}(\lambda)}$ for the spectrum of a point process. The periodogram given by

$$I_{NN}^{(T)}(\lambda) = \frac{1}{2\pi T}\, d_N^{(T)}(\lambda)\overline{d_N^{(T)}(\lambda)} = \frac{1}{2\pi T}\left|\int_0^T e^{-i\lambda t} dN(t)\right|^2 , \; -\infty < \lambda < \infty \tag{3.5.1}$$

has the same symmetry, non-negativity and evenness as the PS of a point process defined by (3.3.2). These properties suggest that the function $I_{NN}^{(T)}(\lambda)$ is an obvious estimate of the spectrum (Bartlett, 1963a; Brillinger, 1969). However, as we shall see, (3.5.1) is not a consistent estimate, and we will consider the problem of constructing consistent estimates of the PS based on $I_{NN}^{(T)}(\lambda)$.

We start by examining the statistical behaviour of $I_{NN}^{(T)}(\lambda)$ in an attempt to clarify the source of difficulty of this estimate. The following theorems discuss the first and second order properties of the periodogram.

Theorem 3.5.1: Let $N(t)$ be a univariate stationary point process on $(0,T]$ with MI p_N and second-order CF $q_{NN}(u)$. Suppose that $q_{NN}(u)$ exists and satisfies (3.3.1). Let also $I_{NN}^{(T)}(\lambda)$ be given by (3.5.1). Then

$$E\{I_{NN}^{(T)}(\lambda)\}=(2\pi T)^{-1}\int_{-\infty}^{\infty}\left\{\frac{\sin T(\lambda-\alpha)/2}{(\lambda-\alpha)/2}\right\}^2 f_{NN}(\alpha)d\alpha+\frac{p_N^2}{2\pi T}\left(\frac{\sin\lambda T/2}{\lambda/2}\right)^2,\quad -\infty<\lambda<\infty\ . \tag{3.5.2}$$

Proof: The periodogram can be written as

$$I_{NN}^{(T)}(\lambda)=\frac{1}{2\pi T}\int_0^T\int_0^T e^{-i\lambda(t-s)}dN(t)\,dN(s)\ .$$

Then

$$E\{I_{NN}^{(T)}(\lambda)\}=\frac{1}{2\pi T}\int_0^T\int_0^T e^{-i\lambda(t-s)}E\{dN(t)\,dN(s)\}$$

$$=\frac{1}{2\pi T}\int_0^T\int_0^T e^{-i\lambda(t-s)}[\,p_N\delta(t-s)+q_{NN}(t-s)\,]\,dt\,ds+\frac{p_N^2}{2\pi T}\int_0^T\int_0^T e^{-i\lambda(t-s)}dt\,ds$$

$$=\frac{1}{2\pi T}\int_{-T}^{T}(T-|u|)\,[\,p_N\delta(u)+q_{NN}(u)\,]\,e^{-i\lambda u}du+\frac{p_N^2}{2\pi T}\left(\frac{\sin\lambda T/2}{\lambda/2}\right)^2,$$

by substituting $u=t-s$ *and* $v=s$

$$=\frac{1}{2\pi T}\int_{-T}^{T}(T-|u|)\left\{\int_{-\infty}^{\infty}e^{i\alpha u}f_{NN}(\alpha)d\alpha\right\}e^{-i\lambda u}du+\frac{p_N^2}{2\pi T}\left(\frac{\sin\lambda T/2}{\lambda/2}\right)^2,$$

by using (3.3.2)

$$= \frac{1}{2\pi T} \int_{-\infty}^{\infty} \left\{ \int_{-T}^{T} (T-|u|)\, e^{-i(\lambda-\alpha)u} du \right\} f_{NN}(\alpha) d\alpha + \frac{p_N^2}{2\pi T} \left(\frac{\sin \lambda T/2}{\lambda/2} \right)^2$$

$$= \frac{1}{2\pi T} \int_{-\infty}^{\infty} \left(\frac{\sin (\lambda-\alpha)T/2}{(\lambda-\alpha)/2} \right)^2 f_{NN}(\alpha)\, d\alpha + \frac{p_N^2}{2\pi T} \left(\frac{\sin \lambda T/2}{\lambda/2} \right)^2,$$

for $-\infty < \lambda < \infty$.

In the case that $\lambda \neq 0$, the final term in the above expression is negligible and we see that $E\left\{ I_{NN}^{(T)}(\lambda) \right\}$ is essentially a weighted average of the $f_{NN}(\lambda)$, with weight concentrated in the neighbourhood of λ .

<u>Corollary 3.5.1</u>: The periodogram $I_{NN}^{(T)}(\lambda)$ is an unbiased estimate of the PS $f_{NN}(\lambda)$ as $T \longmapsto \infty$, $\lambda \neq 0$.

<u>Proof</u>: The function $\frac{1}{2\pi T} \left(\frac{\sin (\lambda-\alpha)T/2}{(\lambda-\alpha)/2} \right)^2$ in expression (3.5.2) becomes a delta function as $T \longmapsto \infty$.
Hence $E\left\{ I_{NN}^{(T)}(\lambda) \right\} \longmapsto f_{NN}(\lambda)$ as $T \longmapsto \infty$.

<u>Proposition 3.5.1</u>: Under the assumptions of Theorem 3.5.1 , and if $\int_{-\infty}^{\infty} |u| |q_{NN}(u)| du < \infty$, then $E\left\{ I_{NN}^{(T)}(\lambda) \right\} = f_{NN}(\lambda) + O(T^{-1})$ for $\lambda \neq 0$.

<u>Proof</u>: The proof is similar to that used for Theorem 3.5.1, and applying Lemma 2.1 of Appendix I.

<u>Theorem 3.5.2</u>: Let N(t) be a univariate stationary point process on (0,T] with MI p_N and CF $q_{NN}(u)$ satisfying $\int_{-\infty}^{\infty} |u| |q_{NN}(u)| du < \infty$. Suppose that $\lambda, \mu, \lambda \pm \mu \neq 0$. Then

$$\operatorname{cov}\left\{I_{NN}^{(T)}(\lambda), I_{NN}^{(T)}(\mu)\right\} = \left\{\left[\frac{\sin(\lambda+\mu)T/2}{T(\lambda+\mu)/2}\right]^2 + \left[\frac{\sin(\lambda-\mu)T/2}{T(\lambda-\mu)/2}\right]^2\right\} f_{NN}^2(\lambda)$$

$$+ O(T^{-1}) .$$

Proof: From the definition of the fourth-order cumulant it follows that

$$\operatorname{cov}\left\{d_N^{(T)}(\lambda)\, d_N^{(T)}(-\lambda),\, d_N^{(T)}(\mu)\, d_N^{(T)}(-\mu)\right\} = \operatorname{cum}\left\{d_N^{(T)}(\lambda), d_N^{(T)}(-\lambda), d_N^{(T)}(\mu), d_N^{(T)}(-\mu)\right\}$$

$$+ \operatorname{cum}\left\{d_N^{(T)}(-\lambda)\right\}\cdot \operatorname{cum}\left\{d_N^{(T)}(\lambda), d_N^{(T)}(\mu), d_N^{(T)}(-\mu)\right\} + \operatorname{cum}\left\{d_N^{(T)}(-\mu)\right\}\cdot \operatorname{cum}\left\{d_N^{(T)}(\lambda), d_N^{(T)}(-\lambda), d_N^{(T)}(\mu)\right\}$$

$$+ \operatorname{cum}\left\{d_N^{(T)}(\lambda)\right\}\cdot \operatorname{cum}\left\{d_N^{(T)}(-\lambda), d_N^{(T)}(\mu), d_N^{(T)}(-\mu)\right\} + \operatorname{cum}\left\{d_N^{(T)}(-\lambda)\right\}\cdot \operatorname{cum}\left\{d_N^{(T)}(\lambda), d_N^{(T)}(\mu), d_N^{(T)}(-\mu)\right\}$$

$$+ \operatorname{cum}\left\{d_N^{(T)}(\lambda)\right\}\cdot \operatorname{cum}\left\{d_N^{(T)}(\mu)\right\}\cdot \operatorname{cum}\left\{d_N^{(T)}(-\lambda), d_N^{(T)}(-\mu)\right\} + \operatorname{cum}\left\{d_N^{(T)}(-\lambda)\right\}\cdot \operatorname{cum}\left\{d_N^{(T)}(-\mu)\right\}\cdot \operatorname{cum}\left\{d_N^{(T)}(\lambda), d_N^{(T)}(\mu)\right\}$$

$$+ \operatorname{cum}\left\{d_N^{(T)}(\lambda)\right\}\cdot \operatorname{cum}\left\{d_N^{(T)}(-\mu)\right\}\cdot \operatorname{cum}\left\{d_N^{(T)}(-\lambda), d_N^{(T)}(\mu)\right\} + \operatorname{cum}\left\{d_N^{(T)}(-\lambda)\right\}\cdot \operatorname{cum}\left\{d_N^{(T)}(\mu)\right\}\cdot \operatorname{cum}\left\{d_N^{(T)}(\lambda), d_N^{(T)}(-\mu)\right\}$$

$$+ \operatorname{cum}\left\{d_N^{(T)}(\lambda), d_N^{(T)}(\mu)\right\}\cdot \operatorname{cum}\left\{d_N^{(T)}(-\lambda), d_N^{(T)}(-\mu)\right\} + \operatorname{cum}\left\{d_N^{(T)}(-\lambda), d_N^{(T)}(\mu)\right\}\cdot \operatorname{cum}\left\{d_N^{(T)}(\lambda), d_N^{(T)}(-\mu)\right\} . \quad (*)$$

Also, from Lemma 2.1 of Appendix I we have

$$\operatorname{cum}\left\{d_N^{(T)}(\lambda), d_N^{(T)}(-\lambda), d_N^{(T)}(\mu), d_N^{(T)}(-\mu)\right\} = (2\pi)^3\, T\, f_{NNNN}(\lambda, -\lambda, \mu) + O(1)$$

$$\text{cum}\{d_N^{(T)}(\lambda), d_N^{(T)}(\mu), d_N^{(T)}(-\mu)\} = (2\pi)^2 \Delta^{(T)}(-\lambda) f_{NNN}(\lambda,\mu) + O(1)$$

$$\text{cum}\{d_N^{(T)}(-\lambda), d_N^{(T)}(-\mu)\} = 2\pi \, \Delta^{(T)}(-\lambda-\mu) f_{NN}(-\lambda) + O(1)$$

$$\text{cum}\{d_N^{(T)}(-\lambda)\} = \Delta^{(T)}(-\lambda) f_N + O(1) \quad .$$

If we substitute these expressions in (*) we get

$$\text{cov}\{d_N^{(T)}(\lambda) d_N^{(T)}(-\lambda), d_N^{(T)}(\mu) d_N^{(T)}(-\mu)\} = (2\pi)^2 \{|\Delta^{(T)}(\lambda+\mu)|^2 f_{NN}^2(\lambda) + |\Delta^{(T)}(\lambda-\mu)|^2 f_{NN}^2(\mu)\} + O(T^{-1})$$

and so,

$$\text{cov}\{I_{NN}^{(T)}(\lambda), I_{NN}^{(T)}(\mu)\} = \frac{1}{T^2}\left\{\left[\frac{\sin(\lambda+\mu)T/2}{(\lambda+\mu)/2}\right]^2 + \left[\frac{\sin(\lambda-\mu)T/2}{(\lambda-\mu)/2}\right]^2\right\} f_{NN}^2(\lambda) + O(T^{-1})$$

for $\lambda, \mu, \lambda \pm \mu \neq 0$.

<u>Corollary 3.5.2</u>: Under the conditions of the above theorem we have

$$\text{Var}\{I_{NN}^{(T)}(\lambda)\} = f_{NN}^2(\lambda) + O(T^{-1}) \quad ,$$

$$\text{cov}\{I_{NN}^{(T)}(\lambda), I_{NN}^{(T)}(\mu)\} = O(T^{-1}) \quad .$$

<u>Proof</u>: The proof easily follows from Theorem 3.5.2.

Brillinger (1975a) uses the same kind of arguments in finding the second-order properties of the periodogram of an ordinary time series.

Corollary 3.5.2 suggests that no matter how large T is taken, the variance of $I_{NN}^{(T)}(\lambda)$ will tend to remain at the same level $f_{NN}^2(\lambda)$. It is also clear from the second part of Cor. 3.5.2 that adjacent periodogram ordinates are uncorrelated for large T. The asymptotic behaviour of the periodogram of a point process is, therefore, similar to that of the periodogram of a continuous time series.

Finally, the following theorem shows that distinct periodogram ordinates are asymptotically independent χ^2 variates.

<u>Theorem 3.5.3:</u> Let N(t) be a univariate stationary point process on (0,T] satisfying conditions of Theorem 3.5.2. Suppose $\lambda_j, \lambda_k, \lambda_j \pm \lambda_k \neq 0$ for $1 \leq j < k \leq J$. Let

$$I_{NN}^{(T)}(\lambda) = \frac{1}{2\pi T} \left| \int_0^T e^{-i\lambda t} dN(t) \right|^2 \quad for \quad -\infty < \lambda < \infty .$$

Then $I_{NN}^{(T)}(\lambda_j)$, j=1,...,J are asymptotically independent $f_{NN}(\lambda_j)\chi_2^2/2$ variates.

<u>Proof:</u> Th. 3.4.1 indicates that $Re\{d_N^{(T)}(\lambda_j)\}$ and $Im\{d_N^{(T)}(\lambda_j)\}$ are asymptotically independent $N(0, \pi T f_{NN}(\lambda_j))$ variates (Koopmans, 1974; ps 26-27). Then it follows in the same way as in Brillinger (1975a), p414, that

$$I_{NN}^{(T)}(\lambda_j) = \frac{1}{2\pi T} \left\{ \left[Re\, d_N^{(T)}(\lambda_j) \right]^2 + \left[Im\, d_N^{(T)}(\lambda_j) \right]^2 \right\} ,$$

is asymptotically $f_{NN}(\lambda_j)\chi_2^2/2$. The asymptotic independence for different ordinates $I_{NN}^{(T)}(\lambda_j)$, j=1,...,J, follows from the asymptotic independence of $d_N^{(T)}(\lambda_j)$.

3.6 EXAMPLE: ASYMPTOTIC PROPERTIES OF THE PERIODOGRAM OF A POISSON POINT PROCESS

Let $N(t)$ be a univariate stationary Poisson point process on $(0,T]$. The first and second-order properties of the periodogram of this process are derived in the following theorems.

Theorem 3.6.1: Under the assumptions of Theorem 3.5.1 the expected value of the periodogram $I_{NN}^{(T)}(\lambda)$ for a Poisson point process is given by

$$(a) \quad E\left\{I_{NN}^{(T)}(\lambda)\right\} = \frac{p_N}{2\pi} + \frac{p_N^2 T}{2\pi}\left(\frac{\sin \lambda T/2}{\lambda T/2}\right)^2 \quad \text{for} \quad \lambda \neq 0 .$$

$$(b) \quad E\left\{I_{NN}^{(T)}(0)\right\} = \frac{p_N}{2\pi}\left(1 + p_N T\right) .$$

Proof: The power spectrum of a Poisson process is a constant given by $f_{NN}(\lambda) = \frac{p_N}{2\pi}$, $-\infty < \lambda < \infty$.

Then applying Theorem 3.5.1 we get

$$(a) \quad E\left\{I_{NN}^{(T)}(\lambda)\right\} = \frac{p_N}{2\pi} + \frac{p_N^2 T}{2\pi}\left(\frac{\sin \lambda T/2}{\lambda T/2}\right)^2 \quad \text{for} \quad \lambda \neq 0 .$$

$$(b) \quad E\left\{I_{NN}^{(T)}(0)\right\} = \frac{p_N}{2\pi} + \frac{p_N^2 T}{2\pi}\left[\lim_{\lambda \to 0}\left(\frac{\sin \lambda T/2}{\lambda T/2}\right)^2\right] = \frac{p_N}{2\pi}\left(1 + p_N T\right) .$$

This result is also given by Cox & Lewis (1968), p128, who use a different kind of argument which may be difficult to extend to processes which are not Poisson.

<u>Theorem 3.6.2</u>: Under the assumptions of Theorems 3.5.1 and 3.5.2 we have for the periodogram of a Poisson process that

$$\text{(a)} \quad Var\left\{ I_{NN}^{(T)}(\lambda) \right\} \simeq \frac{p_N^2}{(2\pi)^2}\left(1+\frac{1}{p_N T}\right) \quad for \quad \lambda \neq 0$$

$$\text{(b)} \quad cov\left\{ I_{NN}^{(T)}(\lambda_1),\, I_{NN}^{(T)}(\lambda_2) \right\} \simeq \frac{p_N^2}{(2\pi)^2}\left(\frac{1}{1+p_N T}\right) \quad for \quad \lambda_1 \neq \lambda_2 \neq 0$$

and

$$\text{(c)} \quad corr\left\{ I_{NN}^{(T)}(\lambda_1),\, I_{NN}^{(T)}(\lambda_2) \right\} \simeq \frac{1}{1+p_N T} \quad for \quad \lambda_1 \neq \lambda_2 \neq 0 \quad .$$

<u>Proof</u>: If we apply the same argument as used in Theorem 3.5.2 we have

$$Var\left\{ I_{NN}^{(T)}(\lambda) \right\} = \frac{1}{(2\pi T)^2}\left[(2\pi)^3 T f_{NNN}(\lambda,-\lambda,\lambda) + (2\pi)^2 T^2 f_{NN}^2(\lambda) + \cdots \right]$$

$$\simeq \frac{2\pi}{T} f_{NNNN}(\lambda,-\lambda,\lambda) + f_{NN}^2(\lambda) + O(T^{-2})$$

$$= \frac{2\pi}{T}\frac{p_N}{(2\pi)^3} + \frac{p_N^2}{(2\pi)^2} + O(T^{-2}) \text{ , since } N \text{ is a Poisson process}$$

$$= \frac{p_N^2}{(2\pi)^2}\left[1+\frac{1}{p_N T}\right] + O(T^{-2}) \simeq \frac{p_N^2}{(2\pi)^2}\left(1+\frac{1}{p_N T}\right) \quad for \quad \lambda \neq 0 \quad .$$

The same argument used in Theorem 3.5.2 also gives

$$\operatorname{cov}\left\{ I_{NN}^{(T)}(\lambda_1), I_{NN}^{(T)}(\lambda_2) \right\} = \frac{1}{(2\pi)^2 T^2}\left\{ (2\pi)^3 T f_{NNNN}(\lambda_1, -\lambda_1, \lambda_2) + \cdots \right\}$$

$$= \frac{2\pi}{T} f_{NNNN}(\lambda_1, -\lambda_1, \lambda_1) + O(T^{-2})$$

$$= \frac{2\pi}{T}\frac{p_N}{(2\pi)^3} + O(T^{-2}) \simeq \frac{p_N}{(2\pi)^2}\frac{1}{T} \quad \text{for } \lambda_1 \neq \lambda_2 \neq 0 .$$

This result can be modified to give

$$\operatorname{cov}\left\{ I_{NN}^{(T)}(\lambda_1), I_{NN}^{(T)}(\lambda_2) \right\} \simeq \frac{p_N}{(2\pi)^2 T}\frac{(1/T + p_N)}{(1/T + p_N)} \simeq \frac{p_N^2}{(2\pi)^2(1+p_N T)} + \frac{p_N}{(2\pi)^2 T(1+p_N T)}$$

$$= \frac{p_N^2}{(2\pi)^2(1+p_N T)} + O(T^{-2}) \simeq \frac{p_N^2}{(2\pi)^2(1+p_N T)} \quad \text{for } \lambda_1 \neq \lambda_2 \neq 0 .$$

By combining the above two relations for the variance and the covariance we find that

$$\operatorname{corr}\left\{ I_{NN}^{(T)}(\lambda_1), I_{NN}^{(T)}(\lambda_2) \right\} = \frac{\operatorname{cov}\left\{ I_{NN}^{(T)}(\lambda_1), I_{NN}^{(T)}(\lambda_2) \right\}}{\left\{ \operatorname{Var}(I_{NN}^{(T)}(\lambda_1))\operatorname{Var}(I_{NN}^{(T)}(\lambda_2)) \right\}^{1/2}} \simeq \frac{p_N T}{(1+p_N T)^2}$$

$$\simeq \frac{1}{1+p_N T} - \frac{1}{(1+p_N T)^2} \simeq \frac{1}{1+p_N T} + O(T^{-2}) \simeq \frac{1}{1+p_N T} \quad \text{for } \lambda_1 \neq \lambda_2 \neq 0 .$$

The results found here for the covariance and correlation of two ordinates of the periodogram of a Poisson point process are also the same as those given by Cox & Lewis (1968) who used an approach, which may only be applicable to Poisson processes, whereas our method may be applied more generally.

We now turn to the problem of smoothing the periodogram and constructing estimates of the PS of a point process with better properties than estimate (3.5.1) considered in section 3.5.

3.7 THE WEIGHTED PERIODOGRAM AS AN ESTIMATE OF THE POWER SPECTRUM OF A POINT PROCESS.

Theorem 3.5.3 indicates that the (2m+1) adjacent periodogram ordinates $I_{NN}^{(T)}(\lambda + \frac{2\pi j}{T})$, j=0, $\pm$ 1,..., $\pm$ m are approximately $f_{NN}(\lambda)\chi_2^2/2$ variates. It is, therefore, obvious that a new class of general estimates of the PS may be presented by

$$f_{NN}^{(T)}(\lambda) = \sum_{j=-m}^{m} W_j I_{NN}^{(T)}\left(\lambda + \frac{2\pi j}{T}\right) \quad \text{if} \quad \lambda \neq 0 \tag{3.7.1}$$

$$= \sum_{j=1}^{m} W_j I_{NN}^{(T)}\left(\lambda + \frac{2\pi j}{T}\right) \Big/ \sum_{j=1}^{m} W_j \quad \text{if} \quad \lambda = 0 , \tag{3.7.2}$$

where W_j, j=0, $\pm$ 1,..., $\pm$ m, are the weights satisfying

$$\sum_{j=-m}^{m} W_j = 1 \quad , \tag{3.7.3}$$

and also $m/T \longmapsto 0$ as $T \longmapsto \infty$.

The following theorems examine the first and second-order properties of estimate (3.7.1) to (3.7.2).

Theorem 3.7.1: Let N(t) be an integer-valued stationary point process on (0,T] with MI p_N and CF $q_{NN}(u)$. Suppose that $q_{NN}(u)$

satisfies $\int |q_{NN}(u)|\, du < \infty$. Let $f_{NN}^{(T)}(\lambda)$ be given by (3.7.1) to (3.7.2). Then

$$E\{f_{NN}^{(T)}(\lambda)\} = \int_{-\infty}^{\infty} A^{(T)}(\alpha) f_{NN}(\lambda-\alpha)d\alpha + \frac{P_N^2}{2\pi T}\sum_{j=-m}^{m} W_j\left[\frac{\sin(\lambda+\frac{2\pi j}{T})T/2}{(\lambda+\frac{2\pi j}{T})/2}\right]^2 \quad \text{if } \lambda \neq 0$$

$$= \int_{-\infty}^{\infty} B^{(T)}(\alpha) f_{NN}(\lambda-\alpha)d\alpha + \frac{P_N^2}{2\pi T}\sum_{j=1}^{m} W_j\left[\frac{\sin(\lambda+\frac{2\pi j}{T})T/2}{(\lambda+\frac{2\pi j}{T})/2}\right]^2 \Big/ \sum_{j=1}^{m} W_j \quad \text{if } \lambda = 0 ,$$

where

$$A^{(T)}(\alpha) = \sum_{j=-m}^{m} W_j F^{(T)}(\alpha - \tfrac{2\pi j}{T})$$

$$B^{(T)}(\alpha) = \sum_{j=1}^{m} W_j F^{(T)}(\alpha - \tfrac{2\pi j}{T}) \Big/ \sum_{j=1}^{m} W_j ,$$

and

$$F^{(T)}(\alpha) = (2\pi T)^{-1}\left[\frac{\sin T\alpha/2}{\alpha/2}\right]^2 , \quad -\infty < \alpha < \infty .$$

<u>Proof:</u> The proof is similar to one given in Theorem 3.5.1 making the appropriate changes to include the weights W_j.

<u>Corollary 3.7.1:</u> Suppose in addition to the assumptions of the theorem 3.7.1, $q_{NN}(u)$ satisfies

$$\int |u||q_{NN}(u)|\, du < \infty \quad \text{and} \quad m/T \longmapsto 0 \quad \text{as} \quad T \longmapsto \infty .$$

Then

$$E\{f_{NN}^{(T)}(\lambda)\} = f_{NN}(\lambda) + O(T^{-1}) \quad \text{for} \quad -\infty < \lambda < \infty .$$

If $T \longmapsto \infty$, $f_{NN}^{(T)}(\lambda)$ is asymptotically an unbiased estimate of $f_{NN}(\lambda)$.

<u>Proof</u>: From expression (3.7.1) we have that

$$E\{f_{NN}^{(T)}(\lambda)\} = \sum_{j=-m}^{m} W_j \; E\{I_{NN}^{(T)}(\lambda + \tfrac{2\pi j}{T})\}$$

$$= \sum_{j=-m}^{m} W_j \frac{1}{2\pi T} E\{d_N^{(T)}(\lambda^*) \overline{d_N^{(T)}(\lambda^*)}\}, \text{ where } \lambda^* = \lambda + \frac{2\pi j}{T}, \; j = 0, \pm 1, \ldots, \pm m$$

$$= \sum_{j=-m}^{m} W_j \frac{1}{2\pi T} \operatorname{cum}\{d_N^{(T)}(\lambda^*), d_N^{(T)}(-\lambda^*)\} + \frac{p_N^2}{2\pi T} \sum_{j=-m}^{m} W_j \left(\frac{\sin \lambda^* T/2}{\lambda^*/2}\right)^2$$

$$= \sum_{j=-m}^{m} W_j \frac{1}{2\pi T}\{2\pi T f_{NN}(\lambda^*) + O(1)\} + \frac{p_N^2}{2\pi T} \sum_{j=-m}^{m} W_j \left(\frac{\sin \lambda^* T/2}{\lambda^*/2}\right)^2,$$

which follows from Lemma 2.1 of Appendix I. Then

$$E\{f_{NN}^{(T)}(\lambda)\} = \sum_{j=-m}^{m} W_j f_{NN}(\lambda^*) + \frac{p_N^2}{2\pi T} \sum_{j=-m}^{m} W_j \left(\frac{\sin \lambda^* T/2}{\lambda^*/2}\right)^2 + O(T^{-1}) \; .$$

For $T \mapsto \infty$ and $\lambda \neq 0$ we have that $\lambda^* \mapsto \lambda$ and $f_{NN}^{(T)}(\lambda)$ is an asymptotically unbiased estimate of $f_{NN}(\lambda)$.

<u>Theorem 3.7.2</u>: Let N(t) be a univariate stationary point process on (0,T] with MI p_N and CF $q_{NN}(u)$ satisfying $\int |u| |q_{NN}(u)| du < \infty$. Let $f_{NN}^{(T)}(\lambda)$ be given by (3.7.1) and (3.7.2). Suppose $\lambda, \mu, \lambda \pm \mu \neq 0$, then

$$\operatorname{Var} f_{NN}^{(T)}(\lambda) = f_{NN}^2(\lambda) \sum_{j=-m}^{m} W_j^2 + O(T^{-1}) \quad \text{if} \quad \lambda \neq 0$$

$$= f_{NN}^2(\lambda) \sum_{j=1}^{m} W_j^2 / \{\sum_{j=1}^{m} W_j\}^2 \quad \text{if} \quad \lambda = 0 \; .$$

Also $\quad \operatorname{cov}\{f_{NN}^{(T)}(\lambda), f_{NN}^{(T)}(\mu)\} = O(T^{-1}) \; .$

<u>Proof</u>: The proof follows from Theorem 3.5.2 and Corollary 3.5.2 and taking into consideration the necessary changes for the weights W_j.

If $\lambda \neq 0$, then for large T the variance of this estimate is proportional to $\sum_{j=-m}^{m} W_j^2$. $\sum_{j=-m}^{m} W_j^2$ will be minimized subject to the constraint $\sum_{j=-m}^{m} W_j = 1$, if we choose $W_j = \frac{1}{2m+1}$ for $j=0, \pm 1, \ldots, \pm m$.

<u>Theorem 3.7.3</u>: Let N(t) be a univariate stationary point process on (0,T] satisfying the assumptions of Theorem 3.7.2. Let $f_{NN}^{(T)}(\lambda)$ be the estimate defined by (3.7.1) to (3.7.2). Suppose $\lambda_j \pm \lambda_k \neq 0$ for $1 \leqslant j < k \leqslant J$. Then $f_{NN}^{(T)}(\lambda_1), \ldots, f_{NN}^{(T)}(\lambda_J)$ are asymptotically independent with $f_{NN}^{(T)}(\lambda)$ asymptotically

$$f_{NN}(\lambda) \sum_{j=-m}^{m} W_j X_2^2(j)/2 \quad \text{if} \quad \lambda \neq 0$$

and

$$f_{NN}(\lambda) \left\{ \sum_{j=1}^{m} W_j X_2^2(j)/2 \right\} \Big/ \left[\sum_{j=1}^{m} W_j \right]^2 \quad \text{if} \quad \lambda = 0 \; . \quad (X_2^2(j) = X_2^2,\; j = 1, \ldots, m).$$

The different chi-squared variates appearing are statistically independent.

<u>Proof</u>: The proof is similar to the one given in Theorem 3.5.3 making the necessary changes for the weights W_j.

The asymptotic distribution of $f_{NN}^{(T)}(\lambda)$ is clearly a weighted combination of independent chi-squared variates.

In practice, there is a standard procedure (Box, 1954)used to approximate the distribution of the weighted sum of independent X^2 variates by a multiple, θX^2_ν, of a chi-squared distribution, where ν are the degrees of freedom. In order to evaluate the mean and the degrees of freedom of the chi-squared distribution we equate the first- and second moments of this distribution with the first and second moments of the quantity $\sum_{j=-m}^{m} W_j X_2^2(j)/2$, i.e.

$$\theta\nu = \sum_{j=-m}^{m} W_j \quad or \quad \theta = \frac{1}{\nu} \tag{3.7.4}$$

$$\theta^2 2\nu = \sum_{j=-m}^{m} W_j^2 \quad or \quad \nu = \frac{2}{\sum_{j=-m}^{m} W_j^2} \ . \tag{3.7.5}$$

Theorem 3.7.2 indicates that the estimate of the form (3.7.1) to (3.7.2) will have minimum variance when

$$W_j = \frac{1}{2m+1} \ , \ j = 0, \pm 1, \ldots, \pm m \ .$$

This choice of the weights then gives the estimates

$$f_{NN}^{(T)}(\lambda) = \frac{1}{2m+1} \sum_{j=-m}^{m} I_{NN}^{(T)}\left(\lambda + \frac{2\pi j}{T}\right) \quad for \quad \lambda \neq 0 \tag{3.7.6}$$

$$= \frac{1}{m} \sum_{j=1}^{m} I_{NN}^{(T)}\left(\lambda + \frac{2\pi j}{T}\right) \quad for \quad \lambda = 0 \ . \tag{3.7.7}$$

Theorem 3.7.4: Let $N(t)$ be a stationary point process on $(0,T]$ satisfying assumptions of Theorem 3.7.1. Suppose that the CF $q_{NN}(u)$ satisfies the condition

$$\int |u||q_{NN}(u)|\, du < \infty .$$

Then the estimate $f_{NN}^{(T)}(\lambda)$ given by (3.7.6) to (3.7.7) is an asymptotically unbiased estimate of the power spectrum $f_{NN}(\lambda)$. Also, the variance and the covariance of this estimate is given by

$$\operatorname{Var} f_{NN}^{(T)}(\lambda) = \frac{f_{NN}^2(\lambda)}{2m+1} + O(T^{-1}) \quad \text{for } \lambda = 0 \tag{3.7.8}$$

$$= \frac{f_{NN}^2(\lambda)}{m} + O(T^{-1}) \quad \text{for } \lambda \neq 0 , \tag{3.7.9}$$

and

$$\operatorname{cov}\left\{ f_{NN}^{(T)}(\lambda), f_{NN}^{(T)}(\mu) \right\} = O(T^{-1}) .$$

Proof: The proof follows from Theorems 3.7.1 and 3.7.2 and Corollary 3.7.1.

The estimate given by (3.7.6) to (3.7.7) is a simple average of the periodogram ordinates in the neighbourhood of λ. The mean value of $f_{NN}^{(T)}(\lambda)$ is a weighted average of the PS in the neighbourhood of λ. If m is not too large compared to T, and $f_{NN}(\lambda)$ is smooth, then $E\left\{ f_{NN}^{(T)}(\lambda) \right\}$ may be close to $f_{NN}(\lambda)$ (see Th. 3.7.1). The bias of this estimate will generally be greater than that of $I_{NN}^{(T)}(\lambda)$. (Brillinger, 1975a; p135, discusses this point in the case of the periodogram of an ordinary time series). However, the variance of the estimate $f_{NN}^{(T)}(\lambda)$ is $\frac{1}{2m+1}$ times that of the unsmoothed periodogram which suggests that by choosing an appropriate value of m we can find an acceptable level of stability in the estimate. Moreover, the bias of the estimate $f_{NN}^{(T)}(\lambda)$ may increase as m gets larger, and so a compromise value of m will have to be found. (For more details about this point refer to Brillinger, 1975a; p136).

We now consider applications of this estimate to the analysis of the data sets described in Chapter 1. In addition, modifications of this estimate to include convergence factors are also considered and then applied to the same data sets.

Finally, the idea of a zero mean point process is introduced. We believe that it further improves the properties of the estimate of the power spectrum.

3.8 FURTHER ESTIMATES OF THE SPECTRUM OF A POINT PROCESS: INTRODUCTION OF THE ZERO MEAN POINT PROCESS AND THE USE OF CONVERGENCE FACTORS: EXAMPLE OF SPECTRAL ESTIMATES BASED ON THE PERIODOGRAM OF ENTIRE RECORD

The suggested method for estimating the power spectrum described in sections 3.5 and 3.7 depends on the calculation of the periodogram, $I_{NN}^{(T)}(\lambda)$, which in turn is based on the finite Fourier-Stieltjes transform of the counting process (ref. section 3.3, equation (3.3.10)). In practice, to be able to apply the fast Fourier transform we approximate equation (3.3.10) by taking the Fourier-Stieltjes transform of the first differences $\{N(t+1)-N(t)\}$ of the counting process (equ. 3.4.3). It is shown in Appendix II how the spectrum of $X(t) = \frac{\{N(t+h)-N(t)\}}{h}$ is related to the spectrum of point process $N(t)$ when $h = 1$ and $|\lambda| \leq \pi$.

The discrete Fourier transform of equation (3.4.3) is then carried out by using the Sande-Tukey Radix 2 form of the Fast Fourier transform (FFT) (Gentleman & Sande, 1966).

Two features of the analytic properties of the discrete Fourier-transform must be kept in mind when applying this transform to the estimation of spectra.

First, the application of the discrete Fourier-transform to a record of length T results in an estimated spectrum which at each frequency is a weighted average of the true spectrum in the neighbourhood of this frequency (Bloomfield, 1976; Brigham, 1974 and Brillinger, 1975a). This phenomenon is called leakage. Since the object of a harmonic analysis is to separate out the effects of different frequencies, and since leakage may obscure interesting features of a spectrum, procedures must be adopted to minimize leakage. One method is to multiply the data by a smoothing function prior to taking its transform. Functions used to multiply the data before taking its Fourier transform are called convergence factors, data windows, or tapers (Brillinger, 1975a; Chapter 3, particularly section 3.3). A second feature of the discrete Fourier transform is a kind of aliasing in the time domain, and is called aliasing of separations (Tukey, 1980). The application of the discrete Fourier transform, using any algorithm, of length M leads to aliasing of time separations of T and T-M. Tukey (1980) demonstrates how aliasing of separations occurs, and shows that it may be broken by adding zeroes to the data. The addition of zeroes to the data is called padding by Tukey.

Therefore, it is clear from above that the application of the discrete Fourier transform inherently carries with it an aliasing of separations, and leakage.

In practice, in order to reduce the leakage and break up possible aliasing of separations we use the following procedures prior to taking the finite Fourier transform of a sample record:

(i) Subtract the mean intensity of $N(t)$ from the differences $\{N(t+1)-N(t)\}$, $t = 0,1,\ldots, T-1$.

(ii) Multiply the data resulting from procedure (i) by a convergence factor.

(iii) Add zeroes to the record so that the length of the sample to be transformed is a power of 2. (In practice this amounts to adding approximately 512 zeroes to the data sets previously described).

A second kind of aliasing called aliasing of frequencies is known to the spectrum analysts for many decades (Blackman & Tukey, 1959). In our case we avoid this problem by taking the sample interval to be 1 msec, which is adequate to ensure that there is no aliasing of frequencies over the range of frequencies for which an estimate of the power spectrum is calculated. The range of frequencies, for which the estimate of the power spectrum is calculated, must be the open interval $(0,\pi)$ since π is the Nyquist frequency.

Continuing in this section, we now consider the sampling distribution of the Fourier-Stieltjes transform of the first differences of the counting process when a convergence factor is contained within the transform. In this case the Fourier-Stieltjes transform can be written as

$$\hat{d}_N^{(T)}(\lambda) = \int_t k_T(t)\, e^{-i\lambda t} dN(t) \simeq \sum_t k_T(t)\, e^{-i\lambda t}\left[N(t+1) - N(t)\right], \qquad (3.8.1)$$

where k(t) is a convergence factor. The function k(t) satisfies the following assumption.

<u>Assumption 3.8.1</u>: The function k(u), $-\infty<u<\infty$, is even, of bounded variation, vanishes for $|u|>1$, and takes the value 1 for u=0 (ref., e.g., Parzen, 1957).

The asymptotic properties of the periodogram based on equation (3.8.1) are also derived.

Furthermore, the fact that we subtract the mean intensity from the data prior to taking its Fourier-Stieltjes transform suggests the consideration of the sampling distribution of the Fourier-Stieltjes transform of the modified counting process (called the zero-mean point process) and the asymptotic properties of the periodogram based on this modification. Finally, we show in the last section of this chapter that the introduction of the zero-mean point process considerably simplifies the derivation of the properties of time domain parameters in terms of frequency domain parameters.

Theorem 3.8.1: Let N(t) be a stationary point process on $(0,T]$ with differential increments dN(t) satisfying Assumption 1.2 of Appendix I. Let also $\hat{d}_N^{(T)}(\lambda)$ be given by (3.8.1). Then, for $\lambda \neq 0$, the distribution of $\hat{d}_N^{(T)}(\lambda)$ is approximately $N_1^C(0, 2\pi T f_{NN}(\lambda)\int k^2(t)dt)$.

Proof: We note that the standardised first order cumulant is given by

$$T^{-1/2}E\left\{\hat{d}_N^{(T)}(\lambda)\right\} = \frac{P_N}{T^{1/2}} \sum_t k_T(t)e^{-i\lambda t} = \frac{P_N}{T^{1/2}} H_1^{(T)}(\lambda) ,$$

$$\text{where} \quad H_1^{(T)}(\lambda) = \sum_t k_T(t)e^{-i\lambda t} .$$

Therefore, the first order cumulant tends to zero for all values of λ $(\lambda \neq 0)$ as $T \longmapsto \infty$.

Next, it follows from Lemma 2.2 Appendix I that

$$T^{-1}\text{cum}\left\{\hat{d}_N^{(T)}(\lambda), \hat{d}_N^{(T)}(\mu)\right\} = \frac{2\pi}{T} H_2^{(T)}(\lambda+\mu) f_{NN}(\lambda) + O(1) .$$

So, cum $(d_N^{(T)}(\lambda), d_N^{(T)}(\mu))$ tends to zero as $T \longmapsto \infty$.

For $\lambda = -\mu$ we have

$$T^{-1}\text{cum}\left\{\hat{d}_N^{(T)}(\lambda), \hat{d}_N^{(T)}(-\lambda)\right\} = T^{-1}\text{cov}\left\{\hat{d}_N^{(T)}(\lambda), \hat{d}_N^{(T)}(\lambda)\right\}$$

$$= T^{-1}\text{Var}\left\{\hat{d}_N^{(T)}(\lambda)\right\} = 2\pi\int k^2(t)dt\, f_{NN}(\lambda) + o(1) ,$$

and

$$\text{Var}\left\{\hat{d}_N^{(T)}(\lambda)\right\} \simeq 2\pi T\int k^2(t)dt\, f_{NN}(\lambda) .$$

Finally

$$T^{-\ell/2}\operatorname{cum}\left\{\hat{d}_N^{(T)}(\lambda_1),\dots,\hat{d}_N^{(T)}(\lambda_\ell)\right\} = T^{-\ell/2}(2\pi)^{\ell-1}H_\ell^{(T)}\left(\sum_{j=1}^{\ell}\lambda_j\right)f_{N\dots N}(\lambda_1,\dots,\lambda_{\ell-1}) + o(T^{1-\ell/2})$$

$$= (2\pi)^{\ell-1}T^{1-\ell/2}\int k^\ell(t)dt\, f_{N\dots N}(\lambda_1,\dots,\lambda_{\ell-1}) + o(T^{1-\ell/2}) \quad .$$

This cumulant of order l tends to zero as $T \mapsto \infty$ for $l > 2$.

<u>Theorem 3.8.2</u>: Let $N(t)$ be a stationary point process on $(0,T]$ satisfying the assumptions of Theorem 3.8.1. Let $k(u), -\infty < u < \infty$, satisfy Assumption 3.8.1 and $\hat{d}_N^{(T)}(\lambda)$ be given by (3.8.1). If we define

$$\hat{I}_{NN}^{(T)}(\lambda) = \frac{1}{2\pi T}\hat{d}_N^{(T)}(\lambda)\overline{\hat{d}_N^{(T)}(\lambda)} \quad ,$$

then for $2\lambda,\ \lambda \pm \mu \neq 0$ we have

$$(i)\quad E\left\{\hat{I}_{NN}^{(T)}(\lambda)\right\} = f_{NN}(\lambda)\int k^2(t)dt + o(1)$$

and

$$(ii)\quad \operatorname{cov}\left\{\hat{I}_{NN}^{(T)}(\lambda),\ \hat{I}_{NN}^{(T)}(\mu)\right\} = o(1) \quad .$$

<u>Proof:</u> (i) For the expectation of the periodogram $\hat{I}_{NN}^{(T)}(\lambda)$ we get

$$E\left\{\hat{I}_{NN}^{(T)}(\lambda)\right\} = \frac{1}{2\pi T}E\left\{\hat{d}_N^{(T)}(\lambda)\hat{d}_N^{(T)}(-\lambda)\right\} = \frac{1}{2\pi T}\operatorname{cum}\left\{\hat{d}_N^{(T)}(\lambda),\ \hat{d}_N^{(T)}(-\lambda)\right\}$$

$$+\frac{1}{2\pi T}E\left\{\hat{d}_N^{(T)}(\lambda)\right\}E\left\{\hat{d}_N^{(T)}(-\lambda)\right\} = \frac{1}{2\pi T}2\pi H_2^{(T)}(0)f_{NN}(\lambda) + \frac{1}{2\pi T}p_N^2\left|H_1^{(T)}(\lambda)\right|^2 + o(1)$$

$$= f_{NN}(\lambda)\int k^2(t)dt + o(1) \quad \text{for } \lambda \neq 0 \quad ,$$

where

$$H_1^T(\lambda) = \sum_t k(t/T)\, e^{-i\lambda t}$$

and

$$H_2^{(T)}(0) = \sum_t k^2(t/T) \simeq T\int k^2(t)\,dt \quad .$$

Also, we have

$$H_\ell^{(T)}\Big(\sum_{j=1}^{\ell}\lambda_j\Big) = \sum_t k^\ell(t/T)\, e^{-i\sum_{j=1}^{\ell}\lambda_j t} \quad .$$

(ii) For the covariance of two different ordinates of the periodogram we find

$$\operatorname{cov}\left\{\hat{I}_{NN}^{(T)}(\lambda),\, \hat{I}_{NN}^{(T)}(\mu)\right\} = \frac{1}{(2\pi T)^2}\operatorname{cov}\left\{\hat{d}_N^{(T)}(\lambda)\hat{d}_N^{(T)}(-\lambda),\, \hat{d}_N^{(T)}(\mu)\, d_N^{(T)}(-\mu)\right\}$$

$$= \frac{1}{(2\pi T)^2}\Big\{ (2\pi)^3 H_4^{(T)}(0) f_{NNNN}^{(T)}(\lambda,-\lambda,\mu) \; + o(T)$$

$$+ \left[\, 2\pi\, H_2^{(T)}(\lambda+\mu) f_{NN}(\lambda) + o(T)\right]\cdot\left[\, 2\pi H_2^{(T)}(-\lambda-\mu) f_{NN}(\lambda) + o(T)\right]$$

$$+ \left[\, 2\pi H_2^{(T)}(\lambda-\mu) f_{NN}(\lambda) + o(T)\right]\left[\, 2\pi H_2^{(T)}(-\lambda+\mu) f_{NN}(\lambda) + o(T)\right] \Big\}$$

$$\left\{\frac{|H_2^{(T)}(\lambda+\mu)|^2}{T^2} + \frac{|H_2^{(T)}(\lambda-\mu)|^2}{T^2}\right\} f_{NN}^2(\lambda) + o(1) \quad .$$

It is clear that

$$\operatorname{cov}\left\{\hat{I}_{NN}^{(T)}(\lambda),\, \hat{I}_{NN}^{(T)}(\mu)\right\} = o(1) \quad \text{for} \quad \lambda \pm \mu \neq 0.$$

Theorem 3.8.2 suggests the following unbiased estimate of the PS of a point process

$$\tilde{I}_{NN}^{(T)}(\lambda) = \frac{\hat{I}_{NN}^{(T)}(\lambda)}{\int K^2(t)\,dt} \tag{3.8.2}$$

Then

Corollary 3.8.1: Under the assumptions of Theorem 3.8.2 we have

(i) $$E\{\tilde{I}_{NN}^{(T)}(\lambda)\} = f_{NN}(\lambda) + o(1)$$

(ii) $$\operatorname{cov}\{\tilde{I}_{NN}^{(T)}(\lambda),\ \tilde{I}_{NN}^{(T)}(\mu)\} = |H_2^{(T)}(0)|^{-2}\{|H_2^{(T)}(\lambda-\mu)|^2 + |H_2^{(T)}(\lambda+\mu)|^2\} f_{NN}^2(\lambda) + o(1),$$

where

$$H_2^{(T)}(\lambda+\mu) = \sum_t K^2(t/T)e^{-i(\lambda+\mu)t} \quad \text{and} \quad H_2^{(T)}(0) = \sum_t K^2(t/T) \sim T\int K^2(t)\,dt \ .$$

Proof: The proof is similar to that given for Theorem 3.8.2.

We now define the Fourier-Stieltjes transform based on the zero mean point process as follows

$$d_{N'}^{(T)}(\lambda) = \int_t K(t/T)e^{-i\lambda t}dN'(t) \ , \tag{3.8.3}$$

where

$$N'(t) = N(t) - p_N t \ . \tag{3.8.4}$$

Theorem 3.8.3: Under the conditions of Theorem 3.8.1 $d_{N'}^{(T)}(\lambda)$ has the same asymptotic distributions as $\hat{d}_N^{(T)}(\lambda)$, $\lambda \neq 0$.

Proof: The proof follows by substituting $N'(t) = N(t) - p_N t$ for $N(t)$ and using the same arguments as in Theorem 3.8.1.

In relation to $d_{N'}^{(T)}(\lambda)$ we may now define the periodogram $I_{N'N'}^{(T)}(\lambda)$ as

$$I_{N'N'}^{(T)}(\lambda) = \frac{1}{2\pi H_1^{(T)}(0)} d_{N'}^{(T)}(\lambda) \overline{d_{N'}^{(T)}(\lambda)} \tag{3.8.5}$$

The asymptotic first- and second-order properties of the modified periodogram are demonstrated in the next theorem.

Theorem 3.8.4: Suppose that the assumptions of Theorem 3.8.2 are satisfied. Then

$$E\left\{ I_{N'N'}^{(T)}(\lambda) \right\} = f_{NN}(\lambda) + o(1)$$

and

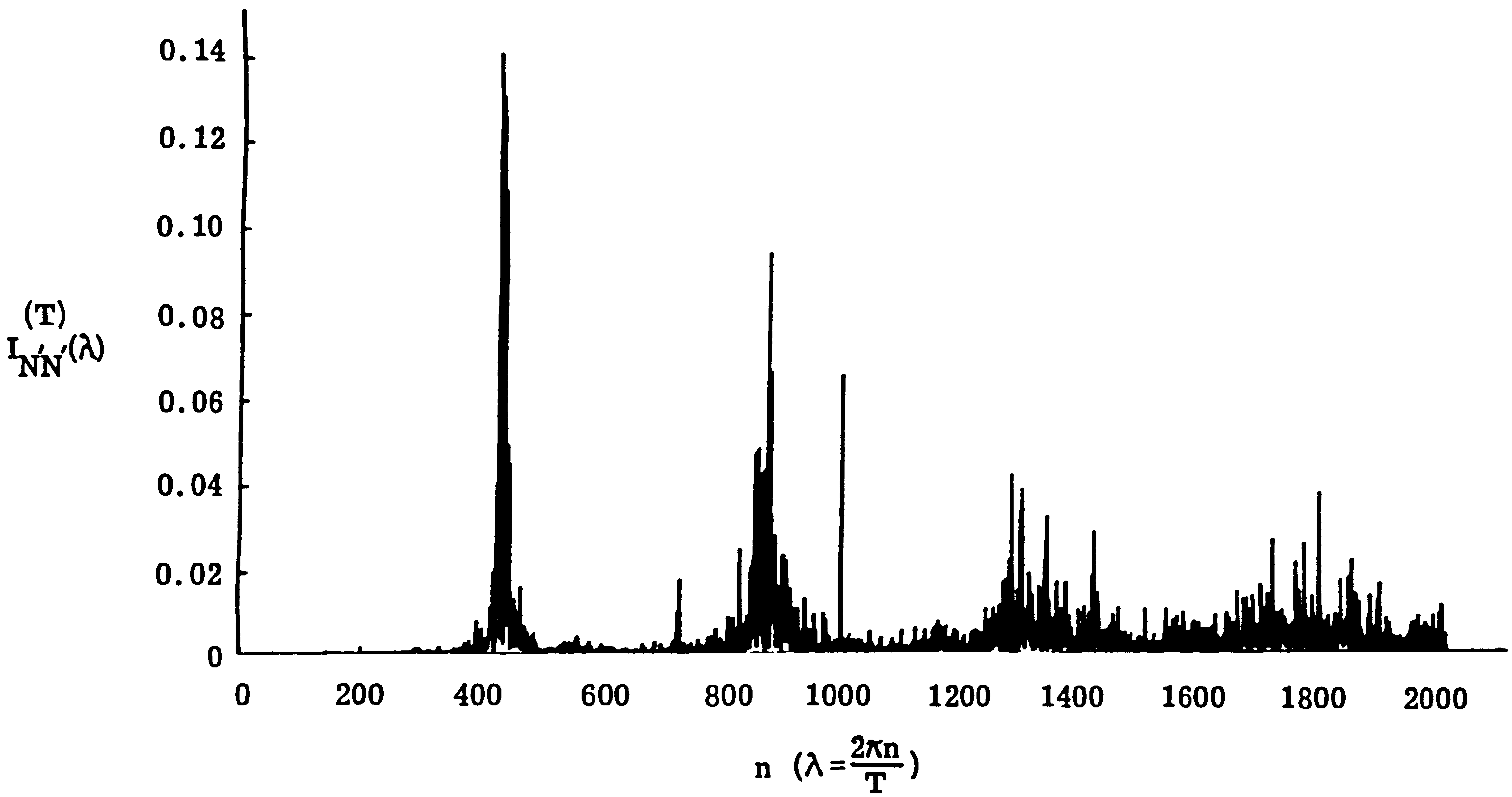
$$\operatorname{cov}\left\{ I_{N'N'}^{(T)}(\lambda), I_{N'N'}^{(T)}(\mu) \right\} = \left| H_2^{(T)}(0) \right|^{-2} \left\{ \left| H_2^{(T)}(\lambda-\mu) \right|^2 + \left| H_2^{(T)}(\lambda+\mu) \right|^2 \right\} f_{NN}^2(\lambda) + o(1) \text{ .}$$

Proof: The proof follows directly from Corollary 3.8.1.

The remainder of this section presents examples of the use of two of the estimates previously described. We first consider the application of estimate (3.8.5)

$$I_{N'N'}^{(T)}(\lambda) = \frac{1}{2\pi H_1^{(T)}(0)} d_{N'}^{(T)}(\lambda) d_{N'}^{(T)}(-\lambda)$$

to the calculation of the spectrum of the spontaneous discharge of the Ia sensory axon (Fig. 3.8.1). A smooth



<u>Fig. 3.8.1</u>: Periodogram of the spontaneous Ia discharge. This function was computed by using the FFT for a record length $T = 2^{14}$

estimate of the spectrum is obtained by using the following expression

$$f_{NN}^{(T)}(\lambda) = \sum_{j=-m}^{m} W_j I_{NN'}^{(T)}(\lambda + \frac{2\pi j}{T}) \quad for \ \lambda \neq 0 \quad . \tag{3.8.6}$$

The first and second order properties and the asymptotic distribution of this estimate are considered in Section 3.9.

Comparisons of the spectral estimates of the spontaneous Ia discharge are given when different weighting schemes are used. In each of the examples we use as a convergence factor a split bell cosine with a 10% tapering at the beginning and the end of the data (Bloomfield, 1976, p194). The first weighting scheme used is a simple averaging procedure of the periodogram ordinates (Figs. 3.8.2 a-d), the second one is a modified Daniell weighting scheme (Figs. 3.8.3 a-d), and finally in Figs. 3.8.4 (a-d) we demonstrate the use of a quadratic weighting scheme.

In the other examples estimate (3.8.6)is used to calculate the spectrum of the Ia discharge when different input conditions are imposed on the muscle spindle. The spectrum of the Ia discharge in the presence of γ_s activity using a simple average weighting scheme (Figs. 3.8.5 a-d) is compared with the same spectral estimates when a quadratic weighting is used (Figs. 3.8.6 a-d). Finally, the spectrum of the Ia discharge in the presence of a length change imposed on the muscle spindle (Fig. 3.8.7) is compared with the spectrum of the Ia discharge in the presence of both a γ_s input to the muscle spindle and an imposed length

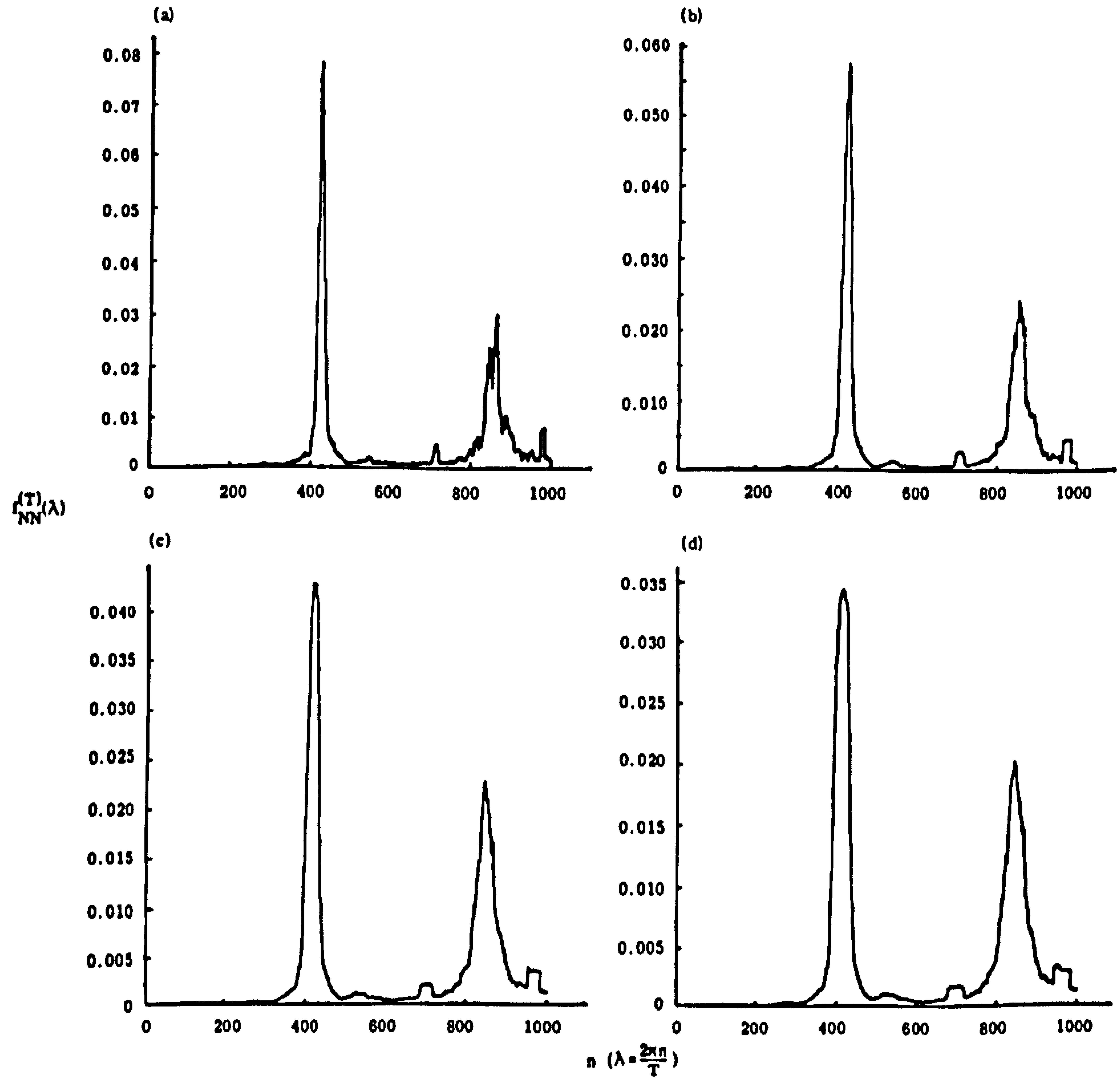
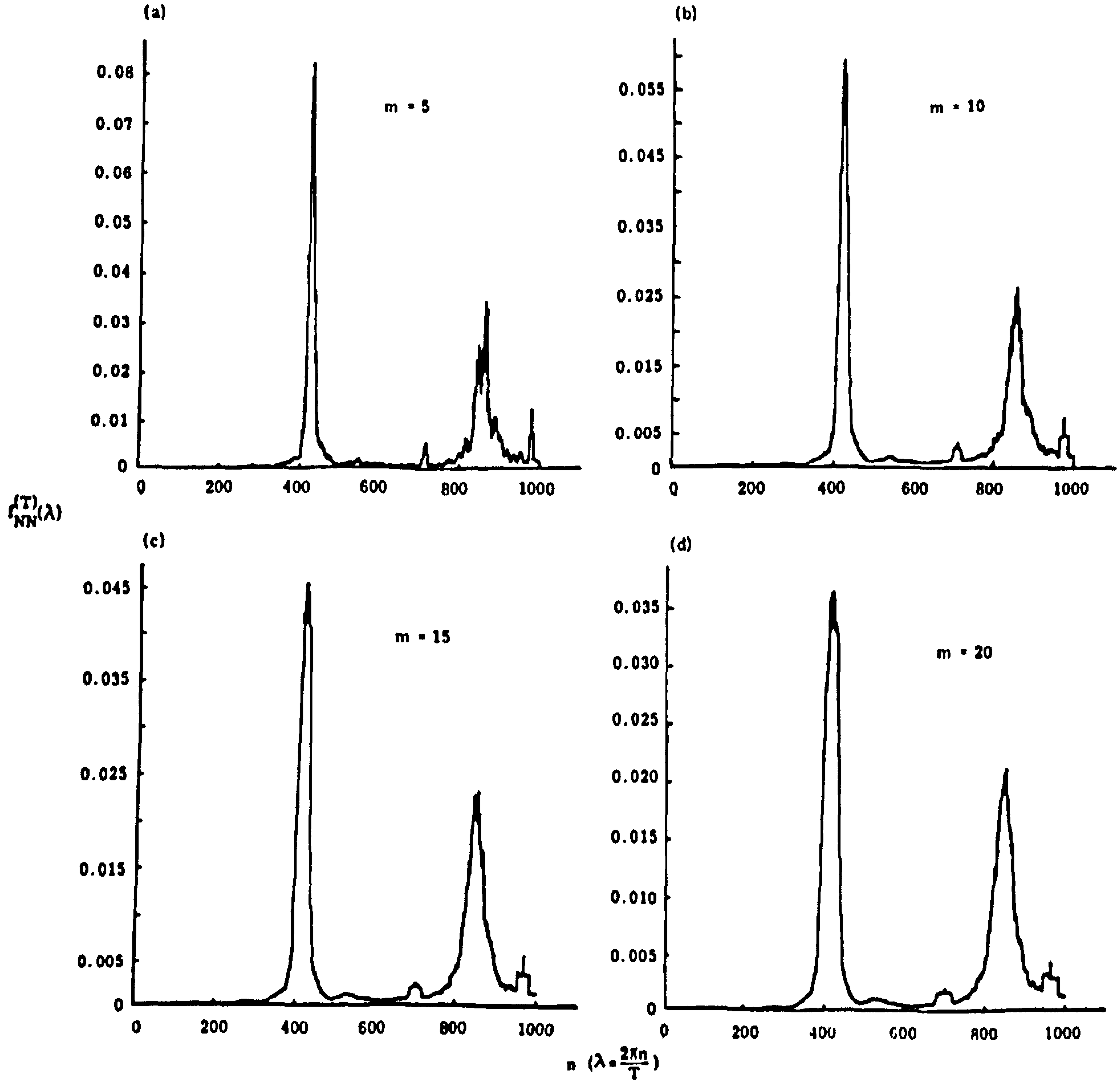


Fig. 3.8.2: Estimated power spectrum of the spontaneous Ia discharge calculated by averaging 2m+1 Periodogram ordinates (a) m = 5, (b) m = 10, (c) m = 15, and (d) m = 20. $T = 2^{14}$.



<u>Fig. 3.8.3:</u> Estimated power spectrum of the spontaneous Ia discharge obtained by smoothing the periodogram with a modified Daniell weighting scheme. (a) m = 5, (b) m = 10, (c) m = 15, and (d) m = 20. $T = 2^{14}$.

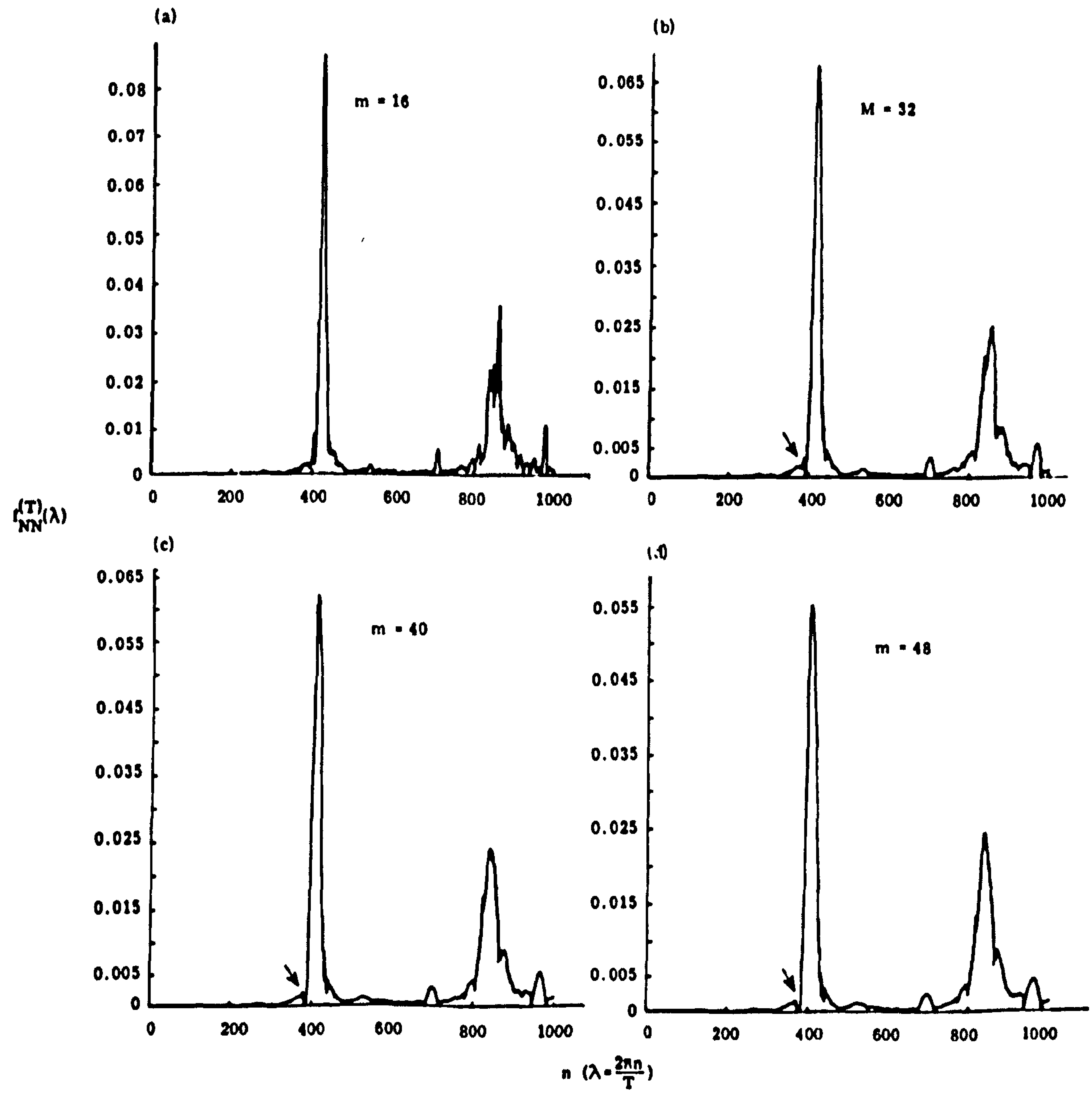


Fig. 3.8.4: Estimated power spectrum of the spontaneous Ia discharge obtained by smoothing the periodogram with a quadratic weighting scheme. (a) m = 16, (b) m = 32, (c) m = 40, and (d) m = 48. $T = 2^{14}$. The arrows in Figures (b), (c) and (d) indicate negative values.

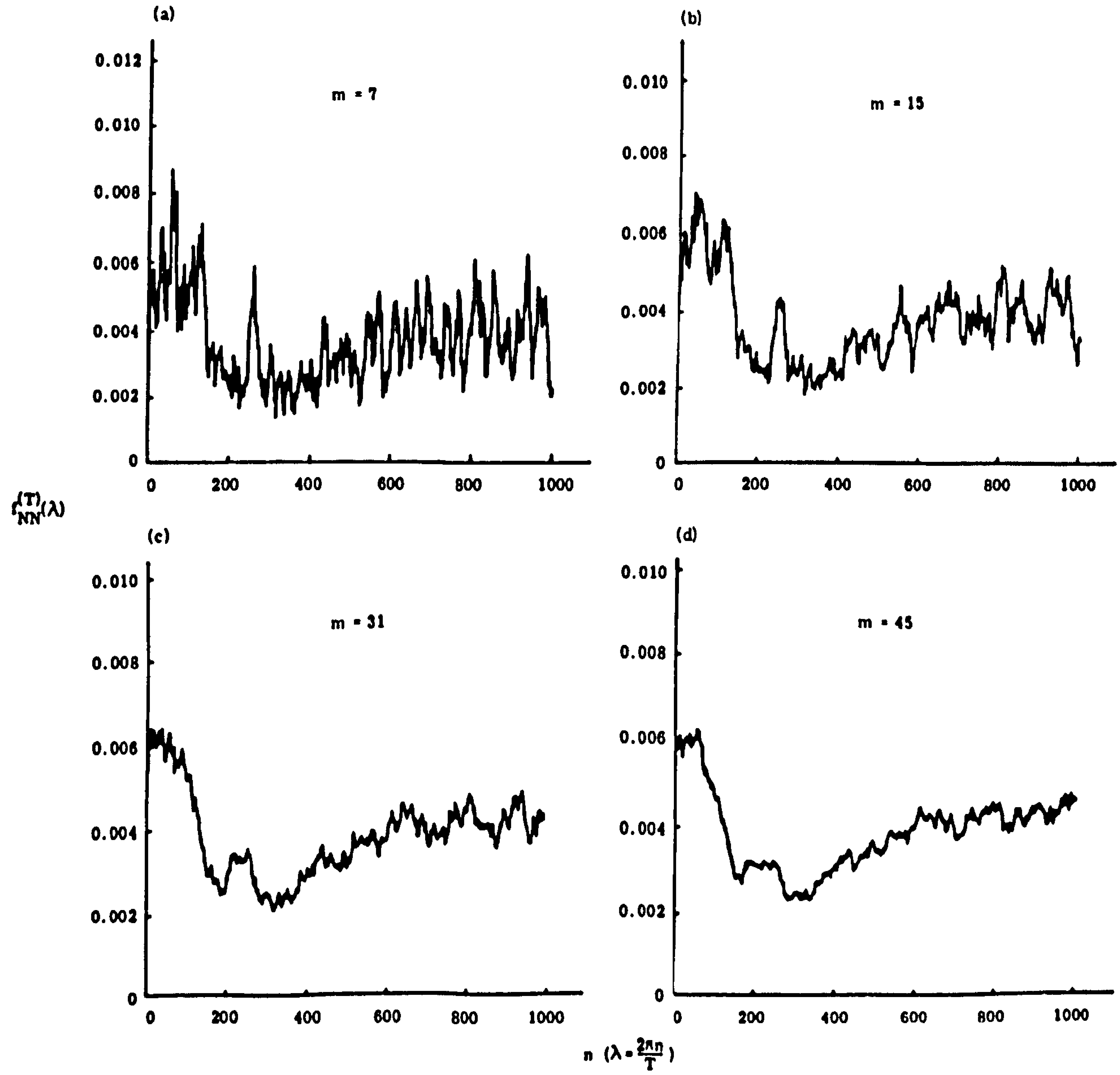


Fig. 3.8.5: Estimated power spectrum of the Ia discharge in presence of a fusimotor input calculated by averaging 2m+1 periodogram ordinates. (a) m = 7, (b) m = 15, (c) m = 31, and (d) m = 45. $T = 2^{14}$.

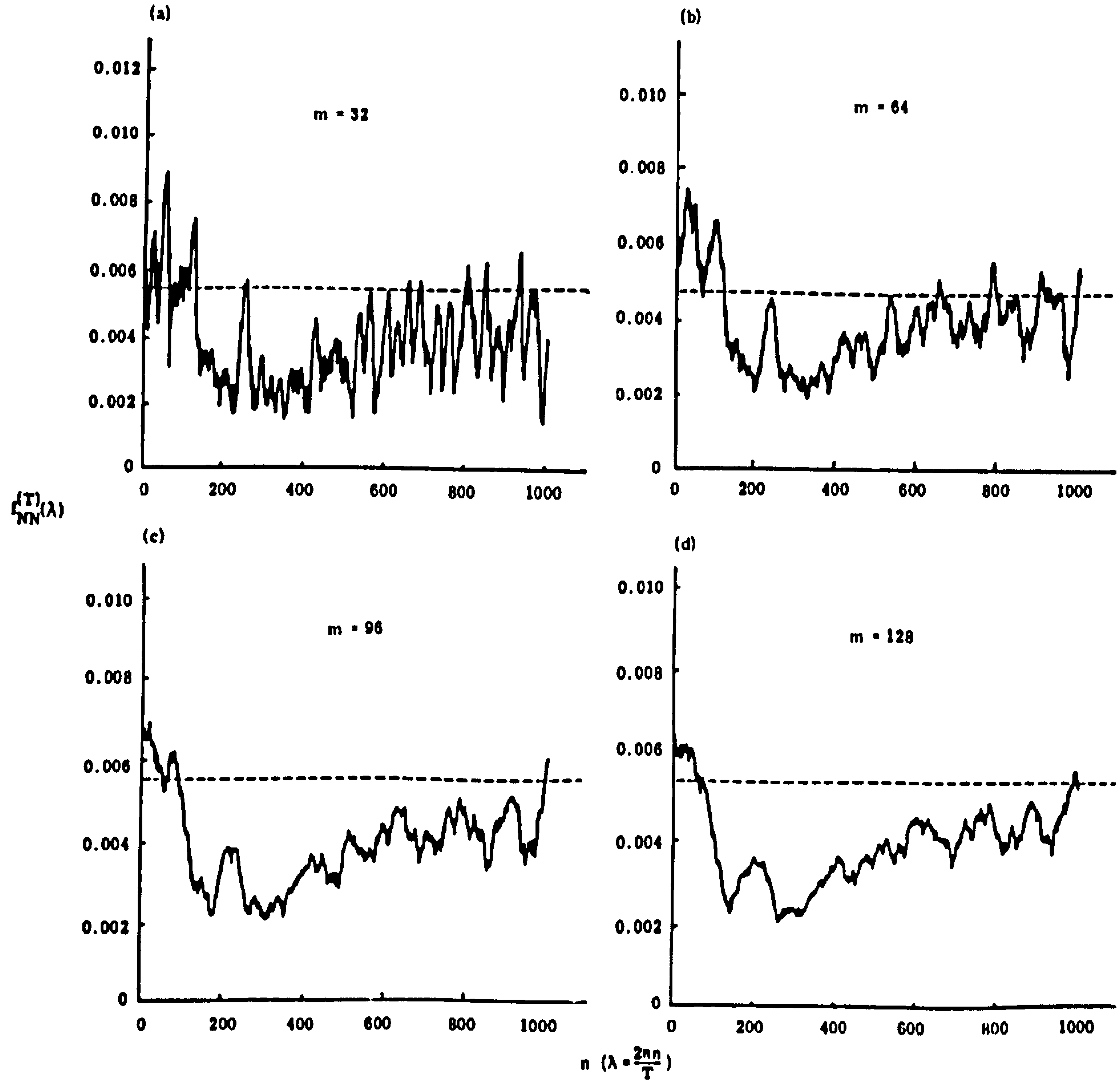


Fig. 3.8.6: Estimated power spectrum of the Ia discharge in presence of a fusimotor input obtained by smoothing the periodogram with a quadratic weighting scheme. (a) $m = 32$, (b) $m = 64$, (c) $m = 96$, and (d) $m = 128$. $T = 2^{14}$. The dotted line corresponds to the estimated power spectrum of a Poisson point process with the same mean rate.

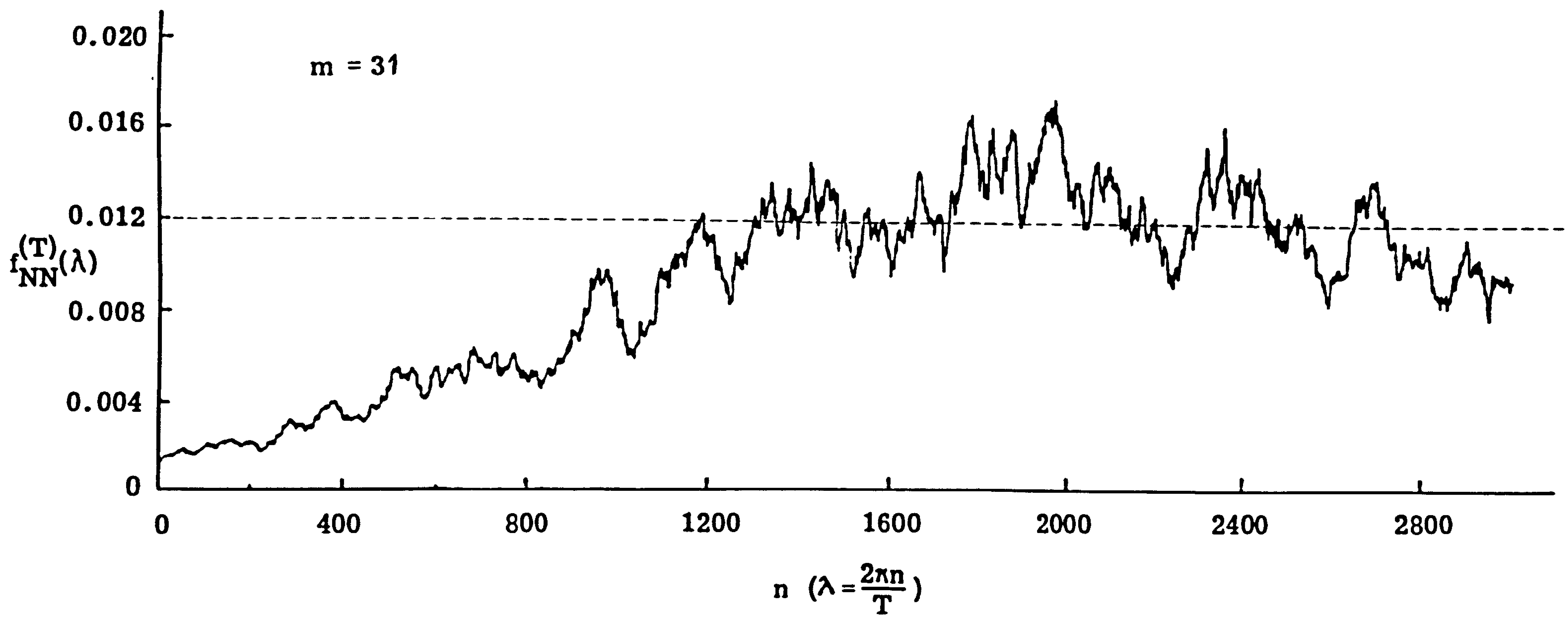


Fig. 3.8.7: Estimated power spectrum of the Ia discharge in presence of a length change calculated by averaging 63 periodogram ordinates. $T = 2^{14}$. The dotted line gives the estimated power spectrum of a Poisson point process with the same mean rate.

change (Fig. 3.8.8). A simple average weighting is used in both of these cases.

The procedures used in the numerical evaluation of the periodogram $I_{N,N'}^{(T)}(\lambda)$ presented in Fig. 3.8.1 are as follows:

(1) The finite Fourier-Stieltjes transform is calculated by using the three steps described in the beginning of this section.

(2) Once the finite Fourier-Stieltjes transform has been calculated, the calculation of $I_{N,N'}^{(T)}(\lambda)$ is implemented by using equation (3.8.5).

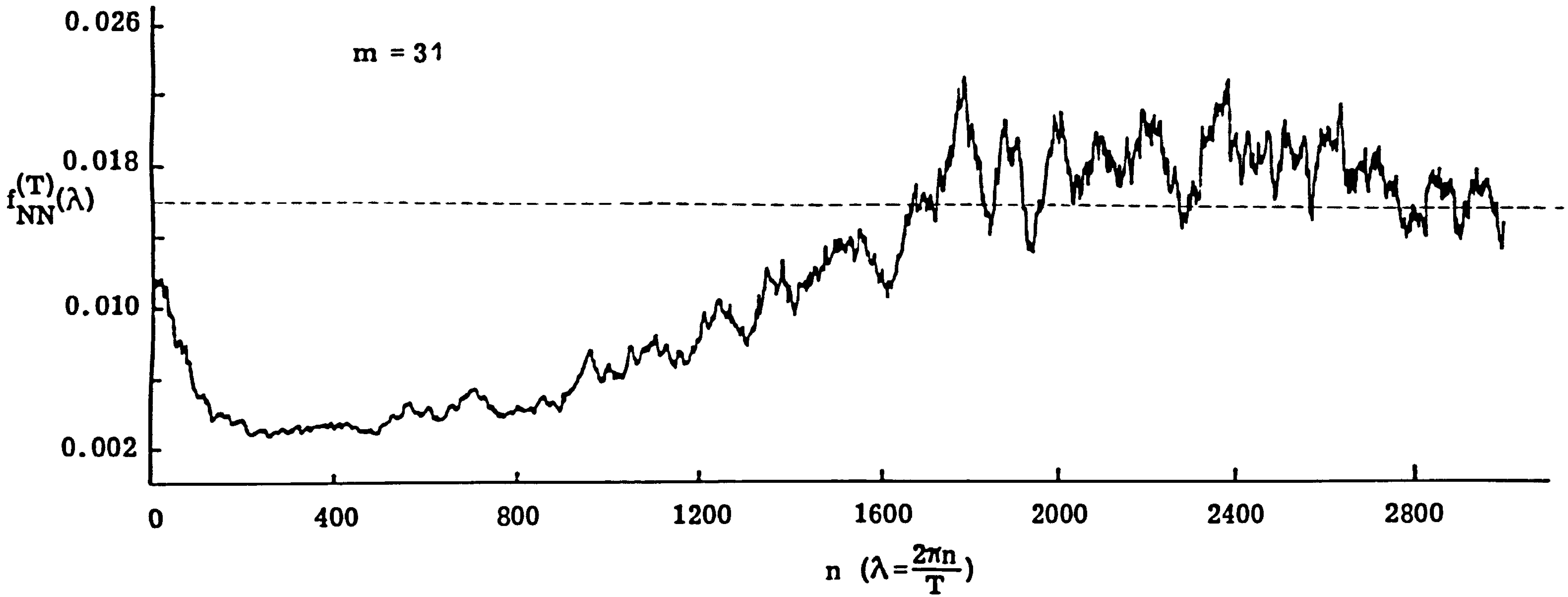
Each periodogram ordinate plotted corresponds to frequency $\lambda_n = \frac{2n\pi}{S}$, $n = 1,2,\ldots, 2000$ where S is the extended sample equal to 2^{14}, since the original sample, $T = 15872$ was extending to $S = 16384$ by adding 512 zeroes.

Although the expected periodic components are present in this spectral estimate, the lack of smoothness of the estimate obscures detailed features of these components. This lack of smoothness of the periodogram can be attributed to

(a) The variance of $I_{N,N'}^{(T)}(\lambda)$ does not decrease as $T \mapsto \infty$, and

(b) Adjacent ordinates of the periodogram are asymptotically independent (see Th. 3.8.4).

(a) and (b) together suggest that the periodogram of a point process does not converge to the true spectrum. (Cox and Lewis (1968) demonstrate a similar behaviour for the periodogram-based estimate of the spectrum of intervals between events, and Bartlett (1950) and Hannan (1960)



<u>Fig. 3.8.8:</u> Estimated power spectrum of the Ia discharge in presence of a fusimotor input and a length change computed by averaging 63 periodogram ordinates. $T = 2^{14}$. The dotted line gives the estimated power spectrum of a Poisson point process with the same mean rate.

demonstrate similar properties for the periodogram of a time series. The cause of the instability of the periodogram of a time series has also been discussed by Jenkins (1961) and Brillinger (1975a)).

The following figures demonstrate the effects on the shape of the estimated spectrum of using different smoothing procedures in the calculation of the spectral estimates.

(i) Figs. 3.8.2 (a-d), illustrate the estimate given by (3.8.6) for weights $W_j = \frac{1}{2m+1}$, $j=0, \pm 1, \pm 2, \ldots, \pm m$, and it was calculated for m = 5,10,15,20. It is obvious that the stability of this estimate improves as m increases. The maxima shown in these figures correspond approximately to the values n = 420,840. The first maximum occurs at 26.5 pulses/second (cps), which is the inverse of the mean value of the inter-spike intervals, while the second one occurs at twice this value.

(ii) Figs. 3.8.3 (a-d) are also estimates of the PS given by (3.8.6). The weights used in these estimates are a modified Daniell weighting scheme and are given by

$$f_{NN}^{(T)}(\lambda) = \frac{1}{m}\left[\frac{1}{2}\sum_{j=1}^{m-1} I_{NN'}^{(T)}(\lambda-\lambda_j) + I_{NN'}^{(T)}(\lambda) + \frac{1}{2}\sum_{j=1}^{m-1} I_{NN'}^{(T)}(\lambda+\lambda_j)\right]. \qquad (3.8.7)$$

For m=2 the weights are $(\frac{1}{4},\frac{1}{2},\frac{1}{4})$. This three-points weighting scheme is called "hanning". These figures indicate the same characteristics as those described in Figs.3.8.2 (a-d) and have been calculated for the same values of m (= 5,10,15,20). The only apparent difference seems to be the presence of small peaks in the middle of each side-band around the second maximum. (This can be explained from expression (3.8.7) since the periodogram at frequency λ is weighted by $1/m$ which corresponds to twice the value of the other weights).

(iii) Figs. 3.8.4 (a-d) illustrate a third method for smoothing the periodogram. This method was introduced by Bartlett (1963b) who extended a method of Rosenblatt (1956b), and derived a quadratic weighting scheme which, under some regularity conditions, minimises the mean square error of the estimate when T is large and m is small ($m/T \mapsto 0$ as $T \mapsto \infty$) (Cox and Lewis, 1968, p 131). The estimates in Figures 3.8.4 (a-d) have been calculated for m = 16,32,40,48.

The weights for m = 16 are

$$-\frac{91}{1344}, -\frac{21}{1344}, \frac{39}{1344}, \frac{89}{1344}, \frac{129}{1344}, \frac{159}{1344}, \frac{179}{1344}, \frac{189}{1344}, \frac{189}{1344}, \ldots$$

while for m = 32 we have,

$$-\frac{87}{2176}, -\frac{57}{2176}, -\frac{29}{2176}, -\frac{3}{2176}, \frac{21}{2176}, \frac{43}{2176}, \frac{63}{2176}, \frac{81}{2176}, \frac{97}{2176}, \frac{111}{2176}, \frac{123}{2176},$$

$$\frac{133}{2176}, \frac{141}{2176}, \frac{147}{2176}, \frac{151}{2176}, \frac{153}{2176}, \frac{153}{2176}, \ldots .$$

Figs. 3.8.2 to 3.8.4 indicate the importance of smoothing the periodogram. The smoothed periodogram clearly demonstrates the periodic nature of the discharge of the Ia sensory axon, and also reveals the presence of side-lobes associated with the main peaks in the periodogram. These side-lobes may be attributed to a sinusoidal modulation of the periodic discharge (Bayly, 1968). These figures also indicate that it is useful to try several different smoothing procedures. For example, the quadratic smoothing procedure (Fig. 3.8.4) possibly suggests the presence of additional side-lobes in the spectrum.

The choice of a particular set of weights for estimating an unknown spectrum, (which is usually the case in practical problems) may be difficult. In practice often little information is available in connection with the real spectrum. Thus the choice of a particular weighting scheme depends on experience. For example, a uniform scheme of weights is better for a flat or linearly increasing (decreasing) spectrum, but poor for peaked spectra (A discussion of this point is given in Lewis, 1970).

The weighting schemes used for the smoothing of the periodogram of the Ia/γs were discussed earlier and are illustrated in Figs. 3.8.5 (a-d) and Fig. 3.8.6 (a-d). The values of m used were 7,15,31,63 and 32,64,96,128, respectively. The comparison of these Figures indicate that the choice of a particular weighting scheme does not appear to affect the shape of the smoothed periodogram.

Finally, the estimates of the power spectra for the Data Sets III and IV (Ch.1,p22) are given by Figures 3.8.7 and 3.8.8. They were calculated for m = 31. The number of ordinates of the estimates plotted were 3000 which correspond to frequencies

$$\lambda_n = \frac{2\pi n}{S}\ , \quad n = 1, 2, \ldots, 3000 \quad \text{where} \quad S = 2^{14}\ .$$

Fig. 3.8.7 clearly suggests that the points of the process are further apart for low frequencies and gradually reach the limiting situation given by expression (3.3.3). Fig. 3.8.8 seems to include features of Figures 3.8.5 and 3.8.7.

3.9 A NEW CLASS OF CONSISTENT ESTIMATES OF THE POWER SPECTRUM OF A UNIVARIATE POINT PROCESS. EXAMPLES OF SPECTRAL ESTIMATES BASED ON THE FOURIER TRANSFORM OF THE AUTOCOVARIANCE FUNCTION

In this section we consider the following estimate of the power spectrum of the univariate point process N,

$$f_{NN}^{(T)}(\lambda) = (2\pi)^{-1}\hat{p}_N + (2\pi)^{-1} b \sum_j k_T(u_j)\,\hat{q}_{NN}(u_j)\,e^{-i\lambda u_j}\ , \tag{3.9.1}$$

where b is some bin width, $u_j = bj$; $j = 0, \pm 1, \ldots$.

The functions $\hat{p}_N$ and $\hat{q}_{NN}(u)$ are the estimates of the mean intensity and second-order cumulant function of the point process N. The function $k_T(u) = k(b_T u)$ is a convergence factor satisfying Assumption 3.8.1.

The constant b_T is called the bandwidth of the estimate (3.9.1) (Parzen, 1957). It is assumed that b_T tends to 0, as $T \mapsto \infty$, in such a way that $b_T T \mapsto \infty$ (Parzen, 1957; Brillinger, 1975a).

Convergence factors are also called data windows or tapers (Tukey, 1967).

The Fourier transform of k(u) is defined as

$$K(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty}k(u)e^{i\alpha u}du \ . \tag{3.9.2}$$

K(a) satisfies the following assumption.

<u>Assumption 3.9.1</u>: K(a), $-\infty<\alpha<\infty$, is real-valued, even, of bounded variation, with

$$\int_{-\infty}^{\infty}K(\alpha)d\alpha=1 \quad \text{and} \quad \int_{-\infty}^{\infty}|K(\alpha)|d\alpha<\infty \ .$$

The estimate (3.9.1) may also be written in the form

$$f_{NN}^{(T)}(\lambda)=\int_{-\infty}^{\infty}K_T(\lambda-\alpha)\,I_{NN}^{(T)}(\alpha)d\alpha \ , \tag{3.9.3}$$

where $I_{NN}^{(T)}(a)$ is the periodogram of the point process N defined by (3.5.1). The function $K_T(a)$ is called a spectral window (Parzen, 1961) and is given by

$$K_T(\alpha)=\frac{1}{2\pi}\int_{-T}^{T}k_T(u)e^{i\alpha u}du \ . \tag{3.9.4}$$

Following Brillinger (1972) we express estimate (3.9.3) as

$$f_{NN}^{(T)}(\lambda)=\frac{2\pi}{b_T T}\sum_{s\neq 0}K\left(b_T^{-1}\left[\lambda-\frac{2\pi s}{T}\right]\right)I_{NN}^{(T)}\left(\frac{2\pi s}{T}\right) \ . \tag{3.9.5}$$

We now turn to the problem of developing the large sample properties of the mean, variance and covariance of the estimate (3.9.5).

<u>Theorem 3.9.1:</u> Let N(t) be a univariate point process with differential increments dN(t) on (0,T] . Suppose that the second-order cumulant function $q_{NN}(u)$ satisfies $\int|u||q_{NN}(u)|du<\infty$

Let $f_{NN}^{(T)}(\lambda)$ be given by (3.9.5). Then

$$E\{f_{NN}^{(T)}(\lambda)\} = \int_{-\infty}^{\infty} K(\alpha) f_{NN}(\lambda - b_T\alpha)d\alpha + O(b_T^{-1}T^{-1}) \quad \text{for } -\infty<\lambda<\infty .$$

Furthermore, as $T \longmapsto \infty$, $f_{NN}^{(T)}(\lambda)$ is asymptotically an unbiased estimate of $f_{NN}(\lambda)$.

Proof: It follows from Lemma 2.3 of Appendix I that

$$f_{NN}^{(T)}(\lambda) = b_T^{-1}\int_{-\infty}^{+\infty} K(b_T^{-1}[\lambda-\alpha])\, I_{NN}^{(T)}(\alpha)d\alpha + O(b_T^{-1}T^{-1})$$

and so,

$$E\{f_{NN}^{(T)}(\lambda)\} = b_T^{-1}\int_{-\infty}^{\infty} K(b_T^{-1}[\lambda-\alpha])\, E(I_{NN}^{(T)}(\alpha))\,d\alpha + O(b_T^{-1}T^{-1})$$

$$= \int_{-\infty}^{\infty} K(\alpha) f_{NN}^{(T)}(\lambda - b_T\alpha)d\alpha + O(b_T^{-1}T^{-1}) ,$$

by using Th. 3.5.1 and Prop. 3.5.1.

Now, for $T \longmapsto \infty$ we have that $b_T \longmapsto 0$ but $b_T T \longmapsto \infty$. Thus

$$E\{f_{NN}^{(T)}(\lambda)\} \longmapsto f_{NN}(\lambda) \quad as \quad T \longmapsto \infty .$$

This implies that $f_{NN}^{(T)}(\lambda)$ is asymptotically an unbiased estimate of $f_{NN}(\lambda)$.

Theorem 3.9.2: Let $N(t)$ be a stationary univariate point process on $(0,T]$ with differential increments $dN(t)$. Suppose that cumulant functions up to the fourth-order exist and satisfy

$$\int \cdots \int |u_j|\,|q_{N\ldots N}(u_1,\ldots,u_{J-1})|\,du_1\ldots du_{J-1} < \infty \quad \text{for } J=2,3,4 \text{ and } j=1,\ldots,J-1 .$$

Let $f_{NN}^{(T)}(\lambda)$ be given by (3.9.5). Then

$$\lim_{T\to\infty} b_T T \operatorname{cov}\{ f_{NN}^{(T)}(\lambda), f_{NN}^{(T)}(\mu)\} = 2\pi\left[\delta\{\lambda-\mu\} + \delta\{\lambda+\mu\}\right] f_{NN}^{2}(\lambda)\int_{-\infty}^{\infty} K^{2}(a)da ,$$

where

$$\delta(u) = \begin{cases} 1 & \text{if } u=0 \\ 0 & \text{otherwise} \end{cases} .$$

<u>Proof:</u> For the covariance of $f_{NN}^{(T)}(\lambda)$ and $f_{NN}^{(T)}(\mu)$ we have that

$$\operatorname{cov}\{ f_{NN}^{(T)}(\lambda), f_{NN}^{(T)}(\mu)\} = b_T^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(b_T^{-1}[\lambda-a])K(b_T^{-1}[\mu-a'])\operatorname{cov}\{I_{NN}^{(T)}(a), I_{NN}(a')\}da\,da'$$

$$+ O(b_T^{-2}T^{-2}) = b_T^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(b_T^{-1}[\lambda-a])\,K(b_T^{-1}[\mu-a'])\left\{\left(\frac{\sin(a+a')T/2}{(a+a')T/2}\right)^2 + \left(\frac{\sin(a-a')T/2}{(a-a')T/2}\right)^2\right\}$$

$$(f_{NN}^{2}(a) + O(T^{-1}))da\,da' , \text{ by using Th. 3.5.2}$$

$$= b_T^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(b_T^{-1}[\lambda-a])K(b_T^{-1}[\mu-a'])\left\{\left(\frac{\sin(a+a')T/2}{(a+a')T/2}\right)^2\right\} f_{NN}^{2}(a)da\,da'$$

$$+ b_T^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(b_T^{-1}[\lambda-a])K(b_T^{-1}[\mu-a'])\left\{\left(\frac{\sin(a-a')T/2}{(a-a')T/2}\right)^2\right\} f_{NN}^{2}(a)da\,da'$$

$$+ O(b_T^{-2}T^{-2}) + O(T^{-1}) \quad (*) .$$

The first term in (*), if we substitute $\beta = b_T^{-1}(\mu - a')$, can be written as

$$b_T^{-2}\iint_{-\infty}^{+\infty} K(b_T^{-1}[\lambda-\alpha])K(b_T^{-1}[\mu-\alpha'])\left\{\frac{\sin(\alpha+\alpha')T/2}{(\alpha+\alpha')T/2}\right\}^2 f_{NN}^2(\alpha)\,d\alpha\,d\alpha'$$

$$= b_T^{-1}\iint_{-\infty}^{+\infty} K(b_T^{-1}[\lambda-\alpha])K(\beta)\left\{\frac{\sin(\alpha+\mu-b_T\beta)T/2}{(\alpha+\mu-b_T\beta)T/2}\right\}^2 f_{NN}^2(\alpha)\,d\alpha\,d\beta$$

$$= b_T^{-1}\iint_{-\infty}^{+\infty} K[b_T^{-1}(\lambda+\mu+\gamma)-\beta]\,K(\beta)\left\{\frac{\sin\gamma T/2}{\gamma T/2}\right\}^2 f_{NN}^2(-\gamma-\mu+b_T\beta)\,d\beta\,d\gamma\,,$$

by setting $\quad -\gamma = \alpha+\mu-b_T\beta$.

For the second term in (*) we get

$$b_T^{-1}\iint_{-\infty}^{+\infty} K[b_T^{-1}(\lambda-\mu+\gamma)\;+\beta]\,K(\beta)\left\{\frac{\sin\gamma T/2}{\gamma T/2}\right\}^2 f_{NN}^2(\mu-\gamma-b_T\beta)\,d\beta\,d\gamma\;.$$

Substituting these two expressions in (*) we obtain

$$b_T T\,\mathrm{cov}\{f_{NN}^{(T)}(\lambda), f_{NN}^{(T)}(\mu)\} = 2\pi\iint_{-\infty}^{+\infty} K[b_T^{-1}(\lambda+\mu+\gamma)+\beta]\,K(\beta)\frac{1}{2\pi T}\left(\frac{\sin\gamma T/2}{\gamma/2}\right)^2 f_{NN}(-\gamma-\mu+b_T\beta)\,d\beta\,d\gamma$$

$$+\,2\pi\iint_{-\infty}^{+\infty} K[b_T^{-1}(\lambda-\mu+\gamma)+\beta]\,K(\beta)\frac{1}{2\pi T}\left(\frac{\sin\gamma T/2}{\gamma/2}\right)^2 f_{NN}^2(-\gamma+\mu-b_T\beta)\,d\beta\,d\gamma + O(b_T^{-1}T^{-1}) + O(b_T)\,.$$

If we take the limit as $T \longmapsto \infty$ we have

$$\lim_{T\to\infty} b_T T\,\mathrm{cov}\{f_{NN}^{(T)}(\lambda), f_{NN}^{(T)}(\mu)\} = 2\pi\iint_{-\infty}^{+\infty} K^2(\beta)\,\delta\{\lambda+\mu+\gamma\}\,\delta(\gamma)\,f_{NN}^2(\lambda)\,d\beta\,d\gamma$$

$$+\,2\pi\iint_{-\infty}^{+\infty} K^2(\beta)\,\delta\{\lambda-\mu+\gamma\}\,\delta(\gamma)\,f_{NN}^2(\lambda)\,d\beta\,d\gamma$$

$$= 2\pi[\delta\{\lambda-\mu\}+\delta\{\lambda+\mu\}]\,f_{NN}^2(\lambda)\int_{-\infty}^{+\infty} K^2(\beta)\,d\beta\;.$$

<u>Corollary 3.9.2</u>: Under the conditions of Theorem 3.9.2 we have for $\lambda = \mu$

$$\lim_{T \to \infty} b_T T \, Var\{ f_{NN}^{(T)}(\lambda) \} = 2\pi f_{NN}^2(\lambda) \int_{-\infty}^{+\infty} K^2(a)\,da \; .$$

<u>Proof</u>: The proof easily follows from Th. 3.9.2 .

<u>Theorem 3.9.3</u>: Let N(t) be a stationary univariate point process on (0,T] with differential increments dN(t). Suppose that N(t) satisfies Assumption 1.2 of Appendix I. Let $f_{NN}^{(T)}(\lambda)$ be given by (3.9.5) with K(a) satisfying Assumption 3.9.2. Let also $f_{NN}(\lambda_j) \neq 0$, j = 1,...,J. Then, as $T \mapsto \infty$, $f_{NN}^{(T)}(\lambda_1),\ldots,$ $f_{NN}^{(T)}(\lambda_J)$ are asymptotically normal with covariance structure given by Theorem 3.9.2.

<u>Proof</u>: The proof of this theorem follows from Cor. 4.6.4 of Chapter 4.

It follows from Cor. 3.9.2 that

$$Var\{ f_{NN}^{(T)}(\lambda) \} \simeq \frac{2\pi}{b_T T} f_{NN}^2(\lambda) \int_{-\infty}^{+\infty} K^2(a)\,da \quad \text{if} \quad \lambda \neq 0 \tag{3.9.6}$$

$$\simeq \frac{4\pi}{b_T T} f_{NN}^2(\lambda) \int_{-\infty}^{+\infty} K^2(a)\,da \quad \text{if} \quad \lambda = 0 \; . \tag{3.9.7}$$

Thus, the variance of $f_{NN}^{(T)}(\lambda)$ tends to 0 as $b_T T \mapsto \infty$. This result and Theorem 3.9.1 implies that

$$\lim_{T \to \infty} E\,| f_{NN}^{(T)}(\lambda) - f_{NN}(\lambda) |^2 = 0 \; .$$

Therefore, under the conditions of Th. 3.9.1, and if $b_T \to 0$, $b_T T \to \infty$ as $T \to \infty$, then $f_{NN}^{(T)}(\lambda)$ is a consistent estimate of $f_{NN}(\lambda)$.

Consistent estimates of the power spectrum for the time series were obtained by Grenander & Rosenblatt (1957) and Parzen (1957, 1958). The asymptotic mean and variance of these estimates was discussed by the same authors and by Blackman and Tukey (1959). The asymptotic normality of the estimates in the case of time series is given by Rosenblatt (1959), Brillinger (1968, 1975a), Hannan (1970) and Anderson (1971).

The estimate (3.9.1) which is based on transforming quantities from the time domain require the use of convergence factors. We introduced convergence factors in section 3.8 of this chapter when we discussed procedures for the reduction of leakage. Some particular convergence factors are given below.

$$k(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ 0 & \text{otherwise} \end{cases}$$ Rectangular window

$$k(u) = \begin{cases} 1 - |u| & \text{if } |u| < 1 \\ 0 & \text{otherwise} \end{cases}$$ Bartlett window

$$k(u) = \begin{cases} 1 - 6|u|^2 + 6|u|^3 & \text{if } 0 \leq |u| < \frac{1}{2} \\ 2(1 - |u|)^3 & \text{if } \frac{1}{2} \leq |u| < 1 \\ 0 & \text{otherwise} \end{cases}$$ Parzen window

$$K(u)=\begin{cases} \frac{1}{2}[1-\cos \pi u/p] & \text{if } 0 \leq u < p/2 \\ 1 & \text{if } p/2 \leq u \leq 1-p/2 \\ \frac{1}{2}[1-\cos \pi(T-u)/p] & \text{if } 1-p/2 \leq u < 1 \end{cases}$$

split-bell cosine window
p is the total proportion of the data tapered .

In the case of the rectangular window the estimate (3.9.1) becomes

$$f_{NN}^{(r)}(\lambda) = (2\pi)^{-1}\hat{p}_N + (2\pi)^{-1} b \sum_{j=-M}^{M} \hat{q}_{NN}(u_j)\, e^{-i\lambda u_j} \qquad (3.9.8)$$

Estimate (3.9.8) is the Fourier transform of the unsmoothed auto-covariance density. The consequences of not smoothing this estimate which is seen to be equivalent to using a rectangular window or a finite record length, is that the problem of leakage is considerable, and since $K_T(a)$ may assume negative values, the estimated spectrum may itself take negative values (Bloomfield, 1976, p82). Brillinger (1975b)indicates that a disadvantage of estimate (3.9.1) is that it possibly leads to negative power spectrum estimates, even if $K(a) \geqslant 0$. An example of this problem is seen in Fig. 3.9.1(a) which gives the estimate of the power spectrum of the spontaneous activity of the Ia sensory axon calculated by using a Tukey window. (The same problem appears when a Parzen window is used). When such problems appear in practice we recommend the use of the procedures discussed in section 3.8.

More details about convergence factors and their properties can be found in Brillinger (1975a), Jenkins &

Watts (1968), Hannan & Thomson (1971) and Neave (1972).

Using Parseval's theorem (Jenkins & Watt, 1968; pp 21-23), (3.9.4) may be written in the equivalent form

$$Var\, f_{NN}^{(T)}(\lambda) \simeq \frac{1}{b_T T} f_{NN}^2(\lambda) \int_{-\infty}^{\infty} k^2(u)\, du \quad for \quad \lambda \neq 0 \tag{3.9.9}$$

Since k(u) = 0 for $|u| > 1$ we further have

$$Var\, f_{NN}^{(T)}(\lambda) \simeq \frac{1}{b_T T} f_{NN}^2(\lambda) \int_{-1}^{1} k^2(u)\, du = \frac{f_{NN}^2(\lambda)}{T} \int_{-M}^{M} k_T^2(u)\, du$$

$$= \frac{f_{NN}^2(\lambda)}{T} \int_{-M}^{M} k^2(b_T v)\, dv \quad for \quad \lambda \neq 0 \tag{3.9.10}$$

where $M = \frac{1}{b_T}$ and b_T is the bandwidth of the estimate.

Priestley (1962) gives a comprehensive discussion of the bandwidth and its role in increasing the precision of spectral density estimates. The bandwidth is, as we saw above, the inverse of the truncation point M which is related to the variance of the estimate of the spectrum through the expression (3.9.10). From this we see that the smaller the value of M, the smaller will be the variance of $f_{NN}^{(T)}(\lambda)$ but the larger will be the bias (Neave, 1971). If M is too small, important features of $f_{NN}(\lambda)$ may be smoothed out, while if M is too large, the shape of $f_{NN}^{(T)}(\lambda)$ becomes more like that of the periodogram with erratic variation. Thus a compromise value must be chosen. A useful approximation is to choose M to be about $2.T^{\frac{1}{2}}$ (Chatfield, 1980, p141). This choice of M will ensure that as $T \mapsto \infty$, $M \mapsto \infty$ in such a way that $M/T \mapsto 0$.

To complete the theoretical part of this section we give the first- and second-order properties and the asymptotic distribution of estimate (3.8.6) in the following theorem.

Theorem 3.9.4: Let N(t) be a univariate stationary point process on (0,T] satisfying the assumptions of Theorem 3.9.3. Let k(u), $-\infty < u < \infty$, be a convergence factor and W_j, $j = 0, \pm 1, \pm 2, \ldots, \pm m$ be weights satisfying relation 3.7.3. We set

$$\hat{I}_{NN}^{(T)}(\lambda) = \left(2\pi T \int k^2(t)dt\right)^{-1} \left| \int_t k_T(t) \exp\{-i\lambda t\} dN(t) \right|^2 , \quad \lambda \neq 0$$

and

$$\hat{f}_{NN}^{(T)}(\lambda) = \sum_{j=-m}^{m} W_j \hat{I}_{NN}^{(T)}\left(\lambda + \frac{2\pi j}{T}\right) . \text{ Let } T \longmapsto \infty, m \longmapsto \infty, \text{ but } \frac{m}{T} \longmapsto 0.$$

Then $\hat{f}_{NN}^{(T)}(\lambda_1), \ldots, \hat{f}_{NN}^{(T)}(\lambda_J)$ are asymptotically jointly normal with

$$\lim_{T \mapsto \infty} E\left\{ \hat{f}_{NN}^{(T)}(\lambda) \right\} = f_{NN}(\lambda) \quad \text{for } \lambda \neq 0$$

$$\mathrm{Var}\left\{ \hat{f}_{NN}^{(T)}(\lambda) \right\} = \frac{\int k^4(t)dt}{\left(\int k^2(t)dt\right)^2} \sum_{j=-m}^{m} W_j^2 f_{NN}^2(\lambda) + O(T^{-1}) \quad \text{for } \lambda \neq 0$$

and

$$\mathrm{cov}\left\{ \hat{f}_{NN}^{(T)}(\lambda), \hat{f}_{NN}^{(T)}(\mu) \right\} = O(T^{-1}) \quad \text{for } \lambda, \mu \neq 0 \text{ and } \lambda \pm \mu \neq 0 .$$

Proof: The proof in general lines is given in Cor. 4.6.4 of Chapter 4.

The variance of this estimate depends on the factor $\frac{\int K^4(t)\,dt}{(\int K^2(t)\,dt)^2}$. From the Schwarz inquality we have that $(\int_0^1 K^2(t)\,dt)^2 \leq \int_0^1 K^4(t)\,dt$ (Koopmans, 1974, p301). In the case of the split bell cosine the factor $\frac{\int K^4(t)\,dt}{(\int K^2(t)\,dt)^2} = 1.055$ when we taper the first and last 5 percent of the data. Bloomfield (1976), ps.82-85, has shown that the convergence factor can reduce the leakage by a factor T ^{3}. The small increase in variance caused by the presence of the convergence factors is therefore a small price to pay for the considerable decrease in leakage afforded by the use of convergence factors.

We now illustrate estimates of the PS based on taking the Fourier transforms of the time domain parameters. Equation (3.9.1) is used to find estimates of the power spectrum for Data Sets I-IV. This procedure requires estimates of the MI and the AIF. These estimates can be rapidly calculated by methods described in section 3.1.

Figs. 3.9.1(a-d) illustrate estimates of the power spectrum which correspond to the four sets of data described in Chapter 1. In these figures each spectrum calculated using 512 lags of the autocovariance density followed by taking the discrete Fourier transform using an FFT algorithm. Fig. 3.9.1(a) shows the periodic character of the spontaneous Ia discharge. Fig. 3.9.1(b) gives the effect of a static gamma stimulation on the Ia discharge. A loss of the strong periodicity of the spontaneous Ia discharge, and a reduction

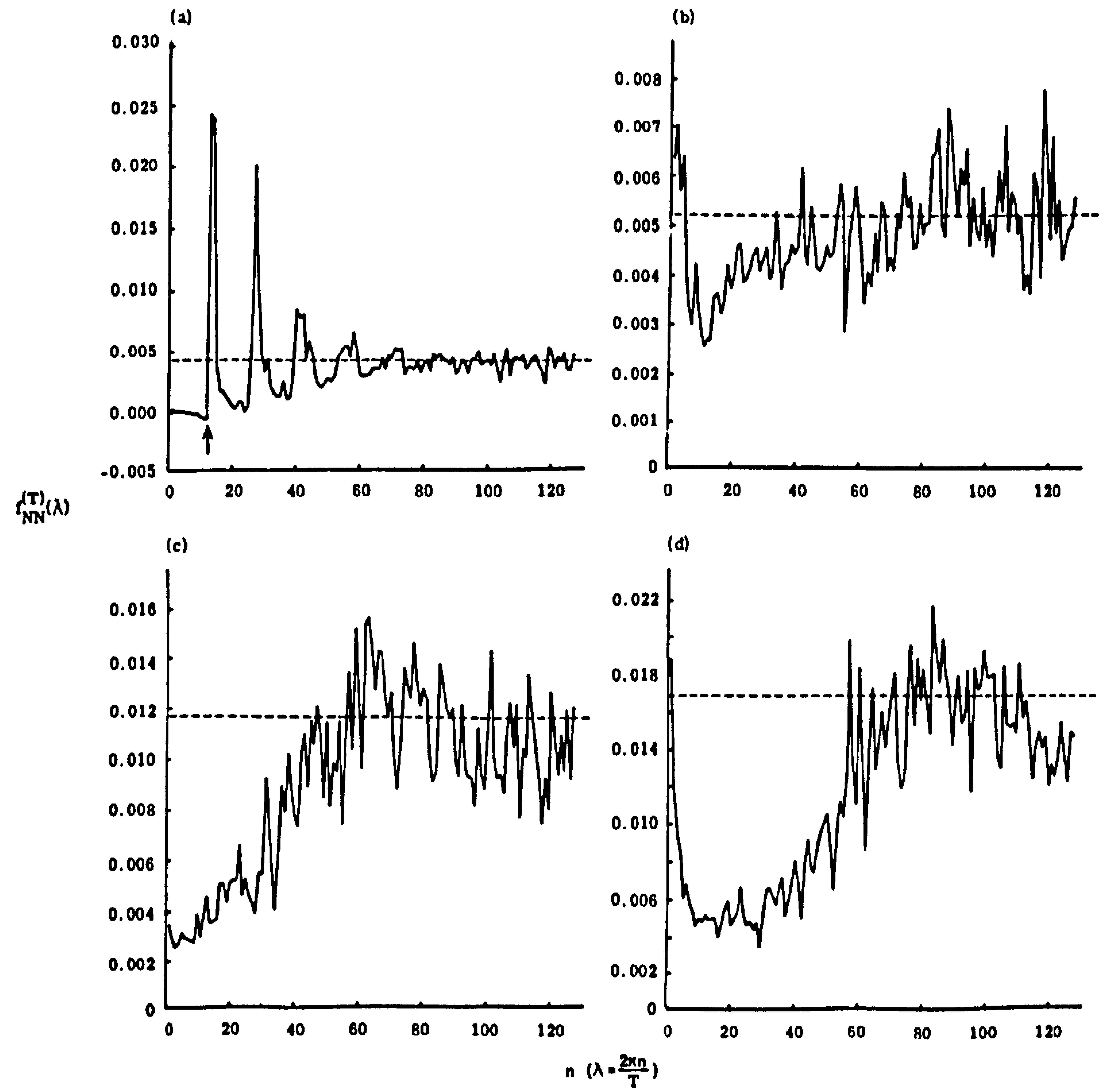


Fig. 3.9.1: Estimated power spectrum of the Ia discharge calculated from the estimate of the AIF. Tukey window was used with T = 512, and the actual calculation carried out by employing the FFT. (a) Spontaneous discharge, (b) Ia discharge in presence of a fusimotor input (γ s), (c) Ia discharge in presence of a length change (l), and (d) Ia discharge in presence of γ s and l. The dotted line in each figure gives the estimated power spectrum of a Poisson point process with the same mean rate. Arrow in Figure (a) indicates negative values.

of the spectral density at low frequencies is quite clear compared with the asymptotic value of the spectrum. Fig. 3.9.1(c) presents the effect on the spontaneous Ia discharge of a length change imposed on the muscle spindle, and clearly shows a reduction of mass at low frequencies which disappears gradually as the spectrum approaches its asymptotic value. Fig. 3.9.1(d) represents the effect of a combined gamma stimulation and a length change on the spontaneous Ia discharge. This spectrum seems to be a combination of the separate effects illustrated in Figs. 3.9.1(b) and 3.9.1(c). As we have already mentioned a disadvantage of this method is that some values of the estimates of the PS are negative (see arrow in Fig. 3.9.1(a)).

Figs. 3.9.2(a-d) again illustrate the application of equation (3.9.1) to estimate the power spectra of Data Sets I-IV. In this example we do not use the FFT, but calculate the spectrum by directly evaluating the cosine transform of the autocovariance density

$$f_{NN}^{(T)}(\lambda) = \frac{\hat{p}_N}{2\pi}\left\{1 + 2\sum_{u=1}^{M} k_T(u)\left[\hat{m}_{NN}(u) - \hat{p}_N\right]\cos\lambda u\right\} \qquad (3.9.11)$$

If we compare Figs. 3.9.1(a-d) with Figs. 3.9.2(a-d) we see that for identical record lengths using the same data window the use of the FFT may shift the entire spectrum downwards. This small shift may account for the presence of negative values in the estimated spectrum, when the value of the true spectrum itself is close to 0. A disadvantage of this last method is that it takes much

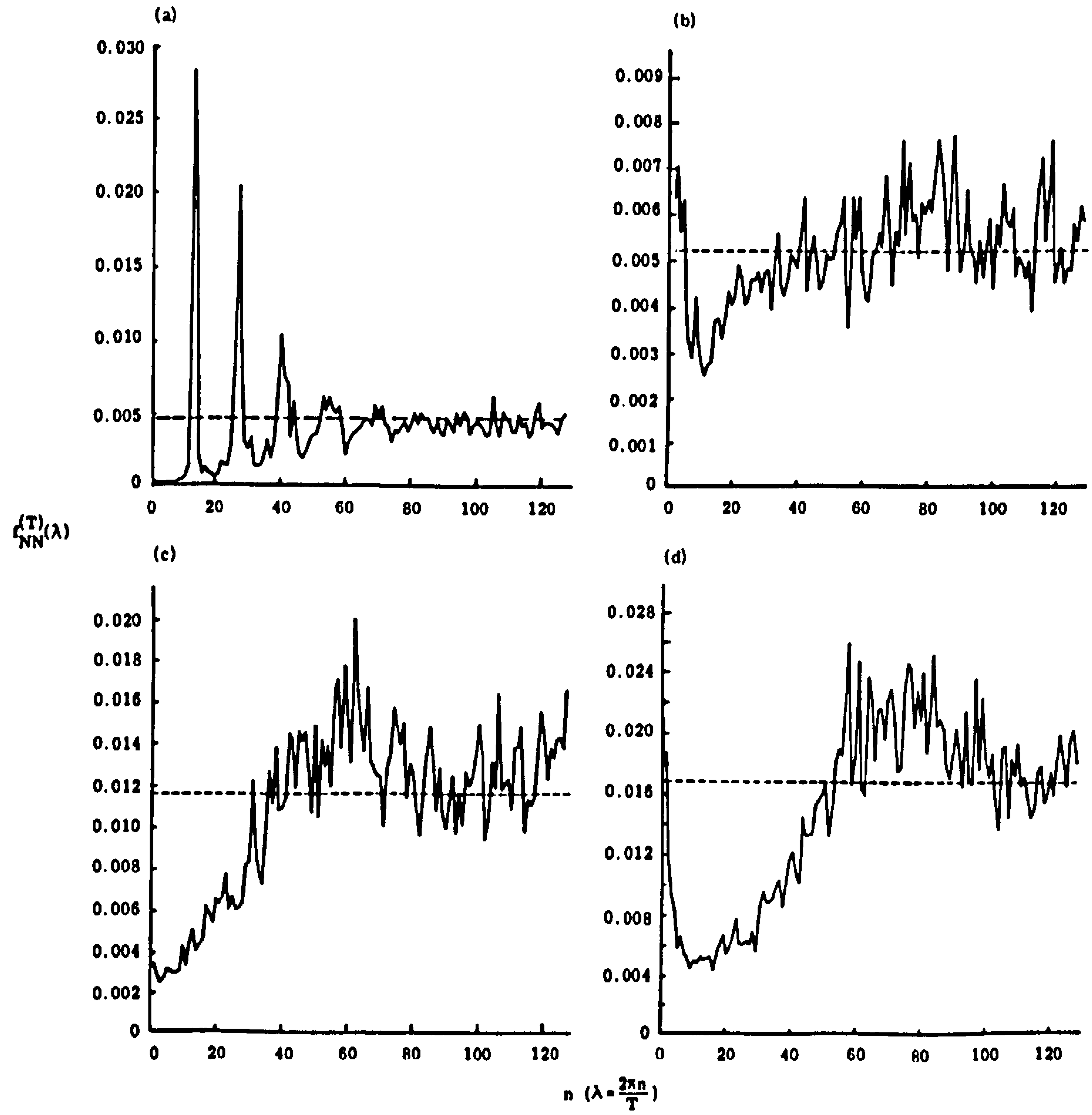


Fig. 3.9.2: Estimated power spectrum of the Ia discharge calculated from the estimate of the AIF. Tukey window was used with T = 512, and the actual calculation carried out by applying the traditional method of adding up a number of cosine terms. (a) Spontaneous discharge, (b) Ia discharge in presence of a fusimotor input (γs), (c) Ia discharge in presence of a length change (l), and (d) Ia discharge in presence of γs and l. The dotted line in each figure gives the estimated power spectrum of a Poisson point process with the same mean rate.

longer to calculate the power spectrum.

Therefore we conclude that it is helpful in having a method which does not give negative values for the estimate of the power spectrum, and does not take long to calculate.

We now proceed to discuss this new method which is similar to the one described in section 3.8 of this chapter, but has the advantage of not requiring large storage space for long data sets and so it is appropriate to be used in small computers.

3.10 ESTIMATION OF THE PS BY DIVIDING THE WHOLE RECORD INTO DISJOINT SECTIONS

Alternatively, the PS of a point process N may be estimated by dividing the whole record T into a number of disjoint sections. Suppose that the number of the disjoint sections are L, each of length R so that T = L R.

We define the periodogram of the jth section, j = 0,...,L-1, as follows

$$I_{NN}^{(R)}(\lambda, j) = (2\pi R)^{-1} d_N^{(R)}(\lambda, j)\overline{d_N^{(R)}(\lambda, j)} \quad \text{for } \lambda \neq 0 \tag{3.10.1}$$

where

$$d_N^{(R)}(\lambda, j) \simeq \sum_{t=jR}^{(j+1)R} e^{-i\lambda t}[N(t+1) - N(t)] \quad \text{for } j = 0, \dots, L-1$$

An estimate of the PS of the point process N may now be given by

$$f_{NN}^{(LR)}(\lambda) = L^{-1} \sum_{j=0}^{L-1} I_{NN}^{(R)}(\lambda, j) \quad \text{for } \lambda \neq 0 \tag{3.10.2}$$

The asymptotic first- and second-order properties of $f_{NN}^{(LR)}(\lambda)$ are examined in the next theorem.

<u>Theorem 3.10.1</u>: Let N(t) be a univariate stationary point process with differential increments dN(t). Suppose that the second-order cumulant $q_{NN}(u)$ satisfies the following condition

$$\int |q_{NN}(u)|\, du < \infty \ .$$

Let $f_{NN}^{(LR)}(\lambda)$ be given by (3.10.2). Then, $f_{NN}^{(LR)}(\lambda)$ is asymptotically an unbiased estimate of $f_{NN}(\lambda)$ as $T \longmapsto \infty$. Also, the variance of $f_{NN}^{(RL)}(\lambda)$ is approximately given by

$$\operatorname{Var} f_{NN}^{(LR)}(\lambda) \simeq \frac{1}{L} f_{NN}^{2}(\lambda) = \frac{R}{T} f_{NN}^{2}(\lambda) \quad \text{for } \lambda \neq 0 \ . \tag{3.10.3}$$

<u>Proof</u>: The proof follows directly from the asymptotic properties of (3.10.1) demonstrated in Ths. (3.5.1) and (3.5.2).

The smoothness of estimate (3.10.2) may be improved by using a "hanning" procedure, which requires the construction of the following estimate

$$\hat{f}_{NN}^{(LR)}(\lambda_k) = \frac{1}{4} f_{NN}^{(LR)}(\lambda_{k-1}) + \frac{1}{2} f_{NN}^{(LR)}(\lambda_k) + \frac{1}{4} f_{NN}^{(LR)}(\lambda_{k+1}) \ , \tag{3.10.4}$$

where

$$\lambda_k = \frac{2\pi k}{R} \quad \text{for } k = 1, \dots, \frac{R-1}{2} \ .$$

The large sample properties of estimate (3.10.4) are discussed in the next theorem.

<u>Theorem 3.10.2</u>: Let N(t) be a univariate stationary point process satisfying the conditions of Theorem 3.10.1. Let also $\hat{f}_{NN}^{(LR)}(\lambda)$ be given by (3.10.4). Then, $\hat{f}_{NN}^{(LR)}(\lambda)$ is an unbiased estimate of $f_{NN}(\lambda)$ with variance given approximately by

$$Var\left\{\hat{f}_{NN}^{(LR)}(\lambda)\right\} \simeq 0.375\,\frac{R}{T}\,f_{NN}^{2}(\lambda) \tag{3.10.5}$$

<u>Proof</u>: The proof follows from the fact that $f_{NN}^{(LR)}(\lambda)$ and $f_{NN}^{(LR)}(\mu)$, $\lambda \neq \mu$, are asymptotically independent.

Theorem 3.9.3 suggests that the estimates $f_{NN}^{(LR)}(\lambda)$ and $\hat{f}_{NN}^{(LR)}(\lambda)$ are asymptotically normally distributed as $T \longmapsto \infty$, $R \longmapsto \infty$ and $R/T \longmapsto 0$.

Figs. 3.10.1 (a-d) illustrate estimates of the power spectra of the Data Sets I-IV calculated according to the procedure described above.

The whole record was divided in 31 sections of length 512 msecs. Clearly the characteristics of these estimates are similar to the ones given by Figures 3.9.1, 3.9.2 and 3.9.3.

We believe that this method of estimating the PS of a point process is the most appropriate one to use in practice since it is fastest, does not give negative values for the estimate, and requires smaller storage space.

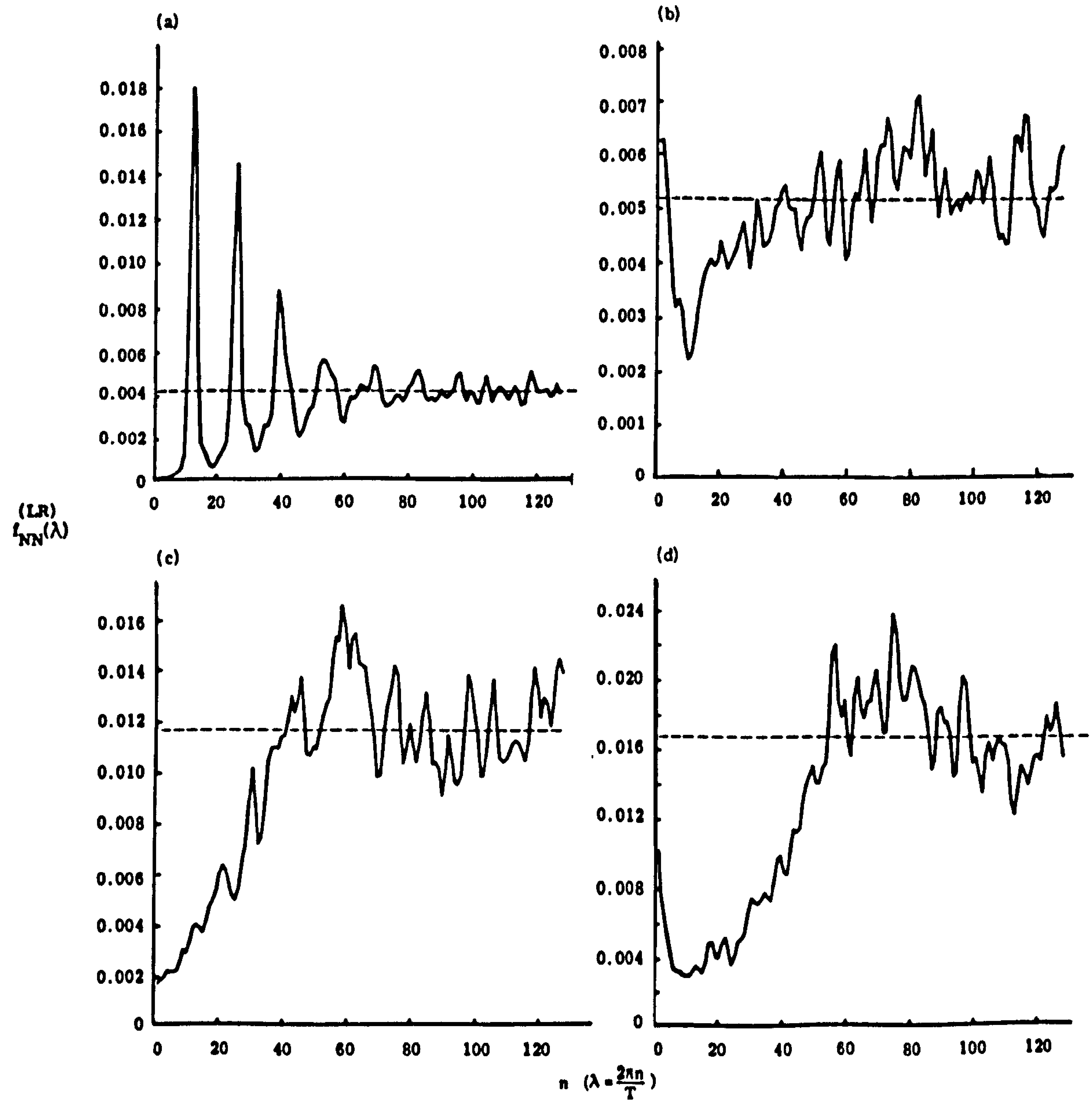


Fig. 3.10.1: Estimated power spectrum of the Ia discharge calculated from the periodogram by dividing the whole record into 31 disjoint sections of length R = 512. (a) Spontaneous discharge, (b) Ia discharge in presence of a fusimotor input (γ s), (c) Ia discharge in presence of a length change (l), and (d) Ia discharge in presence of γ s and l.

3.11 CONFIDENCE INTERVALS FOR THE ESTIMATED SPECTRA

This section considers separately the construction of confidence intervals for each of the procedures used to estimate the spectra of a point process.

Procedure I: Confidence intervals for spectral estimates based on the periodogram of the entire record

Let $f_{NN}^{(T)}(\lambda)$ be the estimate given by (3.8.6). Then

Theorem 3.11.1: Under the assumptions of Theorem 3.9.4, and if $f_{NN}(\lambda) \neq 0$, $\log_{10} f_{NN}^{(T)}(\lambda)$ is asymptotically normal with

$$\mathrm{Var}\, \log_{10} f_{NN}^{(T)}(\lambda) \simeq \frac{\int K^4(t)\,dt}{\left(\int K^2(t)\,dt\right)^2} (\log_{10} e)^2 \sum_{j=-m}^{m} W_j^2 \quad \text{for } \lambda \neq 0 .$$

Proof: The proof follows from Lemma 2.4 of Appendix I.

Theorem 3.11.1 suggests the following approximate confidence interval for the estimate $\log_{10} f_{NN}^{(T)}(\lambda)$,

$$\log_{10} f_{NN}(\lambda) \pm 1.96 \log_{10} e \left\{ c_K \sum_{j=-m}^{m} W_j^2 \right\}^{1/2} , \tag{3.11.1}$$

where

$$c_K = \frac{\int K^4(t)\,dt}{\left(\int K^2(t)\,dt\right)^2} .$$

Under the limiting condition (3.3.3) and if $W_j = \frac{1}{2m+1}$, $j = 0, \pm 1, \ldots, \pm m$, the confidence interval (3.11.1) becomes

$$\log_{10} \hat{P}_N/2\pi \pm 1.96 \log_{10} e \left\{ \frac{c_K}{2m+1} \right\}^{1/2} \quad \text{for } \lambda \neq 0 . \tag{3.11.2}$$

When zeroes are added to the record T the variance of the $\log_{10} f_{NN}^{(T)}(\lambda)$ may be written as

$$Var \log_{10} f_{NN}^{(T)}(\lambda) \simeq \frac{S}{T} c_K (\log_{10} e)^2 \sum_{j=-m}^{m} W_j^2 \quad for \ \lambda \neq 0 \ , \qquad (3.11.3)$$

where S is the extended record.

The factor $\frac{S}{T}$ may be regarded as a correction for the finer spacing of the frequencies (Bloomfield, 1976; p192) which occurs as a consequence of extending the record.

The approximate confidence interval (3.11.2) now becomes

$$\log \hat{P}_{N/2\pi} \pm 1.96 \log_{10} e \left\{ \frac{S}{T} \frac{c_K}{2m+1} \right\}^{1/2} . \qquad (3.11.4)$$

The use of this form of confidence interval is illustrated in Figs. 3.11.1 - 4.

<u>Procedure II</u>: <u>Confidence intervals for spectral estimates based on the autocovariance density</u>

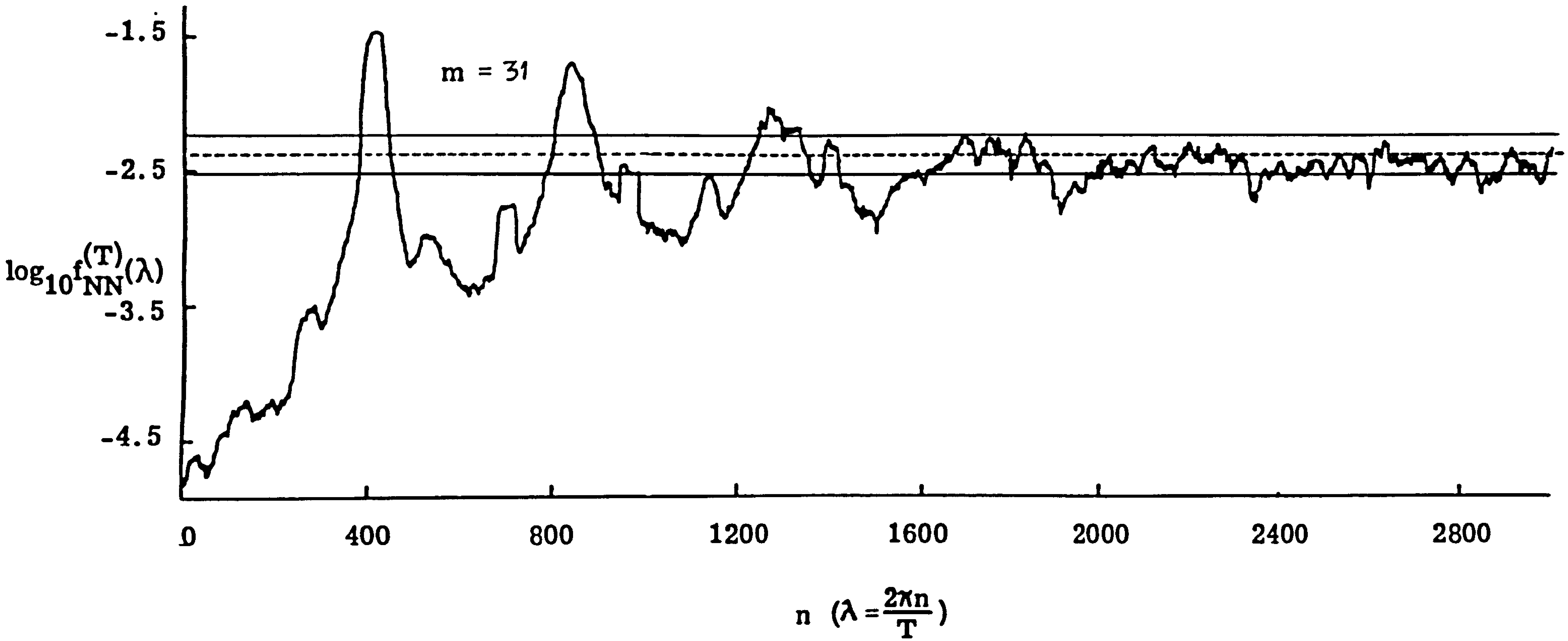
Let $f_{NN}^{(T)}(\lambda)$ be the estimate of the power spectrum given by (3.9.1). Then

<u>Theorem 3.11.2</u>: Under the assumptions of Theorem 3.9.3, and if $f_{NN}(\lambda) \neq 0$, the $\log_{10} f_{NN}^{(T)}(\lambda)$ is asymptotically normal with

$$Var \log_{10} f_{NN}^{(T)}(\lambda) \simeq b_T^{-1} T^{-1} (\log_{10} e)^2 \, 2\pi \int K^2(\alpha) d\alpha \quad if \ \lambda \neq 0$$

$$\qquad (3.11.5)$$

$$\simeq b_T^{-1} T^{-1} (\log_{10} e)^2 \, 4\pi \int K^2(\alpha) d\alpha \quad if \ \lambda = 0 \ .$$



<u>Fig. 3.11.1</u>: Log to base 10 of the estimated power spectrum of the spontaneous Ia discharge calculated by averaging $2m+1$ periodogram ordinates. $T = 2^{14}$. The dotted line gives the estimated power spectrum of a Poisson process with the same mean rate. The horizontal lines are the asymptotic 95% confidence limits.

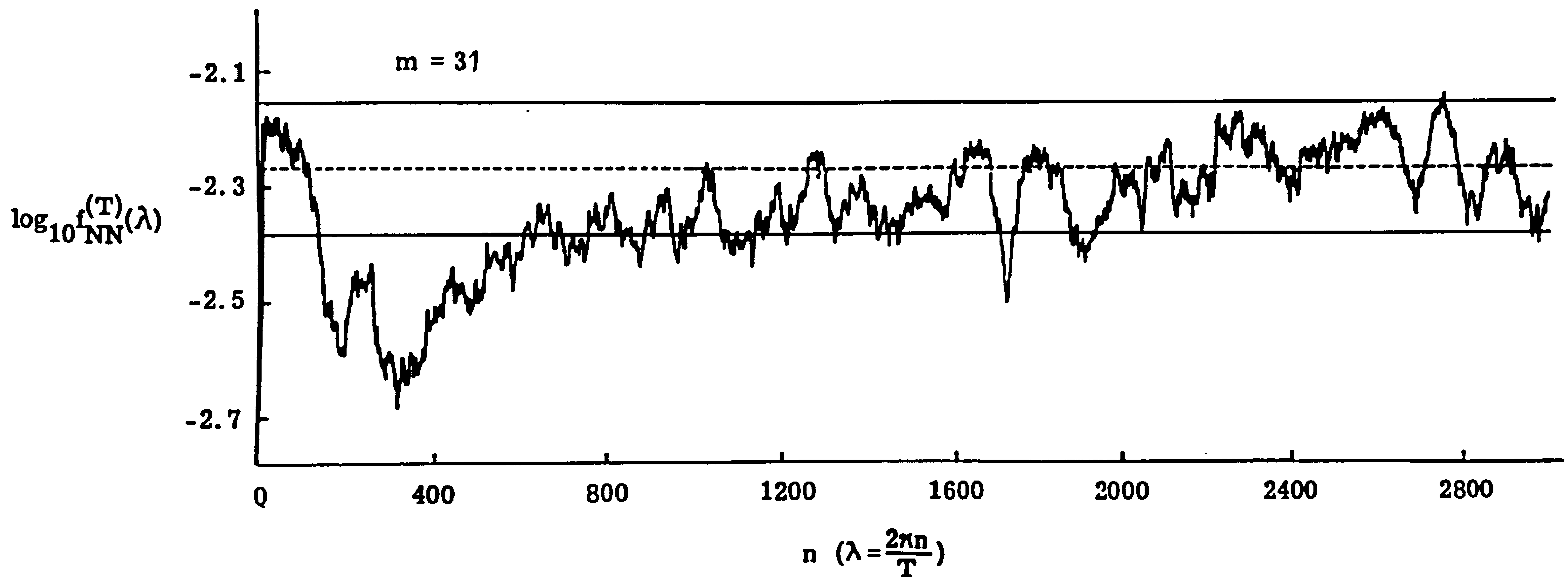


Fig. 3.11.2: Log to base 10 of the estimated power spectrum of the Ia discharge in presence of a fusimotor input calculated by averaging 2m+1 periodogram ordinates. The dotted line gives the estimated power spectrum of a Poisson point process with the same mean rate. The horizontal lines are the asymptotic 95% confidence limits. $T = 2^{14}$.

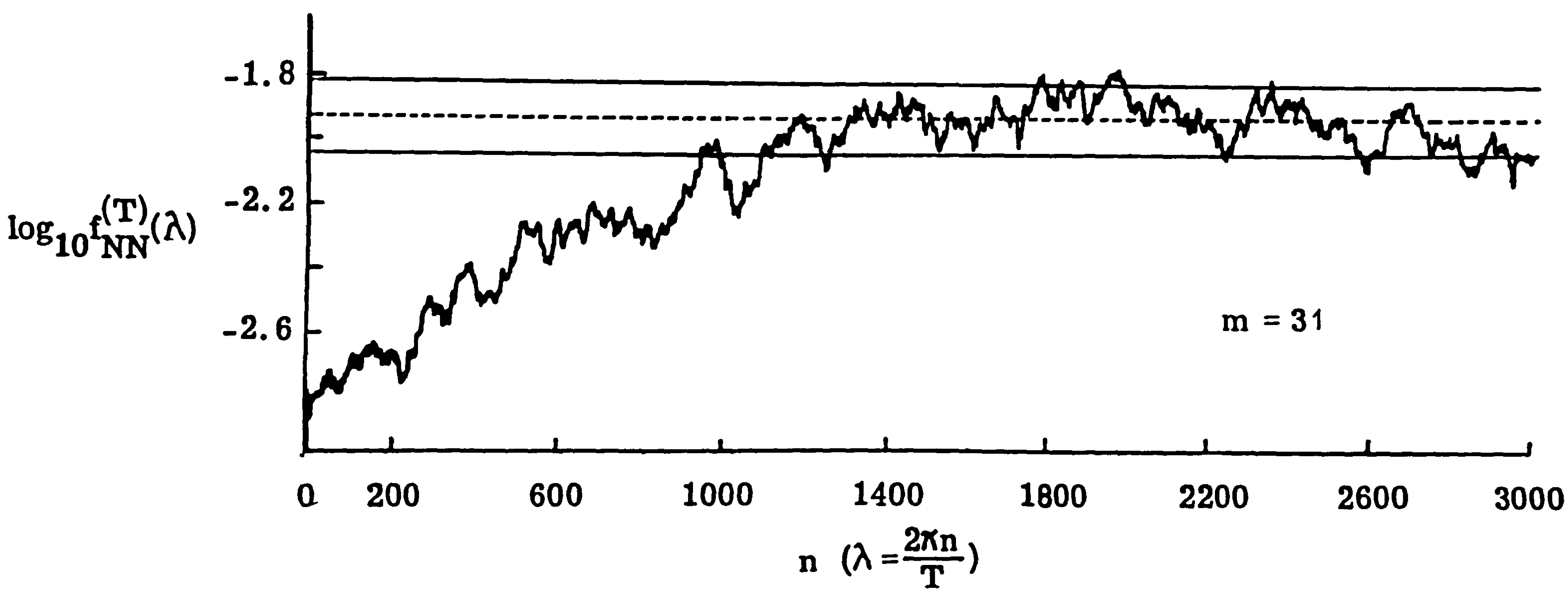


Fig. 3.11.3: Log to base 10 of the estimated power spectrum of the Ia discharge in presence of a length change calculated by averaging 2m+1 periodogram ordinates. The dotted line gives the estimated power spectrum of a Poisson point process with the same mean rate. The horizontal lines are the asymptotic 95% confidence limits. $T = 2^{14}$.

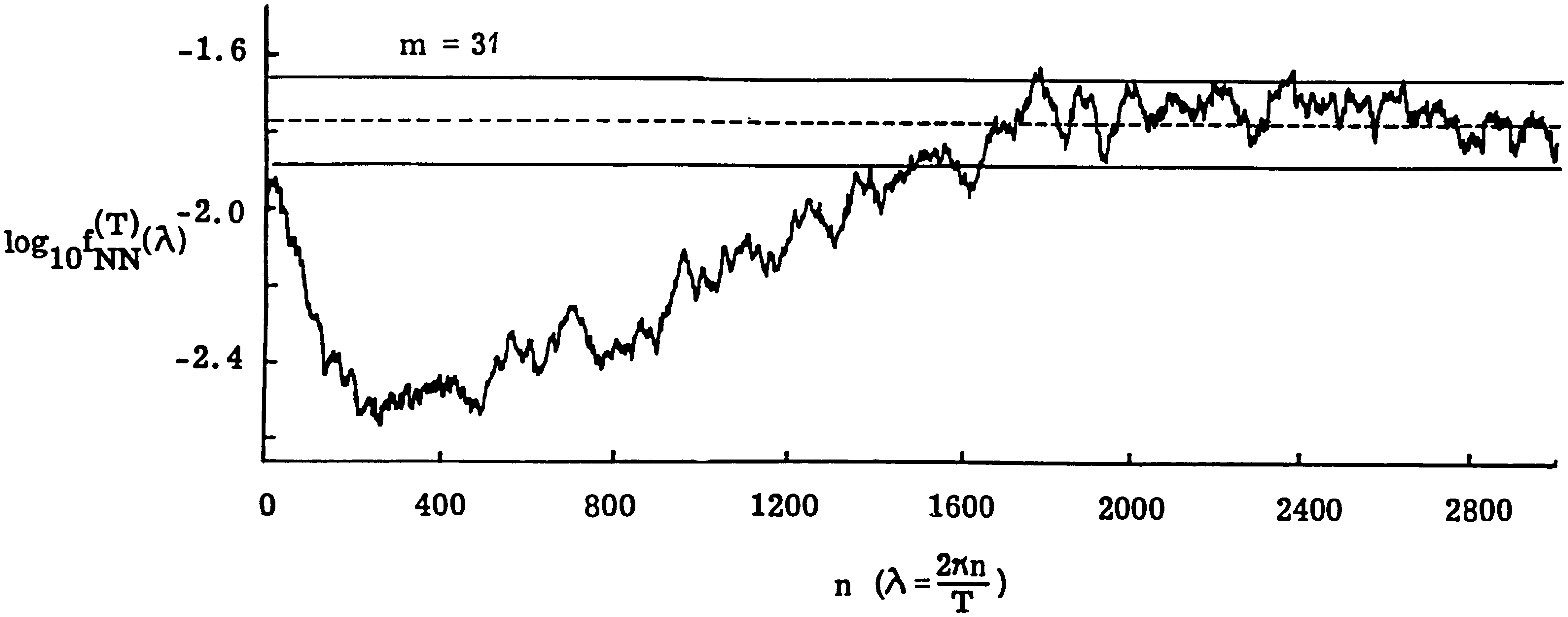


Fig. 3.11.4: Log to base 10 of the estimated power spectrum of the Ia discharge in presence of a fusimotor input and a length change calculated by averaging 2m+1 periodogram ordinates. The dotted line gives the estimated power spectrum of a Poisson point process with the same mean rate. The horizontal lines are the asymptotic 95% confidence limits. $T=2^{14}$.

<u>Proof</u>: The proof follows from Lemma 2.4 of Appendix I.

By using expression (3.9.11) we find a relation for the variance of the $\log_{10} f_{NN}^{(T)}(\lambda)$ which involves the convergence factors $k_T(u)$, that is

$$\begin{aligned} \operatorname{Var} \log_{10} f_{NN}^{(T)}(\lambda) &\simeq \frac{(\log_{10} e)^2}{T} \int_{-M}^{M} k_T^2(u)\,du \quad \text{if } \lambda \neq 0 \\ &\simeq \frac{2(\log_{10} e)^2}{T} \int_{-M}^{M} k_T^2(u)\,du \quad \text{if } \lambda = 0 \;. \end{aligned} \tag{3.11.6}$$

Thus, a 95% approximate confidence interval for $\log_{10} f_{NN}^{(T)}(\lambda)$ is

$$\log_{10} f_{NN}(\lambda) \pm 1.96 \log_{10} e \left\{ \frac{\int_{-M}^{M} k_T^2(u)\,du}{T} \right\}^{1/2} \quad \text{if } \lambda \neq 0 \tag{3.11.7}$$

and under the limiting condition (3.3.3) this becomes

$$\log_{10} \hat{P}_N/2\pi \pm 1.96 \log_{10} e \left\{ \frac{\int_{-M}^{M} k_T^2(u)\,du}{T} \right\}^{1/2} \quad \text{if } \lambda \neq 0 \tag{3.11.8}$$

Applications of this type of confidence interval are illustrated in Figs. 3.11.5 - 7. The structure of this confidence interval depends on the choice of convergence factors. For example, in the case of the Tukey window $\int_{-M}^{M} k_T^2(u)\,du$ is equal to 0.75M. The confidence interval (3.11.8) then becomes

$$\log \hat{P}_N/2\pi \pm 1.96 \log_{10} e \left(0.75\, M/T\right)^{1/2} \tag{3.11.9}$$

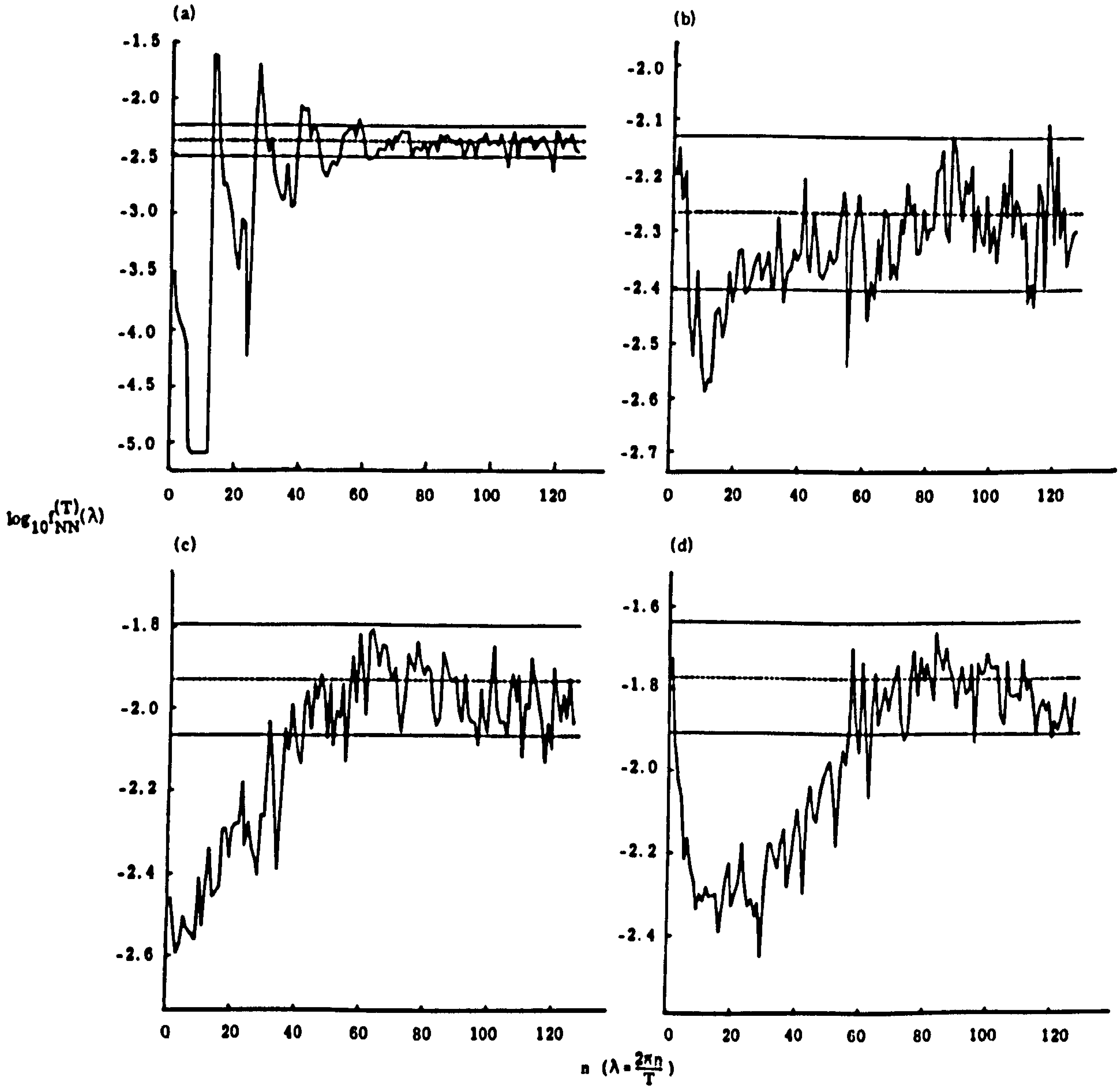


Fig. 3.11.5: Log to base 10 of the estimated power spectrum of the Ia discharge. Tukey window was used with T = 512, and the calculation carried out by employing the FFT. (a) Spontaneous discharge, (b) Ia discharge in presence of a fusimotor input (γs), (c) Ia discharge in presence of a length change (l), and (d) Ia discharge in presence of γs and l. The dotted line in each figure corresponds to the estimated power spectrum of a Poisson point process with the same mean rate. The horizontal lines are the asymptotic 95% confidence limits.

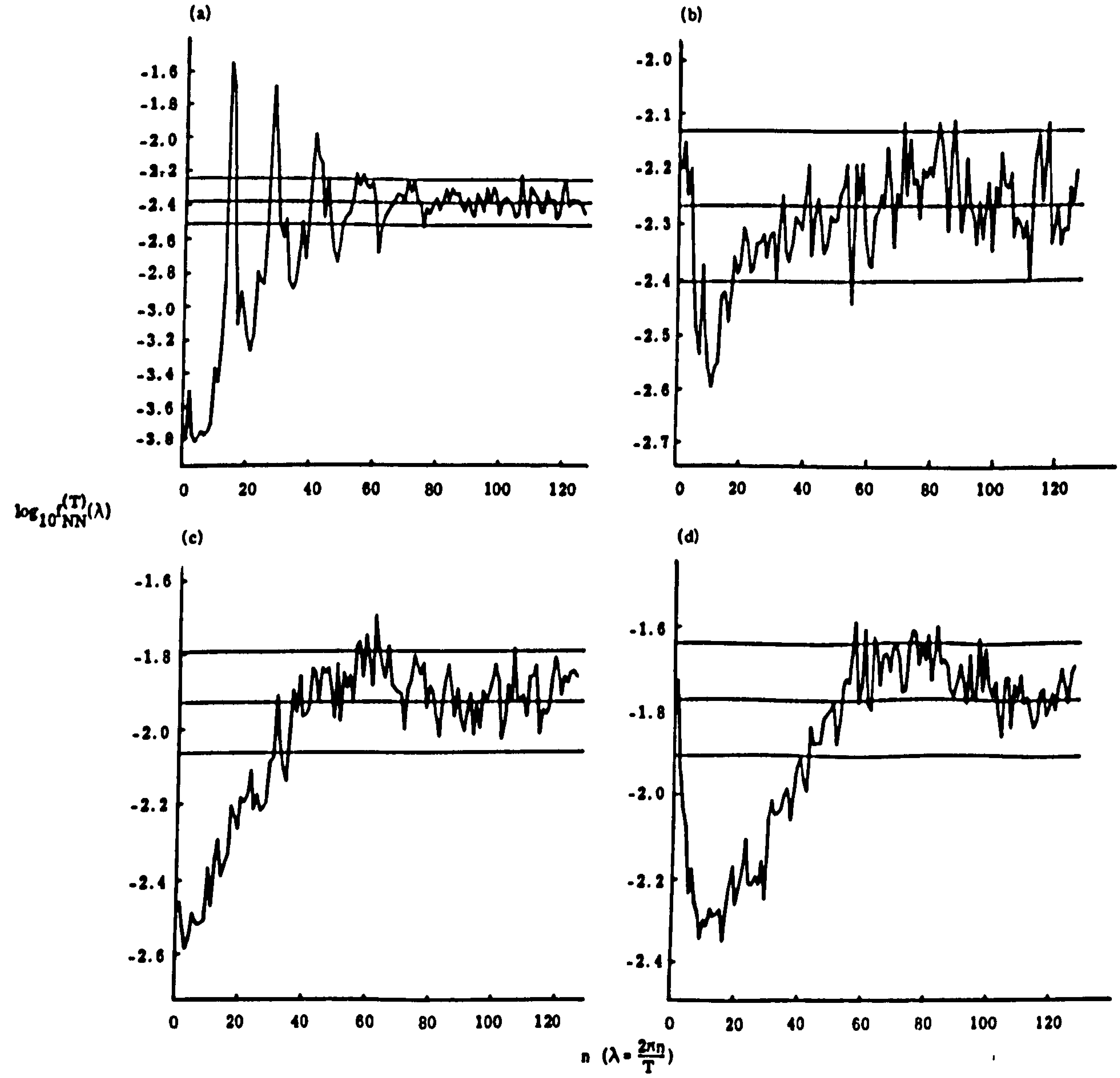


Fig. 3.11.6: Log to base 10 of the estimated power spectrum of the Ia discharge calculated from the AIF. Tukey window was used with T = 512, and the calculation carried out by applying the traditional method of adding up a number of cosine terms. (a) Spontaneous discharge, (b) Ia discharge in presence of a fusimotor input (γs), (c) Ia discharge in presence of a length change (l), and (d) Ia discharge in presence of γs and l. The dotted line in each figure corresponds to the estimated power spectrum of a Poisson process with the same mean rate. The horizontal lines are the asymptotic 95% confidence limits.

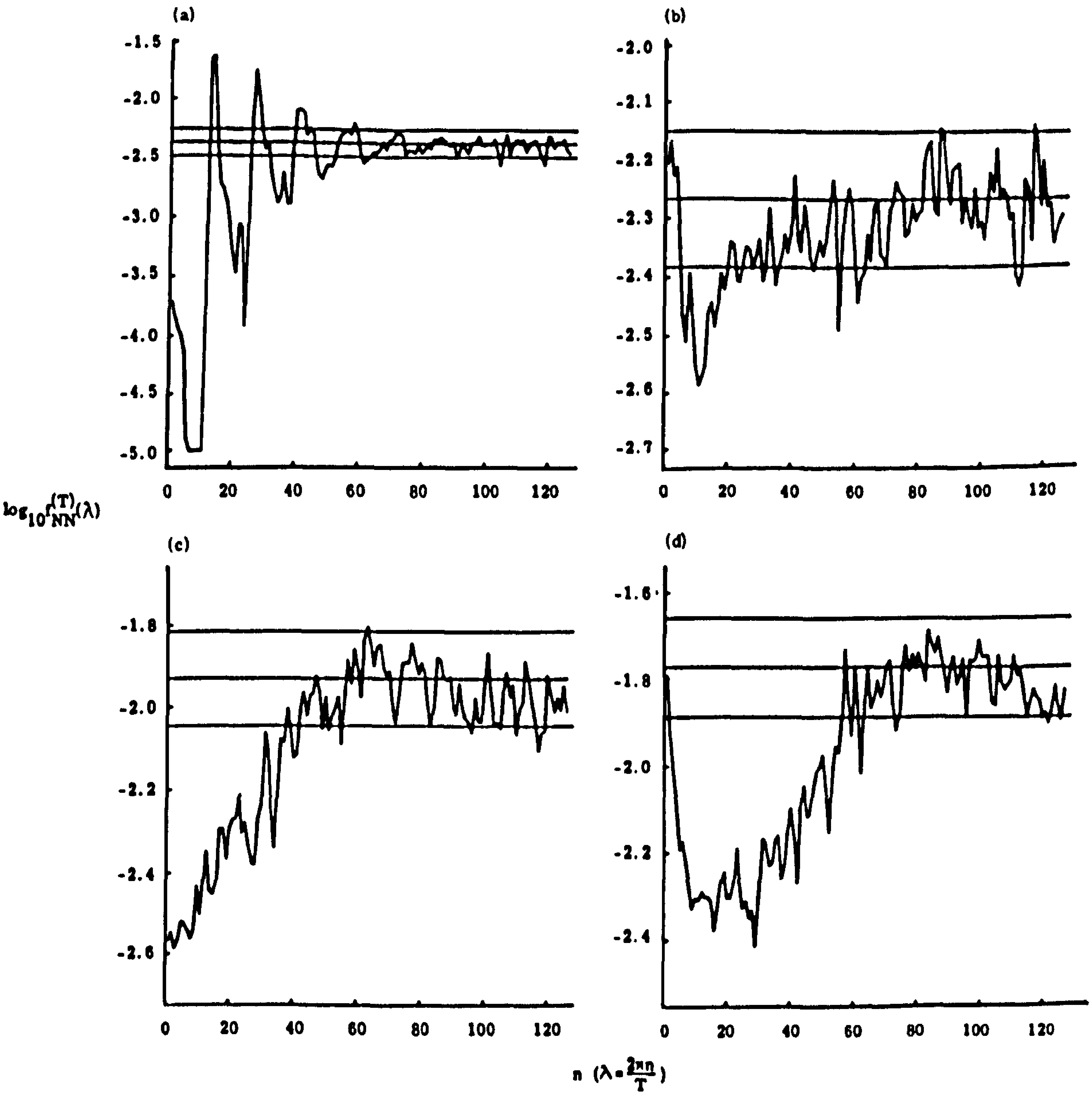


Fig. 3.11.7: Log to base 10 of the estimated power spectrum of the Ia discharge calculated from the AIF. Parzen window was used with T = 512, and the calculation carried out by employing the FFT. (a) Spontaneous discharge, (b) Ia discharge in presence of a fusimotor input (γ s), (c) Ia discharge in presence of a length change (l), and (d) Ia discharge in presence of γ s and l. The dotted line in each figure corresponds to estimated power spectrum of a Poisson point process with the same mean rate. The horizontal lines are the asymptotic 95% confidence limits.

Procedure III: Confidence intervals for spectral estimates based on dividing the record into sections

Let $\hat{f}_{NN}^{(LR)}(\lambda)$ be the estimate given by (3.10.4). Then

Theorem 3.11.3: Under the assumptions of Theorem 3.10.1, and if $f_{NN}(\lambda) \neq 0$, the $\log_{10} \hat{f}_{NN}^{(LR)}(\lambda)$ is asymptotically normal with

$$\mathrm{Var}\, \log_{10} \hat{f}_{NN}^{(LR)}(\lambda) \simeq \frac{R}{T} (\log_{10} e)^2\, 0.375 \,. \tag{3.11.10}$$

Proof: The proof follows from Lemma 2.4 of Appendix I.

An approximate 95% confidence interval of the estimate $\log_{10} \hat{f}_{NN}^{(LR)}(\lambda)$ is given by

$$\log_{10} f_{NN}(\lambda) \pm 1.96 \log_{10} e \left\{ 0.375 \frac{R}{T} \right\}^{1/2}, \tag{3.11.11}$$

and under the limiting condition (3.3.3) this interval becomes

$$\log_{10} \hat{p}_N/2\pi \pm 1.96 \log_{10} e \left\{ 0.375 \frac{R}{T} \right\}^{1/2}. \tag{3.11.12}$$

Fig. 3.11.8 illustrates the use of this approximate 95% confidence interval.

The remainder of this section presents examples of the application of the confidence intervals derived for each of the methods used for the estimation of the spectrum of a point process. Data Sets I-IV again provide the examples

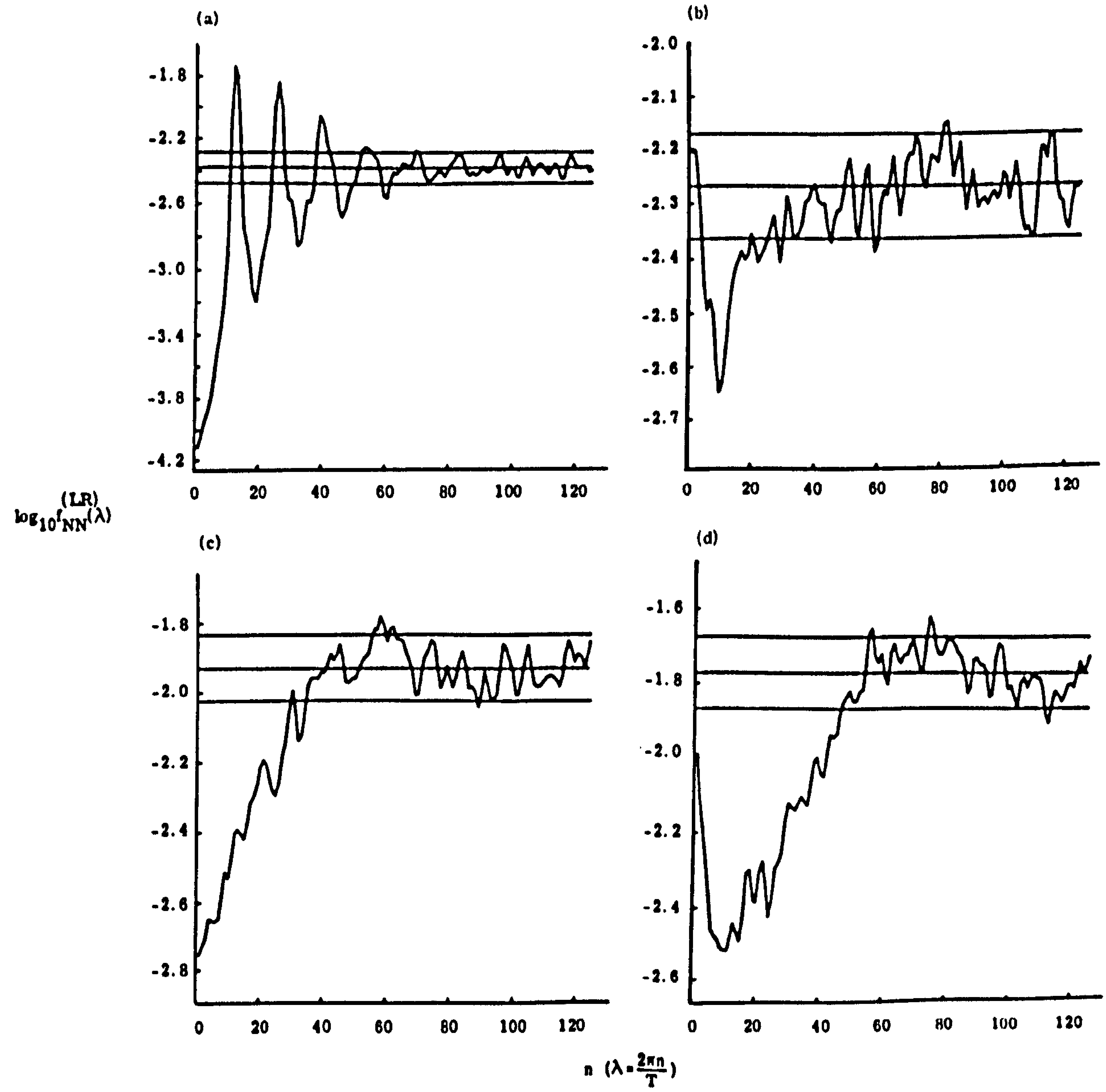


Fig. 3.11.8: Log to base 10 of the estimated power spectrum of the Ia discharge calculated from the periodogram by dividing the whole record into 31 disjoint sections of length R = 512. (a) Spontaneous discharge, (b) Ia discharge in presence of a fusimotor input (γ s), (c) Ia discharge in presence of a length change (l), and (d) Ia discharge in presence of γ s and l. The dotted line in each figure corresponds to the estimated power spectrum of a Poisson point process with the same mean rate. The horizontal lines are the asymptotic 95% confidence limits.

for this section. In each of the following figures the broken line corresponds to the logarithm to the base 10 of the estimated asymptotic value of the spectrum, and the solid horizontal lines represent the approximate 95% confidence limits. The abscissa in each figure corresponds to the value

$$\lambda/2\pi = n/S$$

For the Nyquist frequency the ratio $\lambda/2\pi$ corresponds to 0.5 cycles per unit time (see, for example, Robinson (1967), p85). Now, for a sampling rate of 1000 observations per second the Nyquist frequency would be

$$\lambda_0 = 0.5 \text{ cycles}/0.001 \text{ sec} = 500 \text{ cycles/sec}$$

In the examples given in this section the highest frequency plotted was 250 cycles/sec. It is clear that in all Figures the estimated power spectra have reached their asymptotic values.

Figs. 3.11.1-4 illustrate the application of confidence intervals based on procedure I for the estimation of the spectrum.

Figs. 3.11.5 (a-d) - 7 (a-d) in addition to illustrating the application of confidence intervals based on procedure II for the estimation of a point process spectrum, also allow one to compare the effect of the choice of a particular window on the shape of the estimated spectrum.

The comparison of Figs. 3.11.5 (a-d) with Figs. 3.11.7 (a-d) suggests that the choice of the Parzen or Tukey window does not alter significantly the shape of the estimated spectra, although there are slight differences in the value of the 95% confidence intervals.

Finally, Figs. 3.11.8 (a-d) illustrate the application of confidence intervals based on procedure III for the estimates of the power spectra corresponding to Data Sets I-IV. In each data set the record was divided into 31 sections, and the estimated spectrum taken by averaging the periodograms of the 31 sections.

We now proceed to estimate the AIF by using methods which involve the frequency domain.

3.12 ESTIMATES OF TIME DOMAIN PARAMETERS OBTAINED BY USING FREQUENCY DOMAIN METHODS

Equation (3.3.4) gives the CF as a function of the PS of a stationary point process N(t) defined on (0,T]. This expression is helpful in finding an estimate of the CF through an estimate of the PS. An obvious estimate of the CF will be

$$\hat{q}_{NN}(u) = \int_{-\pi}^{\pi} \left\{ I_{NN}^{(T)}(\lambda) - \frac{\hat{p}_N}{2\pi} \right\} e^{i\lambda u} d\lambda \quad , \tag{3.12.1}$$

where $I_{NN}^{(T)}(\lambda)$ is the periodogram given by (3.5.1) and $\hat{p}_N$ is the estimate of the MI of the point process.

We start by examining the asymptotic first- and second-order properties of the estimate (3.12.1).

<u>Theorem 3.12.1</u>: Let N(t) be a stationary point process on (0,T] with MI p_N and CF $q_{NN}(u)$ satisfying $\int |u||q_{NN}(u)|du < \infty$. Let $\hat{q}_{NN}(u)$ be the estimate given by (3.12.1). Then

$$E\left\{\hat{q}_{NN}(u)\right\} = q_{NN}(u) + O(T^{-1}) ,$$

$$\mathrm{cov}\left\{\hat{q}_{NN}(u), \hat{q}_{NN}(v)\right\} = \frac{2\pi}{T} \left\{ \int_{-\pi}^{\pi} f_{NN}^2(\lambda) e^{i\lambda(u+v)} d\lambda + \int_{-\pi}^{\pi} f_{NN}^2(\lambda) e^{i\lambda(u-v)} d\lambda \right.$$

$$+ \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{NNNN}(\lambda,-\lambda,\mu)\, e^{i(\lambda u-\mu v)} d\lambda\, d\mu \Big\} + O(T^{-2}\log T) \quad \text{for } u,v \neq 0 \;.$$

Proof: Taking expectations in (3.12.1) we have

$$E\{\hat{q}_{NN}(u)\} = \int_{-\pi}^{\pi}\{E[I_{NN}^{(T)}(\lambda)] - E(\hat{P}_N)/2\pi\} e^{i\lambda u} d\lambda$$

and by using Corollary 3.5.1,

$$= \int_{-\pi}^{\pi}\{f_{NN}(\lambda) - P_N/2\pi\} e^{i\lambda u} d\lambda + O(T^{-1})$$

$$= q_{NN}(u) + O(T^{-1}) \quad .$$

This shows that $\hat{q}_{NN}(u)$ is asymptotically an unbiased estimate of $q_{NN}(u)$. The covariance of the estimates $\hat{q}_{NN}(u)$, $\hat{q}_{NN}(v)$ is given by

$$\operatorname{cov}\{\hat{q}_{NN}(u), \hat{q}_{NN}(v)\} = \operatorname{cov}\Big\{\int_{-\pi}^{\pi}\Big(I_{NN}^{(T)}(\lambda) - \frac{\hat{P}_N}{2\pi}\Big) e^{i\lambda u} d\lambda, \int_{-\pi}^{\pi}\Big(I_{NN}^{(T)}(\mu) - \frac{\hat{P}_N}{2\pi}\Big) e^{i\mu v} d\mu\Big\}$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \operatorname{cov}\{I_{NN}^{(T)}(\lambda), I_{NN}^{(T)}(\mu)\} e^{i(\lambda u-\mu v)} d\lambda\, d\mu \quad \text{for } u,v \neq 0$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big\{\frac{2\pi}{T} f_{NNNN}(\lambda,-\lambda,\mu) + |\Delta^{(T)}(\lambda+\mu)|^2 \frac{f_{NN}^2(\lambda)}{T^2} + |\Delta^{(T)}(\lambda-\mu)|^2 \frac{f_{NN}^2(\lambda)}{T^2}\Big\}$$

$$e^{i(\lambda u-\mu v)} d\lambda\, d\mu + O(T^{-2}\log T) \quad (*) \quad , \text{ by considering Th. 3.5.2} \;.$$

The function $\Delta^{(T)}(\lambda)$ has the following property

$$\int_{-\pi}^{\pi} |\Delta^{(T)}(\lambda)|\, d\lambda = O(\log T) \quad (\text{Edwards, 1967; p 80}) \;.$$

Also, we have that

$$\int_0^{2\pi} |\Delta^{(T)}(\lambda+\mu)|^2 d\mu \simeq 2\pi T$$

and

$$\int_0^{2\pi} |\Delta^{(T)}(\lambda+\mu)|^2 e^{i\mu v} d\mu \simeq 2\pi T e^{-i\lambda v}, \quad \int_0^{2\pi} |\Delta^{(T)}(\lambda-\mu)|^2 e^{i\mu v} d\mu \simeq 2\pi T e^{i\lambda v} .$$

By substituting these relations in (*) we get the required result, that is

$$\operatorname{cov}\{\hat{q}_{NN}(u), \hat{q}_{NN}(v)\} = \frac{2\pi}{T}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{NNNN}(\lambda,-\lambda,\mu)\, e^{i(\lambda u-\mu v)} d\lambda\, d\mu$$

$$+\frac{2\pi}{T}\int_{-\pi}^{\pi} f_{NN}^2(\lambda) e^{i\lambda(u-v)} d\lambda + \frac{2\pi}{T}\int_{-\pi}^{\pi} f_{NN}^2(\lambda) e^{i\lambda(u+v)} d\lambda + O(T^{-2}\log T) .$$

In connection with the cumulant function we have defined the product density. An estimate for the product density is given by

$$\hat{p}_{NN}(u) = \hat{p}_N^2 + \int_{-\pi}^{\pi} \left\{ I_{N'N'}^{(T)}(\lambda) - \frac{\hat{p}_N}{2\pi} \right\} e^{i\lambda u} d\lambda \quad . \tag{3.12.2}$$

The first and second order properties of $\hat{p}_{NN}(u)$ are discussed in the next theorem.

<u>Theorem 3.12.2</u>: Let N(t) be a stationary point process on (0,T] satisfying the conditions of Theorem 3.12.1. Let

$\hat{p}_{NN}(u)$ be given by (3.12.2). Then

$$E\{\hat{p}_{NN}(u)\} = p_{NN}(u) + O(T^{-1})$$

and

$$\operatorname{cov}\{\hat{p}_{NN}(u), \hat{p}_{NN}(v)\} = 4\frac{2\pi}{T} f_N^2 f_{NN}(0) + 2\frac{2\pi}{T} f_N \int_{-\pi}^{\pi} f_{NNN}(0,\lambda)e^{i\lambda u}d\lambda$$

$$+ 2\frac{2\pi}{T} f_N \int_{-\pi}^{\pi} f_{NNN}(0,\mu)e^{i\mu v}d\mu + \frac{2\pi}{T}\int_{-\pi}^{\pi} f_{NN}^2(\lambda)e^{i\lambda(u-v)}d\lambda$$

$$+ \frac{2\pi}{T}\int_{-\pi}^{\pi} f_{NN}^2(\lambda)e^{i\lambda(u+v)}d\lambda + \frac{2\pi}{T}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{NNNN}(\lambda,-\lambda,\mu)e^{i(\lambda u-\mu v)}d\lambda d\mu$$

$$+ O(T^{-2}) \quad \text{for } u, v \neq 0.$$

<u>Proof</u>: Taking the expected value in (3.12.2) we find

$$E\{\hat{p}_{NN}(u)\} = E(\hat{p}_N^2) + \int_{-\pi}^{\pi} E\{I_{NN}^{(T)}(\lambda) - \frac{\hat{p}_N}{2\pi}\}e^{i\lambda u}d\lambda$$

$$= \operatorname{Var}\{\hat{p}_N^2\} + p_N^2 + \int_{-\pi}^{\pi}\{f_{NN}(\lambda) - \frac{\hat{p}_N}{2\pi}\}e^{i\lambda u}d\lambda$$

$$= p_N^2 + \frac{1}{T^2}\operatorname{cov}\{d_N^{(T)}(0), d_N^{(T)}(0)\} + q_{NN}(u)$$

$$= p_{NN}(u) + \frac{1}{T^2}[2\pi T f_{NN}(0) + o(1)] = p_{NN}(u) + O(T^{-1}),$$

where $\quad d_N^T(0) = \int_0^T dN(t)$.

For the covariance of $\hat{p}_{NN}(u)$ and $\hat{p}_{NN}(v)$ we have

$$\operatorname{cov}\{\hat{p}_{NN}(u),\hat{p}_{NN}(v)\} = \operatorname{cov}(\hat{p}_N^2,\hat{p}_N^2) + \operatorname{cov}\{\hat{p}_N^2, \int_{-\pi}^{\pi} I_{NN'}^{(T)}(\mu)e^{i\mu v}d\mu\}$$

$$+\operatorname{cov}\{\int_{-\pi}^{\pi} I_{NN'}^{(T)}(\lambda)e^{i\lambda u}d\lambda, \hat{p}_N^2\} + \operatorname{cov}\{\int_{-\pi}^{\pi} I_{NN'}^{(T)}(\lambda)e^{i\lambda u}d\lambda, \int_{-\pi}^{\pi} I_{NN'}^{(T)}(\mu)e^{i\mu v}d\mu\} \text{ for } u,v\neq 0 .$$

We can now prove using properties of the cumulants that

$$\operatorname{cov}(\hat{p}_N^2,\hat{p}_N^2) = \frac{1}{T^4}\left[E\{d_N^{(T)}(0)d_N^{(T)}(0)d_N^{(T)}(0)d_N^{(T)}(0)\} - E\{d_N^{(T)}(0)d_N^{(T)}(0)\}E\{d_N^{(T)}(0)d_N^{(T)}(0)\}\right]$$

$$= \frac{1}{T^4}\left[\operatorname{cum}\{d_N^{(T)}(0), d_N^{(T)}(0), d_N^{(T)}(0), d_N^{(T)}(0)\} + 4\operatorname{cum}\{d_N^{(T)}(0)\}\operatorname{cum}\{d_N^{(T)}(0), d_N^{(T)}(0), d_N^{(T)}(0)\}\right.$$

$$+ 2\operatorname{cum}\{d_N^{(T)}(0), d_N^{(T)}(0)\}\cdot\operatorname{cum}\{d_N^{(T)}(0), d_N^{(T)}(0)\} + 4\operatorname{cum}\{d_N^{(T)}(0)\}\cdot\operatorname{cum}\{d_N^{(T)}(0)\}\cdot$$

$$\left.\operatorname{cum}\{d_N^{(T)}(0), d_N^{(T)}(0)\}\right] \simeq \frac{(2\pi)^3}{T^3} f_{NNNN}(0,0,0) + 4\frac{(2\pi)^2}{T^2} f_N f_{NNN}(0,0)$$

$$+ 2\frac{(2\pi)^2}{T^2} f_{NN}^2(0) + 4\frac{2\pi}{T} f_N^2 f_{NN}(0).$$

Similarly we show that

$$\operatorname{cov}\{\hat{p}_N^2, \int_{-\pi}^{\pi} I_{NN'}^{(T)}(\lambda)e^{i\lambda u}d\lambda\} = \frac{1}{2\pi T^3}\int_{-\pi}^{\pi}\operatorname{cov}\{d_N^{(T)}(0)d_N^{(T)}(0), d_{N'}^{(T)}(\lambda)d_{N'}^{(T)}(-\lambda)\}e^{i\lambda u}d\lambda$$

$$= \frac{1}{2\pi T^3}\int_{-\pi}^{\pi}\left[E\{d_N^{(T)}(0)d_N^{(T)}(0)d_N^{(T)}(\lambda)d_N^{(T)}(-\lambda)\} - p_N\Delta^{(T)}(-\lambda)\, E\{d_N^{(T)}(0)d_N^{(T)}(0)d_N^{(T)}(\lambda)\}\right.$$

$$- p_N \Delta^{(T)}(\lambda)\, E\left\{ d_N^{(T)}(0) d_N^{(T)}(0) d_N^{(T)}(-\lambda) \right\} + p_N^2 |\Delta^{(T)}(\lambda)|^2 E\left\{ d_N^{(T)}(0)\, d_N^{(T)}(0) \right\}$$

$$- E\left\{ d_N^{(T)}(0) d_N^{(T)}(0) \right\} E\left\{ d_N^{(T)}(\lambda) d_N^{(T)}(-\lambda) \right\} + p_N^2 |\Delta^{(T)}(\lambda)|^2 E\left\{ d_N^{(T)}(0) d_N^{(T)}(0) \right\} \Big] e^{i\lambda u} d\lambda$$

$$= \frac{1}{2\pi T^3} \int_{-\pi}^{\pi} \Big[\mathrm{cum}\left\{ d_N^{(T)}(0), d_N^{(T)}(0), d_N^{(T)}(\lambda), d_N^{(T)}(-\lambda) \right\} + \mathrm{cum}\left\{ d_N^{(T)}(0) \right\} \mathrm{cum}\left\{ d_N^{(T)}(0), d_N^{(T)}(\lambda), d_N^{(T)}(-\lambda) \right\}$$

$$+ \mathrm{cum}\left\{ d_N^{(T)}(0) \right\} \mathrm{cum}\left\{ d_N^{(T)}(0), d_N^{(T)}(\lambda), d_N^{(T)}(-\lambda) \right\} + \mathrm{cum}\left\{ d_N^{(T)}(0), d_N^{(T)}(\lambda) \right\} \mathrm{cum}\left\{ d_N^{(T)}(0), d_N^{(T)}(-\lambda) \right\}$$

$$+ \mathrm{cum}\left\{ d_N^{(T)}(0), d_N^{(T)}(-\lambda) \right\} \mathrm{cum}\left\{ d_N^{(T)}(0), d_N^{(T)}(\lambda) \right\} \Big] e^{i\lambda u} d\lambda \simeq \left(\frac{2\pi}{T}\right)^2 \int_{-\pi}^{\pi} f_{NNNN}(0,0,\lambda)\, e^{i\lambda u} d\lambda$$

$$+ 2\left(\frac{2\pi}{T}\right) f_N \int_{-\pi}^{\pi} f_{NN}(0,\lambda)\, e^{i\lambda u} d\lambda + 2\, \frac{2\pi}{T^3} f_{NN}^2(0) \int_{-\pi}^{\pi} |\Delta^{(T)}(\lambda)|^2 e^{i\lambda u} d\lambda$$

$$\simeq \left(\frac{2\pi}{T}\right)^2 \int_{-\pi}^{\pi} f_{NNNN}(0,0,\lambda)\, e^{i\lambda u} d\lambda + 2\left(\frac{2\pi}{T}\right) f_N \int_{-\pi}^{\pi} f_{NNN}(0,\lambda)\, e^{i\lambda u} d\lambda$$

$$+ 2\left(\frac{2\pi}{T}\right)^2 f_{NN}^2(0) .$$

Finally for the last term we get

$$\mathrm{cov}\left\{ \int_{-\pi}^{\pi} I_{N'N'}^{(T)}(\lambda) e^{i\lambda u} d\lambda, \int_{-\pi}^{\pi} I_{N'N'}^{(T)}(\mu)\, e^{i\mu v} d\mu \right\} = \frac{1}{(2\pi T)^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \mathrm{cov}\left\{ d_{N'}^{(T)}(\lambda) d_{N'}^{(T)}(-\lambda), \right.$$

$$\left. d_{N'}^{(T)}(\mu) d_{N'}^{(T)}(-\mu) \right\} e^{i(\lambda u - \mu v)} d\lambda d\mu \simeq \frac{2\pi}{T} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{NNNN}(\lambda, -\lambda, \mu)\, e^{i(\lambda u - \mu v)} d\lambda d\mu$$

$$+\frac{1}{T^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\Delta^{(T)}(\lambda+\mu)|^2 f_{NN}^2(\lambda)e^{i(\lambda u-\mu v)}d\lambda d\mu+\frac{1}{T^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\Delta^{(T)}(\lambda-\mu)|^2 f_{NN}^2(\lambda)e^{i(\lambda u-\mu v)}d\lambda d\mu$$

$$\simeq\frac{2\pi}{T}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f_{NNNN}(\lambda,-\lambda,\mu)e^{i(\lambda u-\mu v)}d\lambda d\mu+\frac{2\pi}{T}\int_{-\pi}^{\pi}f_{NN}^2(\lambda)e^{i\lambda(u+v)}d\lambda$$

$$+\frac{2\pi}{T}\int_{-\pi}^{\pi}f_{NN}^2(\lambda)e^{i\lambda(u-v)}d\lambda\ .$$

Hence the required result for the covariance follows by combining all the above expressions, that is

$$\operatorname{cov}\{\hat{P}_{NN}(u),\hat{P}_{NN}(v)\}=4\frac{2\pi}{T}f_N^2 f_{NN}(0)+2\frac{2\pi}{T}f_N\int_{-\pi}^{\pi}f_{NNN}(0,\lambda)e^{i\lambda u}d\lambda$$

$$+2\frac{2\pi}{T}f_N\int_{-\pi}^{\pi}f_{NNN}(0,\mu)e^{i\mu v}d\mu+\frac{2\pi}{T}\int_{-\pi}^{\pi}f_{NN}^2(\lambda)e^{i\lambda(u-v)}d\lambda$$

$$+\frac{2\pi}{T}\int_{-\pi}^{\pi}f_{NN}^2(\lambda)e^{i\lambda(u+v)}d\lambda+\frac{2\pi}{T}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f_{NNNN}(\lambda,-\lambda,\mu)e^{i(\lambda u-\mu v)}d\lambda d\mu+O(T^{-2}),\ u,v\neq 0\ .$$

Finally we discuss an estimate of the AIF which is based on the modified periodogram of the zero-mean point process $N'(t)$. This estimate is given by

$$\hat{m}_{NN}(u)=\frac{\hat{P}_{NN}(u)}{\hat{P}_N}=\hat{P}_N+\frac{1}{\hat{P}_N}\int_{-\pi}^{\pi}\left\{I_{N'N'}^{(T)}(\lambda)-\frac{\hat{P}_N}{2\pi}\right\}e^{i\lambda u}d\lambda\ . \tag{3.12.3}$$

The approximate first- and second-order properties of $\hat{m}_{NN}(u)$ are developed in the following theorem.

<u>Theorem 3.12.3</u>: Let $N(t)$ be a stationary point process on $(0,T]$ satisfying the assumptions of Theorem 3.12.1. Let $\hat{m}_{NN}(u)$ be given by (3.12.3). Then $\hat{m}_{NN}(u)$ is asymptotically an unbiased estimate of $m_{NN}(u)$ as $T\longrightarrow\infty$. Also the covariance

between $\hat{m}_{NN}(u)$ and $\hat{m}_{NN}(v)$ is given approximately by

$$\operatorname{cov}\{\hat{m}_{NN}(u), \hat{m}_{NN}(v)\} \simeq \frac{2\pi}{T} f_{NN}(0) + \frac{2\pi}{P_N T}\int_{-\pi}^{\pi} f_{NNN}(0,\lambda) e^{i\lambda u} d\lambda + \frac{2\pi}{P_N T}\int_{-\pi}^{\pi} f_{NNN}(0,\mu) e^{i\mu v} d\mu$$

$$- \frac{2\pi f_{NN}(0)}{P_N^2 T}\int_{-\pi}^{\pi} f_{NN}(\mu) e^{i\mu v} d\mu - \frac{2\pi f_{NN}(0)}{P_N^2 T}\int_{-\pi}^{\pi} f_{NN}(\lambda) e^{i\lambda u} d\lambda - \frac{2\pi}{P_N^2 T}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{NNN}(0,\lambda) f_{NN}(\lambda) e^{i(\lambda u - \mu v)} d\lambda d\mu$$

$$- \frac{2\pi}{P_N^2 T}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{NNN}(0,\mu) f_{NN}(\lambda) e^{i(\lambda u - \mu v)} d\lambda d\mu + \frac{2\pi}{P_N^2 T}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{NNNN}(\lambda,-\lambda,\mu) e^{i(\lambda u - \mu v)} d\lambda d\mu$$

$$+ \frac{2\pi}{P_N^2 T}\int_{-\pi}^{\pi} f_{NN}^2(\lambda) e^{i\lambda(u-v)} d\lambda + \frac{2\pi}{P_N^2 T}\int_{-\pi}^{\pi} f_{NN}^2(\lambda) e^{i\lambda(u+v)} d\lambda \quad \text{for} \quad u,v \neq 0 .$$

<u>Proof</u>: Taking expectations in (3.12.3) we have

$$E\{\hat{m}_{NN}(u)\} = E(\hat{P}_N) + \int_{-\pi}^{\pi}\left[E\left\{\frac{I_{N'N'}^{(T)}(\lambda)}{\hat{P}_N}\right\} - \frac{1}{2\pi}\right] e^{i\lambda u} d\lambda \quad .$$

Now, by using the approximate expression for the expected value of a ratio (Kendall & Stuart, 1966, Vol.1, p260), we get

$$E\left\{\frac{I_{N'N'}^{(T)}(\lambda)}{\hat{P}_N}\right\} \simeq \frac{E\{I_{N'N'}^{(T)}(\lambda)\}}{E(\hat{P}_N)} - \frac{1}{P_N^2}\operatorname{cov}\{\hat{P}_N, I_{N'N'}^{(T)}(\lambda)\} + \frac{E\{I_{N'N'}^{(T)}(\lambda)\}}{P_N^3}\operatorname{Var}(\hat{P}_N)$$

$$\simeq \frac{f_{NN}(\lambda)}{P_N} - \frac{1}{P_N^2}\frac{2\pi}{T} f_{NNN}(0,\lambda) + \frac{2\pi}{P_N^3 T} f_{NN}(0) f_{NN}(\lambda)$$

$$= \frac{f_{NN}(\lambda)}{P_N} + O(T^{-1}) \quad ,$$

and the required result for expected value of $\hat{m}_{NN}(u)$ follows, i.e.

$$E\{\hat{m}_{NN}(u)\} \simeq P_N + \frac{1}{P_N}\int_{-\pi}^{\pi}\left\{ f_{NN}(\lambda) - \frac{P_N}{2\pi}\right\} e^{i\lambda u} d\lambda + O(T^{-1})$$

$$= m_{NN}(u) + O(T^{-1}) \quad .$$

The covariance of the two estimates $\hat{m}_{NN}(u)$ and $\hat{m}_{NN}(v)$ can be written as

$$\operatorname{cov}\left[\hat{p}_N+\int_{-\pi}^{\pi}\left\{\frac{I_{N'N'}^{(T)}(\lambda)}{\hat{p}_N}-\frac{1}{2\pi}\right\}e^{i\lambda u}d\lambda,\ \hat{p}_N+\int_{-\pi}^{\pi}\left\{\frac{I_{N'N'}^{(T)}(\mu)}{\hat{p}_N}-\frac{1}{2\pi}\right\}e^{i\mu v}d\mu\right]$$

$$=\operatorname{cov}\left[\hat{p}_N+\int_{-\pi}^{\pi}\frac{I_{N'N'}^{(T)}(\lambda)}{\hat{p}_N}e^{i\lambda u}d\lambda,\ \hat{p}_N+\int_{-\pi}^{\pi}\frac{I_{N'N'}^{(T)}(\mu)}{\hat{p}_N}e^{i\mu v}d\mu\right]\quad \text{for}\quad u,v\neq 0$$

$$=\operatorname{Var}(\hat{p}_N)+\operatorname{cov}\left\{\hat{p}_N,\int_{-\pi}^{\pi}\frac{I_{N'N'}^{(T)}(\lambda)}{\hat{p}_N}e^{i\lambda u}d\lambda\right\}+\operatorname{cov}\left\{\hat{p}_N,\int_{-\pi}^{\pi}\frac{I_{N'N'}^{(T)}(\mu)}{\hat{p}_N}e^{i\mu v}d\mu\right\}$$

$$+\operatorname{cov}\left\{\int_{-\pi}^{\pi}\frac{I_{N'N'}^{(T)}(\lambda)}{\hat{p}_N}e^{i\lambda u}d\lambda,\ \int_{-\pi}^{\pi}\frac{I_{N'N'}^{(T)}(\mu)}{\hat{p}_N}e^{i\mu v}d\mu\right\}$$

$$=\operatorname{Var}(\hat{p}_N)+\int_{-\pi}^{\pi}\operatorname{cov}\left(\hat{p}_N,\frac{I_{N'N'}^{(T)}(\lambda)}{\hat{p}_N}\right)e^{i\lambda u}d\lambda+\int_{-\pi}^{\pi}\operatorname{cov}\left(\hat{p}_N,\frac{I_{N'N'}^{(T)}(\mu)}{\hat{p}_N}\right)e^{i\mu v}d\mu$$

$$+\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\operatorname{cov}\left\{\frac{I_{N'N'}^{(T)}(\lambda)}{\hat{p}_N},\frac{I_{N'N'}^{(T)}(\mu)}{\hat{p}_N}\right\}e^{i(\lambda u-\mu v)}d\lambda d\mu\ .$$

Now we have

$$\operatorname{Var}(\hat{p}_N)=\frac{1}{T^2}\operatorname{cov}\left\{d_N^{(T)}(0),d_N^{(T)}(0)\right\}\simeq\frac{1}{T^2}2\pi T f_{NN}(0)=\frac{2\pi}{T}f_{NN}(0)\ ,$$

$$\operatorname{cov}\left\{\hat{p}_N,\frac{I_{N'N'}^{(T)}(\lambda)}{\hat{p}_N}\right\}\simeq\frac{1}{p_N}\operatorname{cov}\left\{\hat{p}_N,I_{N'N'}^{(T)}(\lambda)\right\}-\frac{f_{NN}(\lambda)}{p_N^2}\operatorname{Var}(\hat{p}_N)$$

$$\simeq\frac{2\pi}{p_N T}f_{NNN}(0,\lambda)-\frac{f_{NN}(\lambda)}{p_N^2}\frac{2\pi f_{NN}(0)}{T}\quad ,$$

$$\text{cov}\left\{\frac{I^{(T)}_{NN'}(\lambda)}{\hat{P}_N}, \frac{I^{(T)}_{NN'}(\mu)}{\hat{P}_N}\right\} \simeq \frac{f_{NN}(\lambda)f_{NN}(\mu)}{P_N^4} \text{Var}(\hat{P}_N) + \frac{1}{P_N^2} \text{cov}\left\{I^{(T)}_{NN'}(\mu), I^{(T)}_{NN'}(\lambda)\right\}$$

$$- \frac{f_{NN}(\mu)}{P_N^3} \text{cov}\left\{\hat{P}_N, I^{(T)}_{NN'}(\lambda)\right\} - \frac{f_{NN}(\lambda)}{P_N^3} \text{cov}\left\{\hat{P}_N, I^{(T)}_{NN'}(\mu)\right\} .$$

The last two expressions follow from Kendall & Stuart, 1966, Vol.1, p247 by applying the Taylor expansion for the covariance of two functions of random variables. Hence the approximate expression for the covariance of $\hat{m}_{NN}(u)$ and $\hat{m}_{NN}(v)$ is given by

$$\text{cov}\left\{\hat{m}_{NN}(u), \hat{m}_{NN}(v)\right\} \simeq \frac{2\pi}{T} f_{NN}(0) + \frac{2\pi}{P_N T}\int_{-\pi}^{\pi} f_{NNN}(0,\lambda) e^{i\lambda u} d\lambda$$

$$+ \frac{2\pi}{P_N T}\int_{-\pi}^{\pi} f_{NNN}(0,\mu) e^{i\mu v} d\mu - \frac{2\pi f_{NN}(0)}{T}\int_{-\pi}^{\pi} \frac{f_{NN}(\lambda)}{P_N^2} e^{i\lambda u} d\lambda$$

$$- \frac{2\pi f_{NN}(0)}{T}\int_{-\pi}^{\pi} \frac{f_{NN}(\mu)}{P_N^2} e^{i\mu v} d\mu - \frac{2\pi}{P_N^2 T}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{NNN}(0,\lambda) f_{NN}(\mu) e^{i(\lambda u - \mu v)} d\lambda d\mu$$

$$- \frac{2\pi}{P_N^2 T}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{NNN}(0,\mu) f_{NN}(\lambda) e^{i(\lambda u - \mu v)} d\lambda d\mu + \frac{2\pi}{P_N^2 T}\int_{-\pi}^{\pi} f_{NN}^2(\lambda) e^{i\lambda(u-v)} d\lambda$$

$$+ \frac{2\pi}{P_N^2 T}\int_{-\pi}^{\pi} f_{NN}^2(\lambda) e^{i\lambda(u+v)} d\lambda + \frac{2\pi}{P_N^2 T}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{NNNN}(\lambda,-\lambda,\mu) e^{i(\lambda u - \mu v)} d\lambda d\mu$$

for $u, v \neq 0$.

3.13 EXAMPLE: DERIVATION OF THE FIRST- AND SECOND-ORDER PROPERTIES OF THE CUMULANT, PRODUCT DENSITY AND AUTO-INTENSITY IN THE CASE OF A POISSON PROCESS BASED ON THE FREQUENCY DOMAIN PARAMETERS

In this section we give as an example the approximate first- and second-order properties of the estimates $\hat{q}_{NN}(u)$, $\hat{p}_{NN}(u)$ and $\hat{m}_{NN}(u)$ under the hypothesis that the point process is Poisson.

We start with the estimate $\hat{q}_{NN}(u)$.

Theorem 3.13.1: Let N(t) be a stationary point process on (0,T] with mean intensity p_N. Suppose that the estimate $\hat{q}_{NN}(u)$ is given by (3.12.1). Then if N(t) is a Poisson point process, we have

$$E\{\hat{q}_{NN}(u)\} = O(T^{-1}) \quad , \; u \neq 0$$

and

$$Var\{\hat{q}_{NN}(u)\} = \frac{p_N^2}{T} + O(T^{-2}\log T) \; , \; u \neq 0 \quad .$$

Proof: The first result follows from Theorem 3.12.1 since the cumulant $q_{NN}(u)$ is zero for a Poisson process.

The second result follows again from Theorem 3.12.1 if we substitute in the expression for the covariance of $\hat{q}_{NN}(u)$ and $\hat{q}_{NN}(v)$

$$f_{NN}(\lambda) = \frac{p_N}{2\pi} \quad \text{and} \quad f_{NNNN}(\lambda, \lambda, \mu) = p_N/(2\pi)^3 \; .$$

These are expressions which hold for the second and fourth order spectra of a Poisson point process.

The next theorem examines the first- and second-order properties of $\hat{p}_{NN}(u)$ under the hypothesis that N is a Poisson point process.

<u>Theorem 3.13.2</u>: Let N(t) be a stationary point process on (0,T] with mean intensity p_N. Suppose that $\hat{p}_{NN}(u)$ is an estimate of the product density given by (3.12.2). Then, if N(t) is a Poisson point process, we have

$$E\left\{\hat{p}_{NN}(u)\right\} = p_N^2 + O(T^{-1})$$

and

$$Var\left\{\hat{p}_{NN}(u)\right\} = \frac{p_N^2}{T} + \frac{4p_N^3}{T} + O(T^{-2}),\ u \neq 0 \quad .$$

<u>Proof</u>: The proof follows from Theorem 3.12.2, since

$$p_{NN}(u) = p_N^2 \ , \ f_{NN}(\lambda) = p_N/2\pi \ \text{ and } \ f_{NNNN}(\lambda,-\lambda,\mu) = p_N/(2\pi)^3$$

hold for a Poisson point process.

The covariance of the estimates $\hat{p}_{NN}(u)$ and $\hat{p}_{NN}(v)$ for a Poisson point process is given by

$$cov\left\{\hat{p}_{NN}(u), \hat{p}_{NN}(v)\right\} = \frac{4p_N^3}{T} + O(T^{-2}) \ \text{ for } \ u,v \neq 0 \quad .$$

Finally the properties of $\hat{m}_{NN}(u)$ are examined when N is assumed to be a Poisson point process.

Theorem 3.13.3: Let N(t) be a stationary point process on $(0,T]$ with mean intensity p_N. Suppose that $\hat{m}_{NN}(u)$ is the estimate of the auto-intensity function given by (3.12.3). Then, under the assumption that N(t) is Poisson point process, we have the following approximate expressions

$$E\left\{\hat{m}_{NN}(u)\right\} = p_N + O(T^{-1})$$

and

$$Var\left\{\hat{m}_{NN}(u)\right\} \simeq \frac{1}{T} + \frac{p_N}{T} \ , \ u \neq 0 \quad .$$

Proof: The proof follows from Theorem 3.12.3, since

$$m_{NN}(u) = p_N \ ; \ f_{NN}(\lambda) = p_N/2\pi \ \text{ and } \ f_{NNNN}(\lambda,-\lambda,\mu) = p_N/(2\pi)^3$$

hold for a Poisson point process.

The approximate covariance of $\hat{m}_{NN}(u)$ and $\hat{m}_{NN}(v)$ is given by

$$\operatorname{cov}\left\{\hat{m}_{NN}(u), \hat{m}_{NN}(v)\right\} \simeq \frac{p_N}{T} \ \text{ for } \ u,v \neq 0 \ .$$

We now use parameters from the frequency domain to estimate the auto-intensity function.

3.14 ESTIMATES OF THE AUTO-INTENSITY FUNCTION OBTAINED BY USING METHODS OF THE FREQUENCY DOMAIN

The use of the frequency domain estimates to derive time domain parameters is illustrated in this section. The procedure required involves the calculation of the periodogram of a point process and the inversion of this to obtain time domain parameters. This procedure may be faster than the one used to estimate time domain parameters directly.

We describe here the calculation of an estimate of the AIF by using the following equation

$$\hat{m}_{NN}(u) = \hat{p}_N + \frac{1}{\hat{p}_N} \int_{-\pi}^{\pi} \left\{ I_{N'N'}^{(T)}(\lambda) - \frac{\hat{p}_N}{2\pi} \right\} e^{i\lambda u} d\lambda$$

In our calculation we use the periodogram $I_{NN}^{(T)}(\lambda)$ of the zero mean point process given by expression (3.8.5).

As indicated in sections 3.8 and 3.10 the periodogram $I_{NN}^{(T)}(\lambda)$ can be calculated in two ways

(a) By using the whole record (see section 3.8) or

(b) By dividing the whole record into sections (see section 3.10)

In the first method the inversion of the periodogram is quickly done by using the Fast Fourier transform. Figs. 3.14.1 (a-d) show the square roots of the estimates of the AIF for the Data Sets I-IV. If we compare them with the ones calculated by using the time domain approach we can see that they are almost identical.

The second method splits the whole record into sections. In each section the periodogram is calculated and is inverted in order to obtain the estimate of the AIF of the particular section. By averaging the estimates of the AIF in all sections we get the required estimate of the AIF. Figs. 3.14.2 (a-d) give the square roots of the estimates of the AIF

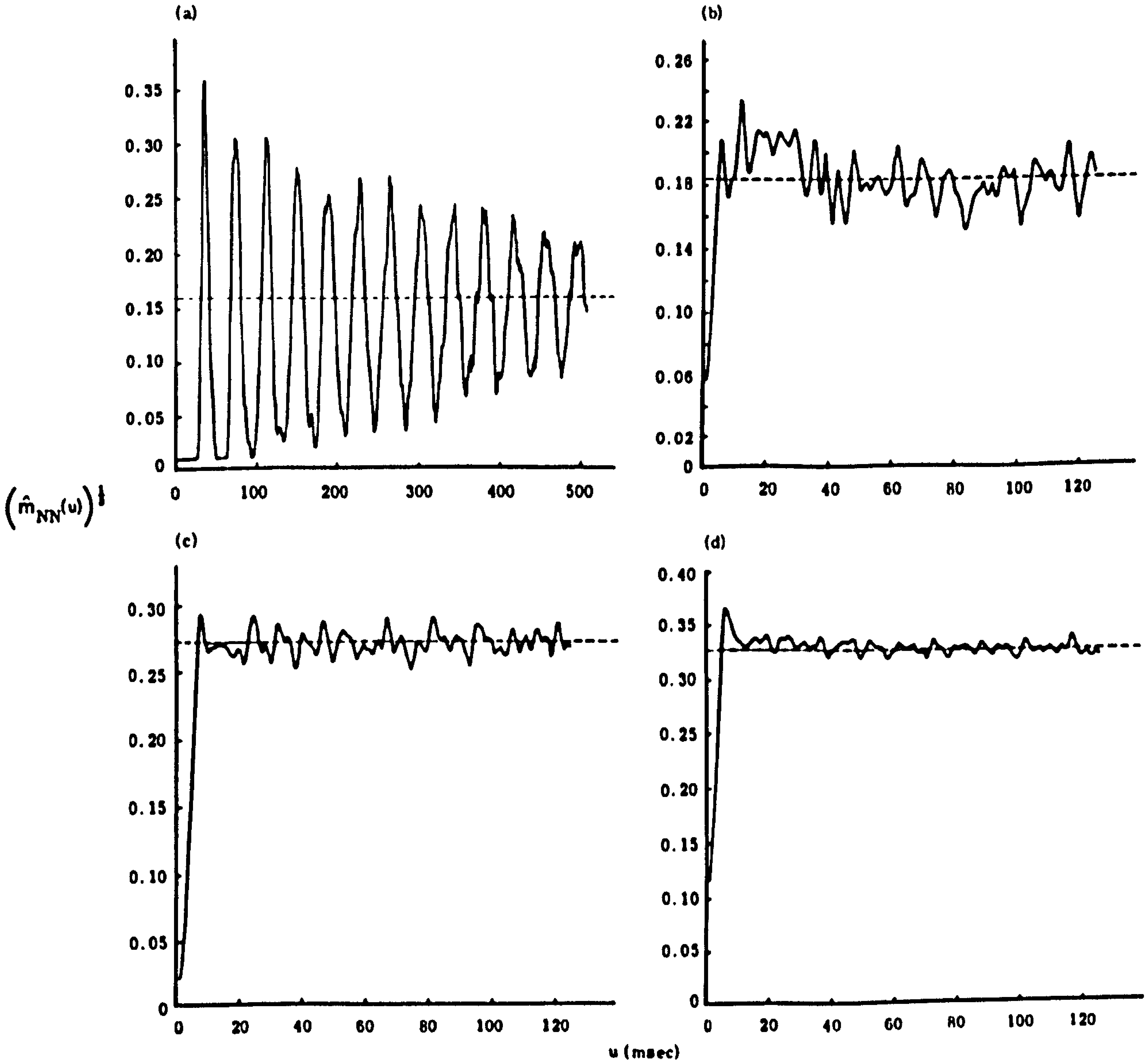


Fig. 3.14.1: Smoothed estimate of the square root of the AIF of the Ia discharge calculated by inverting the periodogram. (a) Spontaneous discharge, (b) Ia discharge in presence of a fusimotor input (γ s), (c) Ia discharge in presence of a length change (l), and (d) Ia discharge in presence of γ s and l. The dotted line in each figure corresponds to the estimated value of the square root of the mean rate of the Ia discharge.

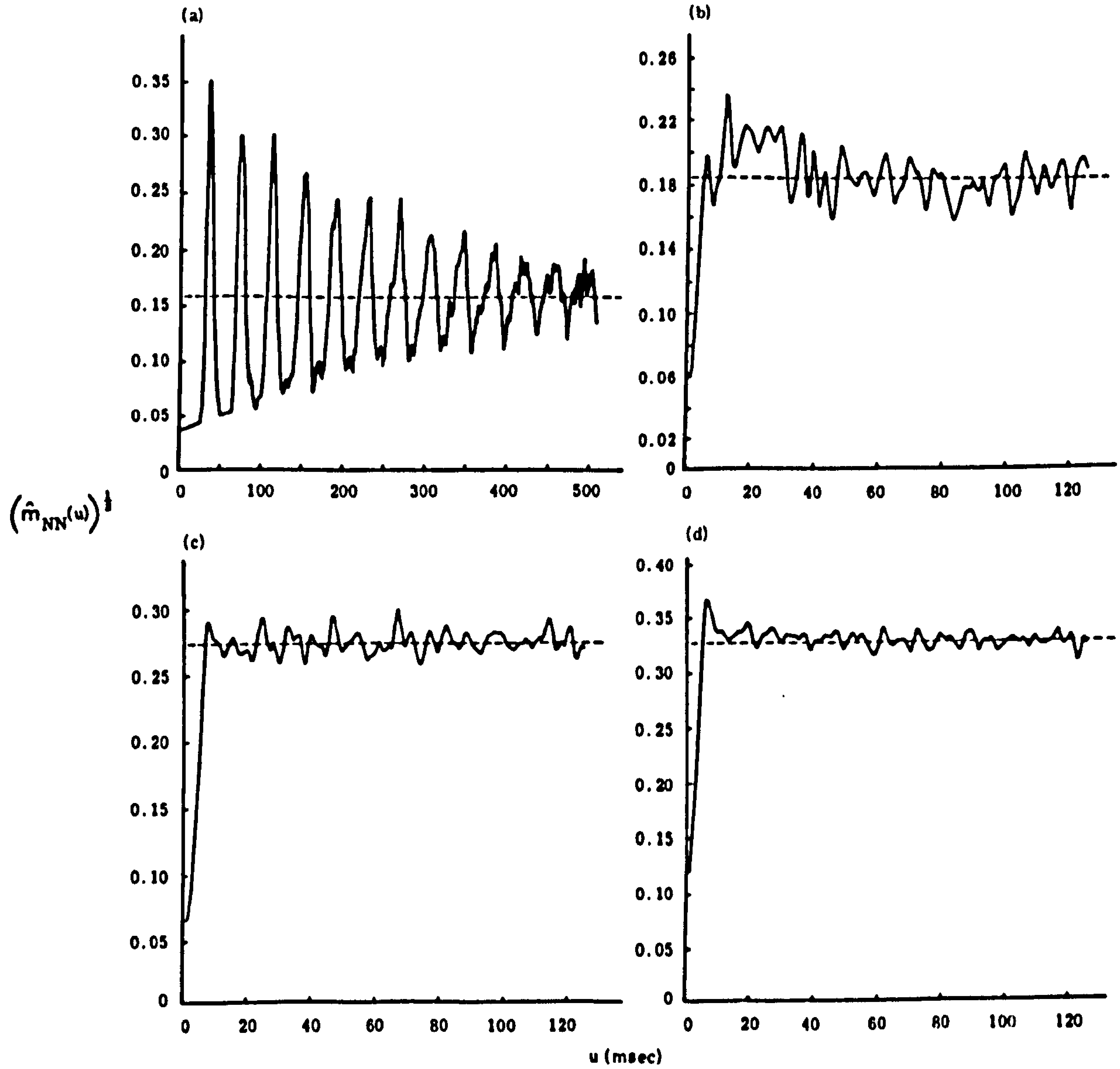


Fig. 3.14.2: Smoothed estimate of the square root of the AIF of the Ia discharge obtained by inverting the periodogram. The periodogram is calculated by dividing the whole record into 31 disjoint sections of length R = 512. (a) Spontaneous discharge, (b) Ia discharge in presence of a fusimotor input (γs), (c) Ia discharge in presence of a length change (l), and (d) Ia discharge in the presence of γs and l. The dotted line in each figure corresponds to the estimated value of the square root of the mean rate of the Ia discharge.

for the Data Sets I-IV. These Figures again are quite similar to those calculated directly from the time domain. This method provides the most rapid calculation of the AIF of a univariate point process.

3.15 CONCLUSIONS

In this chapter we presented certain parameters of the univariate point process in both time and frequency domains. The estimation of these parameters was discussed. The large sample properties of the estimated parameters were examined, and their asymptotic distribution developed. Confidence intervals based on the asymptotic distributions were constructed under the null hypothesis that the univariate point process was random (Poisson point process).

The main emphasis of this chapter is given to frequency domain methods which are based on smoothed periodogram estimates of the power spectrum of a univariate stationary point process.

We may summarise the important and useful properties of the frequency domain methods as follows

(i) Estimates of the power spectrum can be calculated by using two different methods in the frequency domain (sections 3.7 and 3.10) (A third method based on transforming time domain parameters is also discussed and compared with the frequency domain methods (section 3.9)).

(ii) The large sample properties of these estimates based on the periodogram of a point process are developed by proving a number of original theorems, which are shown to be valid for a general stationary point process (sections 3.7 and 3.8). The asymptotic distribution of these estimates is also established (section 3.9).

(iii) The previous steps lead us to the construction of asymptotic confidence intervals for each of the different methods of estimating the power spectrum. In addition, it provides us with information about the range of frequencies in which the estimates of the power spectrum are significantly different than the estimated power spectrum of a Poisson point process with the same mean rate (section 3.11).

(iv) By inverting estimates of certain parameters in the frequency domain we obtain estimates of the corresponding time domain parameters. Original theorems based on the properties of the periodogram of a point process are proved in order to establish the asymptotic properties of these estimates (section 3.12).

(v) Finally, it is shown that it may be faster to calculate estimates of the time domain parameters by using the periodogram analysis of a point process, especially if there is a large number of events of the univariate point process in the interval $(0,T]$ (section 3.14). This is achieved when the original idea of computing the periodogram of a point process by using the Fast Fourier transform is employed as in the case of a discrete time series (section 3.8).

The applications of the steps described above are demonstrated by a large number of illustrations obtained by using the four data sets of Chapter 1. All the Fortran programs involved were written by us with the only exception being the use of a subroutine calculating the Fast Fourier transform. Useful results for the Physiological data can be extracted from the interpretation of the different estimates of the power spectrum, while the auto-intensity function does not seem to provide valuable information in our case.

The clear results, which the estimates of the power spectrum provide, lead us to feel that the use of these methods in other physiological problems involving the analysis of point process data are appropriate and helpful.

CHAPTER FOUR

IDENTIFICATION OF A TIME INVARIANT SYSTEM INVOLVING POINT PROCESSES

4.1 INTRODUCTION

In this chapter the bivariate point-process is introduced. The purpose is to examine the properties of the muscle spindle by relating the response of the Ia sensory axon to the presence of gamma stimulation. This is actually a problem which involves the identification of a point-process system. In order to identify such a system we propose a model analogous to the Volterra expansion for Gaussian processes (Wiener, 1958).

The simplest form of this model describes the linear relation between two point processes. The statistical analysis of the parameters of the linear model requires the definition of new quantities in both time and frequency domain. The estimation of these quantities and the construction of asymptotic confidence intervals are discussed. In the final part of this chapter a more complicated model is examined taking into account the non-linear relation between the two processes as well.

We start the analysis of the bivariate point process by defining certain parameters useful in finding how the univariate point processes are related.

4.2 CERTAIN PARAMETERS RELATED TO THE BIVARIATE POINT PROCESS

Let $(N_1(t),\ N_2(t))$ be a bivariate stochastic process on the interval $(0,T]$ with differential increments at time t given by $\left\{ dN_1(t),\ dN_2(t) \right\} = \left\{ N_1(t,t+dt],\ N_2(t,t+dt] \right\}$.

The bivariate point process is a special case of the r vector-valued point process presented in Appendix I.

We assume that the bivariate point process satisfies the following assumptions

(a) It is completely stationary. This means that the joint distribution of the counts in an arbitrary number of intervals in each process is invariant under translation.

(b) It is orderly. This is a condition which prevents multiple events occurring in a small interval of time for each of the components of the bivariate process, and

(c) It is strong mixing. This additional condition implies that events of the bivariate process well-separated in time are independent.

More general assumptions for an r vector-valued point process are given in Appendix I, section I.1.

The assumptions given above are necessary for the introduction of certain parameters of the bivariate point process.

The first parameter of interest is the second-order cross product density denoted by $p_{21}(u)$ and interpreted as

$$\text{Prob}\left\{ N_2 \text{ event in } (t+u, t+u+du] \text{ and } N_1 \text{ event in } (t, t+dt] \right\} .$$

This function has been defined in Chapter 2 by expression (2.11.4). It is clear from (2.11.4) that

$$p_{21}(u) = p_{12}(-u) \quad . \tag{4.2.1}$$

Also, as a consequence of assumption (c) we have

$$\lim_{|u|\to\infty} p_{21}(u) = p_2\, p_1 \;, \tag{4.2.2}$$

where p_1 and p_2 are the mean-intensities of N_1 and N_2 respectively.

From (4.2.1) it follows that

$$q_{21}(u) = q_{12}(-u) \;, \tag{4.2.3}$$

where $q_{12}(u)$ is the second-order cumulant function defined in Chapter 2 by expression (2.11.7), and abbreviated as CCF.

For large values of u, $q_{21}(u)$ will tend to zero, since increments of the two processes become independent.

Another useful function in practice is the cross-intensity function (CIF) defined by

$$m_{21}(u)\,du = E\left\{ dN_2(t+u) \,/\, \text{event of } N_1 \text{ at } t \right\} , \tag{4.2.4}$$

and interpreted as

$$\text{Prob}\left\{ N_2 \text{ event in } (t+u, t+u+du] \,/\, \text{given an } N_1 \text{ event at } t \right\} .$$

We see from (4.2.4) that the CIF is not an even function. Also, it is clear from (4.2.2) that the CIF will fluctuate around p_2 for large u.

We now go on to consider certain frequency-domain parameters of the bivariate point process as the Fourier transforms of the time-domain parameters discussed above.

Suppose that the CCF exists and satisfies the following condition

$$\int_{-\infty}^{\infty} |q_{21}(u)|\, du < \infty \;. \tag{4.2.5}$$

Then the cross-spectrum (CS) of the bivariate process is given by

$$f_{21}(\lambda) = (2\pi)^{-1}\int_{-\infty}^{\infty} q_{21}(u)\, e^{-i\lambda u}\, du \quad , \quad -\infty < \lambda < \infty \quad . \tag{4.2.6}$$

This function may be interpreted as reflecting how a certain frequency of the process N_2 is associated with one in process N_1.

The CS is generally a complex function which has the property

$$f_{21}(\lambda) = \overline{f_{12}(\lambda)} = f_{12}(-\lambda) \quad , \tag{4.2.7}$$

where $\overline{f_{12}(\lambda)}$ is the complex conjugate function of $f_{12}(\lambda)$.

Also, by applying the Riemann-Lebesgue Lemma (Katznelson, 1968), we have

$$\lim_{|\lambda| \to \infty} f_{21}(\lambda) = 0 \quad . \tag{4.2.8}$$

An approximate expression for the cross-spectrum $f_{21}(\lambda)$ is given by

$$f_{21}(\lambda) \simeq \frac{b}{2\pi} \sum_j q_{21}(u_j)\, e^{-i\lambda u_j} \quad , \tag{4.2.9}$$

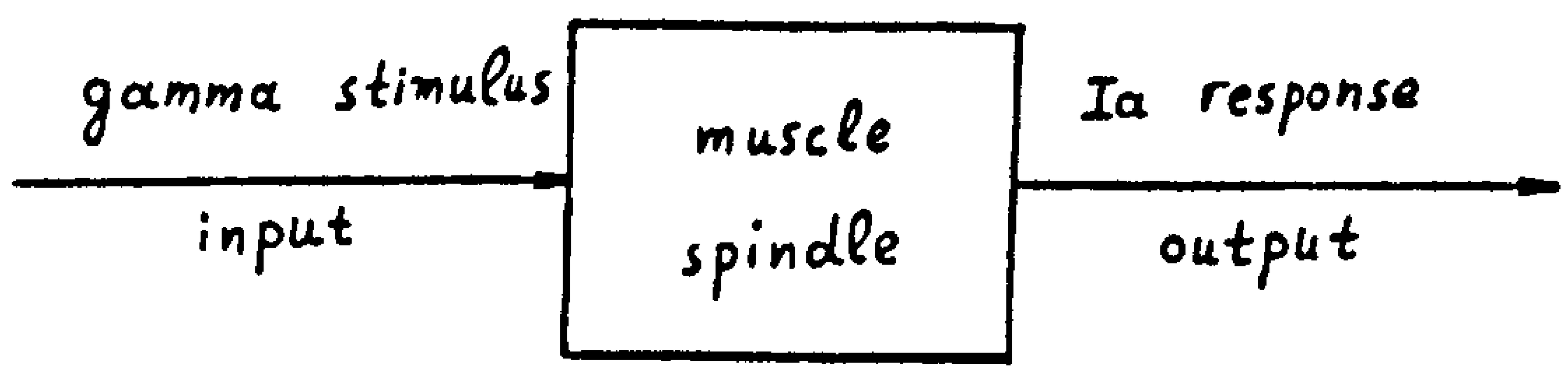
where $u_j = bj$, $j = 0, \pm 1, \pm 2, \ldots$, and b is a scale parameter.

The time and frequency domain parameters discussed above can be found in Cox & Lewis (1972), and Brillinger (1975b, 1976a). Lewis (1972) contains papers which examine the existence of cross product densities and cumulant functions.

4.3 IDENTIFICATION OF A POINT PROCESS SYSTEM

The muscle spindle, i.e. the system under investigation, can be described as a point process system. Specifically, it is assumed that the muscle spindle is a system which gives rise to a point process (response of the Ia sensory axon) when it is influenced by another point process (gamma stimulation). The two point processes involved are called the output and the input of the system respectively.

A simple graphical representation of the system is as follows



We denote the input and output processes by N_1 and N_2. We say that the system is stochastic if it incorporates random features. We also say that the system is time invariant when the bivariate process (N_1, N_2) is stationary.

The problem of the identification of a time invariant point-process system involves the determination of the characteristics of the system from the available information contained in the input and output point processes. In the case of a stochastic system complete identification is not possible and the most we can hope for is to determine average quantities or parameters that characterize the statistical properties of the system.

As we have already mentioned above, for the identification of a point process system it is necessary to consider a model which relates the input and output processes. We start with the following model which describes the linear relation between the two processes N_1 and N_2

$$E\left\{dN_2(t)/N_1\right\} = \left[\, a_0 + \int a_1(t-u)\, dN_1(u)\right] dt \tag{4.3.1}$$

where, by $E\left\{dN_2(t)/N_1\right\}$ we mean approximately the Prob $\left\{ N_2 \text{ event in } (t,t+h] \,/\, N_1 \text{ event at } t \right\}$.
The integral equation (4.3.1) is similar to a model proposed by Hawkes (1972) for investigating mutually exciting point processes. Rice (1973) has introduced a similar kind of model to examine the relation between point processes, and Brillinger (1974a) has extended it to processes with stationary increments.

The function $a_1(\cdot)$ of the linear model (4.3.1) is called the impulse response function and is useful in predicting whether there will be an N_2 event u time units away from an N_1 event. The parameter a_0 is a constant and gives the rate of the process N_2 when N_1 is inactive.

We now turn to the problem of estimating a_0 and $a_1(\cdot)$ in (4.3.1) from the parameters of the bivariate process $\left\{N_1(t),\ N_2(t)\right\}$.

By taking the expected value in (4.3.1) with respect to the process N_1 we have

$$E\left[\, E\left\{dN_2(t)/N_1\right\}\right] = \left[a_0 p_N + \int a_1(t-u)\, E\left\{dN_1(u)\right\}\right] dt$$

which gives, by using the properties of the conditional expectations,

$$p_2 = \alpha_0 + p_1 \int a_1(u)\,du \quad . \tag{4.3.2}$$

Similarly, by multiplying (4.3.1) by $dN_1(t-u)$ and taking the expected value with respect to N_1 we have

$$E\left[E\{dN_2(t)/N_1\}dN_1(t-u)\right] = \left[\alpha_0 E\{dN_1(t-u)\} + \int a_1(t-v)E\{dN_1(v)dN_1(t-u)\}\right]dt \; ,$$

which gives after some calculations

$$q_{21}(u) = p_1 a_1(u) + \int a_1(u-v)q_{11}(v)\,dv \quad . \tag{4.3.3}$$

In the case that N_1 is a Poisson point process we have $q_{11}(u)$ equal to zero and so $a_1(u)$ is given by

$$a_1(u) = q_{21}(u)/p_1 \quad . \tag{4.3.4}$$

For large values of u the function $a_1(u)$ will tend to zero since increments of the bivariate process become independent.

The general equation (4.3.3) is an integral equation with respect to $a_1(\cdot)$. In order to solve this equation we get the Fourier transform of (4.3.3) as follows

$$(2\pi)^{-1}\int q_{21}(u)e^{-i\lambda u}du = (2\pi)^{-1}\int\left[p_1 a_1(u) + \int a_1(u-v)q_{11}(v)dv\right]e^{-i\lambda u}du \; ,$$

and by using Brillinger's arguments (Brillinger, 1975b) we find

$$f_{21}(\lambda) = f_{11}(\lambda)A(\lambda) \quad , \tag{4.3.5}$$

where $f_{21}(\lambda)$ is the cross-spectrum between N_2 and N_1, and $f_{11}(\lambda)$ is the power spectrum of N_1.

The function $A(\lambda)$ is the Fourier transform of $a_1(u)$ defined as

$$A(\lambda) = \int_{-\infty}^{\infty} a_1(u) e^{-i\lambda u} du \quad , \quad -\infty < \lambda < \infty \quad . \tag{4.3.6}$$

We call $A(\lambda)$ the transfer function of the linear system. The equation (4.3.2) can now be written as

$$p_2 = a_0 + p_1 A(0) \quad . \tag{4.3.7}$$

The equations derived above in both time and frequency domain have been given by Brillinger (1975b, 1975c) and Brillinger et al. (1976).

Another way of deriving equations (4.3.2) and (4.3.3) can be achieved by minimising the mean squared error (MSE), given by

$$E\left| dN_2(t) - \left[a_0 + \int a_1(t-u) dN_1(u)\right] dt \right|^2 , \tag{4.3.8}$$

with respect to a_0 and $a_1(\cdot)$.

We now consider the following process with stationary increments

$$d\epsilon(t) = dN_2(t) - \left[a_0 + \int a_1(t-u) dN_1(u)\right] dt \quad . \tag{4.3.9}$$

The process $\epsilon(t)$ is a kind of an error process.
It is clear from (4.3.9) that $E\left\{ d\epsilon(t) \right\} = 0$.
The product density of $d\epsilon(\cdot)$ at t & t' is given by

$$E\left\{ d\epsilon(t) d\epsilon(t') \right\} = \left\{ q_{22}(t-t') + p_2 \delta(t-t') - \int a_1(t-u)\, q_{21}(t'-u) du \right.$$

$$-\int a_1(t'-v)q_{21}(t-v)dv+\iint a_1(t-u)a_1(t'-v)\left[q_{11}(u-v)+p_1\delta(u-v)\right]dudv\Big\}dtdt'. \tag{4.3.10}$$

Alternatively, after some calculations we have

$$p_{\epsilon\epsilon}(t-t')+p_\epsilon\delta(t-t')=\int f_{22}(\lambda)\left[1-|R_{21}(\lambda)|^2\right]e^{i\lambda(t-t')}d\lambda\ , \tag{4.3.11}$$

where $f_{22}(\lambda)$ is the power spectrum of N_2, $p_{\epsilon\epsilon}(\cdot)$ is the product density of ϵ, and p_ϵ is the mean intensity of ϵ.

The quantity $|R_{21}(\lambda)|^2$ is called coherence and defined as

$$|R_{21}(\lambda)|^2=\begin{cases}\dfrac{|f_{21}(\lambda)|^2}{f_{11}(\lambda)f_{22}(\lambda)}\\[2ex] 0 \quad \text{if} \quad f_{11}(\lambda) \text{ or } f_{22}(\lambda)=0\ .\end{cases} \tag{4.3.12}$$

The coherence is a very useful measure of the degree of association between the two processes N_1 and N_2. Equation (4.3.11) can also be written as

$$f_{\epsilon\epsilon}(\lambda)=f_{22}(\lambda)\left[1-|R_{21}(\lambda)|^2\right]\ , \tag{4.3.13}$$

where $f_{\epsilon\epsilon}(\lambda)$ is the power spectrum of process ϵ.

It is obvious from (4.3.13) that $f_{\epsilon\epsilon}(\lambda)$ will be zero when the coherence is one. This further indicates that there is then perfect linear relation between N_1 and N_2.

The problem of estimating quantities defined in the previous sections is now considered. We start with quantities in the time domain.

4.4 ESTIMATES FOR THE TIME-DOMAIN PARAMETERS OF THE BIVARIATE POINT PROCESS

Let $\left\{ N_1(t),\ N_2(t) \right\}$ be a bivariate point process on the interval $(0,T]$ which is stationary, orderly and satisfies a (strong) mixing condition. Let the times of events of the process N_1 be $s_1, s_2, \ldots,$ and the times of events of N_2 be $t_1, t_2, \ldots,$. Let also b be a scale parameter. Then estimates of $p_{21}(u)$ and $m_{21}(u)$ are based on the following variate

$$J_{21}^{(T)}(u) = \#\left\{ u - \frac{b}{2} < t_j - s_k < u + \frac{b}{2}\ ;\ t_j \neq s_k\ ;\ j = 1,\ldots,N_2(T)\ \&\ k = 1,\ldots,N_1(T) \right\}, \qquad (4.4.1)$$

where $N_1(T)$ and $N_2(T)$ are the number of events of the processes N_1 and N_2 respectively. The function $J_{21}^{(T)}(u)$ counts the number of differences, $t_j - s_k$, which fall in a cell of bin width b. Thus $J_{21}^{(T)}(u)$ is a histogram-like statistic considered in Cox & Lewis (1972). By normalizing appropriately this histogram-like statistic we can find estimates of the CPD and CIF given by

$$\hat{p}_{21}(u) = \frac{J_{21}^{(T)}(u)}{bT}, \qquad (4.4.2)$$

$$\hat{m}_{21}(u) = \frac{J_{21}^{(T)}(u)}{b N_1(T)} \qquad (4.4.3)$$

If u is large compared with T, then the unbiased estimate of CPD should be considered since the expression (4.4.2) holds approximately (refer to the alternative definition of $J_{21}^{(T)}(u)$ which is discussed below). The unbiased estimate of $p_{21}(u)$ can be obtained from (4.4.2) by replacing bT with $b(T-|u|)$ in the denominator. Brillinger (1976a) considers the following modified estimate

$$\hat{p}_{21}'(u) = \hat{p}_{21}(u) + |u| \frac{\hat{p}_1 \hat{p}_2}{T} \tag{4.4.4}$$

The estimate (4.4.4) has better overall mean-squared error properties (Parzen, 1961). In our case we only consider estimates of the form (4.4.2) since $\hat{p}_{21}(u)$ and $\hat{p}_{21}'(u)$ have asymptotically the same properties for $|u|$ not large compared to T.

By an extension of Theorem 3.2.1 we find the following asymptotic distributions for $\hat{p}_{21}(u)$ and $\hat{m}_{21}(u)$.

<u>Corollary 4.4.1</u>: The estimate $\hat{p}_{21}(u)$, given by (4.4.2), is asymptotically distributed as $(bT)^{-1}Po(bTp_{21}(u))$ (By Po(a) we mean a Poisson distribution with mean a).

<u>Corollary 4.4.2</u>: The estimate $\hat{m}_{21}(u)$, given by (4.4.3), is asymptotically distributed as $(bT)^{-1}p_1^{-1}Po(bTp_{21}(u))$

The proofs of the Corollaries are given in Brillinger (1976a).

For large bT the estimates $\hat{p}_{21}(u)$ and $\hat{m}_{21}(u)$ will be approximately normally distributed.

An alternative definition for the variable $J_{21}^{(T)}(u)$ is given by

$$J_{21}^{(T)}(u) = \int_{0}^{T-|u|-\frac{b}{2}} \int_{|u|-\frac{b}{2}}^{|u|+\frac{b}{2}} dN_2(t+s)\, dN_1(t) \ , \quad s \neq 0 \ . \tag{4.4.5}$$

For the expected value of $J_{21}^{(T)}(u)$ we have

$$E\left\{ J_{21}^{(T)}(u) \right\} = \int_{0}^{T-|u|-\frac{b}{2}} \int_{|u|-\frac{b}{2}}^{|u|+\frac{b}{2}} p_{21}(s)\, ds\, dt \ , \quad s \neq 0$$

and by expanding $p_{21}(s)$ for values of s in the neighbourhood of u we get

$$E\left\{ J_{21}^{(T)}(u) \right\} \simeq \int_{0}^{T-|u|-\frac{b}{2}} b\, p_{21}(u)\, dt = b\left(T-|u|-\tfrac{b}{2}\right) p_{21}(u) \simeq b\left(T-|u|\right) p_{21}(u) \ , \text{ for small } b \ .$$

Furthermore, for small values of $|u|$ compared to T we have

$$E\left\{ J_{21}^{(T)}(u) \right\} \simeq bT p_{21}(u) \quad . \tag{4.4.6}$$

It is now clear that Corollaries 4.4.1 and 4.4.2 are valid under the assumption (4.4.6).

The estimate (4.4.1) involves the comparison of $N_1(T).N_2(T)$ values. In our situation the number of events $N_1(T)$ and $N_2(T)$ are quite large, which implies that the computation of (4.4.1) takes a long time even for a high speed computer. However, there is an algorithm suggested in Brillinger (1976b) for a direct computation of $J_{21}^{(T)}(u)$ with one pass through the data. This algorithm can be described in our case as follows:

Suppose the data are denoted by (r_j, m_j), $j = 1,2,\ldots,$ $N_1(T)+N_2(T)$, where $r_1 < r_2 < r_3 < \ldots < r_{N_1(T)+N_2(T)}$ and r_j is an s_k if $m_j = 1$, r_j is a t_k if $m_j = 2$. Then for the computation of $J_{21}^{(T)}(u)$ the following steps are required

(1) initialize NP (lh,h) and NN (lh,h) to be 0 for l = 1,2,...

(2) for j = 1,2,... and k = j + 1, j + 2,... if $m_j = 2$, $m_k = 1$ compute $l = u_k - u_j$ and set NP(lh,h) = NP(lh,h) + 1 or if $m_j = 1$, $m_k = 2$ compute $l = u_k - u_j$ and set NN(lh,h) = NN(lh,h) + 1.

Since the statistic $J_{21}^{(T)}(u)$ is not even, the array NP corresponds to positive values of u and the array NN corresponds to negative values of u.

The estimate of the CIF can now be calculated by using relation (4.4.3). The square-root transformation (Proposition 3.2.1) can also be applied in order to calculate approximate confidence limits for the estimate, $\hat{m}_{21}(u)$, of the CIF.

Corollary 4.4.3: The square root of the estimate of the CIF, $\hat{m}_{21}(u)$, is approximately distributed as N ($(m_{21}(u))^{1/2}$, $(4bTp_1)^{-1}$). The variance of $(\hat{m}_{21}(u))^{1/2}$ is stable for all u.

It follows from Corollary 4.4.3 that the approximate 95% confidence limits for $(\hat{m}_{21}(u))^{1/2}$ are

$$(m_{21}(u))^{1/2} \pm (bT\hat{p}_1)^{-1/2}. \qquad (4.4.7)$$

Under the assumption of independence the CIF will be equal to p_2 and the approximate confidence 95% limits given by (4.4.7) become

$$(\hat{p}_2)^{1/2} \pm \{bN_1(T)\}^{-1/2}. \qquad (4.4.8)$$

Fig. 4.4.1(a) gives an estimate of the CIF of the response of the Ia sensory axon in presence of a gamma stimulation (Data Set II). The effect of gamma stimulation is clear when the distance between an Ia spike and a γ_s spike is in the interval 10-30 msecs. The dotted line is the square root of the estimate of the MI of the process N_2 while the solid horizontal lines are the approximate 95% confidence limits.

Fig. 4.4.1(b) is the estimate of the AIF of the N_1 process, i.e. the static gamma stimulation (γ_s). This figure clearly shows that the process N_1 is a Poisson point process.

The estimate of the impulse response function is easily calculated from (4.3.5) by subtracting the estimate $\hat{p}_2$ from the estimate $\hat{m}_{21}(u)$.

We now turn to the frequency domain to calculate estimates of the cross-spectrum by using the estimate of the CCF described in this section. Also, estimates of the transfer function, phase and coherence are discussed and illustrated.

4.5 ESTIMATION OF FREQUENCY DOMAIN PARAMETERS

The cross-spectrum in the case of a bivariate stationary point process is defined by expression (4.2.4) as the Fourier transform of the cross cumulant function $q_{21}(u)$. An estimate of the CS is given by

$$f_{21}^{(T)}(\lambda) = (2\pi)^{-1}\int_{-T}^{T} k_T(u)\hat{q}_{21}(u)e^{-i\lambda u}du$$

$$= (2\pi)^{-1} b \sum_j k_T(u_j)\hat{q}_{21}(u_j)e^{-i\lambda u_j} \quad for\ u_j = bj\ ;\ j = 0, \pm 1, \pm 2, \cdots . \qquad (4.5.1)$$

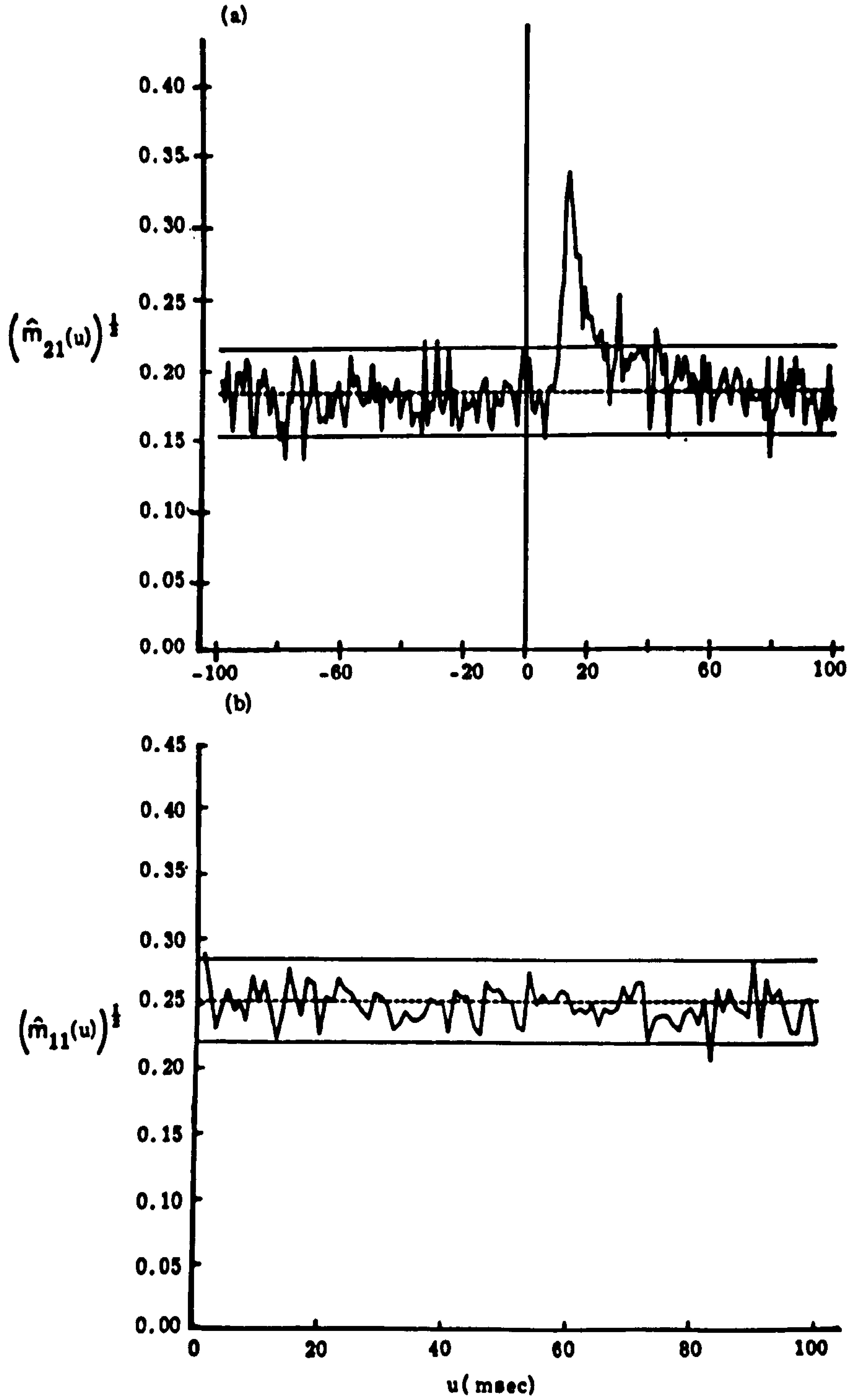


Fig. 4.4.1: Estimates of the time-domain parameters when the muscle spindle is affected by a fusimotor input (γs). (a) Estimate of the square root of the CIF between the Ia discharge and the γs stimulus. The dotted line gives the estimated square root of the mean rate of the Ia discharge. The horizontal lines are the asymptotic 95% confidence limits; (b) Estimate of the square root of the AIF of the fusimotor input. The dotted line in the middle corresponds to the estimated square root of the mean rate of the fusimotor input and the horizontal lines above and below this line are the asymptotic 95% confidence limits. This figure clearly shows that the fusimotor input behaves like a Poisson point process.

where $k_T(u)$ is a convergence factor.

The estimate $f_{21}^{(T)}(\lambda)$ is a complex function since $m_{21}(u) \neq m_{21}(-u)$ and it may be expressed as

$$f_{21}^{(T)}(\lambda) = Re\, f_{21}^{(T)}(\lambda) + i\, Im\, f_{21}^{(T)}(\lambda) \quad , \tag{4.5.2}$$

where Re $f_{21}^{(T)}(\lambda)$ and Im $f_{21}^{(T)}(\lambda)$ are the real and imaginary parts of $f_{21}^{(T)}(\lambda)$. Estimates Re $f_{21}^{(T)}(\lambda)$ and Im $f_{21}^{(T)}(\lambda)$ are calculated when estimates of the CIF $m_{21}(u)$ and the MIs p_1 and p_2 are available.

The transfer function may now be estimated by

$$A^{(T)}(\lambda) = \frac{f_{21}^{(T)}(\lambda)}{f_{11}^{(T)}(\lambda)} \quad , \tag{4.5.3}$$

where $f_{11}^{(T)}(\lambda)$ is the estimate of the power spectrum of the point process N_1.

Two new real-valued functions related to the complex function $A(\lambda)$, the gain and the phase, are defined as follows.

The gain $G(\lambda)$ by

$$G(\lambda) = |A(\lambda)| \quad , \tag{4.5.4}$$

and the phase $\theta(\lambda)$ by

$$\theta(\lambda) = arg\{A(\lambda)\} \quad . \tag{4.5.5}$$

The gain is non-negative and even since

$$G(\lambda) = G(-\lambda) \quad . \tag{4.5.6}$$

The phase is also defined as the argument of the cross-spectrum, that is,

$$\theta(\lambda) = arg\{f_{21}(\lambda)\} \quad . \tag{4.5.7}$$

This result follows from the fact that $f_{11}(\lambda) > 0$.
The phase is an odd function since

$$\theta(\lambda) = -\theta(-\lambda) \quad . \tag{4.5.8}$$

This implies that $\theta(\lambda)$ is zero for $\lambda = 0$.
The functions $G(\lambda)$ and $\theta(\lambda)$ may be estimated by

$$G^{(T)}(\lambda) = |A^{(T)}(\lambda)| = \frac{|f_{21}^{(T)}(\lambda)|}{f_{11}^{(T)}(\lambda)} \tag{4.5.9}$$

and

$$\theta^{(T)}(\lambda) = \arg f_{21}^{(T)}(\lambda) = \tan^{-1}\left\{ Im f_{21}^{(T)}(\lambda) / Re f_{21}^{(T)}(\lambda) \right\} \quad . \tag{4.5.10}$$

Finally, we give an estimate for the coherence by substituting estimates of the cross-spectrum and power spectra of the input and output processes in the expression (4.3.13), i.e.

$$|R_{21}^{(T)}(\lambda)|^2 = \frac{|f_{21}^{(T)}(\lambda)|^2}{f_{11}^{(T)}(\lambda) f_{22}^{(T)}(\lambda)} \quad , \tag{4.5.11}$$

where

$$|f_{21}^{(T)}(\lambda)|^2 = f_{21}^{(T)}(\lambda) \overline{f_{21}^{(T)}(\lambda)} = \left\{ Re f_{21}^{(T)}(\lambda) \right\}^2 + \left\{ Im f_{21}^{(T)}(\lambda) \right\}^2 \quad . \tag{4.5.12}$$

The functions and their estimates considered above can be found in Brillinger (1975a) and Koopmans (1974) for the case of ordinary time series.

We now consider the properties of the estimate $f_{21}^{(T)}(\lambda)$ based on the estimate $\hat{q}_{21}(u)$ of the cross-cumulant function. This estimate, as we shall see in the next section, is equivalent to an estimate of the cross-spectrum which involves the cross-periodogram of the bivariate process. So, it is essential to examine the properties of the cross-periodogram before we go on to refer to the asymptotic properties of $f_{21}^{(T)}(\lambda)$.

4.6 ASYMPTOTIC PROPERTIES OF THE ESTIMATED CROSS-SPECTRUM

The statistic of interest is the estimate of the CS given by (4.5.1). It is possible to express $f_{21}^{(T)}(\lambda)$ in the following equivalent form

$$f_{21}^{(T)}(\lambda) = (2\pi)^{-1} \int_{-T}^{T} k(b_T u)\, \hat{q}_{21}(u)\, e^{-i\lambda u} du$$

$$= (2\pi)^{-1} \int_{-T}^{T} k(b_T u) \left\{ \int_{-\infty}^{\infty} I_{21}^{(T)}(\alpha) e^{i\alpha u} d\alpha \right\} e^{-i\lambda u} du$$

$$= \int_{-\infty}^{\infty} \left\{ \frac{1}{2\pi} \int_{-T}^{T} k(b_T u)\, e^{-i(\lambda-\alpha)u} du \right\} I_{21}^{(T)}(\alpha) d\alpha$$

$$= b_T^{-1} \int_{-\infty}^{\infty} K\left[b_T^{-1}(\lambda-\alpha)\right] I_{21}^{(T)}(\alpha) d\alpha \quad . \qquad (4.6.1)$$

The arguments used for the development of expression (4.6.1) can be found in Parzen (1957, 1961) for the case of ordinary time series. The estimate (4.6.1) is clearly based on the function $I_{21}^{(T)}(a)$.

This function is called the cross periodogram and defined by

$$I_{21}^{(T)}(\alpha) = \frac{1}{2\pi T}\, d_2^{(T)}(\alpha)\, \overline{d_1^{(T)}(\alpha)} \;, \quad -\infty < \alpha < \infty \quad . \qquad (4.6.2)$$

where $d_1^{(T)}(a)$ and $d_2^{(T)}(a)$ are the Fourier-Stieltjes transforms of the processes N_1 and N_2 respectively described in section 3.3 of Chapter 3.

The cross periodogram $I_{21}^{(T)}(a)$ is a complex function whose conjugate function satisfies the relation

$$\overline{I_{21}^{(T)}(a)} = I_{21}^{(T)}(-a) \quad . \tag{4.6.3}$$

The first-, second- and higher-order properties of the cross-periodogram are now examined. We start with the first-order moment of $I_{21}^{(T)}(a)$ presented in the following theorem.

<u>Theorem 4.6.1</u>: Let $\{ N_1(t), N_2(t) \}$ be a stationary bivariate point process on the interval $(0,T]$. Suppose that the second-order cumulant $q_{kl}(u)$, defined for processes with indices k and l, satisfies

$$\int |u||q_{k\ell}(u)|du < \infty \quad for \quad k,\ell = 1,2 \quad .$$

Let $I_{kl}^{(T)}(\lambda)$ be the periodogram for processes with indices k,l. Then

$$E\{ I_{k\ell}^{(T)}(\lambda)\} = \frac{1}{2\pi T}\int_{-\infty}^{+\infty}\left(\frac{\sin(\lambda-\alpha)T/2}{(\lambda-\alpha)/2}\right)^2 f_{k\ell}(\alpha)\,d\alpha + \frac{p_k p_\ell}{2\pi T}\left(\frac{\sin\lambda T/2}{\lambda/2}\right)^2 .$$

Furthermore, $I_{kl}^{(T)}(\lambda)$ is asymptotically an unbiased estimator of $f_{kl}(\lambda)$ as $T \mapsto \infty$.

<u>Proof</u>: It is clear that

$$E\{ I_{k\ell}^{(T)}(\lambda)\} = \frac{1}{2\pi T} E\{ d_k^{(T)}(\lambda)\overline{d_\ell^{(T)}(\lambda)} \} = \frac{1}{2\pi T}\, cum\{ d_k^{(T)}(\lambda), \overline{d_\ell^{(T)}(\lambda)}\} + \frac{1}{2\pi T} E\{ d_k^{(T)}(\lambda)\} E\{\overline{d_\ell^{(T)}(\lambda)}\} ,$$

and the rest of the proof follows in the same way as in Theorem 3.5.1 and Proposition 3.5.1.

The next theorem gives the second-order properties of the $I_{kl}^{(T)}(a)$.

Theorem 4.6.2: Let $\{ N_1(t), N_2(t) \}$ be a stationary bivariate point process on the interval $(0,T]$. Suppose the cumulants up to the fourth order exist and satisfy

$$\int \cdots \int |u_j| \, |q_{a_1 \cdots a_J}(u_1, \ldots, u_{J-1})| \, du_1 \ldots du_{J-1} < \infty \; , \; a_1, \ldots, a_J = 1,2 \; ; \; J = 2,3,4 \; ,$$

and $j = 1, \ldots, J-1$. Let λ, μ, $\lambda \pm \mu \neq 0$ and $I_{kl}^{(T)}(\lambda)$ given by (4.6.2) for $k,l = 1,2$. Then

$$\operatorname{cov}\left\{ I_{k_1 l_1}^{(T)}(\lambda), I_{k_2 l_2}^{(T)}(\mu) \right\} = \frac{2\pi}{T} f_{k_1 l_1 k_2 l_2}(\lambda, -\lambda, -\mu) + \frac{|\Delta^{(T)}(\lambda-\mu)|^2}{T^2} f_{k_1 k_2}(\lambda) f_{l_1 l_2}(-\lambda) + \frac{|\Delta^{(T)}(\lambda+\mu)|^2}{T^2} f_{k_1 l_2}(\lambda) f_{l_1 k_2}(-\lambda) \, ,$$

where

$$\Delta^{(T)}(\lambda) = \int_0^T e^{-i\lambda t} dt \, , \; -\infty < \lambda, \mu < \infty \quad \text{and} \quad k_1, k_2, l_1, l_2 = 1,2 \; .$$

Proof: It follows from the properties of the second-order cumulant that

$$\operatorname{cov}\left\{ I_{k_1 l_1}^{(T)}(\lambda), I_{k_2 l_2}^{(T)}(\mu) \right\} = \operatorname{cum}\left\{ I_{k_1 l_1}^{(T)}(\lambda), \overline{I_{k_2 l_2}^{(T)}(\mu)} \right\} ,$$

and the rest of the proof is completed by applying the same arguments as in Theorem 3.5.2.

The variance of the cross-periodogram $I_{kl}^{(T)}(\lambda)$ follows from Theorem 4.6.2 by setting $\lambda = \mu$ in the formula of the covariance, i.e.

$$Var\left\{ I_{kl}^{(T)}(\lambda) \right\} = f_{kk}(\lambda) f_{ll}(-\lambda) + O(T^{-1}) \quad . \tag{4.6.4}$$

For k = l = 1 we have that

$$Var\left\{ I_{11}^{(T)}(\lambda) \right\} = f_{11}^{2}(\lambda) + O(T^{-1}) \quad ,$$

i.e., the variance of the periodogram of the univariate process N_1 discussed in Chapter 3.

It is now clear that expression (4.6.4) emphasizes two important points

(i) The properties of the periodogram of the univariate point process are special cases of the properties of the cross-periodogram, and

(ii) The variance of the cross-periodogram ($k \neq l$) does not decrease as T becomes larger.

The second point suggests that, as in the case of the periodogram of the univariate point process, the cross-periodogram is not a good estimate of the cross-spectrum. Hence smoothing procedures are needed to improve the properties of this estimate. We refer to these procedures after the next theorem which examines the higher-order properties of the cross-periodogram.

<u>Theorem 4.6.3</u>: Let $\left\{ N_1(t), N_2(t) \right\}$ be a stationary point process on the interval $(0,T]$. Suppose that cumulants up

to the Jth-order of the bivariate point process exist and satisfy

$$\int \cdots \int |u_j| \, |q_{a_1 \cdots a_J}(u_1, \ldots, u_{J-1})| \, du_1 \cdots du_{J-1} < \infty ,$$

for $a_1, \ldots, a_J = 1,2$; $J = 2,3,\ldots$ and $j = 1,\ldots J-1$. Then, if one of $\lambda_{i_1} + \lambda_{i_2} \neq 0$ or one of $\lambda_{i_1} - \lambda_{i_2} \neq 0$ (i_1, $i_2 = 1,\ldots,J$), we have that

$$\operatorname{cum}\left\{ I^{(T)}_{k_1 l_1}(\lambda_1), \ldots, I^{(T)}_{k_J l_J}(\lambda_J) \right\} \longrightarrow 0 \quad \text{for } J > 2 ,$$

where k_1, $l_1, \ldots, k_J$, $l_J = 1,2$.

<u>Proof:</u> The required cumulant takes the following form

$$\operatorname{cum}\left\{ I^{(T)}_{k_1 l_1}(\lambda_1), \ldots, I^{(T)}_{k_J l_J}(\lambda_J) \right\} = (2\pi T)^{-J} \operatorname{cum}\left\{ d^{(T)}_{k_1}(\lambda_1) d^{(T)}_{l_1}(-\lambda_1), \ldots, d^{(T)}_{k_J}(\lambda_J) d^{(T)}_{l_J}(-\lambda_J) \right\}$$

for $J = 2,3,\ldots$.

It follows from Lemma 2.5, Appendix I, that the cumulant of the right hand-side can be expressed as

$$\operatorname{cum}\left\{ d^{(T)}_{k_1}(\lambda_1) d^{(T)}_{l_1}(-\lambda_1), \ldots, d^{(T)}_{k_J}(\lambda_J) d^{(T)}_{l_J}(-\lambda_J) \right\} = \sum_{\nu} \left[(2\pi)^{m_1 - 1} \Delta^{(T)}\left(\sum_{jk \in \nu_1} \mu_{jk}\right) f(\mu'_{jk} ; jk \in \nu_1) \right.$$

$$\left. + O(1) \right] \cdots \left[(2\pi)^{m_p - 1} \Delta^{(T)}\left(\sum_{jk \in \nu_p} \mu_{jk}\right) f(\mu'_{jk} ; jk \in \nu_p) + O(1) \right] , \quad (*)$$

where the summation extends over all indecomposable partitions $\nu = \left\{ \nu_1, \ldots, \nu_p \right\}$ of the following table

$$\begin{array}{cc} (1,1) & (1,2) \\ (2,1) & (2,2) \\ \vdots & \vdots \\ (J,1) & (J,2) \end{array}$$

(ref. Leonov & Shiryaev, 1959).

Also, the following notation has been used in (*)

$$\mu_{\ell 2} = \lambda_\ell \; , \; \mu_{\ell 1} = -\lambda_\ell \; , \; \ell = 1, \dots, T \; .$$

m_i denotes the number of elements in the partition ν_i, $i = 1,\dots,p$, μ'_{jk} means that the last μ is suppressed.

Further, it follows from Brillinger (1969) and the fact that $\Delta^{(T)}(\lambda)/T$ $(\lambda \neq 0)$ tends to zero, as $T \longmapsto \infty$, that the required cumulant will tend to zero for $J > 2$.

We now turn to the problem of finding the asymptotic properties of the estimate $f_{kl}^{(T)}(\lambda)$. This is a weighted estimate of the periodogram $I_{kl}^{(T)}(\lambda)$ in the neighbourhood of λ given by expression (4.6.1) but for general indices $k,l = 1,2$.

<u>Theorem 4.6.4</u>: Let $\{ N_1(t), N_2(t) \}$ be a bivariate stationary point process on $(0,T]$. Suppose that cumulants up to the Jth-order of the bivariate point process exist and satisfy

$$\int \cdots \int |u_j| \, |q_{a_1 \dots a_J}(u_1, \dots, u_{J-1})| \, du_1 \cdots du_{J-1} < \infty \quad ,$$

for $a_1, \dots, a_J = 1,2$; $J = 2,3,\dots$ and $j = 1,\dots,J-1$.
Let $K(a)$ satisfy Assumption 3.9.1 and $f_{kl}^{(T)}(\lambda)$ be the spectrum of the processes N_k and N_l $(k,l = 1,2)$. Then, as $T \longmapsto \infty$, if $b_T \longmapsto 0$ and $b_T T \longmapsto \infty$, we have

$$E\{ f_{k\ell}^{(T)}(\lambda) \} = f_{k\ell}(\lambda) + O(b_T^{-1} T^{-1}) \quad ,$$

$$\lim_{T\to\infty} b_T T \operatorname{cov}\{ f^{(T)}_{k_1 l_1}(\lambda), f^{(T)}_{k_2 l_2}(\mu) \} = 2\pi \int K^2(\alpha)d\alpha \left[\delta\{\lambda-\mu\} f_{k_1 k_2}(\lambda) f_{l_1 l_2}(-\lambda) \right.$$

$$\left. + \delta\{\lambda+\mu\} f_{k_1 l_2}(\lambda) f_{l_1 k_2}(-\lambda) \right]$$

and

$$\operatorname{cum}\{ f^{(T)}_{k_1 l_1}(\lambda_1), \dots, f^{(T)}_{k_J l_J}(\lambda_J) \} = O(b_T^{-J+1} T^{-J+1}) \quad \text{for } k_1, l_1, \dots, k_J, l_J = 1,2 .$$

$$(\delta\{\lambda-\mu\} = 1 \text{ for } \lambda-\mu = 0 \text{ and } = 0 \text{ otherwise}) .$$

<u>Proof</u>: For the expected value of $f^{(T)}_{kl}(\lambda)$ we have

$$E\{ f^{(T)}_{kl}(\lambda) \} = b_T^{-1} \int_{-\infty}^{\infty} K[b_T^{-1}(\lambda-\alpha)] \, E\{ I^{(T)}_{kl}(\alpha) \} d\alpha$$

$$= b_T^{-1} \int_{-\infty}^{\infty} K[b_T^{-1}(\lambda-\alpha)] \{ f_{kl}(\alpha) + O(T^{-1}) \} d\alpha, \text{ by applying Th. 4.6.1}$$

$$= \int_{-\infty}^{\infty} K(\alpha) f_{kl}(\lambda - b_T \alpha) d\alpha + O(b_T^{-1} T^{-1}) ,$$

and as $T \to \infty$

$$E\{ f^{(T)}_{kl}(\lambda) \} \to f_{kl}(\lambda) .$$

Similarly, for the covariance of $f^{(T)}_{k_1 l_1}(\lambda)$ and $f^{(T)}_{k_2 l_2}(\mu)$ we have

$$\operatorname{cov}\{ f^{(T)}_{k_1 l_1}(\lambda), f^{(T)}_{k_2 l_2}(\mu) \} = \operatorname{cum}\{ f^{(T)}_{k_1 l_1}(\lambda), \overline{f^{(T)}_{k_2 l_2}(\mu)} \}$$

$$= b_T^{-2} \iint_{-\infty}^{+\infty} K[b_T^{-1}(\lambda-\alpha_1)] K[b_T^{-1}(\mu-\alpha_2)] \operatorname{cov}\{ I_{k_1 l_1}^{(T)}(\alpha_1), I_{k_2 l_2}^{(T)}(\alpha_2)\} d\alpha_1 d\alpha_2$$

$$= b_T^{-2} \iint_{-\infty}^{+\infty} K[b_T^{-1}(\lambda-\alpha_1)] K[b_T^{-1}(\mu-\alpha_2)] \left\{\left(\frac{\sin(\alpha_1-\alpha_2)T/2}{(\alpha_1-\alpha_2)T/2}\right)^2\right\} f_{k_1 k_2}(\alpha_1) f_{l_1 l_2}(-\alpha_1) d\alpha_1 d\alpha_2$$

$$+ b_T^{-2} \iint_{-\infty}^{+\infty} K[b_T^{-1}(\lambda-\alpha_1)] K[b_T^{-1}(\mu-\alpha_2)] \left\{\left(\frac{\sin(\alpha_1+\alpha_2)T/2}{(\alpha_1+\alpha_2)T/2}\right)^2\right\} f_{k_1 l_2}(\alpha_1) f_{l_1 k_2}(-\alpha_1) d\alpha_1 d\alpha_2$$

$$+ O(b_T^{-2}T^{-2}) + O(T^{-1}) \quad . \quad (*)$$

Now,

$$(b_T T)^{-2} \iint_{-\infty}^{+\infty} K[b_T^{-1}(\lambda-\alpha_1)] K[b_T^{-1}(\mu-\alpha_2)] \left\{\left(\frac{\sin(\alpha_1-\alpha_2)T/2}{(\alpha_1-\alpha_2)/2}\right)^2\right\} f_{k_1 k_2}(\alpha_1) f_{l_1 l_2}(-\alpha_1) d\alpha_1 d\alpha_2$$

by setting $\beta_1 = \frac{\lambda-\alpha_1}{b_T}$ becomes

$$= b_T T^{-2} \iint_{-\infty}^{+\infty} K(\beta_1) K[b_T^{-1}(\mu-\alpha_2)] \left\{\left(\frac{\sin(\lambda-b_T\beta_1-\alpha_2)T/2}{(\lambda-b_T\beta_1-\alpha_2)/2}\right)^2\right\} f_{k_1 k_2}(\lambda-b_T\beta_1) f_{l_1 l_2}(-\lambda+b_T\beta_1) d\alpha_1 d\alpha_2 \;,$$

which gives by substituting $\beta_2 = \lambda - b_T\beta_1 - \alpha_2$

$$= b_T T^{-2} \int_{-\infty}^{+\infty} K(\beta_1) \left\{ \int_{-\infty}^{\infty} K[b_T^{-1}(\mu-\lambda+b_T\beta_1+\beta_2)] \left(\frac{\sin\beta_2 T/2}{\beta_2/2}\right)^2\right\} f_{k_1 k_2}(\lambda-b_T\beta_1) f_{l_1 l_2}(-\lambda+b_T\beta_1) d\beta_2 d\beta_1 \;.$$

A similar expression can be found for the second term in (*). Then, as $T \to \infty$, we have

$$\lim_{T\to\infty} b_T T \operatorname{cov}\{ f_{k_1 l_1}^{(T)}(\lambda), f_{k_2 l_2}^{(T)}(\mu)\} = 2\pi\left[\delta\{\lambda-\mu\} f_{k_1 k_2}(\lambda) f_{l_1 l_2}(-\lambda) + \delta\{\lambda+\mu\} f_{k_1 l_2}(\lambda) f_{l_1 k_2}(-\lambda)\right] \int K^2(\alpha) d\alpha \;,$$

which is the required covariance of $f_{k_1 l_1}^{(T)}(\lambda)$ and $f_{k_2 l_2}^{(T)}(\mu)$.

It can be proved with similar arguments as in the case of the ordinary time series (Brillinger, 1975a; p437) that the cumulant

$$\text{cum}\{f^{(T)}_{k_1 l_1}(\lambda_1),\ldots,f^{(T)}_{k_J l_J}(\lambda_J)\} \text{ is of order } b_T^{-J+1}T^{-J+1}, \text{ that is}$$

$$\text{cum}\{f^{(T)}_{k_1 l_1}(\lambda_1),\ldots,f^{(T)}_{k_J l_J}(\lambda_J)\} = O(b_T^{-J+1}T^{-J+1}) .$$

This completes the proof of the theorem.

<u>Corollary 4.6.4</u>: Under the conditions of Theorem 4.6.3 and if $b_T \longmapsto 0$, $b_T T \longmapsto \infty$ as $T \longmapsto \infty$, then $f^{(T)}_{k_1 l_1}(\lambda_1),\ldots,$ $f^{(T)}_{k_J l_J}(\lambda_J)$ are asymptotically jointly normal.

<u>Proof</u>: The first- and second-order properties of $f^{(T)}_{k_1 l_1}(\lambda_j)$, $j = 1,\ldots,J$, are examined in Theorem 4.6.3. Also, the same theorem suggests that the standardized joint cumulants will be of the following order

$$(b_T T)^{J/2}\text{cum}\{f^{(T)}_{k_1 l_1}(\lambda_1),\ldots,f^{(T)}_{k_J l_J}(\lambda_J)\} = O(b_T^{-J/2+1}T^{-J/2+1}) .$$

Hence, the standardized joint cumulants of order greater than 2 ($J > 2$) tend to zero (as $T \longmapsto \infty$) and the required result follows from Lemma 2.6 given in Appendix I.

The asymptotic first- and second-order properties and the joint probability distribution of the estimates of the cross-spectrum considered by Rosenblatt (1959) in the case of ordinary time series. Parzen (1967) also discussed the asymptotic theory and certain empirical aspects of these estimates in the case of ordinary time series.

The estimate (4.6.1) can also be written as

$$f_{k\ell}^{(T)}(\lambda) = \int_{-\infty}^{\infty} K_T(\lambda-\alpha) I_{k\ell}^{(T)}(\alpha)d\alpha$$

$$\simeq \frac{2\pi}{b_T T} \sum_{s\neq 0} K(b_T^{-1}[\lambda - \tfrac{2\pi s}{T}]) I_{k\ell}^{(T)}(\tfrac{2\pi s}{T}) \quad \text{for } k,\ell = 1,2 \,, \tag{4.6.5}$$

where $K_T(a)$ is the "spectral window" discussed in Chapter 3. In practice this estimate is calculated for $|\lambda| \leq \pi$.

In connection with the properties of the estimate $f_{21}^{(T)}(\lambda)$ of the CS we now discuss the properties of the estimates Re $f_{21}^{(T)}(\lambda)$ and Im $f_{21}^{(T)}(\lambda)$.

These real-valued functions are the estimates of the real and the imaginary parts of the true value of the CS and are useful in estimating the phase.

A discussion of the Re $f_{21}^{(T)}(\lambda)$ and Im $f_{21}^{(T)}(\lambda)$ is given by Jenkins (1965) in the case of ordinary time series.

The next Theorem describes the joint asymptotic distribution of Re $f_{21}^{(T)}(\lambda)$ and Im $f_{21}^{(T)}(\lambda)$.

<u>Theorem 4.6.5</u>: Under the conditions of Theorem 4.6.4 the joint distribution of Re $f_{21}^{(T)}(\lambda)$, Im $f_{21}^{(T)}(\lambda)$ is asymptotically normal with mean Re $f_{21}(\lambda)$, Im $f_{21}(\lambda)$, variances,

$$\frac{2\pi\int K^2(\alpha)d\alpha}{b_T T}\left[1+\delta\{2\lambda\}\right]\left[f_{11}(\lambda)f_{22}(\lambda)+\{\mathrm{Re} f_{21}(\lambda)\}^2-\{\mathrm{Im} f_{21}(\lambda)\}^2\right]/2$$

$$\frac{2\pi\int K^2(\alpha)d\alpha}{b_T T}\left[1-\delta\{2\lambda\}\right]\left[f_{11}(\lambda)f_{22}(\lambda)-\{\mathrm{Re} f_{21}(\lambda)\}^2+\{\mathrm{Im} f_{21}(\lambda)\}^2\right]/2$$

and covariance

$$\frac{2\pi\int K^2(a)da}{b_T T}\left[1-\delta\{2\lambda\}\right]\left[\{Re f_{21}(\lambda)\}\{Im f_{21}(\lambda)\}\right] .$$

Proof: We write Re $f_{21}^{(T)}(\lambda)$ and Im $f_{21}^{(T)}(\lambda)$ as follows

$$Re f_{21}^{(T)}(\lambda) = \frac{f_{21}^{(T)}(\lambda) + f_{21}^{(T)}(-\lambda)}{2}$$

$$Im f_{21}^{(T)}(\lambda) = \frac{f_{21}^{(T)}(\lambda) - f_{21}^{(T)}(-\lambda)}{2i} .$$

It follows from Theorem 4.6.4 that the mean of [Re $f_{21}^{(T)}(\lambda)$, Im $f_{21}^{(T)}(\lambda)$] is asymptotically [Re $f_{21}(\lambda)$, Im $f_{21}(\lambda)$] Also, from Theorem 4.6.4 and the expressions for Re $f_{21}^{(T)}(\lambda)$ and Im $f_{21}^{(T)}(\lambda)$ given above, we have after some computations

$$\lim_{T\to\infty} b_T T \operatorname{cov}\left[Re f_{21}^{(T)}(\lambda), Re f_{21}^{(T)}(\mu)\right]$$

$$= \left[\delta\{\lambda-\mu\} + \delta\{\lambda+\mu\}\right]\left[f_{11}(\lambda)f_{22}(\lambda) + \{Re f_{21}(\lambda)\}^2 - \{Im f_{21}(\lambda)\}^2\right]\pi\int K^2(a)da .$$

So the limit of the variance of Re $f_{21}^{(T)}(\lambda)$ will be given by

$$\lim_{T\to\infty} b_T T \operatorname{Var}\left[Re f_{21}^{(T)}(\lambda)\right] = \left[1+\delta\{2\lambda\}\right]\left[f_{11}(\lambda)f_{22}(\lambda) + \{Re f_{21}(\lambda)\}^2 - \{Im f_{21}(\lambda)\}^2\right]\pi\int K^2(a)da .$$

Similarly for the variance of Im $f_{21}^{(T)}(\lambda)$ we have

$$\lim_{T\to\infty} b_T T \operatorname{Var}\left[Im f_{21}^{(T)}(\lambda)\right] = \left[1-\delta\{2\lambda\}\right]\left[f_{11}(\lambda)f_{22}(\lambda) - \{Re f_{21}(\lambda)\}^2 + \{Im f_{21}(\lambda)\}^2\right]\pi\int K^2(a)da .$$

Finally, for the covariance of Re $f_{21}^{(T)}(\lambda)$ and Im $f_{21}^{(T)}(\lambda)$ we get

$$\lim_{T\to\infty} b_T T \operatorname{cov}\{\operatorname{Re} f_{21}^{(T)}(\lambda), \operatorname{Im} f_{21}^{(T)}(\lambda)\} = -\frac{1}{4i}\left[(1-\delta\{2\lambda\}) f_{21}(\lambda) f_{12}(-\lambda)\right.$$

$$\left. - (1-\delta\{2\lambda\}) f_{12}(\lambda) f_{21}(-\lambda)\right] \; 2\pi \int K^2(\alpha)\,d\alpha$$

$$= 2\pi \int K^2(\alpha)\,d\alpha \left[\{\operatorname{Re} f_{21}(\lambda)\}\{\operatorname{Im} f_{21}(\lambda)\}\right] \quad .$$

The asymptotic normality follows from the fact that $f_{21}^{(T)}(\lambda)$ and $f_{21}^{(T)}(-\lambda)$ are asymptotically normally distributed.
In the last theorem of this section we discuss the asymptotic distribution of the estimate of the modulus of the CS $|f_{21}^{(T)}(\lambda)|$.

<u>Theorem 4.6.6</u>: Under the conditions of Theorem 4.6.4 the distribution of $|f_{21}^{(T)}(\lambda)|$ is asymptotically normal with mean $|f_{21}(\lambda)|$ and variance given by

$$\lim_{T\to\infty} b_T T \operatorname{Var}\{|f_{21}^{(T)}(\lambda)|\} = \pi[1+\delta\{2\lambda\}]\left[f_{11}(\lambda) f_{22}(\lambda) + |f_{21}(\lambda)|^2\right]\int K^2(\alpha)\,d\alpha \quad .$$

<u>Proof</u>: The estimate of $|f_{21}(\lambda)|$ is given by

$$|f_{21}^{(T)}(\lambda)| = \left[\{\operatorname{Re} f_{21}^{(T)}(\lambda)\}^2 + \{\operatorname{Im} f_{21}^{(T)}(\lambda)\}^2\right]^{1/2} \quad .$$

On expanding $|f_{21}^{(T)}(\lambda)|$ in a Taylor Series with respect to Re $f_{21}^{(T)}(\lambda)$ and Im $f_{21}^{(T)}(\lambda)$ in the neighbourhood of Re $f_{21}(\lambda)$ and Im $f_{21}(\lambda)$ we get

$$|f_{21}^{(T)}(\lambda)| = |f_{21}(\lambda)| + \frac{\operatorname{Re} f_{21}(\lambda)}{|f_{21}(\lambda)|}\left[\operatorname{Re} f_{21}^{(T)}(\lambda) - \operatorname{Re} f_{21}(\lambda)\right] + \frac{\operatorname{Im} f_{21}(\lambda)}{|f_{21}(\lambda)|}\left[\operatorname{Im} f_{21}^{(T)}(\lambda) - \operatorname{Im} f_{21}(\lambda)\right] + \cdots .$$

It follows from Theorem 4.6.5 and the above expansion that

$$E(|f_{21}^{(T)}(\lambda)|) = |f_{21}(\lambda)| + O(b_T^{-1}T^{-1}) \; .$$

For the covariance of $|f_{21}^{(T)}(\lambda)|$ and $|f_{21}^{(T)}(\mu)|$ we obtain

$$\operatorname{cov}(|f_{21}^{(T)}(\lambda)|, |f_{21}^{(T)}(\mu)|) \approx \frac{\operatorname{Re} f_{21}(\lambda)\operatorname{Re} f_{21}(\mu)}{|f_{21}(\lambda)||f_{21}(\mu)|} \operatorname{cov}[\operatorname{Re} f_{21}^{(T)}(\lambda), \operatorname{Re} f_{21}^{(T)}(\mu)]$$

$$+ \frac{\operatorname{Im} f_{21}(\lambda)\operatorname{Im} f_{21}(\mu)}{|f_{21}(\lambda)||f_{21}(\mu)|} \operatorname{cov}[\operatorname{Im} f_{21}^{(T)}(\lambda), \operatorname{Im} f_{21}^{(T)}(\mu)] + \frac{\operatorname{Re} f_{21}(\lambda)\operatorname{Im} f_{21}(\mu)}{|f_{21}(\lambda)||f_{21}(\mu)|} \operatorname{cov}[\operatorname{Re} f_{21}^{(T)}(\lambda), \operatorname{Im} f_{21}^{(T)}(\mu)]$$

$$+ \frac{\operatorname{Im} f_{21}(\lambda)\operatorname{Re} f_{21}(\mu)}{|f_{21}(\lambda)||f_{21}(\mu)|} \operatorname{cov}[\operatorname{Im} f_{21}^{(T)}(\lambda), \operatorname{Re} f_{21}^{(T)}(\mu)] \; .$$

Now, it follows from Theorem 4.6.5 that

$$\lim_{T\to\infty} b_T T \operatorname{Var}[|f_{21}^{(T)}(\lambda)|] = \lim_{T\to\infty} b_T T \Big[\frac{\{\operatorname{Re} f_{21}(\lambda)\}^2}{|f_{21}(\lambda)|^2} \operatorname{Var}\{\operatorname{Re} f_{21}^{(T)}(\lambda)\} + \frac{\{\operatorname{Im} f_{21}(\lambda)\}^2}{|f_{21}(\lambda)|^2} \operatorname{Var}\{\operatorname{Im} f_{21}^{(T)}(\lambda)\}\Big]$$

$$+2\, \frac{\operatorname{Re} f_{21}(\lambda)\operatorname{Im} f_{21}(\mu)}{|f_{21}(\lambda)|^2} [1-\delta\{2\lambda\}][\{\operatorname{Re} f_{21}(\mu)\}\{\operatorname{Im} f_{21}(\mu)\}]\, 2\pi \int K^2(a)\,da \; ,$$

which gives after some further calculations

$$\lim_{T\to\infty} b_T T \operatorname{Var}\{|f_{21}^{(T)}(\lambda)|\} = [1+\delta\{2\lambda\}][f_{11}(\lambda) f_{22}(\lambda) + |f_{21}(\lambda)|^2]\, \pi \int K^2(a)\,da \; .$$

The asymptotic normality follows from Theorem 4.6.5 since the joint distribution of $[\operatorname{Re} f_{21}^{(T)}(\lambda), \operatorname{Im} f_{21}^{(T)}(\lambda)]$ is asymptotically normal.

As a consequence of the above Theorem we have the following corollary.

Corollary 4.6.6: Under the conditions of Theorem 4.6.4 the distribution of $\ln|f_{21}^{(T)}(\lambda)|$ is asymptotically normal with mean $\ln|f_{21}(\lambda)|$ and variance given by

$$\lim_{T\to\infty} b_T T \operatorname{Var}\{\ln|f_{21}^{(T)}(\lambda)|\} = [1+\delta\{2\lambda\}][1+|R_{21}(\lambda)|^2]\pi\int K^2(\alpha)d\alpha \;.$$

Proof: The proof follows from Theorem 4.6.6 and by applying Lemma 2.4 of Appendix I.

Corollary 4.6.6 indicates that the distribution of $\log_{10}|f_{21}^{(T)}(\lambda)|$ is normal with variance,

$$\lim_{T\to\infty} b_T T \operatorname{Var}\{\log_{10}|f_{21}^{(T)}(\lambda)|\} = [1+\delta\{2\lambda\}][1+|R_{21}(\lambda)|^2]\pi\log_{10}e\int K^2(\alpha)d\alpha \,.$$

In the case of ordinary time series most of the above results can be found in Brillinger (1975a).

We now turn to the problem of investigating the asymptotic distributions of the estimates of two important functions, the gain and phase.

4.7 ASYMPTOTIC PROPERTIES OF THE ESTIMATES OF GAIN AND PHASE

We start this section by examining the properties of the estimates $A^{(T)}(\lambda)$ of the transfer function $A(\lambda)$.

Theorem 4.7.1: Under the conditions of Theorem 4.6.4 and if $f_{11}(\lambda) \neq 0$ the distribution of $A^{(T)}(\lambda)$ is complex normal with mean A(λ) and variance given by

$$\lim_{T \to \infty} b_T T \, Var\{A^{(T)}(\lambda)\} = f_{22}(\lambda) f_{11}^{-1}(\lambda) [1 - |R_{21}(\lambda)|^2] \, 2\pi \int K^2(\alpha) d\alpha \; .$$

<u>Proof</u>: Using Corollary 3 given in Mann and Wald (1943) we have

$$A^{(T)}(\lambda) = A(\lambda) + \frac{f_{21}^{(T)}(\lambda) - f_{21}(\lambda)}{f_{11}(\lambda)} - \frac{A(\lambda)}{f_{11}(\lambda)} [f_{11}^{(T)}(\lambda) - f_{11}(\lambda)] + O_p(b_T^{-1} T^{-1}) \; .$$

For the expectation of $A^{(T)}(\lambda)$ we get

$$E\{A^{(T)}(\lambda)\} = A(\lambda) + O(b_T^{-1} T^{-1}) \; ,$$

i.e., $A^{(T)}(\lambda)$ is asymptotically an unbiased estimator of A(λ).

The covariance of $A^{(T)}(\lambda)$ and $A^{(T)}(\mu)$ can be written as

$$cov[A^{(T)}(\lambda), A^{(T)}(\mu)] = \frac{1}{f_{11}(\lambda) f_{11}(\mu)} cov[f_{21}^{(T)}(\lambda), f_{21}^{(T)}(\mu)]$$

$$+ \frac{A(\lambda)\overline{A(\mu)}}{f_{11}(\lambda) f_{11}(\mu)} cov[f_{11}^{(T)}(\lambda), f_{11}^{(T)}(\mu)] - \frac{\overline{A(\mu)}}{f_{11}(\lambda) f_{11}(\mu)} cov[f_{21}^{(T)}(\lambda), f_{11}^{(T)}(\lambda)]$$

$$- \frac{A(\lambda)}{f_{11}(\lambda) f_{11}(\mu)} cov[f_{11}^{(T)}(\lambda), f_{21}^{(T)}(\mu)] + O(b_T^{-2} T^{-2}) \; ,$$

and by applying the results of Theorem 4.6.4 we obtain

$$cov[A^{(T)}(\lambda), A^{(T)}(\mu)] = \delta\{\lambda - \mu\} \left[\frac{f_{22}(\lambda)}{f_{11}(\mu)} - \frac{f_{21}(\lambda) f_{21}(-\mu)}{f_{11}^2(\mu)} \right] \frac{2\pi}{b_T T} \int K^2(\alpha) d\alpha + O(b_T^{-2} T^{-2}) \; .$$

Hence, for the variance we get

$$\lim_{T\to\infty} b_T T \,Var\{A^{(T)}(\lambda)\} = f_{22}(\lambda) f_{11}^{-1}(\lambda)[1-|R_{21}(\lambda)|^2]\,2\pi\int K^2(\alpha)d\alpha \quad .$$

The asymptotic normality follows from the fact that $f_{21}^{(T)}(\lambda)$ and $f_{11}^{(T)}(\lambda)$ are asymptotically normal with covariance structure given in Theorem 4.6.4.

As a consequence of this Theorem we have

Corollary 4.7.1: Under the conditions of Theorem 4.6.4 and if $f_{11}(\lambda) \neq 0$ the distribution of $\ln G^{(T)}(\lambda)$ is asymptotically normal with mean $\ln G(\lambda)$ and variance given by

$$\lim_{T\to\infty} b_T T \,Var\{\ln G^{(T)}(\lambda)\} = [\,|R_{21}(\lambda)|^{-2}-1]\,\pi\int K^2(\alpha)d\alpha \quad .$$

Proof: The $\ln |A^{(T)}(\lambda)|$ can be expanded as

$$\ln|A^{(T)}(\lambda)| = \ln|A(\lambda)| + \frac{1}{2}\left[\frac{A^{(T)}(\lambda)-A(\lambda)}{A(\lambda)}\right] + \frac{1}{2}\left[\frac{A^{(T)}(-\lambda)-A(-\lambda)}{A(-\lambda)}\right] + \cdots ,$$

and so we get

$$E\{\ln G^{(T)}(\lambda)\} = \ln G(\lambda) + O(b_T^{-1}T^{-1}) \quad .$$

For the covariance of $\ln G^{(T)}(\lambda)$ and $\ln G^{(T)}(\mu)$ we have

$$cov\{\ln G^{(T)}(\lambda),\ \ln G^{(T)}(\mu)\} \simeq \frac{1}{4}\,\frac{1}{A(\lambda)A(-\mu)}\,cov[A^{(T)}(\lambda), A^{(T)}(\mu)]$$

$$+\frac{1}{4}\,\frac{1}{A(\lambda)A(\mu)}\,cov[A^{(T)}(\lambda), A^{(T)}(\mu)] + \frac{1}{4}\,\frac{1}{A(-\lambda)A(-\mu)}\,cov[A^{(T)}(-\lambda), A^{(T)}(\mu)] + \frac{1}{4}\,\frac{1}{A(-\lambda)A(\mu)}\,cov[A^{(T)}(-\lambda), A^{(T)}(-\mu)] ,$$

and by applying Theorem 4.6.7 we further have

$$\lim_{T\to\infty} b_T T \, Var\{\ln G^{(T)}(\lambda)\} = \frac{1}{|A(\lambda)|^2}\left[\frac{f_{22}(\lambda)}{f_{11}(\lambda)} - \frac{|f_{21}(\lambda)|^2}{f_{11}^2(\lambda)}\right]\pi\int K^2(a)\,da$$

$$= \left[|R_{21}(\lambda)|^{-2} - 1\right]\pi\int K^2(a)\,da \quad .$$

The asymptotic normality of this estimate follows from Theorem 4.7.1.

Corollary 4.7.1 suggests that the variable $\log_{10} G^{(T)}(\lambda)$ is asymptotically normal with mean $\log_{10} G(\lambda)$ and variance

$$Var\{\log_{10} G^{(T)}(\lambda)\} \simeq \frac{\pi \log_{10} e}{b_T T}\int K^2(a)\,da\left[|R_{21}(\lambda)|^{-2} - 1\right] \quad .$$

The variance of $\log_{10} G^{(T)}(\lambda)$ depends on the coherence $|R_{21}(\lambda)|^2$ in such a way that it becomes zero when the coherence is one.

Fig. 4.7.1(a) gives the logarithm to the base 10 of the estimate $G^{(T)}(\lambda)$. The estimated gain is large for small frequencies and decreases almost linearly for higher frequencies. Fig. 4.7.1(b) is the logarithm to the base 10 of the estimate of the power spectrum of the input process. This figure clearly shows that the PS of the input process is a constant and does not differ significantly from the PS of a Poisson process with the same mean rate.

We turn now to the phase $\theta(\lambda)$ which is defined as the argument of the transfer function $A(\lambda)$. An estimate

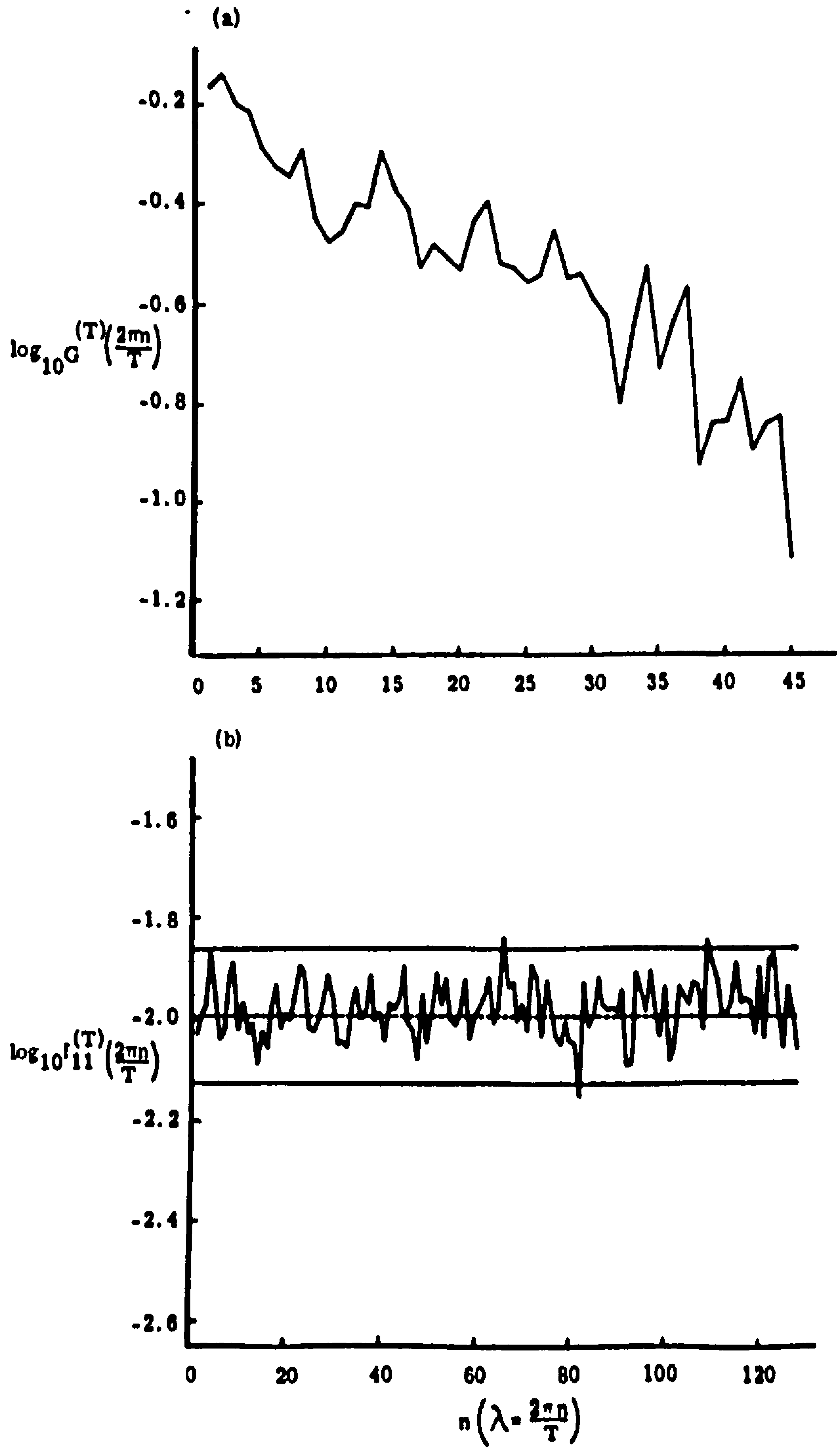


Fig. 4.7.1: Estimates of frequency-domain parameters calculated by using the estimates of the time-domain parameters when the muscle spindle is affected by a fusimotor input (γs). (a) Log to base 10 of the estimate of the gain; (b) Log to base 10 of the estimate of the power spectrum of the fusimotor input. The dotted line in the middle corresponds to the estimated power spectrum of a Poisson point process with the same mean rate, and the two horizontal lines above and below this line are the asymptotic 95% confidence limits. This figure also indicates that the fusimotor input behaves like a Poisson point process. The value of T = 512.

of the phase is given by relation (4.5.10). The asymptotic distribution of this estimate is developed in the next theorem.

<u>Theorem 4.7.2</u>: Under the conditions of Theorem 4.6.4 the estimate $\theta^{(T)}(\lambda)$ is asymptotically normally distributed with mean $\Theta(\lambda)$ and variance given by

$$\lim_{T\to\infty} b_T T \operatorname{Var}\{\theta^{(T)}(\lambda)\} = \pi\left[1-\delta\{2\lambda\}\right]\left[f_{11}(\lambda)f_{22}(\lambda)-|f_{21}(\lambda)|^2\right]|f_{21}(\lambda)|^{-2}\int K^2(a)da$$

$$= \pi\left[1-\delta\{2\lambda\}\right]\left[1-|R_{21}(\lambda)|^2\right]|R_{21}(\lambda)|^{-2}\int K^2(a)da$$

$$= \pi\left[1-\delta\{2\lambda\}\right]\left[|R_{21}(\lambda)|^{-2}-1\right]\int K^2(a)da \ .$$

<u>Proof</u>: The estimate of the phase can be written as

$$\theta^{(T)}(\lambda) = \arg f_{21}^{(T)}(\lambda) \ .$$

By setting $e+\varepsilon = f_{21}^{(T)}(\lambda)$ and $e = f_{21}(\lambda)$ we express $\arg\{e+\varepsilon\}$ in the following way

$$\arg\{e+\varepsilon\} = \arg\left\{e\left(1+\frac{\varepsilon}{e}\right)\right\} = \arg e + \arg\left\{\left(1+\frac{\varepsilon}{e}\right)\right\}$$

$$= \arg e + \frac{1}{2i}\left(\frac{\varepsilon}{e} - \frac{\bar{\varepsilon}}{\bar{e}}\right) - \frac{1}{4i}\left(\frac{\varepsilon^2}{e^2} - \frac{\bar{\varepsilon}^2}{\bar{e}^2}\right) + \cdots \ .$$

The last part of the above expression follows from the fact that

$$\arg\{(1+\frac{\varepsilon}{e})\} = \frac{1}{2i}\left[\log(1+\frac{\varepsilon}{e}) - \log(1+\frac{\bar{\varepsilon}}{e})\right]$$

$$= \frac{1}{2i}\left[\frac{\varepsilon}{e} - \frac{\bar{\varepsilon}}{e}\right] - \frac{1}{4i}\left[\frac{\varepsilon^2}{e^2} - \frac{\bar{\varepsilon}^2}{e^2}\right] + \cdots \quad .$$

Then we have for arg $f_{21}^{(T)}(\lambda)$

$$\arg f_{21}^{(T)}(\lambda) = \arg f_{21}(\lambda) + \frac{1}{2i}\,\frac{f_{21}^{(T)}(\lambda) - f_{21}(\lambda)}{f_{21}(\lambda)} - \frac{1}{2i}\,\frac{f_{21}^{(T)}(-\lambda) - f_{21}(-\lambda)}{f_{21}(-\lambda)} + \cdots \quad ,$$

and further, by using this expansion, we are able to calculate the asymptotic first-, second- and higher-order cumulants of the estimate $\theta^{(T)}(\lambda)$.

For the expected value of $\theta^{(T)}(\lambda)$ we have

$$E\{\theta^{(T)}(\lambda)\} = E\{\arg f_{21}^{(T)}(\lambda)\} = \theta(\lambda) + O(b_T^{-1}T^{-1}) \quad ,$$

since

$$E\{f_{21}^{(T)}(\lambda)\} = f_{21}(\lambda) + O(b_T^{-1}T^{-1}) \quad .$$

Also, for the covariance of $\theta^{(T)}(\lambda)$ and $\theta^{(T)}(\mu)$ we have

$$\operatorname{cov}[\theta^{(T)}(\lambda), \theta^{(T)}(\mu)] = \operatorname{cov}[\arg f_{21}^{(T)}(\lambda), \arg f_{21}^{(T)}(\mu)]$$

$$\simeq \frac{1}{4}\,\frac{1}{f_{21}(\lambda)f_{21}(-\mu)}\operatorname{cov}\{f_{21}^{(T)}(\lambda), f_{21}^{(T)}(\mu)\} - \frac{1}{4}\,\frac{1}{f_{21}(\lambda)f_{21}(\mu)}\operatorname{cov}\{f_{21}^{(T)}(\lambda), f_{21}^{(T)}(-\mu)\}$$

$$-\frac{1}{4}\,\frac{1}{f_{21}(-\lambda)f_{21}(-\mu)}\operatorname{cov}\{f_{21}^{(T)}(-\lambda), f_{21}^{(T)}(\mu)\} + \frac{1}{4}\,\frac{1}{f_{21}(-\lambda)f_{21}(\mu)}\operatorname{cov}\{f_{21}^{(T)}(-\lambda), f_{21}^{(T)}(-\mu)\} \quad .$$

This relation follows from the properties of the covariance for complex variables.
Further, by using Theorem 4.6.4 we get

$$\lim_{T\to\infty} b_T T \, Var\{\theta^{(T)}(\lambda)\} = \frac{1}{4}\frac{1}{|f_{21}(\lambda)|^2}\left[f_{11}(\lambda)f_{22}(\lambda) + \delta\{2\lambda\} f_{21}(\lambda)f_{12}(-\lambda)\right] 2\pi \int K^2(\alpha)d\alpha$$

$$-\frac{1}{4}\frac{1}{f_{21}(\lambda)f_{21}(\lambda)}\left[\delta\{2\lambda\} f_{11}(\lambda)f_{22}(\lambda) + f_{21}(\lambda)f_{12}(-\lambda)\right] 2\pi\int K^2(\alpha)d\alpha$$

$$-\frac{1}{4}\frac{1}{f_{21}(-\lambda)f_{21}(-\lambda)}\left[f_{12}(\lambda)f_{21}(-\lambda) + \delta\{2\lambda\} f_{11}(\lambda)f_{22}(\lambda)\right] 2\pi\int K^2(\alpha)d\alpha$$

$$+\frac{1}{4}\frac{1}{|f_{21}(\lambda)|^2}\left[\delta\{2\lambda\} f_{12}(\lambda)f_{21}(-\lambda) + f_{11}(\lambda)f_{22}(\lambda)\right] 2\pi\int K^2(\alpha)d\alpha$$

$$= \left\{\frac{1}{|f_{21}(\lambda)|^2}\left[1-\delta\{2\lambda\}\right] f_{11}(\lambda)f_{22}(\lambda) - \left[1-\delta\{2\lambda\}\right]\right\} \pi\int K^2(\alpha)d\alpha \;,$$

since $\delta\{2\lambda\}$ is zero everywhere except for $\lambda = 0$

$$= \left[1-\delta\{2\lambda\}\right]\left[f_{11}(\lambda)f_{22}(\lambda) - |f_{21}(\lambda)|^2\right] |f_{21}(\lambda)|^{-2} \pi\int K^2(\alpha)d\alpha$$

$$= \left[1-\delta\{2\lambda\}\right]\left[1-|R_{21}(\lambda)|^2\right] |R_{21}(\lambda)|^{-2} \pi\int K^2(\alpha)d\alpha$$

$$= \left[1-\delta\{2\lambda\}\right]\left[|R_{21}(\lambda)|^{-2} - 1\right] \pi\int K^2(\alpha)d\alpha \quad .$$

The asymptotic normality follows from the fact that

$f_{21}^{(T)}(\lambda_1), \ldots, f_{21}^{(T)}(\lambda_J)$, $J = 2,3\ldots$ are asymptotically normal proved in Theorem 4.6.4.

Parzen (1964), Jenkins (1963b) and Jenkins and Watt (1968) have examined the properties of the estimates described above in the case of ordinary time series. Brillinger (1974a) extends some of these results to the case of Stochastic process with stationary increments.

Figs. 4.7.2 (a-c) give the estimates of the real and imaginary parts of the cross-spectrum as well as the estimate of the phase.

The unconstrained estimate of the phase (Chatfield (1980), p181) is plotted against the value n ($\lambda = \frac{2\pi n}{T}$, $n = 0,1,2,\ldots, 45$ and $T = 512$).

It is clear that the estimated phase can be approximated by a straight line passing through the origin. This straight line indicates a delay between the two point processes equal to the slope of the line. Mathematically we express the delay as the ratio given by

$$t(\lambda) = \theta(\lambda)/\lambda \quad , \tag{4.7.1}$$

and is interpreted as the time delay between the harmonic of the process N_2 at frequency λ over that of N_1 (Koopmans (1974), p138).

An estimate of $t(\lambda)$ is found by using expression (4.7.1) to be approximately 15 msec. This estimated value of $t(\lambda)$ is also in agreement with the one given by the estimate of the CIF (ref. Fig. 4.4.1(a)). The last function in the frequency domain which we have already mentioned in section 4.5 and which is of great importance is the coherence.

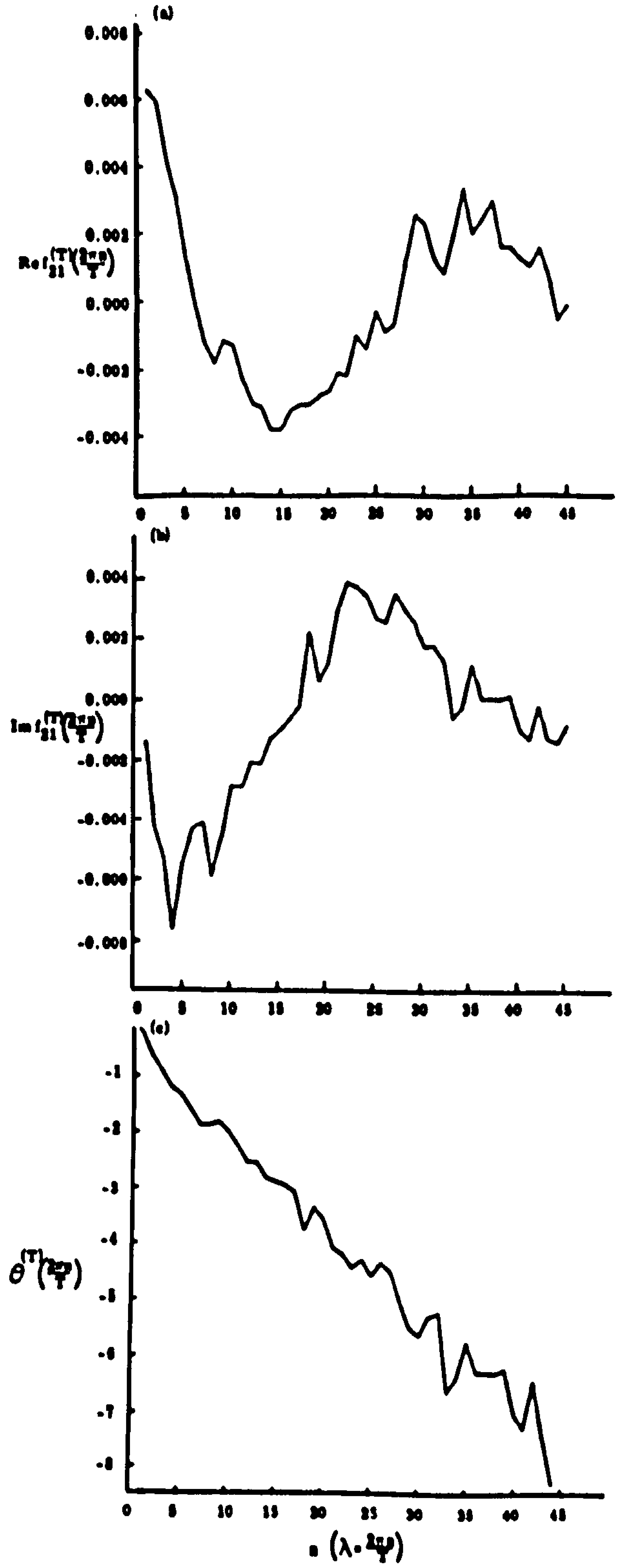


Fig. 4.7.2: Estimates of frequency-domain parameters obtained from the estimate of the CIF in the case when the muscle spindle is affected by a fusimotor input (γs). (a) Estimate of the real part of the cross-spectrum; (b) Estimate of the imaginary part of the cross-spectrum and (c) Estimate of the phase. The value of T = 512.

The coherence provides a measure of the degree of the relationship between the processes N_1 and N_2 and also gives the range of frequencies in which the process N_1 has an effect on the process N_2.

An estimate of the coherence is given by (4.5.11). The calculation of this estimate requires estimates of the cross-spectrum between N_1 and N_2 and the power-spectra of the component processes N_1 and N_2. The density function of the estimate of the coherence, $|R_{21}^{(T)}(\lambda)|^2$, at each frequency $\lambda \neq 0$ is the same as the density function of the ordinary correlation coefficient of two normally distributed random variables. This follows from a general result given by Goodman (1963) which states that the density function of the estimate of the multiple correlation coefficient of the variable Y and the r vector-valued variable $\underset{\sim}{X}$ has the following form

$$(1-|R_{Y\underset{\sim}{X}}|^2)^n {}_2F_1(s,s;r;|R_{Y\underset{\sim}{X}}|^2|\hat{R}_{Y\underset{\sim}{X}}|^2)\frac{\Gamma(s)}{\Gamma(s-r)\Gamma(r)}|\hat{R}_{Y\underset{\sim}{X}}|^{2r-2}(1-|\hat{R}_{Y\underset{\sim}{X}}|^2)^{s-r-1}, \tag{4.7.2}$$

where ${}_2F_1(\cdot)$ is a generalized hypergeometric function (Ref. Abramowitz and Stegun (1964)) and $|\hat{R}_{Y\underset{\sim}{X}}|^2$ is the estimate of the multiple Correlation coefficient $|R_{Y\underset{\sim}{X}}|^2$.

Applying the above result to the case where we have a point process M and an r vector-valued point process $\underset{\sim}{N}$ we see that expression (4.7.2) is now the density function of the estimate of the multiple coherence $|R_{M\underset{\sim}{N}}^{(T)}(\lambda)|^2$ at frequency λ.

In the case $|R_{M\underset{\sim}{N}}(\lambda)|^2 = 0$, expression (4.7.2), for the corresponding estimate of the multiple coherence, becomes

$$\frac{\Gamma(s)}{\Gamma(s-r)\Gamma(r)}\,|R_{MN}^{(T)}(\lambda)|^{2r-2}(1-|R_{MN}^{(T)}(\lambda)|^2)^{s-r-1} \quad for \ \lambda\neq 0 \ , \tag{4.7.3}$$

where $s = \frac{b_T T}{2\pi\int K^2(\alpha)d\alpha} = \frac{b_T T}{\int K^2(u)du}$. (4.7.4)

Expression (4.7.3) shows that the estimate $|R_{MN}^{(T)}(\lambda)|^2$ has a Beta density function with parameter r and s-r under the null hypothesis that the coherence is zero.

In order to be able to test the null hypothesis we need to compute a 100α per cent point of $|R_{MN}^{(T)}(\lambda)|^2$. This can be done by using the following formula given in Abramowitz and Stegun (1964)

$$Prob\{|R_{MN}^{(T)}(\lambda)|^2 < z\} = 1-(1-z)^{n-1}\sum_{j=0}^{r-1}\binom{n-1}{j} z^j(1-z)^{-j} \quad for \ 0<z<1 \ . \tag{4.7.5}$$

For the particular case that $M = N_2$ and $N = N_1$ we get from (4.7.5) that

$$z = 1-(1-\alpha)^{1/s-1}. \tag{4.7.6}$$

Expression (4.7.6) gives the 100α per cent point of $|R_{21}^{(T)}(\lambda)|^2$.

A second way to test the hypothesis $|R_{MN}(\lambda)|^2 = 0$ against the alternative $|R_{MN}(\lambda)|^2 > 0$ is given by the fact that the statistic

$$\frac{s-r}{r}\left(\frac{|R_{MN}^{(T)}(\lambda)|^2}{1-|R_{MN}^{(T)}(\lambda)|^2}\right) = F_{2r,2(s-r)} \ . \tag{4.7.7}$$

has an F-distribution with 2r and 2(s-r) degrees of freedom for $\lambda \neq 0$.

We reject the null hypothesis if

$$|R_{M\underset{\sim}{N}}^{(T)}(\lambda)|^2 \geq rc(\alpha)/[s+r\{c(\alpha)-1\}] \ , \tag{4.7.8}$$

where $c(\alpha) = F_{2r,\ 2(s-r)}^{(\alpha)}$ is the upper α per cent point of the F-distribution. A similar test in the case of ordinary time series can be found in Koopmans (1974), p289.

For the particular case $M = N_2$ and $\underset{\sim}{N} = N_1$ we reject the null hypothesis if

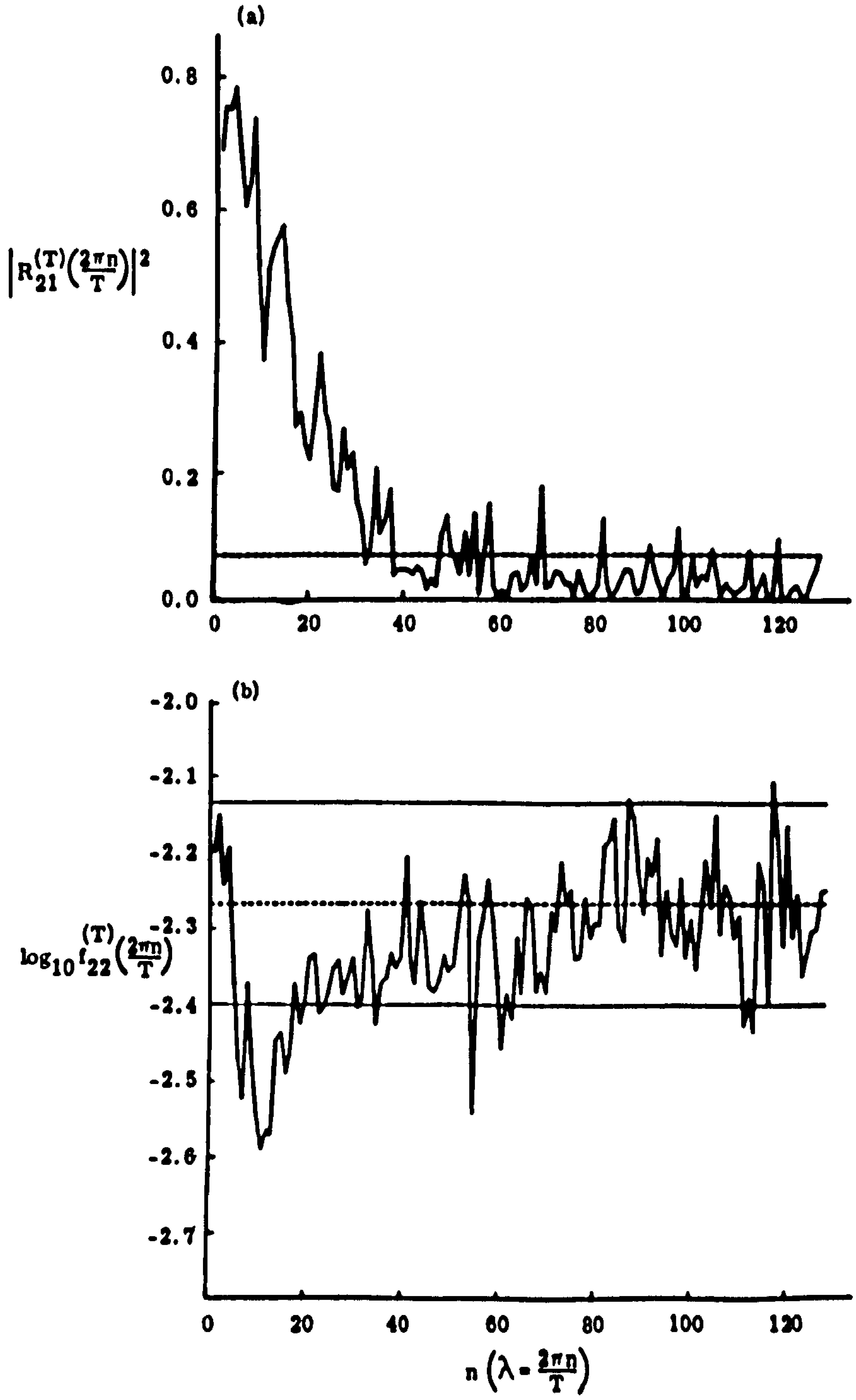
$$|R_{21}^{(T)}(\lambda)|^2 \geq \frac{c(\alpha)}{s+c(\alpha)-1} \ . \tag{4.7.9}$$

Fig. 4.7.3 (a) gives the estimate of the coherence. The dotted line is the 95% point of $|R_{21}^{(T)}(\lambda)|^2$ calculated from (4.7.6) to be 0.072. The value of s is approximately equal to 41 for a Tukey window, T = 15872 and $b_T = 1/512$

We get the same confidence limit for $|R_{21}^{(T)}(\lambda)|^2$ by using expression (4.7.9), in which α is the upper 5% point of the F-distribution and s = 41.

Fig 4.7.3 (b) is the logarithm to base 10 of the PS of the process N_2. The spectrum of the output process N_2 is involved in the calculation of the coherence.

The estimate of the coherence suggests that the gain and phase are significant for frequencies up to about 60 pulses/sec but should be neglected for values higher than that value. It has been shown in Rosenberg, et al (1982) that the estimates of gain and phase are consistent with the transfer function of the form



<u>Fig. 4.7.3</u>: Estimates of frequency-domain parameters obtained from the estimate of the CIF and the estimates of the AIFs when the muscle spindle is affected by a fusimotor input (γs). (a) Estimate of the coherence. The dotted line gives the 95% point of the null distribution of this estimate; (b) Log to base 10 of the Ia discharge in presence of a γs stimulus. The dotted line in the middle corresponds to the estimated power spectrum of a Poisson point process with the same mean rate, and the horizontal lines above and below this line are the asymptotic 95% confidence limits. The value of T = 512.

$$A(\lambda) = k \frac{e^{-\lambda\tau}}{1+\lambda T} \quad , \qquad (4.7.10)$$

for frequencies in the range 0 - 60 pulses/sec.

The estimated values of k, τ and T are 0.73, 0.006 and 0.008 respectively. The pure delay time of 0.006 sec is consistent with the estimates of the delay due to the known conduction velocity of the sensory and fusimotor nerve fibres. The form of the transfer function is in agreement with the transfer function found by Andersson et al (1968) using sinusoidal modulation of the fusimotor inputs. The technique based upon the frequency domain approach, described above, provides an alternative way of the transfer function of the linear system which is simpler and faster to use since in the conventional approach we apply each sinusoidal frequency separately.

4.8 ALTERNATIVE ESTIMATES OF THE FREQUENCY-DOMAIN PARAMETERS

Let $\underset{\sim}{N}(t) = (N_1(t), N_2(t))$ be a stationary bivariate point process on (0,T] with differential increments $\{dN_1(t), dN_2(t)\}$. Suppose that $\underset{\sim}{N}(t)$ is orderly and satisfies a (strong) mixing condition. Then we define the following quantities

$$E\{d\underset{\sim}{N}(t)\} = p_{\underset{\sim}{N}}\, dt \qquad (4.8.1)$$

$$\mathrm{cov}\{d\underset{\sim}{N}(t+u), d\underset{\sim}{N}(t)\} = dC_{\underset{\sim}{N}\underset{\sim}{N}}(u)dt \qquad (4.8.2)$$

Let also the cumulant matrix be given by

$$q_{\underset{\sim}{N}\underset{\sim}{N}}(u) = \left[q_{k\ell}(u)\right] = \begin{bmatrix} q_{11}(u) & q_{12}(u) \\ q_{21}(u) & q_{22}(u) \end{bmatrix} , \tag{4.8.3}$$

with entries $q_{kl}(u)$ satisfying the condition

$$\int_{-\infty}^{+\infty} |q_{k\ell}(u)|du < \infty \quad for \quad k,\ell = 1,2. \tag{4.8.4}$$

The spectral density matrix of the process $\underset{\sim}{N}$ is now defined as

$$f_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{-i\lambda u} dC_{\underset{\sim}{N}\underset{\sim}{N}}(u) , \quad -\infty < \lambda < \infty . \tag{4.8.5}$$

The matrix $f_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)$ may also be written as

$$f_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda) = \left[f_{k\ell}(\lambda)\right] \quad for \quad k,\ell = 1,2 , \tag{4.8.6}$$

where the entry $f_{kl}(\lambda)$ is seen to be the power spectrum of the process N_k if $k = \ell$ and to be the cross-spectrum of the process N_k and the process N_l if $k \neq \ell$.

The fact that the entries of the matrix $dC_{\underset{\sim}{N}\underset{\sim}{N}}(u)$ are real-valued implies the following property

$$\overline{f_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)} = f_{\underset{\sim}{N}\underset{\sim}{N}}(-\lambda) = f_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)^{T} , \tag{4.8.7}$$

where $f_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)^{T}$ is the transpose matrix of $f_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)$.

It follows from (4.8.7) that $f_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)$ is Hermitian.

We consider the following statistic

$$I^{(T)}_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda) = \left[I^{(T)}_{k\ell}(\lambda)\right] = \left[\left\{2\pi H_2^{(T)}(0)\right\}^{-1} d_k^{(T)}(\lambda)\overline{d_\ell^{(T)}(\lambda)}\right] , \tag{4.8.8}$$

as an estimate of the spectral density $f_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)$.

The function $d_{\underset{\sim}{N}}^{(T)}(\lambda)$ defined by

$$d_{\underset{\sim}{N}}^{(T)}(\lambda) = \left[d_k^{(T)}(\lambda) \right] = \left[\int_t k_T(t)\, e^{-i\lambda t} dN_k(t) \right], \quad -\infty < \lambda < \infty , \tag{4.8.9}$$

is the finite Fourier transform of the process $\underset{\sim}{N}$.
Also, the function $H_2^{(T)}(0)$ is related to the window function $k_T(u)$ through the relation

$$H_2^{(T)}(0) = \int k_T^2(u)\, du \quad . \tag{4.8.10}$$

The asymptotic distribution of the statistic $d_{\underset{\sim}{N}}^{(T)}(\lambda)$ is as follows

<u>Theorem 4.8.1</u>: Let $\underset{\sim}{N}(t)$ be a stationary point process on $(0,T]$ with increments $\left\{ dN_1(t), dN_2(t) \right\}$. Let $p_{\underset{\sim}{N}}$ and $q_{\underset{\sim}{N}\underset{\sim}{N}}(u)$ be the MI and the CF of the process $\underset{\sim}{N}$. Suppose that the entries $q_{kl}(u)$ satisfy the condition (4.8.4) for $k,l = 1,2$. Then $d_{\underset{\sim}{N}}^{(T)}(\lambda)$ is asymptotically $N_2^C(\underset{\sim}{0}, 2\pi T [H_2(0) f_{kl}(\lambda)])$, $\lambda \neq 0$. Also, $d_{\underset{\sim}{N}}^{(T)}(0) = \int_t k_T(u) d\underset{\sim}{N}(t)$ is asymptotically $N_2(T[H_1(0)p_k], 2\pi T [H_2(0) f_{kl}(0)])$.

<u>Proof</u>: The proof follows by applying the same arguments as those in Theorem 3.8.1.

<u>Theorem 4.8.2</u>: Let $\underset{\sim}{N}(t)$ be a stationary bivariate point process on $(0,T]$ satisfying the conditions of Theorem 4.8.1. Let also $k(u)$, $-\infty < u < +\infty$, satisfy Assumption 3.8.1 and $I_{\underset{\sim}{N}\underset{\sim}{N}}^{(T)}(\lambda)$ be given by (4.8.8). Then

$$\lim_{T \to \infty} E\left\{ I_{\underset{\sim}{N}\underset{\sim}{N}}^{(T)}(\lambda) \right\} = f_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)$$

and

$$\lim_{T\to\infty} \operatorname{cov}\left\{ I^{(T)}_{k_1\ell_1}(\lambda),\, I^{(T)}_{k_2\ell_2}(\mu)\right\} = \delta\{\lambda-\mu\} f_{k_1k_2}(\lambda) f_{\ell_1\ell_2}(-\lambda)$$

$$+\delta\{\lambda+\mu\} f_{k_1\ell_2}(\lambda) f_{\ell_1 k_2}(-\lambda) \quad .$$

Proof: The proof follows in the same way as that of Theorem 3.8.2.

In the following theorem we indicate the asymptotic distribution of the matrix of the second-order periodograms.

Theorem 4.8.3: Let $\underset{\sim}{N}(t)$ be a stationary bivariate point process satisfying the conditions of Theorem 4.8.1. Let $k(u)$ satisfy Assumption 3.8.1 and $I^{(T)}_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)$ be given by (4.8.8). Suppose $2\lambda j$, $\lambda j \pm \lambda k \neq 0$ for $1 \leqslant j < K \leqslant J$. Then $I^{(T)}_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda j)$, $j = 1,\ldots,J$ are asymptotically independent $W_2{}^C(1,\ f_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda j))$ variates ($W_2{}^C$ denotes Complex Wishart).

Proof: The proof follows from the fact that the variates $d^{(T)}_{\underset{\sim}{N}}(\lambda j)$, $j = 1,\ldots,J$ are asymptotically independent $N_2{}^C$ $(\underset{\sim}{0},\ 2\pi T[H_2(0) f_{kl}(\lambda)])$.

Two important conclusions can be extracted from above:

(i) The limiting distribution of $I^{(T)}_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)$ is a Wishart distribution with 1 degree of freedom which implies that it is well spread out about $f_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)$. This further indicates that $I^{(T)}_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)$ is not a good estimate, and

(ii) In the limit the "window function" seems not to have any effect on the covariance between periodogram ordinates. (Theorem 4.8.2). However, this is not the case since the window function reduces the sample bias before we actually get to the limit.

These two points are discussed in Brillinger (1975a), p238 in the case of ordinary time series.

Theorem 4.8.3 suggests a way of constructing a better estimate of the spectral density $f_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)$.

If

$$I^{(T)}_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda) = \frac{1}{2\pi T H_2(0)}\left(\int_t K_T(t)e^{-i\lambda t}d\underset{\sim}{N}(t)\right)\overline{\left(\int_t K_T(t)e^{-i\lambda t}d\underset{\sim}{N}(t)\right)}^{T}, \tag{4.8.11}$$

then the distribution of the variates $I^{(T)}_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda + \frac{2\pi j}{T})$, $j = 0, \pm 1, \ldots, \pm m$ may be approximated by $2m + 1$ independent $W_2^C(1, f_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda))$ distributions.

This result suggests the consideration of the estimate

$$f^{(T)}_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda) = \frac{1}{2m+1}\sum_{j=-m}^{m} I^{(T)}_{\underset{\sim}{N}\underset{\sim}{N}}\left(\lambda + \frac{2\pi j}{T}\right) \quad for \ \lambda \neq 0 \tag{4.8.12}$$

$$= \frac{1}{m}\sum_{j=1}^{m} Re\, I^{(T)}_{\underset{\sim}{N}\underset{\sim}{N}}\left(\lambda + \frac{2\pi j}{T}\right) \quad for \ \lambda = 0 \ . \tag{4.8.13}$$

The asymptotic properties of $f^{(T)}_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)$ are given in the next theorem.

Theorem 4.8.4: Let $\underset{\sim}{N}(t)$ be a stationary bivariate point process satisfying the conditions of Theorem 4.8.1. Let k(u) satisfy the Assumption 3.8.1 and $f^{(T)}_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)$ be given by (4.8.12). Then

$$\lim_{T\to\infty} E\{f^{(T)}_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)\} = f_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda) \quad \text{for } -\infty<\lambda<\infty$$

and

$$\lim_{T\to\infty} \operatorname{cov}\{f^{(T)}_{k_1\ell_1}(\lambda), f^{(T)}_{k_2\ell_2}(\mu)\} = \frac{\delta\{\lambda-\mu\} f_{k_1k_2}(\lambda) f_{\ell_1\ell_2}(-\lambda) + \delta\{\lambda+\mu\} f_{k_1\ell_2}(\lambda) f_{\ell_1k_2}(-\lambda)}{2m+1} \quad \text{for } \lambda,\mu\neq 0 .$$

Proof: The proof follows from Theorem 4.8.2.

It is clear from the above theorem that the variability of the estimates is reduced as m increases. Hence by choosing an appropriate value of m we can reduce the variability of the estimate $f^{(T)}_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)$ to a desired level.

The asymptotic distribution of $f^{(T)}_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)$ is now considered.

Theorem 4.8.5: Let N(t) be a stationary bivariate point process satisfying the conditions of Theorem 4.8.1. Let k(u) satisfy the Assumption 3.8.1 and $f^{(T)}_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)$ be given by (4.8.12). Then $f^{(T)}_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)$ is asymptotically distributed as $(2m+1)^{-1} W_2^C(2m+1, f_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda))$ if $\lambda \neq 0$.

Proof: The proof follows directly from Theorem 4.8.3 and the fact that the distribution of $f^{(T)}_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)$ is $1/2m+1$ times the sum of 2m + 1 independent $W_2^C(1, f_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda))$ distributions.

The diagonal elements $f_{kk}(\lambda)$, $k = 1,2$, are asymptotically distributed as the chi-squared variates given in Theorem 3.7.3.

It may also be shown that the estimates $f_{kl}^{(T)}(\lambda)$ are normally distributed as $m/T \longrightarrow 0$ $(T \longrightarrow \infty)$.

The approximation of the distribution of the spectral density matrix by a complex Wishart distribution for a vector-valued time series was suggested by Goodman (1963). Brillinger (1972) gives the same approximate distribution for a vector-valued interval function.

In practice we modify the statistic (4.8.11) as follows

$$I_{\underset{\sim}{N}'\underset{\sim}{N}'}^{(T)}(\lambda) = \frac{1}{2\pi T H_2(0)} \left(\int_t K_T(t) e^{-i\lambda t} d\underset{\sim}{N}'(t)\right)\overline{\left(\int_t K_T(t) e^{-i\lambda t} d\underset{\sim}{N}'(t)\right)}^{T}, \tag{4.8.14}$$

where $d\underset{\sim}{N}'(t) = d\underset{\sim}{N}(t) - p_{\underset{\sim}{N}}dt$ is called the bivariate point process with zero-mean.

The asymptotic properties of the new estimate of the spectral density obtained by using the statistic (4.8.14) are the same as those of estimate (4.8.12). The advantage of using the bivariate process with zero-mean is that the bias of the estimate is further reduced.

It is clear from (4.8.12) that estimates of the cross-spectrum of N_1 (input process) and N_2 (output process) and the power spectra of N_1 and N_2 can be obtained in a direct way, since

$$f_{\underset{\sim}{N}\underset{\sim}{N}}^{(T)}(\lambda) = \left[\, f_{k\ell}^{(T)}(\lambda) \right] \quad \text{for} \quad k, \ell = 1,2 \ .$$

Once estimates of $f_{21}^{(T)}(\lambda)$, $f_{11}^{(T)}(\lambda)$ and $f_{22}^{(T)}(\lambda)$ are available then estimates of the gain, phase and coherence may be calculated as well.

The asymptotic distributions of these estimates developed in the same way as the distributions of the estimates described in section 4.7.

The limiting distribution of the estimated coherence is given by expression (4.7.2) with $n = 2m + 1$ ($\lambda \neq 0$). A comparison of Theorems 4.6.4 and 4.8.4 shows that

$$2m+1 = \frac{b_T T}{2\pi \int K^2(\alpha)d\alpha} \quad . \qquad (4.8.15)$$

Figs. 4.8.1 (a-b) illustrate the estimates of $\log_{10}|f_{21}(\lambda)|$ and $|R_{21}(\lambda)|^2$ for the Data Set II ($\gamma_s \longmapsto$ Ia), where $|f_{21}(\lambda)|$ is the modulus of the CS of N_2 and N_1. The 95% point of the estimated coherence is calculated from expression (4.7.6) for $s = 2m + 1$ and $m = 21$.

We now present another way of estimating $f_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda)$ by splitting the sample record into L disjoint segments of length R.

In this case the finite Fourier transform is set to be

$$d_K^{(R)}(\lambda, j) = \int_0^R e^{-i\lambda t} dN_K(t) \quad for\ -\infty < \lambda < \infty;\ j = 0, \dots, L-1 \ , \qquad (4.8.16)$$

where $T = LR$.

From (4.8.16), for the matrix of the second-order periodograms, we have

$$I_{\underset{\sim}{N}\underset{\sim}{N}}^{(R)}(\lambda, j) = \left[\frac{1}{2\pi R} d_K^{(R)}(\lambda, j) \overline{d_\ell^{(R)}(\lambda, j)}\right] \quad for\ j = 0, \dots, L-1 \quad . \qquad (4.8.17)$$

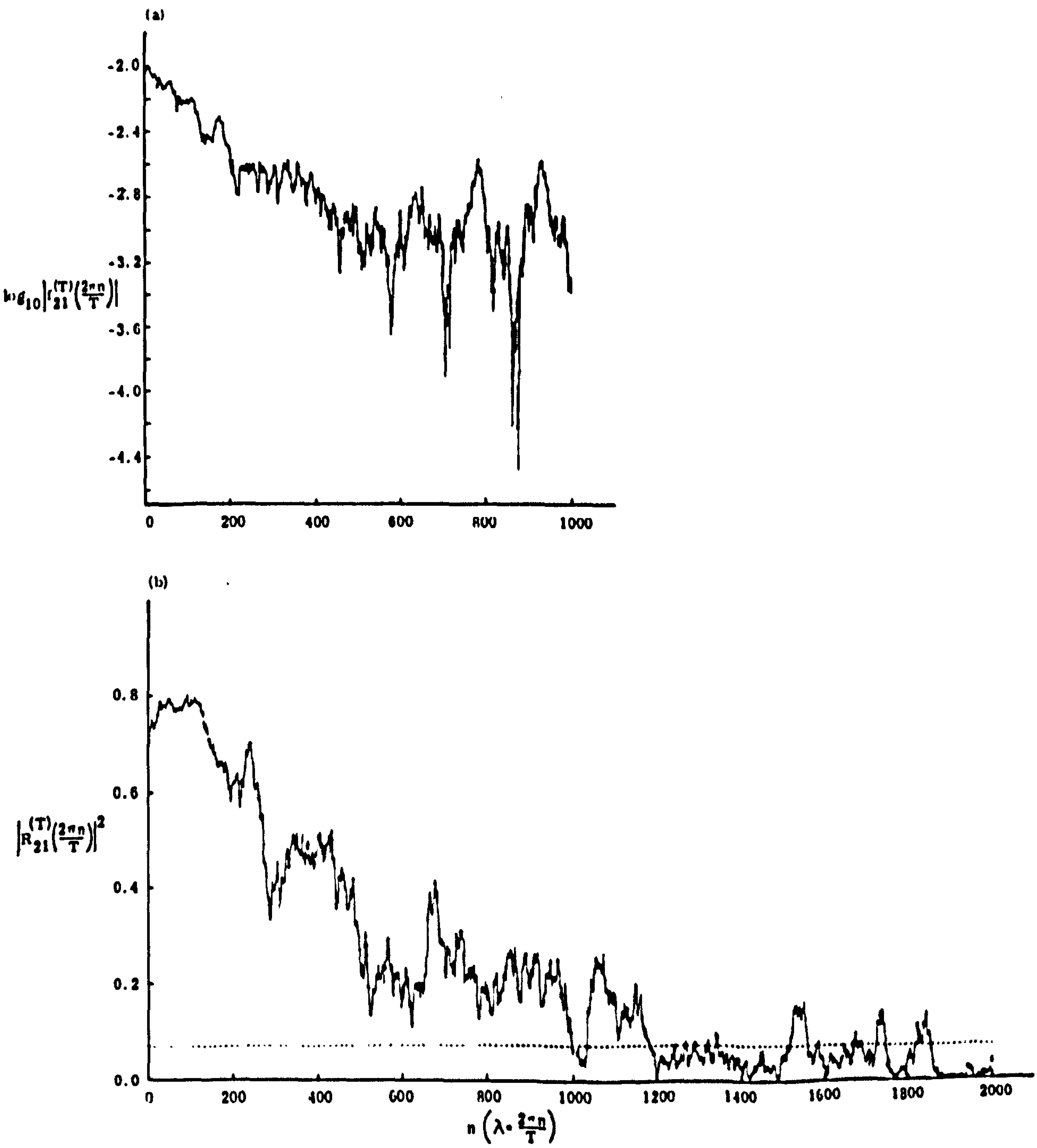


Fig. 4.8.1: Estimates of frequency-domain parameters obtained from the periodogram-matrix by using the whole record in the case that the muscle spindle is affected by a fusimotor input (γs). (a) Log to base 10 of the modulus of the cross-spectrum; (b) Estimate of the coherence. The dotted line gives the 95% point of the null distribution of this estimate. The value of T = 16384.

It follows from Theorem 4.8.3 that the estimates $I_{NN}^{(R)}(\lambda, j)$, $j = 0, \ldots, L-1$ are asymptotically independent $W_2^C(1, f_{NN}(\lambda))$ variates if $\lambda \neq 0$. This suggests that we should consider the estimate

$$f_{NN}^{(T)}(\lambda) = \frac{1}{L} \sum_{j=0}^{L-1} I_{NN}^{(R)}(\lambda, j) \quad , \ -\infty < \lambda < \infty \, ; \ j = 0, \ldots, L-1 \quad . \tag{4.8.18}$$

The asymptotic first- and second-order properties of estimate (4.8.18) are given in the following theorem.

<u>Theorem 4.8.6</u>: Let $N(t)$ be a bivariate stationary point process satisfying the conditions of Theorem 4.8.1. Suppose $k(u)$ satisfies the Assumption 3.8.1. Let also $f_{NN}^{(T)}(\lambda)$ be given by (4.8.18). Then

$$\lim_{T \to \infty} E\left\{ f_{k\ell}^{(T)}(\lambda) \right\} = f_{k\ell}(\lambda) \quad \text{for} \quad \lambda \neq 0 \ \text{and} \ k, \ell = 1, 2$$

and

$$\lim_{T \to \infty} \operatorname{cov}\left\{ f_{k_1\ell_1}^{(T)}(\lambda), f_{k_2\ell_2}^{(T)}(\mu) \right\} = \frac{\delta\{\lambda-\mu\} f_{k_1k_2}(\lambda) f_{\ell_1\ell_2}(-\lambda) + \delta\{\lambda+\mu\} f_{k_1\ell_2}(\lambda) f_{\ell_1k_2}(-\lambda)}{L}$$

$$\text{for} \quad k_1, k_2, \ell_1, \ell_2 = 1, 2 \ .$$

<u>Proof</u>: The proof is similar to the one given in Theorem 4.8.4.

The first-order moments show that $f_{NN}^{(T)}(\lambda)$ is an asymptotically unbiased estimate of $f_{NN}(\lambda)$. The second-order moments depend on the factor $1/L$ which indicates that by choosing a suitable value of L we may reduce the variability of the estimate $f_{NN}^{(T)}(\lambda)$.

For the asymptotic distribution of $f_{\underset{\sim}{N}\underset{\sim}{N}}^{(T)}(\lambda)$ we have

<u>Theorem 4.8.7</u>: Suppose that the conditions of Theorem 4.8.1 are satisfied. Let $f_{\underset{\sim}{N}\underset{\sim}{N}}^{(T)}(\lambda)$ be given by (4.8.18). Then $f_{\underset{\sim}{N}\underset{\sim}{N}}^{(T)}(\lambda)$ is asymptotically $L^{-1}W_2^C(L, f_{\underset{\sim}{N}\underset{\sim}{N}}(\lambda))$ if $\lambda \neq 0$.

<u>Proof</u>: The proof is similar to the one given in 4.8.5.

In the case of Data Set II we estimate the cross-spectrum of the two point processes N_1 and N_2 by splitting the sample record into 62 disjoint segments of length 256. In the same way we estimate the power spectra of processes N_1 and N_2.

Figs. 4.8.2 (a-c) present the estimates of $\log_{10} |f_{21}(\lambda)|$, $\theta(\lambda)$ and $|R_{21}(\lambda)|^2$. The 95% point of the estimate of the coherence is calculated by setting $s = L = 62$ in the expression (4.7.5).

The estimate of the phase decreases almost linearly for mean rates ranging from 4 to 105 pulses/sec. This clearly shows that there is a delay in the system of about 15 msec when it responds to gamma stimulation. The stimulus becomes uncorrelated with the output at frequencies higher than 105 pulses/sec since the estimate of the coherence is close to zero and the estimate of the phase oscillates rather abruptly for $n > 27$.

The methods of estimating the spectral density matrix of the bivariate point processes described in this section can be compared with the method discussed in section 4.7

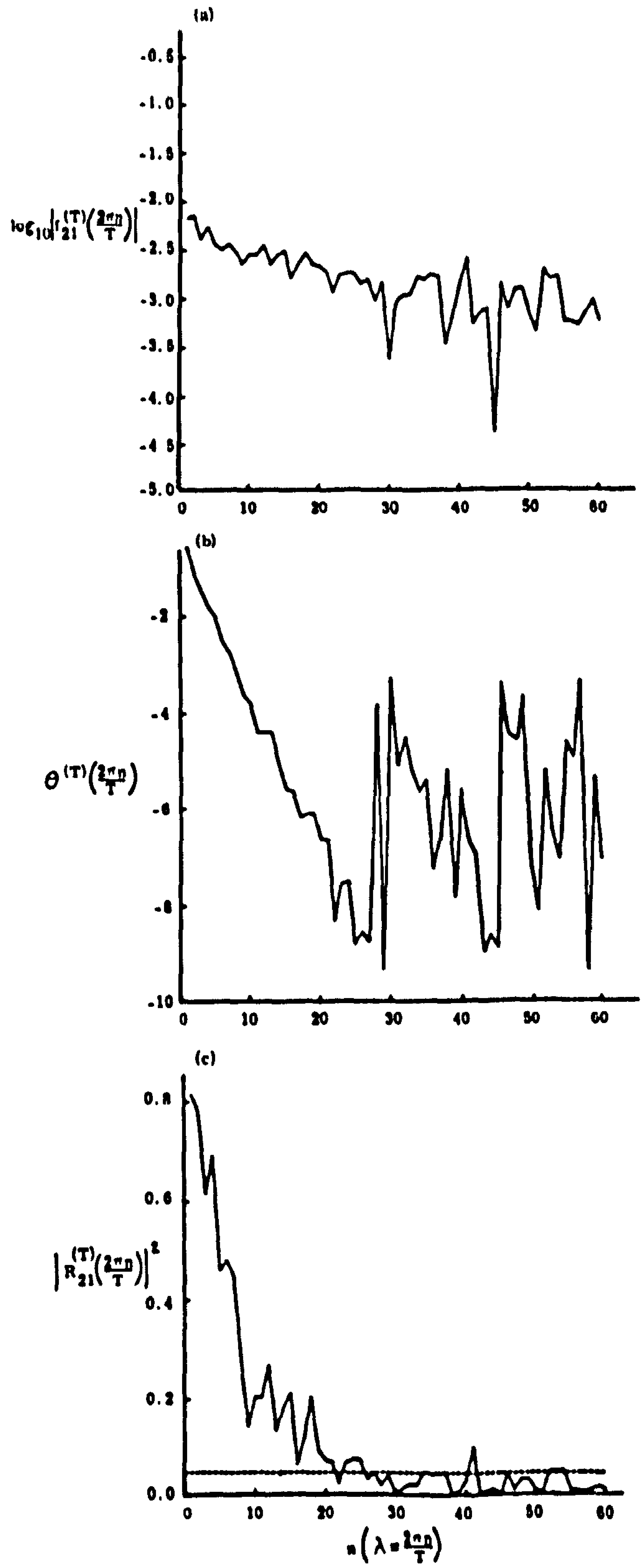


Fig. 4.8.2: Estimates of frequency-domain parameters calculated from the periodogram-matrix by dividing the whole record into 62 disjoint sections in the case that the muscle is affected by a fusimotor input (γs). (a) Log to base 10 of the modulus of the estimate of the cross-spectrum; (b) Estimate of the phase and (c) Estimate of the coherence. The dotted line gives the 95% point of the null distribution of this estimate. The value of T = 256.

since the estimates of the frequency domain parameters are quite similar. Therefore it does not seem to matter which method we use in identifying a linear system involving point processes as input and output.

4.9 ESTIMATION OF THE TIME-DOMAIN PARAMETERS OF THE BIVARIATE POINT PROCESS THROUGH PARAMETERS OF THE FREQUENCY DOMAIN

Let $\underset{\sim}{N}(t)$ be a stationary bivariate point process with MI $p_{\underset{\sim}{N}}$ and CF $q_{\underset{\sim}{N}\underset{\sim}{N}}(u) = [\, q_{kl}(u)]$ such that the entries $q_{kl}(u)$ satisfy the condition

$$\int |u||q_{k\ell}(u)|\, du < \infty \tag{4.9.1}$$

An estimate of the function $q_{kl}(u)$ can be defined by

$$\hat{q}_{k\ell}(u) = \frac{2\pi}{T} \sum_{j \neq 0} I_{k\ell}^{(T)}(\lambda_j)\, e^{i\lambda_j u} \quad for \quad k \neq \ell \tag{4.9.2}$$

where u is assumed to be a real parameter,

$$\lambda_j = \frac{2\pi j}{T} \quad and \quad |\lambda_j| \leq \pi/b \; .$$

The frequency λj is restricted in the interval $(-\frac{\pi}{b}, \frac{\pi}{b}]$ in order to avoid problems of aliasing since

$u_m = bm$, $m = 1,2,\ldots,$ for some scale parameter b.

In the next theorem we examine the properties of the estimate (4.9.2).

Theorem 4.9.1: Let $\underset{\sim}{N}(t)$ be a stationary bivariate point process as described in the beginning of this section. Let $\hat{q}_{kl}(u)$ be given by (4.9.2). Then

$$E\left\{\hat{q}_{k\ell}(u)\right\} = \int_{-\pi/b}^{\pi/b} f_{k\ell}(\lambda) e^{i\lambda u} d\lambda + O(T^{-1})$$

$$= q_{k\ell}(u) + O(T^{-1}) \quad \text{for } k \neq \ell$$

and

$$\operatorname{cov}\left[\hat{q}_{k_1\ell_1}(u), \hat{q}_{k_2\ell_2}(v)\right] = \frac{2\pi}{T} \iint_{-\pi/b}^{\pi/b} f_{k_1\ell_1k_2\ell_2}(\lambda, -\lambda, -\mu) e^{i(\lambda u - \mu v)} d\lambda d\mu$$

$$+ \frac{2\pi}{T}\int_{-\pi/b}^{\pi/b} f_{k_1k_2}(\lambda) f_{\ell_1\ell_2}(-\lambda) e^{i\lambda(u-v)} d\lambda + \frac{2\pi}{T}\int_{-\pi/b}^{\pi/b} f_{k_1\ell_2}(\lambda) f_{\ell_1k_2}(-\lambda) e^{i\lambda(u+v)} d\lambda$$

$$+ O(T^{-2}\log T) .$$

Proof: The first relation follows from the fact that

$$E\left\{I_{k\ell}^{(T)}(\lambda)\right\} = f_{k\ell}(\lambda) + O(T^{-1}) .$$

For the covariance we have

$$\operatorname{cov}\left[\hat{q}_{k_1\ell_1}(u), \hat{q}_{k_2\ell_2}(v)\right] = \operatorname{cum}\left\{\int_{-\pi/b}^{\pi/b} I_{k_1\ell_1}^{(T)}(\lambda) e^{i\lambda u} d\lambda, \overline{\int_{-\pi/b}^{\pi/b} I_{k_2\ell_2}^{(T)}(\mu) e^{i\mu v} d\mu}\right\}$$

$$= \iint_{-\pi/b}^{\pi/b} \operatorname{cum}\left[I_{k_1\ell_1}^{(T)}(\lambda), I_{k_2\ell_2}^{(T)}(-\mu)\right] e^{i(\lambda u - \mu v)} d\lambda d\mu .$$

It follows from Theorem 4.6.2 that

$$\operatorname{cov}\left[\hat{q}_{k_1\ell_1}(u), \hat{q}_{k_2\ell_2}(v)\right] = \iint_{-\pi/b}^{\pi/b} \frac{|\Delta^{(T)}(\lambda-\mu)|^2}{T^2} f_{k_1k_2}(\lambda) f_{\ell_1\ell_2}(-\lambda) e^{i(\lambda u-\mu v)} d\lambda d\mu$$

$$+ \iint_{-\pi/b}^{\pi/b} \frac{|\Delta^{(T)}(\lambda+\mu)|^2}{T^2} f_{k_1\ell_2}(\lambda) f_{\ell_1k_2}(-\lambda)\, e^{i(\lambda u-\mu v)} d\lambda d\mu + \frac{2\pi}{T} \iint_{-\pi/b}^{\pi/b} f_{k_1\ell_1k_2\ell_2}(\lambda, -\lambda, -\mu)\, e^{i(\lambda u-\mu v)} d\lambda d\mu$$

$$+ O(T^{-2}\log T) \ ,$$

where $\Delta^{(T)}(\lambda)$ is approximated by

$$\Delta^{(T)}(\lambda) \simeq \sum_{t=0}^{T-1} e^{-i\lambda t} .$$

The above expression, after some simplifications, becomes

$$\operatorname{cov}\left[\hat{q}_{k_1\ell_1}(u), \hat{q}_{k_2\ell_2}(v)\right] = \frac{2\pi}{bT} \int_{-\pi/b}^{\pi/b} f_{k_1k_2}(\lambda) f_{\ell_1\ell_2}(-\lambda)\, e^{i\lambda(u-v)} d\lambda$$

$$+ \frac{2\pi}{bT} \int_{-\pi/b}^{\pi/b} f_{k_1\ell_2}(\lambda) f_{\ell_1k_2}(-\lambda)\, e^{i\lambda(u+v)} d\lambda + \frac{2\pi}{bT} \iint_{-\pi/b}^{\pi/b} f_{k_1\ell_1k_2\ell_2}(\lambda, -\lambda, -\mu)\, e^{i(\lambda u-\mu v)} d\lambda d\mu$$

$$+ O(T^{-2}\log T) \ .$$

The above theorem indicates that the estimate $\hat{q}_{kl}(u)$ is an asymptotically unbiased estimate of $q_{kl}(u)$. Also, it shows that the variance of $\hat{q}_{kl}(u)$ has the following form

$$\operatorname{Var}\{\hat{q}_{k\ell}(u)\} \simeq \frac{2\pi}{bT}\Big[\int_{-\pi/b}^{\pi/b} f_{kk}(\lambda) f_{\ell\ell}(-\lambda)\, d\lambda + \int_{-\pi/b}^{\pi/b} f_{k\ell}(\lambda) f_{\ell k}(-\lambda)\, e^{i\lambda 2u} d\lambda$$

$$+ \iint_{-\pi/b}^{\pi/b} f_{k\ell k\ell}(\lambda, -\lambda, -\mu)\, e^{i(\lambda-\mu)u} d\lambda d\mu\Big] \quad \text{for } k \neq \ell \ . \tag{4.9.3}$$

Fig. 4.9.1 gives the estimate of $q_{21}(u)$ calculated from (4.9.3) for a bin width b = 1. The horizontal lines are the 95% confidence limits for the estimate of the CCF when we assume that the two point processes are independent. Under the hypothesis of independence the variance (4.9.4) will be approximately given by

$$Var\{\hat{q}_{21}(u)\} \simeq \frac{2\pi}{T}\int_{-\pi}^{\pi} f_{11}(\lambda) f_{22}(\lambda)\, d\lambda \quad . \tag{4.9.4}$$

So an estimate of the variance can be calculated from the expression

$$\left(\frac{2\pi}{T}\right)^2 \sum_{j\neq 0} I_{11}^{(T)}(\lambda_j) I_{22}^{(T)}(\lambda_j) \quad . \tag{4.9.5}$$

In practice we increase the estimate of the variance by multiplying (4.9.5) by a factor which depends on the taper and the finer structure of frequencies used. In the case of the confidence limits of Fig. 4.9.1 we have multiplied (4.9.5) by the factor 1.089.

In connection to the CCF we consider an estimate of the cross-product density (CPD) of the bivariate point process given by

$$\hat{p}_{k\ell}(u) = \hat{p}_k \hat{p}_\ell + \frac{2\pi}{T} \sum_{j\neq 0} I_{k\ell}^{(T)}(\lambda_j)\, e^{i\lambda_j u} \quad for \quad |\lambda_j| \leqslant \pi \quad . \tag{4.9.6}$$

The asymptotic first- and second-order properties of $\hat{p}_{kl}(u)$ are developed in the next theorem.

<u>Theorem 4.9.2</u>: Let $\underset{\sim}{N}(t)$ be a stationary bivariate point process as described in Theorem 4.9.1. Let $\hat{p}_{kl}$ (u) be

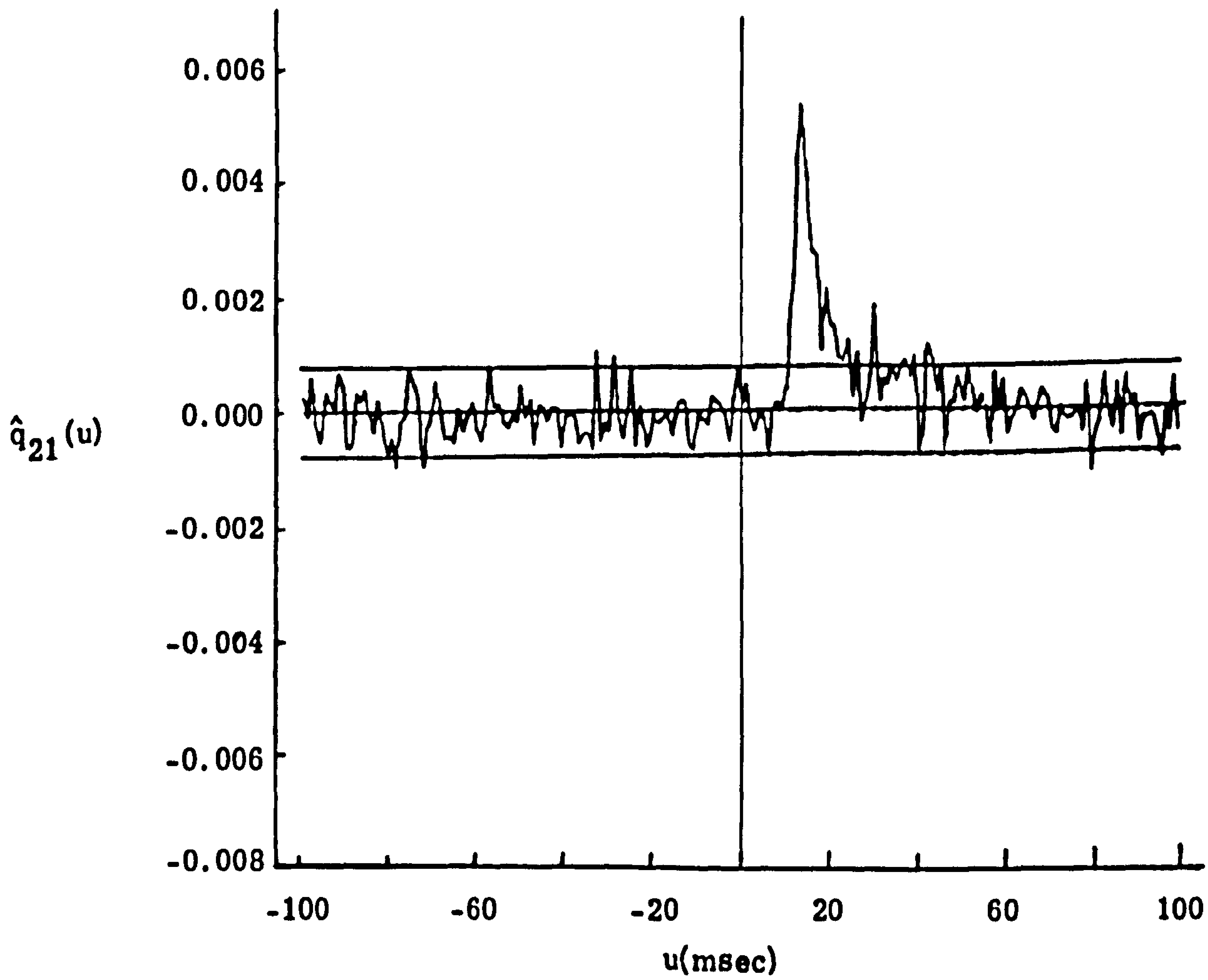


Fig. 4.9.1: Estimate of the second-order cross cumulant function between the Ia discharge and the γs stimulus obtained by inverting the cross-periodogram. The cross-periodogram was calculated by using the whole record (T = 16384). The dotted line in the middle corresponds to the mean value of the estimate under the hypothesis that the Ia discharge and the s stimulus are independent. The horizontal lines are the asymptotic 95% confidence limits.

given by (4.9.7). Then we have

$$E\{\hat{P}_{K\ell}(u)\} = p_K p_\ell + \int_{-\pi}^{\pi} f_{K\ell}^{(T)}(\lambda) e^{i\lambda u} d\lambda + O(T^{-1})$$

$$= p_{K\ell}(u) + O(T^{-1}) ,$$

$$\operatorname{cov}[\hat{P}_{K_1\ell_1}(u), \hat{P}_{K_2\ell_2}(v)] = \frac{2\pi}{T} f_{K_1} f_{K_2} f_{\ell_1\ell_2}(0) + \frac{2\pi}{T} f_{K_1} f_{\ell_2} f_{\ell_1 K_2}(0) + \frac{2\pi}{T} f_{\ell_1} f_{K_2} f_{K_1\ell_2}(0)$$

$$+ \frac{2\pi}{T} f_{\ell_1} f_{\ell_2} f_{K_1K_2}(0) + \frac{f_{K_1}f_{K_2}}{T^2}\int_{-\pi}^{\pi} |\Delta^{(T)}(\mu)|^2 f_{\ell_1\ell_2}(\mu) e^{-i\mu v} d\mu + \frac{f_{\ell_1}f_{\ell_2}}{T^2}\int_{-\pi}^{\pi} |\Delta^{(T)}(\mu)|^2 f_{K_1K_2}(-\mu) e^{-i\mu v} d\mu$$

$$+ \frac{f_{K_1}f_{\ell_2}}{T^2}\int_{-\pi}^{\pi} |\Delta^{(T)}(\mu)|^2 f_{\ell_1K_2}(-\mu) e^{-i\mu v} d\mu + \frac{f_{\ell_1}f_{K_2}}{T^2}\int_{-\pi}^{\pi} |\Delta^{(T)}(\mu)|^2 f_{K_1\ell_2}(\mu) e^{-i\mu v} d\mu$$

$$+ \frac{f_{K_1}f_{K_2}}{T^2}\int_{-\pi}^{\pi} |\Delta^{(T)}(\lambda)|^2 f_{\ell_1\ell_2}(-\lambda) e^{i\lambda u} d\lambda + \frac{f_{\ell_1}f_{\ell_2}}{T^2}\int_{-\pi}^{\pi} |\Delta^{(T)}(\lambda)|^2 f_{K_1K_2}(\lambda) e^{i\lambda u} d\lambda$$

$$+ \frac{f_{K_1}f_{\ell_2}}{T^2}\int_{-\pi}^{\pi} |\Delta^{(T)}(\lambda)|^2 f_{\ell_1K_2}(\lambda) e^{i\lambda v} d\lambda + \frac{f_{\ell_1}f_{K_2}}{T^2}\int_{-\pi}^{\pi} |\Delta^{(T)}(\lambda)|^2 f_{K_1\ell_2}(\lambda) e^{i\lambda u} d\lambda$$

$$+ \frac{2\pi}{T}\int_{-\pi}^{\pi} f_{K_1K_2}(\lambda) f_{\ell_1\ell_2}(-\lambda) e^{i\lambda(u-v)} d\lambda + \frac{2\pi}{T}\int_{-\pi}^{\pi} f_{K_1\ell_2}(\lambda) f_{\ell_1K_2}(-\lambda) e^{i\lambda(u+v)} d\lambda$$

$$+ \frac{2\pi}{T}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{K_1\ell_1K_2\ell_2}(\lambda, -\lambda, -\mu) e^{i(\lambda u - \mu v)} d\lambda d\mu + O(T^{-2}\log T) .$$

Proof: The expected value of $\hat{p}_{kl}(u)$ can be obtained as follows

$$E\{\hat{p}_{k\ell}(u)\} = E\{\hat{p}_k\hat{p}_\ell\} + \int_{-\pi}^{\pi} E\{I_{k\ell}^{(T)}(\lambda)\} e^{i\lambda u} d\lambda$$

$$= p_k p_\ell + \frac{2\pi f_{k\ell}(0)}{T} + \int_{-\pi}^{\pi} f_{k\ell}(\lambda) e^{i\lambda u} d\lambda + O(T^{-1})$$

$$= p_{k\ell}(u) + O(T^{-1}).$$

For the covariance of $\hat{p}_{k_1 l_1}(u)$ and $\hat{p}_{k_2 l_2}(u)$ we have

$$\operatorname{cov}\{\hat{p}_{k_1\ell_1}(u), \hat{p}_{k_2\ell_2}(v)\} = \operatorname{cov}[\hat{p}_{k_1}\hat{p}_{\ell_1}, \hat{p}_{k_2}\hat{p}_{\ell_2}] + \operatorname{cov}[\hat{p}_{k_1}\hat{p}_{\ell_1}, \int_{-\pi}^{\pi} I_{k_2\ell_2}^{(T)}(\mu) e^{i\mu v} d\mu]$$

$$+ \operatorname{cov}[\int_{-\pi}^{\pi} I_{k_1\ell_1}^{(T)}(\lambda) e^{i\lambda u} d\lambda, \hat{p}_{k_2}\hat{p}_{\ell_2}] + \operatorname{cov}[\int_{-\pi}^{\pi} I_{k_1\ell_1}^{(T)}(\lambda) e^{i\lambda u} d\lambda, \int_{-\pi}^{\pi} I_{k_2\ell_2}^{(T)}(\mu) e^{i\mu v} d\mu]. (*)$$

The four terms in the R.H.S. of (*) can be written as

$$(a) \quad \operatorname{cov}\{\hat{p}_{k_1}\hat{p}_{\ell_1}, \hat{p}_{k_2}\hat{p}_{\ell_2}\} = \frac{1}{T^4}\left[E\{d_{k_1}^{(T)}(0) d_{\ell_1}^{(T)}(0) d_{k_2}^{(T)}(0) d_{\ell_2}^{(T)}(0)\} - E\{d_{k_1}^{(T)}(0) d_{\ell_1}^{(T)}(0)\} E\{d_{k_2}^{(T)}(0) d_{\ell_2}^{(T)}(0)\}\right]$$

since

$$\hat{p}_k = \frac{1}{T}\int_0^T dN_k(t) = \frac{d_k^{(T)}(0)}{T} .$$

Now, by using properties of the cumulant functions we find

$$\operatorname{cov}\{\hat{p}_{k_1}\hat{p}_{\ell_1}, \hat{p}_{k_2}\hat{p}_{\ell_2}\} = \frac{2\pi}{T} f_{k_1} f_{k_2} f_{\ell_1\ell_2}(0) + \frac{2\pi}{T} f_{k_1} f_{\ell_2} f_{\ell_1 k_2}(0) + \frac{2\pi}{T} f_{\ell_1} f_{k_2} f_{k_1\ell_2}(0)$$

$$+ \frac{2\pi}{T} f_{\ell_1} f_{\ell_2} f_{k_1 k_2}(0) + O(T^{-2}) .$$

$$(b)\quad \operatorname{cov}\left[\hat{p}_{K_1}\hat{p}_{\ell_1}, \int_{-\pi}^{\pi} I^{(T)}_{K_2\ell_2}(\mu)e^{i\mu v}d\mu\right] = \int_{-\pi}^{\pi}\operatorname{cov}\left[\hat{p}_{K_1}\hat{p}_{\ell_1}, I^{(T)}_{K_2\ell_2}(\mu)\right]e^{-i\mu v}d\mu \ .$$

For the covariance inside the integral we get

$$\operatorname{cov}\left[\hat{p}_{K_1}\hat{p}_{\ell_1}, I^{(T)}_{K_2\ell_2}(\mu)\right] = \frac{1}{2\pi T^3}\operatorname{cov}\left[d^{(T)}_{K_1}(0)d^{(T)}_{\ell_1}(0), d^{(T)}_{K_2}(\mu)d^{(T)}_{\ell_2}(-\mu)\right]$$

$$= \frac{1}{2\pi T^3}\left[E\left\{d^{(T)}_{K_1}(0)d^{(T)}_{\ell_1}(0)d^{(T)}_{K_2}(\mu)d^{(T)}_{\ell_2}(-\mu)\right\} - E\left\{d^{(T)}_{K_1}(0)d^{(T)}_{\ell_1}(0)\right\}E\left\{d^{(T)}_{K_2}(\mu)d^{(T)}_{\ell_2}(-\mu)\right\}\right].$$

This expression after some calculations gives

$$\operatorname{cov}\left[\hat{p}_{K_1}\hat{p}_{\ell_1}, I^{(T)}_{K_2\ell_2}(\mu)\right] = \frac{f_{K_1}f_{K_2}}{T^2}f_{\ell_1\ell_2}(\mu)|\Delta^{(T)}(\mu)|^2 + \frac{f_{\ell_1}f_{\ell_2}}{T^2}f_{K_1K_2}(-\mu)|\Delta^{(T)}(\mu)|^2$$
$$+ \frac{f_{K_1}f_{\ell_2}}{T^2}f_{\ell_1K_2}(-\mu)|\Delta^{(T)}(\mu)|^2 + \frac{f_{\ell_1}f_{K_2}}{T^2}f_{K_1\ell_2}(\mu)|\Delta^{(T)}(\mu)|^2 + O(T^{-2}) \ .$$

So we have

$$\operatorname{cov}\left\{\hat{p}_{K_1}\hat{p}_{\ell_1}, \int_{-\pi}^{\pi} I^{(T)}_{K_2\ell_2}(\mu)e^{i\mu v}d\mu\right\} = \frac{f_{K_1}f_{K_2}}{T^2}\int_{-\pi}^{\pi}|\Delta^{(T)}(\mu)|^2 f_{\ell_1\ell_2}(\mu)e^{-i\mu v}d\mu$$

$$+\frac{f_{\ell_1}f_{\ell_2}}{T^2}\int_{-\pi}^{\pi}|\Delta^{(T)}(\mu)|^2 f_{K_1K_2}(-\mu)e^{-i\mu v}d\mu + \frac{f_{K_1}f_{\ell_2}}{T^2}\int_{-\pi}^{\pi}|\Delta^{(T)}(\mu)|^2 f_{\ell_1K_2}(-\mu)e^{-i\mu v}d\mu + \frac{f_{\ell_1}f_{K_2}}{T^2}\int_{-\pi}^{\pi}|\Delta^{(T)}(\mu)|^2 f_{K_1\ell_2}(\mu)e^{-i\mu v}d\mu + O(T^{-2}) .$$

$$(c)\quad \operatorname{cov}\left\{\int_{-\pi}^{\pi} I^{(T)}_{K_1\ell_1}(\lambda)e^{i\lambda u}d\lambda, \hat{p}_{K_2}\hat{p}_{\ell_2}\right\} = \frac{f_{K_1}f_{K_2}}{T^2}\int_{-\pi}^{\pi}|\Delta^{(T)}(\lambda)|^2 f_{\ell_1\ell_2}(-\lambda)e^{i\lambda u}d\lambda + \frac{f_{\ell_1}f_{\ell_2}}{T^2}\int_{-\pi}^{\pi}|\Delta^{(T)}(\lambda)|^2 f_{K_1K_2}(\lambda)e^{i\lambda u}d\lambda$$

$$+ \frac{f_{K_1}f_{\ell_2}}{T^2}\int_{-\pi}^{\pi}|\Delta^{(T)}(\mu)|^2 f_{\ell_1K_2}(-\lambda)e^{i\lambda v}d\lambda + \frac{f_{\ell_1}f_{K_2}}{T^2}\int_{-\pi}^{\pi}|\Delta^{(T)}(\lambda)|^2 f_{K_1\ell_2}(\lambda)e^{i\lambda u}d\lambda + O(T^{-2}) \ .$$

$$(d)\quad \operatorname{cov}\left\{\int_{-\pi}^{\pi} I^{(T)}_{K_1\ell_1}(\lambda)e^{i\lambda u}d\lambda \ , \ \int_{-\pi}^{\pi} I^{(T)}_{K_2\ell_2}(\mu)e^{i\mu v}d\mu\right\} = \iint_{-\pi}^{\pi}\operatorname{cov}\left\{I^{(T)}_{K_1\ell_1}(\lambda), I^{(T)}_{K_2\ell_2}(\mu)\right\}e^{i(\lambda u+\mu v)}d\lambda d\mu$$

$$= \frac{2\pi}{T}\int_{-\pi}^{\pi} f_{K_1K_2}(\lambda)f_{\ell_1\ell_2}(-\lambda)\,e^{i\lambda(u-v)}d\lambda + \frac{2\pi}{T}\int_{-\pi}^{\pi} f_{K_1\ell_2}(\lambda)f_{\ell_1K_2}(-\lambda)\,e^{i\lambda(u+v)}d\lambda$$

$$+ \frac{2\pi}{T}\iint_{-\pi}^{\pi} f_{K_1\ell_1K_2\ell_2}(\lambda,-\lambda,-\mu)\,e^{i(\lambda u-\mu v)}d\lambda d\mu + O(T^{-2}\log T) \ .$$

If we substitute the results of (a), (b), (c) and (d) into (*) we get the following expression

$$
\begin{aligned}
\operatorname{cov}\left[\hat{p}_{k_1\ell_1}(u), \hat{p}_{k_2\ell_2}(v)\right] &= \frac{2\pi}{T} f_{k_1} f_{k_2} f_{\ell_1\ell_2}(0) + \frac{2\pi}{T} f_{k_1} f_{\ell_2} f_{\ell_1 k_2}(0) + \frac{2\pi}{T} f_{\ell_1} f_{k_2} f_{k_1\ell_2}(0) \\
&+ \frac{2\pi}{T} f_{\ell_1} f_{\ell_2} f_{k_1 k_2}(0) + \frac{f_{k_1} f_{k_2}}{T^2} \int_{-\pi}^{\pi} |\Delta^{(T)}(\mu)|^2 f_{\ell_1\ell_2}(\mu) e^{-i\mu v} d\mu + \frac{f_{\ell_1} f_{\ell_2}}{T^2} \int_{-\pi}^{\pi} |\Delta^{(T)}(\mu)|^2 f_{k_1k_2}(-\mu) e^{-i\mu v} d\mu \\
&+ \frac{f_{k_1} f_{\ell_2}}{T^2} \int_{-\pi}^{\pi} |\Delta^{(T)}(\mu)|^2 f_{\ell_1 k_2}(-\mu) e^{-i\mu v} d\mu + \frac{f_{\ell_1} f_{k_2}}{T^2} \int_{-\pi}^{\pi} |\Delta^{(T)}(\mu)|^2 f_{k_1\ell_2}(\mu) e^{-i\mu v} d\mu \\
&+ \frac{f_{k_1} f_{k_2}}{T^2} \int_{-\pi}^{\pi} |\Delta^{(T)}(\lambda)|^2 f_{\ell_1\ell_2}(-\lambda) e^{i\lambda u} d\lambda + \frac{f_{\ell_1} f_{\ell_2}}{T^2} \int_{-\pi}^{\pi} |\Delta^{(T)}(\lambda)|^2 f_{k_1k_2}(\lambda) e^{i\lambda u} d\lambda \\
&+ \frac{f_{k_1} f_{\ell_2}}{T^2} \int_{-\pi}^{\pi} |\Delta^{(T)}(\lambda)|^2 f_{\ell_1 k_2}(\lambda) e^{i\lambda v} d\lambda + \frac{f_{\ell_1} f_{k_2}}{T^2} \int_{-\pi}^{\pi} |\Delta^{(T)}(\lambda)|^2 f_{k_1\ell_2}(\lambda) e^{i\lambda u} d\lambda \\
&+ \frac{2\pi}{T} \int_{-\pi}^{\pi} f_{k_1k_2}(\lambda) f_{\ell_1\ell_2}(-\lambda) e^{i\lambda(u-v)} d\lambda + \frac{2\pi}{T} \int_{-\pi}^{\pi} f_{k_1\ell_2}(\lambda) f_{\ell_1 k_2}(-\lambda) e^{i\lambda(u+v)} d\lambda \\
&+ \frac{2\pi}{T} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{k_1\ell_1k_2\ell_2}(\lambda, -\lambda, -\mu) e^{i(\lambda u - \mu v)} d\lambda\, d\mu + O(T^{-2}\log T) .
\end{aligned}
$$

The CPDs are very useful in measuring the degree of association of point process when they are suitably standardized.

We now consider an estimate of $a_1(u)$ using the frequency domain. This estimate may be presented as follows

$$\hat{a}_1(u) = \frac{1}{T} \sum_{j \neq 0} A^{(T)}(\lambda_j) e^{i\lambda_j u} \quad for \quad |\lambda_j| \leq \pi \tag{4.9.7}$$

The asymptotic distribution of this estimate is given in the next theorem.

<u>Theorem 4.9.3</u>: Let $\underset{\sim}{N}(t)$ be a stationary bivariate process as given in the beginning of this section. Let also $\hat{a}_1(u)$ be given by (4.9.7). Then $\hat{a}_1(u)$ is asymptotically normally distributed with mean $a_1(u)$ and variance

$$Var\{\hat{a}_1(u)\} \simeq (2\pi T)^{-1} \int_{-\pi}^{\pi} f_{22}(\lambda) f_{11}^{-1}(\lambda) \left[1 - |R_{21}(\lambda)|^2\right] d\lambda$$

$$+ (2\pi T)^{-1} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \Big[f_{2121}(\lambda, -\lambda, -\mu) + f_{1111}(\lambda, -\lambda, -\mu)$$

$$- f_{2111}(\lambda, -\lambda, -\mu) - f_{1121}(\lambda, -\lambda, -\mu) \Big] e^{iu(\lambda-\mu)} d\lambda\, d\mu \ .$$

<u>Proof</u>: It follows from Theorem 4.9.1 that

$$E\{\hat{a}_1(u)\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(\lambda) e^{i\lambda u} d\lambda + O(T^{-1})$$

$$= a_1(u) + O(T^{-1}) \ ,$$

by assuming that $\lambda_j = \frac{2\pi j}{T}$ and $|\lambda_j| \leq \pi$.

Moreover, by using the same arguments of Theorem 4.7.1 we have

$$\operatorname{cov}[\hat{a}_1(u), \hat{a}_1(v)] = (2\pi T)^{-1} \int_{-\pi}^{\pi} f_{22}(\lambda) f_{11}^{-1}(\lambda) [1 - |R_{21}(\lambda)|^2] e^{i\lambda(u-v)} d\lambda$$

$$+ (2\pi T)^{-1} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} [f_{2121}(\lambda, -\lambda, -\mu) + f_{1111}(\lambda, -\lambda, -\mu)$$

$$- f_{2111}(\lambda, -\lambda, -\mu) - f_{1121}(\lambda, -\lambda, -\mu)] e^{i(\lambda u - \mu v)} d\lambda d\mu + O(T^{-2}) .$$

By setting u = v in the above formula we get the result for the variance. The asymptotic normality follows from the fact that the cumulants of $\hat{a}_1(u)$ of order greater than 2 will tend to zero.

In our case we may estimate $A(\lambda)$ by the following expression

$$A^{(T)}(\lambda) = \frac{f_{21}^{(T)}(\lambda)}{f_{11}^{(T)}(\lambda)} = \frac{I_{21}^{(T)}(\lambda)}{\hat{P}_1/2\pi} , \tag{4.9.8}$$

since $f_{11}(\lambda)$ is a Poisson point process.

Under the assumption of independence of the two processes the variance of $\hat{a}_1(u)$ will be approximately given by

$$\operatorname{Var}\{\hat{a}_1(u)\} \simeq (2\pi T)^{-1} \int_{-\pi}^{\pi} f_{22}(\lambda)/(P_1/2\pi)\, d\lambda$$

$$= \frac{1}{T} \int_{-\pi}^{\pi} f_{22}(\lambda)/P_1\, d\lambda \quad \text{for} \quad u \neq 0 . \tag{4.9.9}$$

4.10 QUADRATIC MODEL OF A POINT PROCESS SYSTEM

Let $\underset{\sim}{N}(t) = (N_1(t), N_2(t))$ be a stationary bivariate point process satisfying Assumption 1.2 of Appendix I. In order to take into account the non linear interactive effects of two events of the input process N_1 on the output process N_2, we extend the linear model (4.3.1) in the following way

$$E\left\{dN_2(t)/N_1\right\} = \left\{a_0 + \int a_1(t-u)dN_1(u) + \iint\limits_{u\neq v} a_2(t-u, t-v)dN_1(u)dN_1(v)\right\}dt \quad . \qquad (4.10.1)$$

The function $a_2(\cdot,\cdot)$ is called the second-order kernel of the system and gives the non-linear interactive effects of two input events occurring at different time instances. The model (4.10.1) has been proposed and discussed by Brillinger (1975d).

Before we discuss the solution of equation (4.10.1) and propose an estimate of the second-order kernel, it is necessary to define higher-order point process parameters in the time and frequency domains and examine their asymptotic distributions as well.

It follows from Chapter 2 (section 2.11) that the third-order product density of a bivariate orderly point process is given by

$$E\left\{dN_2(t+u)dN_1(t+v)dN_1(t)\right\} = p_{211}(u,v)dudvdt \ , \quad \begin{matrix} u\neq v \\ u\neq 0, \ v\neq 0 \end{matrix} \quad . \qquad (4.10.2)$$

The third order product density $p_{211}(u,v)$ may be interpreted as

$$\text{Prob}\left\{N_2 \text{ event in } (t+u, t+u+du], \ N_1 \text{ event in } (t+v, t+v+dv] \text{ and } N_1 \text{ event in } (t, t+dt]\right\}.$$

The definition of the function $p_{211}(u,v)$ and the additional assumption of a (strong) mixing condition suggest the following properties

(i) $p_{211}(u,v) = p_{211}(u-v,-v) = p_{112}(v-u,-u) = p_{112}(-u,v-u)$

(ii) $\lim_{v \to \infty} p_{211}(u,v) = p_{21}(u)p_1$.

Another useful function is the third-order conditional density defined by

$$m_{211}(u,v) = \lim_{h \to 0} \operatorname{Prob}\left\{N_2 \text{ event in } (t,t+h] \,/\, N_1 \text{ events at } t-u \text{ and } t-v\right\}/h \,. \quad (4.10.3)$$

Alternatively, since the bivariate process is orderly, we have

$$m_{211}(u,v)\,dt = E\left\{dN_2(t) \,/\, N_1 \text{ events at } t-u \text{ and } t-v\right\} \quad . \qquad (4.10.4)$$

Using the definition of the conditional probability we find that $m_{211}(u,v)$ is connected to $p_{211}(u,v)$ through the expression

$$m_{211}(u,v) = \frac{p_{211}(u,u-v)}{m_{11}(u-v)\,p_1} \quad . \qquad (4.10.5)$$

In the case that N_1 is a Poisson point process (4.10.5) becomes

$$m_{211}(u,v) = \frac{p_{211}(u,u-v)}{p_1^2} \quad . \qquad (4.10.6)$$

It follows from the property (ii) of $p_{211}(u,v)$ that

$$\lim_{v \to \infty} m_{211}(u,v) = m_{21}(u) \quad . \qquad (4.10.7)$$

Following expression (2.11.5) given in Chapter 2 we now define the third-order cumulant function of a bivariate point process as

$$q_{211}(u,v)\,du\,dv\,dt = \mathrm{cum}\left\{dN_2(t+u),\, dN_1(t+v),\, dN_1(t)\right\},\ \begin{smallmatrix} u\neq v \\ u\neq 0,\, v\neq 0\end{smallmatrix} \quad . \tag{4.10.8}$$

The third-order cumulant function measures the relationship between the increments of the process N_2 and the increments of the process N_1 at two different time instances.

The functions $q_{211}(u,v)$ and $p_{211}(u,v)$ are related in the following way

$$q_{211}(u,v) = p_{211}(u,v) - p_1 p_{21}(u) - p_1 p_{21}(v) - p_2 p_{11}(u-v) + 2p_1^2 p_2 \quad . \tag{4.10.9}$$

This formula is a particular case of the general expression (2.11.8) presented in Chapter 2.

As $v \longmapsto \infty$ we have

$$\lim_{v\longmapsto\infty} q_{211}(u,v) = 0 \quad . \tag{4.10.10}$$

In the frequency domain we define the third-order spectrum of the bivariate point process as

$$(2\pi)^2 f_{211}(\lambda,\mu) = \iint_{-\infty}^{+\infty} q_{211}(u,v)e^{-i(\lambda u+\mu v)}du\,dv\,,\ u\neq v \quad . \tag{4.10.11}$$

This definition follows from expression (2.12.5) of Chapter 2. The third-order spectrum is a complex function satisfying the property

$$\overline{f_{211}(\lambda,\mu)} = f_{211}(-\lambda,-\mu) \quad .$$

$f_{211}(\lambda,\mu)$ is also bounded and uniformly continuous since

$$\iint_{-\infty}^{+\infty} |q_{211}(u,v)|\,du\,dv < \infty \quad .$$

By inverting (4.10.10) we get

$$q_{211}(u,v) = \iint_{-\infty}^{+\infty} f_{211}(\lambda,\mu) e^{i(\lambda u+\mu v)} d\lambda d\mu , \quad \begin{matrix} u \neq v \\ u \neq 0, v \neq 0 \end{matrix} . \tag{4.10.12}$$

We now turn to the problem of estimating the time domain parameters of the bivariate point process defined above.

In Brillinger (1975c) an estimate of the third-order product density is given by

$$\hat{P}_{211}(u,v) = \frac{J_{211}^{(T)}(u,v)}{b^2 T} , \tag{4.10.13}$$

for some scale parameter b.

The function $J_{211}^{(T)}(u,v)$ is defined by

$$J_{211}^{(T)}(u,v) = \#\Big\{ u - \tfrac{b}{2} < t_j - s_k < u + \tfrac{b}{2} \text{ and } v - \tfrac{b}{2} < t_j - s_l < v + \tfrac{b}{2}$$

for distinct triples (t_j, s_k, s_l); $j = 1,\ldots,N_2(T)$ and $k \neq l = 1,\ldots,N_1(T) \Big\}$. (4.10.14)

The two ordered sequences $t_1 < t_2 < \cdots < t_{N_2(T)}$ and $s_1 < s_2 < \cdots < s_{N_1(T)}$ are the events of the two processes N_2 and N_1 respectively in the time interval $(0,T]$.

The computation of the function $J_{211}^{(T)}(u,v)$ is exceedingly time consuming for a large record of data. This suggests that, for large data sets, we should split the data into a number of disjoint stretches and calculate estimate (4.10.14) for each stretch. The final estimate is then obtained by averaging the separate estimates. Mathematically, this procedure is expressed as follows

$$J_{211}^{(T)}(u,v) = \frac{1}{l} \sum_{j=1}^{l} J_{211}^{(Rj)}(u,v) , \tag{4.10.15}$$

where $T = Rj$; $j = 1,2,\ldots,l$.

The quantity $m_{211}(u,v)$, under the assumption that N_1 is a Poisson point process, may be estimated by

$$\hat{m}_{211}(u,v) = \frac{\hat{p}_{211}(u,u-v)}{\hat{p}_1^2} \quad . \tag{4.10.16}$$

Theorem 4.10.1: Suppose that the bivariate process $\underset{\sim}{N}$ satisfies the Assumption 1.2 of Appendix I. Let $p_{211}(u,v)$ be continuous at (u,v). Then as $T \longmapsto \infty$,

(i) the variate (4.10.13) is asymptotically Poisson with mean $b^2Tp_{211}(u,v)$ and

(ii) if $b \longmapsto o$, but $b^2T \longmapsto \infty$, the variate (4.10.13) is asymptotically normal with variance $b^2Tp_{211}(u,v)$.

Corollary 4.10.1: Under the conditions of Theorem (4.10.1) we have (i) as $T \longmapsto \infty$, the variate $\hat{p}_{211}(u,v)$ is asymptotically $(b^2T)^{-1}$ Po$(b^2Tp_{211}(u,v)$ and (ii) if $b \longmapsto o$, but $b^2T \longmapsto \infty$, the variate $\hat{p}_{211}(u,v)$ is asymptotically normal with mean $p_{211}(u,v)$ and variance $(b_T^2T)^{-1}p_{211}(u,v)$.
[Po(α) denotes a Poisson distribution with mean α] .

Corollary 4.10.2: Under the conditions of Theorem 4.10.1 the estimate $(\hat{p}_{211}(u,v))^{\frac{1}{2}}$ is normally distributed with mean $(p_{211}(u,v))^{\frac{1}{2}}$ and stable variance $(4b^2T)^{-1}$.

These results are discussed by Brillinger (1975b) An estimate of the third order cumulant function is now given by

$$\hat{q}_{211}(u,v) = \hat{p}_{211}(u,v) - \hat{p}_1\hat{p}_{21}(u) - \hat{p}_1\hat{p}_{21}(v) - \hat{p}_2\hat{p}_{11}(u-v) + 2\hat{p}_2\hat{p}_1^2 . \tag{4.10.17}$$

Alternatively, from (4.10.12) we have

$$\hat{g}_{211}(u,v) = \left(\frac{2\pi}{T}\right)^2 \sum_{\substack{j\neq 0 \\ j\neq -k}} \sum_{k\neq 0} I_{211}^{(T)}(\lambda_j, \lambda_k)\, e^{i(\lambda_j u + \lambda_k v)}, \quad u\neq v,\ u\neq 0,\ v\neq 0, \tag{4.10.18}$$

where $I_{211}^{(T)}(\lambda, \mu)$ is the third-order periodogram defined as

$$I_{211}^{(T)}(\lambda,\mu) = \frac{1}{(2\pi)^2 T}\, d_2^{(T)}(\lambda)\, d_1^{(T)}(\mu)\, \overline{d_1^{(T)}(\lambda+\mu)}, \quad -\infty < \lambda, \mu < \infty . \tag{4.10.19}$$

The definition of the third-order periodogram is considered by Brillinger and Rosenblatt (1967a) in the case of ordinary time series.

We examine now the properties of the modified third-order periodogram given by

$$I_{k_1' k_2' k_3'}^{(T)}(\lambda,\mu) = \frac{1}{(2\pi)^2 T}\, d_{k_1'}^{(T)}(\lambda)\, d_{k_2'}^{(T)}(\mu)\, \overline{d_{k_3'}^{(T)}(\lambda+\mu)}, \quad k_1, k_2, k_3 = 1,2, \tag{4.10.20}$$

where $d_{k_1'}^{(T)}(\lambda)$ is defined as

$$d_{k_1'}^{(T)}(\lambda) = \int_0^T e^{-i\lambda t}\, dN_{k_1'}(t),$$

and $N_{k_1'}(t)$ is the zero-mean point process presented in Chapter 3.

<u>Theorem 4.10.2</u>: Let N(t) be a stationary bivariate point process on (0,T] satisfying Assumption 1.2 of Appendix I. Let $I_{k_1' k_2' k_3'}^{(T)}(\lambda, \mu)$ be given by (4.10.20). Then

$$E\left\{ I_{k_1' k_2' k_3'}^{(T)}(\lambda,\mu)\right\} = f_{k_1 k_2 k_3}(\lambda,\mu) + O(T^{-1}), \quad k_1, k_2, k_3 = 1,2 .$$

<u>Proof</u>: We write $I_{k_1' k_2' k_3'}^{(T)}(\lambda, \mu)$ as follows

$$I^{(T)}_{k_1'k_2'k_3'}(\lambda,\mu) = \frac{1}{(2\pi)^2 T}\, d^{(T)}_{k_1'}(\lambda)\, d^{(T)}_{k_2'}(\mu)\, \overline{d^{(T)}_{k_3'}(\lambda+\mu)}$$

$$= \frac{1}{(2\pi)^2 T}\left[d^{(T)}_{k_1}(\lambda) - p_{k_1}\Delta^{(T)}(\lambda)\right]\left[d^{(T)}_{k_2}(\mu) - p_{k_2}\Delta^{(T)}(\mu)\right]\left[d^{(T)}_{k_3}(-\lambda-\mu) - p_{k_3}\Delta^{(T)}(-\lambda-\mu)\right] ,$$

where

$$\Delta^{(T)}(\lambda) = \int_0^T e^{-i\lambda t}\, dN(t) \quad .$$

So, we have

$$E\left\{ I^{(T)}_{k_1'k_2'k_3'}(\lambda,\mu)\right\} = \frac{1}{(2\pi)^2 T}\, E\left\{\left[d^{(T)}_{k_1}(\lambda) - p_{k_1}\Delta^{(T)}(\lambda)\right]\left[d^{(T)}_{k_2}(\mu) - p_{k_2}\Delta^{(T)}(\mu)\right]\right.$$

$$\left.\left[d^{(T)}_{k_3}(-\lambda-\mu) - p_{k_3}\Delta^{(T)}(-\lambda-\mu)\right]\right\} = \frac{1}{(2\pi)^2 T}\, \mathrm{cum}\left\{d^{(T)}_{k_1}(\lambda),\, d^{(T)}_{k_2}(\mu),\, d^{(T)}_{k_3}(-\lambda-\mu)\right\} ,$$

and by applying Lemma 2.1 of Appendix I we get

$$E\left\{ I^{(T)}_{k_1'k_2'k_3'}(\lambda,\mu)\right\} = \frac{1}{(2\pi)^2 T}\left\{ (2\pi)^2 T f_{k_1k_2k_3}(\lambda,\mu) + O(T^{-1})\right\}$$

$$= f_{k_1k_2k_3}(\lambda,\mu) + O(T^{-1}) \quad .$$

<u>Theorem 4.10.3</u>: Let $\underset{\sim}{N}(t)$ be a stationary process satisfying assumptions of Theorem 4.10.2. Let $I^{(T)}_{k_1' k_2' k_3'}(\lambda , \mu)$ be given by (4.10.20). Then

$$\mathrm{cov}\left\{ I^{(T)}_{k_1'k_2'k_3'}(\lambda_1,\lambda_2),\, I^{(T)}_{\ell_1'\ell_2'\ell_3'}(\mu_1,\mu_2)\right\} = O(T^{-1});\ \lambda_1,\lambda_2,\lambda_3,\lambda_4 \neq 0,\ \lambda_1 \neq -\lambda_2,\ \mu_1 \neq -\mu_2,\ \lambda_1 \neq \pm\mu_1,$$

$$\lambda_1 \neq \pm\mu_2,\ \lambda_2 \neq \pm\mu_1,\ \lambda_2 \neq \pm\mu_2,\ \lambda_1 \neq \pm(\mu_1+\mu_2),\ \lambda_2 \neq \pm(\mu_1+\mu_2),\ \mu_1 \neq \pm(\lambda_1+\lambda_2) \text{ and } \mu_2 \neq \pm(\lambda_1+\lambda_2).$$

Proof: The covariance of $I^{(T)}_{k_1' k_2' k_3'}(\lambda_1, \lambda_2)$ and $I^{(T)}_{l_1' l_2' l_3'}(\mu_1, \mu_2)$ can be written as

$$\operatorname{cov}\left\{ I^{(T)}_{k_1'k_2'k_3'}(\lambda_1,\lambda_2),\ I^{(T)}_{\ell_1'\ell_2'\ell_3'}(\mu_1,\mu_2)\right\}$$

$$= \frac{1}{(2\pi)^4 T^2}\operatorname{cum}\left\{ d^{(T)}_{k_1'}(\lambda_1)\, d^{(T)}_{k_2'}(\lambda_2)\, d^{(T)}_{k_3'}(-\lambda_1-\lambda_2),\ d^{(T)}_{\ell_1'}(-\mu_1)\, d^{(T)}_{\ell_2'}(-\mu_2)\, d^{(T)}_{\ell_3'}(\mu_1+\mu_2)\right\}.$$

Now, by using properties of the cumulant functions and after some extensive calculations we find

$$\operatorname{cov}\left\{ I^{(T)}_{k_1'k_2'k_3'}(\lambda_1,\lambda_2),\ I^{(T)}_{\ell_1'\ell_2'\ell_3'}(\mu_1,\mu_2)\right\}$$

$$= \Big[\operatorname{cum}\left\{ d^{(T)}_{k_1}(\lambda_1),\ d^{(T)}_{k_2}(\lambda_2),\ d^{(T)}_{k_3}(-\lambda_1-\lambda_2),\ d^{(T)}_{\ell_1}(-\mu_1),\ d^{(T)}_{\ell_2}(-\mu_2),\ d^{(T)}_{\ell_3}(\mu_1+\mu_2)\right.$$

$$+ \operatorname{cum}\left\{ d^{(T)}_{k_1}(\lambda_1),\ d^{(T)}_{k_2}(\lambda_2)\right\}\operatorname{cum}\left\{ d^{(T)}_{k_3}(-\lambda_1-\lambda_2),\ d^{(T)}_{\ell_1}(-\mu_1),\ d^{(T)}_{\ell_2}(-\mu_2),\ d^{(T)}_{\ell_3}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{ d^{(T)}_{k_1}(\lambda_1),\ d^{(T)}_{k_3}(-\lambda_1-\lambda_2)\right\}\operatorname{cum}\left\{ d^{(T)}_{k_2}(\lambda_2),\ d^{(T)}_{\ell_1}(-\mu_1),\ d^{(T)}_{\ell_2}(-\mu_2),\ d^{(T)}_{\ell_3}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{ d^{(T)}_{k_1}(\lambda_1),\ d^{(T)}_{\ell_1}(-\mu_1)\right\}\operatorname{cum}\left\{ d^{(T)}_{k_3}(-\lambda_1-\lambda_2),\ d^{(T)}_{k_2}(\lambda_2),\ d^{(T)}_{\ell_2}(-\mu_2),\ d^{(T)}_{\ell_3}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{ d^{(T)}_{k_1}(\lambda_1),\ d^{(T)}_{\ell_2}(-\mu_2)\right\}\operatorname{cum}\left\{ d^{(T)}_{k_3}(-\lambda_1-\lambda_2),\ d^{(T)}_{k_2}(\lambda_2),\ d^{(T)}_{\ell_1}(-\mu_1),\ d^{(T)}_{\ell_3}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{ d^{(T)}_{k_2}(\lambda_2),\ d^{(T)}_{k_3}(-\lambda_1-\lambda_2)\right\}\operatorname{cum}\left\{ d^{(T)}_{k_1}(\lambda_1),\ d^{(T)}_{\ell_1}(-\mu_1),\ d^{(T)}_{\ell_2}(-\mu_2),\ d^{(T)}_{\ell_3}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_2}^{(T)}(\lambda_2), d_{\ell_1}^{(T)}(-\mu_1)\right\} \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_2}^{(T)}(-\mu_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_2}^{(T)}(\lambda_2), d_{\ell_2}^{(T)}(-\mu_2)\right\} \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_1}^{(T)}(-\mu_1), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_1}^{(T)}(-\mu_1)\right\} \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_2}^{(T)}(\lambda_2), d_{\ell_2}^{(T)}(-\mu_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_2}^{(T)}(-\mu_2)\right\} \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_2}^{(T)}(\lambda_2), d_{\ell_1}^{(T)}(-\mu_1), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\} \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_2}^{(T)}(\lambda_2), d_{\ell_1}^{(T)}(-\mu_1), d_{\ell_2}^{(T)}(-\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{\ell_1}^{(T)}(-\mu_1), d_{\ell_2}^{(T)}(-\mu_2)\right\} \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_2}^{(T)}(\lambda_2), d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{\ell_1}^{(T)}(-\mu_1), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\} \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_2}^{(T)}(\lambda_2), d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_2}^{(T)}(-\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{\ell_2}^{(T)}(-\mu_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\} \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_2}^{(T)}(\lambda_2), d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_2}^{(T)}(\lambda_2), d_{\ell_1}^{(T)}(-\mu_1)\right\} \operatorname{cum}\left\{d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_2}^{(T)}(-\mu_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_2}^{(T)}(\lambda_2), d_{\ell_2}^{(T)}(-\mu_2)\right\} \operatorname{cum}\left\{d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_1}^{(T)}(-\mu_1), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_2}^{(T)}(\lambda_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\} \operatorname{cum}\left\{d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_1}^{(T)}(-\mu_1), d_{\ell_2}^{(T)}(-\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_1}^{(T)}(-\mu_1)\right\} \operatorname{cum}\left\{d_{K_2}^{(T)}(\lambda_2), d_{\ell_2}^{(T)}(-\mu_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_2}^{(T)}(-\mu_2)\right\} \operatorname{cum}\left\{d_{K_2}^{(T)}(\lambda_2), d_{\ell_1}^{(T)}(-\mu_1), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\} \operatorname{cum}\left\{d_{K_2}^{(T)}(\lambda_2), d_{\ell_1}^{(T)}(-\mu_1), d_{\ell_2}^{(T)}(-\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{\ell_1}^{(T)}(-\mu_1), d_{\ell_2}^{(T)}(-\mu_2)\right\} \operatorname{cum}\left\{d_{K_2}^{(T)}(\lambda_2), d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{\ell_1}^{(T)}(-\mu_1), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\} \operatorname{cum}\left\{d_{K_2}^{(T)}(\lambda_2), d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_2}^{(T)}(-\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{\ell_2}^{(T)}(-\mu_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\} \operatorname{cum}\left\{d_{K_2}^{(T)}(\lambda_2), d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_1}^{(T)}(-\mu_1)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_2}^{(T)}(\lambda_2)\right\} \operatorname{cum}\left\{d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_1}^{(T)}(-\mu_1)\right\} \operatorname{cum}\left\{d_{\ell_2}^{(T)}(-\mu_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_2}^{(T)}(\lambda_2)\right\} \operatorname{cum}\left\{d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_2}^{(T)}(-\mu_2)\right\} \operatorname{cum}\left\{d_{\ell_1}^{(T)}(-\mu_1), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_2}^{(T)}(\lambda_2)\right\} \operatorname{cum}\left\{d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\} \operatorname{cum}\left\{d_{\ell_1}^{(T)}(-\mu_1), d_{\ell_2}^{(T)}(-\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_3}^{(T)}(-\lambda_1-\lambda_2)\right\} \operatorname{cum}\left\{d_{K_2}^{(T)}(\lambda_2), d_{\ell_1}^{(T)}(-\mu_1)\right\} \operatorname{cum}\left\{d_{\ell_2}^{(T)}(-\mu_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_3}^{(T)}(-\lambda_1-\lambda_2)\right\} \operatorname{cum}\left\{d_{K_2}^{(T)}(\lambda_2), d_{\ell_2}^{(T)}(-\mu_2)\right\} \operatorname{cum}\left\{d_{\ell_1}^{(T)}(-\mu_1), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\}$$

$$+ \operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1), d_{K_3}^{(T)}(-\lambda_1-\lambda_2)\right\} \operatorname{cum}\left\{d_{K_2}^{(T)}(\lambda_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\right\} \operatorname{cum}\left\{d_{\ell_1}^{(T)}(-\mu_1), d_{\ell_2}^{(T)}(-\mu_2)\right\}$$

$$+ \operatorname{cum}\{d_{K_1}^{(T)}(\lambda_1), d_{\ell_1}^{(T)}(-\mu_1)\} \operatorname{cum}\{d_{K_2}^{(T)}(\lambda_2), d_{K_3}^{(T)}(-\lambda_1-\lambda_2)\} \operatorname{cum}\{d_{\ell_2}^{(T)}(-\mu_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\}$$

$$+ \operatorname{cum}\{d_{K_1}^{(T)}(\lambda_1), d_{\ell_1}^{(T)}(-\mu_1)\} \operatorname{cum}\{d_{K_2}^{(T)}(\lambda_2), d_{\ell_2}^{(T)}(-\mu_2)\} \operatorname{cum}\{d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\}$$

$$+ \operatorname{cum}\{d_{K_1}^{(T)}(\lambda_1), d_{\ell_1}^{(T)}(-\mu_1)\} \operatorname{cum}\{d_{K_2}^{(T)}(\lambda_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\} \operatorname{cum}\{d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_2}^{(T)}(-\mu_2)\}$$

$$+ \operatorname{cum}\{d_{K_1}^{(T)}(\lambda_1), d_{\ell_2}^{(T)}(-\mu_2)\} \operatorname{cum}\{d_{K_2}^{(T)}(\lambda_2), d_{K_3}^{(T)}(-\lambda_1-\lambda_2)\} \operatorname{cum}\{d_{\ell_1}^{(T)}(-\mu_1), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\}$$

$$+ \operatorname{cum}\{d_{K_1}^{(T)}(\lambda_1), d_{\ell_2}^{(T)}(-\mu_2)\} \operatorname{cum}\{d_{K_2}^{(T)}(\lambda_2), d_{\ell_1}^{(T)}(-\mu_1)\} \operatorname{cum}\{d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\}$$

$$+ \operatorname{cum}\{d_{K_1}^{(T)}(\lambda_1), d_{\ell_2}^{(T)}(-\mu_2)\} \operatorname{cum}\{d_{K_2}^{(T)}(\lambda_2), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\} \operatorname{cum}\{d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_1}^{(T)}(-\mu_1)\}$$

$$+ \operatorname{cum}\{d_{K_1}^{(T)}(\lambda_1), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\} \operatorname{cum}\{d_{K_2}^{(T)}(\lambda_2), d_{K_3}^{(T)}(-\lambda_1-\lambda_2)\} \operatorname{cum}\{d_{\ell_1}^{(T)}(-\mu_1), d_{\ell_2}^{(T)}(-\mu_2)\}$$

$$+ \operatorname{cum}\{d_{K_1}^{(T)}(\lambda_1), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\} \operatorname{cum}\{d_{K_2}^{(T)}(\lambda_2), d_{\ell_1}^{(T)}(-\mu_1)\} \operatorname{cum}\{d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_2}^{(T)}(-\mu_2)\}$$

$$+ \operatorname{cum}\{d_{K_1}^{(T)}(\lambda_1), d_{\ell_3}^{(T)}(\mu_1+\mu_2)\} \operatorname{cum}\{d_{K_2}^{(T)}(\lambda_2), d_{\ell_2}^{(T)}(-\mu_2)\} \operatorname{cum}\{d_{K_3}^{(T)}(-\lambda_1-\lambda_2), d_{\ell_1}^{(T)}(-\mu_1)\}\Big].$$

This expression can further be written by using Lemma 2.1 of Appendix I as

$$\operatorname{cov}\{I_{K_1'K_2'K_3'}^{(T)}(\lambda_1,\lambda_2), I_{\ell_1'\ell_2'\ell_3'}^{(T)}(\mu_1,\mu_2)\} = \frac{2\pi}{T} f_{K_1K_2K_3\ell_1\ell_2\ell_3}(\lambda_1,\lambda_2,-\lambda_1-\lambda_2,-\mu_1,-\mu_2)$$

$$+\frac{1}{T^2}|\Delta^{(T)}(\lambda_1-\lambda_2)|^2 f_{K_1K_2}(\lambda_1) f_{K_3\ell_1\ell_2\ell_3}(-\lambda_1-\lambda_2,-\mu_1,-\mu_2) + \frac{1}{T^2}|\Delta^{(T)}(\lambda_2)|^2 f_{K_1K_3}(\lambda_1) f_{K_2\ell_1\ell_2\ell_3}(\lambda_2,-\mu_1,-\mu_2)$$

$$+\frac{1}{T^2}\left|\Delta^{(T)}(\lambda_1+\mu_1)\right|^2 f_{k_1\ell_1}(\lambda_1)\, f_{k_2k_3\ell_2\ell_3}(\lambda_2,-\lambda_1-\lambda_2,-\mu_2)+\frac{1}{T^2}\left|\Delta^{(T)}(\lambda_1+\mu_2)\right|^2 f_{k_1\ell_2}(\lambda_1)\, f_{k_2k_3\ell_1\ell_3}(\lambda_2,-\lambda_1-\lambda_2,-\mu_1)$$

$$+\frac{1}{T^2}\left|\Delta^{(T)}(\lambda_1)\right|^2 f_{k_2k_3}(\lambda_2)\, f_{k_1\ell_1\ell_2\ell_3}(\lambda_1,-\mu_1,-\mu_2)+\frac{1}{T^2}\left|\Delta^{(T)}(\lambda_2+\mu_1)\right|^2 f_{k_2\ell_1}(\lambda_2)\, f_{k_1k_3\ell_2\ell_3}(\lambda_1,-\lambda_1-\lambda_2,-\mu_2)$$

$$+\frac{1}{T^2}\left|\Delta^{(T)}(\lambda_2+\mu_2)\right|^2 f_{k_2\ell_2}(\lambda_2)\, f_{k_1k_3\ell_1\ell_3}(\lambda_1,-\lambda_1-\lambda_2,-\mu_1)+\frac{1}{T^2}\left|\Delta^{(T)}(\lambda_1+\lambda_2-\mu_1)\right|^2 f_{k_3\ell_1}(-\lambda_1-\lambda_2)\, f_{k_1k_2\ell_2\ell_3}(\lambda_1,\lambda_2,-\mu_2)$$

$$+\frac{1}{T^2}\left|\Delta^{(T)}(\lambda_1+\lambda_2-\mu_2)\right|^2 f_{k_3\ell_2}(-\lambda_1-\lambda_2)\, f_{k_1k_2\ell_1\ell_3}(\lambda_1,\lambda_2,-\mu_1)+\frac{1}{T^2}\left|\Delta^{(T)}(\lambda_1+\lambda_2-\mu_1-\mu_2)\right|^2 f_{k_3\ell_3}(-\lambda_1-\lambda_2)\, f_{k_1k_2\ell_1\ell_2}(\lambda_1,\lambda_2,-\mu_1)$$

$$+\frac{1}{T^2}\left|\Delta^{(T)}(\mu_1-\mu_2)\right|^2 f_{\ell_1\ell_2}(-\mu_1)\, f_{k_1k_2k_3\ell_3}(\lambda_1,\lambda_2,-\lambda_1-\lambda_2)+\frac{1}{T^2}\left|\Delta^{(T)}(\mu_2)\right|^2 f_{\ell_1\ell_3}(-\mu_1)\, f_{k_1k_2k_3\ell_2}(\lambda_1,\lambda_2,-\lambda_1-\lambda_2)$$

$$+\frac{1}{T^2}\left|\Delta^{(T)}(\mu_1)\right|^2 f_{\ell_2\ell_3}(-\mu_2)\, f_{k_1k_2k_3\ell_1}(\lambda_1,\lambda_2,-\lambda_1-\lambda_2)+\frac{1}{T^2}\left|\Delta^{(T)}(\lambda_1+\lambda_2+\mu_1)\right|^2 f_{k_1k_2\ell_1}(\lambda_1,\lambda_2)\, f_{k_3\ell_2\ell_3}(-\lambda_1-\lambda_2,-\mu_2)$$

$$+\frac{1}{T^2}\left|\Delta^{(T)}(\lambda_1+\lambda_2+\mu_2)\right|^2 f_{k_1k_2\ell_2}(\lambda_1,\lambda_2)\, f_{k_3\ell_1\ell_3}(-\lambda_1-\lambda_2,-\mu_1)+\frac{1}{T^2}\left|\Delta^{(T)}(\lambda_1+\lambda_2+\mu_1+\mu_2)\right|^2 f_{k_1k_2\ell_3}(\lambda_1,\lambda_2)\, f_{k_3\ell_1\ell_2}(-\lambda_1-\lambda_2,-\mu_1)$$

$$+\frac{1}{T^2}\left|\Delta^{(T)}(\lambda_2+\mu_1)\right|^2 f_{k_1k_3\ell_1}(\lambda_1,-\lambda_1-\lambda_2)\, f_{k_2\ell_2\ell_3}(\lambda_2,-\mu_2)+\frac{1}{T^2}\left|\Delta^{(T)}(\lambda_2+\mu_2)\right|^2 f_{k_1k_3\ell_2}(\lambda_1,-\lambda_1-\lambda_2)\, f_{k_2\ell_1\ell_3}(\lambda_2,-\mu_1)$$

$$+\frac{1}{T^2}\left|\Delta^{(T)}(\lambda_1-\mu_1-\mu_2)\right|^2 f_{k_1\ell_1\ell_2}(\lambda_1,-\mu_1)\, f_{k_2k_3\ell_3}(\lambda_2,-\lambda_1-\lambda_2)+\frac{1}{T^2}\left|\Delta^{(T)}(\lambda_1+\mu_2)\right|^2 f_{k_1\ell_1\ell_3}(\lambda_1,-\mu_1)\, f_{k_2k_3\ell_2}(\lambda_2,-\lambda_1-\lambda_2)$$

$$+\frac{1}{T^2}\left|\Delta^{(T)}(\lambda_1+\mu_1)\right|^2 f_{k_1\ell_2\ell_3}(\lambda_1,-\mu_2)\, f_{k_2k_3\ell_1}(\lambda_2,-\lambda_1-\lambda_2)$$

$$+\frac{2\pi}{T^2}\,\Delta^{(T)}(\lambda_1+\lambda_2)\,\Delta^{(T)}(-\lambda_1-\lambda_2-\mu_1)\,\Delta^{(T)}(\mu_1)\, f_{k_1k_2}(\lambda_1)\, f_{k_3\ell_1}(-\lambda_1-\lambda_2)\, f_{\ell_2\ell_3}(-\mu_2)$$

$$+\frac{2\pi}{T^2}\Delta^{(T)}(\lambda_1+\lambda_2)\,\Delta^{(T)}(-\lambda_1-\lambda_2-\mu_2)\,\Delta^{(T)}(\mu_2)\,f_{k_1k_2}(\lambda_1)\,f_{k_3\ell_2}(-\lambda_1-\lambda_2)\,f_{\ell_1\ell_3}(-\mu_1)$$

$$+\frac{2\pi}{T^2}\Delta^{(T)}(\lambda_1+\lambda_2)\,\Delta^{(T)}(-\lambda_1-\lambda_2+\mu_1+\mu_2)\,\Delta^{(T)}(-\mu_1-\mu_2)\,f_{k_1k_2}(\lambda_1)\,f_{k_3\ell_3}(-\lambda_1-\lambda_2)\,f_{\ell_1\ell_2}(-\mu_1)$$

$$+\frac{2\pi}{T^2}\Delta^{(T)}(-\lambda_2)\,\Delta^{(T)}(\lambda_2-\mu_1)\,\Delta^{(T)}(\mu_1)\,f_{k_1k_3}(\lambda_1)\,f_{k_2\ell_1}(\lambda_2)\,f_{\ell_2\ell_3}(-\mu_2)$$

$$+\frac{2\pi}{T^2}\Delta^{(T)}(-\lambda_2)\,\Delta^{(T)}(\lambda_2-\mu_2)\,\Delta^{(T)}(\mu_2)\,f_{k_1k_3}(\lambda_1)\,f_{k_2\ell_2}(\lambda_2)\,f_{\ell_1\ell_3}(-\mu_1)$$

$$+\frac{2\pi}{T^2}\Delta^{(T)}(-\lambda_2)\,\Delta^{(T)}(\lambda_2+\mu_1+\mu_2)\,\Delta^{(T)}(-\mu_1-\mu_2)\,f_{k_1k_3}(\lambda_1)\,f_{k_2\ell_3}(\lambda_2)\,f_{\ell_1\ell_2}(-\mu_1)$$

$$+\frac{2\pi}{T^2}\Delta^{(T)}(\lambda_1-\mu_1)\,\Delta^{(T)}(-\lambda_1)\,\Delta^{(T)}(\mu_1)\,f_{k_1\ell_1}(\lambda_1)\,f_{k_2k_3}(\lambda_2)\,f_{\ell_2\ell_3}(-\mu_2)$$

$$+\frac{2\pi}{T^2}\Delta^{(T)}(\lambda_1-\mu_1)\,\Delta^{(T)}(\lambda_2-\mu_2)\,\Delta^{(T)}(-\lambda_1-\lambda_2+\mu_1+\mu_2)\,f_{k_1\ell_1}(\lambda_1)\,f_{k_2\ell_2}(\lambda_2)\,f_{k_3\ell_3}(-\lambda_1-\lambda_2)$$

$$+\frac{2\pi}{T^2}\Delta^{(T)}(\lambda_1-\mu_1)\,\Delta^{(T)}(\lambda_2+\mu_1+\mu_2)\,\Delta^{(T)}(-\lambda_1-\lambda_2-\mu_2)\,f_{k_1\ell_1}(\lambda_1)\,f_{k_2\ell_3}(\lambda_2)\,f_{k_3\ell_2}(-\lambda_1-\lambda_2)$$

$$+\frac{2\pi}{T^2}\Delta^{(T)}(\lambda_1-\mu_2)\,\Delta^{(T)}(-\lambda_1)\,\Delta^{(T)}(\mu_2)\,f_{k_1\ell_2}(\lambda_1)\,f_{k_2k_3}(\lambda_2)\,f_{\ell_1\ell_3}(-\mu_1)$$

$$+\frac{2\pi}{T^2}\Delta^{(T)}(\lambda_1-\mu_2)\,\Delta^{(T)}(\lambda_2-\mu_1)\,\Delta^{(T)}(\lambda_1-\lambda_2+\mu_1+\mu_2)\,f_{k_1\ell_2}(\lambda_1)\,f_{k_2\ell_1}(\lambda_2)\,f_{k_3\ell_3}(-\lambda_1-\lambda_2)$$

$$+\frac{2\pi}{T^2}\Delta^{(T)}(\lambda_1-\mu_2)\,\Delta^{(T)}(\lambda_2+\mu_1+\mu_2)\,\Delta^{(T)}(-\lambda_1-\lambda_2-\mu_1)\,f_{k_1\ell_2}(\lambda_1)\,f_{k_2\ell_3}(\lambda_2)\,f_{k_3\ell_1}(-\lambda_1-\lambda_2)$$

$$+\frac{2\pi}{T^2}\Delta^{(T)}(\lambda_1+\mu_1+\mu_2)\,\Delta^{(T)}(-\lambda_1)\,\Delta^{(T)}(-\mu_1-\mu_2)\,f_{k_1\ell_3}(\lambda_1)\,f_{k_2k_3}(\lambda_2)\,f_{\ell_1\ell_2}(-\mu_1)$$

$$+ \frac{2\pi}{T^2} \Delta^{(T)}(\lambda_1+\mu_1+\mu_2)\, \Delta^{(T)}(\lambda_2-\mu_1)\, \Delta^{(T)}(-\lambda_1-\lambda_2-\mu_2)\, f_{k_1\ell_3}(\lambda_1) f_{k_2\ell_1}(\lambda_2) f_{k_3\ell_2}(-\lambda_1-\lambda_2)$$

$$+ \frac{2\pi}{T^2} \Delta^{(T)}(\lambda_1+\mu_1+\mu_2)\, \Delta^{(T)}(\lambda_2-\mu_2)\, \Delta^{(T)}(-\lambda_1-\lambda_2-\mu_1)\, f_{k_1\ell_3}(\lambda_1) f_{k_2\ell_2}(\lambda_2) f_{k_3\ell_1}(-\lambda_1-\lambda_2)$$

$$= O(T^{-1}) \quad \text{if} \quad \lambda_1, \lambda_2, \mu_1, \mu_2 \neq 0, \ \lambda_1 \neq -\lambda_2, \ \mu_1 \neq -\mu_2, \ \lambda_1 \neq \pm\mu_1,$$

$$\lambda_1 \neq \pm\mu_2, \ \lambda_2 \neq \pm\mu_1, \ \lambda_2 \neq \pm\mu_2, \ \lambda_1 \neq \pm(\mu_1+\mu_2), \ \lambda_2 \neq \pm(\mu_1+\mu_2), \ \mu_1 \neq \pm(\lambda_1+\lambda_2)$$

$$\text{and} \quad \mu_2 \neq \pm(\lambda_1+\lambda_2).$$

In the next two theorems we develop the first- and second-order properties of the estimate of $\hat{q}_{k_1k_2k_3}(u_1,u_2)$ given by (4.10.18).

<u>Theorem 4.10.4</u>: Let $\underset{\sim}{N}(t)$ be a stationary bivariate point process on (0,T] satisfying Assumption 1.2 of Appendix I. Let $\hat{q}_{k_1k_2k_3}(u_1,u_2)$ be given by (4.10.18) for general indices k_1,k_2,k_3. Then

$$E\left\{\hat{q}_{k_1k_2k_3}(u_1,u_2)\right\} = q_{k_1k_2k_3}(u_1,u_2) + O(T^{-1}) .$$

<u>Proof</u>: The estimate of $q_{k_1k_2k_3}(u_1,u_2)$ can be written as a function of the modified periodogram in the following way

$$\hat{q}_{k_1k_2k_3}(u_1,u_2) = \iint_{-\pi/b}^{\pi/b} I^{(T)}_{k_1'k_2'k_3'}(\lambda_1,\lambda_2)\, e^{i(\lambda_1u_1+\lambda_2u_2)} d\lambda_1 d\lambda_2 \ , \quad \begin{matrix} u_1 \neq u_2 \\ u_1 \neq 0, u_2 \neq 0 . \end{matrix}$$

The expected value will be then given by

$$E\{\hat{q}_{k_1k_2k_3}(u_1,u_2)\} = \iint_{-\pi/b}^{\pi/b} E\{I^{(T)}_{k_1k_2k_3}(\lambda_1,\lambda_2)\} e^{i(\lambda_1u_1+\lambda_2u_2)} d\lambda_1 d\lambda_2 ,$$

and by using Theorem 4.10.2 we obtain

$$E\{\hat{q}_{k_1k_2k_3}(u_1,u_2)\} = \iint f_{k_1k_2k_3}(\lambda_1,\lambda_2) e^{i(\lambda_1u_1+\lambda_2u_2)} d\lambda_1 d\lambda_2 + O(T^{-1})$$

$$= q_{k_1k_2k_3}(u_1,u_2) + O(T^{-1}) .$$

This result follows from the fact that $\hat{q}_{k_1k_2k_3}(u_1,u_2)$ is sampled in such a way that

$$u = bj \text{ and } v = bj' \text{ for } j,j' = 1,2,\ldots \; (j \neq j') .$$

<u>Theorem 4.10.5</u>: Let $\underset{\sim}{N}(t)$ be a stationary bivariate point process satisfying the conditions of Theorem 4.10.4. Let $\hat{q}_{k_1k_2k_3}(u_1,u_2)$ be given by (4.10.18). Then

$$\operatorname{cov}\{\hat{q}_{k_1k_2k_3}(u_1,u_2), \hat{q}_{\ell_1\ell_2\ell_3}(v_1,v_2)\}$$

$$= \frac{2\pi}{bT} \iiint f_{k_1\ell_1}(\lambda_1) f_{k_2k_3\ell_2\ell_3}(-\lambda_2,-\lambda_1-\lambda_2,-\mu_2) e^{i\lambda_1(u_1-v_1)} e^{i(\lambda_2u_2-\mu_2v_2)} d\lambda_1 d\lambda_2 d\mu_2$$

$$+ \frac{2\pi}{bT} \iiint |\Delta^{(T)}(\lambda_1)|^2 f_{k_1\ell_1}(\lambda_1) f_{k_2\ell_2}(\lambda_2) f_{k_3\ell_3}(-\mu_2) e^{i\lambda_1(u_1-v_1)} e^{i(\lambda_2u_2-\mu_2v_2)} d\lambda_1 d\lambda_2 d\mu_2$$

$$+\frac{2\pi}{bT}\iint f_{K_1\ell_1}(\lambda_1)f_{K_2\ell_2}(\lambda_2)f_{K_3\ell_3}(-\lambda_1-\lambda_2)\,e^{i\lambda_1(u_1-v_1)}\,e^{i\lambda_2(u_2-v_2)}d\lambda_1 d\lambda_2$$

$$+\frac{2\pi}{bT}\iint f_{K_1\ell_1}(\lambda_1)f_{K_2\ell_3}(\lambda_2)f_{K_3\ell_2}(-\lambda_1-\lambda_2)\,e^{-i\lambda_1 u_2}d\lambda_1 d\lambda_2 + O(T^{-1})\,.$$

<u>Proof</u>: We have that

$$\mathrm{cov}\left\{\hat{q}_{K_1K_2K_3}(u_1,u_2)\,,\ \hat{q}_{\ell_1\ell_2\ell_3}(v_1,v_2)\right\} = \iiiint \mathrm{cov}\left\{I^{(T)}_{K_1'K_2'K_3'}(\lambda_1,\lambda_2),\ I^{(T)}_{\ell_1'\ell_2'\ell_3'}(\mu_1,\mu_2)\right\}$$

$$e^{i(\lambda_1u_1+\lambda_2u_2)}e^{-i(\mu_1v_1+\mu_2v_2)}d\lambda_1 d\lambda_2 d\mu_1 d\mu_2$$

$$=\frac{1}{(2\pi)^4T^2}\iiiint \mathrm{cov}\left\{d^{(T)}_{K_1'}(\lambda_1)d^{(T)}_{K_2'}(\lambda_2)d^{(T)}_{K_3'}(-\lambda_1-\lambda_2)\,,\ d^{(T)}_{\ell_1'}(\mu_1)d^{(T)}_{\ell_2'}(\mu_2)d^{(T)}_{\ell_3'}(-\mu_1-\mu_2)\right\}$$

$$e^{i(\lambda_1u_1-\mu_1v_1)}\,e^{i(\lambda_2u_2-\mu_2v_2)}d\lambda_1 d\lambda_2 d\mu_1 d\mu_2\ .$$

Now, by applying the arguments of Theorem 4.10.3 we get

$$\mathrm{cov}\left\{\hat{q}_{K_1K_2K_3}(u_1,u_2)\,,\ \hat{q}_{\ell_1\ell_2\ell_3}(v_1,v_2)\right\} = \frac{1}{(2\pi)^4T^2}\Big[\iiiint \left|\Delta^{(T)}(\lambda_1-\mu_1)\right|^2 f_{K_1\ell_1}(\lambda_1)$$

$$f_{K_2K_3\ell_2\ell_3}(\lambda_2,-\lambda_1-\lambda_2,-\mu_2)\,e^{i(\lambda_1u_1-\mu_1v_1)}\,e^{i(\lambda_2u_2-\mu_2v_2)}d\lambda_1 d\mu_1 d\lambda_2 d\mu_2$$

$$+\iiiint \Delta^{(T)}(\lambda_1-\mu_1)\Delta^{(T)}(-\lambda_1)\Delta^{(T)}(\mu_1)f_{K_1\ell_1}(\lambda_1)f_{K_2K_3}(\lambda_2)f_{\ell_2\ell_3}(-\mu_2)\,e^{i(\lambda_1u_1-\mu_1v_1)}\,e^{i(\lambda_2u_2-\mu_2v_2)}d\lambda_1 d\lambda_2 d\mu_1 d\mu_2$$

$$+\iiiint \Delta^{(T)}(\lambda_1-\mu_1)\Delta^{(T)}(\lambda_2-\mu_2)\Delta^{(T)}(-\lambda_1-\lambda_2+\mu_1+\mu_2)\, f_{K_1\ell_1}(\lambda_1)\, f_{K_2\ell_2}(\lambda_2) f_{K_3\ell_3}(-\lambda_1-\lambda_2)$$

$$e^{i(\lambda_1u_1-\mu_1v_1)}\, e^{i(\lambda_2u_2-\mu_2v_2)}\, d\lambda_1 d\lambda_2 d\mu_1 d\mu_2$$

$$+\iiiint \Delta^{(T)}(\lambda_1-\mu_1)\Delta^{(T)}(\lambda_2+\mu_1+\mu_2)\Delta^{(T)}(-\lambda_2-\mu_1-\mu_2)\, f_{K_1\ell_1}(\lambda_1) f_{K_2\ell_3}(\lambda_2) f_{K_3\ell_2}(-\lambda_1-\lambda_2)$$

$$e^{i(\lambda_1u_1-\mu_1v_1)}\, e^{i(\lambda_2u_2-\mu_2v_2)} d\lambda_1 d\lambda_2 d\mu_1 d\mu_2 \Big] + O(T^{-1})$$

$$=\frac{2\pi}{bT}\iiint f_{K_1\ell_1}(\lambda_1) f_{K_2K_3\ell_2\ell_3}(\lambda_2, -\lambda_1-\lambda_2, -\mu_2)\, e^{i\lambda_1(u_1-v_1)} e^{i(\lambda_2u_2-\mu_2v_2)} d\lambda_1 d\lambda_2 d\mu_2$$

$$+\frac{2\pi}{bT}\iiint |\Delta^{(T)}(\lambda_1)|^2 f_{K_1\ell_1}(\lambda_1) f_{K_2K_3}(\lambda_2) f_{\ell_2\ell_3}(-\mu_2)\, e^{i\lambda_1(u_1-v_1)} e^{i(\lambda_2u_2-\mu_2v_2)} d\lambda_1 d\lambda_2 d\mu_2$$

$$+\frac{2\pi}{bT}\iint f_{K_1\ell_1}(\lambda_1) f_{K_2\ell_2}(\lambda_2) f_{K_3\ell_3}(-\lambda_1-\lambda_2)\, e^{i\lambda_1(u_1-v_1)} e^{i\lambda_2(u_2-v_2)} d\lambda_1 d\lambda_2$$

$$+\frac{2\pi}{bT}\iint f_{K_1\ell_1}(\lambda_1) f_{K_2\ell_3}(\lambda_2) f_{K_3\ell_2}(-\lambda_1-\lambda_2)\, e^{-i\lambda_1u_2} d\lambda_1 d\lambda_2 + O(T^{-1}) .$$

It follows from the above estimate that the variance of $\hat{q}_{211}(u_1,u_2)$ is given by

$$Var\left\{\hat{q}_{211}(u_1,u_2)\right\} = \frac{2\pi}{bT}\iiint f_{22}(\lambda_1) f_{1111}(\lambda_2, -\lambda_1-\lambda_2, -\mu_2)\, e^{i(\lambda_2-\mu_2)u_2} d\lambda_1 d\lambda_2 d\mu_2$$

$$+\frac{2\pi}{bT}\iiint|\Delta^{(T)}(\lambda_1)|^2 f_{22}(\lambda_1)f_{11}(\lambda_2)f_{11}(-\mu_2)e^{i(\lambda_2-\mu_2)v_2}d\lambda_1 d\lambda_2 d\mu_2$$

$$+\frac{2\pi}{bT}\iint f_{22}(\lambda_1)f_{11}(\lambda_2)f_{11}(-\lambda_1-\lambda_2)\,d\lambda_1 d\lambda_2$$

$$+\frac{2\pi}{bT}\iint f_{22}(\lambda_1)f_{11}(\lambda_2)f_{11}(-\lambda_1-\lambda_2)e^{-i\lambda_1 u_2}d\lambda_1 d\lambda_2+O(T^{-1})\ . \tag{4.10.21}$$

When u_2 and v_2 tend to infinity the two processes N_1 and N_2 become independent and so the estimate of the variance becomes

$$Var\{\hat{q}_{211}(u_1,u_2)\}=\frac{2\pi}{bT}\iint f_{22}(\lambda_1)f_{11}(\lambda_2)f_{11}(-\lambda_1-\lambda_2)d\lambda_1 d\lambda_2+O(T^{-1})\ . \tag{4.10.22}$$

In the case that the process N_1 is Poisson (4.10.22) further simplifies as

$$Var\{\hat{q}_{211}(u_1,u_2)\}\simeq\frac{p_1^2}{2\pi bT}\int f_{22}(\lambda_1)d\lambda_1\ . \tag{4.10.23}$$

Hence, an estimate of the variance in practice will be given by

$$Var\{\hat{q}_{211}(u_1,u_2)\}\simeq\frac{\hat{p}_1^2}{b^2T^2}\sum_{j\neq 0} I_{22}^{(T)}(\lambda_j) \tag{4.10.24}$$

and

$$|\lambda_j|\leq\pi/b\ .$$

We now turn to the problem of solving the non-linear equation given by (4.10.1).

From Appendix III we see that an analytical solution of the second-order kernel is only possible when N_1 is a Poisson point process. There is no obvious way of solving equation (1.10) of Appendix III in general.

In the case that N_1 is a Poisson point process the solution of the second-order kernel can be expressed as follows

$$a_2(u,v) = \frac{q_{211}(u,u-v)}{2p_1^2} \quad , \; u \neq v \; . \tag{4.10.25}$$

The second-order kernel may now be estimated by

$$\hat{a}_2(u,v) = \frac{\hat{q}_{211}(u,u-v)}{2\hat{p}_1^2} \quad , \; u \neq v \; . \tag{4.10.26}$$

Next, we define the process $\varepsilon(t)$ with stationary increments as follows

$$d\varepsilon(t) = dN_2(t) - \left\{a_0 + \int a_1(t-u)\,dN_1(u) + \iint\limits_{u\neq v} a_2(t-u,t-v)\,dN_1(u)\,dN_1(v)\right\}dt \; . \tag{4.10.27}$$

Alternatively we write

$$d\varepsilon(t) = dN_2(t) - \left\{s_0 + \int s_1(t-u)\,dN_1'(u) + \iint\limits_{u\neq v} s_2(t-u,t-v)\,dN_1'(u)\,dN_1'(v)\right\}dt \, , \tag{4.10.28}$$

where $dN_1'(u) = dN_1(u) - p_1\,dt$ and the new kernels s_0, $s_1(\cdot)$, $s_2(\cdot,\cdot)$ are related to the old kernels by the following relations

$$a_0 = s_0 - p_1 \int s_1(u)\,du + p_1^2 \iint\limits_{u\neq v} s_2(u,v)\,du\,dv \tag{4.10.29}$$

$$a_1(u) = s_1(u) - 2p_1 \int s_2(u,v)\,dv \tag{4.10.30}$$

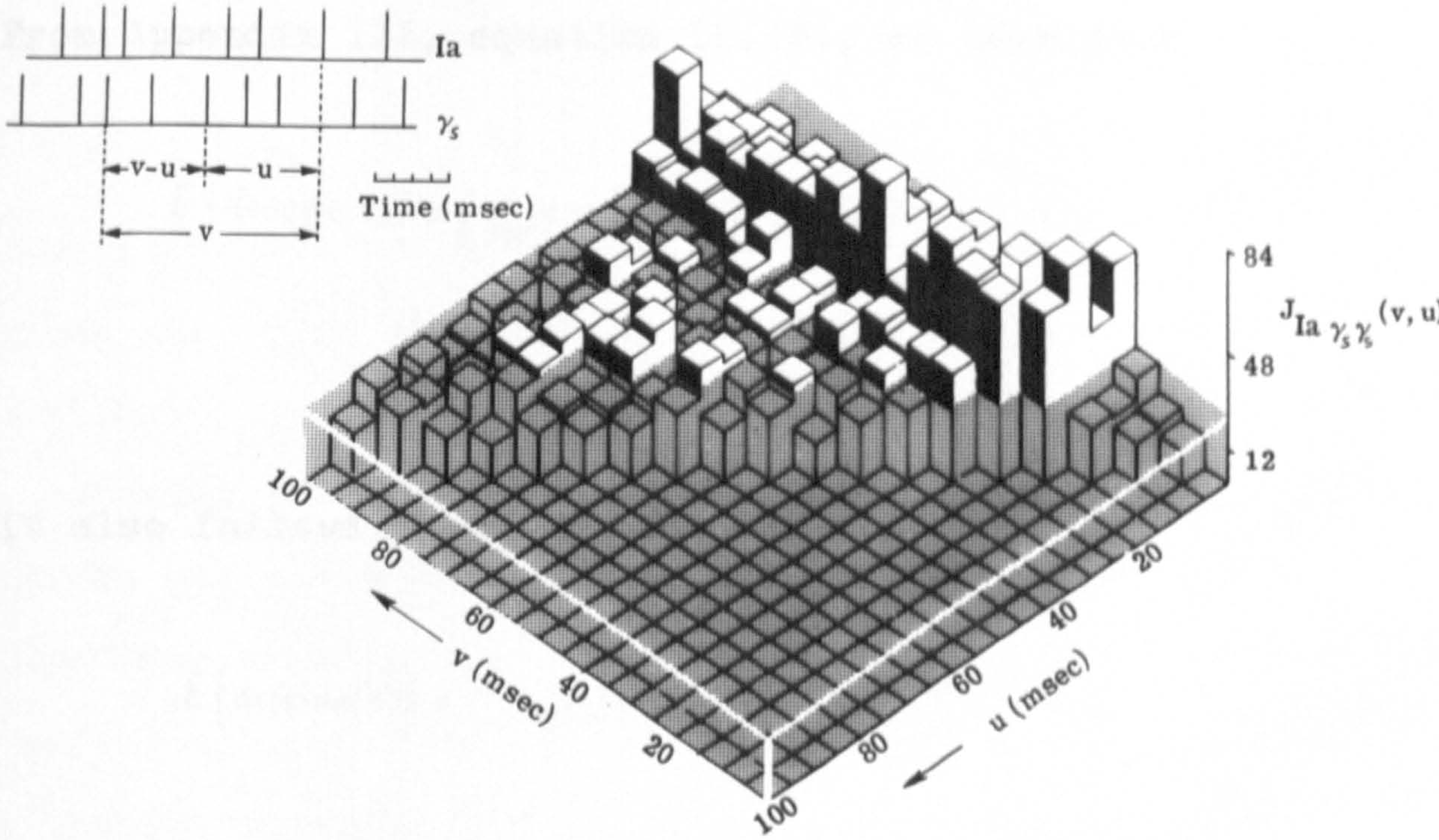


Fig. 4.10.1: Three-dimensional histogram estimate presenting

$J^{(T)}_{Ia\,\gamma s\,\gamma s}(v,u) = \{$ number of Ia events at time t preceded by γs events at times (t-u) and (t-v). $v > u.\}$.

and

$$\alpha_2(u,v) = s_2(u,v) \quad . \tag{4.10.31}$$

From Appendix III, equation (2.12), we have that

$$E\{d\epsilon(t)d\epsilon(t')\} = \{ q_{22}(t-t') + p_2\delta(t-t') - p_1 \int s_1(t-u)s_1(t'-u)du$$

$$- 2p_1^2 \iint_{u\neq v} s_2(t-u,t-v)\, s_2(t'-u,t'-v)\, du dv \} dt dt' \quad . \tag{4.10.32}$$

It also follows from Appendix III that

$$E\{d\epsilon(t)d\epsilon(t')\} = \{ \int f_{22}(\lambda) e^{i\lambda(t-t')} d\lambda - \int \frac{|f_{21}(\lambda)|^2}{P_1/2\pi} e^{i\lambda(t-t')} d\lambda$$

$$- \frac{1}{2} \iint \frac{|f_{211}(-\lambda-\mu,\mu)|^2}{(P_1/2\pi)^2} e^{i(\lambda+\mu)(t-t')} d\lambda d\mu \} dt dt' \quad . \tag{4.10.33}$$

Finally, the power spectrum of $\epsilon(\cdot)$ can be expressed as

$$f_{\epsilon\epsilon}(\lambda) = f_{22}(\lambda) - \frac{|f_{21}(\lambda)|^2}{P_1/2\pi} - \frac{1}{2} \int \left(\frac{\sin \mu h/2}{\mu h/2} \right)^2 \frac{|f_{211}(-\lambda-\mu,\mu)|^2}{(P_1/2\pi)^2} d\mu \quad , \tag{4.10.34}$$

where $h \mapsto 0$.

Fig. 4.10.1 presents the estimate $J_{211}^{(T)}(v,u)$. This estimate gives the number of events of the Ia response in the interval $(t,t+h]$, $t = 0,1,\ldots$ and $h = 5$, when two γs spikes occur at times $t-u$ and $t-v$. The areas given by the unstippled, black and white indicate time intervals during which a previous input pulse exerts an influence on the effect of a currently applied pulse on the Ia discharge.

By using expressions (4.10.13), (4.10.17) and (4.10.26) we can also obtain estimates of the third-order product density, the third-order cumulant function and the second-order kernel. The graphical representation and the interpretation of these estimates will be left as a subject of further research.

4.11 SUMMARY

In this chapter we discussed certain parameters of the bivariate point process in both time and frequency domains. Some of these parameters may be assumed as an extension of the parameters of the univariate point process presented in Chapter 3. Estimates of these parameters were given and their asymptotic distributions were developed. Confidence intervals based on the asymptotic distributions were constructed under the null hypothesis of independence (the two univariate point processes N_1 and N_2 are assumed to be independent). The identification of the point process system considered was carried out by using the linear and quadratic models.

The frequency domain methods are again the main object of this chapter and they are related to theory and applications.

We summarize the use of the frequency domain methods as follows

(i) Estimates of the cross-spectrum based on the cross-periodogram may be calculated in two different ways (section 4.8). Also, estimates of the gain, phase and coherence are obtained.

(ii) These estimates are compared with the ones obtained by transforming time domain parameters of the bivariate point process to the frequency domain (sections 4.6, 4.7 and 4.8).

(iii) By inverting estimates of certain parameters in the frequency domain we obtain estimates of the corresponding time domain parameters. Original theorems based on the properties of the cross-periodogram and the third-order periodogram are proved in order to develop the asymptotic properties of the second- and third-order cumulants (sections 4.9 and 4.10).

(iv) It may be faster to calculate estimates of the time domain of the bivariate point process by using the second- and third-order periodograms. This is achieved by using the Fast Fourier transform as it has been described in Chapter 3.

(v) Finally, the estimates of the second- and third-order cumulants are used in identifying the linear and the quadratic models of a point process system under the assumption that the input process to the system is a Poisson point process.

The steps mentioned above are demonstrated by a number of illustrations which were obtained by using the Data Set II. The Fortran programs which were used for the analysis of the data sets given in Chapter 3 were extended to enable us to calculate the estimates of the frequency and time domain parameters. Estimates of both time and frequency domain parameters seem to provide useful information about how the system responds to the presence of a fusimotor input (γ s).

These methods may be extended to systems with a number of different input point processes.

CHAPTER FIVE

IDENTIFICATION OF A TIME INVARIANT SYSTEM INVOLVING POINT PROCESSES AND TIME SERIES

5.1 INTRODUCTION

In Chapter 4 we have considered the case of an input point process (γs stimulus) giving rise to another point process (Ia response) by influencing the muscle spindle. By relating the response of the Ia sensory axon to the γs stimulus we were able to extract useful information about the mechanical properties of the muscle spindle.

In this chapter we concentrate on examining (i) the effect of an imposed length change on the muscle spindle, and (ii) the combined effect of the simultaneous application of a γs stimulus and an imposed length change.

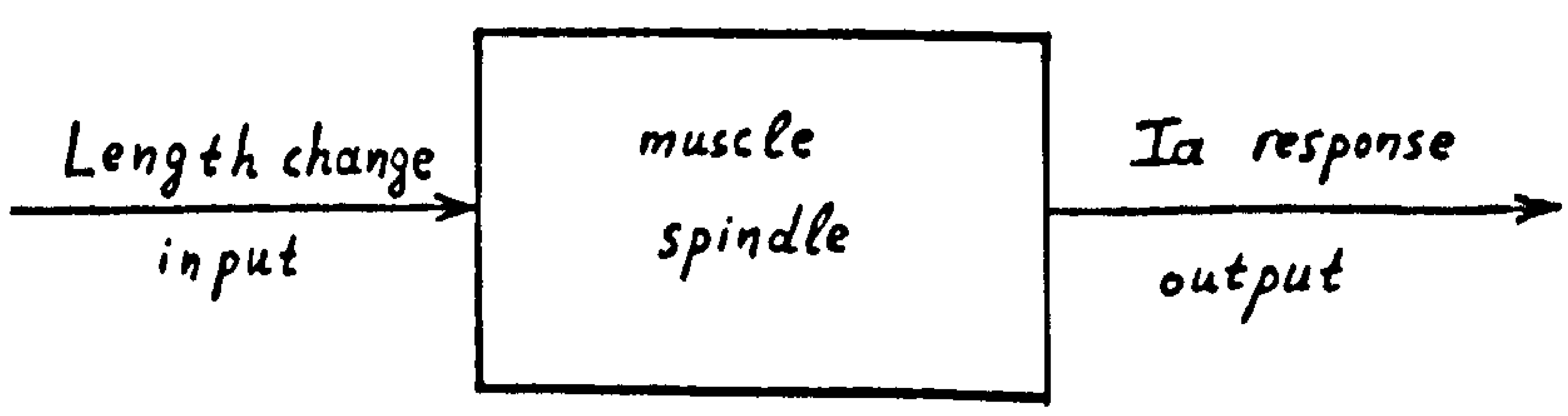
It is known that the behaviour of the muscle spindle is very sensitive to the length changes imposed on the parent muscle. In many situations the muscle spindle may be considered as a stretch receptor which provides the central nervous system with information about the current length of the parent muscle (Matthews, 1981). Traditionally the problem of the identification of the muscle spindle has been considered in terms of linearised transfer function descriptions relating the Ia output to imposed length changes. A brief review of these studies may be found in a recent paper by Rosenberg et al (1982).

Our approach to this problem makes use of methods involving the spectral analysis of hybrid processes, i.e., a continuous signal and a point process.

The first function in the frequency domain to be defined is the cross-periodogram between a continuous signal and a point process as suggested by Jenkins (1963a) in the discussion of Bartlett's paper on the spectral analysis of point processes. By inverting the cross-periodogram we show that we have the "spike-triggered average" which is a measure commonly used by neurophysiologists to detect functional connections in the nervous system (Kirkwood, 1979). In addition we present estimates for the transfer function of the system, phase and coherence. These functions provide invaluable information about the behaviour of the muscle spindle in the cases (i) and (ii) mentioned above.

5.2 IDENTIFICATION OF A LINEAR HYBRID SYSTEM INVOLVING A CONTINUOUS SIGNAL AND A POINT PROCESS

A graphical representation of the hybrid system of the case (i) described in the introduction can be given as follows



Suppose that X(t) is a stationary time series on the interval (O,T] representing the length change imposed on the muscle. Suppose also that N(t) is a stationary univariate point process on (O,T] with differential increments dN(t) representing the response of the Ia sensory axon.

By extending the linear model (4.3.1) of the previous chapter we may model the hybrid system presented above in the following way

$$dN(t) = \left\{ a_0 + \int a_1(t-u)\,X(u)\,du \right\} dt + d\varepsilon(t) \quad , \tag{5.2.1}$$

where a_0 and $a_1(\cdot)$ are the zero- and first-order kernels of the hybrid system. The process $\varepsilon(t)$ has stationary increments with zero-mean and is independent of the continuous signal X(t).

On taking the expected value in (5.2.1) we obtain

$$p_N = a_0 + \mu_X \int a_1(u)\,du \quad , \tag{5.2.2}$$

where μ_X is the mean value of X(t) and p_N is the mean intensity of N(t).

Next, on multiplying (5.2.1) by X(t-v) and taking the expectation we have

$$E\left\{ dN(t)\,X(t-v) \right\} = \left\{ a_0\mu_X + \int a_1(t-u)\,E\left\{ X(u)X(t-v) \right\} du \right\} dt \ . \tag{5.2.3}$$

The increments dN(t) for different values of t may be expressed as

$$dN(t) = \sum_{i=1}^{N(T)} \delta(t-t_i)\,dt \tag{5.2.4}$$

(Beutler & Leneman, 1968; Brillinger, 1974b), where t_i, i = 1,..., N(T), are the time instances at which the events of the process N(t) occur in the interval (0,T] and δ(t) is the dirac delta function.

The expected value $E\left\{dN(t)X(t-v)\right\}$ may then be written as

$$\mu_{NX}(v)\,dt = \sum_{i=1}^{N(T)} E\left\{\delta(t-t_i)\,X(t-v)\right\}dt \ , \quad \text{and}$$

$$\hat{\mu}_{NX}(v) \simeq \frac{1}{T}\sum_{i=1}^{N(T)} X(t_i - v) \ , \qquad (\text{ref. Marmarelis \& Marmarelis, 1978}) \qquad (5.2.5)$$

and equation (5.2.3) becomes

$$\mu_{NX}(v) = \alpha_0\mu_X + \int \alpha_1(v-u)\,\mu_{XX}(u)\,du \ . \qquad (5.2.6)$$

Equation (5.2.5) is defined as the "spike-triggered average". Thus, by substituting a_0 from (5.2.2) into (5.2.6) we get

$$c_{NX}(v) = \int \alpha_1(v-u)\,c_{XX}(u)\,du \ , \qquad (5.2.7)$$

where $C_{NX}(v)$ is the cross-covariance of N(t) & X(t) and $C_{XX}(u)$ is the auto-covariance of the signal X(t).

The equation (5.2.7) is an integral equation and it may be solved by taking the Fourier transform as follows

$$(2\pi)^{-1}\int_{-\infty}^{+\infty} c_{NX}(v)\,e^{-i\lambda v}dv = (2\pi)^{-1}\int_{-\infty}^{\infty}\left\{\int \alpha_1(v-u)\,c_{XX}(u)\,du\right\}e^{-i\lambda v}dv \ ,$$

which further gives

$$f_{NX}(\lambda) = A(\lambda)\,f_{XX}(\lambda) \ . \qquad (5.2.8)$$

The function $f_{NX}(\lambda)$, $-\infty < \lambda < \infty$, is the cross-spectrum of N(t) and X(t), while $f_{XX}(\lambda)$ is the power-spectrum of X(t). Also, $A(\lambda)$ is the transfer function of the system defined by

$$A(\lambda)=\int a_1(u)e^{-i\lambda u}du \quad , \ -\infty<\lambda<\infty \ . \tag{5.2.9}$$

It can be shown by arguments similar to those in Chapter 4 that the auto-covariance of $\varepsilon(t)$ is given by

$$E\left\{d\varepsilon(t)\,d\varepsilon(t')\right\}=\left\{\int f_{NN}(\lambda)\left[1-\frac{|f_{NX}(\lambda)|^2}{f_{XX}(\lambda)f_{NN}(\lambda)}\right]e^{i\lambda(t-t')}d\lambda\right\}dt\,dt'$$

$$=\int f_{NN}(\lambda)\left[1-|R_{NX}(\lambda)|^2\right]e^{i\lambda(t-t')}d\lambda \ dt\,dt' \ , \tag{5.2.10}$$

where $|R_{NX}(\lambda)|^2$ is the coherence of X(t) and N(t) defined by

$$|R_{NX}(\lambda)|^2=\begin{cases}\dfrac{|f_{NX}(\lambda)|^2}{f_{XX}(\lambda)f_{NN}(\lambda)} & \\ 0 \ , & \text{if} \quad f_{XX}(\lambda)=0 \text{ or } f_{NN}(\lambda)=0 \ .\end{cases} \tag{5.2.11}$$

The coherence measures the degree of association between the continuous signal X(t) and the point process N(t).

It follows from (5.2.10) that

$$f_{\varepsilon\varepsilon}(\lambda)=f_{NN}(\lambda)\left[1-|R_{NX}(\lambda)|^2\right] \ . \tag{5.2.12}$$

Expression (5.2.12) clearly shows that the linear model given by (5.2.1) will be adequate if the coherence is close to one.

We now proceed to define the cross-periodogram between a continuous signal and a point process.

5.3 THE CROSS-PERIODOGRAM OF A TIME SERIES AND A POINT PROCESS

Let X(t) be a stationary time series and N(t) be a stationary point process defined on (0,T]. By considering Jenkins's suggestion (1963a) we define the cross-periodogram of a continuous time series and a point process as

$$I_{NX}^{(T)}(\lambda) = \frac{1}{2\pi T} d_N^{(T)}(\lambda)\, \overline{d_X^{(T)}(\lambda)} \;, \; -\infty < \lambda < \infty \tag{5.3.1}$$

where $d_N^{(T)}(\lambda)$ is the finite Fourier-Stieltjes transform of N(t) considered in Chapter 3 and $d_X^{(T)}(\lambda)$ is the finite Fourier transform of X(t) defined by

$$d_X^{(T)}(\lambda) = \int_0^T X(t) e^{-i\lambda t} dt \quad . \tag{5.3.2}$$

In practice, in order to improve the properties of $I_{NX}^{(T)}(\lambda)$ we use convergence factors inside the Fourier transforms $d_N^{(T)}(\lambda)$ and $d_X^{(T)}(\lambda)$ in the same way as described in Chapter 3.

However, by inserting convergence factors inside the finite Fourier transforms we emphasize the problem of bias in the estimate of the cross-spectrum (Chapter 3, p 90) and so we should modify (5.3.1) in the following way

$$\tilde{I}_{NX}^{(T)}(\lambda) = \frac{1}{2\pi H_2^{(T)}(0)} d_N^{(T)}(\lambda)\, \overline{d_X^{(T)}(\lambda)} \;, \tag{5.3.3}$$

where

$$H_2^{(T)}(0) = T \int k^2(u)\, du \;, \tag{5.3.4}$$

and k(u) is a convergence factor satisfying Assumption 3.8.1 of Chapter 3.

We now discuss the problem of smoothing the cross-periodogram and thus finding an estimate of the cross-spectrum of X(t) and N(t).

5.4 ESTIMATES OF THE CROSS-SPECTRUM OF A CONTINUOUS TIME SERIES AND A POINT PROCESS

An estimate of the cross-spectrum of X(t) and N(t) can be calculated from the periodogram in two ways.

(a) By averaging consecutive ordinates of the cross-periodogram. Mathematically this can be written as

$$f_{NX}^{(T)}(\lambda) = \frac{1}{2m+1} \sum_{j=-m}^{m} I_{NX}^{(T)}\left(\lambda + \frac{2\pi j}{T}\right) \quad for \quad \lambda \neq 0 \ . \tag{5.4.1}$$

(b) By splitting the whole record into a number of subrecords and calculating the cross-periodogram in each subrecord. The estimate of the cross-spectrum is then calculated by averaging the separate estimates. Mathematically we can express this estimate as follows

$$\hat{f}_{NX}^{(T)}(\lambda) = \frac{1}{l} \sum_{j=1}^{l} I_{NX}^{(R)}(\lambda, j) \quad for \quad \lambda \neq 0 \ .$$

The function $I_{NX}^{(R)}(\lambda, j)$ is the cross-periodogram of X(t) and N(t) for the jth subsample of length R and $T = lR$.

The cross-spectrum $f_{NX}(\lambda)$ is generally a complex function and it may be written in the usual way as

$$f_{NX}(\lambda) = Re\, f_{NX}(\lambda) + i\, Im\, f_{NX}(\lambda) \quad . \qquad (5.4.2)$$

The modulus of $f_{NX}(\lambda)$ is then given by

$$|f_{NX}(\lambda)|^2 = f_{NX}(\lambda)\overline{f_{NX}(\lambda)}$$

$$= \{Re\, f_{NX}(\lambda)\}^2 + \{Im\, f_{NX}(\lambda)\}^2 \quad . \qquad (5.4.3)$$

The functions $Ref_{NX}(\lambda)$ and $Imf_{NX}(\lambda)$ can be estimated by using either procedure (i) or (ii) and writing $I_{NX}^{(T)}(\lambda)$ as follows

$$I_{NX}^{(T)}(\lambda) = Re\, I_{NX}^{(T)}(\lambda) + i\, Im\, I_{NX}^{(T)}(\lambda) \quad . \qquad (5.4.4)$$

An estimate of the modulus of the cross-spectrum may now be found by using expression (5.4.3).

5.5 ESTIMATES OF THE TRANSFER FUNCTION, PHASE AND COHERENCE

Expression (5.2.8) suggests that the transfer function $A(\lambda)$ of the hybrid system may be estimated by

$$A^{(T)}(\lambda) = \frac{f_{NX}^{(T)}(\lambda)}{f_{XX}^{(T)}(\lambda)} \quad , \qquad (5.5.1)$$

where $f_{NX}^{(T)}(\lambda)$ is an estimate of the cross-spectrum and $f_{XX}^{(T)}(\lambda)$ is an estimate of the power spectrum of the time series $X(t)$. The methods which we use to estimate the power spectrum of a univariate time series are well-known (Bloomfield, 1976; Jenkins & Watts, 1968).

An estimate of the gain is given by

$$G^{(T)}(\lambda) = |A^{(T)}(\lambda)| \quad .$$

The phase is defined as the argument of the cross-spectrum (Chapter 4, section 4.5) and it is estimated by

$$\theta^{(T)}(\lambda) = \arg\left\{ f_{NX}^{(T)}(\lambda) \right\} = \tan^{-1}\left\{ \frac{\operatorname{Im} f_{NX}^{(T)}(\lambda)}{\operatorname{Re} f_{NX}^{(T)}(\lambda)} \right\} \quad . \tag{5.5.2}$$

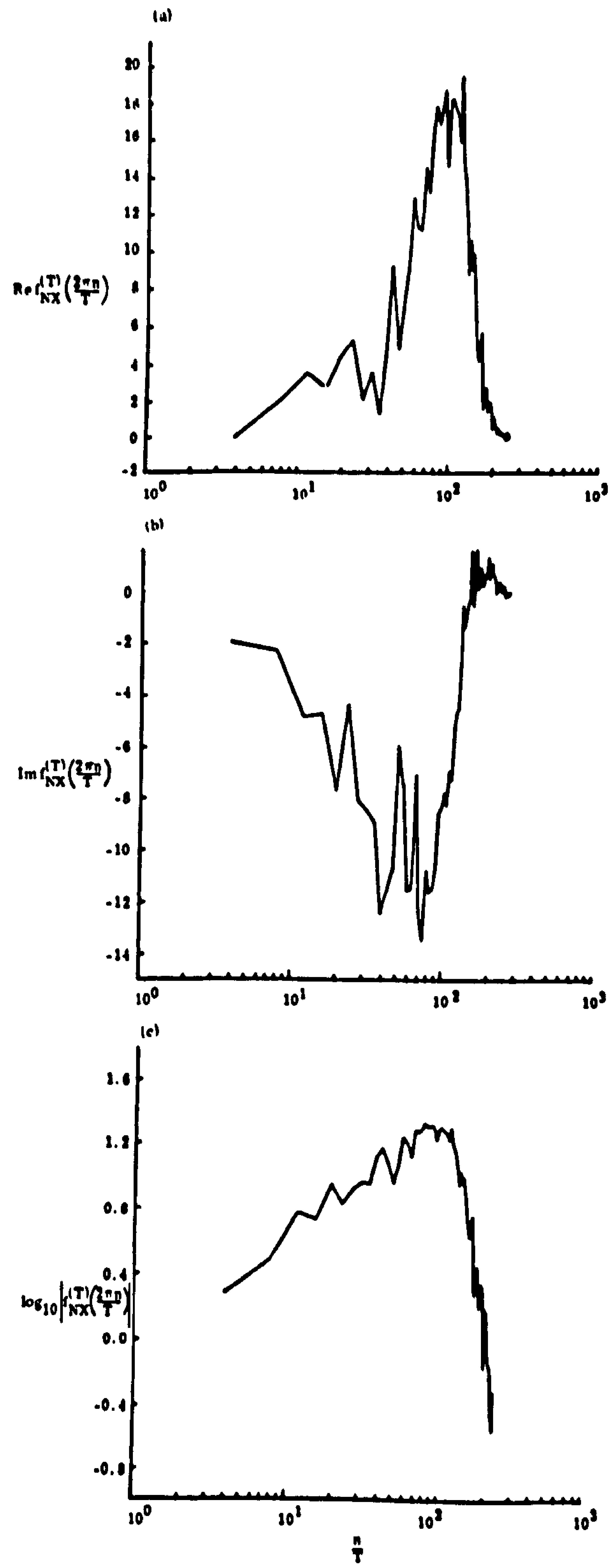
The linear relationship between the continuous signal X(t) and the point process N(t) can be found by using expression (5.2.1). This expression suggests the following estimate for the coherence

$$|R_{NX}^{(T)}(\lambda)|^2 = \frac{\left| f_{NX}^{(T)}(\lambda) \right|^2}{f_{NN}^{(T)}(\lambda) f_{XX}^{(T)}(\lambda)} \quad . \tag{5.5.3}$$

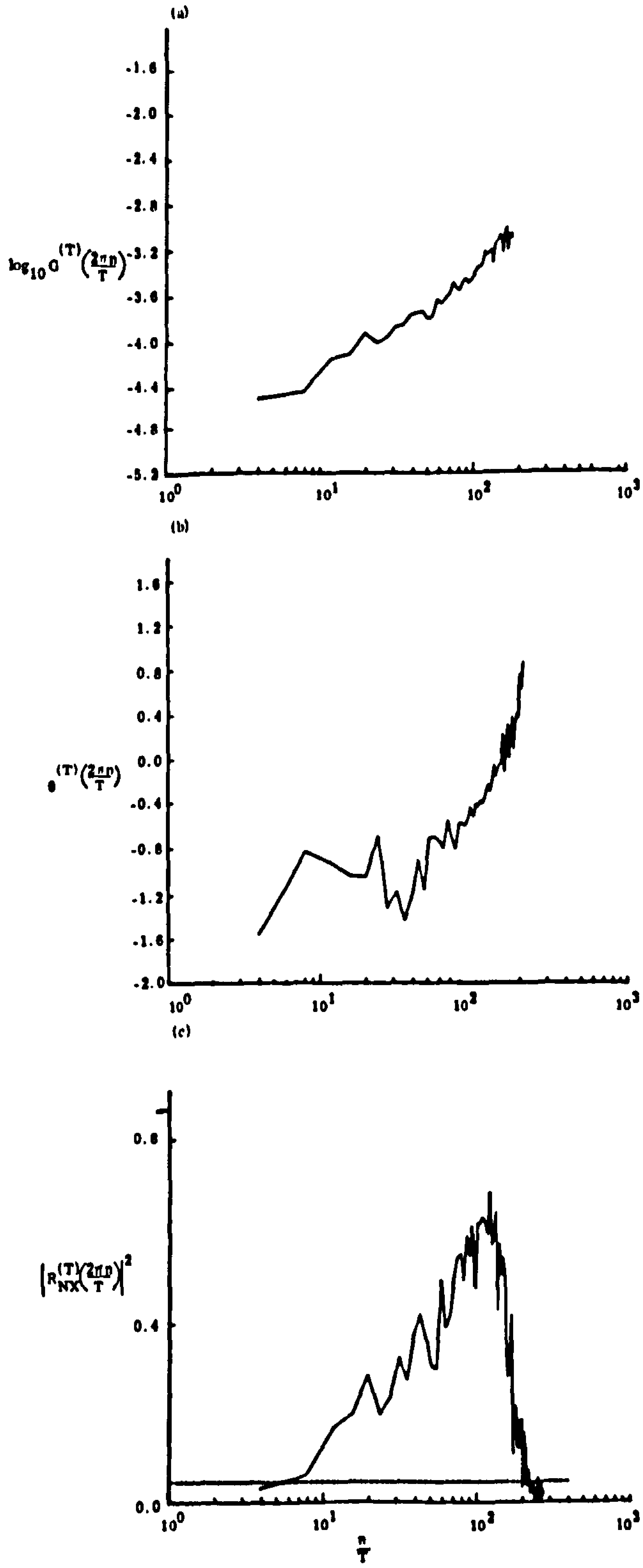
Figs. 5.5.1 (a-c) are the estimates of $\operatorname{Re} f_{NX}(\lambda)$, $\operatorname{Im} f_{NX}(\lambda)$ and $\log_{10}|f_{NX}(\lambda)|$. The estimate of the modulus of the cross-spectrum indicates that there is substantial covariance between the continuous signal X(t) and the point process N(t), most of which is concentrated at high frequencies.

Figs. 5.5.2 (a-c) are the estimates $\log_{10} G^{(T)}(\lambda)$, $\theta^{(T)}(\lambda)$ and $|R_{NX}^{(T)}(\lambda)|^2$. Fig. 5.5.2 (b) clearly shows that for low frequencies the estimate of the phase is negative while for high frequencies becomes positive which is consistent with the estimate of the logarithm to the base 10 of the gain (Fig. 5.5.2(a)). Figures 5.5.2 (a) and 5.5.2 (c) indicate that our system behaves like a high pass filter (Jenkins, 1965).

We may further smooth the above estimates by using a hanning procedure. Such smoothed estimates of $\operatorname{Re} f_{NX}(\lambda)$, $\operatorname{Im} f_{NX}(\lambda)$ and $\log_{10}|f_{NX}(\lambda)|$ are given in Figs. 5.5.3 (a-c).



<u>Fig. 5.5.1</u>: Estimates of frequency-domain parameters calculated from the cross-periodogram when the muscle spindle is affected by a length change. The whole record was divided into 62 disjoint sections. (a) Estimate of the real part of the cross-spectrum between the Ia discharge and the length change X(t); (b) Estimate of the imaginary part of the cross-spectrum and (c) Log to base 10 of the estimate of the modulus of the cross-spectrum.



<u>Fig. 5.5.2</u>: Estimates of frequency-domain parameters calculated from the periodogram-matrix when the muscle spindle is affected by a length change. The whole record was divided into 62 disjoint sections. (a) Log to base 10 of the estimate of the gain; (b) Estimate of the phase and (c) Estimate of the coherence. The dotted line gives the 95% point of the null distribution of this estimate.

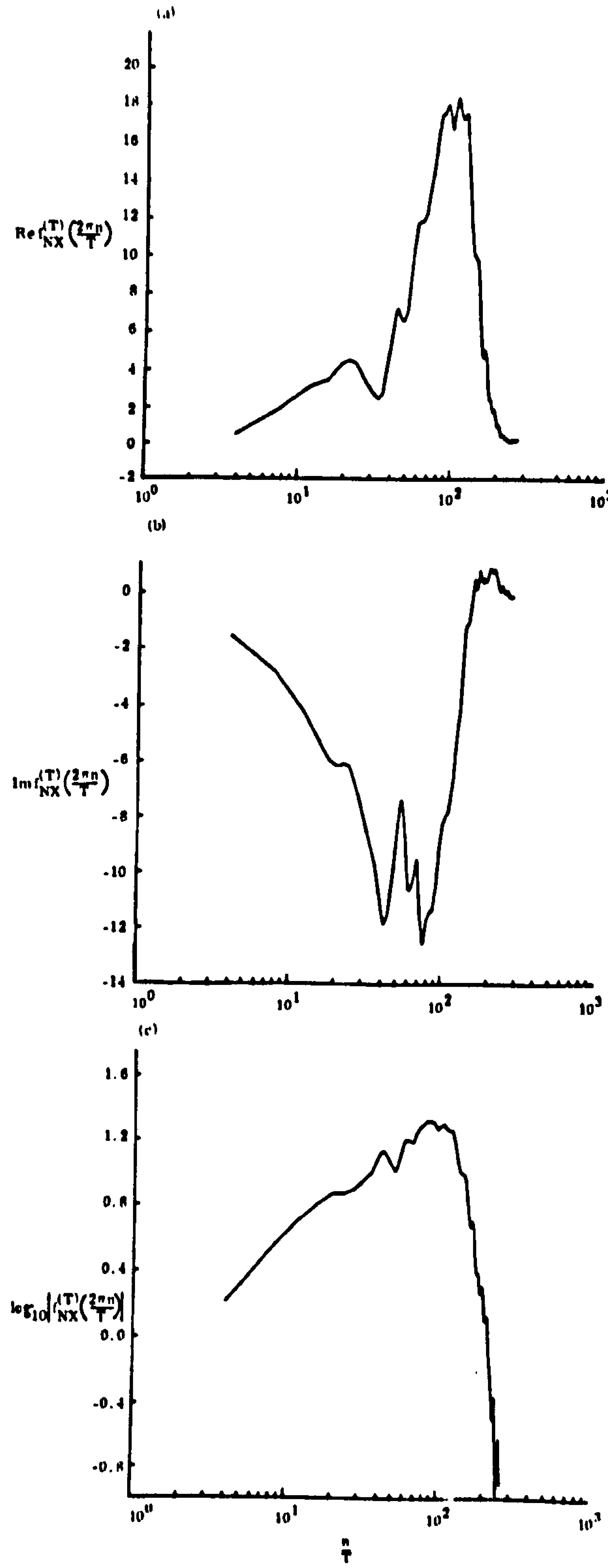


Fig. 5.5.3: Estimates of frequency-domain parameters calculated from the cross-periodogram when the muscle spindle is affected by a length change. These estimates were obtained by dividing the whole record into 62 disjoint sections and were further smoothed by using a hanning window. (a) Estimate of the real part of the cross-spectrum; (b) Estimate of the imaginary part of the cross-spectrum and (c) Log to base 10 of the estimate of the modulus of the cross-spectrum.

Also, improved estimates of the phase, the logarithm to base 10 of the gain and the coherence are shown in Figs. 5.5.4 (a-c). The approximate 5% point of the coherence is calculated by using the methods described in Chapter 4.

Following Brillinger (1974a) we may express the linear model (5.2.1) as

$$dN(t) = \left\{ s_0 + \int s_1(t-u)\, dX(u) \right\} dt + d\varepsilon(t) \tag{5.5.4}$$

The finite Fourier transform of dX(u) can be written as

$$d_X^{(T)}(\lambda) = \int_0^T e^{-i\lambda t}\, dX(t)$$

$$\simeq \sum_{t=0}^{T-1} e^{-i\lambda t}\left[X(t+1) - X(t)\right] \quad . \tag{5.5.5}$$

An examination of (5.5.5) shows that it corresponds to carrying out a spectral analysis on the time series of first differences (Brillinger, 1972). This procedure is common in the analysis of economic time series.

By defining similar quantities in the frequency domain as in section 4 of this chapter we find estimates of $\mathrm{Re} f_{NX}(\lambda)$, $\mathrm{Im} f_{NX}(\lambda)$, $\log_{10}|f_{NX}(\lambda)|$, $\theta(\lambda)$, $\log_{10} G(\lambda)$ and $|R_{NX}(\lambda)|^2$. They are shown in Figs. 5.5.5 (a-c) and Figs. 5.5.6 (a-c).

It is obvious from Fig. 5.5.6 (c) that the estimate of the coherence does not change at all except slightly for low frequencies.

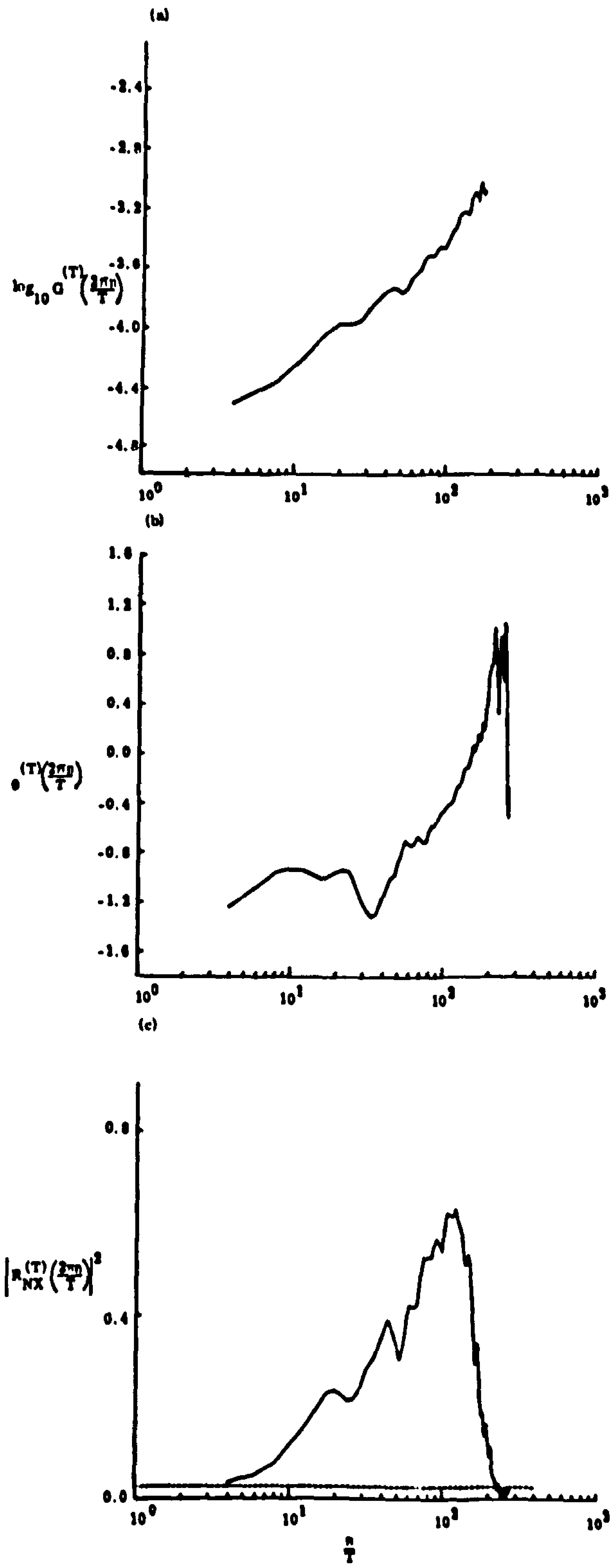


Fig. 5.5.4: Estimates of frequency-domain parameters obtained from the periodogram-matrix when the muscle spindle is affected by a length change. These estimates were calculated by dividing the whole record into 62 disjoint sections and were further smoothed by using a hanning window. (a) Log to base 10 of the estimate of the gain; (b) Estimate of the phase and (c) Estimate of the coherence. The dotted line corresponds to the 95% point of the null distribution of this estimate.

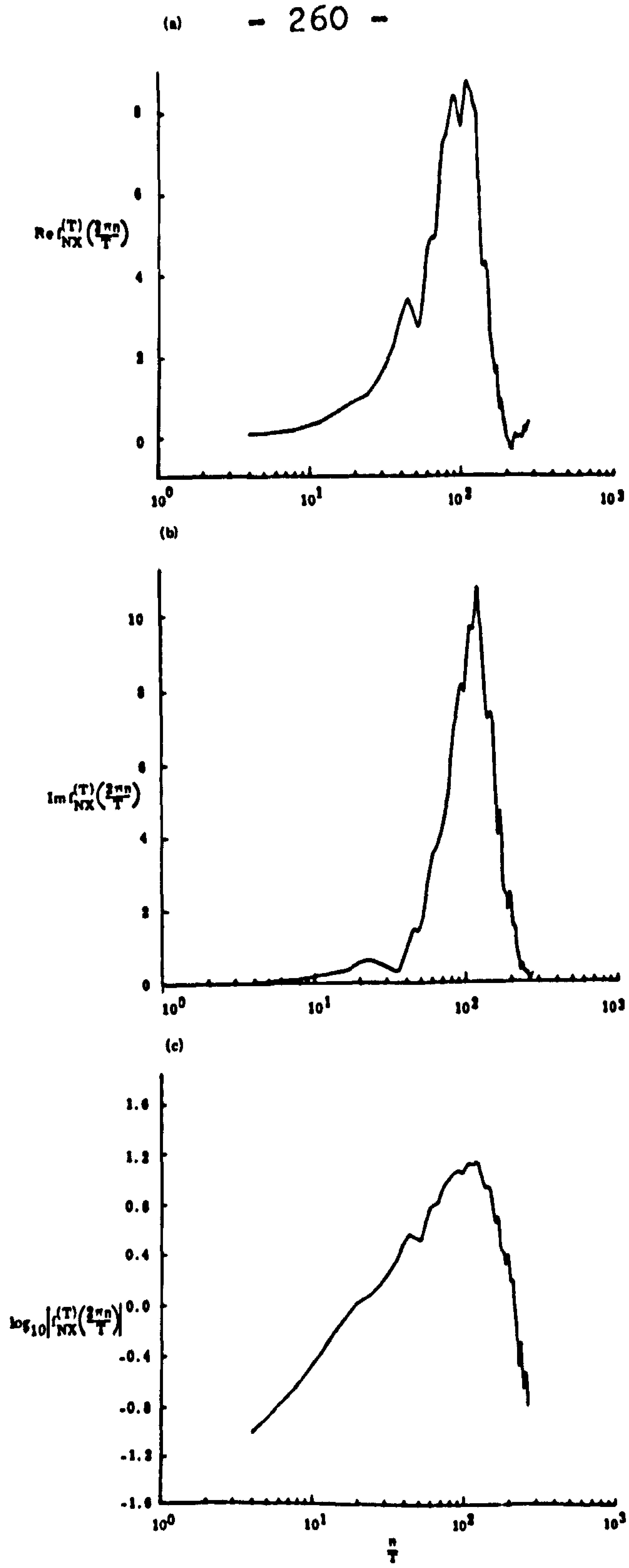


Fig. 5.5.5: Estimates of the frequency-domain parameters calculated from the cross-periodogram when the muscle spindle is affected by a length change. The whole record was divided into 62 disjoint sections. The first differences of the length change were taken for the computation of these estimates and a hanning procedure was used for further smoothing. (a) Estimate of the real part of the cross-spectrum; (b) Estimate of the imaginary part of the cross-spectrum and (c) Log to base 10 of the estimate of the modulus of the cross-spectrum.

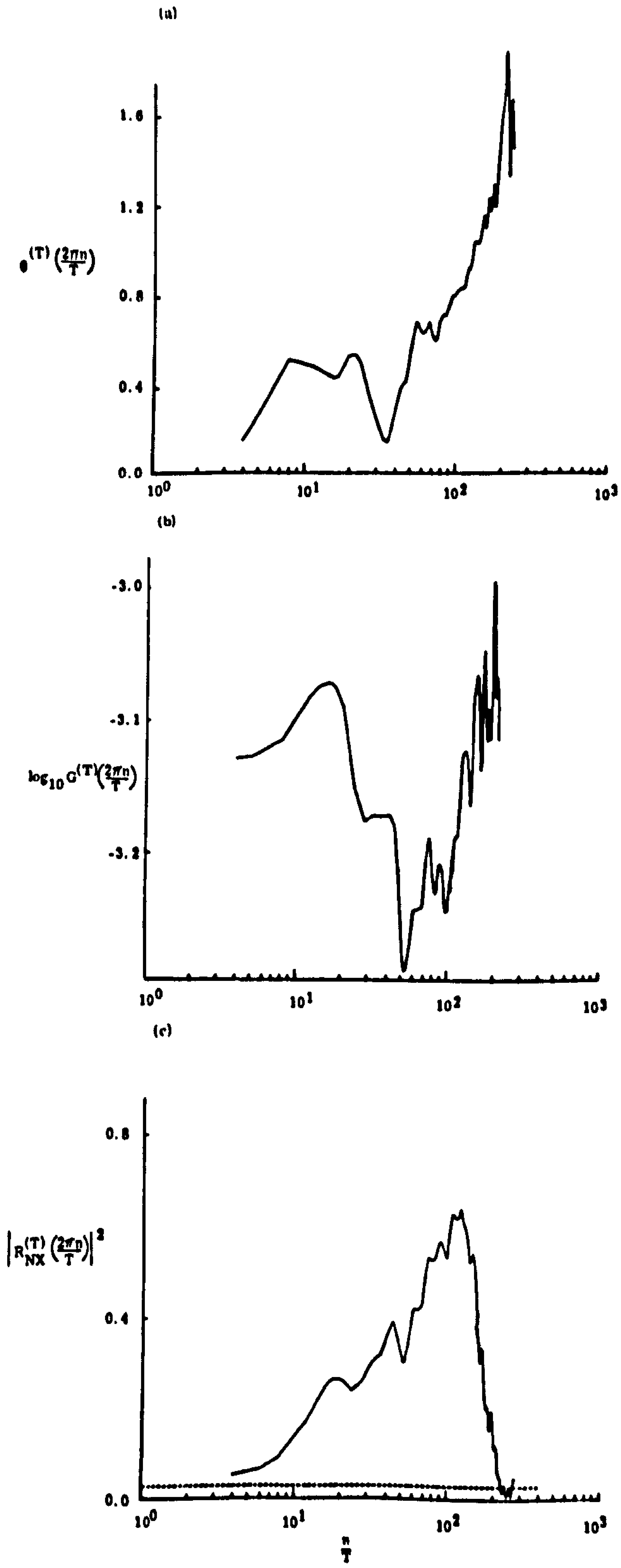


Fig. 5.5.6: Estimates of frequency-domain parameters obtained from the periodogram-matrix when the muscle spindle is affected by a length change. The whole record was divided into 62 disjoint sections. The first differences of the length change were taken for the computation of these estimates and a hanning procedure was used for further smoothing. (a) Estimate of the phase; (b) Log to base 10 of the estimate of the gain and (c) Estimate of the coherence. The dotted line gives the 95% point of the null distribution of this estimate.

Model (5.5.4) presents an alternative approach to the problem of identifying our system.

5.6 ESTIMATION OF THE TIME DOMAIN PARAMETERS BY USING METHODS OF THE FREQUENCY DOMAIN

The cross-periodogram $I_{NX}^{(T)}(\lambda)$ can be expressed as

$$I_{NX}^{(T)}(\lambda) = \frac{1}{2\pi T} d_N^{(T)}(\lambda) \overline{d_X^{(T)}(\lambda)}$$

$$= \frac{1}{2\pi T} \int_0^T e^{-i\lambda t} dN(t) \int_0^T e^{i\lambda t} X(t) dt$$

$$= \frac{1}{2\pi T} \sum_{i=1}^{N(T)} e^{-i\lambda \tau_i} \int_0^T e^{i\lambda t} X(t) dt$$

$$= \frac{1}{2\pi T} \sum_{i=1}^{N(T)} \int_0^T e^{i\lambda(t-\tau_i)} X(t) dt \quad . \tag{5.6.1}$$

Taking the inverse Fourier transform in (5.6.1) we have

$$\int_{-\infty}^{\infty} I_{NX}^{(T)}(\lambda) e^{i\lambda v} d\lambda = \int_{-\infty}^{\infty} \left\{ \frac{1}{2\pi T} \sum_{i=1}^{N(T)} \int_0^T e^{i\lambda(t-\tau_i)} X(t) dt \right\} e^{i\lambda v} d\lambda$$

$$= \frac{1}{T} \sum_{i=1}^{N(T)} \int_0^T \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda(t+v-\tau_i)} d\lambda \right\} X(t) dt$$

$$= \frac{1}{T} \sum_{i=1}^{N(T)} \int_0^T X(t) \delta(t+v-\tau_i) dt$$

$$\simeq \frac{1}{T} \sum_{i=1}^{N(T)} X(\tau_i - v) \quad . \tag{5.6.2}$$

Expression (5.6.2) is the "spike triggered average" defined in section 5.2.

In practice an estimate of the "spike-triggered average" can be obtained by using the cross-periodogram in the following way

$$\hat{\mu}_{NX}(v) = \frac{2\pi}{T} \sum_{s \neq 0} I_{NX}^{(T)}\left(\frac{2\pi s}{T}\right) e^{i\frac{2\pi s}{T}v} \quad . \tag{5.6.3}$$

Under the hypothesis that N(t) and X(t) are independent it follows from Chapter 4 that $\hat{\mu}_{NX}(v)$ is approximately normally distributed with zero mean and estimated variance

$$\left(\frac{2\pi}{T}\right)^2 \sum_{j \neq 0} I_{NN}^{(T)}(\lambda_j)\, I_{XX}^{(T)}(\lambda_j) \quad . \tag{5.6.4}$$

This estimate of the variance must be increased by a factor 1.089 in order to take into account the use of a taper, and the extension of the sample record by adding a number of zeroes.

Fig. 5.6.1 is the estimate of the spike-triggered average. The horizontal lines are the 95% confidence limits.

An alternative estimate of the spike-triggered average can be calculated from the following expression

$$\hat{\mu}'_{NX}(v) = \frac{2\pi}{S} \sum_{s \neq 0} \hat{f}_{NX}^{(T)}\left(\frac{2\pi s}{S}\right) e^{i\frac{2\pi s}{S}v} \tag{5.6.5}$$

where $\hat{f}_{NX}^{(T)}(\lambda)$ is the estimate of the cross-spectrum of X(t) and N(t) described in procedure (ii) of section 5.4 of this chapter. It follows from Theorem 4.8.4 that the

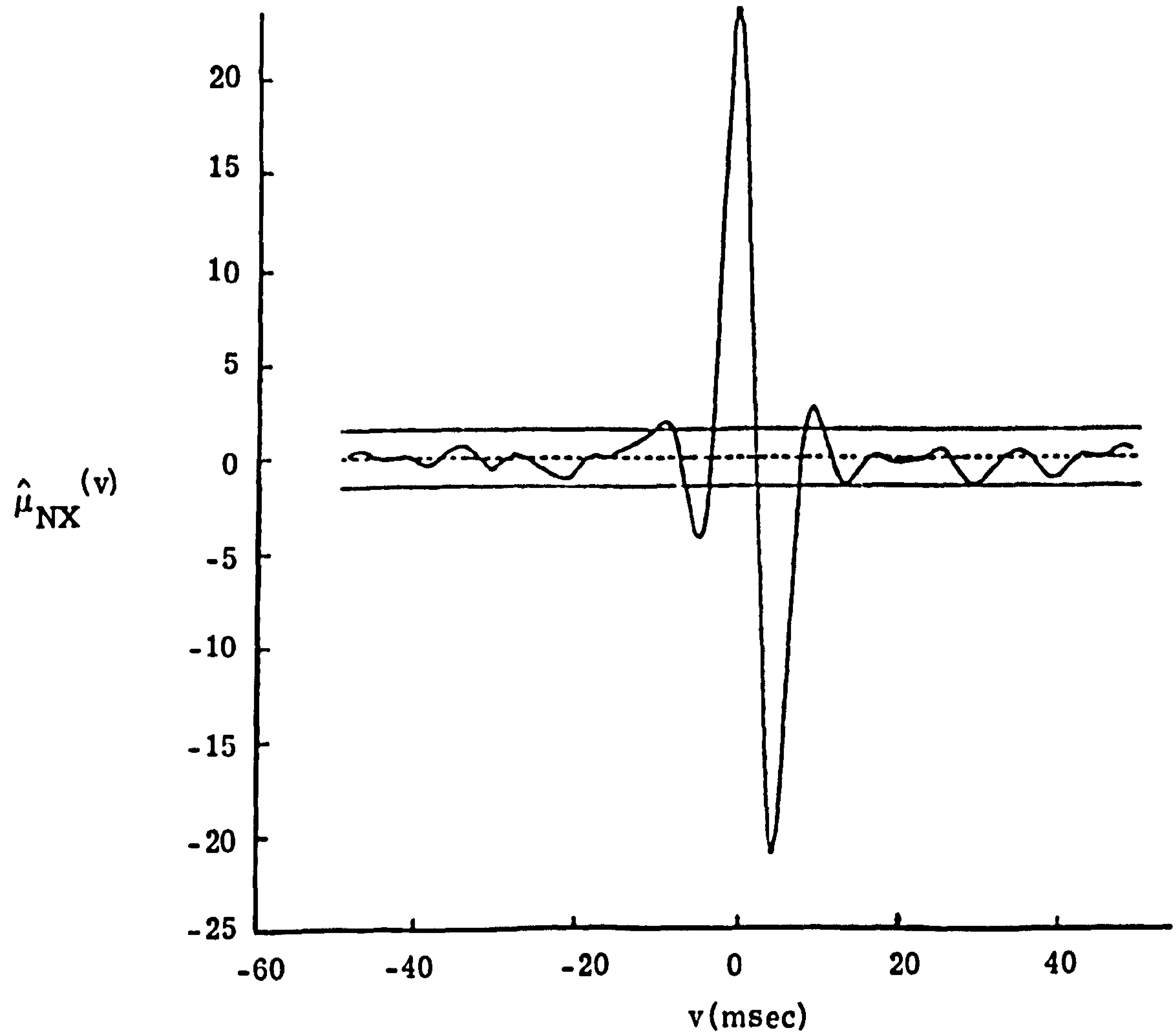


Fig. 5.6.1: Estimate of the second-order cross cumulant function between the Ia discharge and the length change calculated from the cross-periodogram by using the whole record. The dotted line in the middle is the mean value of this estimate under the assumption that the Ia discharge and the length change are independent. The horizontal lines are the asymptotic 95% confidence limits.

estimate of the asymptotic variance of $\hat{\mu}'_{NX}(v)$ is given by

$$Var\left\{\hat{\mu}'_{NX}(v)\right\} \simeq \left(\frac{2\pi}{S}\right)^2 \frac{1}{\ell} \sum_{s\neq 0} \hat{f}^{(T)}_{NN}\left(\frac{2\pi s}{S}\right) \hat{f}^{(T)}_{XX}\left(\frac{2\pi s}{S}\right) . \tag{5.6.6}$$

It also follows from the arguments of Chapter 4 that the distribution of $\hat{\mu}'_{NX}(v)$ is asymptotically normal.

Fig. 5.6.2 gives the estimate $\hat{\mu}'_{NX}(v)$.

An estimate of the first-order kernel, $a_1(u)$, can be obtained in the same way as that of the spike-triggered average, that is

$$\hat{a}_1(u) = \frac{1}{S} \sum_{j\neq 0} A^{(T)}\left(\frac{2\pi j}{S}\right) e^{i\frac{2\pi j}{S}u} . \tag{5.6.7}$$

Following the arguments of Chapter 4 we have that $a_1(u)$ is normally distributed with mean $\hat{a}_1(u)$ and variance

$$\frac{(2\pi)^{-1}}{S^2} \frac{1}{\ell} \int_{-\pi}^{\pi} f_{NN}(\lambda) f^{-1}_{XX}(\lambda) \left\{1 - |R_{NX}(\lambda)|^2\right\} d\lambda . \tag{5.6.8}$$

Under the hypothesis of independence $\hat{a}_1(u)$ is normally distributed with zero mean and variance given by

$$\frac{1}{S^2} \frac{1}{\ell} \sum_{j\neq 0} f_{NN}\left(\frac{2\pi j}{S}\right) \Big/ f_{XX}\left(\frac{2\pi j}{S}\right) . \tag{5.6.9}$$

Fig. 5.6.3 presents the estimate of $a_1(u)$.

5.7 ESTIMATES OF THE CROSS-SPECTRUM BASED ON TIME-DOMAIN PARAMETERS

The cross-spectrum between X(t) and N(t) may be defined by

$$f_{NX}(\lambda) = (2\pi)^{-1} \int_{-\infty}^{\infty} \varsigma_{NX}(v) e^{-i\lambda v} dv , \quad -\infty < \lambda < \infty \tag{5.7.1}$$

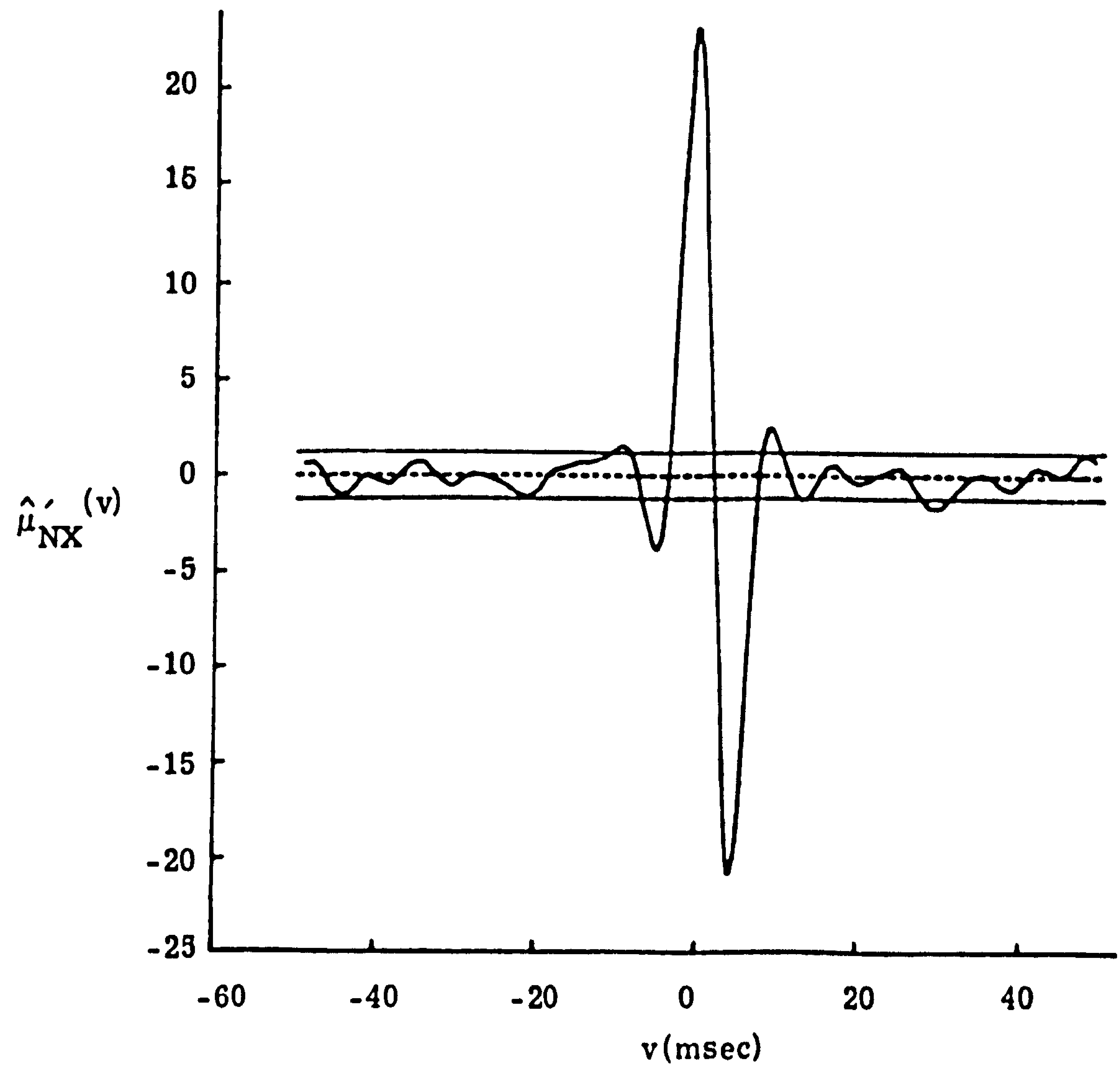


Fig. 5.6.2: Estimate of the cross cumulant function between the Ia discharge and the length change calculated from the cross-periodogram by dividing the whole record into 62 disjoint sections. The dotted line in the middle is the mean value of this estimate under the assumption of independence and the horizontal lines above and below this line are the asymptotic confidence limits.

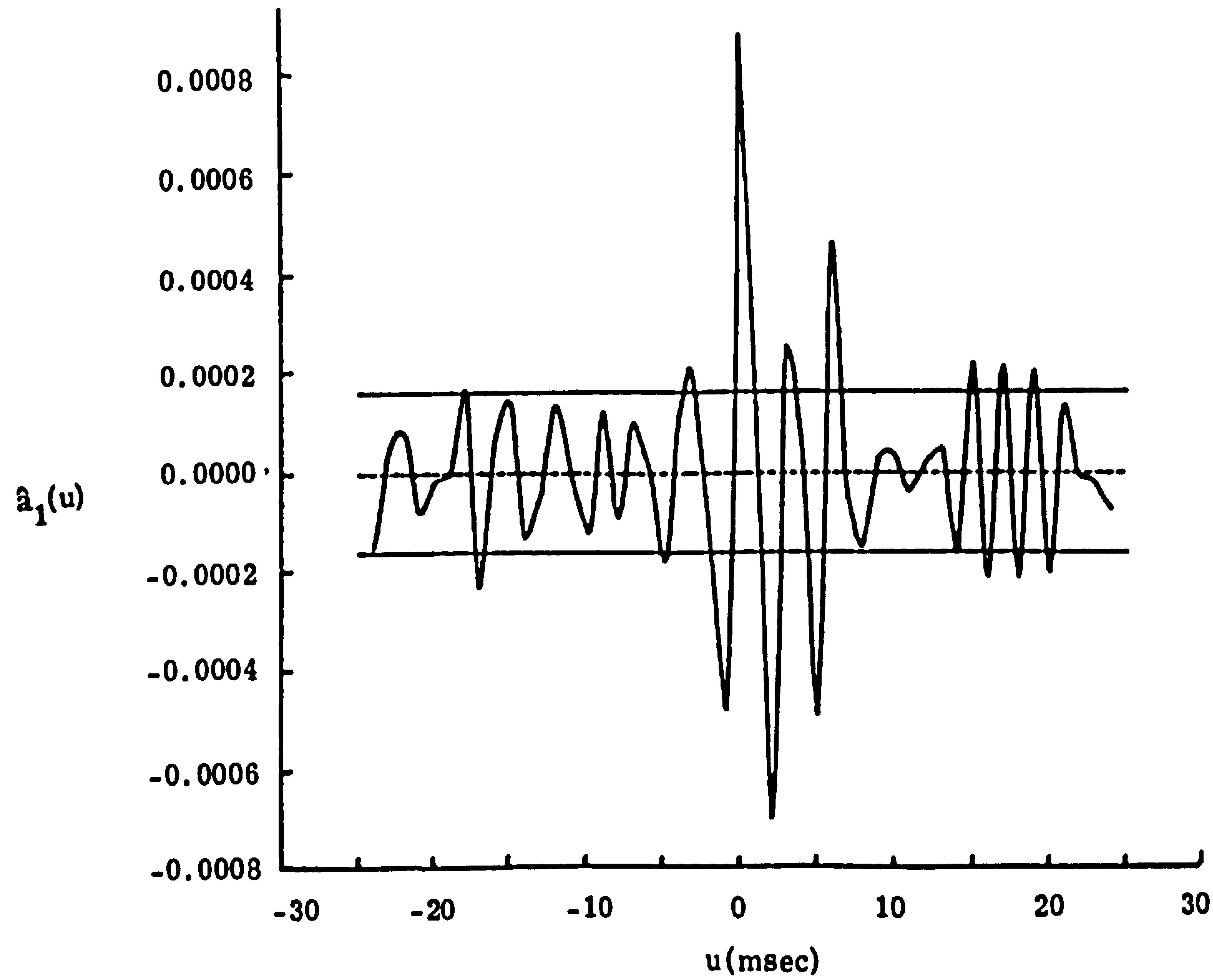


Fig. 5.6.3: Estimate of the first-order kernel of the linear model which relates the Ia discharge to the length change. This estimate was computed by inverting the estimate of the transfer function of the system. The dotted line in the middle corresponds to the mean value of this estimate under the assumption of independence between the Ia discharge and the length change. The horizontal lines are the 95% confidence limits.

where $C_{NX}(v)$ is the cross-covariance of $X(t)$ and $N(t)$. The cross-spectrum $f_{NX}(\lambda)$ exists if $C_{NX}(v)$ exists and satisfies the following relation

$$\int_{-\infty}^{+\infty} |C_{NX}(v)|\,dv < \infty \quad . \tag{5.7.2}$$

Expression (5.7.1) suggests that an estimate of $f_{NX}(\lambda)$ may be given by

$$f_{NX}^{(T)}(\lambda) = \frac{b}{2\pi} \sum_{j} k_T(v_j)\, \hat{C}_{NX}(v_j)\, e^{-i\lambda v_j} \tag{5.7.3}$$

where b is some bin width and $v_j = bj$; $j = 0,1,2,\ldots$.
The estimate of the cross-covariance, $\hat{C}_{NX}(v)$, is expressed as

$$\hat{C}_{NX}(v) = \hat{\mu}_{NX}(v) - \hat{p}_N \hat{\mu}_X \quad . \tag{5.7.4}$$

In the case that the mean of the series $X(t)$ is zero the cross-covariance becomes the "spike-triggered average".

The convergence factor $k_T(v) = k(b_T v)$ improves the properties of the estimate of the cross-spectrum.

Since the cross-spectrum is a complex function with respect to λ, the estimate (5.7.3) can also be written as

$$f_{NX}^{(T)}(\lambda) = Re\, f_{NX}^{(T)}(\lambda) + i\, Im\, f_{NX}^{(T)}(\lambda) \quad , \tag{5.7.5}$$

where $Ref_{NX}^{(T)}(\lambda)$ and $Imf_{NX}^{(T)}(\lambda)$ are the estimates of the real and imaginary parts of the cross-spectrum respectively.

We can now find estimates of the modulus of the cross-spectrum, the phase, the gain and the coherence. These estimates are given by the following relations

$$|f_{NX}^{(T)}(\lambda)| = \left[\left\{ Re f_{NX}^{(T)}(\lambda) \right\}^2 + \left\{ Im f_{NX}^{(T)}(\lambda) \right\}^2 \right]^{1/2} \tag{5.7.6}$$

$$\theta^{(T)}(\lambda) = \tan^{-1} \left\{ Im f_{NX}^{(T)}(\lambda) / Re f_{NX}^{(T)}(\lambda) \right\} \tag{5.7.7}$$

$$G^{(T)}(\lambda) = |A^{(T)}(\lambda)| = \frac{|f_{NX}^{(T)}(\lambda)|}{f_{XX}^{(T)}(\lambda)} \tag{5.7.8}$$

and

$$|R_{NX}^{(T)}(\lambda)|^2 = \frac{|f_{NX}^{(T)}(\lambda)|^2}{f_{NN}^{(T)}(\lambda)\, f_{XX}^{(T)}(\lambda)} \quad . \tag{5.7.9}$$

The estimates of the power spectra of the series X(t) and the point process N(t) are expressed as

$$f_{XX}^{(T)}(\lambda) = \frac{b}{2\pi} \sum_j k_T(u_j)\, \hat{c}_{XX}(u_j)\, e^{-i\lambda u_j} \tag{5.7.10}$$

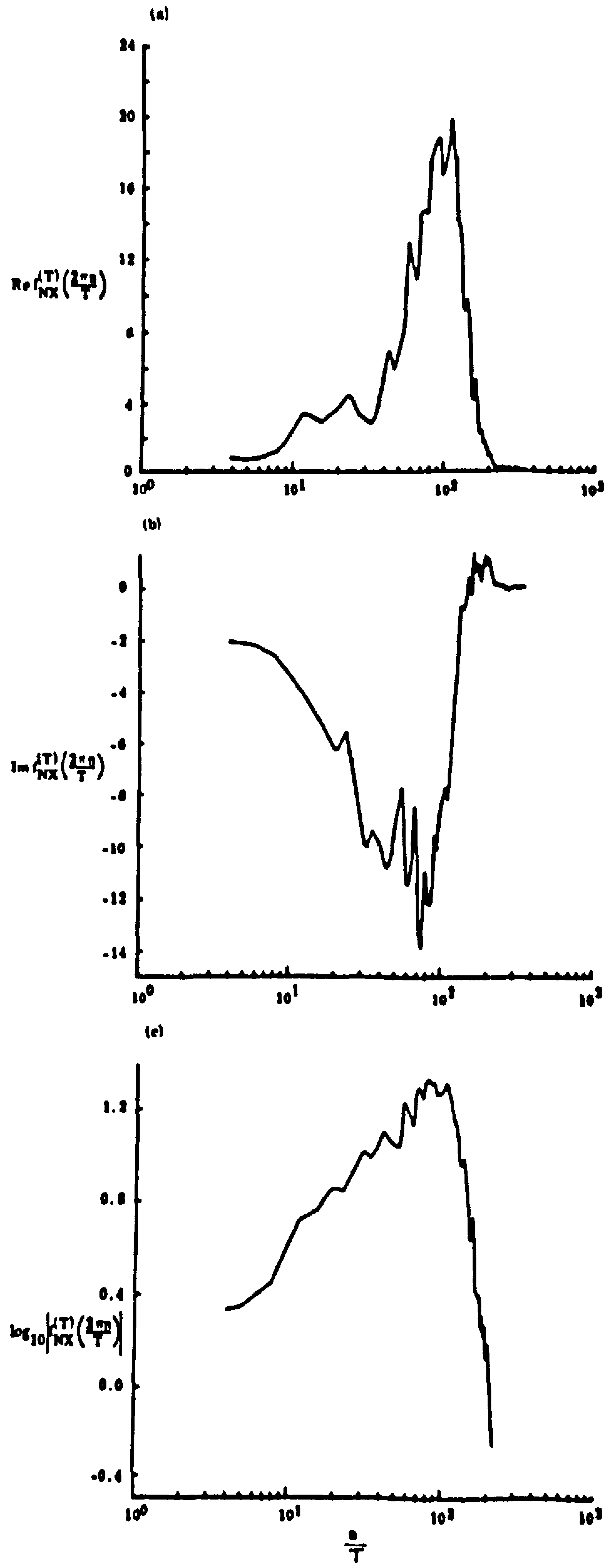
and

$$f_{NN}^{(T)}(\lambda) = \frac{\hat{p}_N}{2\pi} + \frac{b}{2\pi} \sum_j k_T(u_j)\, \hat{q}_{NN}(u_j)\, e^{-i\lambda u_j} \quad , \tag{5.7.11}$$

where $\hat{C}_{XX}(u_j)$ and $\hat{q}_{NN}(u_j)$ are the estimates of the auto-covariance of X(t) and the second-order cumulant of N(t), respectively.

Figs. 5.7.1 (a-c) give the estimates of the real and imaginary parts of the cross-spectrum and the logarithm to the base 10 of the modulus of the cross-spectrum.

Figs. 5.7.2 (a-c) give the estimate of the logarithm to the base 10 of the gain, the estimate of the phase,



<u>Fig. 5.7.1</u>: Estimates of frequency-domain parameters calculated from the cross covariance function between the Ia discharge and the length change. A Parzen window, with a bandwidth $b_T = \frac{1}{256}$, was used for the improvement of the properties of these estimates.
(a) Estimate of the real part of the cross-spectrum; (b) Estimate of the imaginary part of the cross-spectrum and (c) Log_{10} of the estimate of the modulus of the cross-spectrum.

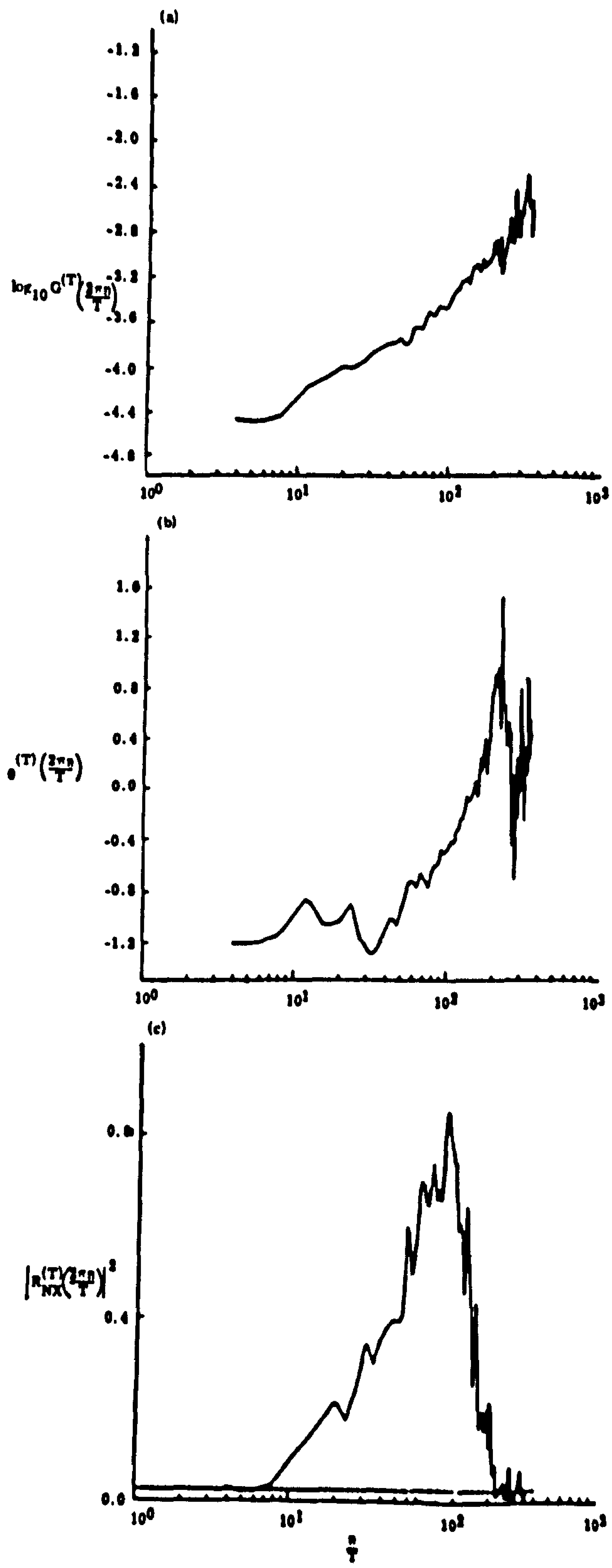


Fig. 5.7.2: Estimates of frequency-domain parameters calculated from the cross covariance matrix when the muscle spindle is affected by a length change. A Parzen window, with a bandwidth $b_T = \frac{1}{256}$, was used for the improvement of the properties of these estimates. (a) Log_{10} of the estimate of the gain; (b) Estimate of the phase and (c) Estimate of the coherence. The dotted line gives the 95% point of the null distribution of this estimate.

and the estimate of the coherence.

The results found here are similar to the ones described in section 5 of this chapter.

It is clear from the estimate of the coherence that the input and output processes are strongly related for frequencies between 10 - 200 Hz.

5.8 THE QUADRATIC MODEL OF HYBRID SYSTEMS

In order to find the non-linear relation between the point process N(t) and the time series X(t), we extend the model (5.2.1) as follows

$$dN(t) = \left[a_0 + \int a_1(t-u)X(u)du + \iint_{u \neq v} a_2(t-u, t-v)X(u)X(v)du\,dv\right]dt + d\varepsilon(t) \, . \tag{5.8.1}$$

The function $a_2(\cdot,\cdot)$ is the second-order kernel which gives the interactive effect of two input points at different times on the output. The model (5.8.1) is a generalization of similar models considered in the case of time series by Tick (1961) and Brillinger (1970a).

The identification of the model (5.8.1) can be carried out by writing (5.8.1) as

$$dN(t) = \left[s_0 + \int s_1(t-u)Y(u)du + \iint_{u \neq v} s_2(t-u, t-v)Y(u)Y(v)du\,dv\right]dt + d\varepsilon(t) \, , \tag{5.8.2}$$

where

$$Y(u) = X(u) - \mu_X \quad . \tag{5.8.3}$$

Now, by using the same arguments of section 4.10 of Chapter 4 and Appendix III, we get

$$p_N = s_0 + \iint_{u \neq v} s_2(t-u, t-v)\, \mu_{YY}(u-v)\, du\, dv \qquad (5.8.4)$$

$$\mu_{NY}(\tau) = \int s_1(\tau - v)\, \mu_{YY}(v)\, dv \qquad (5.8.5)$$

$$\mu_{NYY}(\tau, \sigma) - p_N \mu_{YY}(\sigma - \tau) = \iint_{u \neq v} s_2(t-u, \sigma - v)\, \mu_{YY}(u)\, \mu_{YY}(v)\, du\, dv$$

$$+ \iint_{u \neq v} s_2(\sigma - v, \tau - u)\, \mu_{YY}(u)\, \mu_{YY}(v)\, du\, dv \ . \qquad (5.8.6)$$

Expression (5.8.6) follows from the fact that Y(t) is a Gaussian process with zero mean.

In the frequency domain the relations (5.8.5) and (5.8.6) become

$$f_{NY}(\lambda) = S_1(\lambda)\, f_{YY}(\lambda) \qquad (5.8.7)$$

$$f_{NYY}(\lambda, \mu) = 2\, S_2(-\lambda, -\mu)\, f_{YY}(\lambda)\, f_{YY}(\mu) \ , \qquad (5.8.8)$$

where $f_{NYY}(\lambda, \mu)$ is the third-order spectrum defined by

$$f_{NYY}(\lambda, \mu) = (2\pi)^{-2} \iint_{-\infty}^{+\infty} \left[\mu_{NYY}(\tau, \sigma) - p_N \mu_{YY}(\sigma - \tau) \right] e^{-i(\lambda\tau + \mu\sigma)} d\sigma\, d\tau \ , \ -\infty < \lambda, \mu < \infty \ . \qquad (5.8.9)$$

The cross bi-spectrum between two time series has been considered by Tick (1961) who generalized the definition of the bi-spectrum of a single time series (Tukey, 1959a).

The transfer function of the quadratic model (5.8.2) is defined by

$$S_2(\lambda,\mu) = \iint s_2(u,v)\, e^{-i(\lambda u+\mu v)} du dv \ , \quad -\infty < \lambda,\mu < \infty \ , \tag{5.8.10}$$

that is, the Fourier transform of $s_2(u,v)$.
It also follows from (5.8.8) that

$$S_2(\lambda,\mu) = \frac{f_{NYY}(-\lambda,-\mu)}{2 f_{YY}(\lambda) f_{YY}(\mu)} \ , \quad \text{if } f_{YY}(\lambda) \text{ and } f_{YY}(\mu) \neq 0 \ . \tag{5.8.11}$$

In order to be able to identify the model in practice, we require estimates of the cumulant spectra based on the data of the time series Y(t) and the point process N(t) in the time interval (0,T] .

An estimate of $f_{NYY}(\lambda, \mu)$ is given by

$$f^{(T)}_{NYY}(\lambda,\mu) = \frac{b^2}{(2\pi)^2} \sum_j \sum_k w_T(\tau_j, \sigma_k) \left[\hat{\mu}_{NYY}(\tau_j,\sigma_k) - \hat{p}_N \mu_{YY}(\sigma_k - \tau_j) \right] e^{-i(\lambda\tau_j + \mu\sigma_k)}, \tag{5.8.12}$$

where $\tau_j = bj$ and $\sigma_k = bk$; j, k = 1,2,... .

The function $w_T(\tau, \sigma)$ is a convergence factor which improves the properties of the double sum of (5.8.12). Such convergence factors are discussed by Brillinger (1965).

The estimate $\hat{\mu}_{NYY}(\tau,\sigma)$ is approximately given by

$$\hat{\mu}_{NYY}(\tau,\sigma) \simeq \frac{1}{T} \sum_{i=1}^{N(T)} Y(t_i - \tau) Y(t_i - \sigma) \ . \tag{5.8.13}$$

Thus, an estimate of $S_2(\lambda, \mu)$ can be written as

$$S^{(T)}_2(\lambda,\mu) = \frac{f^{(T)}_{NYY}(-\lambda,-\mu)}{2 f^{(T)}_{YY}(\lambda) f^{(T)}_{YY}(\mu)} \ . \tag{5.8.14}$$

Now, the increments of the error process can be expressed in the following manner

$$d\epsilon(t) = dN(t) - \left[s_0 + \int s_1(t-u)Y(u)du + \iint_{u \neq v} s_2(t-u, t-v)Y(u)Y(v)du\,dv \right] dt \, , \qquad (5.8.15)$$

and by considering the arguments of Tick (1961) and Appendix III we obtain

$$f_{\epsilon\epsilon}(\lambda) = f_{NN}(\lambda) - f_{YY}(\lambda)|S_1(\lambda)|^2 - 2\int |S_2(\lambda-\mu, \mu)|^2 f_{YY}(\lambda-\mu) f_{YY}(\mu) d\mu \, . \qquad (5.8.16)$$

By substituting $S_1(\lambda)$ from (5.8.7) and $S_2(\lambda, \mu)$ from (5.8.11) into (5.8.16) we further have

$$f_{\epsilon\epsilon}(\lambda) = f_{NN}(\lambda)\left[1 - |R_{NY}(\lambda)|^2 - \frac{0.5}{f_{NN}(\lambda)} \int \frac{|f_{NYY}(\lambda-\mu, \mu)|^2}{f_{YY}(\lambda-\mu) f_{YY}(\mu)} d\mu \right], \qquad (5.8.17)$$

where by the quantity

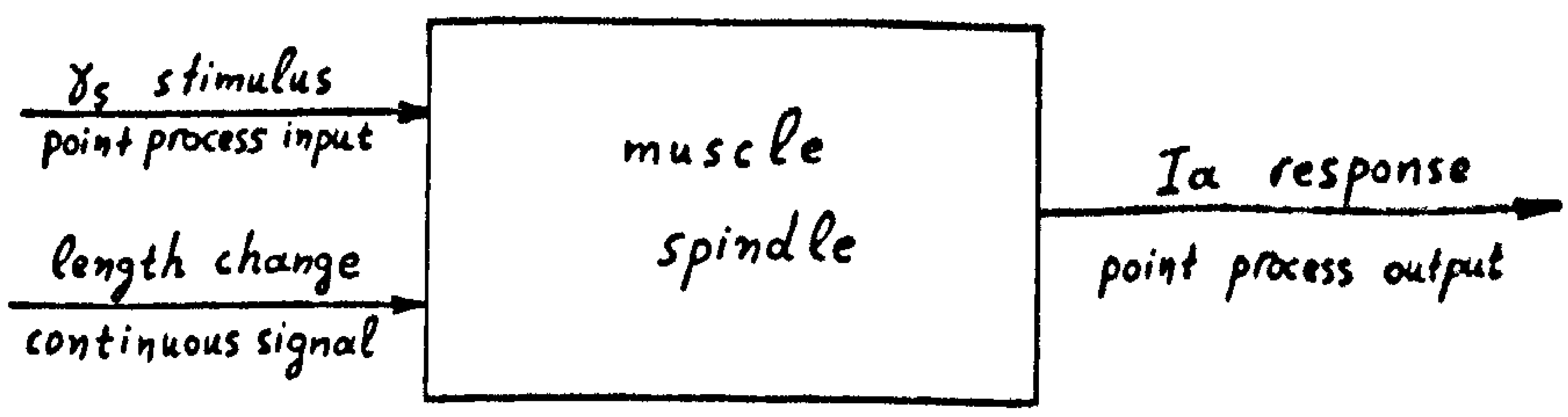
$$|R_{NY}(\lambda)|^2 + \frac{0.5}{f_{NN}(\lambda)} \int \frac{|f_{NYY}(\lambda-\mu, \mu)|^2}{f_{YY}(\lambda-\mu) f_{YY}(\mu)} d\mu \quad , \qquad (5.8.18)$$

we define the quadratic coherence.

The quadratic coherence takes values between 0 and 1. Equation (5.8.18) will be one when the system is quadratic.

5.9 IDENTIFICATION OF A LINEAR HYBRID SYSTEM HAVING A CONTINUOUS SIGNAL AND A POINT PROCESS AS INPUTS

In this section we examine the situation which arises when a length change and a γs stimulus are simultaneously applied to the muscle spindle. A simple graphical representation of this system may be given as



We now assume that the Ia response is a stationary point process N(t) on the interval (0,T]. The two inputs, which are applied independently to the muscle spindle, consist of a point process M(t) (γs stimulus), and a time series X(t) (length change) where t is defined on the interval (0,T].

We may consider the Ia response as a linear combination of the two inputs which can be described mathematically as

$$dN(t) = \left\{ a_0 + \int a_1(t-u)\,dM(u) + \int a_2(t-v)X(v)\,dv \right\} dt + d\varepsilon(t) \quad , \tag{5.9.1}$$

where the error process $\varepsilon(t)$ has stationary increments with zero mean, and it is assumed to be independent of the series X(t) and the point process M(t). The functions a_0, $a_1(\cdot)$ and $a_2(\cdot)$ are the zero- and the first-order kernels of the system.

Following the arguments of Chapter 4 (section 4.3) and of this chapter (section 5.2) we have

$$p_N = a_0 + p_M \int a_1(u)\,du + \mu_X \int a_2(v)\,dv \tag{5.9.2}$$

$$q_{NM}(u) = p_M a_1(u) + \int a_1(u-u_1)\, q_{MM}(u_1)\,du_1 \tag{5.9.3}$$

and

$$c_{NX}(v) = \int a_2(v-v_1)\, c_{XX}(v_1)\,dv_1 \quad . \tag{5.9.4}$$

In the frequency domain expressions (5.9.3) and (5.9.4) become

$$f_{NM}(\lambda) = A_1(\lambda) f_{MM}(\lambda) \qquad (5.9.5)$$

and

$$f_{NX}(\lambda) = A_2(\lambda) f_{XX}(\lambda) \quad , \qquad (5.9.6)$$

where

$$A_1(\lambda) = \int a_1(u) e^{-i\lambda u} d\lambda \ , \ -\infty < \lambda < \infty \qquad (5.9.7)$$

and

$$A_2(\lambda) = \int a_2(u) e^{-i\lambda u} d\lambda \ , \ -\infty < \lambda < \infty \quad . \qquad (5.9.8)$$

It can be shown that the power spectrum of the error process is given by

$$f_{\epsilon\epsilon}(\lambda) = f_{NN}(\lambda)\left[1 - |R_{NM}(\lambda)|^2 - |R_{NX}(\lambda)|^2\right] \quad , \qquad (5.9.9)$$

where the coherences $|R_{NM}(\lambda)|^2$ and $|R_{NX}(\lambda)|^2$ have previously been defined.

The linear model 5.9.1 will be adequate if the power spectrum of the error process is close to zero.

The estimation of the parameters discussed above can be carried out by considering the methods of estimation described in Chapters 3 and 4, and the first part of this chapter.

Fig. 5.9.1 indicates that the two inputs are uncorrelated. Fig. 5.9.1 (a) is the estimate of $\mu_{MX}(u)$, that is, the "spike-triggered average" of M(t) and X(t).

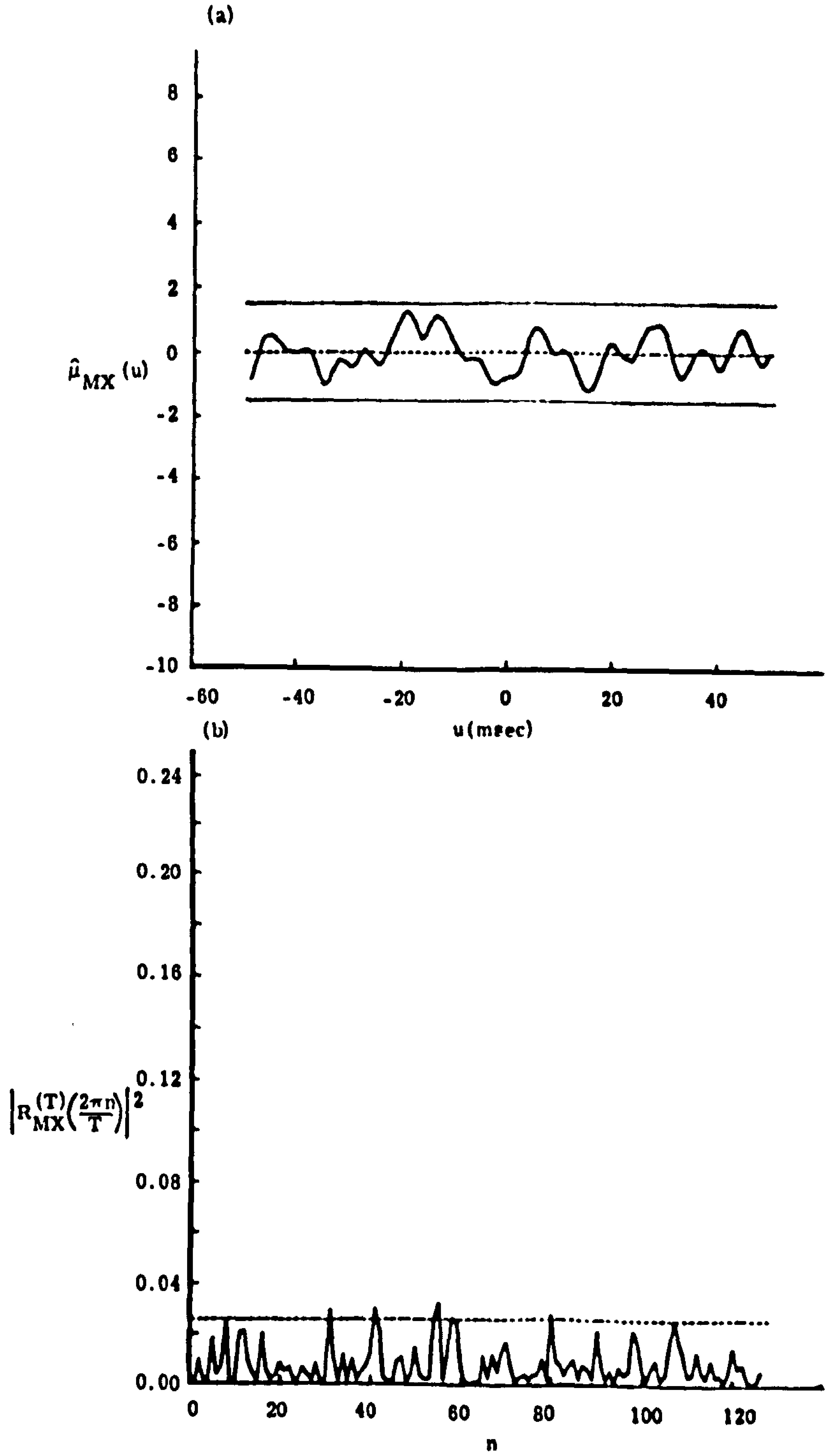


Fig. 5.9.1: Estimates of time- and frequency-domain parameters between the fusimotor input (γs) and the length change (l) when the muscle spindle is affected by γs and l simultaneously. (a) Estimate of the "spike-triggered average" between the fusimotor input and the length change. The dotted line in the middle gives the mean value of this estimate under the hypothesis of independence between the two inputs, and the horizontal lines above and below this line are the asymptotic 95% confidence limits. (b) Estimate of the coherence between the fusimotor input and the length change. The dotted line gives the 95% point of the null distribution of this estimate. Both figures clearly show that the two inputs are uncorrelated.

Fig. 5.9.1 (b) is the estimate $|R_{MX}(\lambda)|^2$. The coherence is not significant for all frequencies indicating that there is no correlation between the two inputs M(t) and X(t).

Fig. 5.9.2 gives estimates of the frequency domain parameters which illustrate the relation between the two point processes. Fig. 5.9.2 (a) is the estimate of the real part of the cross-spectrum between N(t) and M(t).

Fig. 5.9.2 (b) is the estimate of the imaginary part of the $f_{MN}(\lambda)$. Fig. 5.9.2 (c) is the estimate of the phase. This estimate is called the constrained estimate of the phase (Chatfield, 1980; p180). Fig. 5.9.2 (d) is the unconstrained estimate of the phase. This figure illustrates the effect of the fusimotor input on the muscle spindle when a length change is present. For frequencies greater than 40 cycles/sec the phase becomes very irregular which indicates that the input and output point processes become independent.

Figs. 5.9.3 (a-b) are estimates of the $\log_{10}|f_{MN}(\lambda)|$ and $|R_{MN}(\lambda)|^2$ between M(t) and N(t) when a length change is present. The estimate of the coherence clearly shows that the two processes are uncorrelated for frequencies greater than 40 cycles/sec.

Fig. 5.9.4 is the estimate of the first-order kernel $a_1(u)$ under the assumption that the point process M(t) is a Poisson process. Under the hypothesis of independence the mean value of this estimate is zero. The mean value of $a_1(u)$ is indicated by the dotted line while the

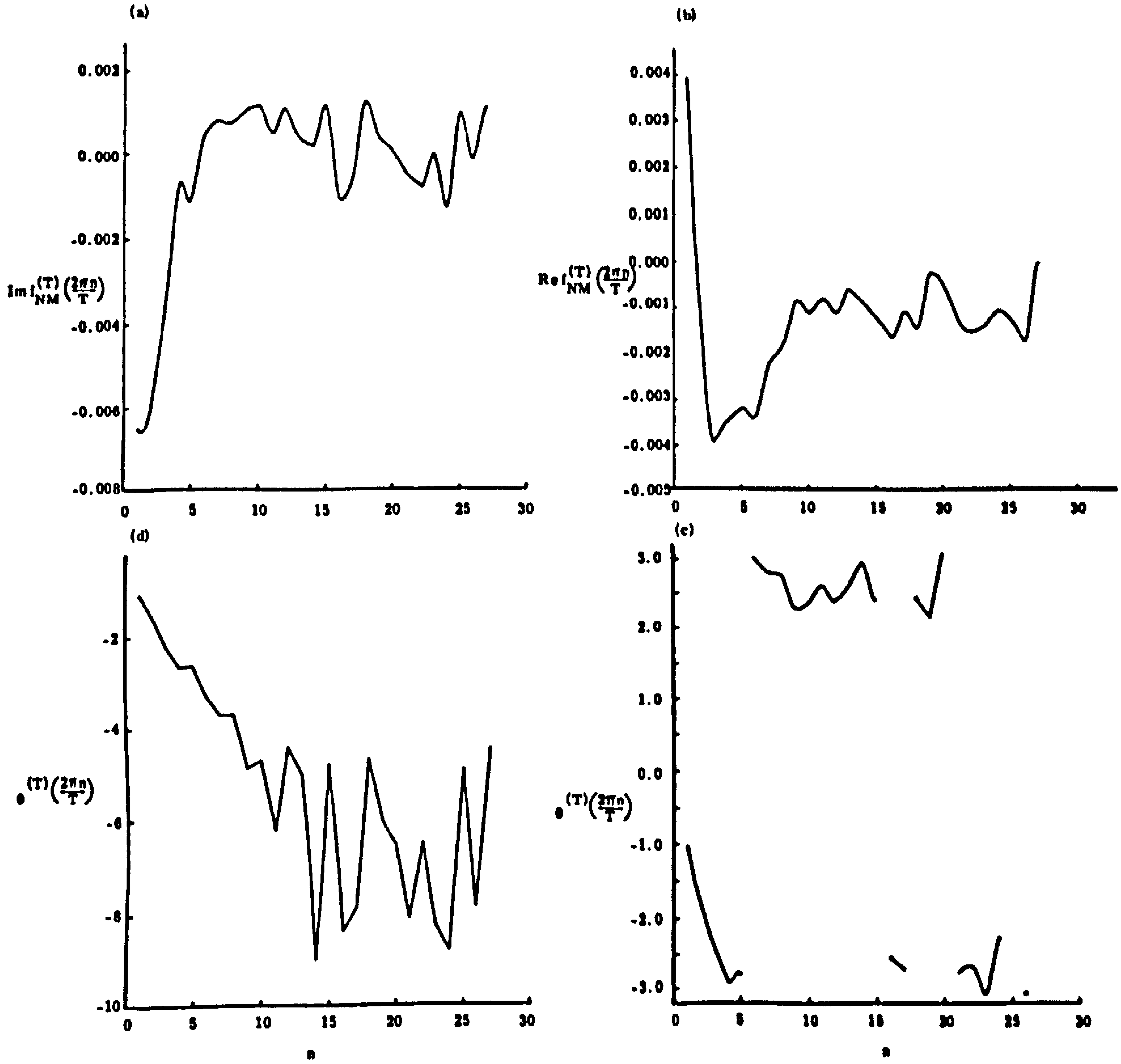
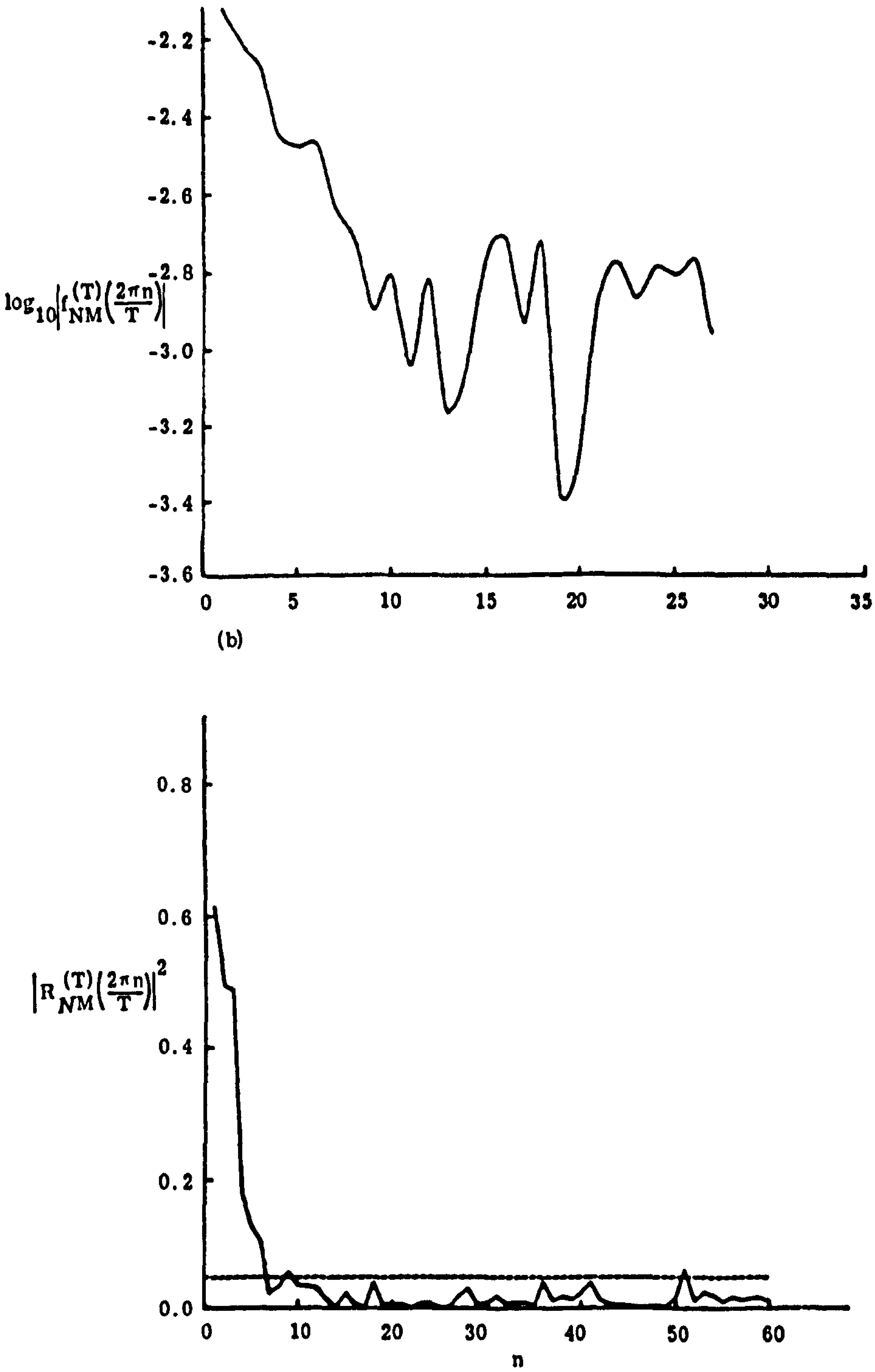


Fig. 5.9.2: Estimates of frequency-domain parameters calculated from the estimate of the CIF between the Ia discharge and the fusimotor input when the muscle spindle is affected simultaneously by a length change and a fusimotor input. A Parzen window, with a bandwidth $b_T = \frac{1}{256}$, was used for the improvement of the properties of these estimates. (a) Estimate of the real part of the cross-spectrum; (b) Estimate of the imaginary part of the cross-spectrum ; (c) Estimate of the constrained phase and (d) Estimate of the unconstrained phase.



<u>Fig. 5.9.3:</u> Estimates of frequency-domain parameters between the Ia discharge and the fusimotor input when the muscle spindle is affected simultaneously by a length change and a fusimotor input. These estimates were calculated from the time-domain parameters by using a Parzen window with a bandwidth $b_T = \frac{1}{256}$. (a) Log_{10} of the estimate of the modulus of the cross-spectrum and (b) Estimate of the coherence. The dotted line is the 95% point of the null distribution of this estimate.

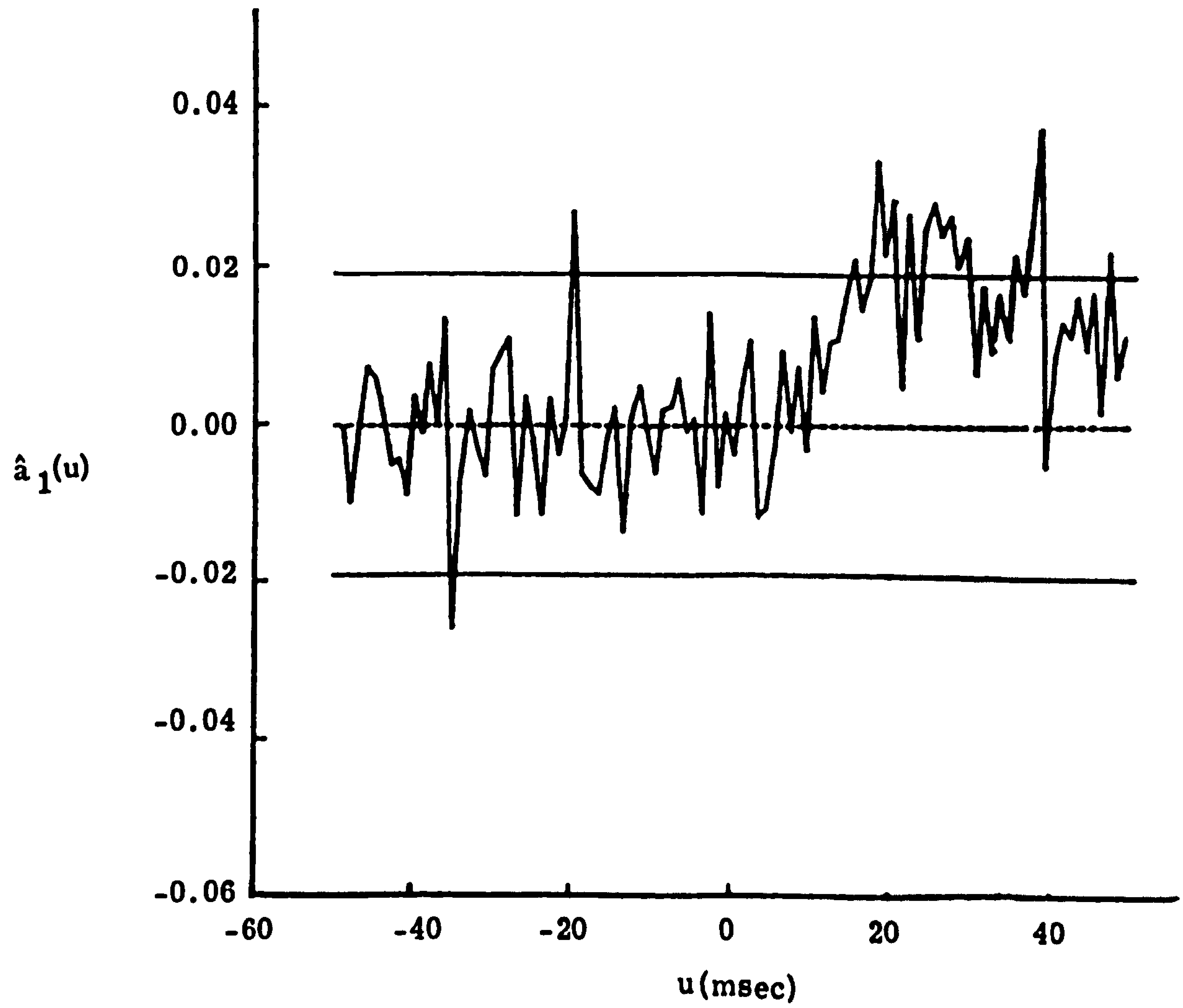


Fig. 5.9.4: Estimate of the first-order kernel of the linear model which relates the Ia discharge to the fusimotor input and the length change. This estimate describes the effect of the fusimotor input on the Ia discharge when a length change is present. The dotted line in the middle is the mean value of this estimate under the hypothesis that the Ia discharge and the fusimotor input are independent. The two horizontal lines are the asymptotic 95% confidence limits.

horizontal lines are the 95% confidence limits.

Figs. 5.9.5 (a-c) are the estimates of the $\mathrm{Re}f_{NX}(\lambda)$, $\mathrm{Im}f_{NX}(\lambda)$ and $\theta^{(T)}(\lambda)$ between the point process N(t) and the continuous signal X(t) when a fusimotor process is present.

Figs. 5.9.6 (a-c) are the estimates of the $\log_{10}|f_{NX}(\lambda)|$, $\log_{10} G_2(\lambda)$ and $|R_{NX}(\lambda)|^2$. The estimate of the $\log_{10} G_2(\lambda)$ is defined as the estimate of $\log_{10}|A_2(\lambda)|$. Fig. 5.9.6 (c) clearly shows that the output and the input are uncorrelated for frequencies up to 30 cycles/sec, while for higher frequencies they are strongly related. The coherence becomes zero for frequencies greater than 200 cycles/sec.

Figs. 5.9.7 (a-b) give the estimates of the "spike-triggered average" between N(t) and X(t) and the first-order kernel $a_2(u)$ of the linear system 5.9.1.

The application of the methods described in Chapters 3,4 and 5 for calculating the spectra of point processes, and the cross-spectrum between a continuous signal and a point process may reveal interesting features in the behaviour of the muscle spindle when two inputs are present simultaneously. For example, the coherences illustrated in figures 5.9.3 (b) and 5.9.6 (c) show that the presence of the length change reduces the effect of the fusimotor input for frequencies less than 40 cycles/sec, while the length change starts to be related strongly with the Ia response for frequencies greater than 40 cycles/sec.

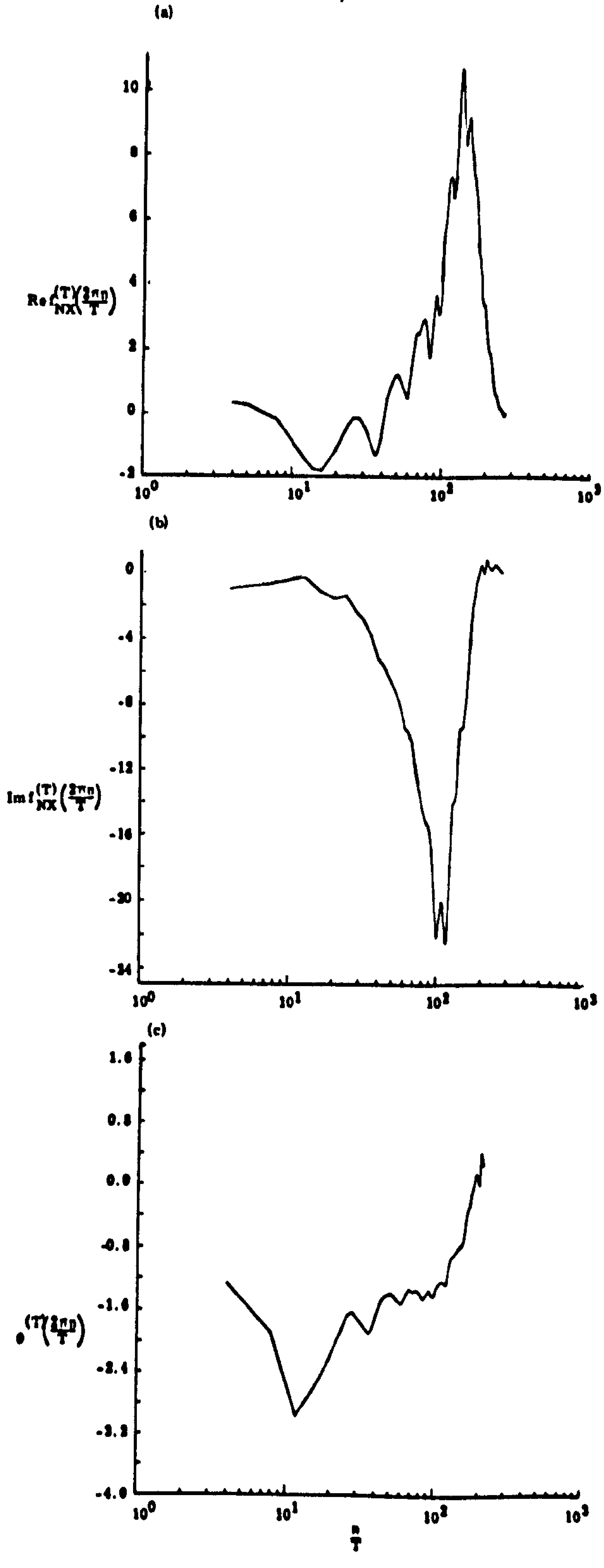


Fig. 5.9.5: Estimates of frequency-domain parameters between the Ia discharge and the length change when the muscle spindle is affected simultaneously by a length change and a fusimotor input. These estimates were obtained from the cross-periodogram by dividing the whole record into 62 disjoint sections and were further smoothed by using a hanning window. (a) Estimate of the real part of the cross-spectrum; (b) Estimate of the imaginary part of the cross-spectrum and (c) Estimate of the phase. The value of $T = 256$.

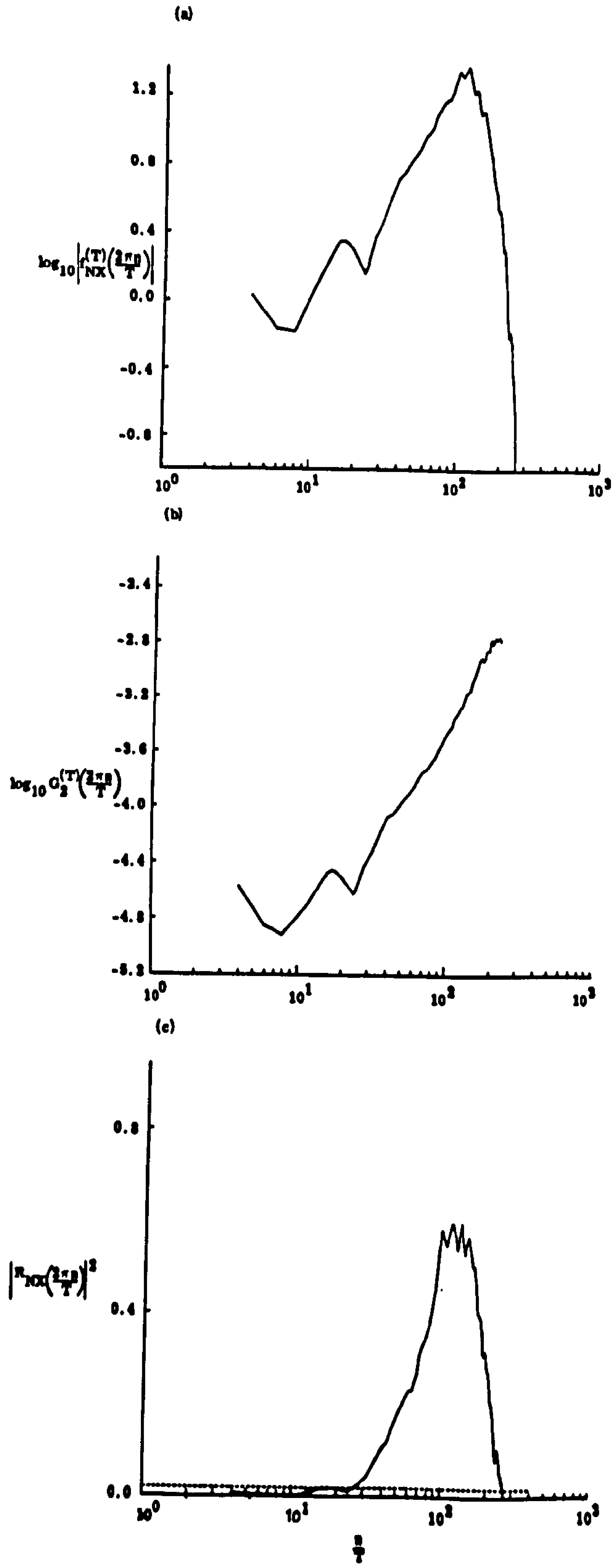


Fig. 5.9.6: Estimates of frequency-domain parameters between the Ia discharge and the length change when the muscle spindle is affected simultaneously by a length change and a fusimotor input. These estimates were calculated from the periodogram matrix by dividing the whole record into 62 disjoint sections and were further smoothed by using a hanning window. (a) Log_{10} of the estimate of the modulus of the cross-spectrum; (b) Log_{10} of the estimate of the gain and (c) Estimate of the coherence. The dotted line is the 95% point of the null distribution of this estimate.

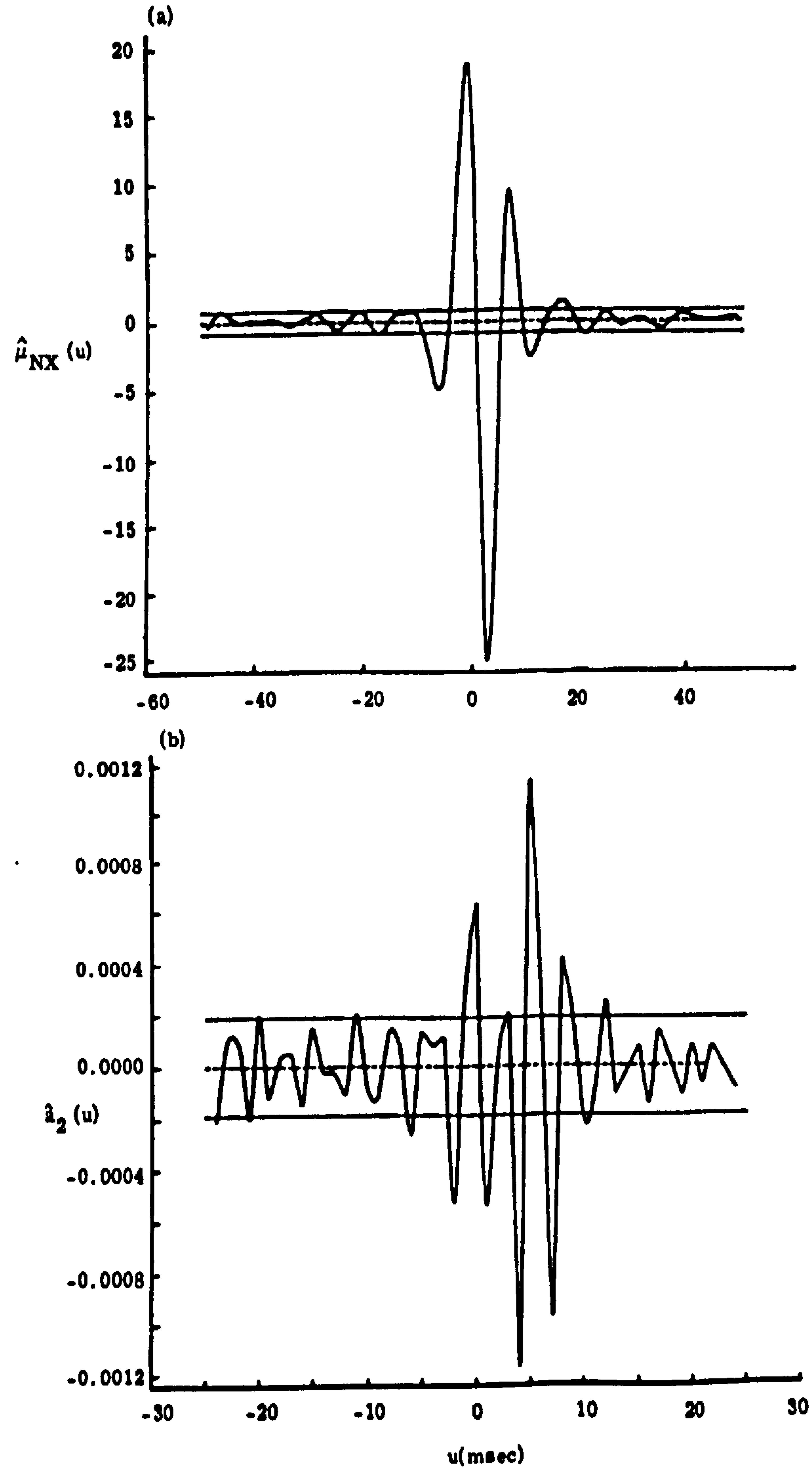


Fig. 5.9.7: Estimates of the time-domain parameters between the Ia discharge and the length change when the muscle spindle is affected simultaneously by a length change and a fusimotor input. (a) Estimate of the "spike-triggered average" between the Ia discharge and the length change when a fusimotor input is present. (b) Estimate of the first-order kernel of the linear model which relates the Ia discharge to the length change and the fusimotor input. This estimate describes the effect of the length change on the Ia discharge when a fusimotor input is present. The dotted lines in these figures are the mean values of these estimates under the hypothesis that the Ia discharge and the length change are independent. The horizontal lines are the asymptotic 95% confidence limits.

This is a very interesting result because it clearly shows that the effect of the fusimotor input on the Ia response takes place at low frequencies while the effect of the length change on the Ia response takes place at higher frequencies, and therefore, a single Ia input to the central nervous system may simultaneously carry information related to two different inputs. The power spectra illustrated in Chapter 3 (Figs. 3.11.7 (b-d)) support and extend this suggestion.

5.10 SUMMARY

In this chapter we introduced a bivariate process which is a mixture of a time series and a point process. Estimates of certain parameters of this bivariate process were discussed. The asymptotic distributions of these parameters were approximately the same with the ones given in Chapter 4 for analogous parameters of the bivariate point process. Confidence intervals based on the asymptotic distributions were constructed under the null hypothesis that the time series and the point process were independent. The identification of the hybrid system involving a time series and a point process was carried out by using a linear model. The extension of this model was also discussed by considering the quadratic model which requires the definition of higher order parameters in time and frequency domains. In the last part of this chapter the identification of a more complicated system was carried out in which two inputs were involved, a time series and a point process.

The practical results of this chapter, which are presented as a number of illustrations verify the usefulness of the frequency domain methods. These methods clearly show the separate areas of frequencies at which the two inputs have an effect on the output (Fig. 5.9.3 (b) and Fig. 5.9.6 (c)). Moreover, by taking into account the results of Chapter 4, we are able to see how the presence of a length change modifies the effect of a fusimotor input on the Ia response. Finally, by comparing the results of this chapter with the results of Chapter 3, we realise the importance of constructing an estimate for the power spectrum of a univariate point process.

The statistical methods of the previous chapters were developed for the identification of the muscle spindle in the cases that we had either a point process input (fusimotor input) or a continuous input (length change), or a point process input and a continuous input together.

We now discuss a number of other cases which arise in practice and require further identification studies of the muscle spindle. The definition and estimation of the parameters for the models of the new cases can be considered as problems for future work.

1. The response of the Ia afferent axon when the muscle spindle is assumed to have two point processes as inputs.

In this case it is assumed that the muscle spindle is affected by two different point processes (inputs). The two inputs may be, for example, the effects of two different fusimotor axons or the direct effect of a fusimotor axon and the indirect effect of an alpha motoneuron. If we denote the two inputs as the point processes M_1 and M_2 and the output response as the point process N, then in order to identify the muscle spindle we propose the following linear model.

$$E\{dN(t)/M_1,M_2\} = \left\{\alpha_0 + \int \alpha_1(t-u)dM_1(u) + \int \alpha_2(t-u)dM_2(u)\right\}dt \quad . \qquad (6.1)$$

The point processes N, M_1, M_2 are assumed to be stationary, orderly and satisfy a (strong) mixing condition.

If the two inputs are independent then, in order to solve equation (6.1) with respect to the functions $a_1(u)$ and $a_2(u)$, we follow the methods described in Chapters 4 and 5. However, if the two inputs are not independent, as occurs in practice, then arguments similar to the ones discussed by Tick (1963) in the case of time series must be applied. These arguments will require the definition and estimation of multiple coherences and partial coherences of point processes (see Brillinger (1975), Ch. 8, for the definition of these functions in the case of time series).

An extension of the linear model (6.1) to take into account the non-linearities of the system can be expressed as

$$E\{dN(t)/M_1, M_2\} = \{\alpha_0 + \int \alpha_1(t-u)\,dM_1(u) + \int \alpha_2(t-u)\,dM_2(u)$$

$$+ \iint_{u \neq v} \alpha_{11}(t-u, t-v)\,dM_1(u)\,dM_1(v) + \iint_{u \neq v} \alpha_{22}(t-u, t-v)\,dM_2(u)\,dM_2(v)$$

$$+ \iint \alpha_{12}^*(t-u, t-v)\,dM_1(u)\,dM_2(v)\}\,dt, \tag{6.2}$$

where $\qquad \alpha_{12}^*(u,v) = \alpha_{12}(u,v) + \alpha_{21}(u,v)$

is the cross term giving the non-linear interaction between the two inputs M_1 and M_2. This model is discussed by Marmarelis (1975). In order to be able to solve the equation (6.2) with respect to $a_{11}(u,v)$, $a_{22}(u,v)$ and $a_{12}^*(u,v)$ the two inputs M_1 and M_2 must be

Poisson point processes with independent increments.

2. Identification of the muscle spindle when it is affected by r point processes simultaneously.

This is a generalization of the previous case taking into account more simultaneous inputs to the muscle spindle. This case gives a more realistic picture of what is happening in practice, since the muscle spindle may be affected by many fusimotor axons directly, and a number of alpha motorneurons indirectly at the same time (ref. Chapter 1). By denoting the r inputs to the system as $\underset{\sim}{M}(t) = \left\{ M_1(t), \ldots, M_r(t) \right\}$, and the output of the system as N(t), we examine the linear relationship between the inputs and the output by using the following model

$$E\left\{ dN(t)/\underset{\sim}{M} \right\} = \left\{ \alpha_0 + \sum_{i=1}^{r} \int \alpha_i(t-u)\, dM_i(u) \right\} dt, \tag{6.3}$$

where $\underset{\sim}{M}(t)$ and N(t) are assumed to be point processes with stationary increments.

In the case that the point processes ($M_i(t)$, $i = 1, \ldots, r$) are not independent we will need to define partial coherences which give the effect of single or multiple input components on the output in presence of different input components.

The linear model (6.3) can be extended to take into account the nonlinearities of the system as follows

$$E\left\{ dN(t)/\underset{\sim}{M} \right\} = \left\{ \alpha_0 + \sum_{\ell=1}^{L} \sum_{K_1 \cdots K_\ell}^{r} \int \cdots \int \alpha_{K_1 \cdots K_\ell}(t-u_1, \ldots, t-u_\ell) \prod_{j=1}^{\ell} dM_{K_j}(u_j) \right\} dt \,. \tag{6.4}$$

This model is considered by Marmarelis (1975) in the multiple input case, and is obviously a generalization of model (6.2). For purposes of identification it is convenient to choose $M_1(t),\ldots,M_r(t)$ to be Poisson point processes with independent increments. General expressions can then be obtained for the kernels $a_{k_1\ldots k_l}(t-u_1,\ldots,t-u_l)$ in terms of the cumulant functions of point processes. Relations involving higher order spectral functions can also be formulated by arguments similar to the ones described in Chapter 4 and 5.

3. Identification of the muscle spindle when it is affected simultaneously by r point processes and a length change.

A more general case arising in practice occurs when the muscle spindle is acted upon simultaneously by a number of discrete inputs (point processes) and a continuous input (length change). By denoting the r discrete inputs to the system as $\underset{\sim}{M}(t) = \{M_1(t),\ldots,M_r(t)\}$, the continuous input as X(t) and the output (point process) as N(t), we may examine the linear relationship between the inputs and the output by considering the following model

$$E\{dN(t)/\underset{\sim}{M},X\} = \{a_0 + \sum_{i=1}^{r}\int a_i(t-u)dM_i(u) + \int b(t-u)X(u)du\}dt \quad . \qquad (6.5)$$

If the inputs are independent then, in order to solve equation (6.5) with respect to $a_1(u),\ldots,a_r(u)$ and b(u), we apply the procedures described in Chapter 5.

If, on the other hand, some of the inputs are not independent, then the definition and estimation of partial coherences between point processes and time series is needed in order to be able to obtain estimates of first-order kernels of the linear model (6.5).

The model (6.5) can also be extended to take into consideration nonlinearities of the system by adding more terms which involve higher order kernels. For identification purposes it would be appropriate to choose $M_1(t),\ldots,M_r(t)$ to be Poisson point processes with independent increments, and X(t) to be a stationary Gaussian process independent of the point processes. Statistical methods which involve time- and frequency-domain parameters must be developed in order to be able to obtain the higher order kernels.

4. <u>Identification of the muscle spindle when we have two outputs and a number of discrete and continuous inputs</u>.

In this case the two outputs are the Ia primary axon and the II secondary axon (ref. Chapter 1). The inputs may consist of a number of fusimotor axons, a number of alpha motorneurons and a length change. For purposes of identification of the system we need to consider more complicated models with multiple outputs and multiple inputs. A model involving only point processes has been described by Brillinger (1975d).

The simplest case of this model with two outputs and one input has been examined in some detail by Brillinger et al (1976).

As we have already mentioned, the four problems described above for future work will involve higher order parameters in both time and frequency domains. Statistical methods requiring estimates of these parameters and their asymptotic properties must be developed. Applications of these methods to the existing data sets should reveal more interesting features of the system under investigation.

APPENDIX I

THE r VECTOR-VALUED POINT PROCESS AND SOME USEFUL LEMMAS

1. Definitions

Let $\underset{\sim}{N}(t) = (N_1(t),\ldots, N_r(t))$, $t \in \mathbb{R}$, be an r vector-valued variable. Then we have

Definition 1.1: We say that the variable $\underset{\sim}{N}(t)$, $t \in \mathbb{R}$, is an r vector-valued stochastic point process, if the individual components $N_1,\ldots, N_r$ are random, non-negative, integer-valued variables (Brillinger, 1975c).

A more general notation for the stochastic point process $\underset{\sim}{N}(t)$ may be given by $\underset{\sim}{N}(I,s)$ where $I \in B_{\mathbb{R}}$, $s \in S$. $B_{\mathbb{R}}$ is the σ-algebra of Borel sets of the real line and (S, B_S, P) is the basic probability space. Suppose that N_k is the k-th component of $\underset{\sim}{N}$. Then $N_k(I,s)$ gives the number of events of the individual process k for the realization s and $k = 1,\ldots, r$. Since we refer to one realization s, we suppress the dependence of N_k on s.

The process $\underset{\sim}{N}$ is assumed to be orderly, that is, the probability of more than two events occurring in a small interval is zero. It is also assumed to be stationary under a more general approach which requires that the distribution of the variate $\left\{N_{k_1}(I_1),\ldots,N_{k_l}(I_l)\right\}$ is invariant under translation, for $k_1,\ldots,k_l = 1,\ldots,r$; $l = 1,2,\ldots$.

Definition 1.2: The process $\underset{\sim}{N}$ is said to be strong mixing with

$$g(u) = \sup\left\{ \left|P(E\cap F) - P(E)P(F)\right| : E \in \mathcal{B}_{\underset{\sim}{N}(-\infty,t]}\ ,\ F \in \mathcal{B}_{\underset{\sim}{N}[t+u,\infty)} \right\},$$

tending to 0 as $u \longmapsto \infty$.

Rosenblatt (1956a) and Hannan (1970) discuss the strong mixing condition in the case of stationary time series while Brillinger (1975c) applies the same condition to point processes.

Assumption 1.1: Let $\underset{\sim}{N}(t) = (N_1(t),\ldots,N_r(t))$ be an r vector-valued point process. Suppose that $t_1,\ldots,t_l$ are distinct real parameters. Then,
Prob $\{$ event of process N_{k_1} in $(t_1,t_1+h_1]$,..., event of process N_{k_l} in $(t_l,t_l+h_l]$ $\}$ $\sim p_{k_1\ldots k_l}(t_1,\ldots,t_l)h_1\ldots h_l$ as $h_1,\ldots,h_l$ tend to 0; $l = 1,2,\ldots$. The function $p_{k_1\ldots k_l}$ $(t_1,\ldots,t_l)$ is called a product density function of order l.

Definition 1.3: Let $\underset{\sim}{N}(t)$, $t\in\mathbb{R}$, be an r vector-valued stochastic point process satisfying Assumption 1.1. Then

$$E\{dN_{k_1}(t_1)\ldots dN_{k_l}(t_l)\} = \sum_{\alpha=1}^{l}\ \sum_{k_1\ldots k_\alpha=1}^{r} \Big[\prod_{j\in\nu_1}\delta\{k_1-k_j\}\Big]\cdots\Big[\prod_{j\in\nu_\alpha}\delta\{k_\alpha-k_j\}\Big]$$

$$\Big[\prod_{j\in\nu_1}\delta\{t_1-t_j\}\Big]\cdots\Big[\prod_{j\in\nu_\alpha}\delta\{t_\alpha-t_j\}\Big]\, p_{k_1\ldots k_\alpha}(t_1,\ldots,t_\alpha)\,dt_1\cdots dt_\alpha\ , \qquad (1.1)$$

where the summation extends over all partitions $(\nu_1,\ldots,\nu_l)$ of the set $(1,\ldots,r)$.

Particular cases, which follow from expression (1.1), include

$$E\{dN_K(t)\} = p_K(t)dt$$

$$E\{dN_{K_1}(t_1)dN_{K_2}(t_2)\} = p_{K_1K_2}(t_1,t_2)dt_1dt_2 \qquad \text{if } K_1 \neq K_2$$

$$= p_{KK}(t_1,t_2)\,dt_1dt_2 + p_K(t_1)dt_1 \quad .$$

In relation to the product density function we define the cumulant density function, $q_{k_1 \ldots k_l}(t_1,\ldots,t_l)$, as

$$q_{K_1\ldots K_\ell}(t_1,\ldots,t_\ell) = \sum_{\alpha=1}^{\ell}(-1)^{\ell-1}(\ell-1)!\,[p_{K_j}(t_j)\,;\, j\in\nu_1]\cdots[p_{K_j}(t_j)\,;\, j\in\nu_\alpha] \;, \tag{1.2}$$

where the summation extends over all partitions $(\nu_1,\ldots,\nu_l)$ of $(1,\ldots r)$. This function measures the degree of dependence between the different components of the process $\underset{\sim}{N}$ at times t_j, $j = 1,\ldots l$. The inverse relation of (1.2) is

$$p_{K_1\ldots K_\ell}(t_1,\ldots,t_\ell) = \sum_{\alpha=1}^{\ell}[q_{K_j}(t_j);\, j\in\nu_1]\cdots[q_{K_j}(t_j)\,;\, j\in\nu_\alpha] \;. \tag{1.3}$$

<u>Definition 1.4</u>: Let $\underset{\sim}{N}(t)$ be an r vector-valued point process satisfying Assumption 1.1. Then

$$\operatorname{cum}\{dN_{K_1}(t_1),\ldots,dN_{K_\ell}(t_\ell)\} = \sum_{\alpha=1}^{\ell}\sum_{K_1\ldots K_\alpha=1}^{r}\Big[\prod_{j\in\nu_1}\delta\{K_1-K_j\}\Big]\cdots\Big[\prod_{j\in\nu_\alpha}\delta\{K_\alpha-K_j\}\Big]$$

$$\Big[\prod_{j\in\nu_1}\delta\{t_1-t_j\}\Big]\cdots\Big[\prod_{j\in\nu_\alpha}\delta\{t_\alpha-t_j\}\Big]\, q_{K_1\ldots K_\alpha}(t_1,\ldots,t_\alpha)\,dt_1\cdots dt_\alpha \;, \tag{1.4}$$

where the summation extends over all partitions $(\nu_1,\ldots,\nu_l)$ of $(1,\ldots,r)$. Particular cases of this definition include

$$E\{dN_K(t)\} = q_K(t)dt = p_K(t)dt$$

$$\operatorname{cov}\{dN_{K_1}(t_1), dN_{K_2}(t_2)\} = q_{K_1K_2}(t_1,t_2)\,dt_1dt_2 \qquad \text{if } K_1 \neq K_2$$

$$= q_{KK}(t_1,t_2)\,dt_1dt_2 + q_K(t_1)dt_1 \ .$$

These definitions for the product and cumulant density functions are particular cases of more general situations which arise when interval functions are defined (Brillinger, 1972).

If the point process $\underset{\sim}{N}(t)$ is stationary, then

$$p_{K_1\ldots K_l}(t_1,\ldots,t_l) = p_{K_1\ldots K_l}(t_1-t_l,\ldots,t_{l-1}-t_l,0) \tag{1.5}$$

and

$$q_{K_1\ldots K_l}(t_1,\ldots,t_l) = q_{K_1\ldots K_l}(t_1-t_l,\ldots,t_{l-1}-t_l,0) \ . \tag{1.6}$$

Under the assumption of stationarity the first-order cumulant function becomes a constant while the second-order cumulant function only depends on the difference $t_1 - t_2$.

<u>Assumption 1.2</u>: Let $\underset{\sim}{N}(t) = (N_1(t),\ldots,N_r(t))$, $t \in \mathbb{R}$, be an r vector-valued stationary point process with differential increments $(dN_1(t),\ldots,dN_r(t))$ satisfying Assumption (1.1).

Then we assume that the cumulants of lth order exist and satisfy

$$\int \cdots \int |u_j| \, |q_{k_1 \ldots k_\ell}(u_1, \ldots, u_{\ell-1})| \, du_1 \cdots du_{\ell-1} < \infty$$

for $\quad k_1, \ldots, k_\ell = 1, \ldots, r \, ; \; \ell = 2, 3, \ldots \quad .$

2. Important Lemmas

Lemma 2.1: Let $\underset{\sim}{N}(t)$, $t \in \mathbb{R}$, be an r vector-valued stationary stochastic process whose cumulant functions up to the l-th order exist and satisfy Assumption 1.2. Let $d^{(T)}_{k_j}(\lambda)$ be the Fourier-Stieltjes transform of the component N_{k_j}, j = 1, ..., r, defined by

$$d^{(T)}_{k_j}(\lambda) = \int_0^T e^{-i\lambda t} dN_{k_j}(t) \quad , \; j = 1, \ldots, r \quad \text{and} \quad -\infty < \lambda < \infty \quad .$$

Then

$$\operatorname{cum}\left\{ d^{(T)}_{k_1}(\lambda_1), \ldots, d^{(T)}_{k_\ell}(\lambda_\ell) \right\} = (2\pi)^{\ell-1} \Delta^{(T)}\left(\sum_{j=1}^{\ell} \lambda_j \right) f_{k_1 \ldots k_\ell}(\lambda_1, \ldots, \lambda_{\ell-1}) + O(1) \, ,$$

where

$$\Delta^{(T)}\left(\sum_{j=1}^{\ell} \lambda_j \right) = \int_0^T e^{-i \sum_{j=1}^{\ell} \lambda_j t} dt \quad .$$

Proof: The required cumulant may be written as follows

$$\operatorname{cum}\left\{ \int_0^T e^{-i\lambda_1 t_1} dN_{k_1}(t_1), \ldots, \int_0^T e^{-i\lambda_\ell t_\ell} dN_{k_\ell}(t_\ell) \right\} = \int_0^T \cdots \int_0^T e^{-i \sum_{j=1}^{\ell} \lambda_j t_j} \operatorname{cum}\left\{ dN_{k_1}(t_1), \ldots, dN_{k_\ell}(t_\ell) \right\}$$

$$= \int_0^T \cdots \int_0^T e^{-i \sum_{j=1}^{\ell} \lambda_j t_j} dQ_{k_1 \ldots k_\ell}(t_1 - t_\ell, \ldots, t_{\ell-1} - t_\ell) \, dt_\ell = \int_{-T}^T \cdots \int_{-T}^T e^{-i \sum_{j=1}^{\ell-1} \lambda_j u_j} dQ_{k_1 \ldots k_\ell}(u_1, \ldots, u_{\ell-1}) \left[\int_t e^{-it \sum_{j=1}^{\ell} \lambda_j} dt \right]$$

where

$$\operatorname{cum}\left\{dN_1(t_1),\ldots,dN_{K_\ell}(t_\ell)\right\} = dQ_{K_1\ldots K_\ell}(t_1-t_\ell,\ldots,t_{\ell-1}-t_\ell)\,dt_\ell,$$

$$t_j - t_\ell = u_j\,,\quad j=1,\ldots,\ell-1 \quad \text{and} \quad t_\ell = t\ .$$

We now need to prove that

$$\left|\int_t e^{-it\sum_{j=1}^{\ell}\lambda_j}dt - \Delta^{(T)}\Big(\sum_{j=1}^{\ell}\lambda_j\Big)\right| \le 2\left(|u_1|+\cdots+|u_{\ell-1}|\right) \qquad .$$

We have

$$\left|\int_t e^{-i\sum_{j=1}^{\ell}\lambda_j t}dt - \Delta^{(T)}\Big(\sum_{j=1}^{\ell}\lambda_j\Big)\right| = \left|\int_0^T e^{-i\sum_{j=1}^{\ell}\lambda_j t}dt - \int_{t_{max}}^T e^{-i\sum_{j=1}^{\ell}\lambda_j t}dt - \int_0^{t_{min}} e^{-i\sum_{j=1}^{\ell}\lambda_j t}dt\right.$$

$$\left.-\Delta^{(T)}\Big(\sum_{j=1}^{\ell}\lambda_j\Big)\right| = \left|\int_{t_{max}}^T e^{-i\sum_{j=1}^{\ell}\lambda_j t}dt + \int_0^{t_{min}} e^{-i\sum_{j=1}^{\ell}\lambda_j t}dt\right|$$

$$\le \left|\int_{t_{max}}^T e^{-i\sum_{j=1}^{\ell}\lambda_j t}dt\right| + \left|\int_0^{t_{min}} e^{-i\sum_{j=1}^{\ell}\lambda_j t}dt\right| < T - t_{max} + t_{min}$$

$$\le \left|T - t_{max} + t_{min}\right| \le 2\left(|u_1|+\cdots+|u_{\ell-1}|\right)\,,\ \text{since}\quad t_{max} = T - \max(u_1,\ldots,u_{\ell-1},0)\,,$$

$$t_{min} = -\min(u_1,\ldots,u_{\ell-1},0) \ \text{ and } \ |t_{min}|,|t_{max}| \le |u_1|+\cdots+|u_{\ell-1}|\,,\ |u_j|\le T,\ j=1,\ldots,\ell-1\ .$$

Thus the required cumulant becomes

$$\operatorname{cum}\left\{d_{K_1}^{(T)}(\lambda_1),\ldots,d_{K_\ell}^{(T)}(\lambda_\ell)\right\} = \Delta^{(T)}\Big(\sum_{j=1}^{\ell}\lambda_j\Big)\int_{-T}^{T}\!\!\cdots\!\int_{-T}^{T} e^{-i\sum_{j=1}^{\ell-1}\lambda_j u_j}dQ_{K_1\ldots K_\ell}(u_1,\ldots,u_{\ell-1}) + \varepsilon$$

where

$$|\varepsilon| \leqslant 2\int_{-T}^{T}\cdots\int_{-T}^{T}(|u_1|+\cdots+|u_{\ell-1}|)\,|dQ_{K_1\ldots K_\ell}(u_1,\ldots,u_{\ell-1})| \quad .$$

We set $\varepsilon = O(1)$ because

$$\int_{-T}^{T}\cdots\int_{-T}^{-T}(|u_1|+\cdots+|u_{\ell-1}|)\,|dQ_{K_1\ldots K_\ell}(u_1,\ldots,u_{\ell-1})|$$

is bounded under the Assumption 1.2.

Assumption 2.1: $h(u)$, $-\infty < u < \infty$, is bounded, is of bounded variation and vanishes for $|u| > 1$.

Lemma 2.2: Let $\underset{\sim}{N}(t)$, $t \in \mathbb{R}$, be an r vector-valued stationary stochastic point process satisfying the assumptions of Lemma 2.1. Let the Fourier-Stieltjes transform be defined by

$$d_{K_j}^{(T)}(\lambda) = \int_t h_{K_j}(t/T)\, e^{-i\lambda t} dN_{K_j}(t) \ , \ j=1,\ldots,r \ , \ -\infty < \lambda < \infty \ ,$$

where the function $h(u)$, $-\infty < u < \infty$, satisfies Assumption 2.1. Then

$$\mathrm{cum}\{d_{K_1}^{(T)}(\lambda_1),\ldots,d_{K_\ell}^{(T)}(\lambda_\ell)\} = (2\pi)^{\ell-1} H^{(T)}\Big(\sum_{j=1}^{\ell}\lambda_j\Big) f_{K_1\ldots K_\ell}(\lambda_1,\ldots,\lambda_{\ell-1}) + O(1) \ ,$$

where

$$H^{(T)}\Big(\sum_{j=1}^{\ell}\lambda_j\Big) = \int_t h_{K_1}(t/T)\cdots h_{K_\ell}(t/T)\, e^{-i\sum_{j=1}^{\ell}\lambda_j t} dt \quad .$$

Proof: The required cumulant can be written as

$$\mathrm{cum}\{dN_{K_1}(\lambda_1),\ldots,dN_{K_\ell}(\lambda_\ell)\} = \int_t\cdots\int_t h_{K_1}(t_1/T)\cdots h_{K_\ell}(t_\ell/T)\,e^{-i\sum_{j=1}^{\ell}\lambda_j t_j}\,\mathrm{cum}\{dN_{K_1}(t_1),\ldots,dN_{K_\ell}(t_\ell)\}$$

$$= \int_t\cdots\int_t h_{K_1}(t_1/T)\cdots h_{K_\ell}(t_\ell/T)\,e^{-i\sum_{j=1}^{\ell}\lambda_j t_j}\,dQ_{K_1\ldots K_\ell}(t_1-t_\ell,\ldots,t_{\ell-1}-t_\ell)\,dt_\ell$$

$$= \int_{-T}^{T}\cdots\int_{-T}^{T} e^{-i\sum_{j=1}^{\ell-1}\lambda_j u_j}\,dQ_{K_1\ldots K_\ell}(u_1,\ldots,u_{\ell-1})\Big[\int_t h_{K_1}(t+u_1/T)\cdots h_{K_\ell}(t/T)\,e^{-i\sum_{j=1}^{\ell}\lambda_j t}\,dt\Big],$$

by substituting $u_j = t_j - t_l$, $j = 1,\ldots,l$, and $t = t_l$.
Now, the following relation is satisfied

$$\Big|\int_t h_{K_1}(t+u_1/T)\cdots h_{K_\ell}(t/T)\,e^{-i\sum_{j=1}^{\ell}\lambda_j t}\,dt - H^{(T)}_{K_1\ldots K_\ell}\Big(\sum_{j=1}^{\ell}\lambda_j\Big)\Big| < L'(|u_1|+\cdots+|u_{\ell-1}|),$$

where

$$H^{(T)}_{K_1\ldots K_\ell}\Big(\sum_{j=1}^{\ell}\lambda_j\Big) = \int_t h_{K_1}(t/T)\cdots h_{K_\ell}(t/T)\,e^{-i\sum_{j=1}^{\ell}\lambda_j t}\,dt \quad .$$

This result can be proved as follows

$$\Big|\int_t h_{K_1}(t+u_1/T)\cdots h_{K_\ell}(t/T)\,e^{-i\sum_{j=1}^{\ell}\lambda_j t}\,dt - \int h_{K_1}(t/T)\cdots h_{K_\ell}(t/T)\,e^{-i\sum_{j=1}^{\ell}\lambda_j t}\,dt\Big|$$

$$\leqslant \Big|\int_t\{h_{K_1}(t+u_1/T)\cdots h_{K_{\ell-1}}(t+u_{\ell-1}/T) - h_{K_1}(t/T)\cdots h_{K_{\ell-1}}(t/T)\}\Big|\,\big|h_{K_\ell}(t/T)\big|\,dt$$

$$\leqslant L\sum_{j=1}^{\ell-1}\int_t\big|h_{K_j}(t+u_j/T) - h_{K_j}(t/T)\big|\,dt \leq L'(|u_1|+\cdots+|u_{\ell-1}|) \quad .$$

Hence we have

$$\mathrm{cum}\{d^{(T)}_{K_1}(\lambda_1),\ldots,d^{(T)}_{K_{\ell-1}}(\lambda_\ell)\} = \int_{-T}^{T}\cdots\int_{-T}^{T} e^{-i\sum_{j=1}^{\ell}\lambda_j u_j}\,H^{(T)}_{K_1\ldots K_{\ell-1}}\Big(\sum_{j=1}^{\ell}\lambda_j\Big)\,dQ_{K_1\ldots K_\ell}(u_1,\ldots,u_{\ell-1}) + \varepsilon$$

$$= (2\pi)^{\ell-1} H^{(T)}_{K_1\cdots K_\ell}\Big(\sum_{j=1}^{\ell}\lambda_j\Big) f_{K_1\cdots K_\ell}(\lambda_1,\cdots,\lambda_{\ell-1}) + \varepsilon \quad ,$$

where $\varepsilon = O(1)$.

<u>Lemma 2.3</u>: (a) If a function g(x) has finite total variation, V, on [a,b], then

$$\Big| \int_a^b g(x)dx - \frac{b-a}{n}\sum_{k=1}^{n} g\Big[a+(k-1)\frac{b-a}{n}\Big]\Big| \leqslant \frac{V}{n}$$

(b) Moreover, if the first derivative of the function g exists, then the term V/n may be replaced by

$$\int_a^b |g'(x)|dx/n \quad .$$

<u>Proof</u>: The proof of this lemma can be found in Polya and Szegö (1925).

<u>Lemma 2.4</u>: Let $\{T_n\}$, n = 1,2,..., be a sequence of variables such that $\sqrt{n}\,(T_n-\theta) \xrightarrow{L} N(0, \sigma^2(\theta))$ (The symbol L implies convergence in distribution). Let g be a function of a single variable admitting the first derivative. Then

$$\sqrt{n}\,(g(T_n)-g(\theta)) \xrightarrow{L} X \sim N(0, [g'(\theta)\sigma(\theta)]^2) \quad .$$

<u>Proof</u>: The proof of this lemma is given in Rao (1965), p321.

Lemma 2.5: Let X_{ij} be a two-way array of random variables, where $i = 1,\ldots,I$ and $j = 1,2$. Define I random variables as follows

$$Y_i = \prod_{j=1}^{2} X_{ij} \ , \ i=1,\ldots,I \ .$$

The joint cumulant, $\mathrm{cum}(Y_1,\ldots,Y_I)$, is then given by

$$\sum_{\nu} \mathrm{cum}(X_{ij}\,;\, ij\in\nu_1)\ldots\mathrm{cum}(X_{ij}\,;\, ij\in\nu_p)$$

where the summation is over all indecomposable partitions $\nu = \nu_1 \cup \ldots \cup \nu_p$ of the following Table

$$\begin{array}{cc} (1,1) & (1,2) \\ (2,1) & (2,2) \\ \vdots & \\ (I,1) & (I,2) \end{array}$$

Proof: This lemma is given by Brillinger (1975a) and its proof is a particular case of a theorem developed by Leonov and Shiryaev (1959).

Lemma 2.6: Let $\underset{\sim}{A}^{(T)}$, $T = 1,2,\ldots$ be a sequence of r vector-valued random variables, with complex components, and such that all cumulants of the variate $[A_1^{(T)}, \bar{A}_1^{(T)}, \ldots, A_r^{(T)}, \bar{A}_r^{(T)}]$ exist and tend to the corresponding cumulants of a variate $[A_1,\bar{A}_1,\ldots, A_r, \bar{A}_r]$ that is determined by its moments. Then $\underset{\sim}{A}^{(T)}$ tends in distribution to a variate having components $A_1,\ldots,A_r$.

Proof: The proof is given in Brillinger (1975a), p403.

APPENDIX II

RELATION BETWEEN THE SPECTRUM OF THE SERIES $X(t) = \frac{N(t,t+h)}{h}$ AND THE SPECTRUM OF THE POINT PROCESS N(t)

We prove in this Appendix that the spectrum of the series X(t) given by

$$X(t) = \frac{N(t,t+h)}{h} \quad , \quad t = kh \quad and \quad k = 0, \pm 1, \pm 2, \cdots, \tag{II.1}$$

is related to the spectrum of the point process N through the expression

$$f_{XX}(\lambda) = \sum_j f_{NN}(\lambda + \tfrac{2\pi j}{h}) \left\{ \frac{\sin[(\lambda + 2\pi j/h)\, h/2]}{(\lambda + 2\pi j/h)h/2} \right\}^2 \quad . \tag{II.2}$$

Before we go on to prove (II.2) we define the spectral representation of the process N and then from (II.1) we find the spectral representation of the series X(t).

The spectral representation of the process N which is assumed to have stationary increments is given by

$$N(t) = \int_{-\infty}^{+\infty} \left\{ \frac{e^{i\lambda t} - 1}{i\lambda} \right\} dZ(\lambda) \quad , \tag{II.3}$$

where $Z(\lambda)$ is a process with orthogonal increments (Cramèr & Leadbetter, 1967, p109).

<u>Proposition 1:</u> The spectral representation of the series X(t) is given by

$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda(t + h/2)} \frac{\sin \lambda h/2}{\lambda h/2} \, dZ(\lambda) \quad . \tag{II.4}$$

<u>Proof</u>: It follows from (II.1) that

$$X(t) = \frac{N(t,t+h)}{h} = \int_{-\infty}^{+\infty} \frac{e^{i\lambda(t+h)} - e^{i\lambda t}}{i\lambda h}\, dZ(\lambda)$$

$$= \int_{-\infty}^{+\infty} \frac{e^{i\lambda(t+h/2)}}{i\lambda h}\left(e^{i\lambda h/2} - e^{-i\lambda h/2}\right) dZ(\lambda)$$

$$= \int_{-\infty}^{+\infty} e^{i\lambda(t+h/2)}\left(\frac{\sin \lambda h/2}{\lambda h/2}\right) dZ(\lambda) \;,$$

which is the required expression.

<u>Lemma 1</u>: The spectrum of the series X(t) is given by (II.2).

<u>Proof</u>: We start by calculating the auto-covariance of the series X(t).

$$\operatorname{cov}\left\{X(t+u), X(t)\right\} = \operatorname{cov}\left\{\int_{-\infty}^{+\infty} e^{i\lambda(t+u+h/2)} \frac{\sin \lambda h/2}{\lambda h/2}\, dZ(\lambda),\ \int_{-\infty}^{+\infty} e^{i\mu(t+h/2)} \frac{\sin \mu h/2}{\mu h/2}\, dZ(\mu)\right\}$$

$$= \operatorname{cum}\left\{\int_{-\infty}^{+\infty} e^{i\lambda(t+u+h/2)} \frac{\sin \lambda h/2}{\lambda h/2}\, dZ(\lambda),\ \overline{\int_{-\infty}^{+\infty} e^{i\mu(t+h/2)} \frac{\sin \mu h/2}{\mu h/2}\, dZ(\mu)}\right\}$$

$$= \int_{-\infty}^{+\infty} e^{i\lambda(t+u+h/2)} \frac{\sin \lambda h/2}{\lambda h/2} \int_{-\infty}^{+\infty} e^{-i\mu(t+h/2)} \frac{\sin \mu h/2}{\mu h/2} \operatorname{cum}\left\{dZ(\lambda), dZ(-\mu)\right\}$$

$$= \iint_{-\infty}^{+\infty} e^{i\lambda(t+u+h/2)}\, e^{-i\mu(t+h/2)} \frac{\sin \lambda h/2}{\lambda h/2}\, \frac{\sin \mu h/2}{\mu h/2}\, \delta(\lambda-\mu)\, f_{NN}(\lambda)\, d\lambda\, d\mu$$

$$= \int_{-\infty}^{+\infty} e^{i\lambda u}\left(\frac{\sin \lambda h/2}{\lambda h/2}\right)^2 f_{NN}(\lambda)\, d\lambda \;,$$

which implies that

$$c_{XX}(u) = \operatorname{cov}\left\{X(t+u), X(t)\right\} = \int_{-\infty}^{+\infty} e^{i\lambda u}\left(\frac{\sin \lambda h/2}{\lambda h/2}\right)^2 f_{NN}(\lambda)\,d\lambda\,, \quad u = \ell h,\; \ell = 0, \pm 1, \pm 2, \ldots \,. \qquad (II.5)$$

It now follows from (II.5) that

$$\frac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty} c_{XX}(\ell.h)\,e^{-i\ell v} = \frac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} e^{i\mu u}\left(\frac{\sin \mu h/2}{\mu h/2}\right)^2 f_{NN}(\mu)\,d\mu\right\} e^{-i\ell v}$$

This expression can also be written as

$$\frac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty} c_{XX}(\ell.h)\,e^{-i\frac{v}{h}\ell h} = \int_{-\infty}^{+\infty}\left(\frac{\sin \mu h/2}{\mu h/2}\right)^2 f_{NN}(\mu)\left\{\frac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty} e^{-i\ell(v-\mu h)}\right\}d\mu\,,$$

and by using Poisson's summation formula (Brillinger, 1975a, p47) we get

$$\frac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty} c_{XX}(\ell.h)\,e^{-i\frac{v}{h}\ell h} = \int_{-\infty}^{+\infty}\left(\frac{\sin \mu h/2}{\mu h/2}\right)^2 f_{NN}(\mu)\left\{\frac{1}{h}\sum_{j=-\infty}^{+\infty}\delta\left(\mu - \frac{v}{h} - \frac{2\pi j}{h}\right)\right\}d\mu \qquad (II.6)$$

Relation (II.6) is further expressed as

$$\frac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty} c_{XX}(\ell.h)\,e^{-i\frac{v}{h}\ell h}.h = \sum_{j=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\frac{\sin \mu h/2}{\mu h/2}\right)^2 f_{NV}(\mu)\,\delta\left(\mu - \frac{v}{h} - \frac{2\pi j}{h}\right)d\mu$$

$$= \sum_{j=-\infty}^{+\infty} f_{NN}\left(\frac{v}{h} + \frac{2\pi j}{h}\right)\left\{\frac{\sin\left[\left(\frac{v}{h} + \frac{2\pi j}{h}\right)h/2\right]}{\left(\frac{v}{h} + \frac{2\pi j}{h}\right)h/2}\right\}^2 . \qquad (II.7)$$

Finally, from (II.7) we have

$$f_{XX}\left(\frac{v}{h}\right) = \sum_{j=-\infty}^{+\infty} f_{NN}\left(\frac{v}{h} + \frac{2\pi j}{h}\right)\left\{\frac{\sin\left[\left(\frac{v}{h} + \frac{2\pi j}{h}\right)h/2\right]}{\left(\frac{v}{h} + \frac{2\pi j}{h}\right)h/2}\right\}^2$$

or

$$f_{XX}(\lambda) = \sum_{j=-\infty}^{+\infty} f_{NN}\left(\lambda + \frac{2\pi j}{h}\right)\left\{\frac{\sin\left[\left(\lambda + \frac{2\pi j}{h}\right)h/2\right]}{\left(\lambda + \frac{2\pi j}{h}\right)h/2}\right\}^2 . \qquad (II.8)$$

If we now put h = 1 in the expression (II.8) we find

$$f_{XX}(\lambda) = \sum_{j=-\infty}^{+\infty} f_{NN}(\lambda + 2\pi j)\left\{\frac{\sin(\lambda + 2\pi j)/2}{(\lambda + 2\pi j)/2}\right\}^2$$

and if the point process has no components with frequency greater than π (Nyquist frequency), then

$$f_{XX}(\lambda) = \frac{p_N}{2\pi} + \left(f_{NN}(\lambda) - \frac{p_N}{2\pi}\right)\left(\frac{\sin \lambda/2}{\lambda/2}\right)^2 .$$

This shows that the spectrum of the series X(t) is close to the spectrum of the point process N(t) for h = 1 and $|\lambda| \leqslant \pi$.

APPENDIX III

THE QUADRATIC MODEL

1. Solution of the quadratic model

Our purpose is to solve the equation given by

$$E\left\{dN(t)/M\right\} = \left\{a_0 + \int a_1(t-u)\,dM(u) + \iint\limits_{u\neq v} a_2(t-u,\, t-v)\,dM(u)\,dM(v)\right\}dt \quad . \tag{1.1}$$

with respect to the functions a_0, $a_1(\cdot)$ & $a_2(\cdot,\cdot)$, that is, the zero-, the first- and the second-order kernels.

The expected value, $E\left\{dN(t)/M\right\}$, is the conditional probability of an N event occurring in the interval $(t,\, t+dt]$ given that the process M is present.

On taking the expectation with respect to M in (1.1) we have the following equation

$$p_N = a_0 + p_M\int a_1(u)\,du + \iint\limits_{u\neq v} a_2(t-u,\, t-v)\, p_{MM}(u-v)\,du\,dv$$

$$= a_0 + p_M\int a_1(u)\,du + \iint\limits_{u\neq v} a_2(u-v)\, q_{MM}(u-v)\,du\,dv + p_M^2\iint\limits_{u\neq v} a_2(u,v)\,du\,dv \tag{1.2}$$

since

$$q_{MM}(u) = p_{MM}(u) - p_M^2 \quad \text{and} \quad q_{MM}(u) = q_{MM}(-u) \quad .$$

Similarly, on multiplying by $dM(t-\tau)$ both sides of (1.1) and taking the expectation with respect to M we obtain

$$p_{NM}(\tau) = a_0 p_M + \int a_1(t-u)\left[\, p_{MM}(u-t+\tau) + p_M\,\delta(u-t+\tau)\right]du$$

$$+ \iint\limits_{u\neq v} a_2(t-u,\, t-v)\left[\, p_{MMM}(u-t+\tau,\, v-t+\tau) + p_{MM}(u-t+\tau)\,\delta(v-t+\tau)\right.$$

$$\left. + \, p_{MM}(v-t+\tau)\,\delta(u-t+\tau)\right]du\,dv\,, \tag{1.3}$$

where $p_{MMM}(u,v)$ is the third-order product density of the point process M.

Substituting (1.2) into (1.3) we get

$$q_{NM}(\tau) = \alpha(\tau)\, p_M + \int a_1(\tau-u)\, q_{MM}(u)du + \int a_2(\tau-u,\tau)\, p_{MM}(u)du$$

$$+ \int a_2(\tau,\tau-v)\, p_{MM}(v)dv + \iint\limits_{u\neq v} a_2(\tau-u,\tau-v)\left[p_{MMM}(u,v) - p_M p_{MM}(u-v)\right]dudv\,. \quad (1.4)$$

Expression (1.4) can also be written as

$$q_{NM}(\tau) = \alpha(\tau)\, p_M + \int a_1(t-u)\, q_{MM}(u)du + \int a_2(\tau-u,\tau)\, q_{MM}(u)du + p_M^2 \int a_2(\tau-u,\tau)du$$

$$+ \int a_2(\tau,\tau-v)\, q_{MM}(v)dv + p_M^2 \int a_2(\tau,\tau-v)dv + p_M \iint\limits_{u\neq v} a_2(\tau-u,\tau-v)\, q_{MM}(u)dudv$$

$$+ p_M \iint\limits_{u\neq v} a_2(\tau-u,\tau-v)\, q_{MM}(v)dudv + \iint\limits_{u\neq v} a_2(\tau-u,\tau-v)\, q_{MMM}(u,v)dudv\,, \quad (1.5)$$

where $q_{MMM}(u,v)$ is the third-order cumulant function given by

$$q_{MMM}(u,v) = p_{MMM}(u,v) - p_M p_{MM}(u-v) - p_M p_{MM}(u) - p_M p_{MM}(v) + 2p_M^3\,.$$

A third and final equation involving a_0, $a_1(\cdot)$ & $a_2(\cdot,\cdot)$ can be obtained by multiplying (1.1) through by $dM(t-\tau)dM(t-\sigma)$ and taking the expectation with respect to M, i.e.

$$p_{NMM}(\sigma,\sigma-\tau) = a_0\, p_{MM}(\sigma-\tau) + \int a_1(t-u)\left[p_{MMM}(u-t+\sigma,\sigma-\tau)\right.$$

$$\left. + \delta(u-t+\sigma)\, p_{MM}(\sigma-\tau) + \delta(u-t+\sigma)\, p_{MM}(u-t+\tau)\right]du$$

$$+ \iint\limits_{u\neq v} a_2(t-u,t-v)\left[p_{MMMM}(u-t+\sigma,\, v-t+\sigma,\, \sigma-\tau)\right.$$

$$+ \delta(u-t+\tau)\, p_{MMM}(v-t+\tau,\sigma-\tau) + \delta(u-t+\sigma)\, p_{MMM}(v-t+\tau,\sigma-\tau)$$

$$+ \delta(v-t+\tau)\, p_{MMM}(u-t+\sigma,\sigma-\tau) + \delta(v-t+\sigma)\, p_{MMM}(u-t+\sigma,\sigma-\tau)$$

$$\left. + \delta(u-t+\tau)\delta(v-t+\sigma)\, p_{MM}(\sigma-\tau) + \delta(u-t+\sigma)\delta(v-t+\tau)\, p_{MM}(\sigma-\tau)\right]dudv\,. \quad (1.6)$$

Expression (1.6) can now be written as

$$
\begin{aligned}
p_{NMM}(\sigma,\sigma-\tau) &= \alpha_0 p_{MM}(\sigma-\tau) + \int \alpha_1(\sigma-u)\, p_{MMM}(u,\sigma-\tau)\,du + p_{MM}(\sigma-\tau)\,\alpha_1(\tau) \\
&+ p_{MM}(\sigma-\tau)\,\alpha_1(\sigma) + \iint_{u\neq v} \alpha_2(\sigma-u,\sigma-v)\, p_{MMMM}(u,v,\sigma-\tau)\,du\,dv \\
&+ \int \alpha_2(\tau,\sigma-v)\, p_{MMM}(v,\sigma-\tau)\,dv + \int \alpha_2(\sigma,\sigma-v)\, p_{MMM}(v,\sigma-\tau)\,dv \\
&+ \int \alpha_2(\sigma-u,\tau)\, p_{MMM}(u,\sigma-\tau)\,du + \int \alpha_2(\sigma-u,\sigma)\, p_{MMM}(u,\sigma-\tau)\,du \\
&+ p_{MM}(\sigma-\tau)\,\alpha_2(\tau,\sigma) + p_{MM}(\sigma-\tau)\,\alpha_2(\sigma,\tau),
\end{aligned}
$$

and by setting τ in the position of σ and σ in the position of $\sigma-\tau$, we have

$$
\begin{aligned}
p_{NMM}(\tau,\sigma) &= \alpha_0 p_{MM}(\sigma) + \int \alpha_1(\tau-u)\, p_{MMM}(u,\sigma)\,du + p_{MM}(\sigma)\,\alpha_1(\tau) + p_{MM}(\sigma)\,\alpha_1(\tau-\sigma) \\
&+ \iint_{u\neq v} \alpha_2(\tau-u,\tau-v)\, p_{MMMM}(u,v,\sigma)\,du\,dv + \int \alpha_2(\tau-\sigma,\tau-u)\, p_{MMM}(u,\sigma)\,du \\
&+ \int \alpha_2(\tau,\tau-u)\, p_{MMM}(u,\sigma)\,du + \int \alpha_2(\tau-u,\tau-\sigma)\, p_{MMM}(u,\sigma)\,du \\
&+ \int \alpha_2(\tau-u,\tau)\, p_{MMM}(u,\sigma)\,du + p_{MM}(\sigma)\,\alpha_2(\tau-\sigma,\tau) + p_{MM}(\tau)\,\alpha_2(\tau,\tau-\sigma)\,. \qquad (1.7)
\end{aligned}
$$

Substituting (1.2) into (1.7) we obtain

$$
\begin{aligned}
p_{NMM}(\tau,\sigma) - p_N p_{MM}(\sigma) &= p_{MM}(\sigma)\,\alpha_1(\tau) + p_{MM}(\sigma)\,\alpha_1(\tau-\sigma) + \int \alpha_1(\tau-u)\, q_{MMM}(u,\sigma)\,du \\
&+ p_M \int \alpha_1(\tau-u)\, q_{MM}(u-\sigma)\,du + p_M \int \alpha_1(\tau-v)\, q_{MM}(v)\,dv \\
&+ 2\alpha_2(\tau-\sigma,\tau)\, p_{MM}(\sigma) + 2\int \alpha_2(\tau-u,\tau)\, p_{MMM}(u,\sigma)\,du \\
&+ \iint_{u\neq v} \alpha_2(\tau-u,\tau-v)\left[p_{MMMM}(u,v,\sigma) - p_{MM}(\sigma)\, p_{MM}(u-v)\right] du\,dv\,, \qquad (1.8)
\end{aligned}
$$

since we assume that $a_2(u,v)$ is symmetric with respect to u and v.

The function $p_{MMMM}(u,v,\sigma)$ is the fourth-order product density of the process M.

Expression (1.8), after some more calculations, becomes

$$p_{NMM}(\tau,\sigma) - p_N p_{MM}(\sigma) - p_M p_{NM}(\tau) - p_M p_{NM}(\tau-\sigma) + 2p_N p_M^2$$

$$= \alpha_1(\tau)\, q_{MM}(\sigma) + \alpha_1(\tau-\sigma)\, q_{MM}(\sigma) + \int \alpha_1(\tau-u)\, q_{MMM}(u,\sigma)\,du + 2\alpha_2(\tau-\sigma,\tau)\, q_{MM}(\sigma)$$

$$+ 2\alpha_2(\tau-\sigma,\tau)\, p_M^2 - 2p_M^3 \int \alpha_2(\tau,\tau-v)\,dv - 2p_M^2 \iint_{u\neq v} \alpha_2(\tau-u,\tau-v)\, q_{MM}(u)\,du$$

$$- 2p_M^3 \int \alpha_2(\tau-\sigma,\tau-v)\,dv - 2p_M^2 \iint_{u\neq v} \alpha_2(\tau-\sigma-u,\tau-\sigma-v)\, q_{MM}(u)\,du\,dv$$

$$+ 2p_M^2 \iint_{u\neq v} \alpha_2(\tau-u,\tau-v)\, q_{MM}(v)\, q_{MM}(u-\sigma)\,du\,dv + 2\int \alpha_2(\tau-u,\tau)\, q_{MMM}(u,\sigma)\,du$$

$$+ 2p_M p_{MM}(\tau) \int \alpha_2(\tau-u,\tau-\sigma)\,du + 2\int \alpha_2(\tau-u,\tau-\sigma)\, q_{MMM}(u,\sigma)\,du$$

$$+ 2p_M p_{MM}(\sigma) \int \alpha_2(\tau-u,\tau)\,du + \iint_{u\neq v} \alpha_2(\tau-u,\tau-v)\, q_{MMMM}(u,v,\sigma)\,du\,dv\,, \qquad (1.9)$$

where $q_{MMMM}(u,v,t)$ is the fourth-order cumulant of the process M given by

$$q_{MMMM}(u,v,t) = p_{MMMM}(u,v,t) - p_M p_{MMM}(v,t) - p_M p_{MMM}(u,v) - p_M p_{MMM}(u-t,v-t)$$

$$- p_{MM}(u-v)\, p_{MM}(t) - p_{MM}(u-t)\, p_{MM}(v) - p_{MM}(u)\, p_{MM}(v-t) + 2p_M^2 p_{MM}(t) + 2p_M^2 p_{MM}(v)$$

$$+ 2p_M^2 p_{MM}(v-t) + 2p_M^2 p_{MM}(u) + 2p_M^2 p_{MM}(u-v) + 2p_M^2 p_{MM}(u-t) - 6p_M^4 \;.$$

Finally, equation (1.9) takes the form

$$q_{NMM}(\tau,\sigma) = a_1(\tau)q_{MM}(\sigma) + a_1(\tau-\sigma)q_{MM}(\sigma) + \int a_1(\tau-u)q_{MMM}(u,\sigma)du$$

$$+ 2a_2(\tau-\sigma,\tau)q_{MM}(\sigma) + 2a_2(\tau-\sigma,\tau)p_M^2 + 2p_M\iint_{u\neq v} a_2(\tau-u,\tau-v)q_{MMM}(v,\sigma)dudv$$

$$+ 2p_M q_{MM}(\sigma)\int a_2(\tau-u,\tau)du + 2p_M q_{MM}(\sigma)\int a_2(\tau-u,\tau-\sigma)du$$

$$+ 2\int a_2(\tau-u,\tau)q_{MMM}(u,\sigma)du + 2\int a_2(\tau-u,\tau-\sigma)q_{MMM}(u,\sigma)du$$

$$+ 2\iint_{u\neq v} a_2(\tau-u,\tau-v)q_{MM}(v)q_{MM}(u-v)dudv + \iint_{u\neq v} a_2(\tau-u,\tau-v)q_{MMMM}(u,v,\sigma)dudv, \qquad (1.10)$$

where $q_{NMM}(\tau,\sigma)$ is the third-order cumulant of the two processes M and N.

From equations (1.2), (1.5) and (1.10) we see that there is no obvious solution to the functions a_0, $a_1(\cdot)$ & $a_2(\cdot,\cdot)$. However, in the case that M is a Poisson process, we have that the cumulants for M of order greater than two will be zero and so expression (1.10) simplifies as follows

$$q_{NMM}(\tau,\sigma) = 2p_M^2 a_2(\tau-\sigma,\tau) \quad . \qquad (1.11)$$

Also, from (1.5) and (1.2) we get

$$q_{NM}(\tau) = a_1(\tau)p_M + 2p_M^2\int a_2(\tau-u,\tau)du \qquad (1.12)$$

$$p_N = a_0 + p_M\int a_1(u)du + p_M^2\iint_{u\neq v} a_2(u,v)dudv \quad . \qquad (1.13)$$

We have further from (1.11) that

$$a_2(\tau,\sigma) = \frac{q_{NMM}(\tau,\tau-\sigma)}{2p_M^2} \quad . \qquad (1.14)$$

Relation (1.14) provides a solution for the second-order kernel in terms of the cumulant $q_{NMM}(\tau,\tau-\sigma)$ when M is assumed to be a Poisson point process.

The first- and zero-order kernels may now be calculated from expression (1.12) and (1.13).

2. The Mean Squared-Error of the quadratic model

A modification of the model (1.1) has been discussed in Brillinger (1975d). The modified model is given by

$$E\left\{dN(t)/M\right\} = \left\{ s_0 + \int s_1(t-u)\,dM'(u) + \iint_{u\neq v} s_2(t-u,t-v)\,dM'(u)\,dM'(v)\right\} dt\,, \tag{2.1}$$

where

$$dM'(u) = dM(u) - p_M\,du \quad .$$

The new kernels s_0, $s_1(\cdot)$ & $s_2(\cdot,\cdot)$ are connected to the old ones a_0, $a_1(\cdot)$ & $a_2(\cdot,\cdot)$ by the following relations,

$$a_0 = s_0 - p_M \int s_1(u)\,du + p_M^2 \iint_{u\neq v} s_2(u,v)\,du\,dv \tag{2.2}$$

$$a_1(u) = s_1(u) - 2p_M \int s_2(u,v)\,dv \tag{2.3}$$

$$a_2(u,v) = s_2(u,v) \quad . \tag{2.4}$$

If we substitute $a_1(u)$ from (2.3) into (2.2) we obtain

$$a_0 = s_0 - p_M \int a_1(u)\,du - p_M^2 \iint_{u\neq v} a_2(u,v)\,du\,dv\,, \tag{2.5}$$

and by comparing it with (1.13) we get

$$s_0 = p_N \quad . \tag{2.6}$$

Also, by inserting $a_1(u)$ from (2.3) into (1.12) we find

$$s_1(\tau) = g_{NM}(\tau)/p_M \quad . \tag{2.7}$$

Expression (2.6), (2.7) and (1.14) are the analytical solutions of the kernels s_0, $s_1(\cdot)$ and $s_2(\cdot,\cdot)$ in the case that M is a Poisson point process.

The computation of the Mean Squared-Error (MSE) can be done quicker by using the model (2.1). Expression (2.1) suggests that we have to define the error process with stationary increments as

$$d\epsilon(t) = dN(t) - \left\{ s_0 + \int s_1(t-u)\,dM'(u) + \iint_{u\neq v} s_2(t-u, t-v)\,dM'(u)\,dM'(v) \right\} dt \tag{2.8}$$

It is obvious that $E\left\{ d\epsilon(t) \right\} = 0$ when M is a Poisson point process. The product density of $d\epsilon(\cdot)$ at two different times t & t' is given by

$$E\left\{ d\epsilon(t)\,d\epsilon(t') \right\} = E\left\{ dN(t)\,dN(t') - s_0\,dN(t')\,dt - \int s_1(t-u)\,dM'(u)\,dN(t')\,dt \right.$$

$$- \iint_{u\neq v} s_2(t-u, t-v)\,dM'(u)\,dM'(v)\,dN(t')\,dt - s_0\,dN(t)\,dt' + s_0^2\,dt\,dt' + s_0 \int s_1(t-u)\,dM'(u)\,dt\,dt'$$

$$+ s_0 \iint_{u\neq v} s_2(t-u, t-v)\,dM'(u)\,dM'(v)\,dt\,dt' - \int s_1(t'-u)\,dM'(u)\,dN(t)\,dt + s_0 \int s_1(t'-u)\,dM'(u)\,dt\,dt'$$

$$+ \int s_1(t-u)\,dM'(u) \int s_1(t'-v)\,dM'(v)\,dt\,dt' + \int s_1(t'-\tau)\,dM'(\tau) \iint_{u\neq v} s_2(t-u, t-v)\,dM'(u)\,dM'(v)\,dt\,dt'$$

$$- dN(t) \iint_{u\neq v} s_2(t'-u, t'-v)\,dM'(u)\,dM'(v)\,dt\,dt' + s_0 \iint_{u\neq v} s_2(t'-u, t'-v)\,dM'(u)\,dM'(v)\,dt\,dt'$$

$$+ \int s_1(t-u)\,dM'(u) \iint_{u\neq v} s_2(t'-\tau, t'-\sigma)\,dM'(\tau)\,dM'(\sigma)\,dt\,dt' + \iint_{u\neq v} s_2(t-u, t-v)\,dM'(u)\,dM'(v).$$

$$\left. \iint_{\tau\neq\sigma} s_2(t'-\tau, t'-\sigma)\,dM'(\tau)\,dM'(\sigma)\,dt\,dt' \right\} = \left\{ p_{NN}(t-t') + p_N\delta(t-t') - p_N^2 - \int s_1(t-u)\,q_{NM}(t'-u)\,du \right.$$

$$- \iint_{u\neq v} s_2(t-u, t-v)\left[p_{NMM}(t'-v, u-v) - p_M p_{NM}(t'-u) - p_M p_{NM}(t'-v) + p_N p_M^2 \right] - p_N^2 + p_N^2$$

$$- p_N \iint_{u\neq v} s_2(t-u, t-v)\,q_{MM}(u-v)\,du\,dv + \iint s_1(t-u)\,s_1(t'-v)\left[q_{MM}(u-v) + p_M\delta(u-v) \right] du\,dv$$

$$+ \int s_1(t-\tau) \iint_{u\neq v} s_2(t-u, t-v)\left[q_{MMM}(\tau-v, u-v) + q_{MM}(\tau-v)\delta(\tau-u) + q_{MM}(\tau-u)\delta(\tau-v) \right] du\,dv\,d\tau$$

$$-\int s_1(t'-u)q_{NM}(t-u)du-\iint\limits_{u\neq v} s_2(t'-u,t'-v)\left[p_{NMM}(t-u,u-v)-p_Mp_{NM}(t-u)-p_Mp_{NM}(t-v)+p_Np_M^2\right]dudv$$

$$+p_N\iint\limits_{u\neq v} s_2(t'-u,t'-v)q_{MM}(u-v)dudv+\int s_1(t-u)\iint\limits_{\tau\neq\sigma} s_2(t'-\tau,t'-\sigma)\left[q_{MMM}(u-\sigma,\tau-\sigma)\right.$$

$$\left.+q_{MM}(u-\sigma)\delta(u-\tau)+q_{MM}(u-\tau)\delta(u-\sigma)\right]dud\tau d\sigma+\iint\limits_{u\neq v}\iint\limits_{\tau\neq\sigma} s_2(t-u,t-v)\,s_2(t'-\tau,t'-\sigma)$$

$$\left[q_{MMMM}(\tau-v,\sigma-v,u-v)+q_{MMM}(\tau-u,\sigma-u)\delta(\tau-v)+q_{MMM}(\tau-v,\sigma-v)\delta(\tau-u)\right.$$

$$+q_{MMM}(\tau-u,\sigma-u)\delta(\sigma-v)+q_{MMM}(\tau-v,\sigma-v)\delta(\sigma-u)+p_Mq_{MM}(\sigma-u)\delta(\tau-v)$$

$$+p_Mq_{MM}(\sigma-v)\delta(\tau-u)+p_Mq_{MM}(\tau-v)\delta(\sigma-u)+p_Mq_{MM}(\tau-u)\delta(\sigma-v)+q_{MM}(\tau-\sigma)q_{MM}(u-v)$$

$$+q_{MM}(\tau-u)q_{MM}(\sigma-v)+q_{MM}(\tau-v)q_{MM}(\sigma-u)+q_{MM}(\tau-\sigma)\delta(\tau-u)\delta(\sigma-v)$$

$$\left.\left.+q_{MM}(\tau-\sigma)\delta(\tau-v)\delta(\sigma-u)+p_M^2\delta(\tau-u)\delta(\sigma-v)+p_M^2\delta(\tau-v)\delta(\sigma-u)\right]dudvd\tau d\sigma\right\}dtdt', \quad (2.9)$$

since $s_0 = p_N$ and $E\left\{dM'(u)\right\} = 0$

Also, we have after some rearrangements in (2.9)

$$E\left\{d\epsilon(t)d\epsilon(t')\right\}=\left\{q_{NN}(t-t')+p_N\delta(t-t')-\int s_1(t-u)q_{NM}(t'-u)du-\int s_1(t'-u)q_{NM}(t-u)du\right.$$

$$+\iint s_1(t-u)s_1(t'-v)\left[q_{MM}(u-v)+p_M\delta(u-v)\right]dudv-\iint\limits_{u\neq v} s_2(t-u,t-v)\,q_{NMM}(t'-u,u-v)dudv$$

$$-\iint\limits_{u\neq v} s_2(t'-u,t'-v)q_{NMM}(t-u,u-v)dudv+\iiint\limits_{u\neq v} s_1(t-\tau)s_2(t-u,t-v)q_{MMM}(\tau-v,u-v)dudvd\tau$$

$$+\iint\limits_{u\neq v} s_1(t-u)\,s_2(t-u,t-v)q_{MM}(u-v)dudv+\iint\limits_{u\neq v} s_1(t-v)\,s_2(t-u,t-v)\,q_{MM}(v-u)\,dudv$$

$$+\iiint\limits_{\tau\neq\sigma} s_1(t-u)\,s_2(t'-\tau,t'-\sigma)q_{MMM}(u-\sigma,\tau-\sigma)\,dud\tau d\sigma+\iint\limits_{\tau\neq\sigma} s_1(t-\tau)s_2(t'-\tau,t'-\sigma)q_{MM}(\tau-\sigma)d\tau d\sigma$$

$$+\iint\limits_{\tau\neq\sigma} s_1(t-\sigma)s_2(t'-\tau,t'-\sigma)q_{MM}(\sigma-\tau)d\tau d\sigma+\iint\limits_{u\neq v}\iint\limits_{\tau\neq\sigma} s_2(t-u,t-v)s_2(t'-\tau,t'-\sigma)q_{MMMM}(\tau-v,\sigma-v,u-v)dudvd\sigma d\tau$$

$$+\iint\limits_{u\neq v}\iint\limits_{\tau\neq\sigma}\Big[q_{MMM}(\tau-u,\sigma-u)\delta(\tau-v)+q_{MMM}(\tau-v,\sigma-v)\delta(\tau-u)+q_{MMM}(\tau-u,\sigma-u)\delta(\sigma-v)$$

$$+q_{MMM}(\tau-v,\sigma-v)\delta(\sigma-u)+p_M q_{MM}(\sigma-u)\delta(\tau-v)+p_M q_{MM}(\sigma-v)\delta(\tau-u)+p_M q_{MM}(\tau-v)\delta(\sigma-u)$$

$$+p_M q_{MM}(\tau-u)\delta(\sigma-v)+q_{MM}(\tau-\sigma)q_{MM}(u-v)+q_{MM}(\tau-u)q_{MM}(\sigma-v)+q_{MM}(\tau-v)q_{MM}(\sigma-u)$$

$$+q_{MM}(\tau-\sigma)\delta(\tau-u)\delta(\sigma-v)+q_{MM}(\tau-\sigma)\delta(\tau-v)\delta(\sigma-u)\Big]dudvd\sigma d\tau+p_M^2\iint\limits_{u\neq v}s_2(t-u,t-v)s_2(t'-u,t'-v)dudv$$

$$+p_M^2\iint\limits_{u\neq v}s_2(t-v,t-u)s_2(t'-u,t'-v)\,dudv\Big\}dt\,dt'\,. \tag{2.10}$$

This result holds for M being a general point process. If we take M to be a Poisson point process, then $q_{MM}(u)$, $q_{MMM}(u,v)$ & $q_{MMMM}(u,v,z)$ are zero and expression (2.10) simplifies as follows

$$E\{d\epsilon(t)d\epsilon(t')\}=\Big\{q_{NN}(t-t')+p_N\delta(t-t')-\int s_1(t-u)q_{NM}(t'-u)du-\int s_1(t'-u)q_{NM}(t-u)du$$

$$+p_M\int s_1(t-u)s_1(t'-u)du-\iint\limits_{u\neq v}s_2(t-u,t-v)q_{NMM}(t'-u,u-v)dudv-\iint\limits_{u\neq v}s_2(t'-u,t'-v)q_{NMM}(t-u,u-v)dudv$$

$$+p_M^2\iint\limits_{u\neq v}s_2(t-u,t-v)\,s_2(t'-u,t'-v)dudv+p_M^2\iint\limits_{u\neq v}s_2(t-v,t-u)\,s_2(t'-v,t'-u)dudv\Big\}dt\,dt'\,. \tag{2.11}$$

Expression (2.11) can also be written as

$$E\{d\epsilon(t)d\epsilon(t')\}=\Big\{q_{NN}(t-t')+p_N\delta(t-t')-p_M\int s_1(t-u)\,s_1(t'-u)\,du$$

$$-2p_M^2\iint\limits_{u\neq v}s_2(t-u,t-v)\,s_2(t'-u,t'-v)\,dudv\Big\}dt\,dt'\ , \tag{2.12}$$

since

$$q_{NM}(u)=p_M s_1(u)\ ,\quad q_{NMM}(u,u-v)=2p_M^2 s_2(u,v)\quad\text{and}\quad s_2(u,v)=s_2(v,u)\ .$$

By using quantities of the frequency domain in (2.12) we get

$$E\{d\varepsilon(t)d\varepsilon(t')\} = \Big\{ \int f_{NN}(\lambda) e^{i\lambda(t-t')} d\lambda - (2\pi) P_M \int |S_1(\lambda)|^2 e^{i\lambda(t-t')} d\lambda$$

$$- 2(2\pi)^2 P_M^2 \iint |S_2(\lambda,\mu)|^2 e^{i(\lambda+\mu)(t-t')} d\lambda d\mu \Big\} dt dt' , \qquad (2.13)$$

where $S_1(\lambda)$ and $S_2(\lambda, \mu)$ are the Fourier Transforms of the kernels $s_1(u)$ and $s_2(u,v)$.

The functions $S_1(\lambda)$ and $S_2(\lambda, \mu)$ from (1.14), (2.4) & (2.7) can be expressed as

$$S_1(\lambda) = \frac{f_{NM}(\lambda)}{P_M/2\pi} \qquad (2.14)$$

$$S_2(\lambda,\mu) = \frac{f_{NMM}(-\lambda-\mu,\mu)}{2\,(P_M/2\pi)^2} \quad . \qquad (2.15)$$

Hence we obtain from (2.13)

$$E\{d\varepsilon(t)d\varepsilon(t')\} = \Big\{ \int f_{NN}(\lambda) e^{i\lambda(t-t')} d\lambda - \int \frac{|f_{NM}(\lambda)|^2}{P_M/2\pi} e^{i\lambda(t-t')} d\lambda$$

$$- \frac{1}{2} \iint \frac{|f_{NMM}(-\lambda-\mu,\mu)|^2}{(P_M/2\pi)^2} e^{i(\lambda+\mu)(t-t')} d\lambda d\mu \Big\} dt dt' . \qquad (2.16)$$

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