Every 3-connected, locally connected, claw-free graph is Hamilton-connected

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ABSTRACT

A graph G is locally connected if the subgraph induced by the neighbourhood of each vertex is connected. We prove that a locally connected graph G of order $p \geq 4$, containing no induced subgraph isomorphic to $K_{1,3}$, is Hamilton-connected if and only if G is 3-connected.

1. INTRODUCTION

We use [1] for terminology and notation not defined here and consider finite simple graphs only. Let $V(G)$ and $E(G)$ denote, respectively, the vertex set and edge set of a graph G. For each vertex u of G, the neighbourhood $N(u)$ is the set of all vertices adjacent to u and $M(u) = N(u) \cup \{u\}$. If W is a nonempty subset of $V(G)$, then we denote by $\langle W \rangle$ the subgraph of G induced by W. A graph G is called claw-free if G has no induced subgraph isomorphic to $K_{1,3}$.

A graph G is said to be hamiltonian if it has a cycle containing all the vertices of G. A path with end vertices x and y is called an (x, y)-path. An (x, y)-path is a Hamilton path of G if it contains all the vertices of G. A graph G is Hamilton-connected if every two vertices x, y are connected by a Hamilton (x, y)-path.

The following concept of local connectivity was defined in [4]: A graph G is locally n-connected, $n \geq 1$, if $\langle N(u) \rangle$ is n-connected for each $u \in V(G)$. Later, Oberly and Sumner [8] proved that every connected, locally connected, claw-free graph G with $|V(G)| \geq 3$ is hamiltonian. Clark [6] improved this result by showing that in a graph G satisfying the Oberly-Sumner condition, each vertex of G lies on a cycle of every length from 3 to $|V(G)|$ inclusive.

When is a locally connected, claw-free graph Hamilton-connected? This problem was investigated first by Chartrand, Gould and Polimeni [3]. They proved that a connected, locally 3-connected, claw-free graph is Hamilton-connected. Later, Kanetkar and Rao [7] improved this result, by showing that the condition of 3-connectedness can be changed to 2-connectedness. Moreover, they proved that even in this case each pair of distinct vertices x and y of G is connected by a path of every length from $d(x,y)$ to $|V(G)|$-1 inclusive.

In this paper we give a complete solution of the problem. We prove that a locally connected, claw-free graph G with $|V(G)| \geq 4$ is Hamilton-connected if and only if G is 3-connected. This result was conjectured by Broersma and Veldman [2].

2. NOTATIONS AND PRELIMINARY RESULTS

Let P be a path of G. We denote by $\vec{P}$ the path P with a given orientation and by $\overleftarrow{P}$ the path P with the reverse orientation. If $u, v \in V(P)$, then $u\vec{P}v$ denotes the consecutive vertices of P from u to v in the direction specified by $\vec{P}$. The same vertices, in reverse order, are given by $v\overleftarrow{P}u$. We use w^+ to denote the successor of w on $\vec{P}$ and w^- to denote its predecessor. We assume that an (x,y)-path $\vec{P}$ has an orientation from x to y. We will denote by $k(G)$ and $\alpha(G)$ the connectivity and the independence number of a graph G, respectively. Let H be a graph with $V(H) = A \cup \{u, v\}$ where $\langle A \rangle$ is a complete subgraph of H and $uz, vz \in E(H)$ for each $z \in A$. In this situation we let $u[A]v$ denote a Hamiltonian (u,v)-path of H.

Proposition 2.1. Let G be a connected, locally connected, claw-free graph, and u a vertex of G. If there exist two non-adjacent vertices $z_1, z_2 \in N(u)$ such that $N(u) \cap N(z_1) \cap N(z_2) = \emptyset$, then the sets $A_1 = \{z_1\} \cup (N(z_1) \cap N(u))$ and $A_2 = \{z_2\} \cup (N(z_2) \cap N(u))$ have the following properties:

(1) $A_1 \cup A_2 = N(u), A_1 \cap A_2 = \emptyset$ and $|A_i| \geq 2$ for $i = 1, 2$.

(2) The graphs $H_1 = \langle A_1 \rangle$ and $H_2 = \langle A_2 \rangle$ are complete and there exists an edge v_1v_2 where $v_1 \in A_1$ and $v_2 \in A_2$.

Proof. Clearly, $A_1 \cap A_2 = \emptyset$. If $A_1 \cup A_2 \neq N(u)$ then there is a vertex $z_3 \in N(u)$ such that $z_3z_1, z_3z_2 \notin E(G)$ and the set $\{z_1, z_2, z_3, u\}$ induces $K_{1,3}$; a contradiction. So $A_1 \cup A_2 = N(u)$. If one of the graphs H_1 and H_2, say H_1, contains two nonadjacent vertices s and t then the set $\{u, s, t, z_2\}$ induces $K_{1,3}$; a contradiction. So, H_1 and H_2 are complete graphs. The connectedness of $\langle N(u) \rangle$ implies that there exists an edge v_1v_2 where $v_1 \in A_1$ and $v_2 \in A_2$. Then $d(u) > 2, |A_1| \geq 2$ and $|A_2| \geq 2$. ∎

Proposition 2.2. Let G be a connected, locally connected, claw-free graph, and let u be a vertex of G. Furthermore, let w be a cut vertex of $H = \langle N(u) \rangle$. Then the following properties hold:

(1) The graph $H - w$ has two components and each of them is a complete graph.

(2) The graph H has at most two cut vertices. Moreover, if H has two cut vertices then they are adjacent.

Proof. The first property follows from the fact that G is claw-free. Let H_1 and H_2 be components of $H - w$. Then for some $i \in \{1, 2\}$, say $i = 1, w$ is adjacent to all the vertices of H_i. Since w is a cut vertex of H, we can deduce that $z_1 = w$ for each edge $z_1z_2 \in E(G)$ with $z_1 \in V(H_1) \cup \{w\}, z_2 \in V(H_2)$. This means that H has at most two cut vertices. Moreover, H has two cut vertices if and only if w is adjacent to only one vertex from $V(H_2)$. This vertex is the second cut vertex of H. ∎

Proposition 2.3. Let G be a connected, locally connected, claw-free graph. If $v \in N(u)$ and v is not a cut vertex of $H = \langle N(u) \rangle$ then there is a Hamilton (u, v)-path of $\langle M(u) \rangle$.

Proof. If H has no cut vertex then $2 \leq k(H)$. Since G is claw-free, $\alpha(H) \leq 2$. Hence, by a theorem of Chvatál and Erdös [5], H is hamiltonian. This implies that there exists a Hamilton (u, v)-path of $\langle M(u) \rangle$.

If H has a cut vertex w, then by Proposition 2.2, $H - w$ has two components and each of them is a complete graph. Since v is not a cut vertex, the existence of a Hamilton (u, v)-path of $\langle M(u) \rangle$ is evident. ∎

Let z be an internal vertex of an (x, y)-path $P, x \neq y$. We say that P has a local detour of z if there exists a path in $\langle N(z) \setminus \{x, y\} \rangle$ with origin outside P and terminus a neighbour of z on P. The following result was obtained in [6].

Proposition 2.4 [6]. Let G be a claw-free graph with $|V(G)| \geq 3$ and P be an (x, y)-path of length $n, x \neq y, 3 \leq n \leq |V(G)| - 2$. If P has a local detour then G contains an (x, y)-path Q of length $n + 1$ with $V(P) \subset V(Q)$.

Theorem 2.5. Let G be a connected, locally connected, claw-free graph and x, y be two distinct vertices of G. If there exists an (x, y)-path of length at least 3 including the set $N(x) \cup N(y)$ then there exists a Hamilton (x, y)-path of G.

Proof. It is sufficient to prove that if P is an (x, y)-path of length $n < |V(G)| - 1$ and $N(x) \cup N(y) \subseteq V(P)$ then there exists an (x, y)-path Q of length $n+1$ with $V(P) \subset V(Q)$.

Let $P = x_0x_1 \cdots x_n$, where $x_0 = x$ and $x_n = y$. Since G is connected and $n < |V(G)| - 1$, the set $U = \cup_{i=1}^{n-1} N(x_i) \setminus V(P)$ is not empty. If P has a local detour at x_j for some $1 \leq j \leq n - 1$ then, by Proposition 2.4, there exists an (x, y)-path Q of length $n + 1$ such that $V(P) \subset V(Q)$. Assume now that

(1) for each $j = 1, \ldots, n - 1, P$ has no local detour at x_j.

Consider a vertex $v \in U$. Since G is claw-free, $x_{j-1}x_{j+1} \in E(G)$ for each $x_j \in N(v) \cap V(P)$. Let $i_1 = \min_{x_i v \in E(G)} i$ and $u_1u_2 \cdots u_r$ be a shortest (v, x_{1+i_1})-path in the graph $\langle N(x_{i_1}) \rangle$, where $u_1 = v$ and $u_r = x_{1+i_1}$. Since G is claw-free, $r \leq 4$. Furthermore, since $N(x) \cup N(y) \subseteq V(P)$, (1) implies that $r \geq 4$. So, $r = 4, u_3 \in \{x_0, x_n\}$ and $u_2 \in V(P) \setminus \{x_0, x_n\}$.

Let $u_2 = x_{i_2}$ for some $i_2, 1 \leq i_1 < i_2 \leq n - 1$.

Case 1. $u_3 = x_0$. We have $x_{i_2}x_{1+i_2}, vx_{i_2}, x_{i_2}x_0 \in E(G)$ and $vx_0, vx_{1+i_2} \notin E(G)$. Then $x_{1+i_2}x_0 \in E(G)$, because G is claw-free. If $i_1 = 1$ then there is an (x, y)-path Q of length $n+1$, where $Q = x_0x_{i_2}vx_1x_2 \cdots x_{i_2-1}x_{i_2+1} \cdots x_n$. Let $i_1 \geq 2$ that is $vx_1 \notin E(G)$. Since G is claw-free, $E(G) \cap \{x_1x_{1+i_1}, x_1x_{1+i_2}, x_{1+i_1}x_{1+i_2}\} \neq \emptyset$. Hence there exists an (x, y)-path Q of length $n + 1$ where

$$Q = \begin{cases} x_0x_{i_2}vx_{i_1}\overleftarrow{P}x_1x_{1+i_1}\vec{P}x_{i_2-1}x_{1+i_2}\vec{P}x_n & \text{if } x_1x_{1+i_1} \in E(G), \\ x_0x_{i_1}vx_{i_2}\overleftarrow{P}x_{1+i_1}x_{i_1-1}\overleftarrow{P}x_1x_{1+i_2}\vec{P}x_n & \text{if } x_1x_{1+i_2} \in E(G), \\ x_0\vec{P}x_{i_1}vx_{i_2}\overleftarrow{P}x_{1+i_1}x_{1+i_2}\vec{P}x_n & \text{if } x_{1+i_1}x_{1+i_2} \in E(G). \end{cases}$$

Case 2. $u_3 = x_n$. If $vx_{n-1} \in E(G)$ then $Q = x_0\vec{P}x_{i_1}vx_{n-1}\overleftarrow{P}x_{1+i_1}x_n$ is the (x, y)-path of length $n+1$.

Let $vx_{n-1} \notin E(G)$. Then $i_2 < n-1$ and we have $x_{i_j}x_{i_j-1}, vx_{i_j}, x_nx_{i_j} \in E(G)$ and $vx_n, vx_{i_j-1} \notin E(G)$ for $j = 1, 2$. This implies $x_nx_{i_1-1}, x_nx_{i_2-1} \in E(G)$ because G is claw-free. We have now that $x_nx_{n-1}, x_nx_{i_1-1}, x_nx_{i_2-1} \in E(G)$. Therefore $E(G) \cap \{x_{n-1}x_{i_1-1}, x_{n-1}x_{i_2-1}, x_{i_1-1}x_{i_2-1}\} \neq \emptyset$ since G is claw-free. Then G has an (x, y)-path Q of length $n+1$, where

$$Q = \begin{cases} x_0\vec{P}x_{i_1-1}x_{i_1+1}\vec{P}x_{i_2-1}x_{n-1}\overleftarrow{P}x_{i_2}vx_{i_1}x_n & \text{if } x_{n-1}x_{i_2-1} \in E(G), \\ x_0\vec{P}x_{i_1-1}x_{n-1}\overleftarrow{P}x_{i_2+1}x_{i_2-1}\overleftarrow{P}x_{i_1}vx_{i_2}x_n & \text{if } x_{n-1}x_{i_1-1} \in E(G), \\ x_0\vec{P}x_{i_1-1}x_{i_2-1}\overleftarrow{P}x_{i_1}vx_{i_2}\vec{P}x_n & \text{if } x_{i_1-1}x_{i_2-1} \in E(G). \end{cases}$$

∎

Theorem 2.6. Let u, v be two distinct vertices of a 3-connected, locally connected, claw-free graph G with $d(u, v) \leq 2$. If $N(v) \cap N(w_1) \cap N(w_2) \neq \emptyset$ for each pair of non-adjacent vertices $w_1, w_2 \in N(v)$ then there exists a Hamilton (u, v)-path of G.

Proof. Taking Theorem 2.5 into consideration, it is sufficient to prove that there exists a (u, v)-path Q with $N(u) \cup N(v) \subseteq V(Q)$. First we prove that there is a (u, v)-path P with $N(u) \subseteq V(P)$. Let $H = \langle N(u) \rangle$.

Case 1. $d(u, v) = 2$. Since G is 3-connected, by a theorem of Whitney [9], there are three (u, v)-paths Q_1, Q_2, Q_3 such that $Q_i = uP_iv, |V(P_i) \cap N(u)| = 1, i = 1, 2, 3$, and P_1, P_2, P_3 are vertex disjoint. If $V(P_i) \cap N(u) = \{u_i\}$ for $i = 1, 2, 3$ then, by Proposition 2.2, one of the vertices u_1, u_2, u_3, say u_1, is not a cut vertex of $\langle N(u) \rangle$. Hence, by Proposition 2.3, there is a Hamilton (u, u_1)-path $P' = uP_4u_1$ of $\langle M(u) \rangle$. Then $N(u) \subseteq V(P)$ for the (u, v)-path $P = uP_4P_1v$.

Case 2. $d(u, v) = 1$. If v is not a cut vertex of H then, by Proposition 2.3, there exists a Hamilton (u, v)-path P of $\langle M(u) \rangle$ and $N(u) \subseteq V(P)$.

Now let v be a cut vertex of H. Then, by Proposition 2.2, $N(u) = A \cup B \cup \{v\}$ where $A \cap B = \emptyset, v \notin A \cup B$ and $\langle A \rangle, \langle B \rangle$ are complete graphs. Since G is 3-connected, in $G - \{u, v\}$ there is a path $z_2P_0z_1$ such that $z_1 \in A, z_2 \in B$ and $V(P_0) \cap M(u) = \emptyset$. Consider a Hamilton (z_1, z_2)-path P' of H. Let $P' = z_1P_1vbP_2z_2$, where z_1P_1 is a Hamilton path of $\langle A \rangle$ and bP_2z_2 is a Hamilton path of $\langle B \rangle$. Then the (u, v)-path $P = ubP_2z_2P_0z_1P_1v$ satisfies $N(u) \subseteq V(P)$.

So, in each case there exits a (u, v)-path P with $N(u) \subseteq V(P)$. Consider a (u, v)-path Q with $V(P) \subseteq V(Q)$ which has the maximum number of vertices from $N(v)$. Suppose that $N(v) \setminus V(Q) \neq \emptyset$ and $z \in N(v) \setminus V(Q)$. Clearly, $zv^- \notin E(G)$ where v^- is the predecessor of v in Q. Then there exists $z_1 \in N(v) \cap N(z) \cap N(v^-)$. Clearly, $z_1 \in V(Q)$ (otherwise there is a (u, v)-path $Q' = u\vec{Q}v^-z_1zv$ which satisfies $V(P) \subset V(Q')$ and $|N(v) \cap V(Q')| > |N(v) \cap V(Q)|$; a contradiction).

Since $N(u) \subseteq V(Q), z_1 \neq u$. Hence $z_1^-z_1^+ \in E(G)$ in Q since G is claw-free. Then there is a (u, v)-path $Q' = u\vec{Q}z_1^-z_1^+\vec{Q}v^-z_1zv$ satisfying $V(P) \subset V(Q')$ and $|N(v) \cap V(Q')| > |N(u) \cap V(Q)|$; a contradiction. So, $N(v) \cup N(v) \subseteq V(Q)$. ∎

3. MAIN RESULT

Theorem 3.1. Let G be a 3-connected, locally connected, claw-free graph. Then for any pair of vertices x, y with $d(x,y) \geq 3$ there is a Hamilton (x,y)-path of G.

Proof. Taking Theorem 2.5 into consideration, it is sufficient to prove that there is an (x,y)-path Q with $N(x) \cup N(y) \subseteq V(Q)$.

Case 1. There is an (x,y)-path xP_0y such that $|N(x) \cap V(P_0)| = |N(y) \cap V(P_0)| = 1$, the unique vertex $x_1 \in N(x) \cap V(P_0)$ is not a cut vertex of $\langle N(x) \rangle$ and the unique vertex $y_1 \in N(y) \cap V(P_0)$ is not a cut vertex of $\langle N(y) \rangle$. Such a path we call a convenient (x,y)-path.

Then, by Proposition 2.3, there is a Hamilton (x,x_1)-path xQ_1x_1 of $\langle M(x) \rangle$ and a Hamilton (y_1,y)-path y_1Q_2y of $\langle M(y) \rangle$. The path $Q = xQ_1P_0Q_2y$ satisfies the condition $N(x) \cup N(y) \subseteq V(Q)$.

Case 2. There does not exist a convenient (x,y)-path.

Since G is 3-connected, there exist three (x,y)-paths xP_1y, xP_2y, xP_3y such that $V(P_i) \cap V(P_j) = \emptyset$ for $1 \leq i \leq j \leq 3$. We can assume that $|V(P_i) \cap N(x)| = |V(P_i) \cap N(y)| = 1$ for $i = 1,2,3$. Let $V(P_i) \cap N(x) = \{x_i\}$ and $V(P_i) \cap N(y) = \{y_i\}$.

Since xP_iy is not a convenient (x,y)-path, either x_i is a cut vertex of $\langle N(x) \rangle$ or y_i is a cut vertex of $\langle N(y) \rangle, i = 1,2,3$. This implies, by Proposition 2.2, that one of the graphs $\langle N(x) \rangle$ and $\langle N(y) \rangle$, say $\langle N(x) \rangle$, contains exactly two cut vertices and the other, $\langle N(y) \rangle$, contains at least one cut vertex. We assume that y_1 is a cut vertex of $\langle N(y) \rangle$ and x_2, x_3 are cut vertices of $\langle N(x) \rangle$. By Proposition 2.2, $x_2x_3 \in E(G)$. Furthermore, $N(x) = A_1 \cup A_2, N(y) = B_1 \cup B_2$ where $A_1 \cap A_2 = \emptyset = B_1 \cap B_2$ and $\langle A_i \rangle, \langle B_i \rangle$ are complete graphs for $i = 1,2$. Without loss of generality we assume that $x_1, x_2 \in A_1, x_3 \in A_2$ and $y_1 \in B_2$. So, $|A_1| \geq 2, |A_2| \geq 2$ and $|B_2| \geq 2$. Let $P_i = Q_iv_iy_i$ for $i = 1,2,3$. Then

$$v_1z \notin E(G) \text{ for each } z \in N(y) \cap N(y_1) \text{ which is not a cut vertex of } \langle N(y) \rangle. \tag{2}$$

(Otherwise we obtain a convenient (x,y)-path $Q = xQ_1v_1zy$). Let $u_1 \in B_1 \cap N(y_1)$. Since G is claw-free, (2) implies that u_1 is adjacent to v_1, u_1 is the second cut vertex of $\langle N(y) \rangle$ and $|B_1| \geq 2$.

Subcase 2.1. y_2 and y_3 belong to different B_i's.

Then we can produce an (x,y)-path with $N(x) \cup N(y) \subseteq V(Q)$ in the following way. If $y_2 \in B_1$ and $y_3 \in B_2$ then

$$Q = x[A_2 \setminus \{x_3\}]x_3P_3y_3[B_2 \setminus \{y_1,y_3\}]y_1v_1\overleftarrow{Q}_1x_1[A_1 \setminus \{x_1,x_2\}]x_2P_2y_2[B_1 \setminus \{u_1,y_2\}]u_1y.$$

If $y_2 \in B_2$ and $y_3 \in B_1 \setminus \{u_1\}$ then

$$Q = x[A_2 \setminus \{x_3\}]x_3P_3y_3[B_1 \setminus \{u_1,y_3\}]u_1v_1\overleftarrow{Q}_1x_1[A_1 \setminus \{x_1,x_2\}]x_2P_2y_2[B_2 \setminus \{y_2\}]y.$$

If $y_2 \in B_2, y_3 = u_1$ and $v_3z \in E(G)$ for some $z \in B_1 \setminus \{u_1\}$ then by considering the path $P_3' = Q_3v_3z$ instead P_3 will obtain the previous situation.

Now let $y_2 \in B_2, y_3 = u_1$ and $N(v_3) \cap B_1 = \{u_1\}$. This implies that $v_3y_1 \in E(G)$ because otherwise the set $\{y_1, u_1, v_3, g\}$ induces $K_{1,3}$ where $g \in B_1 \setminus \{u_1\}$. Then

$$Q = x[A_2 \setminus \{x_3\}]x_3Q_3v_3y_1[B_2 \setminus \{y_1,y_2\}]y_2\overleftarrow{P}_2x_2[A_1 \setminus \{x_1,x_2\}]x_1Q_1v_1y_3[B_1 \setminus \{y_3\}]y.$$

Subcase 2.2. y_2 and y_3 belong to the same B_i. If $y_2, y_3 \in B_2$ then the path

$$Q = x[A_2 \setminus \{x_3\}]x_3P_3y_3[B_2 \setminus \{y_2, y_3\}]y_2\widetilde{P}_2x_2[A_1 \setminus \{x_1, x_2\}]x_1Q_1v_1u_1[B_1 \setminus \{u_1\}]y$$

satisfies the condition $N(x) \cup N(y) \subseteq V(Q)$. If $y_2, y_3 \in B_1$ then by considering u_1 instead y_1 we will obtain the same situation since $v_1u_1, v_1y_1 \in E(G)$. ∎

Theorem 3.2. Let G be a 3-connected, locally connected, claw-free graph. Then for each pair of adjacent vertices x, y there is a Hamilton (x, y)-path.

Proof. If for some $v \in \{x, y\}, N(v) \cap N(w_1) \cap N(w_2) \neq \emptyset$ for each pair of non-adjacent vertices $w_1, w_2 \in N(v)$ then, by Theorem 2.6, there is a Hamilton (x, y)-path. Suppose now that there exist non-adjacent vertices $z_1, z_2 \in N(x)$ and non-adjacent vertices $v_1, v_2 \in N(y)$ such that $N(x) \cap N(z_1) \cap N(z_2) = \emptyset = N(y) \cap N(v_1) \cap N(v_2)$. Then, by Proposition 2.1, $N(x) = A_1 \cup A_2$ and $N(y) = B_1 \cup B_2$ where $A_1 \cap A_2 = \emptyset = B_1 \cap B_2, |A_i| \geq 2, |B_i| \geq 2$ and $\langle A_i \rangle, \langle B_i \rangle$ are complete graphs for $i = 1, 2$. Without loss of generality we assume that $y \in A_1$. Suppose there does not exist a Hamilton (x, y)-path of G. Then, by Theorem 2.5,

$$\text{there does not exist an } (x, y) - \text{path } Q \text{ with } N(x) \cup N(y) \subseteq V(Q). \qquad (3)$$

Case 1. y is not a cut vertex of $\langle N(x) \rangle$.

By Proposition 2.3, there is a Hamilton (x, y)-path of $\langle M(x) \rangle$. Then (3) implies that $N(y) \setminus M(x) \neq \emptyset$ and the vertex x and the set $N(y) \setminus M(x)$ are in different B_i's in $N(y)$. Without loss of generality we assume that $x \in B_2$ and $N(y) \setminus M(x) \subseteq B_1$. Since $\langle N(y) \rangle$ is connected,

$$\text{there exists an edge } zu \text{ with } u \in N(x) \setminus \{y\} \text{ and } z \in N(y) \setminus M(x). \qquad (4)$$

Furthermore,

there does not exist an edge zu such that

$$z \in N(y) \setminus M(x), u \in N(x) \setminus \{y\} \text{ and } u \text{ is not a cut vertex of } \langle N(x) \rangle. \qquad (5)$$

Assuming the contrary, we can produce a path Q contradicting (3) in the following way. Let a_1a_2 be an edge such that $a_1 \in A_1, a_2 \in A_2$ and $u \notin \{a_1, a_2\}$.

If $u \in A_1$ and $y \neq a_1$ then

$$Q = x[A_2 \setminus \{a_2\}]a_2a_1[A_1 \setminus \{a_1, y, u\}]uz[(N(y) \setminus M(x)) \setminus \{z\}]y.$$

If $u \in A_2$ and $y \neq a_1$ then

$$Q = x[A_1 \setminus \{y, a_1\}]a_1a_2[A_2 \setminus \{u, a_2\}]uz[(N(y) \setminus M(x)) \setminus \{z\}]y.$$

Now suppose that y is the only choice for a_1. Then $\{u, y\}$ is a cut set of $\langle N(x) \rangle$. If $u \in A_1$ then u has a neighbour v in A_2 and (3) implies that there is a vertex $s \in A_1 \setminus \{u, y\}$.

If $s \in B_1$ then $Q = x[A_2 \setminus \{v\}]vuz[(N(y) \setminus M(x)) \setminus \{z\}]s[A_1 \setminus \{s, y, u\}]y$. If $s \in B_2$ and $a_2 \in B_1$ then

$$Q = x[A_2 \setminus \{a_2\}]a_2[(N(y) \setminus M(x)) \setminus \{z\}]zu[A_1 \setminus \{y, u\}]y.$$

If $s \in B_2$ and $a_2 \in B_2$ then $sa_2 \in E(G)$ and

$$Q = x[A_2 \setminus \{a_2\}]a_2s[A_1 \setminus \{s, y, u\}]uz[(N(y) \setminus M(x)) \setminus \{z\}]y.$$

In each case we obtained a contradiction to (3). So, (5) is proved.

Now consider an edge zu with $u \in N(x) \setminus \{y\}$ and $z \in N(y) \setminus M(x)$. Then, by (5), u is a cut vertex of $\langle N(x) \rangle$. (Note that if $u \in A_1$ then (3) implies $|A_1| \geq 3$). Choose vertices $g_1 \in N(u) \cap (A_1 \setminus \{y\})$ and $g_2 \in N(u) \cap A_2$. Then $E(G) \cap \{z_{g_1}, z_{g_2}\} \neq \emptyset$ since G is claw-free. This and (5) imply that $\langle N(x) \rangle$ has two cut vertices, u_1 and u_2, and $N(z) \cap (N(x) \setminus \{y\}) = \{u_1, u_2\}$ for each vertex $z \in N(y) \setminus M(x)$ having neighbours in $N(x) \setminus \{y\}$. The last property and (3) imply that $|N(u_1) \cap (N(y) \setminus M(x))| = 1$.

Let $N(u_1) \cap (N(y) \setminus M(x)) = \{z_0\}$. Clearly, (3) implies that $|N(y) \setminus M(x)| \geq 2$. Since G is 3-connected, in $G - \{z_0, y\}$ there exists a path $s_1 P_0 s_2$ such that $s_2 \in N(y) \setminus (M(x) \cup \{z_0\}), s_1 \in N(x) \setminus \{y\}$ and $V(P_0) \cap (N(x) \cup N(y)) = \emptyset$. But now we can indicate an (x, y)-path Q with $N(x) \cup N(y) \subseteq V(Q)$, contradicting (3).

If $s_1 \in A_2$ then

$$Q = x[A_2 \setminus \{s_1\}]s_1 P_0 s_2[(N(y) \setminus M(x)) \setminus \{z_0, s_2\}]z_0 u_1[A_1 \setminus \{u_1, y\}]y.$$

If $s_1 \in A_1$ then $Q = x[A_2 \setminus \{u_2\}]u_2 z_0[(N(y) \setminus M(x)) \setminus \{z_0, s_2\}]s_2 \overleftarrow{P}_0 s_1[A_1 \setminus \{s_1, y\}]y$.

Remark 1. By using the same argument, we will obtain a contradiction in the case when x is not a cut vertex of $\langle N(y) \rangle$.

Case 2. x is a cut vertex of $\langle N(y) \rangle$ and y is a cut vertex of $\langle N(x) \rangle$.

By Proposition 2.1, $|A_1| \geq 2$. Hence, $N(x) \cap N(y) \supseteq A_1 \setminus \{y\} \neq \emptyset$. Without loss of generality we assume that $A_1 \cap B_1 \neq \emptyset$. Then $A_1 \setminus \{y\} \subseteq B_1$ and $A_1 \cap B_2 = \emptyset$. Furthermore, $A_2 \cap B_2 \neq \emptyset$ because x is a cut vertex of $\langle N(y) \rangle$.

Since G is 3-connected, in $G - \{x, y\}$ there exists a path $g_2 P_0 g_1$ such that $g_2 \in A_2, g_1 \in A_1$ and $V(P_0) \cap M(x) = \emptyset$. Now we shall produce an (x, y)-path Q with $N(x) \cup N(y) \subseteq V(Q)$, contradicting (3).

If $x \in B_2$ then $B_2 \subseteq A_2 \cup \{x\}$ and $Q = x[A_2 \setminus \{g_2\}]g_2 P_0 g_1[B_1 \setminus V(P_0)]y$.

Now let $x \in B_1$. Then $B_1 = (A_1 \setminus \{y\}) \cup \{x\}$. Choose a vertex $a \in A_2 \cap B_2$. If $V(P_0) \cap (B_2 \setminus A_2) = \emptyset$ then $Q = x[A_1 \setminus \{g_1, y\}]g_1 \overleftarrow{P}_0 g_2[A_2 \setminus \{a, g_2\}]a[B_2 \setminus A_2]y$. If $V(P_0) \cap (B_2 \setminus A_2) \neq \emptyset$ and b is the last common vertex of P_0 and $B_2 \setminus A_2$ then

$$Q = x[A_2 \setminus \{a\}]a[B_2 \setminus (A_2 \cup \{b\})]b\vec{P}_0 g_1[A_1 \setminus \{g_1, y\}]y.$$

In each case we obtained a contradiction to (3). So, there exists a Hamilton (x, y)-path of G. The proof of the theorem is complete. ∎

Theorem 3.3. Let G be a 3-connected, locally connected, claw-free graph. Then for any pair of vertices x, y with $d(x, y) = 2$ there is a Hamilton (x, y)-path of G.

Proof. If for some $v \in \{x, y\}, N(v) \cap N(w_1) \cap N(w_2) \neq \emptyset$ for each pair of non-adjacent vertices $w_1, w_2 \in N(v)$, then, by Theorem 2.6, there is a Hamilton (x, y)-path of G. Suppose now that there exist non-adjacent vertices $z_1, z_2 \in N(x)$ and non-adjacent vertices $y_1, y_2 \in N(y)$ such that

$$N(x) \cap N(z_1) \cap N(z_2) = \emptyset = N(y) \cap N(y_1) \cap N(y_2). \tag{6}$$

By Proposition 2.1, $N(x) = A_1 \cup A_2, N(y) = B_1 \cup B_2$, where $A_1 \cap A_2 = \emptyset = B_1 \cap B_2, |A_i| \geq 2, |B_i| \geq 2$ and $\langle A_i \rangle, \langle B_i \rangle$ are complete graphs for $i = 1, 2$. Taking Theorem 2.5 into consideration, it is sufficient to prove that there exists an (x, y)-path Q with $N(x) \cup N(y) \subseteq V(Q)$. Without loss of generality we assume that $A_1 \cap B_1 \neq \emptyset$.

Case 1. $A_1 \cap B_2 \neq \emptyset$ or $B_1 \cap A_2 \neq \emptyset$.

We assume that $A_1 \cap B_2 \neq \emptyset$. Let $u_i \in A_1 \cap B_i$ for $i = 1, 2$.

Subcase 1.1. One of the sets $A_1 \cap B_1, A_1 \cap B_2$ contains a cut vertex of $\langle N(x) \rangle$.

Let, for example, u_2 be a cut vertex of $\langle N(x) \rangle$ and $q \in N(u_2) \cap A_2$. If $|A_1 \cap B_1| \geq 2$ and $u_3 \in (A_1 \cap B_2) \setminus \{u_2\}$ then the following path Q includes $N(x) \cup N(y)$: $Q = x[A_2 \setminus \{g\}]gu_2[B_2 \setminus N(x)]u_3[A_1 \setminus \{u_1, u_2, u_3\}]u_1[B_1 \setminus N(x)]y$. Now let $A_1 \cap B_2 = \{u_2\}$. We shall show that there is a vertex $v_0 \in B_2 \setminus \{u_2\}$ such that $N(v_0) \cap A_2 \neq \emptyset$. Assuming the contrary, we obtain that $B_2 \cap A_2 = \emptyset$. Furthermore, $vu_1 \in E(G)$ for each $v \in B_2 \setminus \{u_2\}$, because G is claw-free and u_2 is a cut vertex of $\langle N(x) \rangle$. But then $u_1 \in N(y) \cap N(w_1) \cap N(w_2)$ for each pair of non-adjacent vertices $w_1, w_2 \in N(y)$, which contradicts (6).

So, there are vertices $z_0 \in A_2$ and $v_0 \in B_2 \setminus \{u_2\}$ which are adjacent. Then the following path Q includes $N(x) \cup N(y)$:

$$Q = x[A_2 \setminus \{z_0\}]z_0v_0[B_2 \setminus \{u_2, v_0\}]u_2[A_1 \setminus \{u_1, u_2\}]u_1[B_1 \setminus N(x)]y.$$

Subcase 1.2. $A_1 \cap B_1$ and $A_1 \cap B_2$ contain no cut vertex of $\langle N(x) \rangle$ and $|A_1 \cap B_j| \geq 2$ for some $j \in \{1, 2\}$.

Let $u_0, u_1 \in A_1 \cap B_1$ and $u_2 \in B_2 \cap A_1$. Clearly, there is an edge a_1a_2 such that $a_1 \in A_1 \setminus \{u_2\}$ and $a_2 \in A_2$. Then there is an (x, y)-path Q,

$$Q = x[A_2 \setminus \{a_2\}]a_2R[A_1 \setminus \{u_0, u_1, u_2, a_1\}]u_2[B_2 \setminus N(x)]y$$

with $N(x) \cup N(y) \subseteq V(Q)$ where

$$R = \begin{cases} a_1u_0[B_1 \setminus N(x)]u_1 & \text{if } a_1 \notin \{u_0, u_1\} \\ a_1[B_1 \setminus N(x)]u_1 & \text{if } a_1 = u_0 \\ a_1[B_1 \setminus N(x)]u_0 & \text{if } a_1 = u_1 \end{cases}$$

Subcase 1.3. $A_1 \cap B_i = \{u_i\}$ and u_i is not a cut vertex of $\langle N(x) \rangle$ for $i = 1, 2$.

First we consider the situation when $A_2 \cap (B_1 \cup B_2) \neq \emptyset$. W.l.o.g. we assume that $A_2 \cap B_2 \neq \emptyset$. Let $s \in A_2 \cap B_2$. Then the following (x, y)-path Q includes the set $N(x) \cup N(y)$:

$$Q = x[A_2 \setminus \{s\}]s[(B_2 \setminus A_2) \setminus \{u_2\}]u_2[A_1 \setminus \{u_1, u_2\}]u_1[(B_1 \setminus A_2) \setminus \{u_1\}]y.$$

Now let $A_2 \cap (B_1 \cup B_2) = \emptyset$. Since G is 3-connected, in $G - \{u_1, u_2\}$ there is an (x, y)-path xs_1Ps_2y, where $s_1 \in N(x), s_2 \in N(y)$ and $V(P) \cap (N(x) \cup N(y)) = \emptyset$. W.l.o.g. we assume that $s_2 \in B_2$. Clearly, there is an edge a_1a_2 with $a_1 \in A_1 \setminus \{u_1\}$ and $a_2 \in A_2$. Now we will produce an (x, y)-path Q with $N(x) \cup N(y) \subseteq V(Q)$.

If $s_1 \in A_2$ then

$$Q = x[A_2 \setminus \{s_1\}]s_1Ps_2[B_2 \setminus \{s_2, u_2\}]u_2[A_1 \setminus \{u_1, u_2\}]u_1[B_1 \setminus \{u_1\}]y.$$

If $s_1 \in A_1$ then $Q = x[A_2 \setminus \{a_2\}]a_2R[A_1 \setminus \{a_1, u_1, u_2, s_1\}]u_1[B_1 \setminus \{u_1\}]y$, where

$$R = \begin{cases} a_1u_2[B_2 \setminus \{u_2, s_2\}]s_2\overleftarrow{P}s_1 & \text{if } a_1 \notin \{s_1, u_2\} \\ a_1[B_2 \setminus \{a_1, s_2\}]s_2\overleftarrow{P}s_1 & \text{if } a_1 = u_2 \\ a_1Ps_2[B_2 \setminus \{u_2, s_2\}]u_2 & \text{if } a_1 = s_1. \end{cases}$$

Remark 2. By symmetry, the case $B_1 \cap A_2 \neq \emptyset$ requires the same argument but for sets $B_1 \cap A_1$ and $B_1 \cap A_2$.

Case 2. $A_1 \cap B_2 = B_1 \cap A_2 = \emptyset$ and also $A_2 \cap B_2 = \emptyset$. Then

$$A_1 \cap B_1 \text{ contains a vertex } u_1 \text{ which is not a cut vertex of } \langle N(x) \rangle. \qquad (7)$$

This is evident if $|A_1 \cap B_1| \geq 2$. If $|A_1 \cap B_1| = 1$ then (7) follows from the fact that G is claw-free. Clearly, (7) implies that there is an edge $a_1 a_2$ such that $a_1 \in A_1 \setminus \{u_1\}$ and $a_2 \in A_2$. Furthermore, we have that $\langle N(y) \rangle$ is connected.

If there exists an edge $v_1 v_2$ with $v_1 \in B_1 \setminus A_1$ and $v_2 \in B_2$ then there is an (x, y)-path $Q = x[A_2 \setminus \{a_2\}]a_2 a_1[A_1 \setminus \{a_1, u_1\}]u_1[B_1 \setminus (N(x) \cup \{v_1\})]v_1 v_2[B_2 \setminus \{v_2\}]y$ with $N(x) \cup N(y) \subseteq V(Q)$.

Suppose now that $v_1 \in A_1$ for each edge $v_1 v_2$ with $v_1 \in B_1$ and $v_2 \in B_2$ and consider one of these edges, $v_1 v_2$. Clearly, $B_1 \subseteq A_1$. (Otherwise a set $\{x, v_1, v_2, g_1\}$ induces $K_{1,3}$, where $g_1 \in (B_1 \setminus A_1) \cap N(v_1)$). If v_1 is not a cut vertex of $\langle N(x) \rangle$ then there is an edge $w_1 w_2$ such that $w_1 \in A_1 \setminus \{v_1\}$ and $w_2 \in A_2$. Then the path $Q = x[A_2 \setminus \{w_2\}]w_2 w_1[A_1 \setminus \{w_1, v_1\}]v_1 v_2[B_2 \setminus \{v_2\}]y$ satisfies the condition $N(x) \cup N(y) \subseteq V(Q)$.

Now we assume that v_1 is a cut vertex of $\langle N(x) \rangle$. Let $s_0 \in N(v_1) \cap A_2$. Then $v_2 s_0 \in E(G)$. (Otherwise $v_2 z \in E(G)$ for each $z \in B_1$ since G is claw-free. But then $v_2 \in N(y) \cap N(b_1) \cap N(b_2)$ for each pair of non-adjacent vertices $b_1, b_2 \in N(y)$, which contradicts (6)).

Subcase 2.1. s_0 is not a cut vertex of $N(x)$. Then there is an edge $v_1 a_0$ with $a_0 \in A_2 \setminus \{s_0\}$ and an (x, y)-path $Q = x[A_1 \setminus \{v_1\}]v_1 a_0[A_2 \setminus \{a_0, s_0\}]s_0 v_2[B_2 \setminus \{v_2\}]y$ with $N(x) \cup N(y) \subseteq V(Q)$.

Subcase 2.2. v_1 is not a cut vertex of $\langle N(y) \rangle$.

Clearly, there is an edge $b_1 b_2$ such that $b_1 \in B_1 \setminus \{v_1\}$ and $b_2 \in B_2$. Then there is an (x, y)-path $Q = x[A_2 \setminus \{s_0\}]s_0 v_1[A_1 \setminus \{v_1, b_1\}]b_1 b_2[B_2 \setminus \{b_2\}]y$ with $N(x) \cup N(y) \subseteq V(Q)$.

Subcase 2.3. v_2 is not a cut vertex of $\langle N(y) \rangle$.

Then there is an edge $v_1 v_3$, where $v_3 \in B_2 \setminus \{v_2\}$. By (7), the set $A_1 \cap B_1$ contains a vertex u_1 which is not a cut vertex of $\langle N(x) \rangle$. Clearly, $u_1 \neq v_1$. Then there is an (x, y)-path $Q = x[A_2 \setminus \{s_0\}]s_0 v_2[B_2 \setminus \{v_2, v_3\}]v_3 v_1[A_1 \setminus \{v_1, u_1\}]u_1 y$ with $N(x) \cup N(y) \subseteq V(Q)$.

Subcase 2.4. $\langle N(y) \rangle$ has two cut vertices, v_1 and v_2, and $\langle N(x) \rangle$ has two cut vertices, v_1 and s_0.

Since G is 3-connected, in $G - \{x, s_0\}$ there is a path $s_1 P s_2$ where $s_1 \in A_2, s_2 \in N(y) \cup A_1$ and $V(P) \cap (N(x) \cup N(y)) = \emptyset$. Then we can produce an (x, y)-path Q with $N(x) \cup N(y) \subseteq V(Q)$ in the following way.

If $s_2 \in B_2$ then $Q = x[A_1 \setminus \{v_1\}]v_1 s_0[A_2 \setminus \{s_0, s_1\}]s_1 P s_2[B_2 \setminus \{s_2\}]y$ and if $s_2 \in A_1$ then $Q = x[A_1 \setminus \{s_2\}]s_2 \overleftarrow{P} s_1[A_2 \setminus \{s_1, s_0\}]s_0 v_2[B_2 \setminus \{v_2\}]y$.

Case 3. $A_1 \cap B_2 = A_2 \cap B_1 = \emptyset$ and $A_2 \cap B_2 \neq \emptyset$.

Subcase 3.1. For each $i \in \{1, 2\}$ the set $A_i \cap B_i$ contains a vertex u_i which is not a cut vertex of $\langle N(x) \rangle$.

(a) $\{u_1, u_2\}$ is a cut set of $\langle N(x) \rangle$. Then there is an edge $a_1 u_2$ where $a_1 \in A_1 \setminus \{u_1\}$ and an edge $u_1 a_2$ where $a_2 \in A_2 \setminus \{u_2\}$. Consider the set $\{a_1, a_2, y, u_2\}$. Since $\{u_1, u_2\}$ is a cut set of $\langle N(x) \rangle, a_1 a_2 \notin E(G)$. Then $y a_1 \in E(G)$ or $y a_2 \in E(G)$ because G is claw-free. Since these situations are similar, we consider the case $y a_1 \in E(G)$ only. We have that $a_1 \in A_1$ and $A_1 \cap B_2 = \emptyset$. Therefore $a_1 \in B_1$. Then

$$Q = x[A_1 \setminus \{a_1, u_1\}]a_1[B_1 \setminus \{u_1\}]u_1 a_2[A_2 \setminus \{a_2, u_2\}]u_2[B_2 \setminus \{u_2\}]y$$

is an (x, y)-path Q with $N(x) \cup N(y) \subseteq V(Q)$.

(b) $\{u_1, u_2\}$ is not a cut set of $\langle N(x) \rangle$.

Then there is an edge a_1a_2 where $a_i \in A_i \setminus \{u_i\}$ for $i = 1,2$. If one of the sets $A_j \cap B_j, 1 \leq j \leq 2$, say $A_1 \cap B_1$, contains a vertex $u_0 \notin \{u_1, u_2\}$, then the path $Q = xu_0[B_1 \setminus A_1]u_1[A_1 \setminus \{u_0, u_1, a_1\}]a_1a_2[A_2 \setminus \{a_2, u_2\}]u_2[B_2 \setminus A_2]y$ satisfies the condition $N(x) \cup N(y) \subseteq V(Q)$.

Now let $A_i \cap B_i = \{u_i\}$ for $i = 1,2$. Since G is 3-connected, in $G - \{u_1, u_2\}$ there is an (y,x)-path ys_1Ps_2x, where $s_1 \in N(y) \setminus \{u_1, u_2\}, s_2 \in N(x) \setminus \{u_1, u_2\}$ and $V(P) \cap (N(x) \cup N(y)) = \emptyset$. We assume that $s_1 \in B_2$. Now we will produce an (x,y)-path Q satisfying the condition $N(x) \cup N(y) \subseteq V(Q)$. If $s_2 \in A_1$ then $Q = x[A_2 \setminus \{u_2\}]u_2[B_2 \setminus \{u_2, s_1\}s_1Ps_2[A_1 \setminus \{s_2, u_1\}]u_1[B_1 \setminus \{u_1\}]y$.

If $s_2 \in A_2$ and $a_2 \neq s_2$ then

$$Q = x[A_2 \setminus \{s_2, a_2, u_2\}]u_2[B_2 \setminus \{u_2, s_1\}]s_1Ps_2a_2a_1[A \setminus \{a_1, u_1\}]u_1[B_1 \setminus \{u_1\}]y.$$

The same path, but with a_2 deleted, corresponds to the case $s_2 \in A_2$ and $s_2 = a_2$.

Subcase 3.2. For some $i \in \{1,2\}$ the set $A_i \cap B_i$ consists of the unique vertex u_i which is a cut vertex of $\langle N(x) \rangle$.

We assume that $A_1 \cap B_1 = \{u_1\}$ and u_1 is a cut vertex of $\langle N(x) \rangle$. Let u_2 be a vertex from $A_2 \cap N(u_1)$. Since G is claw-free, $yu_2 \in E(G)$. So, $u_2 \in B_2$. Then $v_1z_1 \in E(G)$ for some $v_1 \in A_1 \setminus \{u_1\}$ and $z_1 \in B_1 \setminus \{u_1\}$. (Otherwise $zu_2 \in E(G)$ for each $z \in B_1$ since G is claw-free. Therefore, $u_2 \in N(y) \cap N(w_1) \cap N(w_2)$ for each pair of non-adjacent vertices $w_1, w_2 \in N(y)$, which contradicts (6)).

If $N(u_1) \cap A_2 = \{u_2\}$ then u_2 is a cut vertex of $\langle N(x) \rangle$. Hence, by symmetry, $v_2z_2 \in E(G)$ for some $v_2 \in A_2 \setminus \{u_2\}$ and $z_2 \in B_2 \setminus \{u_2\}$. Then $N(x) \cup N(y) \subseteq V(Q)$ for an (x,y)-path Q,

$$Q = x[A_1 \setminus \{v_1, u_1\}]v_1z_1[B_1 \setminus \{u_1, z_1\}]u_1u_2[A_2 \setminus \{u_2, v_2\}]v_2z_2[B_2 \setminus \{z_2, u_2\}]y.$$

Now let $|N(u_1) \cap A_2| \geq 2, u_3 \in N(u_1) \cap A_2$ and $u_3 \neq u_2$. Then for an (x,y)-path

$$Q = x[A_1 \setminus \{u_1, v_1\}]v_1z_1[B_1 \setminus \{u_1, z_1\}]u_1u_3[A_2 \setminus \{u_2, u_3\}]u_2[B_2 \setminus A_2]y$$

the condition $N(x) \cup N(y) \subseteq V(Q)$ holds. The proof of Theorem 3.3 is complete. ∎

Theorem 3.4. A locally connected, claw-free graph G with $|V(G)| \geq 4$ is Hamilton-connected if and only if G is 3-connected.

Proof. Clearly, if G is Hamilton-connected and has at least 4 vertices then it is also 3-connected. Conversely, if G is a 3-connected, claw-free graph with $|V(G)| \geq 4$ then it follows from Theorems 3.1–3.3 that G is Hamilton-connected. ∎

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