Bifurcation and vibration of a surface-coated elastic block under flexure

by

Gillian Dryburgh

A thesis submitted to the Faculty of Science, University of Glasgow, for the degree of Doctor of Philosophy

Department of Mathematics,

University of Glasgow

August 1999

Preface

This thesis was submitted to the University of Glasgow in accordance with the requirements for the degree of Doctor of Philosophy.

I wish to sincerely thank my supervisor, Professor Ray Ogden, for his patience, encouragement and constant support throughout the period of this research.

My thanks also go to the rest of the staff in the Mathematics Department at the University of Glasgow for their help during both my undergraduate and postgraduate studies.

For additional assistance with this thesis, particularly in obtaining numerical results, I am most grateful to Dr. David Haughton and Dr. David Roxburgh.

Thanks are also due to the U.K. Engineering and Physical Sciences Research Council for financing my research by way of a studentship.

My deepest thanks are reserved for my late grandmother, Catherine Rogan, who never doubted my ability, and for my parents, Jean and Tommy Dryburgh, who have had the unenviable task of supporting me throughout my 21 years in education, I am indebted to them!

Summary

The behaviour of a surface-coated rectangular, non-linearly elastic block subject to (plane-strain) flexure is investigated in this thesis. We consider a rectangular block of incompressible, isotropic elastic material coated with a thin elastic film on part of its boundary. Initially, the bulk material undergoes a non-homogeneous deformation and the equilibrium of the coated body is examined on the basis of the elastic surface coating theory derived by Steigmann and Ogden (1997a). Incremental displacements are then superimposed on the finitely deformed configuration in order to study possible bifurcation of the deformed block. Numerical bifurcation results pertaining to two particular strain-energy functions (for the bulk material) and a general energy function (for the coating material) are subsequently obtained. These results allow the influence of the surface coating on the bifurcation behaviour of the block to be determined and assessed with reference to corresponding results for an uncoated block. Next, use is made of the dynamic equivalent of the static surface coating theory, developed by Ogden and Steigmann (1999), to establish incremental equations of motion for the coated block. Corresponding incremental governing equations for an uncoated, pre-flexed block then emerge as a special case. The resulting frequency equations are solved numerically, again on specialization of

the form of strain-energy function. The numerical vibration results then provide evidence of the effect of surface coating on the dynamic behaviour of the considered coated block relative to the uncoated case. Finally, we turn our attention to the (non-linear) shear response of bonded elastic bodies. We examine the plane-strain problem of a rectangular compressible isotropic elastic block bonded to two rigid parallel plates. The deformation behaviour of the block is described by applying minimum energy and maximum complementary energy principles to obtain upper and lower bounds on the shear stress-strain relationship. Although maximum and minimum principles are not generally justifiable in non-linear elasticity we show that under certain conditions they are applicable and, for a particular form of strain-energy function, derive explicit energy bounds which we illustrate graphically.

Contents

Chapter 1

Introduction

This thesis is concerned mainly with examining the effect of elastic surface coating on the behaviour of an incompressible isotropic elastic block subject to non-homogeneous (plane-strain) flexure. The surface coating, taken to be infinitesimally thin and effectivley bonded to the bulk material, can occupy one or both of the boundaries of the block that become curved on deformation and coating of the same type is prescribed on these boundaries. In the first instance, we consider finite flexure in which the block undergoes a (symmetric) deformation about the X_1-axis so that it deforms into a sector of a circular cylinder. Thereafter, we concentrate on deriving and solving bifurcation and frequency equations pertaining to (linearized) incremental displacements superimposed on the finitely deformed configuration. Hence, with reference to the corresponding uncoated results, we are able to quantify the effect of the coating on the bifurcation and vibration behaviour of the block. The influence of certain material and geometrical parameters on this behaviour is also assessed.

The non-linear (plane-strain) theory of elastic surface coating, recently developed by Steigmann and Ogden (1997a), is fundamental to the research herein. In this paper, the authors used the example of a coated elastic half-space to illustrate their theory for the plane deformation of an elastic body coated with a thin elastic film on its surface. In a subsequent paper, Steigmann and Ogden (1997b), they derived necessary conditions for the existence of energy minimizing solutions of such bodies. A further illustration of the theory was provided, in Ogden, Steigmann and Haughton (1997), by its application to the analysis of the stability of a pressurized coated circular annulus. This thesis provides a further illustration of the theory through its examination of the possible bifurcation behaviour of an incompressible rectangular elastic block subject to non-homogeneous finite flexure. Although bending of an *uncoated* isotropic incompressible non-linearly elastic rectangular block has been described by Ogden (1984) and Green and Zerna (1968), the corresponding bifurcation problem has not been examined previously in the literature.

In a further development of their static surface coating theory, Steigmann and Ogden (1999) derived a theory for the (plane-strain) dynamics of incompressible coated elastic bodies. In this paper, the authors applied their dynamic theory to the problem of surface wave propagationon on a pre-stressed incompressible isotropic coated elastic half-space, paying particular attention to the role of the rotatory inertia term. This dynamic theory forms the basis for the study of vibration behaviour of a pre-flexed surface-coated incompressible elastic block considered in this thesis. The results which then emerge, as a special case, for the dynamic behaviour of a corresponding *uncoated* pre-flexed block are also new.

We now turn our attention to the final problem investigated in this thesis: shear of a bonded elastic block. We consider the (non-homogeneous) shear deformation of a rectangular compressible isotropic non-linearly elastic block, in plane strain.

The block is deformed by displacing one plate, parallel to itself, relative to the other while the lateral surfaces are maintained free of traction. As there is no known *exact* solution to this problem an alternative means of solution is required. Hence, we make use of maximum and minimum energy principes to obtain (general) upper and lower bounds on the total energy stored in the block as a result of the deformation. We demonstrate that such extremum principles are justifiable in a non-linear setting provided the strain-energy function satisfies certain convexity and growth conditions. On choosing an appropriate strain-energy function we are then able to derive explicit energy bounds.

The approach adopted in solving this problem follows closely that of Haddow and Ogden (1988). In this paper, the authors sought to obtain energy bounds in relation to compression of bonded elastic bodies in terms of both linear and non-linear elasticity. The concept of seeking energy bounds by way of solution owes much to the contribution of Prager and Synge (1947) who obtained approximate solutions to boundary value problems, in the linear theory, on application of minimum and maximum energy arguments.

In Chapter 2, following the approach of standard texts such as Ogden (1984) and Atkin and Fox (1980), we introduce the notation that will be used throughout the thesis as well as setting up the equations of equilibrium and motion and the constitutive relations used in finite elasticity for both compressible and incompressible materials. We then introduce the corresponding incremental equations for small deformation superimposed on a nonlinearly, finitely deformed state, as well as discussing the strong-ellipticity condition and, in the two-dimensional case, giving necessary and sufficient conditions for it to hold with respect to plane strain.

In Chapter 3 we consider plane-strain finite flexure of an *uncoated* incompressible elastic block and then, on application of the basic surface coating theory, extend our findings to flexure of a surface-coated elastic block. In Section 3.1, we outline the theory developed in Steigmann and Ogden (1997a) by summarizing the equations and boundary conditions governing the equilibrium of a coated elastic body and the crucial conditions coupling the response of the bulk solid to that of the coating. The basic theory is applied, in Section 3.2, to the bending of a coated rectangular elastic block, and use of the neo-Hookean strain-energy function illustrates the deformation behaviour of the block.

In Chapter 4, the bifurcation behaviour of coated and uncoated elastic blocks is examined. Bifurcation equations are derived and numerical methods are used to determine the onset of bifurcation in respect of specific forms of strain energy. Section 4.1 provides the incremental equilibrium equations and boundary conditions linearized relative to the underlying deformation. These are applied, in Section 4.2, to the finite deformation discussed in Chapter 3. The incompressibility condition is used to express the incremental equilibrium equations as a single equation for a scalar function ψ, and the boundary conditions are also cast in terms of ψ. The possibility of bifurcation from a finitely deformed configuration is then illustrated, graphically, in Section 4.3, after seeking solutions for ψ of the form $\phi(r)\cos q\theta$ and solving numerically the relevant bifurcation equations.

Chapter 5 extends the static case of Chapters 3 and 4 to consideration of the vibration behaviour of the coated and uncoated blocks examined previously. Using (plane-strain) surface coating dynamic theory, we establish incremental equations of motion and boundary conditions applicable to vibration of a pre-flexed block. In

Section 5.1 we summarize the plane-strain dynamic theory derived by Ogden and Steigmann (1999) and set out the equations governing the motion of the substrate and the film as well as their incremental counterparts. These equations are applied, in Section 5.2, to the case of a surface-coated elastic block. Equations governing the motion of the film then provide boundary conditions for the displacement of the bulk material which, as in Chapter 4, is governed by a single scalar equation in ψ.

In Chapter 6, we investigate the dynamic behaviour of the coated and uncoated elastic blocks. The relevant frequency equations are solved numerically and a comprehensive analysis of the vibration results is conducted. Section 6.1 involves seeking solutions, of the form $\phi(r)\cos q\theta e^{i\omega t}$, to identify the frequency of vibrations of the surface-coated block. Numerical results are then obtained, in Section 6.2, and the effect of the coating on the vibration behaviour of the block is assessed and illustrated, graphically, taking account of the influence of the material properties of the coating and the relative densities of the bulk and film materials.

In Chapter 7, we focus on determining the (non-linear) shear response of bonded elastic blocks. Given the lack of a closed form solution for this problem, we apply maximum and minimum principles to derive general upper and lower bounds on the stored energy which are then made explicit by their specialization to particular strain and stress fields and to a semi-linear material. In view of the sudden change in direction from that of the previous chapters, Section 7.1 gives a brief description of the shear problem to be studied. In Section 7.2 we set out some variational arguments and establish the allied extremum principles, justifying their use in both linear and non-linear elasticity (subject to certain restrictions). Then, in Section 7.3, we define the theory of minimum energy and the theory of maximum complementary energy, for future use. These theories are applied, in Section 7.4, to the shear of a bonded elastic block and general energy bounds are obtained. The choice of particular kinematically admissible deformation fields and statically admissible stress fields, in Section 7.5, allows specific energy bounds to be artificially constructed and further use of the semi-linear strain-energy function makes these bounds explicit.

Results from Chapters 3 and 4 have been published in *Zeitschrift für Angewandte Mathematik und Mechanik* [see Ogden, Dryburgh and Steigmann (1998)] and further results from Chapter 4 appear in *Zeitschrift für Angewandte Mathematik und Physik* [see Dryburgh and Ogden (1999)]. The work of Chapters 5 and 6 will also be written up for publication.

Chapter 2

Basic theory

In this chapter we summarize the basic equations of non-linear elasticity theory which are essential for the subsequent chapters. Full details are given in standard texts such as Ogden (1984).

2.1 Deformation and strain

An unstressed elastic body is taken to occupy the region $\mathcal{B}_0$ in three dimensional Euclidean space. Points in this reference configuration are denoted by their position vectors $\mathbf{X}$ relative to some arbitrarily chosen origin. In time t the body is deformed into a new configuration $\mathcal{B}_t$ in which the material point $\mathbf{X}$ moves to position $\mathbf{x}$ according to

$$\mathbf{x} = \chi_t(\mathbf{X}) \qquad \mathbf{X} \in \mathcal{B}_0, \tag{2.1.1}$$

where the bijective mapping χ_t: $\mathcal{B}_0 \to \mathcal{B}_t$ describes the motion from the reference configuration $\mathcal{B}_0$ to the current configuration $\mathcal{B}_t$. We may, alternatively, write (2.1.1) in the form

$$\mathbf{x} = \chi(\mathbf{X}, t), \quad \chi \in \mathcal{B}_0, \quad t \in \mathcal{I}. \tag{2.1.2}$$

We take χ as a twice-continuously differentiable function of space and time with explicit dependence on t, where $\mathcal{I}$ is a suitable interval of time. When (2.1.2) is independent of t, χ is called the *deformation* from $\mathcal{B}_0$ to the deformed configuration $\mathcal{B}$ ($\mathcal{B}_t$ without the subscript t) and we write the deformation of the body in the form

$$\mathbf{x} = \chi(\mathbf{X}) \qquad \mathbf{X} \in \mathcal{B}_0. \tag{2.1.3}$$

The *deformation gradient* tensor $\mathbf{A}$ is defined by

$$\mathbf{A} = \mathrm{Grad}\chi, \tag{2.1.4}$$

where Grad is the gradient operator with respect to $\mathbf{X}$. When $\mathbf{A}$ is independent of $\mathbf{X}$, the deformation is said to be *homogeneous.*

In addition to the assumed regularity conditions mentioned above, namely that χ is a bijective mapping and twice continuously differentiable, we impose the usual requirement that, for all deformations χ,

$$J = \det \mathbf{A} > 0 \tag{2.1.5}$$

holds. This ensures that $\mathbf{A}$ has inverse $\mathbf{A}^{-1}$.

Differentials $\mathrm{d}\mathbf{X}$ in $\mathcal{B}_0$ and $\mathrm{d}\mathbf{x}$ in $\mathcal{B}_t$ are related by

$$\mathrm{d}\mathbf{x} = \mathbf{A}\mathrm{d}\mathbf{X}, \tag{2.1.6}$$

which describes how a line element $\mathbf{dX}$ of material deforms into $\mathbf{dx}$ under the deformation. Area elements transform according to *Nanson's formula* as

$$\mathbf{n}\mathrm{d}s = J\mathbf{A}^{-T}\mathbf{N}\mathrm{d}S, \tag{2.1.7}$$

where $\mathrm{dS} = \mathbf{N}\mathrm{d}S$ and $\mathrm{ds} = \mathbf{n}\mathrm{d}s$ represent infinitesimal surface elements in $\mathcal{B}_0$ and $\mathcal{B}_t$ with unit outward normals $\mathbf{N}$ and $\mathbf{n}$ respectively, ^{T} denotes transpose and $^{-T}$ is $^{(-1)T}$. Similarly,

$$\mathrm{d}v = J\mathrm{d}V, \tag{2.1.8}$$

describes the transformation of a volume element $\mathrm{d}V$ in $\mathcal{B}_0$ into the volume element $\mathrm{d}v$ in $\mathcal{B}_t$. Hence, if the volume in $\mathcal{B}_0$ is unchanged by a deformation it follows that $J = 1$ in (2.1.8). Materials which satisfy the constraint $J = 1$ are said to be *incompressible.*

Since, by (2.1.5), $\mathbf{A}$ is non-singular it may be decomposed according to the *Polar Decomposition Theorem* in the form

$$\mathbf{A} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}, \tag{2.1.9}$$

where $\mathbf{R}$ is proper orthogonal and $\mathbf{U}$ and $\mathbf{V}$ are symmetric, positive definite *right* and *left stretch tensors* respectively such that

$$\mathbf{U}^2 = \mathbf{A}^T\mathbf{A} \qquad \mathbf{V}^2 = \mathbf{A}\mathbf{A}^T. \tag{2.1.10}$$

Since $\mathbf{U}$ is symmetric and positive definite, its principal values λ_i $(i = 1,\ 2,\ 3)$ are positive. Let $\mathbf{u}^{(i)}$ $(i = 1,\ 2,\ 3)$ be the principal axes of $\mathbf{U}$. Then, we have the spectral decomposition

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}. \tag{2.1.11}$$

The λ_i are also the principal values of $\mathbf{V}$ corresponding to principal axes $\mathbf{v}^{(i)} = \mathbf{R}\mathbf{u}^{(i)}$ $(i = 1,\ 2,\ 3)$, and we therefore have

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}. \tag{2.1.12}$$

We refer to λ_i as the *principal stretches*, and $\mathbf{u}^{(i)}$ and $\mathbf{v}^{(i)}$ as *Lagrangian* and *Eulerian principal axes* of the deformation respectively. It follows that

$$\mathbf{A} = \sum_{i=1}^{3} \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{u}^{(i)}. \tag{2.1.13}$$

2.2 Stress and equilibrium equations

The load, or traction, $\mathbf{t}$ per unit area acting on the surface of an elastic material in the current configuration is

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}, \tag{2.2.1}$$

where $\mathbf{n}$ is the unit outward normal of the surface in $\mathcal{B}_t$ and $\boldsymbol{\sigma}$ is a second order tensor called the *Cauchy stress tensor.* Similarly, when defined per unit area of the reference configuration the traction $\mathbf{t}_0$ can be expressed as

$$\mathbf{t}_0 = \mathbf{S}^T\mathbf{N}, \tag{2.2.2}$$

where $\mathbf{N}$ denotes the unit outward normal of the surface in $\mathcal{B}_0$ and $\mathbf{S}$ is the *nominal stress tensor.* Then, from Nanson's formula (2.1.7) it follows that

$$\mathbf{S} = J\mathbf{A}^{-1}\boldsymbol{\sigma}. \tag{2.2.3}$$

The balance of angular momentum forces $\boldsymbol{\sigma}$ to be symmetric and hence

$$\mathbf{AS} = \mathbf{S}^T\mathbf{A}^T, \tag{2.2.4}$$

so that, in general, $\mathbf{S} \neq \mathbf{S}^T$. For an isotropic material $\mathbf{SR}$ is a symmetric tensor and we can write

$$\mathbf{S} = \mathbf{TR}^T, \tag{2.2.5}$$

where $\mathbf{T}$ is the *Biot stress tensor.*

The (Eulerian) *equation of motion* is

$$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{x}_{,tt}\,, \tag{2.2.6}$$

where ρ is the material density per unit volume in $\mathcal{B}_t$, $\mathbf{b}$ is the body force per unit mass and div is the divergence operator in the current configuration. Note that $(\cdot)_{,t}$ represents the material time derivative.

Mass conservation relates the density ρ at $\mathbf{x}$ to the density ρ_r at $\mathbf{X}$ through

$$J = \rho_r/\rho. \tag{2.2.7}$$

The Lagrangian alternative to the equation of motion (2.2.6) is

$$\operatorname{Div} \mathbf{S} + \rho_r \mathbf{b} = \rho_r \mathbf{x}_{,tt}\,, \tag{2.2.8}$$

where Div is the divergence operator in $\mathcal{B}_0$, and (2.2.8) is equivalent to (2.2.6)

In the static case, when $\mathbf{x}_{,tt} = 0$, we obtain the *equilibrium equations.* In the absence of body force, the equilibrium equations are

$$\operatorname{Div} \mathbf{S} = \mathbf{0} \tag{2.2.9}$$

or, in terms of Cauchy Stress,

$$\operatorname{div} \boldsymbol{\sigma} = \mathbf{0}. \tag{2.2.10}$$

2.3 Constitutive relations and strain energy

For an elastic material, the stress depends only on the deformation and we assume the constitutive relation

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{A}), \tag{2.3.1}$$

where $\mathbf{g}$ is a tensor-valued function called the *response function* of the material. To ensure physically reasonable behaviour, we require this constitutive relation to be *objective* under rigid-body motions. This means the material response is invariant under superimposed rotations and $\mathbf{g}$ must satisfy

$$\mathbf{g}(\mathbf{QA}) = \mathbf{Qg}(\mathbf{A})\mathbf{Q}^T, \tag{2.3.2}$$

where $\mathbf{Q}$ is any proper orthogonal tensor.

We also assume that the elastic solid is isotropic, i.e. its material properties have no preferred direction; then we must have

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{A}) = \mathbf{g}(\mathbf{AP}), \tag{2.3.3}$$

for all rotations $\mathbf{P}$. On using the polar decomposition (2.1.9) and setting $\mathbf{P} = \mathbf{R}^T$, we obtain

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{A}) = \mathbf{g}(\mathbf{AR}^T) = \mathbf{g}(\mathbf{VRR}^T) = \mathbf{g}(\mathbf{V}). \tag{2.3.4}$$

We also obtain

$$\mathbf{g}(\mathbf{QVQ}^T) = \mathbf{Qg}(\mathbf{V})\mathbf{Q}^T. \tag{2.3.5}$$

Taken together, objectivity and isotropy imply that $\boldsymbol{\sigma}$ and $\mathbf{V}$ are coaxial, i.e. have the same principal directions. Hence, the stress $\boldsymbol{\sigma}$ can be expressed in the form

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{2.3.6}$$

analogously to the spectral decomposition (2.1.12) of $\mathbf{V}$, where the σ_i are principal values of $\boldsymbol{\sigma}$. Similarly, we can show that $\mathbf{T}$ is coaxial with $\mathbf{U}$ and, analogously to (2.1.11), using (2.2.3) and (2.2.5) given $\mathbf{v}^{(i)} = \mathbf{Ru}^{(i)}$, the Biot stress tensor can be expressed as

$$\mathbf{T} = \sum_{i=1}^{3} t_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \tag{2.3.7}$$

where $t_i = J\sigma_i\lambda_i^{-1}$ and t_i are the *principal Biot stresses*. From (2.2.5), it follows that

$$\mathbf{S} = \sum_{i=1}^{3} t_i \mathbf{u}^{(i)} \otimes \mathbf{v}^{(i)}. \tag{2.3.8}$$

An elastic material is said to be *hyperelastic* if there exists a scalar function $W(\mathbf{A})$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} W(\mathbf{A}) = J\mathrm{tr}(\boldsymbol{\sigma}\mathbf{L}), \tag{2.3.9}$$

where $\mathbf{L} = \partial\mathbf{v}/\partial\mathbf{x}$ is the velocity gradient and $\mathbf{v}(\mathbf{x}, t)$ is the velocity of the material particle occupying position $\mathbf{x}$ at time t. Since the function $W(\mathbf{A})$ is a measure of the energy stored in the material as a result of the deformation, it is called the *strain-energy* (or *stored-energy*) function per unit reference volume. When such a W exists, the Cauchy stress tensor can be expressed in the form

$$J\boldsymbol{\sigma} = \mathbf{A}\frac{\partial W}{\partial \mathbf{A}} \tag{2.3.10}$$

for an unconstrained hyperelastic material, and, with reference to (2.2.3), it follows that

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{A}}, \tag{2.3.11}$$

or in Cartesian component form,

$$S_{ij} = \frac{\partial W}{\partial A_{ji}}. \tag{2.3.12}$$

If $W(\mathbf{A})$ is taken to be objective and isotropic then

$$W(\mathbf{A}) = W(\mathbf{QA}) = W(\mathbf{AP}), \tag{2.3.13}$$

for arbitrary rotations $\mathbf{Q}$ and $\mathbf{P}$. We also have

$$W(\mathbf{QVQ}^T) = W(\mathbf{V}), \tag{2.3.14}$$

which holds for all proper orthogonal $\mathbf{Q}$. Now taking $\mathbf{Q} = \mathbf{R}$, we find that

$$W(\mathbf{A}) = W(\mathbf{U}) = W(\mathbf{V}). \tag{2.3.15}$$

Then, with reference to (2.3.11) and (2.2.5), we obtain

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{U}}. \tag{2.3.16}$$

It follows from (2.3.15) and (2.1.11) that W depends solely on the principal stretches and we have

$$W = W(\lambda_1, \lambda_2, \lambda_3), \tag{2.3.17}$$

where the order in which λ_1, λ_2 and λ_3 appear is irrelevant. Consequently, with reference to (2.3.6) and (2.3.7), when evaluated along the principal axes, for a compressible material, (2.3.10) and (2.3.16) can be written in component form as

$$J\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i}, \qquad i = 1,\ 2,\ 3 \tag{2.3.18}$$

and

$$t_i = \frac{\partial W}{\partial \lambda_i}, \qquad i = 1,\ 2,\ 3 \tag{2.3.19}$$

respectively, where no summation is implied by the repetition of the index i in (2.3.18).

For the case of an incompressible material, the constraint

$$J = \lambda_1 \lambda_2 \lambda_3 = 1 \tag{2.3.20}$$

removes the independence of the principal stretches and we introduce a Lagrange multiplier p (which may be viewed as a hydrostatic pressure term) and write

$$\sigma = \mathbf{A}\frac{\partial W}{\partial \mathbf{A}} - p\mathbf{I} \tag{2.3.21}$$

instead of (2.3.10), with (2.3.11) and (2.3.16) similarly being replaced by

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{A}} - p\mathbf{A}^{-1} \tag{2.3.22}$$

and

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{U}} - p\mathbf{U}^{-1} \tag{2.3.23}$$

respectively. Then, for an isotropic, incompressible hyperelastic material (2.3.21) and (2.3.23) have component forms

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p \tag{2.3.24}$$

and

$$t_i = \frac{\partial W}{\partial \lambda_i} - p{\lambda_i}^{-1} \tag{2.3.25}$$

evaluated on the principal axes.

2.4 Incremental deformations and constitutive relations

Previously we considered the deformation of a body into the configuration $\mathcal{B}$ relative to $\mathcal{B}_0$ such that

$$\mathbf{x} = \chi(\mathbf{X}), \tag{2.4.1}$$

where, for the purposes of this Section, the explicit dependence on time t has been suppressed. We now consider that a small displacement $\mathbf{u}$ is superimposed on this finite deformation in which case

$$\mathbf{u} = \delta\mathbf{x} = \delta\chi(\mathbf{X}), \tag{2.4.2}$$

where δ indicates a small increment in the quantity concerned and small implies that quantities of second order can be neglected. The change in the deformation gradient brought about by the incremental deformation $\mathbf{u} = \delta\mathbf{x}$ is then given (exactly) by

$$\delta\mathbf{A} = \delta(\text{Grad } \mathbf{x}) = \text{Grad } (\mathbf{x} + \mathbf{u}) - \text{Grad } \mathbf{x} = \text{Grad } \mathbf{u}. \tag{2.4.3}$$

The corresponding change in the determinant has the linear approximation

$$\delta J = J\text{tr } \{(\delta\mathbf{A})\mathbf{A}^{-1}\}. \tag{2.4.4}$$

For an unconstrained material, the increment in the nominal stress (to the first order) is

$$\dot{\mathbf{S}} = \mathcal{A}\dot{\mathbf{A}}, \tag{2.4.5}$$

where we have introduced the alternative notation $\dot{\mathbf{S}} = \delta\mathbf{S}$, $\dot{\mathbf{A}} = \delta\mathbf{A}$ and

$$\mathcal{A} = \frac{\partial\mathbf{S}}{\partial\mathbf{A}} \tag{2.4.6}$$

is the fourth-order tensor of elastic moduli associated with the stress-deformation pair (S, A).

For incompressible materials, since $J = 1$, (2.4.4) becomes

$$\mathrm{tr}\,\{(\dot{\mathbf{A}})\mathbf{A}^{-1}\} = 0 \tag{2.4.7}$$

and, following from (2.3.22), the corresponding form of (2.4.5) is

$$\dot{\mathbf{S}} = \mathcal{A}\dot{\mathbf{A}} - \dot{p}\mathbf{A}^{-1} + p\mathbf{A}^{-1}\dot{\mathbf{A}}\mathbf{A}^{-1}, \tag{2.4.8}$$

where the formula $(\dot{\mathbf{A}^{-1}}) = -\mathbf{A}^{-1}\dot{\mathbf{A}}\mathbf{A}^{-1}$ has been used and $\dot{p}$ denotes the increment in p.

If we now update the reference configuration to the deformed state (2.4.1), then relative to this new reference configuration the deformation gradient takes the form

$$\dot{\mathbf{A}}_0 = \mathrm{grad}\ \mathbf{u} = \dot{\mathbf{A}}\mathbf{A}^{-1}, \tag{2.4.9}$$

where grad is the gradient operator with respect to $\mathbf{x}$ and the subscript $_0$ indicates a quantity evaluated in $\mathcal{B}$.

Similar to (2.4.9), updating the nominal stress increment, on use of Nanson's formula (2.1.7), yields

$$\dot{\mathbf{S}}_0 = J^{-1}\mathbf{A}\dot{\mathbf{S}}, \tag{2.4.10}$$

and corresponding to (2.4.5) we write

$$\dot{\mathbf{S}}_0 = \mathcal{A}_0\dot{\mathbf{A}}_0, \tag{2.4.11}$$

where $\mathcal{A}_0$ is the tensor of *instantaneous elastic moduli* referred to $\mathcal{B}$. On comparing (2.4.10) and (2.4.11) we find that $\mathcal{A}$ and $\mathcal{A}_0$ are related in component form by

$$\mathcal{A}_{0ijkl} = J^{-1}A_{is}A_{kt}\mathcal{A}_{sjtl}. \tag{2.4.12}$$

Updating equations (2.4.7) and (2.4.8) for an incompressible material yields

$$\mathrm{tr}(\dot{\mathbf{A}}_0) = \mathrm{div}\mathbf{u} = 0 \tag{2.4.13}$$

and

$$\dot{\mathbf{S}}_0 = \mathcal{A}_0\dot{\mathbf{A}}_0 - \dot{p}\mathbf{I} + p\dot{\mathbf{A}}_0 \tag{2.4.14}$$

respectively. If the material is hyperelastic, $\mathcal{A}$ is expressible as

$$\mathcal{A} = \frac{\partial^2 W}{\partial \mathbf{A}^2}, \tag{2.4.15}$$

or in Cartesian component form

$$\mathcal{A}_{ijkl} = \frac{\partial^2 W}{\partial A_{ji}\partial A_{lk}}. \tag{2.4.16}$$

For an isotropic, hyperelastic, unconstrained material the components of $\mathcal{A}_0$ relative to the Eulerian principal axes take the form

$$\mathcal{A}_{0iijj} = J^{-1}\lambda_i\lambda_j W_{ij}, \tag{2.4.17}$$

$$\mathcal{A}_{0ijij} = J^{-1}\frac{(\lambda_i W_i - \lambda_j W_j)\,\lambda_i^2}{\lambda_i^2 - \lambda_j^2} \qquad i \neq j,\ \lambda_i \neq \lambda_j \tag{2.4.18}$$

$$= \frac{1}{2}(\mathcal{A}_{0iiii} - \mathcal{A}_{0iijj} + J^{-1}\lambda_i W_i) \quad i \neq j,\ \lambda_i = \lambda_j \tag{2.4.19}$$

$$\mathcal{A}_{0ijji} = \mathcal{A}_{0jiij} = \mathcal{A}_{0ijij} - J^{-1}\lambda_i W_i \quad i \neq j \tag{2.4.20}$$

with $i,\ j = 1,\ 2,\ 3$, where $W_i = \partial W/\partial\lambda_i$ and $W_{ij} = \partial^2 W/\partial\lambda_i\partial\lambda_j$, and the summation convention does not apply.

In the case of no deformation, the components of $\mathcal{A}_0$ in an unstrained configuration and for an unconstrained material take the values

$$\mathcal{A}_{0iiii} = \lambda + 2\mu, \qquad \mathcal{A}_{0iijj} = \lambda, \tag{2.4.21}$$

$$\mathcal{A}_{0ijij} = \mathcal{A}_{0ijji} = \mu \tag{2.4.22}$$

corresponding to the classical theory of linear elasticity, where $i \neq j$ and λ and μ are the classical Lamé constants.

We obtain the components of $\mathcal{A}_0$ for incompressible materials by substituting $J = 1$ into equations (2.4.17)-(2.4.20) and the equivalent components for an unstressed configuration become

$$\mathcal{A}_{0iiii} = \mathcal{A}_{0ijij} = \mu, \tag{2.4.23}$$

$$\mathcal{A}_{0iijj} = \mathcal{A}_{0ijji} = 0, \tag{2.4.24}$$

where $i \neq j$ and μ (> 0) is now the shear modulus of the material.

In the absence of body forces, the incremental form of the equation of motion (2.2.8) is

$$\mathrm{Div}\,\dot{\mathbf{S}} = \rho_r \mathbf{u}_{,tt}\,. \tag{2.4.25}$$

When updated to the current configuration this becomes

$$\mathrm{div}\,\dot{\mathbf{S}}_0 = \rho \mathbf{u}_{,tt} \tag{2.4.26}$$

and likewise the equilibrium equation without body force in the current configuration takes the form

$$\mathrm{div}\,\dot{\mathbf{S}}_0 = \mathbf{0}. \tag{2.4.27}$$

2.5 The strong ellipticity condition

A second-order tensor $\mathbf{D}$ is said to have rank one if its nullspace is two-dimensional. Thus, such a tensor can be written in the form $\mathbf{D} = \mathbf{m} \otimes \mathbf{N}$ for some non-zero vectors $\mathbf{m}$ and $\mathbf{N}$. Given a rank-one tensor $\mathbf{m} \otimes \mathbf{N}$, where $\mathbf{m}$ and $\mathbf{N}$ are Eulerian and Lagrangian vectors respectively, we can define the *Strong Ellipticity Condition* as

$$\mathrm{tr}[\mathcal{A}(\mathbf{m}\otimes\mathbf{N})(\mathbf{m}\otimes\mathbf{N})] > 0, \tag{2.5.1}$$

and this should hold for all $\mathbf{m}\otimes\mathbf{N} \neq \mathbf{0}$. When the relevant strain-energy function (and thus the constitutive relation) is strongly elliptic, then the equilibrium equations - for global or incremental problems - form a strongly elliptic system of partial differential equations. Note that by relaxing the strict inequality in (2.5.1), we obtain the *Legendre-Hadamard condition* [see, for example, Truesdell and Noll (1965)].

Introduction of the acoustic tensor $\mathbf{Q}(\mathbf{N})$, where

$$Q_{ij} = \mathcal{A}_{\alpha i\beta j}N_\alpha N_\beta \tag{2.5.2}$$

allows (2.5.1) to take the form

$$[\mathbf{Q}(\mathbf{N})\mathbf{m}]\cdot\mathbf{m} > \mathbf{0}. \tag{2.5.3}$$

Thus, necessary and sufficient conditions for strong ellipticity to hold are

$$Q_{ii}(\mathbf{N}) > 0, \qquad i \in \{1,2,3\} \tag{2.5.4}$$

$$Q_{ii}(\mathbf{N})Q_{jj}(\mathbf{N}) - Q_{ij}(\mathbf{N})Q_{ij}(\mathbf{N}) > 0, \qquad i \neq j \in \{1,2,3\} \tag{2.5.5}$$

$$\det\mathbf{Q}(\mathbf{N}) > 0, \tag{2.5.6}$$

for all $\mathbf{N} \neq \mathbf{0}$.

Equivalent conditions can be given in terms of $\mathcal{A}_0$ although, to reduce algebraic complexity, we only record the two-dimensional case. Thus, necessary and sufficient conditions for strong ellipticity to hold in compressible materials when restricted to the (1,2)-plane are

$$\mathcal{A}_{01111} > 0, \quad \mathcal{A}_{02222} > 0, \quad \mathcal{A}_{01212} > 0, \quad \mathcal{A}_{02121} > 0, \tag{2.5.7}$$

$$(\mathcal{A}_{01111}\mathcal{A}_{02222})^{1/2} + (\mathcal{A}_{01212}\mathcal{A}_{02121})^{1/2} \pm (\mathcal{A}_{01122} + \mathcal{A}_{02112}) > 0. \tag{2.5.8}$$

For the incompressible case the corresponding inequalities are

$$\mathcal{A}_{01212} > 0, \quad \mathcal{A}_{02121} > 0, \tag{2.5.9}$$

$$\mathcal{A}_{01111} + \mathcal{A}_{02222} + 2(\mathcal{A}_{01212}\mathcal{A}_{02121})^{1/2} - 2(\mathcal{A}_{01122} + \mathcal{A}_{02112}) > 0. \tag{2.5.10}$$

2.6 Polar coordinate equations

For later use, we note that in polar coordinate form, with restriction to plane strain, the equilibrium equations (2.2.9) become

$$\begin{aligned} \frac{\partial S_{Rr}}{\partial R} + \frac{1}{R}\frac{\partial S_{\Theta r}}{\partial \Theta} + \frac{1}{R}(S_{Rr} - S_{\Theta\theta}) = 0, \\ \frac{\partial S_{R\theta}}{\partial R} + \frac{1}{R}\frac{\partial S_{\Theta\theta}}{\partial \Theta} + \frac{1}{R}(S_{R\theta} + S_{\Theta r}) = 0, \end{aligned} \tag{2.6.1}$$

where the polar coordinates (R, Θ) and (r, θ) correspond to the reference and current configurations respectively. The displacement $\mathbf{u}$, in polar coordinates, is defined to be

$$\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta, \tag{2.6.2}$$

where u and v are functions of r and θ and $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the usual polar coordinate basis vectors. The displacement gradient $\mathbf{\Gamma} \equiv \text{grad}\ \mathbf{u}$ becomes

$$\begin{aligned} & \frac{\partial}{\partial r}(u\mathbf{e}_r + v\mathbf{e}_\theta) \otimes \mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}(u\mathbf{e}_r + v\mathbf{e}_\theta) \otimes \mathbf{e}_\theta \\ = \ & \frac{\partial u}{\partial r}\mathbf{e}_r \otimes \mathbf{e}_r + \frac{1}{r}(\frac{\partial u}{\partial \theta} - v)\mathbf{e}_r \otimes \mathbf{e}_\theta + \frac{\partial v}{\partial r}\mathbf{e}_\theta \otimes \mathbf{e}_r + \frac{1}{r}(\frac{\partial v}{\partial \theta} + u)\mathbf{e}_\theta \otimes \mathbf{e}_\theta. \end{aligned} \tag{2.6.3}$$

Hence, the components of $\mathbf{\Gamma}$, referred to polar coordinates, are

$$\Gamma_{rr} = \frac{\partial u}{\partial r}, \qquad \Gamma_{r\theta} = \frac{1}{r}(\frac{\partial u}{\partial \theta} - v),$$
$$\Gamma_{\theta r} = \frac{\partial v}{\partial r}, \qquad \Gamma_{\theta\theta} = \frac{1}{r}(\frac{\partial v}{\partial \theta} + u). \tag{2.6.4}$$

The incompressibility condition (2.4.13) follows in the form

$$\Gamma_{rr} + \Gamma_{\theta\theta} = \frac{1}{r}(r\frac{\partial u}{\partial r} + u + \frac{\partial v}{\partial \theta}) = 0. \tag{2.6.5}$$

In the (1,2)-plane, the incremental equations of motion (2.4.26), in polar coordinate form, are given by

$$\frac{\partial \dot{S}_{0rr}}{\partial r} + \frac{1}{r}\frac{\partial \dot{S}_{0\theta r}}{\partial \theta} + \frac{1}{r}\left(\dot{S}_{0rr} - \dot{S}_{0\theta\theta}\right) = \rho u_{,tt}\,,$$
$$\frac{\partial \dot{S}_{0r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \dot{S}_{0\theta\theta}}{\partial \theta} + \frac{1}{r}\left(\dot{S}_{0r\theta} + \dot{S}_{0\theta r}\right) = \rho v_{,tt}\,, \tag{2.6.6}$$

and the corresponding incremental equilibrium equations (2.4.27) are

$$\frac{\partial \dot{S}_{0rr}}{\partial r} + \frac{1}{r}\frac{\partial \dot{S}_{0\theta r}}{\partial \theta} + \frac{1}{r}\left(\dot{S}_{0rr} - \dot{S}_{0\theta\theta}\right) = 0,$$
$$\frac{\partial \dot{S}_{0r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \dot{S}_{0\theta\theta}}{\partial \theta} + \frac{1}{r}\left(\dot{S}_{0r\theta} + \dot{S}_{0\theta r}\right) = 0. \tag{2.6.7}$$

Chapter 3

Flexure of a surface-coated elastic block

3.1 Basic surface coating theory

This chapter examines the plane-strain flexure of a rectangular incompressible isotropic elastic block with an elastic surface coating on part of its boundary. In this section, we outline the *surface coating theory* developed by Steigmann and Ogden (1997a) by summarizing the equations and boundary conditions governing the equilibrium of a coated elastic body and the crucial conditions coupling the response of the bulk solid to that of the coating.

We consider the plane deformation of a body whose plane section occupies the two-dimensional region $\mathcal{B}_0$ in its natural (stress-free) configuration and whose image in the current configuration is $\mathcal{B}$. The deformation of the body is then described by (2.1.3). The arclength, S, parametrizes the boundary $\partial\mathcal{B}_0$ of $\mathcal{B}_0$ so that $\mathbf{X}(S) = X_i(S)\mathbf{e}_i$ is taken in the sense that the unit outward normal $\mathbf{N}$ on $\partial\mathcal{B}_0$ has components defined by

$$N_i = e_{ij}\frac{\partial X_j}{\partial S}, \tag{3.1.1}$$

where e_{ij} is the unit alternator ($e_{12} = -e_{21} = 1\,, e_{11} = e_{22} = 0$) with indices i and j taking values in $\{1, 2\}$ and $\mathbf{e}_1$ and $\mathbf{e}_2$ are in-plane Cartesian basis vectors .

We assume that a subset P of $\partial\mathcal{B}_0$ is coated with a thin elastic film that deforms as a material curve. Then, if $\hat{P}$ is the image of P in $\mathcal{B}$, its position vector may be written

$$\mathbf{r}(S) = \chi(\mathbf{X}(S)) \tag{3.1.2}$$

for those S for which $\mathbf{X}(S)$ is on P. The unit tangent $\mathbf{T}(S) \equiv \mathbf{X}'(S)$ to P maps to

$$\mathbf{r}'(S) = \mathbf{A}(\mathbf{X}(S))\mathbf{T}(S) \equiv \lambda(S)\boldsymbol{\tau}(S) \tag{3.1.3}$$

under the deformation, where $\boldsymbol{\tau}(S)$ is the unit tangent to $\hat{P}$ at S and

$$\lambda(S) = |\mathbf{A}(\mathbf{X}(S))\mathbf{X}'(S)| > 0 \tag{3.1.4}$$

is the stretch of P induced by χ. If, on $\hat{P}$, the arclength parameter is denoted by $s(S)$ then

$$s'(S) = \lambda(S). \tag{3.1.5}$$

Let $\vartheta(S)$ be the angle which defines the direction of $\boldsymbol{\tau}(S)$ such that

$$\boldsymbol{\tau}(S) = \cos\vartheta(S)\mathbf{e}_1 + \sin\vartheta(S)\mathbf{e}_2. \tag{3.1.6}$$

Then,

$$\boldsymbol{\nu}(S) = \mathbf{k} \times \boldsymbol{\tau}(S) \tag{3.1.7}$$

is the leftward unit normal to $\hat{P}$, where $\mathbf{k} \equiv \mathbf{e}_1 \times \mathbf{e}_2$ is the unit normal to the plane of $\mathcal{B}_0$ and $\mathcal{B}$. It follows that

$$\boldsymbol{\tau}'(S) = \kappa(S)\boldsymbol{\nu}(S), \quad \mathbf{r}'' = \lambda'(S)\boldsymbol{\tau}(S) + \lambda(S)\kappa(S)\boldsymbol{\nu}(S), \tag{3.1.8}$$

where

$$\kappa(S) = \vartheta'(S), \tag{3.1.9}$$

and the physical curvature of $\hat{P}$ is $\lambda^{-1}\kappa$.

The strain energy per unit area is denoted by $W(\mathbf{A})$ for the bulk solid in $\mathcal{B}_0$. Then the total strain energy of $\mathcal{B}_0$ is

$$\int_{\mathcal{B}_0} W(\mathbf{A}) dA. \tag{3.1.10}$$

For the boundary coating on P, we take a strain energy, B, per unit length. This accounts for the elastic resistence to extension and flexure of P. It suffices to take B to depend on $\mathbf{r}'$ and $\mathbf{r}''$; then the contribution of P to the overall strain energy is

$$\int_P B(\mathbf{r}', \mathbf{r}'')\, dS. \tag{3.1.11}$$

It is necessary that $W(\mathbf{A})$ and $B(\mathbf{r}', \mathbf{r}'')$ be invariant under rigid body motions superimposed upon $\boldsymbol{\chi}$. For W, such objectivity requirements are dealt with in (2.3.13) and for B we require

$$B(\mathbf{r}', \mathbf{r}'') = B(\mathbf{Q}\mathbf{r}', \mathbf{Q}\mathbf{r}'') \tag{3.1.12}$$

for all proper orthogonal $\mathbf{Q}$ with axis $\mathbf{k}$ such that $\mathbf{Qk} = \mathbf{k}$.

With reference to Cauchy's Theorem on isotropic functions [see Tadjbakhsh (1966) for details] (3.1.12) is satisfied if and only if B depends on $\mathbf{r}'$ and $\mathbf{r}''$ through the scalar invariants

$$I_1 = \mathbf{r}' \, . \, \mathbf{r}', \quad I_2 = \mathbf{r}' \times \mathbf{r}'' \, . \, \mathbf{k}, \quad I_3 = \mathbf{r}' \, . \, \mathbf{r}'', \quad I_4 = \mathbf{r}'' \, . \, \mathbf{r}''. \tag{3.1.13}$$

On use of (3.1.3) and (3.1.8), it follows that

$$I_1 = \lambda^2, \quad I_2 = \lambda^2 \kappa, \quad I_3 = \lambda \lambda', \quad I_4 = (\lambda')^2 + \lambda^2 \kappa^2. \tag{3.1.14}$$

Since $I_4 = (I_2^2 + I_3^2)/I_1$, only three of the invariants are independent and I_4 can be excluded from the arguments of B. We simplify the model by ignoring possible strain-gradient effects caused by the dependence of B on $\lambda'(S)$, and consequently, take B to be a function of I_1 and I_2 only. Then, there is no loss of generality in taking B to be a function of λ and κ, i.e.

$$B = B(\lambda, \kappa). \tag{3.1.15}$$

A variational method [see Steigmann and Ogden (1997a)] is used to derive the equilibrium equations. On application of the stationary energy principle we identify

$$F = \frac{\partial B}{\partial \lambda} \equiv B_\lambda \tag{3.1.16}$$

as the tangential component of force on $\hat{P}$, and

$$M = \frac{\partial B}{\partial \kappa} \equiv B_\kappa \tag{3.1.17}$$

as the bending moment associated with the flexure of P. The force acting on a point of P has, in general, both a tangential and a normal component and we write it in the form

$$\mathbf{F} = F\boldsymbol{\tau} + G\boldsymbol{\nu}, \tag{3.1.18}$$

where G is a Lagrange multiplier, analogous to p, which can be interpreted as a workless reaction force arising from the geometrical constraint on $\hat{P}$. This corresponds to suppression of transverse shear strain in the coating being modelled.

Energy considerations [see Steigmann and Ogden (1997b)] demand that if B is independent of κ, then, necessarily,

$$B_\lambda \geq 0, \quad B_{\lambda\lambda} \geq 0, \tag{3.1.19}$$

in an energy-minimizing configuration. On the other hand, when flexural stiffness is present, the Hessian

$$\begin{bmatrix} B_{\lambda\lambda} & B_{\lambda\kappa} \\ B_{\lambda\kappa} & B_{\kappa\kappa} \end{bmatrix}$$

is positive semi-definite in an energy-minimizing configuration and the first inequality in (3.1.19) is not required.

For the bulk material the equilibrium equations, in the absence of body forces are

$$\mathrm{Div}\,\mathbf{S} = 0 \quad \text{in } \mathcal{B}_0, \tag{3.1.20}$$

where Div is the divergence operator in $\mathcal{B}_0$. The boundary conditions are given by

$$\mathbf{x} = \boldsymbol{\xi}(\mathbf{X}) \quad \text{on } \partial\mathcal{B}_0^x, \tag{3.1.21}$$

$$\mathbf{S}^T\mathbf{N} = \mathbf{t}_0(\mathbf{X}) \quad \text{on } \partial\mathcal{B}_0^t, \tag{3.1.22}$$

where $\partial\mathcal{B}_0^x$ and $\partial\mathcal{B}_0^t$ are parts of $\partial\mathcal{B}_0$ on which, respectively, position $\boldsymbol{\xi}$ and *dead load* traction $\mathbf{t}_0$ are prescribed. Additionally, for the coating, we have the equations

$$\mathbf{F}'(S) = \mathbf{S}^T\mathbf{N} \quad \text{on } P, \tag{3.1.23}$$

$$M'(S) + \lambda G = 0 \quad \text{on } P. \tag{3.1.24}$$

Equation (3.1.23) couples the response of the bulk solid to that of the coating while (3.1.24) determines the Lagrange multiplier G. If P has ends, we must include

$$\mathbf{F}|_{\partial P} = \mathbf{0}, \quad \mathbf{M}|_{\partial P} = \mathbf{0}, \tag{3.1.25}$$

where ∂P denotes the end points of P. [See Steigmann and Ogden (1997a) for more details.]

3.2 Flexure of a surface-coated rectangular elastic block

The basic theory of Section 3.1 is now applied to the bending of an incompressible, coated rectangular elastic block, and use of the neo-Hookean strain-energy function illustrates the deformation behaviour of the block.

3.2.1 Bending of a rectangular elastic block

The undeformed block $\mathcal{B}_0$ is defined in Cartesian coordinates by

$$-A \leq X_1 \leq A, \quad -H \leq X_2 \leq H. \tag{3.2.1}$$

The bending deformation is given by

$$r = f(X_1), \quad \theta = g(X_2), \tag{3.2.2}$$

as expressed in Ogden (1984), where, in polar coordinates $(r,\, \theta)$, the deformed block $\mathcal{B}$ is described by

$$f(-A) \leq r \leq f(A), \quad g(-H) \leq \theta \leq g(H). \tag{3.2.3}$$

The (in-plane) deformation gradient takes the form

$$\mathbf{A} = f'(X_1)\mathbf{e}_r \otimes \mathbf{E}_1 + f(X_1)g'(X_2)\mathbf{e}_\theta \otimes \mathbf{E}_2, \tag{3.2.4}$$

which may be written in the (two-dimensional) polar decomposed form (2.1.9) with $\mathbf{R} = \mathbf{e}_r \otimes \mathbf{E}_1 + \mathbf{e}_\theta \otimes \mathbf{E}_2$ and $\mathbf{V} = f'(X_1)\mathbf{e}_r \otimes \mathbf{e}_r + f(X_1)g'(X_2)\mathbf{e}_\theta \otimes \mathbf{e}_\theta$. Then the principal stretches λ_1 and λ_2 are given by

$$\lambda_1 = f'(X_1), \quad \lambda_2 = f(X_1)g'(X_2), \tag{3.2.5}$$

and are associated with the Eulerian principal directions $\mathbf{e}_r$ and $\mathbf{e}_\theta$ respectively.

Since we are considering isotropic elastic materials, the strain-energy per unit area of the considered plane may be written as a symmetric function of the stretches, namely $W(\lambda_1, \lambda_2)$ for the present plane-strain context. With reference to (2.3.8) we may write the nominal stress as

$$\mathbf{S} = t_1\, \mathbf{E}_1 \otimes \mathbf{e}_r + t_2\, \mathbf{E}_2 \otimes \mathbf{e}_\theta, \tag{3.2.6}$$

where

$$\mathbf{e}_r = \cos\theta\, \mathbf{E}_1 + \sin\theta\, \mathbf{E}_2, \quad \mathbf{e}_\theta = -\sin\theta\, \mathbf{E}_1 + \cos\theta\, \mathbf{E}_2, \tag{3.2.7}$$

and t_1 and t_2 denote the principal values of the Biot stress $\mathbf{T}$ given in (2.3.25) such that

$$t_1 = \frac{\partial W}{\partial \lambda_1} - p\lambda_1^{-1}, \quad t_2 = \frac{\partial W}{\partial \lambda_2} - p\lambda_2^{-1}. \tag{3.2.8}$$

Substituting (3.2.7) with (3.2.2) into (3.2.6), reduces the equilibrium equation (3.1.20), in the absence of body forces, to the scalar equations

$$\frac{\partial t_1}{\partial X_1} - t_2 g'(X_2) = 0, \tag{3.2.9}$$

$$\frac{\partial t_2}{\partial X_2} = 0. \tag{3.2.10}$$

Observing that λ_1 is independent of X_2, it follows from (3.2.10) that

$$\frac{\partial t_2}{\partial \lambda_2} f(X_1) g''(X_2) = 0.$$

We make the assumption that $f(X_1) \neq 0$ and $\partial t_2/\partial\lambda_2 \neq 0$. Hence $g''(X_2) = 0$, which, using the fact that the deformation is symmetric about $X_2 = 0$, leads to

$$g(X_2) = \alpha X_2, \tag{3.2.11}$$

where α is a constant of integration. Since the material is incompressible, the constraint (2.3.20) forces $\lambda_1\lambda_2 = 1$ and thus

$$f(X_1) f'(X_1) g'(X_2) = 1, \tag{3.2.12}$$

which, on separation, can be integrated. As a result, the deformation may be written in the explicit form

$$r^2 = \beta + \frac{2X_1}{\alpha}, \quad \theta = \alpha X_2, \tag{3.2.13}$$

where β is another constant. The constants may be prescribed or determined by suitably chosen boundary conditions.

The principal stretches can now be rewritten as

$$\lambda_1 = \lambda^{-1}, \quad \lambda_2 = \lambda = \alpha r, \tag{3.2.14}$$

which defines the notation λ. With reference to (3.2.11), the equilibrium equations then reduce to the single equation

$$\frac{\mathrm{d}t_1}{\mathrm{d}X_1} = \alpha t_2, \tag{3.2.15}$$

which does not involve X_2. Integration of (3.2.15) using (3.2.8) yields

$$t_1 = (\hat{W} + \gamma)\lambda, \tag{3.2.16}$$

where γ is a constant and $\hat{W}(\lambda)$ is defined by

$$\hat{W}(\lambda) = W(\lambda,\, \lambda^{-1}). \tag{3.2.17}$$

On the curved boundaries of the deformed body the traction is defined to be

$$\mathbf{S}^T\mathbf{N} = \pm t_1 \mathbf{e}_r \;\text{ on } X_1 = \pm A. \tag{3.2.18}$$

The moment, $\mathcal{M}$ say, of the normal stresses t_2 on $\theta = \pm\alpha H$ about the origin and the resultant (normal) load, $\mathcal{L}$ say, on $\theta = \pm\alpha H$ are given by

$$\mathcal{M} = \int_{-A}^{A} r t_2 \,\mathrm{d}X_1\, \mathbf{k}, \tag{3.2.19}$$

and

$$\mathcal{L} = \pm \int_{-A}^{A} t_2 \,\mathrm{d}X_1\, \mathbf{e}_\theta = \pm\frac{1}{\alpha}\left[t_1\right]_{-A}^{A} \mathbf{e}_\theta. \tag{3.2.20}$$

3.2.2 Bending of a surface-coated elastic block

Let the boundaries $X_1 = \pm A$ be coated with a homogeneous material associated with the strain energy B. These boundaries are denoted by P^- and P^+, so that $P = P^- \cup P^+$. Parametrization of P is set so that

$$S = H - X_2 \quad \text{on } P^-, \qquad S = H + X_2 \quad \text{on } P^+. \tag{3.2.21}$$

Consequently, for the given problem, since $\vartheta = \theta + \frac{\pi}{2}$ on P^+ and $\vartheta = \theta + \frac{3\pi}{2}$ on P^-, $\kappa = \vartheta'(S)$ yields, with reference to $(3.2.13)_2$

$$\kappa_- = -\theta'(X_2) = -\alpha \quad \text{on } P^-, \qquad \kappa_+ = \theta'(X_2) = \alpha \quad \text{on } P^+. \tag{3.2.22}$$

Let $\lambda = \lambda_-$ on P^- and $\lambda = \lambda_+$ on P^+, where λ is the principal stretch associated with the principal direction $\mathbf{e}_\theta$. In the present context,

$$\mathbf{n} = -\boldsymbol{\nu} = -\mathbf{e}_r,\; \boldsymbol{\tau} = -\mathbf{e}_\theta \quad \text{on } P^-, \qquad \mathbf{n} = -\boldsymbol{\nu} = \mathbf{e}_r,\; \boldsymbol{\tau} = \mathbf{e}_\theta \quad \text{on } P^+. \tag{3.2.23}$$

The kinematics of the deformation ensure that λ and κ are constant on P^- and P^+, and, since the coating material is homogeneous, it follows from (3.1.16) and (3.1.17) that

$$F'(S) = \frac{dB_\lambda}{dS} = 0, \quad M'(S) = \frac{dB_\kappa}{dS} = 0 \tag{3.2.24}$$

and hence, from (3.1.24), that $G = 0$ on each coated boundary. Thus, (3.1.18) reduces to

$$\mathbf{F}(S) = F\boldsymbol{\tau} = B_\lambda \boldsymbol{\tau}, \tag{3.2.25}$$

and hence, on use of $(3.1.8)_1$,

$$\mathbf{F}'(S) = \kappa B_\lambda \boldsymbol{\nu} \quad \text{on } P^- \text{ or } P^+. \tag{3.2.26}$$

Equating (3.1.23) and (3.2.26) with reference to (3.2.18), it follows that

$$t_1 = -\kappa B_\lambda \quad \text{on } P^- \text{ or } P^+, \tag{3.2.27}$$

unlike the corresponding case of an uncoated block, for which t_1 vanishes at the curved boundaries. Boundary conditions coupling the response of the bulk solid to that of the elastic coating are obtained by combining (3.2.16) and (3.2.27) to give

$$t_1 = (\hat{W} + \gamma)\lambda = -\kappa B_\lambda \quad \text{on } P^- \text{ or } P^+. \tag{3.2.28}$$

These conditions, together with (3.2.19) and (3.2.20), provide sufficient information to determine α, β and γ (or $\mathcal{M}$).

3.2.3 Results for the neo-Hookean strain-energy function

The bulk material is characterized by a two-dimensional form of the neo-Hookean strain-energy function, namely

$$\hat{W}(\lambda) = \frac{1}{2}\mu(\lambda^2 + \lambda^{-2} - 2), \tag{3.2.29}$$

μ being the ground-state shear modulus of the material. The extensibility and flexibility of the boundary coating are accounted for by use of the simple energy function

$$B(\lambda,\, \kappa) = \frac{1}{2}m(\lambda - 1)^2 + \frac{1}{2}n\kappa^2 \tag{3.2.30}$$

introduced by Steigmann and Ogden (1997a), where m and n are positive material constants. Substituting (3.2.29) and (3.2.30) into (3.2.28) gives

$$t_1 = \frac{1}{2}\mu(\lambda^{-1} + \lambda^3 - 2\lambda) + \gamma\lambda = -\kappa m(\lambda - 1) \quad \text{on } P^- \text{ or } P^+, \tag{3.2.31}$$

which, using (3.2.22), can be rearranged to obtain

$$\gamma = -\mu\frac{(\lambda_-^2 - 1)^2}{2\lambda_-^2} + \alpha m\frac{(\lambda_- - 1)}{\lambda_-} = -\mu\frac{(\lambda_+^2 - 1)^2}{2\lambda_+^2} - \alpha m\frac{(\lambda_+ - 1)}{\lambda_+}, \tag{3.2.32}$$

so that

$$\mu\lambda_+^2(\lambda_-^2 - 1)^2 - \mu\lambda_-^2(\lambda_+^2 - 1)^2 - 2\alpha m\lambda_+\lambda_-^2(\lambda_+ - 1) - 2\alpha m\lambda_-\lambda_+^2(\lambda_- - 1) = 0,$$

which, on use of the definition $\lambda = \alpha r$ applied at each boundary, leads to

$$\mu\alpha^6(r_-^4 r_+^2 - r_+^4 r_-^2) + \mu\alpha^2(r_+^2 - r_-^2) - 4\alpha^5 m r_+^2 r_-^2 + 2\alpha^4 m(r_+ r_-^2 + r_- r_+^2) = 0.$$

Substitution of (3.2.13) for r provides an expression relating α and β implicitly in the form

$$\begin{aligned} &2\alpha^{\frac{3}{2}} m\{(\beta\alpha{+}2A)^{\frac{1}{2}}(\beta\alpha - 2A) + (\beta\alpha - 2A)^{\frac{1}{2}}(\beta\alpha + 2A)\} \\ &\qquad - 4\alpha^2 m(\beta^2\alpha^2 - 4A^2) + 4A\mu(1 - \beta^2\alpha^4 + 4A^2\alpha^2) = 0. \end{aligned} \tag{3.2.33}$$

Similarly, $\mathcal{M}$ and $\mathcal{L}$ can be expressed in terms of α and β. From (3.2.19), it follows that

$$\mathcal{M} = \frac{1}{\alpha}[rt_1]_{-A}^{A} - \frac{1}{\alpha}\int_{r_-}^{r_+} t_1\, dr.$$

After integration and use of (3.2.31) this becomes

$$\mathcal{M} = -\alpha m({r_+}^2 + {r_-}^2) + m(r_+ + r_-) - \left[\frac{\mu \log r}{2\alpha^2} + \frac{\mu\alpha^2 r^4}{8} - \frac{\mu r^2}{2} + \frac{\gamma r^2}{2}\right]_{r_-}^{r_+}.$$

Finally, substitution from (3.2.13) and (3.2.32) for r and γ plus some further manipulation yields

$$\begin{aligned}\mathcal{M} = {} & \frac{\mu}{4\alpha^2}\log\left[(\beta\alpha - 2A)(\beta\alpha + 2A)^{-1}\right] + \frac{m}{2\alpha^{\frac{1}{2}}}(\beta\alpha + 2A)^{\frac{1}{2}} \\ & + \frac{m}{2\alpha^{\frac{1}{2}}}(\beta\alpha - 2A)^{\frac{1}{2}} - \beta\alpha(m - A\mu).\end{aligned} \tag{3.2.34}$$

Again, following from (3.2.20), on use of (3.2.13) and (3.2.31), we obtain

$$\mathcal{L} = -m\alpha^{\frac{1}{2}}\left[(\beta\alpha + 2A)^{\frac{1}{2}} + (\beta\alpha - 2A)^{\frac{1}{2}} - 2\alpha^{-\frac{1}{2}}\right]. \tag{3.2.35}$$

Now, if α is prescribed, (3.2.33), (3.2.34) and (3.2.35) provide expressions from which β, γ, $\mathcal{L}$ and $\mathcal{M}$ can be determined. Similarly, if one of β, γ, $\mathcal{L}$ or $\mathcal{M}$ is prescribed instead of α, the others are determined. In particular, prescribing α allows us to specify the angle through which the block is being bent.

For the uncoated block, equation (3.2.32) reduces to $\lambda_-\lambda_+ = 1$ and hence β is given explicitly in terms of α through

$$\beta\alpha^2 = (1 + 4A^2\alpha^2)^{\frac{1}{2}}, \tag{3.2.36}$$

in which case $m = 0$, $\mathcal{L} = 0$ and $\mathcal{M}$ simplifies accordingly.

For detailed discussion of the bending problem for an uncoated block we refer to Green and Zerna (1968) and Ogden (1984). The results for a coated block given here are new.

Chapter 4

Bifurcation of a surface-coated elastic block subject to bending

4.1 Incremental surface coating theory

An incremental deformation is now superimposed on the finite deformation described in Chapter 3. The displacement is given by $\mathbf{u} = \dot{\mathbf{x}}$, following the notation of Section 2.4. Taking $\mathbf{u}(S) = \mathbf{u}(\mathbf{X}(S))$ as the restriction of $\mathbf{u}$ to P, then using (3.1.3) and (3.1.6), we obtain

$$\mathbf{u}'(S) = \dot{\lambda}\boldsymbol{\tau} + \lambda a \boldsymbol{\nu}, \tag{4.1.1}$$

where $a(S) = \dot{\vartheta}$ is the incremental rotation of the tangent $\boldsymbol{\tau}$ at the point (3.1.2) on $\hat{P}$.

Recalling Nanson's formula (2.1.7), which takes the form $\lambda\mathbf{n} = \mathbf{A}^{-T}\mathbf{N}$ in the present context, we define the tensor $\boldsymbol{\Sigma}$ by

$$\dot{\mathbf{S}}^T\mathbf{N} = \lambda\boldsymbol{\Sigma}^T\mathbf{n}, \tag{4.1.2}$$

where $\dot{\mathbf{S}}$ is the nominal stress increment and λ is given by (3.1.5) in terms of arclength parameters S and s associated with the boundary normals $\mathbf{N}$ and $\mathbf{n}$ respectively. In linearized form $\boldsymbol{\Sigma}$ is given by

$$\boldsymbol{\Sigma} = \mathcal{A}_0\boldsymbol{\Gamma} + p\boldsymbol{\Gamma} - \dot{p}\mathbf{I}, \tag{4.1.3}$$

where $\mathbf{I}$ is the (two-dimensional) identity tensor and, in terms of (2.4.9), $\boldsymbol{\Gamma} = \operatorname{grad}\mathbf{u}$. Thus, $\boldsymbol{\Sigma}$ is the two-dimensional specialization of $\dot{\mathbf{S}}_0$, as given by (2.4.14). When linearized, the incremental incompressibility condition becomes

$$\operatorname{tr}\boldsymbol{\Gamma} = 0. \tag{4.1.4}$$

The incremental equilibrium equations (2.4.27) are written in the form

$$\operatorname{div}\boldsymbol{\Sigma} = 0 \quad \text{in } \mathcal{B}, \tag{4.1.5}$$

where div is the divergence operator in $\mathcal{B}$.

Incremental coupling equations are obtained by taking increments of (3.1.23) and (3.1.24) and rewriting these equations referred to the current configuration $\mathcal{B}$. Thus, we derive

$$\dot{\mathbf{F}}'(s) = \boldsymbol{\Sigma}^T\mathbf{n} \quad \text{on } \hat{P}, \tag{4.1.6}$$

$$\dot{M}'(s) + \dot{G} + \lambda^{-1}\dot{\lambda}G = 0 \quad \text{on } \hat{P}, \tag{4.1.7}$$

where the prime now indicates differentiation with respect to s, use having been made of $\lambda = ds/dS$ in changing the independent variable from S to s.

In (4.1.6) and (4.1.7) we require

$$\dot{F} = B_{\lambda\lambda}\dot{\lambda} + B_{\lambda\kappa}\dot{\kappa}, \quad \dot{M} = B_{\lambda\kappa}\dot{\lambda} + B_{\kappa\kappa}\dot{\kappa} \tag{4.1.8}$$

and

$$\dot{\mathbf{F}} = \dot{F}\boldsymbol{\tau} + F\dot{\boldsymbol{\tau}} + \dot{G}\boldsymbol{\nu} + G\dot{\boldsymbol{\nu}}, \tag{4.1.9}$$

where $\dot{G}$ can be determined from (4.1.7). For use in (4.1.9), we relate $\dot{\lambda}$ and $\dot{\kappa}$ to $\mathbf{u}(s)$, with reference to (4.1.1) through

$$\lambda^{-1}\dot{\lambda} = \boldsymbol{\tau} \cdot \mathbf{u}'(s), \quad \dot{\kappa} = \lambda a'(s) = \lambda \frac{d}{ds}[\boldsymbol{\nu} \cdot \mathbf{u}'(s)]. \tag{4.1.10}$$

After substitution of $\boldsymbol{\nu}'(s) = -\lambda^{-1}\kappa\boldsymbol{\tau}$, the second expression in (4.1.10) reduces to

$$\dot{\kappa} = \lambda\boldsymbol{\nu} \cdot \mathbf{u}''(s) - \kappa\boldsymbol{\tau} \cdot \mathbf{u}'(s). \tag{4.1.11}$$

In (4.1.9) we also require

$$\dot{\boldsymbol{\tau}} = [\boldsymbol{\nu} \cdot \mathbf{u}'(s)]\boldsymbol{\nu}, \quad \dot{\boldsymbol{\nu}} = -[\boldsymbol{\nu} \cdot \mathbf{u}'(s)]\boldsymbol{\tau}, \tag{4.1.12}$$

which follow from use of $\dot{\boldsymbol{\tau}} = a\boldsymbol{\nu}$ and $\dot{\boldsymbol{\nu}} = \mathbf{k} \times \dot{\boldsymbol{\tau}}$. Now, substitution from (4.1.7), (4.1.10) and (4.1.12), equation (4.1.9) becomes

$$\dot{\mathbf{F}} = \{\dot{F} + M'(\boldsymbol{\nu} \cdot \mathbf{u}')\}\boldsymbol{\tau} + \{F(\boldsymbol{\nu} \cdot \mathbf{u}') - \dot{M}' + \lambda^{-1}\dot{\lambda}M'\}\boldsymbol{\nu}. \tag{4.1.13}$$

Differentiating (4.1.13) with respect to s and making use of $\boldsymbol{\tau}' = \lambda^{-1}\kappa\boldsymbol{\nu}$ and $\boldsymbol{\nu}' = -\lambda^{-1}\kappa\boldsymbol{\tau}$, we obtain

$$\begin{aligned} \dot{\mathbf{F}}'(s) &= \{\dot{F}' + M''(\boldsymbol{\nu} \cdot \mathbf{u}') + M'(\boldsymbol{\nu} \cdot \mathbf{u}'') - \lambda^{-1}\kappa M'(\boldsymbol{\tau} \cdot \mathbf{u}') - \lambda^{-1}\kappa[F(\boldsymbol{\nu} \cdot \mathbf{u}') - \dot{M}' \\ &+ \lambda^{-1}\dot{\lambda}M']\}\boldsymbol{\tau} + \{\lambda^{-1}\kappa[\dot{F} + M'(\boldsymbol{\nu} \cdot \mathbf{u}')] + F'(\boldsymbol{\nu} \cdot \mathbf{u}') + F(\boldsymbol{\nu} \cdot \mathbf{u}'') \\ &- \lambda^{-1}\kappa F(\boldsymbol{\tau} \cdot \mathbf{u}') - \dot{M}'' + \lambda^{-1}\dot{\lambda}M'' + \lambda^{-1}\dot{\lambda}'M' - \lambda^{-2}\lambda'\dot{\lambda}M'\}\boldsymbol{\nu} \end{aligned} \tag{4.1.14}$$

on each coated boundary. It is taken as implicit that the prime indicates differentiation with respect to s.

For the problem considered in Chapter 3, recalling (3.2.24), we have $F' = M' = G = 0$, λ and κ are constant and (4.1.14) reduces to

$$\begin{aligned} \dot{\mathbf{F}}'(s) &= \{\dot{F}' - \lambda^{-1}\kappa F(\boldsymbol{\nu}\cdot\mathbf{u}') + \lambda^{-1}\kappa \dot{M}'\}\boldsymbol{\tau} \\ &+ \{\lambda^{-1}\kappa\dot{F} + F(\boldsymbol{\nu}\cdot\mathbf{u}'') - \lambda^{-1}\kappa F(\boldsymbol{\tau}\cdot\mathbf{u}') - \dot{M}''\}\boldsymbol{\nu}, \end{aligned} \tag{4.1.15}$$

which, on substitution from (4.1.8) and use of (4.1.11), becomes

$$\begin{aligned} \dot{\mathbf{F}}'(s) &= \left\{B_{\lambda\lambda}\dot{\lambda}' + B_{\lambda\kappa}\dot{\kappa}' - \lambda^{-1}\kappa[B_\lambda(\boldsymbol{\nu}\cdot\mathbf{u}') - B_{\lambda\kappa}\dot{\lambda}' - B_{\kappa\kappa}\dot{\kappa}']\right\}\boldsymbol{\tau} \\ &+ \left\{\lambda^{-1}\kappa[B_{\lambda\lambda}\dot{\lambda} + B_{\lambda\kappa}\dot{\kappa}] + \lambda^{-1}B_\lambda\dot{\kappa} - B_{\lambda\kappa}\dot{\lambda}'' - B_{\kappa\kappa}\dot{\kappa}''\right\}\boldsymbol{\nu} \end{aligned} \tag{4.1.16}$$

on each coated boundary.

4.2 Incremental equations for the coated block

4.2.1 Incremental equilibrium equations

The displacement is given by $\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta$ and the components of the displacement gradient $\boldsymbol{\Gamma}$ with respect to the polar coordinate axes are, from (2.6.4),

$$\Gamma_{rr} = u_{,r}\,, \quad \Gamma_{r\theta} = (u_{,\theta} - v)/r, \quad \Gamma_{\theta r} = v_{,r}\,, \quad \Gamma_{\theta\theta} = (u + v_{,\theta})/r, \tag{4.2.1}$$

where $u_{,r} = \partial u/\partial r$, $u_{,\theta} = \partial u/\partial\theta$ etc..

From the incompressibility condition, as in (2.6.5), we have

$$\Gamma_{rr} + \Gamma_{\theta\theta} \equiv (ru_{,r} + u + v_{,\theta})/r = 0. \tag{4.2.2}$$

Use of (4.2.2) in (4.1.3) yields the components of Σ in the form

$$\begin{aligned}
\Sigma_{rr} &= (\mathcal{A}_{01111} - \mathcal{A}_{01122} + p)\Gamma_{rr} - \dot{p}, \\
\Sigma_{r\theta} &= \mathcal{A}_{01212}\Gamma_{\theta r} + (\mathcal{A}_{01212} - \sigma_1)\Gamma_{r\theta}, \\
\Sigma_{\theta r} &= \mathcal{A}_{02121}\Gamma_{r\theta} + (\mathcal{A}_{02121} - \sigma_2)\Gamma_{\theta r}, \\
\Sigma_{\theta\theta} &= -(\mathcal{A}_{02222} - \mathcal{A}_{02211} + p)\Gamma_{rr} - \dot{p}.
\end{aligned} \tag{4.2.3}$$

Using (2.6.7) we may express the incremental equilibrium equations (4.1.5) in the polar coordinate form

$$\frac{\partial \Sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \Sigma_{\theta r}}{\partial \theta} + \frac{1}{r}(\Sigma_{rr} - \Sigma_{\theta\theta}) = 0, \tag{4.2.4}$$

$$\frac{\partial \Sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \Sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r}(\Sigma_{r\theta} + \Sigma_{\theta r}) = 0. \tag{4.2.5}$$

Then substituting (4.2.3) into the constituent parts of (4.2.4) and (4.2.5), on use of (4.2.1) and (4.2.2), we obtain

$$\begin{aligned}
\frac{\partial \Sigma_{rr}}{\partial r} &= (\mathcal{A}'_{01111} - \mathcal{A}'_{01122} + p')u_{,r} + (\mathcal{A}_{01111} - \mathcal{A}_{01122} + p)u_{,rr} - \dot{p}_{,r}, \\
\frac{1}{r}\frac{\partial \Sigma_{\theta r}}{\partial \theta} &= \frac{1}{r}(\mathcal{A}_{02121} - \sigma_2)v_{,r\theta} + \frac{1}{r^2}\mathcal{A}_{02121}(u_{,\theta\theta} - v_{,\theta}), \\
\frac{1}{r}(\Sigma_{rr} - \Sigma_{\theta\theta}) &= \frac{1}{r}(\mathcal{A}_{01111} - 2\mathcal{A}_{01122} + \mathcal{A}_{02222} + 2p)u_{,r}
\end{aligned} \tag{4.2.6}$$

and

$$\begin{aligned}
\frac{\partial \Sigma_{r\theta}}{\partial r} &= \mathcal{A}'_{01212}v_{,r} + \mathcal{A}_{01212}v_{,rr} + \frac{1}{r}(\mathcal{A}'_{01212} - \sigma'_1)(u_{,\theta} - v) \\
&+ \frac{1}{r}(\mathcal{A}_{01212} - \sigma_1)(u_{,\theta r} - v_{,r}) - \frac{1}{r^2}(\mathcal{A}_{01212} - \sigma_1)(u_{,\theta} - v), \\
\frac{1}{r}\frac{\partial \Sigma_{\theta\theta}}{\partial \theta} &= -\frac{1}{r}(\mathcal{A}_{02222} - \mathcal{A}_{01122} + p)u_{,r\theta} - \frac{1}{r}\dot{p}_{,\theta}, \\
\frac{1}{r}(\Sigma_{r\theta} + \Sigma_{\theta r}) &= \frac{1}{r}(\mathcal{A}_{01212} + \mathcal{A}_{02121} - \sigma_2)v_{,r} + \frac{1}{r^2}(\mathcal{A}_{01212} + \mathcal{A}_{02121} - \sigma_1)(u_{,\theta} - v),
\end{aligned} \tag{4.2.7}$$

where a prime denotes differentiation with respect to r.

Now making use of (2.4.17)-(2.4.20) and the fact that $\sigma_1' + (\sigma_1 - \sigma_2)/r = 0$ in substituting (4.2.6) and (4.2.7) into (4.2.4) and (4.2.5), we obtain

$$\begin{aligned} r^2\dot{p}_{,r} &= [r(\mathcal{A}'_{01111} - \mathcal{A}'_{01122} + p') + \mathcal{A}_{01111} + \mathcal{A}_{02222} - 2\mathcal{A}_{01122}]\, ru_{,r} \\ &+ (\mathcal{A}_{01111} - \mathcal{A}_{01122})r^2 u_{,rr} + \mathcal{A}_{02121}(u_{,\theta\theta} - v_{,\theta}) + \mathcal{A}_{02112} r v_{,r\theta}\,, \end{aligned} \tag{4.2.8}$$

$$\begin{aligned} r\dot{p}_{,\theta} &= (r\mathcal{A}'_{01212} + \mathcal{A}_{01212})(rv_{,r} + u_{,\theta} - v) + \mathcal{A}_{01212} r^2 v_{,rr} \\ &+ (\mathcal{A}_{02112} + \mathcal{A}_{01122} - \mathcal{A}_{02222}) r u_{,r\theta}\,, \end{aligned} \tag{4.2.9}$$

where $\mathcal{A}_{0ijkl}$, $i, j, k, l \in \{1, 2\}$, are the components of $\boldsymbol{\mathcal{A}}_0$ relative to the principal axes $\mathbf{e}_r$ and $\mathbf{e}_\theta$ associated with λ_1 and λ_2 respectively and are given by (2.4.12). The prime in (4.2.8) and (4.2.9) indicates differentiation with respect to r, and $\dot{p}_{,r} \equiv \partial\dot{p}/\partial r$, $\dot{p}_{,\theta} \equiv \partial\dot{p}/\partial\theta$.

From the incompressibility condition (4.2.2) we deduce the existence of a function $\psi = \psi(r, \theta)$ such that

$$u = \frac{1}{r}\psi_{,\theta}\,, \quad v = -\psi_{,r}\,, \tag{4.2.10}$$

with the subscripts r and θ denoting partial derivatives. Substitution of (4.2.10) into (4.2.8) and (4.2.9) and elimination of $\dot{p}$ allows the equilibrium equations to be written as a single equation for ψ. To this end, we differentiate (4.2.8) and (4.2.9) with respect to θ and r, respectively, and substitute for (4.2.10) to obtain

$$\begin{aligned} \dot{p}_{,r\theta} &= -\frac{1}{r^2}(\mathcal{A}'_{01111} - \mathcal{A}'_{01122} + p')\psi_{,\theta\theta} + \frac{1}{r}(\mathcal{A}'_{01111} - \mathcal{A}_{01122} + p')\psi_{,\theta\theta r} + \frac{1}{r^3}\mathcal{A}_{01111}\psi_{,\theta\theta} \\ &- \frac{1}{r^2}\mathcal{A}_{01111}\psi_{,\theta\theta r} - \frac{1}{r^3}\mathcal{A}_{02222}\psi_{,\theta\theta} + \frac{1}{r^2}\mathcal{A}_{02222}\psi_{,\theta\theta r} + \frac{1}{r}\mathcal{A}_{01111}\psi_{,\theta\theta rr} \\ &- \frac{1}{r}\mathcal{A}_{01122}\psi_{,\theta\theta rr} + \frac{1}{r^3}\mathcal{A}_{02121}\psi_{,\theta\theta\theta\theta} + \frac{1}{r^2}\mathcal{A}_{02121}\psi_{,r\theta\theta} - \frac{1}{r}\mathcal{A}_{02112}\psi_{,rr\theta\theta} \end{aligned} \tag{4.2.11}$$

and

$$
\begin{aligned}
\dot{p}_{,\theta r} &= -r\mathcal{A}''_{01212}\psi_{,rr} + \frac{1}{r}\mathcal{A}''_{01212}\psi_{,\theta\theta} + \mathcal{A}''_{01212}\psi_{,r} - \mathcal{A}'_{01212}\psi_{,rr} \\
&- 2r\mathcal{A}'_{01212}\psi_{,rrr} - 2\mathcal{A}_{01212}\psi_{,rrr} - \frac{2}{r^3}\mathcal{A}_{01212}\psi_{,\theta\theta} + \frac{1}{r^2}\mathcal{A}_{01212}\psi_{,\theta\theta r} \\
&+ \frac{1}{r}\mathcal{A}'_{01212}\psi_{,\theta\theta r} + \frac{1}{r}\mathcal{A}'_{01212}\psi_{,r} + \frac{1}{r}\mathcal{A}_{01212}\psi_{rr} - \frac{1}{r^2}\mathcal{A}_{01212}\psi_{,r} \\
&- r\mathcal{A}_{01212}\psi_{,rrrr} - \frac{1}{r^2}\mathcal{A}'_{02112}\psi_{,\theta\theta} - \frac{1}{r^2}\mathcal{A}'_{01122}\psi_{,\theta\theta} + \frac{1}{r^2}\mathcal{A}'_{02222}\psi_{,\theta\theta} \\
&+ \frac{1}{r}\mathcal{A}'_{02112}\psi_{,\theta\theta r} + \frac{1}{r}\mathcal{A}'_{01122}\psi_{,\theta\theta r} - \frac{1}{r}\mathcal{A}'_{02222}\psi_{,\theta\theta r} + \frac{2}{r^3}\mathcal{A}_{02112}\psi_{,\theta\theta} \\
&+ \frac{2}{r^3}\mathcal{A}_{01122}\psi_{,\theta\theta} - \frac{2}{r^3}\mathcal{A}_{02222}\psi_{,\theta\theta} - \frac{2}{r^2}\mathcal{A}_{02112}\psi_{\theta\theta r} - \frac{2}{r^2}\mathcal{A}_{01122}\psi_{,\theta\theta r} \\
&+ \frac{2}{r^2}\mathcal{A}_{02222}\psi_{,\theta\theta r} + \frac{1}{r}\mathcal{A}_{02112}\psi_{,\theta\theta rr} + \frac{1}{r}\mathcal{A}_{01122}\psi_{,\theta\theta rr} - \frac{1}{r}\mathcal{A}_{02222}\psi_{,\theta\theta rr}\,.
\end{aligned}
\tag{4.2.12}
$$

Combining (4.2.11) and (4.2.12) in the expression $\dot{p}_{,r\theta} - \dot{p}_{,\theta r} = 0$, with reference to (2.4.17)-(2.4.20), allows the incremental equilibrium equations to take the form

$$
\begin{aligned}
& ar^4\psi_{,rrrr} + 2br^2\psi_{,rr\theta\theta} + c\psi_{,\theta\theta\theta\theta} + 2(ra' + a)r^3\psi_{,rrr} + 2(rb' - b)r\psi_{,r\theta\theta} \\
+ \quad & (r^2a'' + ra' - a)(r^2\psi_{,rr} - r\psi_{,r}) - (2rb' - 2b + r^2a'' + ra' - a - c)\psi_{,\theta\theta} = 0,
\end{aligned}
\tag{4.2.13}
$$

where the prime again indicates differentiation with respect to r, and the notation a, b, c is defined by

$$a = \mathcal{A}_{01212}, \quad c = \mathcal{A}_{02121}, \quad 2b = \mathcal{A}_{01111} + \mathcal{A}_{02222} - 2\mathcal{A}_{01122} - 2\mathcal{A}_{02112}. \tag{4.2.14}$$

Using (2.4.17)-(2.4.20) and (3.2.17), a, b, c can be expressed in terms of the strain-energy function such that

$$c = \lambda^5\hat{W}_\lambda/(\lambda^4 - 1) = \lambda^4 a, \quad 2b + 2a = \lambda^2\hat{W}_{\lambda\lambda}, \tag{4.2.15}$$

and on use of (3.2.14) it follows that

$$2b = r(\lambda^4 - 1)a' + (3\lambda^4 - 1)a. \tag{4.2.16}$$

The strong ellipticity condition (2.5.1) takes the form of the inequalities

$$c > 0 \ (a > 0), \quad b > -\sqrt{ac}, \tag{4.2.17}$$

and in terms of a strain-energy function in the form (3.2.17) these may be expressed as

$$\frac{\hat{W}_\lambda}{\lambda^2 - 1} > 0, \qquad \lambda^2 \hat{W}_{\lambda\lambda} > -\frac{\lambda \hat{W}_\lambda}{\lambda^2 + 1}. \tag{4.2.18}$$

4.2.2 Boundary conditions

Recalling (4.1.6), the boundary conditions on $X_1 = \pm A$ are obtained by combining (4.1.16) with

$$\mathbf{\Sigma}^T \mathbf{n} = \pm(\Sigma_{rr}\mathbf{e}_r + \Sigma_{r\theta}\mathbf{e}_\theta) \tag{4.2.19}$$

to obtain

$$-\Sigma_{rr} = \lambda^{-1}\kappa(B_{\lambda\lambda}\dot{\lambda} + B_{\lambda\kappa}\dot{\kappa}) + \lambda^{-1}B_\lambda\dot{\kappa} - B_{\lambda\kappa}\dot{\lambda}'' - B_{\kappa\kappa}\dot{\kappa}'', \tag{4.2.20}$$

$$\Sigma_{r\theta} = B_{\lambda\lambda}\dot{\lambda}' + B_{\lambda\kappa}\dot{\kappa}' - \lambda^{-1}\kappa\left[B_\lambda(\boldsymbol{\nu}\cdot\mathbf{u}') - B_{\lambda\kappa}\dot{\lambda}' - B_{\kappa\kappa}\dot{\kappa}'\right], \tag{4.2.21}$$

where a prime indicates differentiation with respect to s and each of (4.2.20) and (4.2.21) holds on P^- and P^+.

Given that the boundary conditions involve the components Σ_{rr} and $\Sigma_{r\theta}$ of Σ, expressing them in terms of ψ requires the elimination of $\dot{p}$ from Σ_{rr}, in $(4.2.3)_1$, by forming $\Sigma_{rr,\theta} \equiv \partial\Sigma_{rr}/\partial\theta$ and substitution for $\dot{p}_{,\theta}$ from (4.2.9). This yields

$$r^2\Sigma_{rr,\theta} = (2b + a - \sigma_1)(r\dot{\psi}_{,r\theta\theta} - \dot{\psi}_{,\theta\theta}) + ar^3\dot{\psi}_{,rrr} + (ra' + a)(r^2\dot{\psi}_{,rr} - r\dot{\psi}_{,r} - \dot{\psi}_{,\theta\theta}),$$
(4.2.22)

and, from $(4.2.3)_2$, with reference to $\sigma_1 = \lambda t_1 = -\lambda^{-1}\kappa B_\lambda$ we obtain

$$r^2\Sigma_{r\theta} = a(\dot{\psi}_{,\theta\theta} + r\dot{\psi}_{,r} - r^2\dot{\psi}_{,rr}) + \lambda^{-1}\kappa B_\lambda(\dot{\psi}_{,\theta\theta} + r\dot{\psi}_{,r}). \tag{4.2.23}$$

Expressions for $\dot{\lambda}$, $\dot{\kappa}$ and $\boldsymbol{\nu}\cdot\mathbf{u}'$, required in (4.2.20) and (4.2.21), are obtained as follows. Using the fact that $\partial/\partial s = \kappa\lambda^{-1}\partial/\partial\theta$, differentiation of $\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta$ (expressed in terms of $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$) provides

$$\mathbf{u}'(s) = -\frac{1}{r}\left[(u_{,\theta} - v)\boldsymbol{\nu} - (u + v_{,\theta})\boldsymbol{\tau}\right] \tag{4.2.24}$$

and

$$\mathbf{u}''(s) = -\frac{1}{r^2}[(u_{,\theta\theta} - 2v_{,\theta} - u)\boldsymbol{\nu} - (v_{,\theta\theta} + 2u_{,\theta} - v)\boldsymbol{\tau}]. \tag{4.2.25}$$

Then, making use of (4.2.24) and (4.2.25) in (4.1.10)-(4.1.12), we obtain

$$\dot{\lambda} = -\lambda u_{,r}, \quad \dot{\kappa} = \kappa(v_{,\theta} - u_{,\theta\theta})/r, \tag{4.2.26}$$

$$\boldsymbol{\nu}\cdot\mathbf{u}' = (v - u_{,\theta})/r. \tag{4.2.27}$$

After differentiation of (4.2.20) with respect to θ, we can equate this equation with (4.2.22) and equate (4.2.21) with (4.2.23) to express each of the boundary conditions in terms of ψ in the form

$$
\begin{aligned}
& \lambda^2 B_{\lambda\lambda}(\psi_{,\theta\theta} - r\psi_{,r\theta\theta}) - \lambda\kappa B_{\lambda\kappa}(r\psi_{,r\theta\theta} + 2\psi_{,\theta\theta\theta\theta} - r\psi_{,r\theta\theta\theta\theta}) \\
+ \quad & \kappa^2 B_{\kappa\kappa}(\psi_{,\theta\theta\theta\theta\theta\theta} + r\psi_{,r\theta\theta\theta\theta}) - \lambda B_{\lambda}(\psi_{,\theta\theta\theta\theta} + \psi_{,\theta\theta}) \\
= \quad & \lambda^2\kappa^{-1}\left[(ra' + a)(\psi_{,\theta\theta} + r\psi_{,r} - r^2\psi_{,rr}) + (2b + a)(\psi_{,\theta\theta} - r\psi_{,r\theta\theta}) - ar^3\psi_{,rrr}\right] \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } P^-, P^+.
\end{aligned}
\tag{4.2.28}
$$

$$
\begin{aligned}
(\lambda^2 B_{\lambda\lambda} + \lambda\kappa B_{\lambda\kappa})(\psi_{,\theta\theta} - r\psi_{,r\theta\theta}) - (\lambda\kappa B_{\lambda\kappa} + \kappa^2 B_{\kappa\kappa})(\psi_{,\theta\theta\theta\theta} + r\psi_{,r\theta\theta}) \\
= \lambda^2 a\kappa^{-1}(\psi_{,\theta\theta} + r\psi_{,r} - r^2\psi_{,rr}) \qquad \text{on } P^-, P^+.
\end{aligned}
\tag{4.2.29}
$$

Boundary conditions on $X_2 = \pm H$ are provided by the incremental moment and load

$$
\dot{\mathcal{M}} = \int_{\delta\mathcal{B}_0} (\dot{\mathbf{x}} \times \mathbf{S}^T\mathbf{N} + \ \mathbf{x} \times \dot{\mathbf{S}}^T\mathbf{N})\,\mathrm{d}S, \qquad \dot{\mathcal{L}} = \int_{\delta\mathcal{B}_0} (\dot{\mathbf{S}}^T\mathbf{N})\,\mathrm{d}S, \tag{4.2.30}
$$

which, on substitution of $\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta$, become

$$
\dot{\mathcal{M}} = \int_{r_-}^{r_+} (u\lambda^{-1}t_2 + r\Sigma_{\theta\theta})\,\mathrm{d}r\,\mathbf{k}, \qquad \dot{\mathcal{L}} = \int_{r_-}^{r_+} (\Sigma_{\theta r}\mathbf{e}_r + \Sigma_{\theta\theta}\mathbf{e}_\theta)\,\mathrm{d}r, \tag{4.2.31}
$$

where $r_\pm$ are the values of r on $X_1 = \pm A$. Note that $\dot{\mathcal{L}}$ has both normal and shear components while $\mathcal{L}$ itself, as given by (3.2.20), has no shear component.

4.2.3 Boundary conditions for the uncoated case

For an uncoated block, the incremental equilibrium equation (4.2.13) is unchanged, but the incremental boundary conditions (4.2.28) and (4.2.29) simplify to

$$
\begin{aligned}
& ar^3\psi_{,rrr} - (ra' + a)(\psi_{,\theta\theta} + r\psi_{,r} - r^2\psi_{,rr}) - (2b + a)(\psi_{,\theta\theta} - r\psi_{,r\theta\theta}) = 0 \\
& a(\psi_{,\theta\theta} + r\psi_{,r} - r^2\psi_{,rr}) = 0,
\end{aligned}
\tag{4.2.32}
$$

on each boundary. Since $a \neq 0$ follows from (4.2.17), equations (4.2.32) simplify to

$$ar^3\psi,_{rrr} - (2b+a)(\psi,_{\theta\theta} - r\psi,_{r\theta\theta}) = 0$$

$$\psi,_{\theta\theta} + r\psi,_r - r^2\psi,_{rr} = 0, \tag{4.2.33}$$

on each boundary $r = r_{\pm}$.

4.3 Bifurcation behaviour

We consider separable solutions of the form

$$\psi = \phi(r)\cos q\theta, \tag{4.3.1}$$

where ϕ is a real function of r, and q is a real number to be determined. Substitution into (4.2.13) leads to the following fourth-order differential equation for $\phi(r)$ with coefficients depending on r

$$\begin{aligned} & ar^4\phi'''' + 2(ra'+a)r^3\phi''' + (r^2a'' + ra' - a - 2bq^2)r^2\phi'' \\ - \ & (2rb'q^2 - 2bq^2 + r^2a'' + ra' - a)r\phi' \\ + \ & (2rb' - 2b + r^2a'' + ra' - a - \lambda^4 a + \lambda^4 aq^2)q^2\phi = 0. \end{aligned} \tag{4.3.2}$$

The boundary conditions (4.2.28) and (4.2.29) become

$$\begin{aligned} & \lambda^2\kappa^{-1}ar^3\phi''' + \lambda^2\kappa^{-1}(ra'+a)r^2\phi'' \\ + \ & [\lambda^2 B_{\lambda\lambda}q^2 + \lambda\kappa B_{\lambda\kappa}q^4 + \lambda\kappa B_{\lambda\kappa}q^2 + \kappa^2 B_{\kappa\kappa}q^4 \\ - \ & \lambda^2\kappa^{-1}(ra'+a) - \lambda^2\kappa^{-1}(2b+a)q^2]r\phi' \\ + \ & [\lambda B_{\lambda}q^2 - \lambda B_{\lambda}q^4 - \lambda^2 B_{\lambda\lambda}q^2 - 2\lambda\kappa B_{\lambda\kappa}q^4 \\ - \ & \kappa^2 B_{\kappa\kappa}q^6 + \lambda^2\kappa^{-1}(ra' + 2a + 2b)q^2]\phi = 0, \end{aligned} \tag{4.3.3}$$

on P^-, P^+ and

$$\lambda^2\kappa^{-1}ar^2\phi'' + (\lambda^2 B_{\lambda\lambda}q^2 + 2\lambda\kappa B_{\lambda\kappa}q^2 + \kappa^2 B_{\kappa\kappa}q^2 - \lambda^2\kappa^{-1}a)r\phi'$$

$$+ \quad (\lambda^2\kappa^{-1}aq^2 - \lambda^2 B_{\lambda\lambda}q^2 - \lambda\kappa B_{\lambda\kappa}q^2 - \lambda\kappa B_{\lambda\kappa}q^4 - \kappa^2 B_{\kappa\kappa}q^4)\phi = 0, \quad (4.3.4)$$

on P^-, P^+. When the general solution of (4.3.2) is inserted into (4.3.3) and (4.3.4) we obtain a *bifurcation equation* identifying configurations in which non-trivial incremental solutions are possible. In general, the equation for $\phi(r)$ must be solved numerically.

Firstly, the parameter q may be determined by setting appropriate incremental boundary conditions on the ends of the block. Several sets of boundary conditions are possible and can be expected to yield broadly similar results. For the sake of tractability, we consider zero incremental displacement in the radial direction and vanishing normal incremental traction. Thus, at $\theta = \pm\alpha H$, we have

$$u = 0, \qquad \Sigma_{\theta\theta} = 0 \tag{4.3.5}$$

and, consequently, the incremental moment $(4.2.31)_1$ vanishes. The first condition in (4.3.5) requires

$$\psi_{,\theta} = 0 \quad \text{at} \quad \theta = \pm\alpha H, \tag{4.3.6}$$

and it follows from (4.3.1) that

$$q = \frac{k\pi}{\alpha H} \tag{4.3.7}$$

where k is a (positive) integer. The condition $\Sigma_{\theta\theta} = 0$ is consistent with $u = 0$ since it follows by differentiating along the boundary that $u_{,r} = 0$ and hence, from $(4.2.3)_4$, that $\Sigma_{\theta\theta} = -\dot{p}$ on $\theta = \pm\alpha H$. By integrating (4.2.8) along the boundary and setting the constant of integration to zero we deduce that $\Sigma_{\theta\theta} = 0$.

Since H and k appear in equation (4.3.2) and the boundary conditions (4.3.3) and (4.3.4) only through q, it suffices to set $k = 1$ in (4.3.7). For any given value of H, results for $k > 1$ are then obtained from those for the value H/k on the H-scale. To allow for all possible mode numbers we therefore take the H-scale to run from zero upwards. We note that (4.3.1) corresponds to a solution antisymmetric with respect to θ, while a symmetric solution is obtained if $\cos q\theta$ is replaced by $\sin q\theta$ in (4.3.1), in which case k is replaced by $k-\frac{1}{2}$ in (4.3.7).

Numerical calculations have been carried out in respect of the neo-Hookean strain-energy function, (3.2.29), and for the class of strain-energy functions introduced by Ogden in (1972), namely

$$W(\lambda_1, \lambda_2, \lambda_3) = \sum_{p=1}^{3} \mu_p(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3)/\alpha_p, \qquad (4.3.8)$$

for the bulk material, along with the coating energy function. In the present problem, (4.3.8) is used with the restriction $\lambda_3 = 1$ and with values of the material constants given (in dimensionless form) by

$$\alpha_1 = 1.3, \quad \alpha_2 = 5.0, \quad \alpha_3 = -2.0,$$

$$\hat{\mu}_1 = 1.491, \quad \hat{\mu}_2 = 0.003, \quad \hat{\mu}_3 = -0.0237, \qquad (4.3.9)$$

where $\hat{\mu}_i = \mu_i/\mu$, $i \in \{1, 2, 3\}$, μ being the shear modulus of the material.

For the coating material we use the energy function defined in (3.2.30).

For numerical purposes, we also non-dimensionalize the elastic moduli m, n associated with the coating using the lengthscale A. As discussed in Steigmann and Ogden (1997a) and Ogden and Steigmann (1999), $(n/m)^{\frac{1}{2}}$ may be interpreted as a measure of the effective thickness of the coating in the sense of engineering plate theory [see Timoshenko and Goodier (1951)]. In this way we make the identifications

$$m = \frac{Et_r}{(1-\nu^2)}, \quad n = \frac{Et_r^3}{12(1-\nu^2)}, \tag{4.3.10}$$

where E is Young's Modulus, ν is Poisson's Ratio and t_r is the film thickness in the reference configuration. This implies that

$$\left(\frac{n}{m}\right)^{\frac{1}{2}} = \frac{t_r}{2\sqrt{3}}. \tag{4.3.11}$$

Since m has the dimension of $\mu\times$(length) and n has the dimension of $\mu \times (\text{length})^3$, we rewrite the (scaled) material constants in dimensionless form

$$\hat{m} = \frac{m}{\mu A}, \quad \hat{n} = \frac{n}{\mu A^3} \tag{4.3.12}$$

so that $\hat{n}/\hat{m}$ is of order (coating thickness/A)^2. Given that the theory of surface coating is designed to treat problems in which coating thickness is very small compared with any other lengthscale, we restrict attention to values of $\hat{m}$, $\hat{n}$ such that

$$\frac{\hat{n}}{\hat{m}} \ll 1. \tag{4.3.13}$$

4.3.1 Numerical results

The solution of the system (4.3.2)-(4.3.4) leads to the requirement that a 4×4 determinant should be zero. We solve this numerically using the method of *compound matrices*. This method (see Section 4.4 for details) replaces the fourth-order system and a 4×4 determinantal condition by a sixth-order system of differential equations and a simple algebraic condition, and can be more accurate than the determinantal method.

Calculations have been performed for coating of the same type on both curved boundaries (P^- and P^+), coating on P^- or P^+ only and for the situation where no coating is present.

For numerical purposes, we set $A = 1$. Calculations have been carried out for values of $\hat{m}$ ranging from 0.01 to 10. The qualitative difference between results for different values of $\hat{m}$ is not significant and we therefore illustrate the results mainly for $\hat{m} = 1$. For comparison, results are also shown for $\hat{m} = 0.1$ and $\hat{m} = 5$ to demonstrate the effect of changing the relative shear stiffnesses of the coating and bulk materials. Smaller (larger) values of $\hat{m}$ correspond to relatively soft (stiff) elastic coatings. In this connection we note that by combining $(4.3.12)_1$ and (4.3.10), where $E = 2\mu_c(1+\nu)$, we may deduce that $\hat{m}$ is of order $(\mu_c/\mu) \times$ (coating thickness)$/A$, where μ_c is the shear modulus of the coating material.

Note that in each of the following figures the curves are cut off on the right at a value of H/A for which the block is deformed into a sector of a circular cylinder such that $\alpha H = \pi$, which point is reached before bifurcation occurs.

Results for the neo-Hookean material

In Fig. 4.1, for the neo-Hookean material, the critical value of the stretch λ_- (denoted λ_c) for which bifurcation first occurs is plotted against the aspect ratio H/A of the block for $\hat{n} = 0.001, 0.005, 0.01, 0.015, 0.02, 0.025, 0.03$ separately (solid curves), and for $\hat{n} = 0$, when no bending stiffness is present in the coating (dashed curve), with $\hat{m} = 1$ in each case. These results are compared with that corresponding to the uncoated block (dotted curve).

Clearly, except for small values of H/A, bifurcation is promoted by the presence of a surface coating, i.e. it becomes possible at a value of λ_c closer to unity than in the uncoated case. However, an increase in the bending stiffness mitigates this effect, although, in view of the requirement (4.3.13), caution should be exercised in interpreting the results for the larger values of $\hat{n}$.

In respect of the $k = 1$ mode the dependence of bifurcation on the aspect ratio H/A is apparent from Fig. 4.1. For a given value of H/A, bifurcation can occur in some mode $k > 1$ at a value of the stretch λ_- nearer to 1 than for $k = 1$. This can be seen by reading off the value of the critical stretch at the location H/kA $(k > 1)$ on the H/A axis.

For relatively small values of H/A, bifurcation is, exceptionally, delayed relative to the uncoated case. This is accounted for by the bending stiffness in the coating suppressing bifurcation in the $k = 1$ mode, for which the wavelength is relatively long compared with H.

Figure 4.2, also for the neo-Hookean material, illustrates the results when a surface coating is present on the P^- boundary only (solid curves) and on the P^+ boundary only (dot-dashed curve). Again, comparison is made with the uncoated case (dotted curve).

We note that when the coating is on the P^- boundary only, the bifurcation behaviour is very similar to that observed when both curved boundaries are coated. By contrast, when only the P^+ boundary is coated the bifurcation behaviour is very similar to that for an uncoated block and this result is virtually unaffected by a change in the value of the bending stiffness. The difference is explained by noting that on P^- the coating is in compression and therefore susceptible to Euler-type buckling, while on P^+ the coating is in tension.

In Figs 4.1 and 4.2, results for the case of no bending stiffness in the coating are shown (dashed curves). When $\hat{n} = 0$, the coating behaves in a membrane-like manner and, as illustrated, bifurcation occurs immediately that P^- is placed in compression ($\lambda_c = 1$), the inequality $(3.1.19)_1$ then being violated. This effect is associated with the short wavelength limit corresponding to $k \to \infty$ and would be manifested as wrinkling instability in the absence of bending stiffness.

Results for an Ogden material

For the strain-energy function (4.3.8) with material constants (4.3.12), results corresponding to those in Figs 4.1 and 4.2 are shown in Figs 4.3 and 4.4 respectively. The qualitative features are the same and there are only very small numerical differences.

To illustrate the effect of reducing the value of $\hat{m}$, the analogue of Fig. 4.3 is shown in Fig. 4.5 for $\hat{m} = 0.1$ and with $\hat{n} = 0.0001, 0.0005, 0.001, 0.002, 0.003$. The results are much closer to those for the uncoated case than for $\hat{m} = 1$ except for small values of H/A. Thus, if the shear modulus of the coating is close to that of the block then a coating (with bending stiffness) has little effect on bifurcation.

By contrast, the effect of increasing $\hat{m}$ is demonstrated in Fig.4.6 for $\hat{m} = 5$ and $\hat{n} = 0.005, 0.03, 0.05, 0.07, 0.1$. Clearly, a (relatively) stiff coating has a profound effect on bifurcation of the block: quickening its onset to a large extent.

Conclusions

Overall, the results show that, relative to the uncoated situation, bifurcation is generally advanced by the presence of a surface coating on either or both curved boundaries, that is bifurcation occurs at a smaller strain. This effect is less marked for a coating material whose shear modulus is comparable with that of the block material than for one whose shear modulus is much larger than that of the block. Bifurcation is delayed by an increase in magnitude of the bending stiffness of the coating at fixed values of the other parameters.

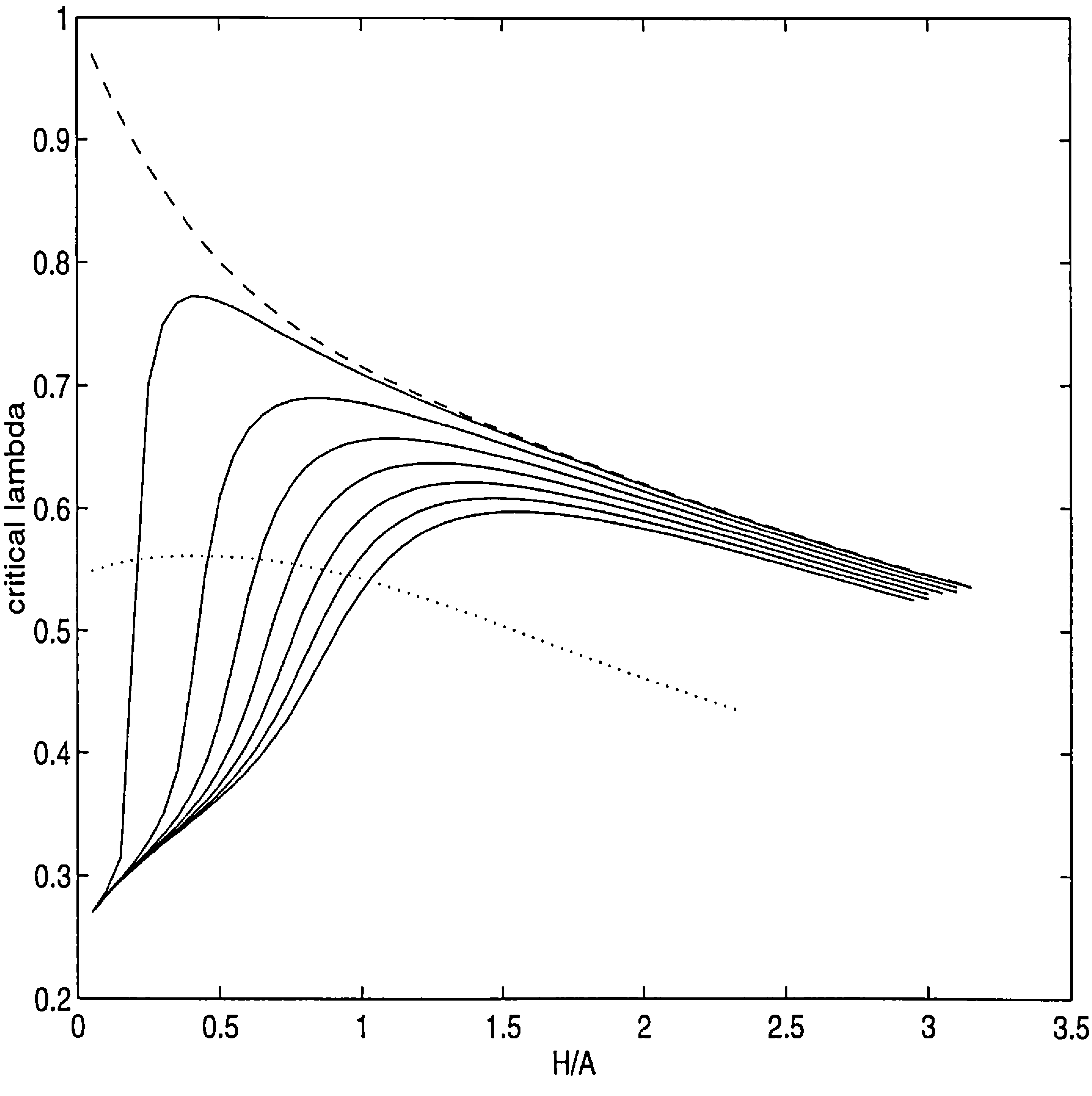

Figure 4.1: Plot of the critical stretch λ_c against H/A for the neo-Hookean material: uncoated block (dotted curve); membrane coating on P^- and P^+ (dashed curve); coating with $\hat{n} = 0.001, 0.005, 0.01, 0.015, 0.02, 0.025, 0.03$ on P^- and P^+ (continuous curves with the maximum decreasing as $\hat{n}$ increases); $\hat{m} = 1$.

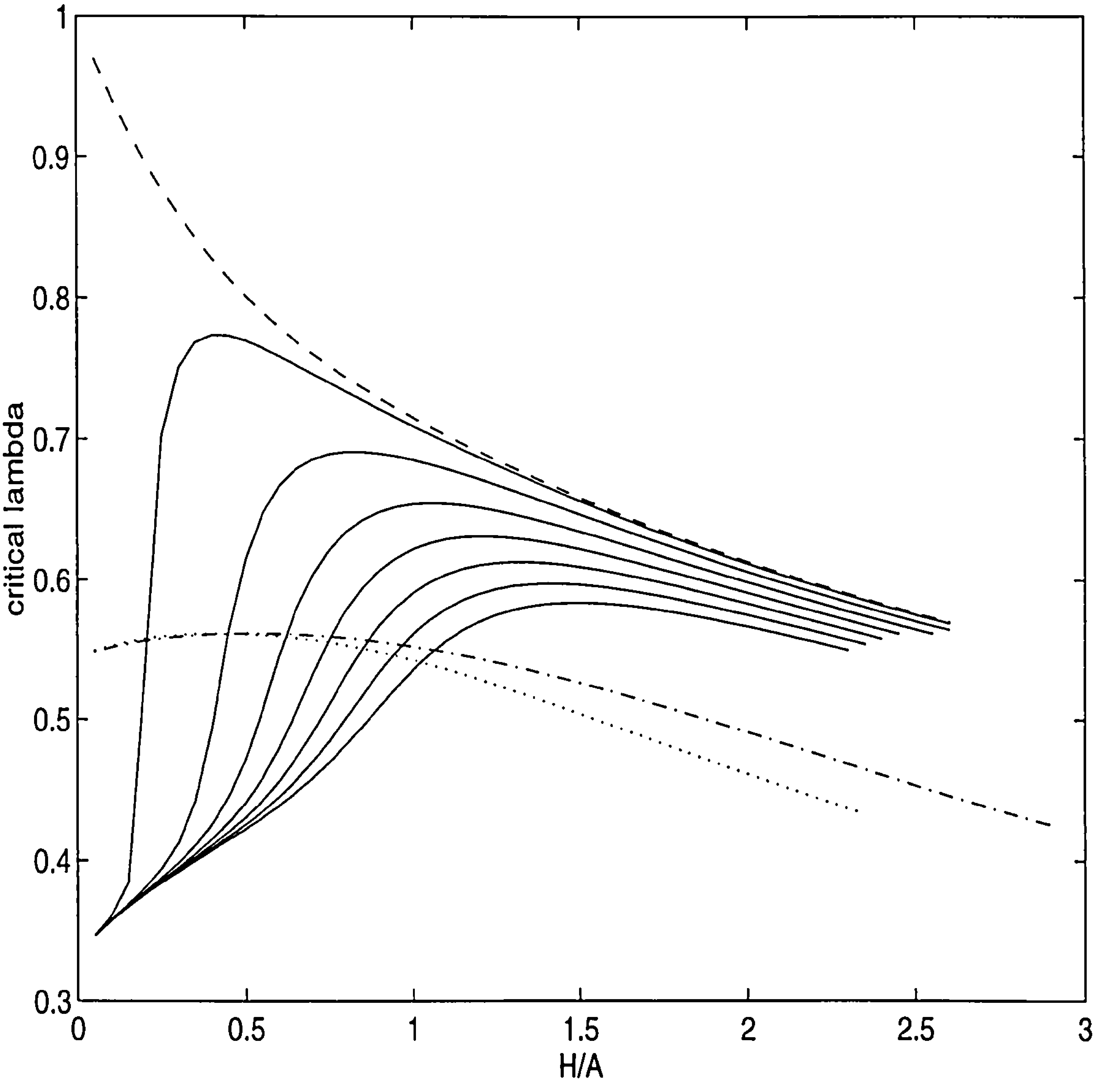

Figure 4.2: Plot of the critical stretch against H/A for the neo-Hookean material: uncoated block (dotted curve); membrane coating on P^- (dashed curve); coating with $\hat{n} = 0.001, 0.005, 0.01, 0.015, 0.02, 0.025, 0.03$ on P^- (continuous curves with the maximum decreasing as $\hat{n}$ increases); coating on P^+ (dot-dashed curve); $\hat{m} = 1$.

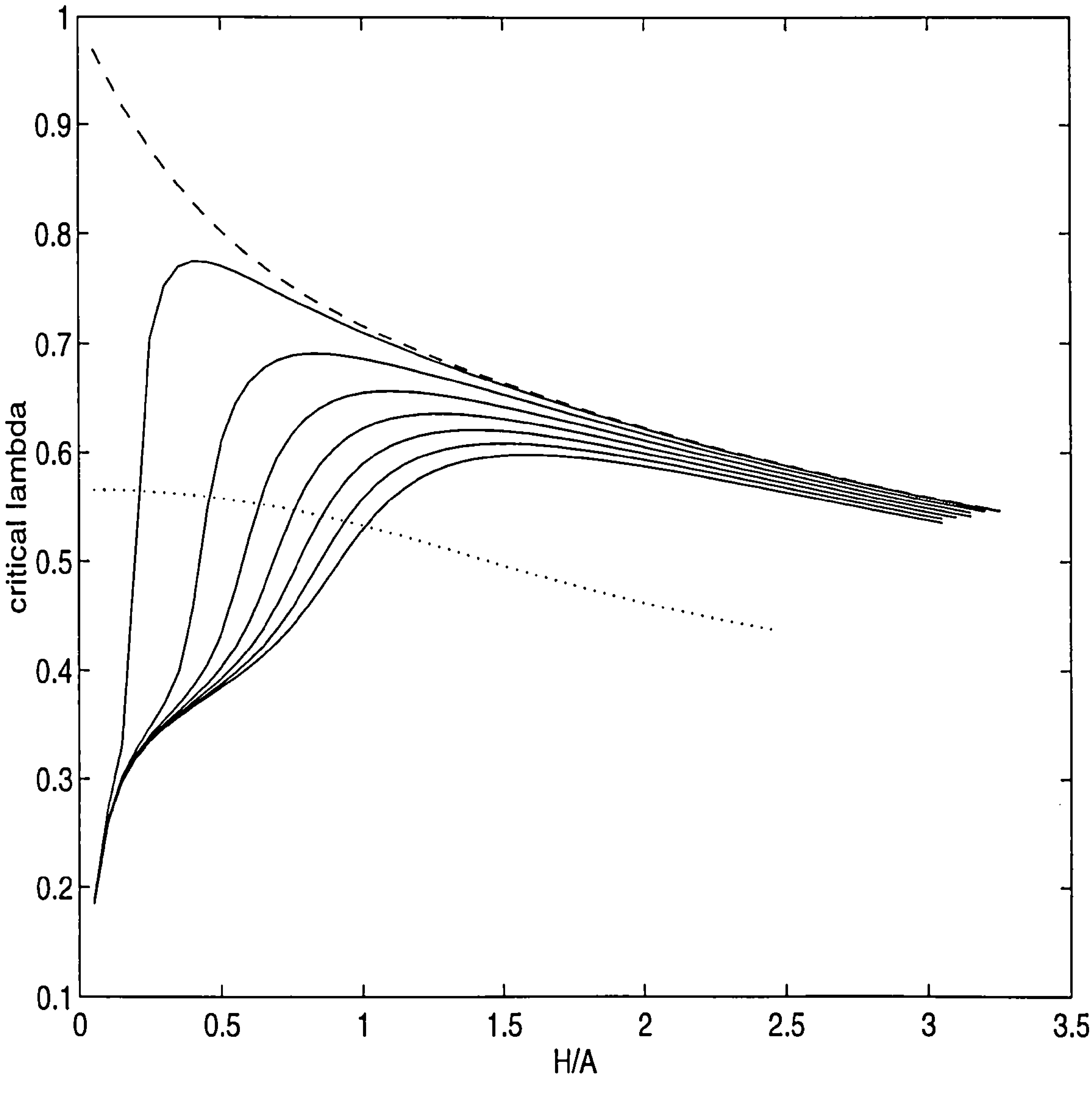

Figure 4.3: Plot of the critical stretch against H/A for an Ogden material: uncoated block (dotted curve); membrane coating on P^- and P^+ (dashed curve); coating with $\hat{n} = 0.001, 0.005, 0.01, 0.015, 0.02, 0.025, 0.03$ on P^- and P^+ (continuous curves with the maximum decreasing as $\hat{n}$ increases); $\hat{m} = 1$.

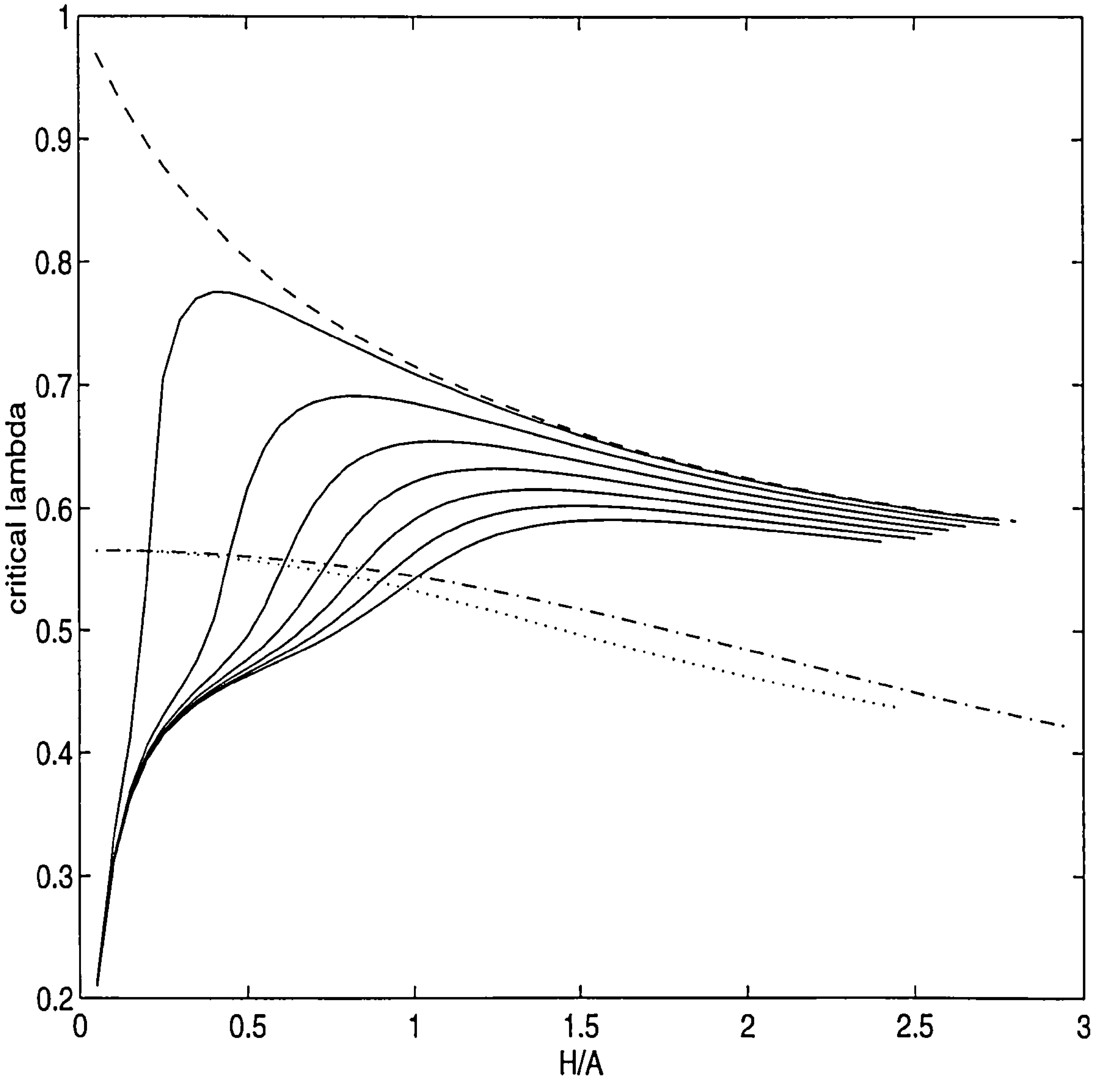

Figure 4.4: Plot of the critical stretch against H/A for an Ogden material: uncoated block (dotted curve); membrane coating on P^- (dashed curve); coating with $\hat{n} = 0.001, 0.005, 0.01, 0.015, 0.02, 0.025, 0.03$ on P^- (continuous curves with the maximum decreasing as $\hat{n}$ increases); coating on P^+ (dot-dashed curve); $\hat{m} = 1$.

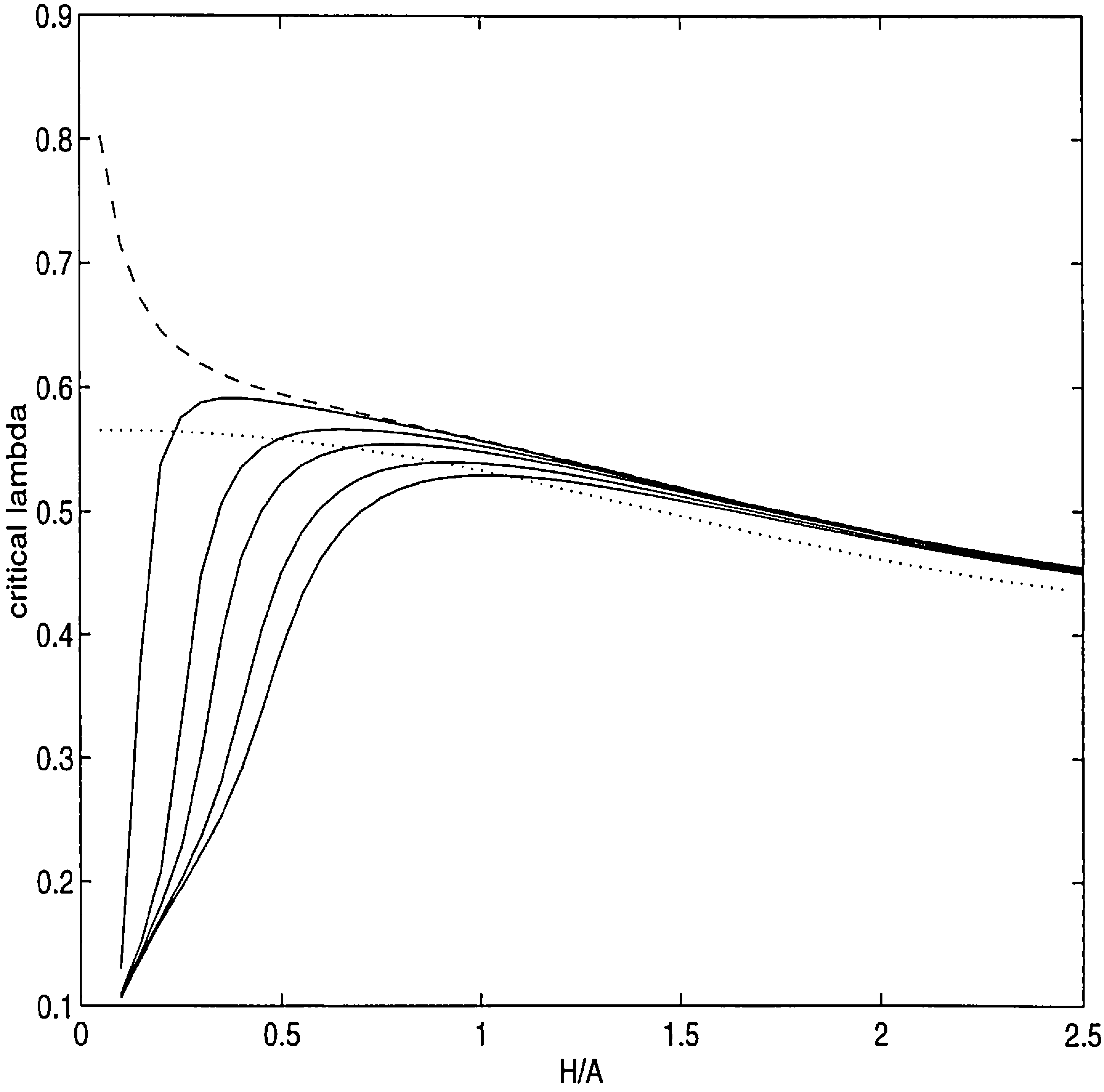

Figure 4.5: Plot of the critical stretch against H/A for an Ogden material: uncoated block (dotted curve); membrane coating on P^- and P^+ (dashed curve); coating with $\hat{n} = 0.0001, 0.0005, 0.001, 0.002, 0.003$ on P^- and P^+ (continuous curves with the maximum decreasing as $\hat{n}$ increases); $\hat{m} = 0.1$.

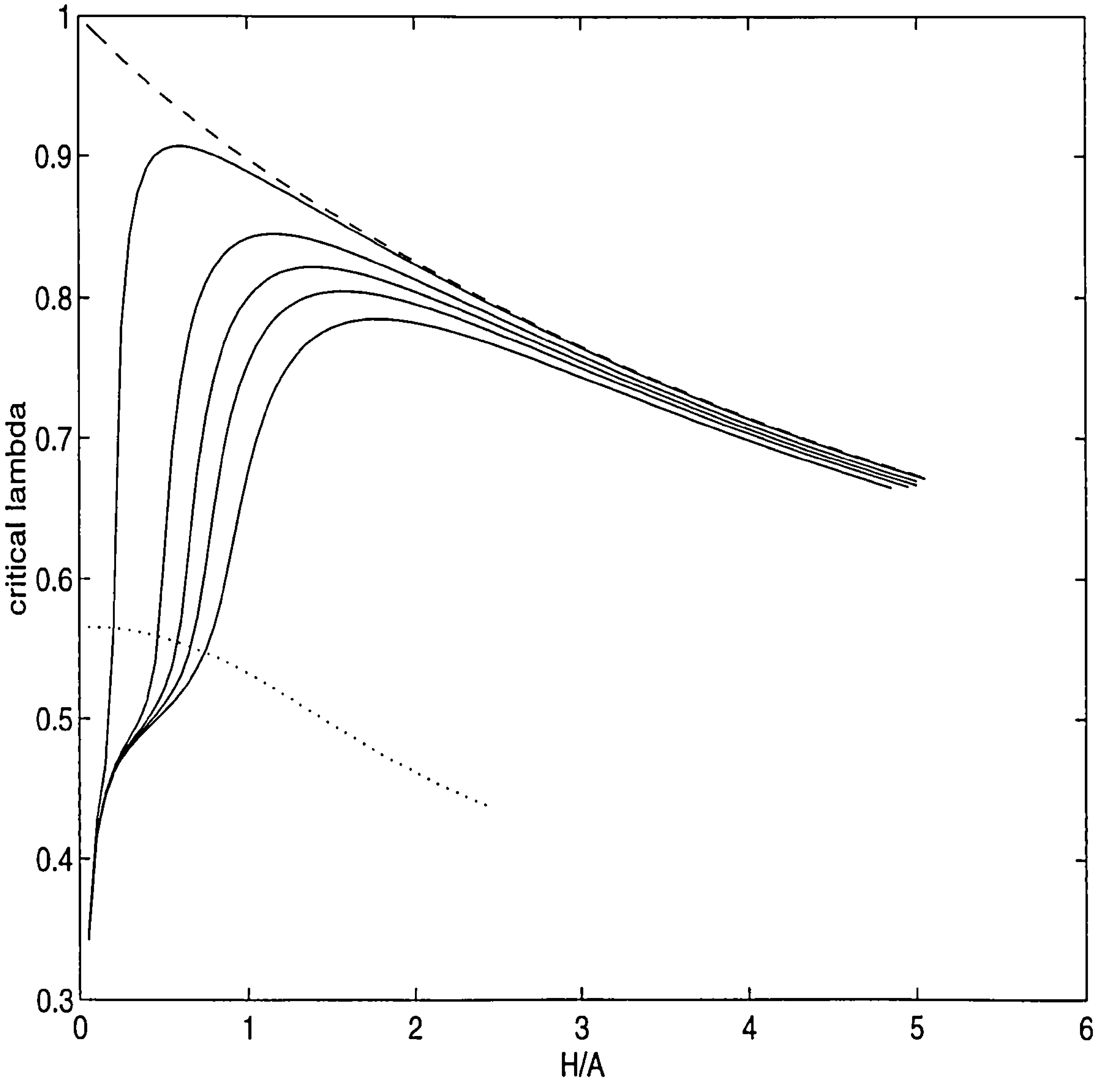

Figure 4.6: Plot of the critical stretch against H/A for an Ogden material: uncoated block (dotted curve); membrane coating on P^- and P^+ (dashed curve); coating with $\hat{n}$ = 0.005, 0.03, 0.05, 0.07, 0.1 on P^- and P^+ (continuous curves with the maximum decreasing as $\hat{n}$ increases); $\hat{m} = 5$.

4.4 The compound matrix method

In this section we describe the compound matrix method as it is used to solve the incremental equilibrium equation (4.3.2) subject to the boundary conditions (4.3.3) and (4.3.4).

A detailed description of the method is given in Lindsay and Rooney (1992) and effectively demonstrated in Haughton and Orr (1997).

First, we set out the problem to be solved.

We choose

$$\mathbf{y} = (\phi, \phi', \phi'', \phi''')^T, \tag{4.4.1}$$

and let

$$\mathbf{y}^{(i)} = (y_1^{(i)}, y_2^{(i)}, y_3^{(i)}, y_4^{(i)})^T = (\phi_i, \phi_i', \phi_i'', \phi_i''')^T, \tag{4.4.2}$$

for $i = \{1, 2\}$.

Then we can rewrite the incremental equilibrium equation (4.3.2) as

$$\mathbf{y}' = \mathbf{A}(r, \alpha)\mathbf{y} \tag{4.4.3}$$

where $\mathbf{A}$ depends on the variable r and the parameter α and the components of matrix $\mathbf{A}$ take the form

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -D & -C & -B & -A \end{bmatrix}$$

and the non-zero components are obtained from (4.3.2). Similarly, the boundary conditions become

$$\mathbf{C}\mathbf{y} = \mathbf{0} \tag{4.4.4}$$

on the P^- and P^+ boundaries, where

$$\mathbf{C} = \begin{bmatrix} G & F & E & 1 \\ K & J & 1 & 0 \end{bmatrix}$$

and the non-zero components of $\mathbf{C}$ are obtained from (4.3.3) and (4.3.4).

Now, consider two independent solutions $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$ to (4.3.2), satisfying the required initial conditions (4.3.3) and (4.3.4) on P^-. Then the general solution to any initial value problem satisfying the two initial conditions is

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} \tag{4.4.5}$$

for arbitrary c_1 and c_2.

Hence, from (4.4.4) using (4.4.2), we need to satisfy

$$(c_1\phi_1''' + c_2\phi_2''') + E(c_1\phi_1'' + c_2\phi_2'') + F(c_1\phi_1' + c_2\phi_2') + G(c_1\phi_1 + c_2\phi_2) = 0 \tag{4.4.6}$$

$$(c_1\phi_1'' + c_2\phi_2'') + J(c_1\phi_1' + c_2\phi_2') + K(c_1\phi_1 + c_2\phi_2) = 0 \tag{4.4.7}$$

on P^+.

This requires that we find c_1 and c_2 such that

$$\begin{bmatrix} \phi_1''' + E\phi_1'' + F\phi_1' + G\phi_1 & \phi_2''' + E\phi_2'' + F\phi_2' + G\phi_2 \\ \phi_1'' + J\phi_1' + K\phi_1 & \phi_2'' + J\phi_2' + K\phi_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0 \tag{4.4.8}$$

on P^+. We need c_1 and c_2 non-zero, which leads to

$$\begin{vmatrix} \phi_1''' + E\phi_1'' + F\phi_1' + G\phi_1 & \phi_2''' + E\phi_2'' + F\phi_2' + G\phi_2 \\ \phi_1'' + J\phi_1' + K\phi_1 & \phi_2'' + J\phi_2' + K\phi_2 \end{vmatrix} = 0 \tag{4.4.9}$$

on P^+.

The direct approach (i.e. the determinantal method) uses the solutions of the initial value problem to construct entries of the determinant in (4.4.9) which is then calculated arithmetically from these entries. Since the parameter, α, is chosen to make the determinant zero, unacceptable levels of inaccuracy may result from numerical cancellation of significant figures.

By contrast, the compound matrix method calculates the determinant indirectly without the need for additional arithmetic, thus avoiding possible rounding-off errors, as we now show.

As this problem involves four equations with two boundary conditions at each end of the interval, we require ${}^4C_2 = 6$ compound variables and, hence, have to solve a system of 6 equations.

The compound variables are then defined, using (4.4.2), as follows

$$z_1 = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} \\ y_2^{(1)} & y_2^{(2)} \end{vmatrix} = \phi_1\phi_2' - \phi_1'\phi_2,$$

$$z_2 = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} \\ y_3^{(1)} & y_3^{(2)} \end{vmatrix} = \phi_1\phi_2'' - \phi_1''\phi_2,$$

and, similarly,

$$\begin{aligned} z_3 &= \phi_1\phi_2''' - \phi_1'''\phi_2, \\ z_4 &= \phi_1'\phi_2'' - \phi_1''\phi_2', \\ z_5 &= \phi_1'\phi_2''' - \phi_1'''\phi_2', \\ z_6 &= \phi_1''\phi_2''' - \phi_1'''\phi_2''. \end{aligned} \tag{4.4.10}$$

After some work, using (4.4.2) and (4.4.3), we obtain

$$\begin{aligned}
z_1' &= z_2, \\
z_2' &= z_4 + z_3, \\
z_3' &= z_5 - Az_3 - Bz_2 - Cz_1, \\
z_4' &= z_5, \\
z_5' &= z_6 - Az_5 - Bz_4 + Dz_1, \\
z_6' &= -Az_6 + Cz_4 + Dz_2,
\end{aligned} \tag{4.4.11}$$

which can be expressed in the form

$$\mathbf{z}' = \mathbf{A}\mathbf{z}. \tag{4.4.12}$$

The initial conditions are obtained directly from the boundary conditions (4.4.4). We can set z_1, on P^- boundary, to be constant, say $z_1(P^-) = 1$, and express all the initial conditions in terms of $z_1(P^-) = y_1^{(1)}y_2^{(2)} - y_1^{(2)}y_2^{(1)}$.

Then, for example,

$$z_2(P^-) = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} \\ y_3^{(1)} & y_3^{(2)} \end{vmatrix} = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} \\ -Jy_2^{(1)} - Ky_1^{(1)} & -Jy_2^{(2)} - Ky_1^{(2)} \end{vmatrix} = -J,$$

and similarly for the other initial conditions. Hence, the completed initial vector $\mathbf{z}(P^-)$ takes the form

$$\mathbf{z}(P^-) = (1,\ -J,\ EJ - F,\ K,\ G - EK,\ FK - JG)^T. \tag{4.4.13}$$

The target condition is obtained using the boundary conditions on the P^+ boundary. As discussed earlier, we require that (4.4.9) holds on P^+. Expanding the determinant in (4.4.9), using (4.4.10), yields the target condition

$$(GJ - FK)z_1 + (G - EK)z_2 - Kz_3 + (F - EJ)z_4 - Jz_5 - z_6 = 0 \tag{4.4.14}$$

on the P^+ boundary.

The compound matrix method then consists of solving the system of six first order equations (4.4.11) subject to the initial conditions (4.4.13). The bifurcation parameter, α, is then adjusted until the target condition (4.4.14) is satisfied. Consequently, the accuracy of the compound matrix method is only dependent on the accuracy obtained in the solution of (4.4.12).

Chapter 5

Vibration of a surface-coated elastic block subject to bending

5.1 Surface coating dynamics

5.1.1 Kinematics and equations of motion

This chapter extends the static case, of Chapters 3 and 4, to allow for the dynamics of the pre-flexed, uncoated and coated elastic blocks considered previously.

Firstly, we set out the plane-strain dynamic theory for surface-coated elastic solids, developed by Ogden and Steigmann (1999). This involves establishing equations of motion for a surface-coated elastic body and including the effects of the inertia of the coating. Equations governing the motion of the film provide boundary conditions for the displacement of the bulk solid. Corresponding governing equations for vibration of an uncoated, pre-stressed, elastic block then emerge as a special case.

The preliminaries required in this dynamic context follow the pattern of Section 3.1, now with the inclusion of time dependence.

As in Chapter 3, we consider the plane deformation of a body whose plane section occupies the two-dimensional region $\mathcal{B}_0$ in its natural (stress-free) configuration and whose image in the current configuration is $\mathcal{B}_t$. The deformation of the body is now described by (2.1.2) such that

$$\mathbf{x} = \chi(\mathbf{X}, t), \quad \chi \in \mathcal{B}_t, \quad t \in \mathcal{I}, \tag{5.1.1}$$

where t is time and $\mathcal{I}$ an appropriate time interval. For each $t \in \mathcal{I}$, χ is invertible and possesses the required regularity properties.

The deformation gradient tensor (2.1.4) now takes the form

$$\mathbf{A} = \text{Grad}\chi(\mathbf{X}, \text{t}). \tag{5.1.2}$$

The arclength, S, describes the boundary $\partial\mathcal{B}_0$ of $\mathcal{B}_0$, as before, so that $\mathbf{X}(S)$ is the parametrization of points on $\partial\mathcal{B}_0$.

We assume that a subset P of $\partial\mathcal{B}_0$ is coated with a thin elastic film that deforms as a material curve. Then, if $\hat{P}$ is the image of P in $\mathcal{B}_t$, its position vector may be written

$$\mathbf{r}(S, t) = \chi(\mathbf{X}(S), t) \tag{5.1.3}$$

for those S for which $\mathbf{X}(S)$ is on P. The unit tangent $\mathbf{T}(S) \equiv \mathbf{X}'(S)$ to P maps to

$$\mathbf{r}'(S, t) = \mathbf{A}(\mathbf{X}(S), t)\mathbf{T}(S) \equiv \lambda(S, t)\boldsymbol{\tau}(S, t) \tag{5.1.4}$$

under the deformation, where $\boldsymbol{\tau}(S, t)$ is the unit tangent to $\hat{P}$ at S,

$$\lambda(S, t) = |\mathbf{A}(\mathbf{X}(S), t)\mathbf{X}'(S)| > 0 \tag{5.1.5}$$

is the stretch of P induced by χ and a prime indicates differentiation with respect to S. If, on $\hat{P}$, the arclength parameter is denoted by $s(S,t)$ then

$$s'(S,t) = \lambda(S,t). \tag{5.1.6}$$

Let $\vartheta(S,t)$ be the angle which defines the direction of $\boldsymbol{\tau}$ such that

$$\boldsymbol{\tau}(S,t) = \cos\vartheta(S,t)\mathbf{e}_1 + \sin\vartheta(S,t)\mathbf{e}_2. \tag{5.1.7}$$

Then,

$$\boldsymbol{\nu}(S,t) = \mathbf{k} \times \boldsymbol{\tau}(S,t) \tag{5.1.8}$$

is the leftward unit normal to $\hat{P}$, where $\mathbf{k} \equiv \mathbf{e}_1 \times \mathbf{e}_2$ is the unit normal to the plane of $\mathcal{B}_0$ and $\mathcal{B}_t$. It follows that

$$\boldsymbol{\tau}'(S,t) = \kappa(S,t)\boldsymbol{\nu}(S,t), \quad \mathbf{r}'' = \lambda'(S,t)\boldsymbol{\tau}(S,t) + \lambda(S,t)\kappa(S,t)\boldsymbol{\nu}(S,t), \tag{5.1.9}$$

where

$$\kappa(S,t) = \vartheta'(S,t) \tag{5.1.10}$$

and the physical curvature of $\hat{P}$ is $\lambda^{-1}\kappa$.

The strain energy per unit area is denoted by $W(\mathbf{A})$ for the bulk solid in $\mathcal{B}_0$. As in Section 3.1, we take the strain energy of the coating, B, per unit length of P to depend only on λ and κ so that

$$B = B(\lambda, \kappa). \tag{5.1.11}$$

As in the static case, variational arguments yield

$$F = \frac{\partial B}{\partial \lambda} \equiv B_\lambda \tag{5.1.12}$$

as the tangential component of force on $\hat{P}$, and

$$M = \frac{\partial B}{\partial \kappa} \equiv B_\kappa \tag{5.1.13}$$

as the bending moment associated with the flexure of P. The force acting on a point of P is again defined by

$$\mathbf{F} = F\boldsymbol{\tau} + G\boldsymbol{\nu}, \tag{5.1.14}$$

where G is a Lagrange multiplier, described in Section 3.1.

In the absence of body forces, the equation of motion (2.2.8), of the bulk material, takes the standard form

$$\mathrm{Div}\,\mathbf{S} = \rho_r \mathbf{x}_{,tt} \quad \text{in } \mathcal{B}_0, \tag{5.1.15}$$

where ρ_r is the mass density per unit area of $\mathcal{B}_0$ and $(\cdot)_{,t}$ represents the material time derivative. The boundary conditions derived for the static case are carried over to the present context. Thus,

$$\mathbf{x} = \boldsymbol{\xi} \quad \text{on } \partial\mathcal{B}_0^x, \tag{5.1.16}$$

$$\mathbf{S}^T\mathbf{N} = \mathbf{t}_0 \quad \text{on } \partial\mathcal{B}_0^t, \tag{5.1.17}$$

where $\partial\mathcal{B}_0^x$ and $\partial\mathcal{B}_0^t$ are parts of $\mathcal{B}_0$ on which, respectively, position $\boldsymbol{\xi}$ and traction $\mathbf{t}_0$ are prescribed. Further boundary conditions are provided by derivation of the equations governing the motion of the surface coating, which we now discuss. In considering the linear momentum balance of the film, we take an arbitrary section $[S_1, S_2] \in P$ where S measures arclength along P and $S_2 > S_1$. We let $\mathbf{F}_1$ and $\mathbf{F}_2$ be forces applied to the ends S_1 and S_2, respectively, of the section, where $\mathbf{F}_2 = \mathbf{F}(S_2, t)$

and $\mathbf{F}(S,t)$, from (5.1.14), is defined to be the force exerted by the material in $(S, S_2]$ on that in $[S_1, S]$. Then, the balance of linear momentum for the film is given by

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{B} = \frac{d}{dt}\int_{S_1}^{S_2} \rho_0 \mathbf{r}_{,t}\, dS, \tag{5.1.18}$$

where ρ_0 in the mass density of the film per unit reference length and

$$\mathbf{B} = -\int_{S_1}^{S_2} \mathbf{S}^T\mathbf{N}\, dS, \tag{5.1.19}$$

the force exerted by the substrate on the film, is equal and opposite to that transmitted to the substrate by the film.

Now, assuming the integrands in (5.1.18) and (5.1.19) are bounded and that $\mathbf{F}(S,t)$ is a continuous function of S, we let the length of the interval tend to zero to obtain $\mathbf{F}(S_1,t) + \mathbf{F}_1 = \mathbf{0}$.

Hence, we can write (5.1.18) in the form

$$\int_{S_1}^{S_2} (\mathbf{F}'(S) - \mathbf{S}^T\mathbf{N} - \rho_0\mathbf{r}_{,tt})\, dS = \mathbf{0}. \tag{5.1.20}$$

Since the interval is arbitrary, (5.1.20) reduces to the local equation of motion

$$\mathbf{F}'(S) = \mathbf{S}^T\mathbf{N} + \rho_0\mathbf{r}_{,tt}. \tag{5.1.21}$$

We also consider the moment-of-momentum balance for the film which we take to be

$$\begin{aligned}(M_1 + M_2)\mathbf{k} + \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \int_{S_1}^{S_2} \mathbf{r} \times (-\mathbf{S}^T\mathbf{N})\, dS \\ = \frac{d}{dt}\int_{S_1}^{S_2} \rho_0 \mathbf{r} \times \mathbf{r}_{,t}\, dS + \mathbf{k}\frac{d}{dt}\int_{S_1}^{S_2} I\vartheta_{,t}\, dS,\end{aligned} \tag{5.1.22}$$

where M_1 and M_2 are the moments applied to the ends of the interval, $\mathbf{k} = \boldsymbol{\tau} \times \boldsymbol{\nu}$ is the (fixed) unit normal to the plane of motion, and $I(S)$ is the *mass moment* *of inertia* per unit reference length. The final integral in (5.1.22) is the rotatory inertia term. We assume that $M_2 = M(S_2, t)$, where $M(S, t)\mathbf{k}$ is the moment exerted by the material in $(S, S_2]$ on that in $[S_1, S]$. This is the value of the constitutive function (5.1.13) at (S, t). With reference to appropriate boundness and continuity assumptions, we obtain $M(S_1, t) + M_1 = 0$, and with our previous results the leading terms on the left-hand side of (5.1.22) may then be combined into the expression

$$\mathbf{k}\int_{S_1}^{S_2} M'\,dS + [\mathbf{r}\times\mathbf{F}]_{S_1}^{S_2}. \tag{5.1.23}$$

The second term in (5.1.23) can be written as the integral of $\mathbf{r}'\times\mathbf{F}+\mathbf{r}\times\mathbf{F}'$ in which $\mathbf{r}'$ and $\mathbf{F}'$ are replaced by (5.1.4) and (5.1.21) respectively and $\mathbf{F}$ is given by (5.1.14). The resulting form of the global balance law is equivalent to the local equation

$$M'(S) + \lambda G = I\vartheta_{,tt} \tag{5.1.24}$$

which effectivley determines, in terms of the motion, the otherwise arbitrary function $G(S, t)$ and, in the present formulation, the inertia coefficient, I, is taken to be independent of the motion.

Consequently, the equations governing the motion of the film are (5.1.21) and (5.1.24) along with (5.1.12), (5.1.13) and (5.1.14). In effect, (5.1.21) couples the film behaviour to that of the bulk material and can be regarded as a boundary condition for the solution of (5.1.15) while (5.1.24) provides an expression for $G(S,\ t)$. [These details can be found in Ogden and Steigmann (1999)].

5.1.2 Incremental equations

As in Chapter 4, an incremental deformation is now superimposed on the initial quasi-static finite deformation, which is described by the equations in Chapter 3 with the time dependence omitted, with (5.1.21) and (5.1.24) replacing (3.1.23) and (3.1.24), in this context. The displacement is now given by

$$\mathbf{u}(\mathbf{x},\, t) = \dot{\chi} = \dot{\mathbf{x}}(\mathbf{X},\, t). \tag{5.1.25}$$

Following from Section 4.1, Nanson's formula again takes the form $\lambda\mathbf{n} = \mathbf{A}^{-T}\mathbf{N}$ and $\mathbf{\Sigma}$ is defined

$$\dot{\mathbf{S}}^{T}\mathbf{N} = \lambda\mathbf{\Sigma}^{T}\mathbf{n}, \tag{5.1.26}$$

where $\dot{\mathbf{S}}$ is the nominal stress increment and λ is given by (5.1.6). Linearized, $\mathbf{\Sigma}$ becomes

$$\mathbf{\Sigma} = \mathcal{A}_0\mathbf{\Gamma} + p\mathbf{\Gamma} - \dot{p}\mathbf{I}, \tag{5.1.27}$$

where $\mathbf{I}$ is the (two-dimensional) identity tensor, $\mathbf{\Gamma}$ is the displacement gradient and the linearized incremental incompressibility condition is

$$\operatorname{tr}\mathbf{\Gamma} = 0. \tag{5.1.28}$$

The incremental equation of motion (2.4.26) can now be written in the form

$$\operatorname{div}\mathbf{\Sigma} = \rho\mathbf{u}_{,tt}\,, \tag{5.1.29}$$

where div is the divergence operator in $\mathcal{B}_t$ and ρ is the density of the bulk material per unit area.

Incremental coupling equations, obtained by taking increments of (5.1.21) and (5.1.24) and updating to the current configuration $\mathcal{B}_t$, provide

$$\dot{\mathbf{F}}'(s,t) = \mathbf{\Sigma}^{T}\mathbf{n} + \lambda^{-1}\rho_0\mathbf{u}_{,tt}\,(s,t), \tag{5.1.30}$$

and

$$\dot{M}'(s,t) + \dot{G} + \lambda^{-1}\dot{\lambda}G = \lambda^{-1}I\dot{\vartheta}_{,tt}(s,t) \tag{5.1.31}$$

where $\mathbf{u} = \mathbf{u}(s,t)$ when evaluated on $\hat{P}$ and the prime now indicates differentiation with respect to s, use having been made of $\lambda = ds/dS$ in changing the independent variable from S to s. The form of the inertia term in (5.1.31) is appropriate whether or not I is strain dependent since the equations are linearized with respect to a fixed configuration of the film.

Similarly to (4.1.10), (4.1.11) and (4.1.12) we derive

$$\lambda^{-1}\dot{\lambda} = \boldsymbol{\tau}\cdot\mathbf{u}'(s,t), \quad \dot{\kappa} = \lambda\boldsymbol{\nu}\cdot\mathbf{u}''(s,t) - \kappa\boldsymbol{\tau}\cdot\mathbf{u}'(s,t), \tag{5.1.32}$$

$$\dot{\boldsymbol{\tau}} = [\boldsymbol{\nu}\cdot\mathbf{u}'(s,t)]\boldsymbol{\nu}, \quad \dot{\boldsymbol{\nu}} = -[\boldsymbol{\nu}\cdot\mathbf{u}'(s,t)]\boldsymbol{\tau}, \tag{5.1.33}$$

as well as

$$\dot{\vartheta} = \boldsymbol{\nu}\cdot\mathbf{u}', \qquad \mathbf{u}' = \lambda^{-1}\dot{\lambda}\boldsymbol{\tau} + (\boldsymbol{\nu}\cdot\mathbf{u}')\boldsymbol{\nu}. \tag{5.1.34}$$

These expressions are required in

$$\dot{F} = B_{\lambda\lambda}\dot{\lambda} + B_{\lambda\kappa}\dot{\kappa}, \quad \dot{M} = B_{\lambda\kappa}\dot{\lambda} + B_{\kappa\kappa}\dot{\kappa} \tag{5.1.35}$$

and

$$\dot{\mathbf{F}} = \dot{F}\boldsymbol{\tau} + F\dot{\boldsymbol{\tau}} + \dot{G}\boldsymbol{\nu} + G\dot{\boldsymbol{\nu}}, \tag{5.1.36}$$

where $\dot{G}$ is determined from (5.1.31). After substitution from (5.1.31) and (5.1.33), we obtain

$$\begin{aligned} \dot{\mathbf{F}} &= \left\{\dot{F} + M'(\boldsymbol{\nu}\cdot\mathbf{u}') - \lambda^{-1}I\vartheta_{,tt}(\boldsymbol{\nu}\cdot\mathbf{u}')\right\}\boldsymbol{\tau} \\ &+ \left\{F(\boldsymbol{\nu}\cdot\mathbf{u}') - \dot{M}' + \lambda^{-1}\dot{\lambda}M' + \lambda^{-1}I\dot{\vartheta}_{,tt} - \lambda^{-2}\dot{\lambda}I\vartheta_{,tt}\right\}\boldsymbol{\nu}. \end{aligned} \tag{5.1.37}$$

Note that (5.1.37) differs from (4.1.13) by inclusion of the rotatory inertia terms. Differentiating with respect to s and making use of $\boldsymbol{\tau}' = \lambda^{-1}\kappa\boldsymbol{\nu}$, $\boldsymbol{\nu}' = -\lambda^{-1}\kappa\boldsymbol{\tau}$, yields

$$
\begin{aligned}
\dot{\mathbf{F}}'(s) &= \{\dot{F}' + M''(\boldsymbol{\nu}\cdot\mathbf{u}') + M'(\boldsymbol{\nu}\cdot\mathbf{u}'') - \lambda^{-1}\kappa M'(\boldsymbol{\tau}\cdot\mathbf{u}') \\
&- \lambda^{-1}\kappa[F(\boldsymbol{\nu}\cdot\mathbf{u}') - \dot{M}' + \lambda^{-1}\dot{\lambda}M'] + \lambda^{-2}\lambda' I\vartheta_{,tt}(\boldsymbol{\nu}\cdot\mathbf{u}') \\
&- \lambda^{-1}I\vartheta'_{,tt}(\boldsymbol{\nu}\cdot\mathbf{u}') + \lambda^{-2}\kappa I\vartheta_{,tt}(\boldsymbol{\tau}\cdot\mathbf{u}') - \lambda^{-1}I\vartheta_{,tt}(\boldsymbol{\nu}\cdot\mathbf{u}'') \\
&- \lambda^{-2}\kappa I\dot{\vartheta}_{,tt} + \lambda^{-3}\kappa\dot{\lambda}I\vartheta_{,tt}\}\boldsymbol{\tau} \\
&+ \{\lambda^{-1}\kappa[\dot{F} + M'(\boldsymbol{\nu}\cdot\mathbf{u}')] + F'(\boldsymbol{\nu}\cdot\mathbf{u}') + F(\boldsymbol{\nu}\cdot\mathbf{u}'') \\
&- \lambda^{-1}\kappa F(\boldsymbol{\tau}\cdot\mathbf{u}') - \dot{M}'' + \lambda^{-1}\dot{\lambda}M'' + \lambda^{-1}\dot{\lambda}'M' - \lambda^{-2}\lambda'\dot{\lambda}M' \\
&- \lambda^{-2}\lambda' I\dot{\vartheta}_{,tt} + \lambda^{-1}I\dot{\vartheta}'_{,tt} + 2\lambda^{-3}\lambda'\dot{\lambda}I\vartheta_{,tt} - \lambda^{-2}\dot{\lambda}'I\vartheta_{,tt} - \lambda^{-2}\dot{\lambda}I\vartheta'_{,tt} \\
&- \lambda^{-2}\kappa I\vartheta_{,tt}(\boldsymbol{\nu}\cdot\mathbf{u}')\}\boldsymbol{\nu} \qquad (5.1.38)
\end{aligned}
$$

on each coated boundary. It is taken as implicit that the prime indicates differentiation with respect to s.

Further substitutions from (5.1.32)-(5.1.34) results in

$$
\begin{aligned}
\dot{\mathbf{F}}'(s) &= \{\dot{F}' + M''\dot{\vartheta} + \lambda^{-1}\dot{\kappa}M' - \lambda^{-2}\kappa\dot{\lambda}M' \\
&- \lambda^{-1}\kappa F\dot{\vartheta} + \lambda^{-1}\kappa\dot{M}' + \lambda^{-2}\lambda' I\vartheta_{,tt}\dot{\vartheta} \\
&- \lambda^{-1}I\vartheta'_{,tt}\dot{\vartheta} - \lambda^{-2}\dot{\kappa}I\vartheta_{,tt} - \lambda^{-2}\kappa I\dot{\vartheta}_{,tt} \\
&+ \lambda^{-3}\kappa\dot{\lambda}I\vartheta_{,tt}\}\boldsymbol{\tau} \\
&+ \{\lambda^{-1}\kappa\dot{F} + \lambda^{-1}\kappa M'\dot{\vartheta} + F'\dot{\vartheta} + \lambda^{-1}\dot{\kappa}F \\
&- \dot{M}'' + \lambda^{-1}\dot{\lambda}M'' + \lambda^{-1}\dot{\lambda}'M' - \lambda^{-2}\lambda'\dot{\lambda}M' \\
&- \lambda^{-2}\lambda' I\dot{\vartheta}_{,tt} + \lambda^{-1}I\dot{\vartheta}'_{,tt} + 2\lambda^{-3}\lambda'\dot{\lambda}I\vartheta_{,tt} \\
&- \lambda^{-2}\dot{\lambda}'I\vartheta_{,tt} - \lambda^{-2}\dot{\lambda}I\vartheta'_{,tt} - \lambda^{-2}\kappa I\vartheta_{,tt}\dot{\vartheta}\}\boldsymbol{\nu}. \qquad (5.1.39)
\end{aligned}
$$

In the present context, λ and κ are constant on the coated boundaries so $\lambda' = 0$ and, recalling (3.2.24), $F' = M' = 0$. Hence

$$\begin{aligned}\dot{\mathbf{F}}'(s) &= \{\dot{F}' - \lambda^{-1}\kappa F\dot{\vartheta} + \lambda^{-1}\kappa\dot{M}' - \lambda^{-1}I\vartheta'_{,tt}\dot{\vartheta}\\ &- \lambda^{-2}\dot{\kappa}I\vartheta_{,tt} - \lambda^{-2}\kappa I\dot{\vartheta}_{,tt} + \lambda^{-3}\kappa\dot{\lambda}I\vartheta_{,tt}\}\boldsymbol{\tau}\\ &+ \{\lambda^{-1}\kappa\dot{F} + \lambda^{-1}\dot{\kappa}F - \dot{M}'' + \lambda^{-1}I\dot{\vartheta}'_{,tt}\\ &- \lambda^{-2}\dot{\lambda}'I\vartheta_{,tt} - \lambda^{-2}\dot{\lambda}I\vartheta'_{,tt} - \lambda^{-2}\kappa I\vartheta_{,tt}\dot{\vartheta}\}\boldsymbol{\nu}\end{aligned} \tag{5.1.40}$$

which on substitution of (5.1.35) becomes

$$\begin{aligned}\dot{\mathbf{F}}'(s) &= \{B_{\lambda\lambda}\dot{\lambda}' + B_{\lambda\kappa}\dot{\kappa}' - \lambda^{-1}\kappa B_{\lambda}\dot{\vartheta} + \lambda^{-1}\kappa B_{\lambda\kappa}\dot{\lambda}' + \lambda^{-1}\kappa B_{\kappa\kappa}\dot{\kappa}'\\ &- \lambda^{-1}I\vartheta'_{,tt}\dot{\vartheta} - \lambda^{-2}\dot{\kappa}I\vartheta_{,tt} - \lambda^{-2}\kappa I\dot{\vartheta}_{,tt} + \lambda^{-3}\kappa\dot{\lambda}I\vartheta_{,tt}\}\boldsymbol{\tau}\\ &+ \{\lambda^{-1}\kappa B_{\lambda\lambda}\dot{\lambda} + \lambda^{-1}\kappa B_{\lambda\kappa}\dot{\kappa} + \lambda^{-1}B_{\lambda}\dot{\kappa} - B_{\lambda\kappa}\dot{\lambda}'' - B_{\kappa\kappa}\dot{\kappa}''\\ &+ \lambda^{-1}I\dot{\vartheta}'_{,tt} - \lambda^{-2}\dot{\lambda}'I\vartheta_{,tt} - \lambda^{-2}\dot{\lambda}I\vartheta'_{,tt} - \lambda^{-2}\kappa I\vartheta_{,tt}\dot{\vartheta}\}\boldsymbol{\nu}\end{aligned} \tag{5.1.41}$$

on each coated boundary. This equation is the dynamic counterpart of (4.1.16) and differs from (4.1.16) by the inclusion of terms in I and ϑ.

5.2 Derivation of frequency equations for a surface-coated elastic block

5.2.1 Incremental equations of motion

As in Chapter 4, the displacement is given by $\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta$, the components of the displacement gradient $\boldsymbol{\Gamma}$ with respect to the polar coordinate axes are

$$\Gamma_{rr} = u_{,r}, \quad \Gamma_{r\theta} = (u_{,\theta} - v)/r, \quad \Gamma_{\theta r} = v_{,r}, \quad \Gamma_{\theta\theta} = (u + v_{,\theta})/r, \tag{5.2.1}$$

and the incompressibility condition is given by

$$\Gamma_{rr} + \Gamma_{\theta\theta} \equiv (ru_{,r} + u + v_{,\theta})/r = 0. \tag{5.2.2}$$

The components of $\boldsymbol{\Sigma}$ are again given by (4.2.3).

The incremental equations of motion (5.1.29) are expressed in polar coordinate form as

$$\frac{\partial \Sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \Sigma_{\theta r}}{\partial \theta} + \frac{1}{r}(\Sigma_{rr} - \Sigma_{\theta\theta}) = \rho u_{,tt}\,, \tag{5.2.3}$$

$$\frac{\partial \Sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \Sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r}(\Sigma_{r\theta} + \Sigma_{\theta r}) = \rho v_{,tt}\,. \tag{5.2.4}$$

Substitution of (4.2.3) into (5.2.3) and (5.2.4) yields

$$\begin{aligned} r^2\dot{p}_{,r} &= [r(\mathcal{A}'_{01111} - \mathcal{A}'_{01122} + p') + \mathcal{A}_{01111} + \mathcal{A}_{02222} - 2\mathcal{A}_{01122}]\, ru_{,r} \\ &+ (\mathcal{A}_{01111} - \mathcal{A}_{01122})r^2u_{,rr} + \mathcal{A}_{02121}(u_{,\theta\theta} - v_{,\theta}) + \mathcal{A}_{02112}rv_{,r\theta} - r^2\rho u_{,tt}\,, \end{aligned} \tag{5.2.5}$$

$$\begin{aligned} r\dot{p}_{,\theta} &= (r\mathcal{A}'_{01212} + \mathcal{A}_{01212})(rv_{,r} + u_{,\theta} - v) + \mathcal{A}_{01212}r^2v_{,rr} \\ &+ (\mathcal{A}_{02112} + \mathcal{A}_{01122} - \mathcal{A}_{02222})ru_{,r\theta} - r^2\rho v_{,tt}\,, \end{aligned} \tag{5.2.6}$$

where $\mathcal{A}_{0ijkl}$, $i, j, k, l \in \{1, 2\}$, are the components of $\boldsymbol{\mathcal{A}}_0$ relative to the principal axes $\mathbf{e}_r$ and $\mathbf{e}_\theta$ associated with λ_1 and λ_2 respectively and the prime in (5.2.5) and (5.2.6) indicates differentiation with respect to r. Equations (5.2.5) and (5.2.6) differ from (4.2.8) and (4.2.9) by inclusion of the terms in ρ.

Substitution of

$$u = \frac{1}{r}\psi_{,\theta}\,, \quad v = -\psi_{,r}\,, \tag{5.2.7}$$

into (5.2.5) and (5.2.6), following differentiation of (5.2.5) and (5.2.6) with respect to θ and r respectively, and then combining the resulting equations in the expression

$\dot{p}_{,r\theta} - \dot{p}_{,\theta r} = 0$, allows the incremental equations of motion to be written as a single equation for ψ. This is

$$
\begin{aligned}
& ar^4\psi_{,rrrr} + 2br^2\psi_{,rr\theta\theta} + c\psi_{,\theta\theta\theta\theta} + 2(ra' + a)r^3\psi_{,rrr} + 2(rb' - b)r\psi_{,r\theta\theta} \\
+ & \ (r^2a'' + ra' - a)(r^2\psi_{,rr} - r\psi_{,r}) - (2rb' - 2b + r^2a'' + ra' - a - c)\psi_{,\theta\theta} \\
- & \ \rho r^2(\psi_{,\theta\theta tt} + r\psi_{,rtt} + r^2\psi_{,rrtt}) = 0, \qquad (5.2.8)
\end{aligned}
$$

where the prime again indicates differentiation with respect to r, and a, b and c are again given by (4.2.14). The strong ellipticity conditions then follow from (4.2.17).

5.2.2 Boundary conditions

The boundary conditions on $X_1 = \pm A$ are obtained by combining (5.1.30) with $\mathbf{\Sigma}^T\mathbf{n} = \pm(\Sigma_{rr}\mathbf{e}_r + \Sigma_{r\theta}\mathbf{e}_\theta)$ to obtain

$$
\begin{aligned}
\dot{\mathbf{F}}'(s) &= -(\Sigma_{rr} + \lambda^{-1}\rho_0 u_{,tt})\boldsymbol{\nu} + (\Sigma_{r\theta} + \lambda^{-1}\rho_0 v_{,tt})\boldsymbol{\tau} \quad \text{on } P^+ \\
& \quad -(\Sigma_{rr} - \lambda^{-1}\rho_0 u_{,tt})\boldsymbol{\nu} + (\Sigma_{r\theta} - \lambda^{-1}\rho_0 v_{,tt})\boldsymbol{\tau} \quad \text{on } P^- \qquad (5.2.9)
\end{aligned}
$$

where a prime indicates differentiation with respect to s. Since $\vartheta = \theta + \frac{\pi}{2}$ on P^+ and $\vartheta = \theta + \frac{3\pi}{2}$ on P^-, it follows that $\dot{\vartheta} = \dot{\theta}$ and since the underlying deformation is static, $\vartheta_{,tt} = 0$. Then (5.1.41) can be rewritten as

$$
\begin{aligned}
\dot{\mathbf{F}}'(s) &= \{B_{\lambda\lambda}\dot{\lambda}' + B_{\lambda\kappa}\dot{\kappa}' - \lambda^{-1}\kappa B_\lambda\dot{\theta} + \lambda^{-1}\kappa B_{\lambda\kappa}\dot{\lambda}' \\
&+ \ \lambda^{-1}\kappa B_{\kappa\kappa}\dot{\kappa}' - \lambda^{-2}\kappa I\dot{\theta}_{,tt}\}\boldsymbol{\tau} \\
&+ \ \{\lambda^{-1}\kappa B_{\lambda\lambda}\dot{\lambda} + \lambda^{-1}\kappa B_{\lambda\kappa}\dot{\kappa} + \lambda^{-1}B_\lambda\dot{\kappa} \\
&- \ B_{\lambda\kappa}\dot{\lambda}'' - B_{\kappa\kappa}\dot{\kappa}'' + \lambda^{-1}I\dot{\theta}'_{,tt}\}\boldsymbol{\nu} \qquad (5.2.10)
\end{aligned}
$$

on each coated boundary. Now, equating (5.2.10) with (5.2.9) yields

$$
\begin{aligned}
-(\Sigma_{rr} + \lambda^{-1}\rho_0 u_{,tt}) &= \{\lambda^{-1}\kappa B_{\lambda\lambda}\dot{\lambda} + \lambda^{-1}\kappa B_{\lambda\kappa}\dot{\kappa} + \lambda^{-1}B_{\lambda}\dot{\kappa} \\
&- B_{\lambda\kappa}\dot{\lambda}'' - B_{\kappa\kappa}\dot{\kappa}'' + \lambda^{-1}I\dot{\theta}'_{,tt}\}\boldsymbol{\nu} \qquad (5.2.11)
\end{aligned}
$$

$$
\begin{aligned}
(\Sigma_{r\theta} + \lambda^{-1}\rho_0 v_{,tt}) &= \{B_{\lambda\lambda}\dot{\lambda}' + B_{\lambda\kappa}\dot{\kappa}' - \lambda^{-1}\kappa B_{\lambda}\dot{\theta} + \lambda^{-1}\kappa B_{\lambda\kappa}\dot{\lambda}' \\
&+ \lambda^{-1}\kappa B_{\kappa\kappa}\dot{\kappa}' - \lambda^{-2}\kappa I\dot{\theta}_{,tt}\}\boldsymbol{\tau} \qquad (5.2.12)
\end{aligned}
$$

which hold on P^+, and

$$
\begin{aligned}
-(\Sigma_{rr} - \lambda^{-1}\rho_0 u_{tt}) &= \{\lambda^{-1}\kappa B_{\lambda\lambda}\dot{\lambda} + \lambda^{-1}\kappa B_{\lambda\kappa}\dot{\kappa} + \lambda^{-1}B_{\lambda}\dot{\kappa} \\
&- B_{\lambda\kappa}\dot{\lambda}'' - B_{\kappa\kappa}\dot{\kappa}'' + \lambda^{-1}I\dot{\theta}'_{,tt}\}\boldsymbol{\nu} \qquad (5.2.13)
\end{aligned}
$$

$$
\begin{aligned}
(\Sigma_{r\theta} - \lambda^{-1}\rho_0 v_{,tt}) &= \{B_{\lambda\lambda}\dot{\lambda}' + B_{\lambda\kappa}\dot{\kappa}' - \lambda^{-1}\kappa B_{\lambda}\dot{\theta} + \lambda^{-1}\kappa B_{\lambda\kappa}\dot{\lambda}' \\
&+ \lambda^{-1}\kappa B_{\kappa\kappa}\dot{\kappa}' - \lambda^{-2}\kappa I\dot{\theta}_{,tt}\}\boldsymbol{\tau} \qquad (5.2.14)
\end{aligned}
$$

which hold on P^-.

Given that the boundary conditions involve the components Σ_{rr} and $\Sigma_{r\theta}$ of $\boldsymbol{\Sigma}$, expressing them in terms of ψ requires the elimination of $\dot{p}$ from Σ_{rr}, using $(4.2.3)_1$, by forming $\Sigma_{rr,\theta}$ and substitution for $\dot{p}_{,\theta}$ from (5.2.6), as in Chapter 4. This yields

$$
\begin{aligned}
r^2\Sigma_{rr,\theta} &= (2b + a + \lambda^{-1}\kappa B_{\lambda})(r\psi_{,r\theta\theta} - \psi_{,\theta\theta}) + ar^3\psi_{,rrr} \\
&+ (ra' + a)(r^2\psi_{,rr} - r\psi_{,r} - \psi_{,\theta\theta}) - r^3\rho\psi_{,rtt}, \qquad (5.2.15)
\end{aligned}
$$

and, as in the static case (4.2.23), we also obtain

$$
r^2\Sigma_{r\theta} = a(\psi_{,\theta\theta} + r\psi_{,r} - r^2\psi_{,rr}) + \lambda^{-1}\kappa B_{\lambda}(r\psi_{,r} + \psi_{,\theta\theta}). \qquad (5.2.16)
$$

Following differentiation of (5.2.11) and (5.2.13) with respect to θ we substitute (5.2.15) into (5.2.11) and (5.2.13) and substitute (5.2.16) into (5.2.12) and (5.2.14).

Hence, on use of (5.2.7), we can express each of the boundary conditions in terms of ψ. It is necessary to make use of the results

$$\dot{\lambda} = -\lambda u_{,r}\,, \quad \dot{\kappa} = \kappa(v_{,\theta} - u_{,\theta\theta})/r, \tag{5.2.17}$$

$$\dot{\theta} = (v - u_{,\theta})/r. \tag{5.2.18}$$

Equations (5.2.11)-(5.2.14) now yield

$$\begin{aligned} & \lambda^2 B_{\lambda\lambda}(\psi_{,\theta\theta} - r\psi_{,r\theta\theta}) - \lambda\kappa B_{\lambda\kappa}(r\psi_{,r\theta\theta} + 2\psi_{,\theta\theta\theta\theta} - r\psi_{,r\theta\theta\theta\theta}) \\ + \;& \kappa^2 B_{\kappa\kappa}(\psi_{,\theta\theta\theta\theta\theta\theta} + r\psi_{,r\theta\theta\theta\theta}) - \lambda B_{\lambda}(\psi_{,\theta\theta\theta\theta} + \psi_{,\theta\theta}) - I(r\psi_{,\theta\theta tt} + \psi_{,\theta\theta\theta\theta tt}) \\ = \;& \lambda^2\kappa^{-1}\{(ra' + a)(\psi_{,\theta\theta} + r\psi_{,r} - r^2\psi_{,rr}) + (2b + a)(\psi_{,\theta\theta} - r\psi_{,r\theta\theta}) - ar^3\psi_{,rrr} \\ + \;& r^3\rho\psi_{,rtt} - \lambda^{-1}r\rho_0\psi_{,\theta\theta tt}\} \end{aligned} \tag{5.2.19}$$

$$\begin{aligned} & (\lambda^2 B_{\lambda\lambda} + \lambda\kappa B_{\lambda\kappa})(\psi_{,\theta\theta} - r\psi_{,r\theta\theta}) - (\lambda\kappa B_{\lambda\kappa} + \kappa^2 B_{\kappa\kappa})(\psi_{,\theta\theta\theta\theta} + r\psi_{,r\theta\theta}) \\ + \;& I(r\psi_{,rtt} + \psi_{,\theta\theta tt}) = \lambda^2 a\kappa^{-1}(\psi_{,\theta\theta} + r\psi_{,r} - r^2\psi_{,rr}) - \lambda\kappa^{-1}r^2\rho_0\psi_{,rtt} \end{aligned} \tag{5.2.20}$$

on the coated boundary P^+, and

$$\begin{aligned} & \lambda^2 B_{\lambda\lambda}(\psi_{,\theta\theta} - r\psi_{,r\theta\theta}) - \lambda\kappa B_{\lambda\kappa}(r\psi_{,r\theta\theta} + 2\psi_{,\theta\theta\theta\theta} - r\psi_{,r\theta\theta\theta\theta}) \\ + \;& \kappa^2 B_{\kappa\kappa}(\psi_{,\theta\theta\theta\theta\theta\theta} + r\psi_{,r\theta\theta\theta\theta}) - \lambda B_{\lambda}(\psi_{,\theta\theta\theta\theta} + \psi_{,\theta\theta}) - I(r\psi_{,\theta\theta tt} + \psi_{,\theta\theta\theta\theta tt}) \\ = \;& \lambda^2\kappa^{-1}\{(ra' + a)(\psi_{,\theta\theta} + r\psi_{,r} - r^2\psi_{,rr}) + (2b + a)(\psi_{,\theta\theta} - r\psi_{,r\theta\theta}) - ar^3\psi_{,rrr} \\ + \;& r^3\rho\psi_{,rtt} + \lambda^{-1}r\rho_0\psi_{,\theta\theta tt}\} \end{aligned} \tag{5.2.21}$$

$$\begin{aligned} & (\lambda^2 B_{\lambda\lambda} + \lambda\kappa B_{\lambda\kappa})(\psi_{,\theta\theta} - r\psi_{,r\theta\theta}) - (\lambda\kappa B_{\lambda\kappa} + \kappa^2 B_{\kappa\kappa})(\psi_{,\theta\theta\theta\theta} + r\psi_{,r\theta\theta}) \\ + \;& I(r\psi_{,rtt} + \psi_{,\theta\theta tt}) = \lambda^2 a\kappa^{-1}(\psi_{,\theta\theta} + r\psi_{,r} - r^2\psi_{,rr}) + \lambda\kappa^{-1}r^2\rho_0\psi_{,rtt} \end{aligned} \tag{5.2.22}$$

on the coated boundary P^-.

In the above boundary conditions, we take the moment of inertia I (per unit length in P) to be that of a rod about the axis $\mathbf{k}$ through its centre. Thus I has the standard form

$$I = \frac{\rho_0 t_d^2}{12} \tag{5.2.23}$$

where t_d is the deformed coating thickness. Let $\rho_0 = \rho_c t_r$, where ρ_c is the mass density of the film per unit reference area and t_r is the film thickness in the reference configuration. Then, I may also be written

$$I = \frac{\rho_c t_d^2 t_r}{12}. \tag{5.2.24}$$

The boundary conditions $u = \Sigma_{\theta\theta} = 0$ on $\theta = \pm\alpha H$ are unaffected by the inclusion of time dependence, as are the expressions (4.2.31) for the incremental moment and load.

5.2.3 Boundary conditions for the uncoated case

For the case of vibration of an uncoated block, the incremental equation of motion (5.2.8) is unchanged, but the incremental boundary conditions (5.2.19)-(5.2.22) become

$$\begin{aligned} & ar^3\psi_{,rrr} - (2b+a)(\psi_{,\theta\theta} - r\psi_{,r\theta\theta}) - r^3\rho\psi_{,rtt} = 0 \\ & \psi_{,\theta\theta} + r\psi_{,r} - r^2\psi_{,rr} = 0 \end{aligned} \tag{5.2.25}$$

on each boundary $r = r_\pm$, similarly to (4.2.33) but now with a ρ term in $(5.2.25)_1$.

Chapter 6

Dynamic behaviour

6.1 Frequency equations

We consider separable solutions of the form

$$\psi = \phi(r)\cos q\theta e^{i\omega t}, \tag{6.1.1}$$

where ϕ is a real function of r, and q is a real number to be determined. Substitution into (5.2.8) leads to the following fourth-order differential equation for $\phi(r)$ with coefficients depending on r

$$\begin{aligned} & ar^4\phi'''' + 2(ra' + a)r^3\phi''' + (r^2a'' + ra' - a - 2bq^2 + r^2\rho\omega^2)r^2\phi'' \\ - & (2rb'q^2 - 2bq^2 + r^2a'' + ra' - a - r^2\rho\omega^2)r\phi' \\ + & (2rb' - 2b + r^2a'' + ra' - a - \lambda^4 a + \lambda^4 aq^2 - r^2\rho\omega^2)q^2\phi = 0. \end{aligned} \tag{6.1.2}$$

The boundary conditions (5.2.19)-(5.2.22) now become

$$\begin{aligned}
& \lambda^2\kappa^{-1}ar^3\phi''' + \lambda^2\kappa^{-1}(ra'+a)r^2\phi'' \\
+ \quad & [\lambda^2 B_{\lambda\lambda}q^2 + \lambda\kappa B_{\lambda\kappa}q^4 + \lambda\kappa B_{\lambda\kappa}q^2 + \kappa^2 B_{\kappa\kappa}q^4 \\
- \quad & \lambda^2\kappa^{-1}(ra'+a) - \lambda^2\kappa^{-1}(2b+a)q^2 + \lambda^2\kappa^{-1}r^2\rho\omega^2 - I\omega^2q^2]r\phi' \\
+ \quad & [\lambda B_\lambda q^2 - \lambda B_\lambda q^4 - \lambda^2 B_{\lambda\lambda}q^2 - 2\lambda\kappa B_{\lambda\kappa}q^4 - \kappa^2 B_{\kappa\kappa}q^6 \\
+ \quad & \lambda^2\kappa^{-1}(ra'+2a+2b)q^2 + \lambda\kappa^{-1}r\rho_0\omega^2q^2 + I\omega^2q^4]\phi = 0, \qquad (6.1.3)
\end{aligned}$$

$$\begin{aligned}
& \lambda^2\kappa^{-1}ar^2\phi'' + (\lambda^2 B_{\lambda\lambda}q^2 + 2\lambda\kappa B_{\lambda\kappa}q^2 + \kappa^2 B_{\kappa\kappa}q^2 - \lambda^2\kappa^{-1}a - \lambda\kappa^{-1}r\rho_0\omega^2 - I\omega^2)r\phi' \\
+ \quad & (\lambda^2\kappa^{-1}aq^2 - \lambda^2 B_{\lambda\lambda}q^2 - \lambda\kappa B_{\lambda\kappa}q^2 - \lambda\kappa B_{\lambda\kappa}q^4 - \kappa^2 B_{\kappa\kappa}q^4 + I\omega^2q^2)\phi = 0, \qquad (6.1.4)
\end{aligned}$$

on P^+, and

$$\begin{aligned}
& \lambda^2\kappa^{-1}ar^3\phi''' + \lambda^2\kappa^{-1}(ra'+a)r^2\phi'' \\
+ \quad & [\lambda^2 B_{\lambda\lambda}q^2 + \lambda\kappa B_{\lambda\kappa}q^4 + \lambda\kappa B_{\lambda\kappa}q^2 + \kappa^2 B_{\kappa\kappa}q^4 \\
- \quad & \lambda^2\kappa^{-1}(ra'+a) - \lambda^2\kappa^{-1}(2b+a)q^2 + \lambda^2\kappa^{-1}r^2\rho\omega^2 - I\omega^2q^2]r\phi' \\
+ \quad & [\lambda B_\lambda q^2 - \lambda B_\lambda q^4 - \lambda^2 B_{\lambda\lambda}q^2 - 2\lambda\kappa B_{\lambda\kappa}q^4 - \kappa^2 B_{\kappa\kappa}q^6 \\
+ \quad & \lambda^2\kappa^{-1}(ra'+2a+2b)q^2 - \lambda\kappa^{-1}r\rho_0\omega^2q^2 + I\omega^2q^4]\phi = 0, \qquad (6.1.5)
\end{aligned}$$

$$\begin{aligned}
& \lambda^2\kappa^{-1}ar^2\phi'' + (\lambda^2 B_{\lambda\lambda}q^2 + 2\lambda\kappa B_{\lambda\kappa}q^2 + \kappa^2 B_{\kappa\kappa}q^2 - \lambda^2\kappa^{-1}a + \lambda\kappa^{-1}r\rho_0\omega^2 - I\omega^2)r\phi' \\
+ \quad & (\lambda^2\kappa^{-1}aq^2 - \lambda^2 B_{\lambda\lambda}q^2 - \lambda\kappa B_{\lambda\kappa}q^2 - \lambda\kappa B_{\lambda\kappa}q^4 - \kappa^2 B_{\kappa\kappa}q^4 + I\omega^2q^2)\phi = 0, \qquad (6.1.6)
\end{aligned}$$

on P^-.

When the general solution of (6.1.2) is inserted into (6.1.3)-(6.1.6) we obtain the *frequency equation* identifying the frequency of vibrations of the elastic surface-coated block. In general, the equation for $\phi(r)$ must be solved numerically.

The parameter q is determined by imposing incremental boundary conditions on the ends of the block. Thus, at $\theta = \pm\alpha H$, we have

$$u = 0, \qquad \Sigma_{\theta\theta} = 0 \tag{6.1.7}$$

as in Chapter 4, and, it follows that

$$\psi_\theta = 0 \quad \text{at} \quad \theta = \pm\alpha H, \tag{6.1.8}$$

Hence

$$q = \frac{k\pi}{\alpha H} \tag{6.1.9}$$

where k is a (positive) integer. As before, $k = 1$ suffices in (6.1.9) and for a given value of H, results for $k > 1$ are then obtained from those for the value H/k.

Numerical calculations have been carried out in respect of the neo-Hookean strain-energy function (3.2.29) and for the class of strain-energy functions (4.3.8), introduced by Ogden (1972), along with the coating energy function (3.2.30).

Previously we considered the non-dimensionalization of material constants in relation to numerical bifurcation results, namely

$$\hat{m} = \frac{m}{\mu A}, \quad \hat{n} = \frac{n}{\mu A^3}. \tag{6.1.10}$$

In obtaining numerical vibration results, further non-dimensionalization of material constants appearing in equations (6.1.2)-(6.1.6) is required. As the relevant terms only appear in these equations in conjunction with ω, we non-dimensionalize $\rho\omega^2$, $\rho_0\omega^2$ and $I\omega^2$ as follows. A measure of the frequency of vibrations is given by

$$\hat{\Omega} = \rho\omega^2 A^2/\mu. \tag{6.1.11}$$

Similarly, with reference to $\rho_0 = \rho_c t_r$, we define

$$\hat{\Omega}_0 = \rho_0 \omega^2 A/\mu = \rho_c \hat{t}_r \hat{\Omega}/\rho \tag{6.1.12}$$

where $\hat{t}_r = t_r/A$.

The moment of inertia term, in non-dimensional form, becomes

$$\widehat{I\omega^2} = I\omega^2/\mu A = \hat{I}\hat{\Omega} \tag{6.1.13}$$

where, from (5.2.24),

$$\hat{I} = I/\rho A^3 = \lambda^{-2}\rho_c \hat{t}_r^3/12\rho \tag{6.1.14}$$

making use of the assumed incompressibility of the film material, so that $t_d = \lambda^{-1} t_r$.

Note, from (4.3.11) and (4.3.12), that t_r is defined as

$$t_r = 2\sqrt{3}A\left(\frac{\hat{n}}{\hat{m}}\right)^{1/2} \tag{6.1.15}$$

and, as before, we limit the values of $\hat{m}$ and $\hat{n}$ such that

$$\frac{\hat{n}}{\hat{m}} \ll 1. \tag{6.1.16}$$

6.2 Numerical results

The system (6.1.2)-(6.1.6) is solved using the method of compound matrices, as in Chapter 4.

The frequency of vibration of a surface-coated block depends on the deformation of the bulk material, the material properties of the bulk and film materials and the relative densities of the two materials. This dependence is now illustrated graphically.

Calculations have been performed for coating of the same type on both curved boundaries (P^- and P^+), coating on P^- or P^+ only, and for the situation where no coating is present. For illustrative purposes, we concentrate on results where the surface coating occupies only one boundary, P^- or P^+.

As in Chapter 4, we set $A = 1$, for numerical simplicity, and give results mainly for $\hat{m} = 1$ and for the range $\hat{n} = 0.001, 0.005, 0.01, 0.02$. Results for lower (0.0005) and higher (0.03) values of $\hat{n}$ show little qualitative difference from the range considered here. The influence of the relative densities of the bulk and film materials is examined by considering a range of density ratios $\rho_c/\rho = 0.1, 0.5, 1, 3, 5$.

Numerical results are illustrated graphically by plotting frequency, $\hat{\Omega}$, (vertical scale) against stretch, λ_+, (horizontal scale) for fixed heights $H = 1$ and $H = 2$. Results for blocks of height greater than $H = 2$ were calculated. However, it was found that setting $H = 3$, and above, leads to little change (from the $H = 2$ case) in the underlying character of the graphs but does result in proportionate decreases in frequency as height increases.

The first four possible solution branches are used to demonstrate vibration behaviour of the block and the cut-off point for the solution branches on the right of the figures corresponds to the value of λ_+ for which the block is deformed into a sector of a circular cylinder such that $\alpha H = \pi$.

We note that for a given value of H/A, vibration results for modes $k > 1$ can be viewed by reading the value of the frequency for a given λ_+ from the H/kA graph pertaining to the same parameter values. For example, a result for $H/A = 5$ (and $k = 5$) is the same as that for $H/A = 1$ (with $k = 1$).

Vibration results for uncoated blocks of various heights, obtained for a neo-

Hookean material, are shown in Fig.6.1. Clearly, the height of the block has a significant effect on its dynamic behaviour as the frequency of vibration falls markedly with increased height.

Solution branches tending to zero in Fig.6.1 correspond to the bifurcation results of Chapter 4 only up to $H/A = 2$. For aspect ratios $H/A = 2.5, 3, 5$ the earlier bifurcation results show that bifurcation of the uncoated block has not occurred before $\alpha H = \pi$ is reached, yet the vibration results register the respective bifurcation points as occurring at $\alpha H = \pi$, which is obviously wrong (see dashed solution branches in Fig.6.1). This discrepancy can be explained by obtaining analytical bifurcation results for an uncoated block for the specific case when $\alpha H = \pi$. This is achieved by setting $q = 1$ in (4.3.2) and we find that the resulting bifurcation criterion never equals zero. Hence, we conclude that if bifurcation does not occur before $\alpha H = \pi$, it does not occur at all. So, although as $H/A \to \infty$ the numerical method gives spurious critical points at $\alpha H = \pi$, we regard such results as being mathematically correct but lacking a physical interpretation. This effect has also been observed in a recent paper, Haughton (1998).

Figure 6.2 illustrates corresponding uncoated results for an Ogden material where, as in Fig.6.1, the dashed solution branches represent the erroneous bifurcation results obtained for $H/A = 2.5$ and above. As the results in Fig.6.2 are almost identical to those in Fig.6.1, all subsequent results make use of the neo-Hookean strain-energy function only.

Figs 6.3-6.8 illustrate how increasing flexural stiffness in the coating effects the vibration behaviour of blocks of height $H = 1$ and $H = 2$ when surface coating is on the P^+ boundary only, for a range of density ratios. Figs 6.3 and 6.6 show that when the bulk material dominates the density ratio, increased flexural stiffness has no effect on the frequency of vibration for either height. Indeed, the results are almost identical to those for the equivalent uncoated blocks in Fig.6.1. When the substrate and film are of equal density, as in Figs 6.4 and 6.7, frequency is slightly reduced with increased flexural stiffness, for each height. In contrast to the results for $\rho_c/\rho = 0.1$, when the film density is dominant Figs 6.5 and 6.8 show that increasing flexural stiffness has the effect of greatly reducing the frequency of vibration, for small amounts of stretch. In Fig 6.5 (graph D) two solution branches appear to almost intersect but closer inspection (graph E) proves they do not.

The solution branches tending to zero in Figs 6.3-6.8 correspond to the previously obtained bifurcation results and we note that, as in Chapter 4, when a coating is present only on the P^+ boundary flexural stiffness has no effect on the onset of bifurcation.

Now we consider blocks of height $H = 1$ and $H = 2$ coated only on the P^- boundary. Figs 6.9-6.14 illustrate these results and a similar pattern emerges to that observed in the corresponding results for a block coated on P^+ only. When the density ratio is small, in Figs 6.9 and 6.12, increasing flexural stiffness has no effect on vibration behaviour for either height. Figures 6.10 and 6.13 show that equating the two material densities results in a decrease in frequency as flexural stiffness increases. This trend is repeated in Figs 6.11 and 6.14 when the film density dominates and increasing flexural stiffness corresponds to decreasing frequency of vibration. In Fig.6.11, two solution branches appear close to intersection (graph C) but reference to graph E shows this is not the case.

The solution branches tending to zero frequency in Figs 6.9-6.14, echoing the bifurcation behaviour of Chapter 4, show that increasing flexural stiffness delays the onset of bifurcation in a block with coating present on the P^- boundary only.

For fixed height and flexural stiffness, Figs 6.15-6.17 demonstrate how the relative densities of the two materials influence the dynamic behaviour of a block coated on P^+ only. In Fig.6.15 and Fig.6.17 (graphs A, C and E) the coating has minimal flexural stiffness and we find that frequency steadily decreases as the density ratio increases, for each height. A coating with a large amount of flexural stiffness present, as in Fig.6.16 and Fig.6.17 (graphs B, D and F), results in a more pronounced decrease in frequency as the film density becomes more dominant, particularly for small amounts of stretch for each height.

The solution branches tending to $\hat{\Omega} = 0$ in Figs 6.15-6.17 highlight the fact that bifurcation behaviour of the block is independent of the relative densities of the two materials - as would be expected.

Comparison of Figs 6.17 and 6.18, pertaining to $H = 2$ and $H = 3$ respectively, illustrates the fact that increased height has only a small effect on the frequency of vibration. Note that the dashed solution branches in Fig.6.18 are those pertaining to the false bifurcation results discussed previously.

When a coating with little flexural stiffness is present on the P^- boundary only, the effect of the relative densities of the two materials on vibration behaviour is shown in Fig.6.19 and Fig.6.21 (graphs A, C and E): the frequency decreases as the density ratio increases. A more pronounced reduction in frequency, corresponding to increasing ρ_c/ρ, is observed in Figs 6.20 and Fig.6.21 (graphs B, D and F) where a large amount of flexural stiffness is present in the coating.

Again, as in the P^+ only case, the solution branches which tend to $\hat{\Omega} = 0$ in

Figs 6.19-6.22 demonstrate that bifurcation results for a block coated on P^- only are independent of the relative material densities.

The minimal effect caused by further increasing the height of a block coated on the P^- boundary only is shown in Fig.6.22, the analogue of Fig.6.18.

Note that Figs 6.23-6.25 can be compared with Figs 6.15-6.17 (for the P^+ only case) and Figs 6.19-6.21 (for the P^- only case) to show the effect of having a surface coating on both boundaries.

To determine the effect of changing the values of $\hat{m}$ results pertaining to relatively soft ($\hat{m} = 0.1$) and stiff ($\hat{m} = 5$) coatings were examined. These results are illustrated, in Figs 6.26 and 6.27, when the film density is dominant for values of $\hat{n}$ which ensure the restriction (6.1.16) holds. Clearly, whether the coating on P^+ or P^- is more or less destabilizing depends on the relative shear stiffnesses of the two materials and on the level of flexural stiffness present.

Setting $\hat{n} = 0$ and $\hat{I} = 0$ allowed us to obtain results for a membrane-type coating. Since the P^- boundary is in compression, the theory is not valid in the membrane case. However, when the membrane is in tension, on the P^+ boundary, the results are essentially identical to those for an uncoated block. Hence, no figures are included here.

As rotatory inertia is sometimes omitted in classical elasticity theory, we now consider the effect of setting $\hat{I} = 0$ (but not $\hat{n} = 0$) in the present context. Numerical calculations show that omission of the rotatory inertia term has no effect on results for a block coated on the P^+ boundary only, hence, no figures are included here. When coating is present on the P^- boundary only, as the film density increases relative to the bulk material density, a slight deviation (from the results with rotatory inertia) is recorded in the uppermost solution branch at each height. The deviation becomes significant when $\rho_c/\rho = 5$ as the flexural stiffness increases. This effect is illustrated in Figs 6.28 and 6.29 for $H/A = 1$ and $H/A = 2$, where the dotted curves are the solutions *without* rotatory inertia and the solid curves represent those uppermost solution branches, in the corresponding results *with* rotatory inertia, which deviate from the results pertaining to $\hat{I} = 0$.

Conclusions

Overall, the numerical results show that frequency of vibration is consistently lower when a block is coated on the P^- boundary compared to the P^+ only case. This is due to the fact that the P^- boundary is in compression and therefore closer to instability. Increasing the height of the block or the flexural stiffness in the coating both result in reduced frequency. As does an increase in film density relative to the bulk material density. Reducing the relative shear stiffnesses of the two materials also reduces frequency of vibration while increasing the shear stiffness ratio ensures stiffer elastic coatings on the P^- and P^+ boundaries and, consequently, higher levels of frequency. Omission of the rotatory inertia term does appear to be justified, except when surface coating occupies the P^- boundary *and* the film density equates to, or is greater than, the bulk material density.

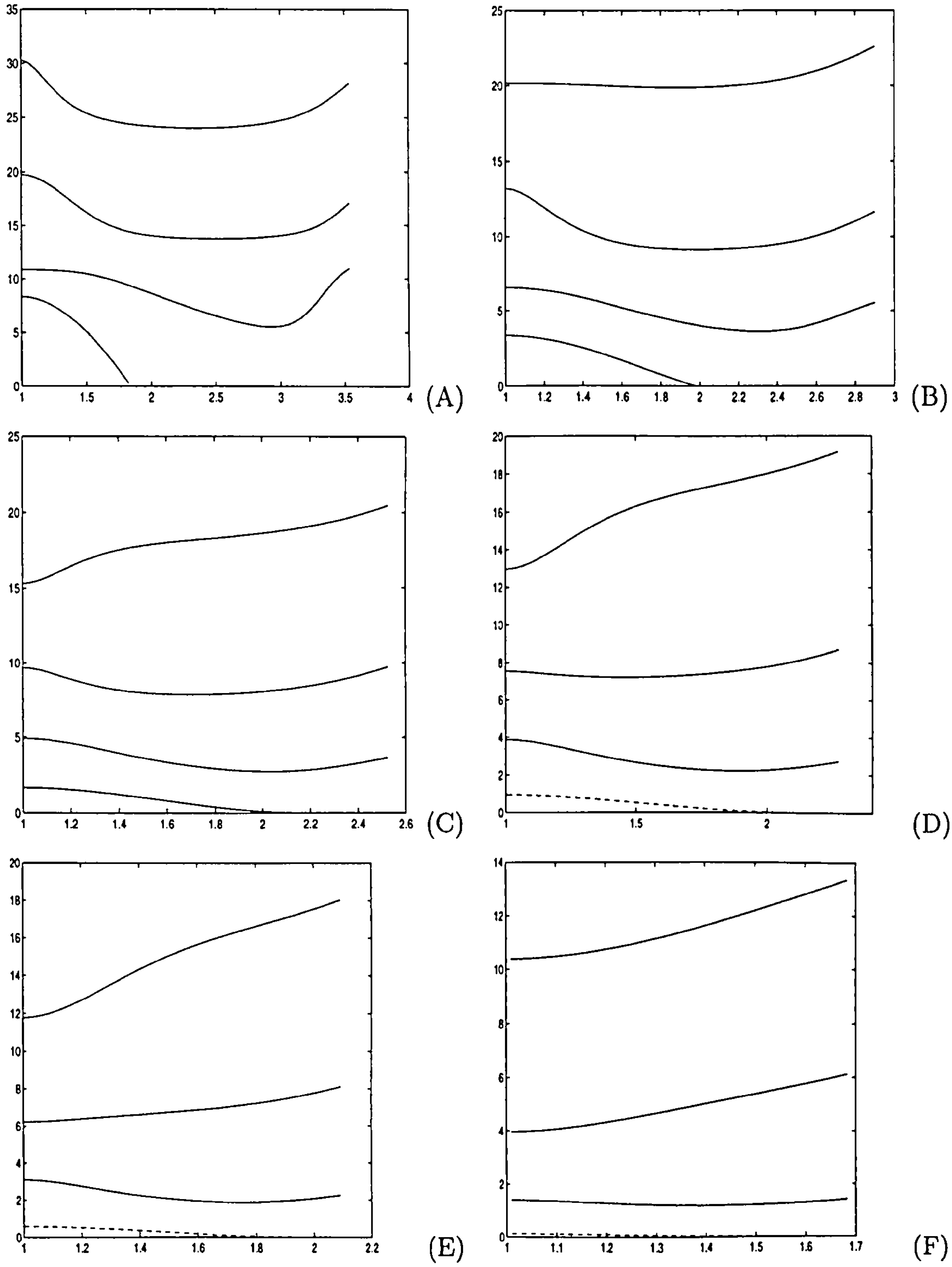

Figure 6.1: Frequency-stretch plot: uncoated block where (A) $H/A = 1$, (B) $H/A = 1.5$, (C) $H/A = 2$, (D) $H/A = 2.5$, (E) $H/A = 3$, (F) $H/A = 5$ for the neo-Hookean material.

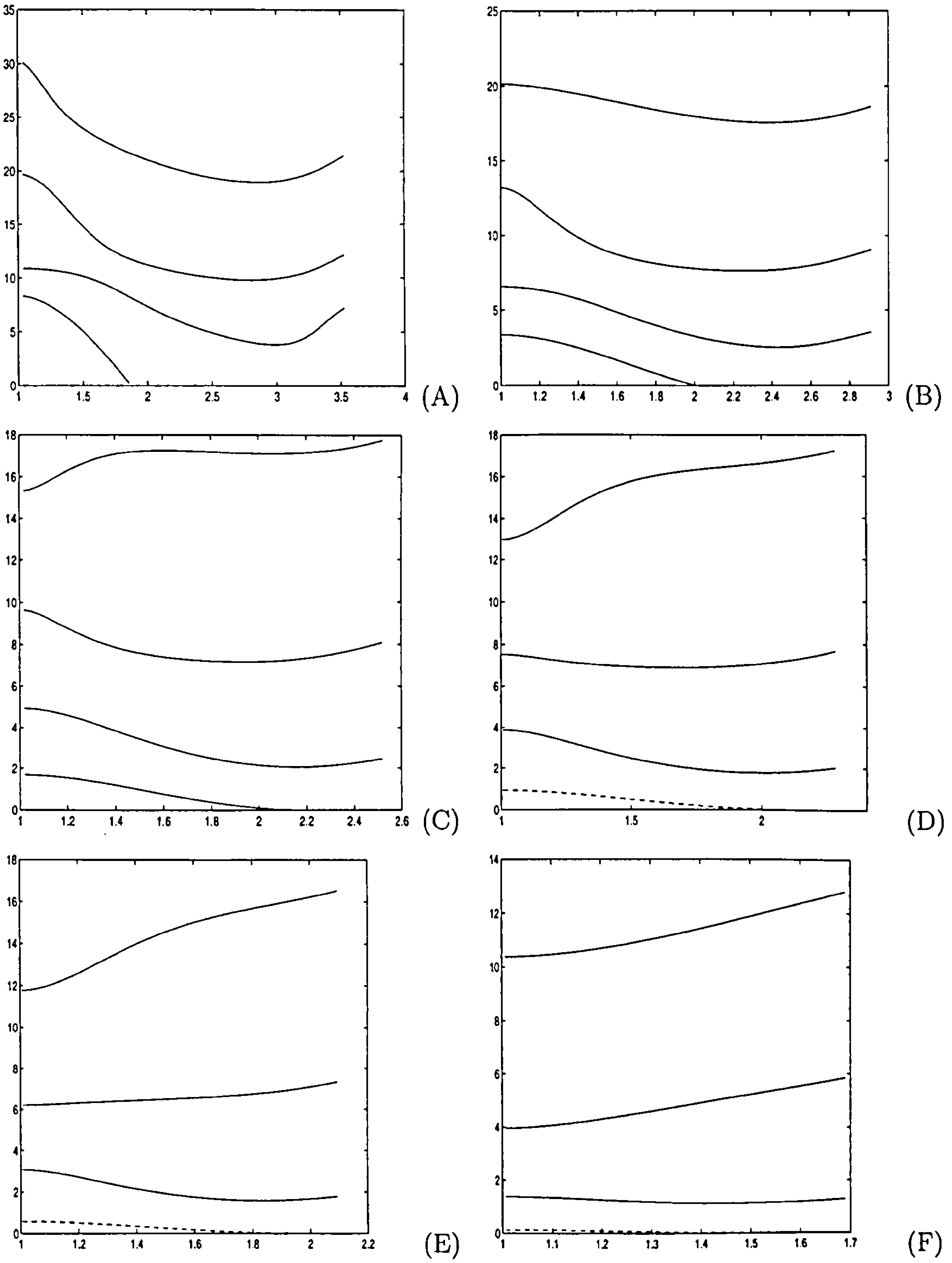

Figure 6.2: Frequency-stretch plot: uncoated block where (A) $H/A = 1$, (B) $H/A = 1.5$, (C) $H/A = 2$, (D) $H/A = 2.5$, (E) $H/A = 3$, (F) $H/A = 5$ for an Ogden material.

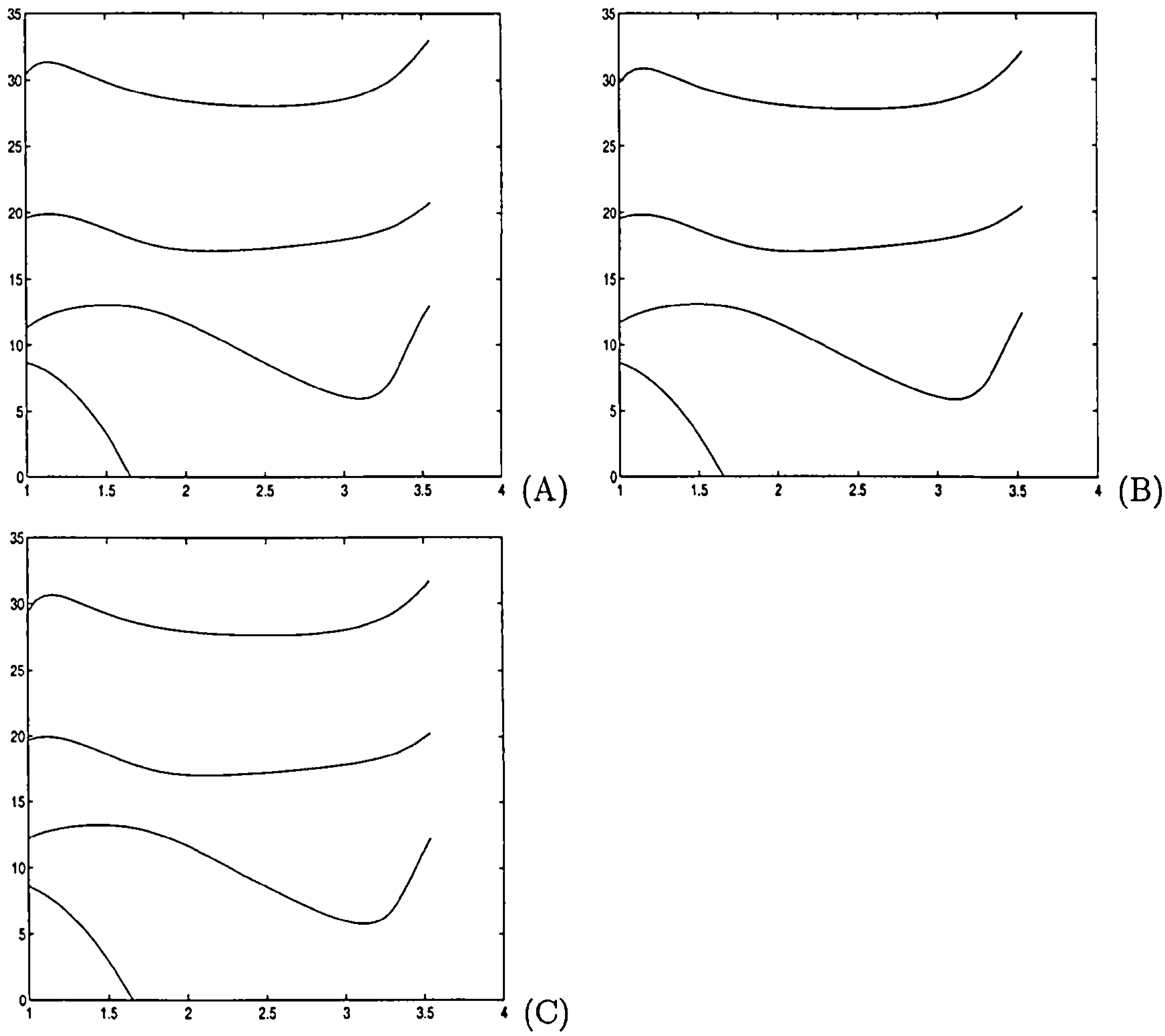

Figure 6.3: Frequency-stretch plot: coating on P^+ for $H/A = 1$ and $\rho_c/\rho = 0.1$ where (A) $\hat{n} = 0.001$, (B) $\hat{n} = 0.01$, (C) $\hat{n} = 0.02$.

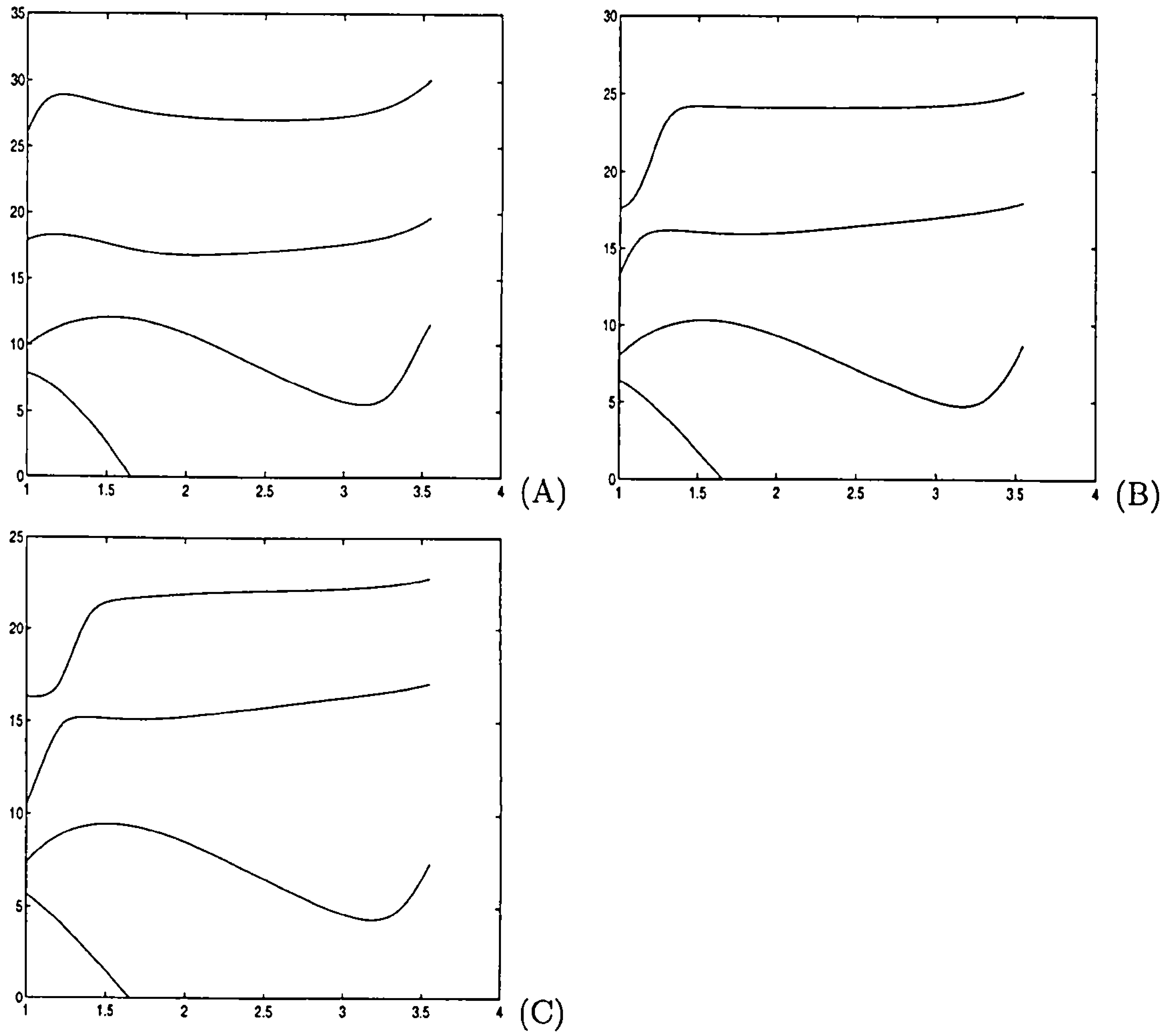

Figure 6.4: Frequency-stretch plot: coating on P^+ for $H/A = 1$ and $\rho_c/\rho = 1$ where (A) $\hat{n} = 0.001$, (B) $\hat{n} = 0.01$, (C) $\hat{n} = 0.02$.

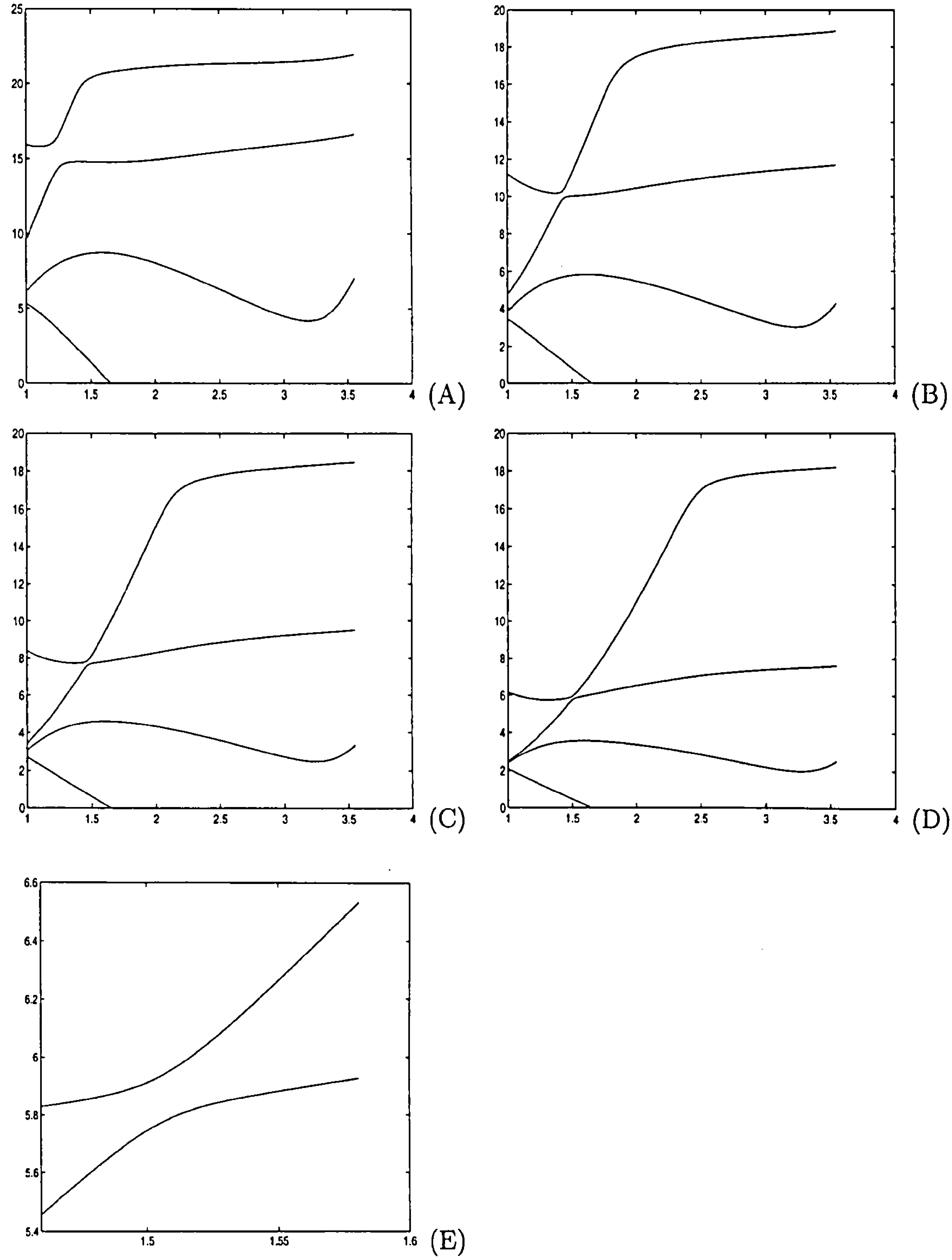

Figure 6.5: Frequency-stretch plot: coating on P^+ for $H/A = 1$ and $\rho_c/\rho = 5$ where (A) $\hat{n} = 0.001$, (B) $\hat{n} = 0.005$, (C) $\hat{n} = 0.01$, (D) $\hat{n} = 0.02$, (E) close up of plot D.

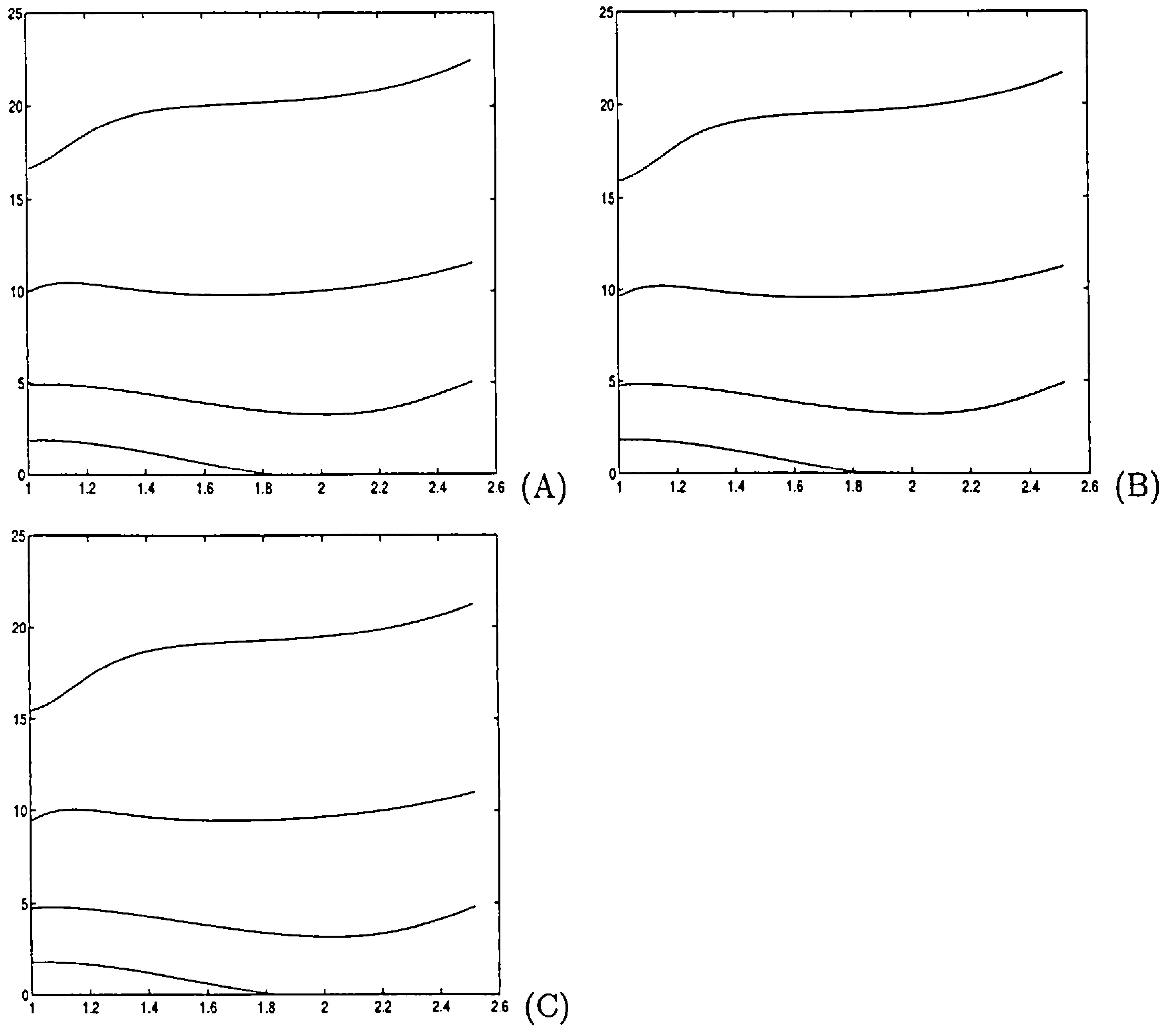

Figure 6.6: Frequency-stretch plot: coating on P^+ for $H/A = 2$ and $\rho_c/\rho = 0.1$ where (A) $\hat{n} = 0.001$, (B) $\hat{n} = 0.01$, (C) $\hat{n} = 0.02$.

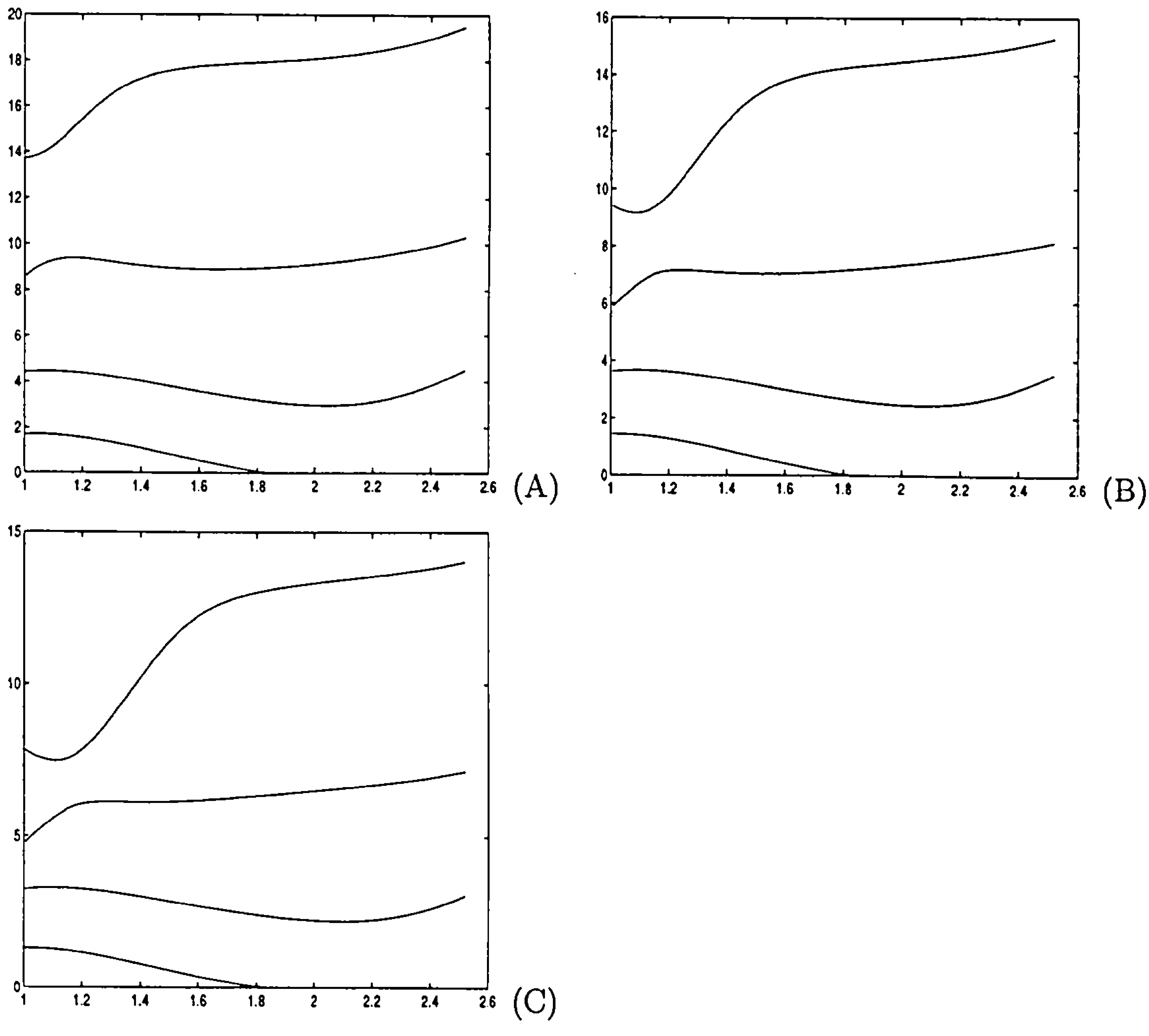

Figure 6.7: Frequency-stretch plot: coating on P^+ for $H/A = 2$ and $\rho_c/\rho = 1$ where (A) $\hat{n} = 0.001$, (B) $\hat{n} = 0.01$, (C) $\hat{n} = 0.02$.

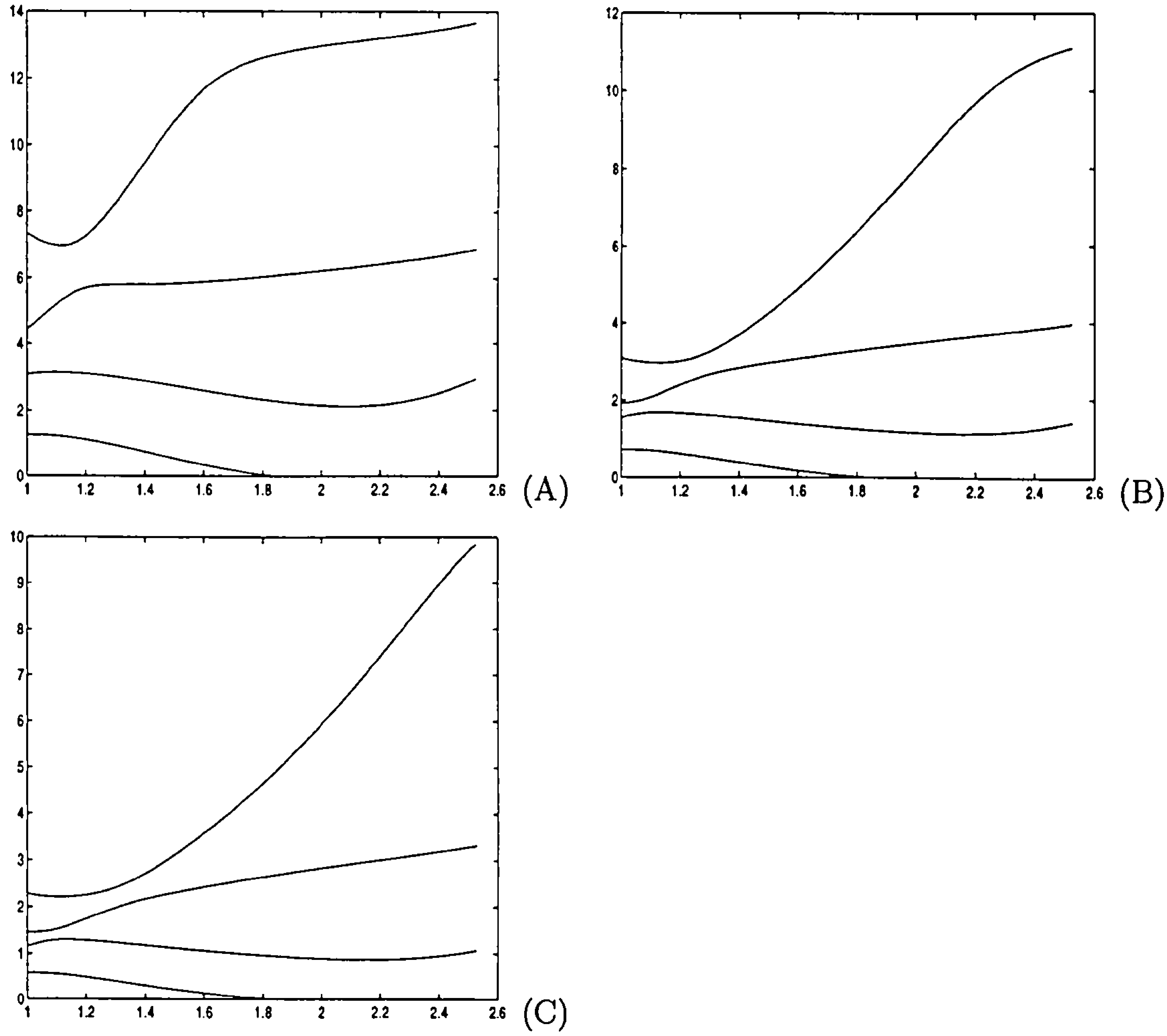

Figure 6.8: Frequency-stretch plot: coating on P^+ for $H/A = 2$ and $\rho_c/\rho = 5$ where (A) $\hat{n} = 0.001$, (B) $\hat{n} = 0.01$, (C) $\hat{n} = 0.02$.

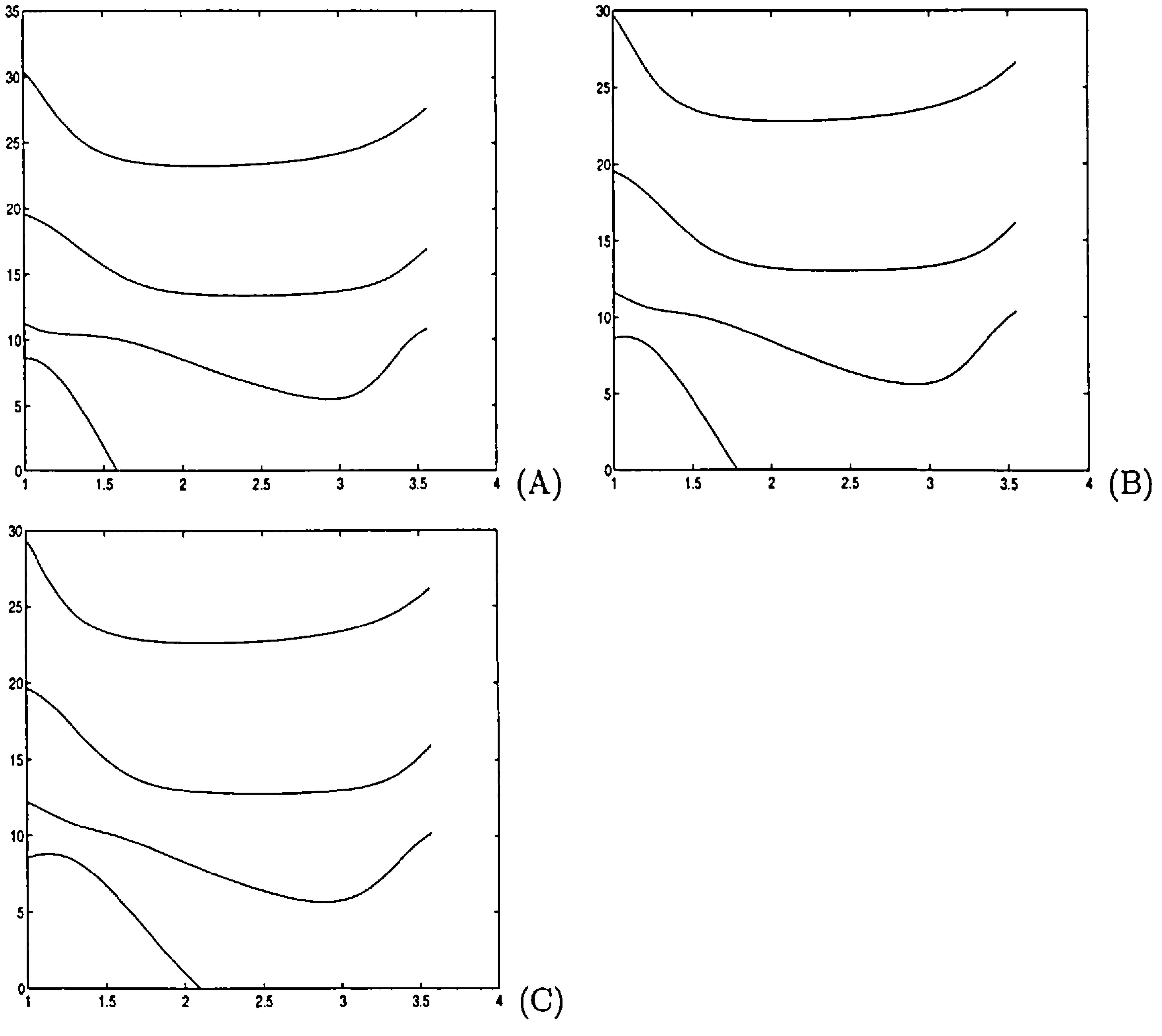

Figure 6.9: Frequency-stretch plot: coating on P^- for $H/A = 1$ and $\rho_c/\rho = 0.1$ where (A) $\hat{n} = 0.001$, (B) $\hat{n} = 0.01$, (C) $\hat{n} = 0.02$.

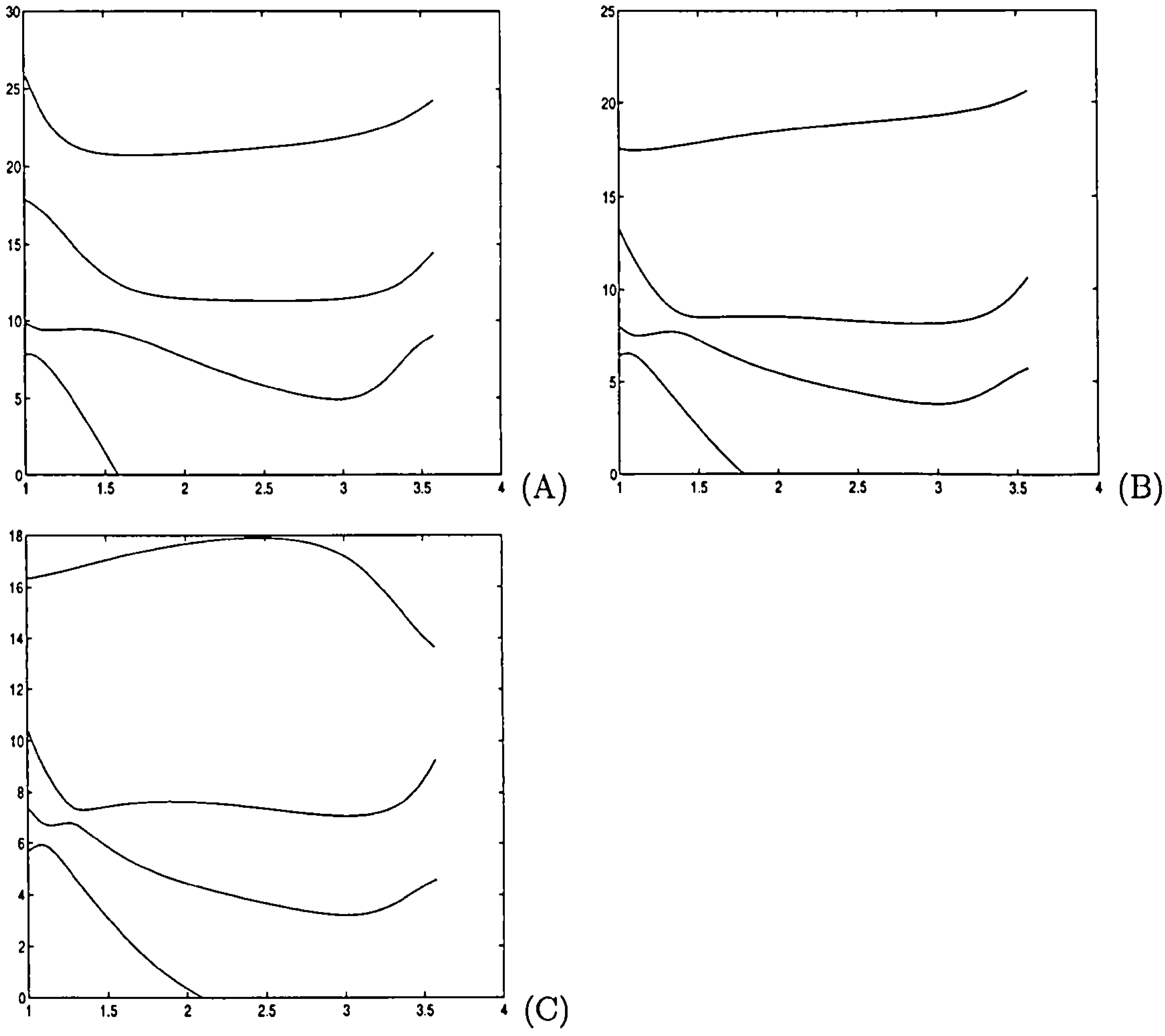

Figure 6.10: Frequency-stretch plot: coating on P^- for $H/A = 1$ and $\rho_c/\rho = 1$ where (A) $\hat{n} = 0.001$, (B) $\hat{n} = 0.01$, (C) $\hat{n} = 0.02$.

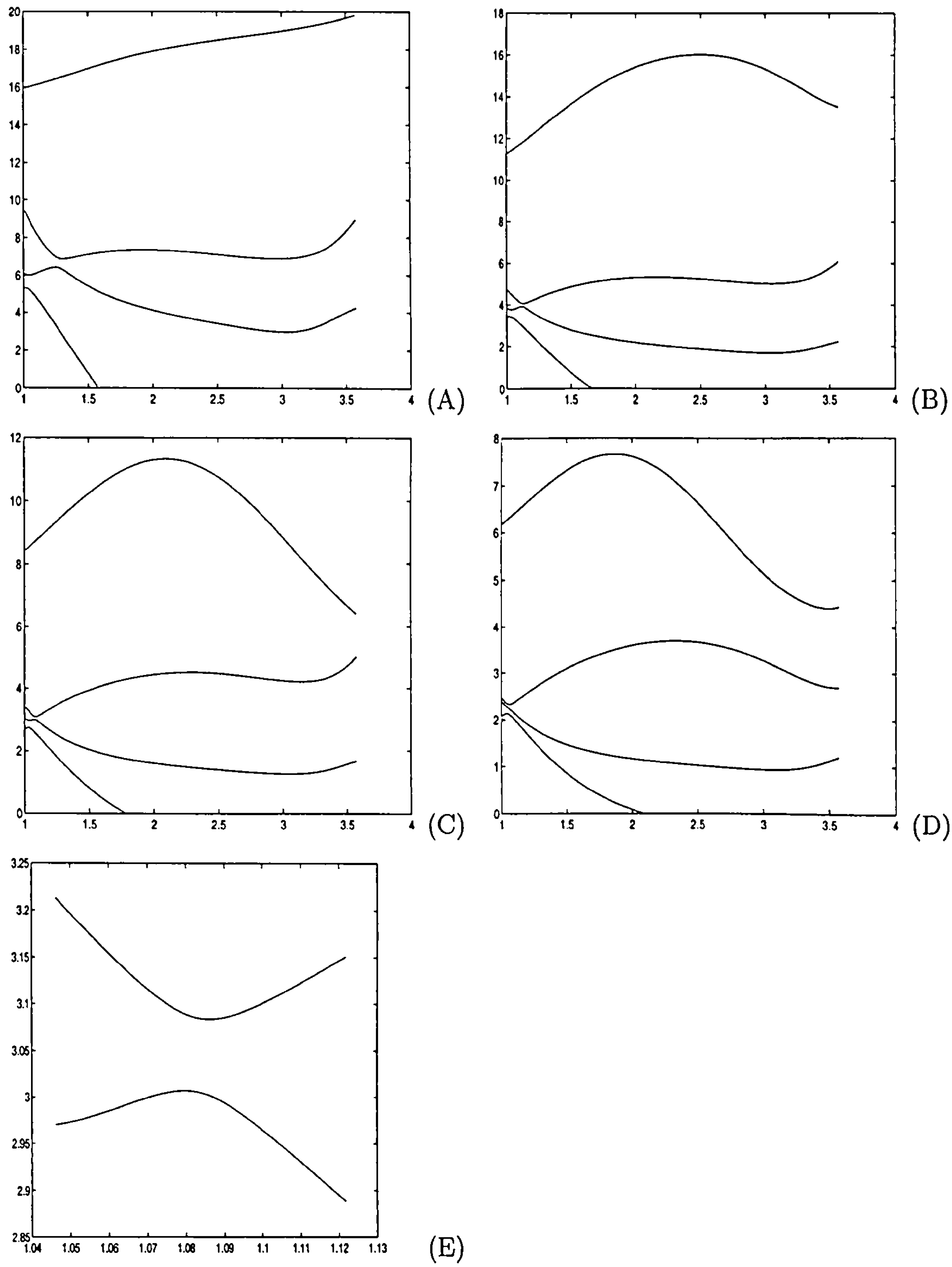

Figure 6.11: Frequency-stretch plot: coating on P^- for $H/A = 1$ and $\rho_c/\rho = 5$ where (A) $\hat{n} = 0.001$, (B) $\hat{n} = 0.005$, (C) $\hat{n} = 0.01$, (D) $\hat{n} = 0.02$, (E) close up of plot C.

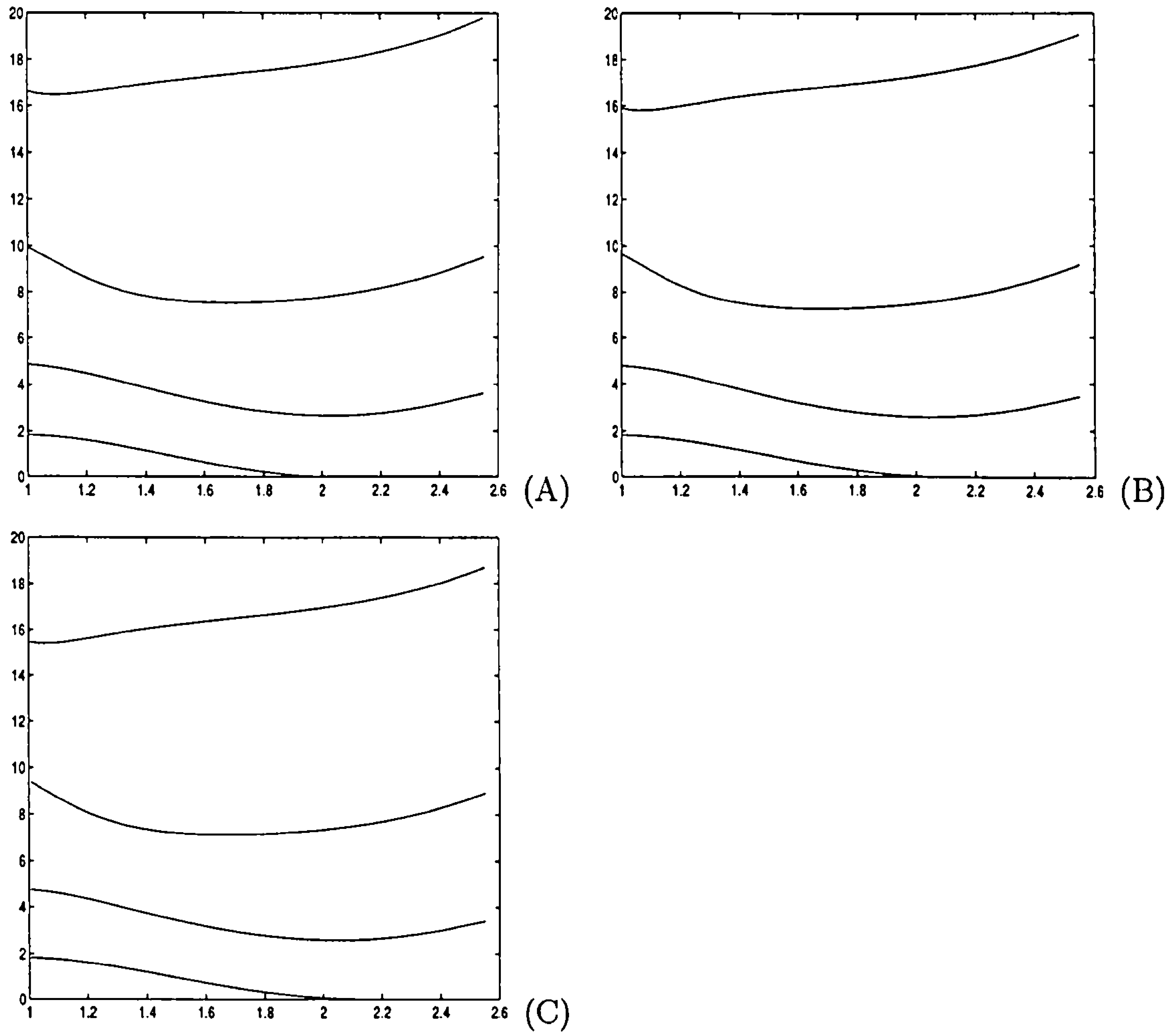

Figure 6.12: Frequency-stretch plot: coating on P^- for $H/A = 2$ and $\rho_c/\rho = 0.1$ where (A) $\hat{n} = 0.001$, (B) $\hat{n} = 0.01$, (C) $\hat{n} = 0.02$.

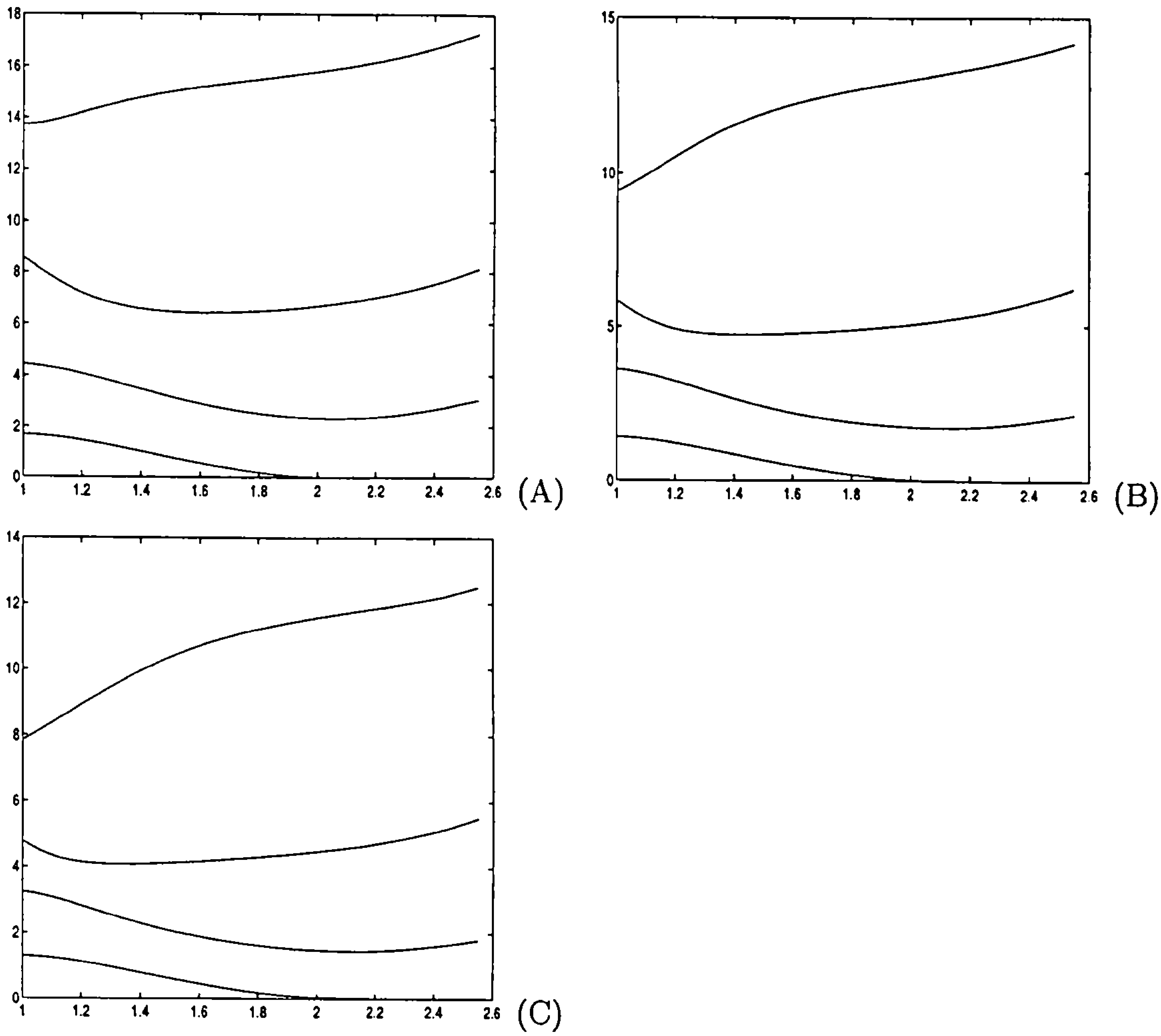

Figure 6.13: Frequency-stretch plot: coating on P^- for $H/A = 2$ and $\rho_c/\rho = 1$ where (A) $\hat{n} = 0.001$, (B) $\hat{n} = 0.01$, (C) $\hat{n} = 0.02$.

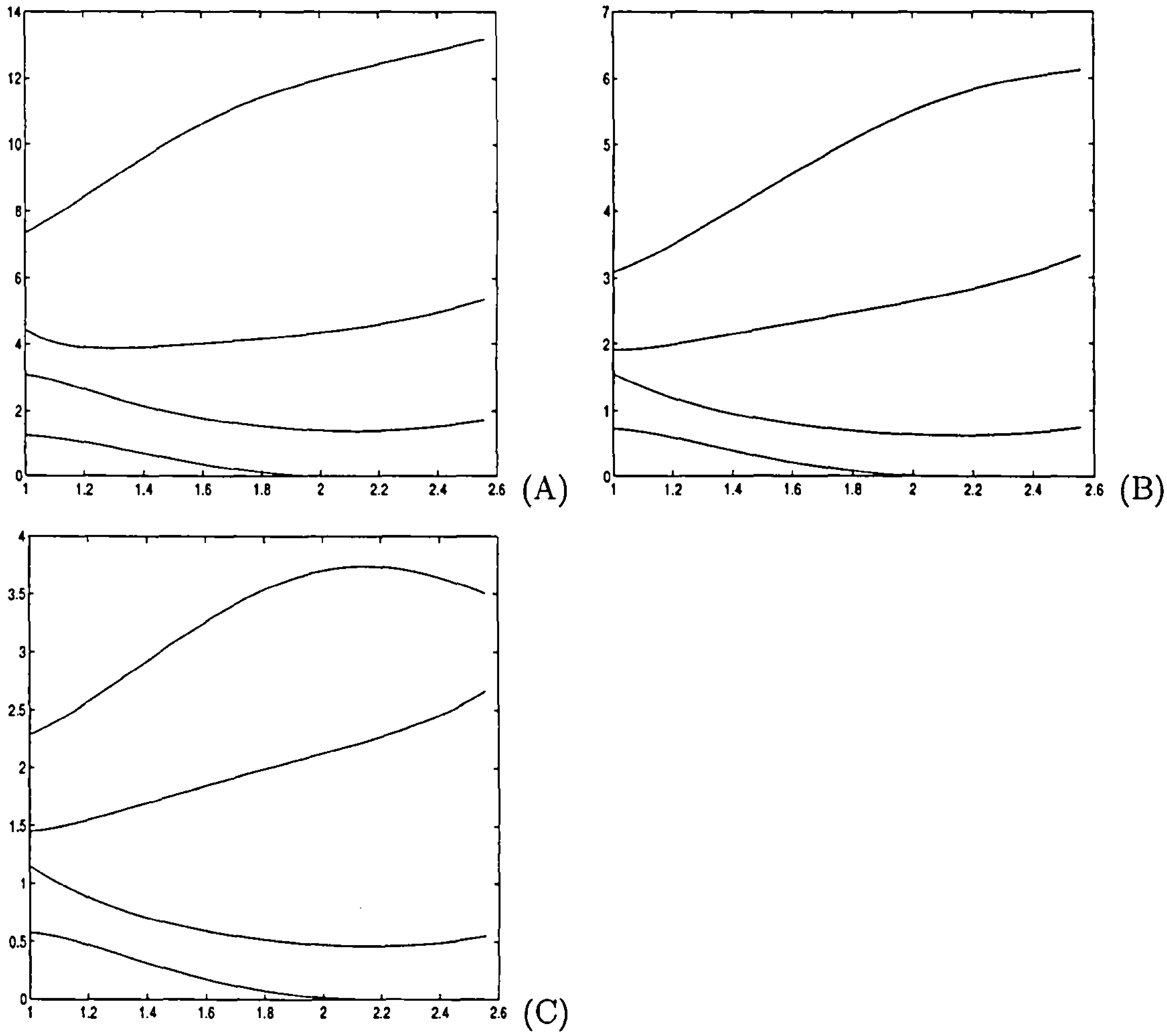

Figure 6.14: Frequency-stretch plot: coating on P^- for $H/A = 2$ and $\rho_c/\rho = 5$ where (A) $\hat{n} = 0.001$, (B) $\hat{n} = 0.01$, (C) $\hat{n} = 0.02$.

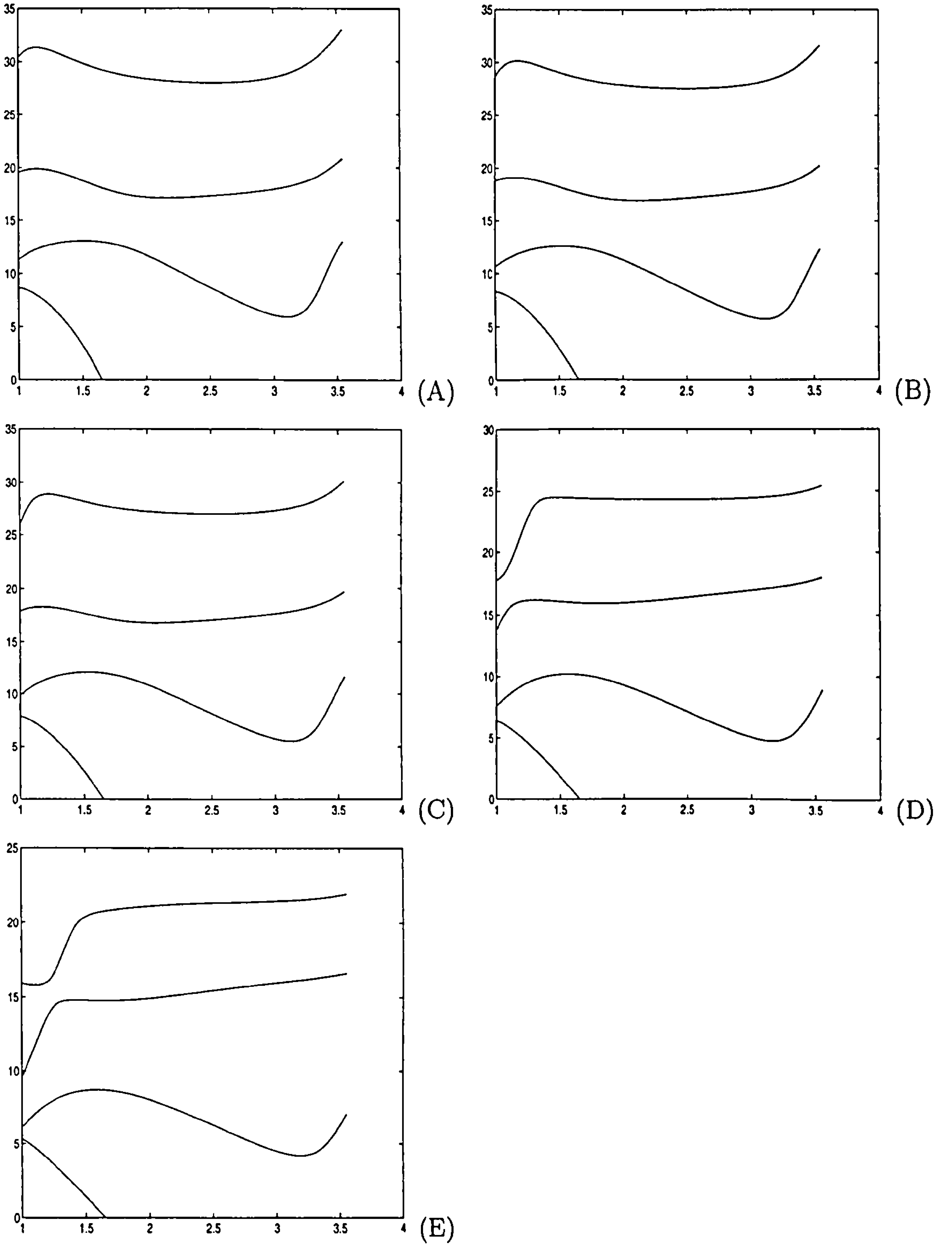

Figure 6.15: Frequency-stretch plot: coating on P^+ for $H = 1$ and $\hat{n} = 0.001$ where (A) $\rho_c/\rho = 0.1$, (B) $\rho_c/\rho = 0.5$, (C) $\rho_c/\rho = 1$,(D) $\rho_c/\rho = 3$, (E) $\rho_c/\rho = 5$.

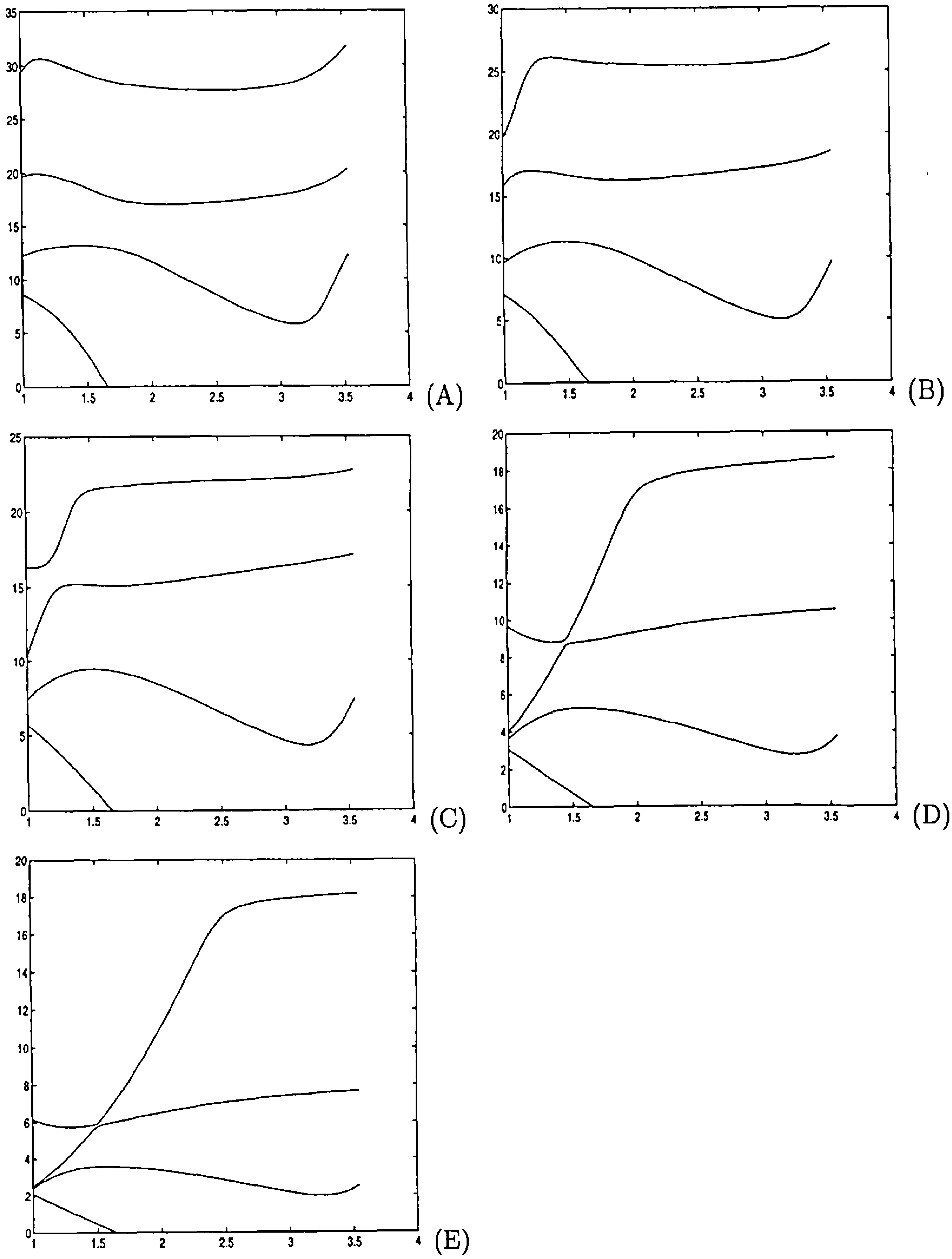

Figure 6.16: Frequency-stretch plot: coating on P^+ for $H/A = 1$ and $\hat{n} = 0.02$ where (A) $\rho_c/\rho = 0.1$, (B) $\rho_c/\rho = 0.5$, (C) $\rho_c/\rho = 1$,(D) $\rho_c/\rho = 3$, (E) $\rho_c/\rho = 5$.

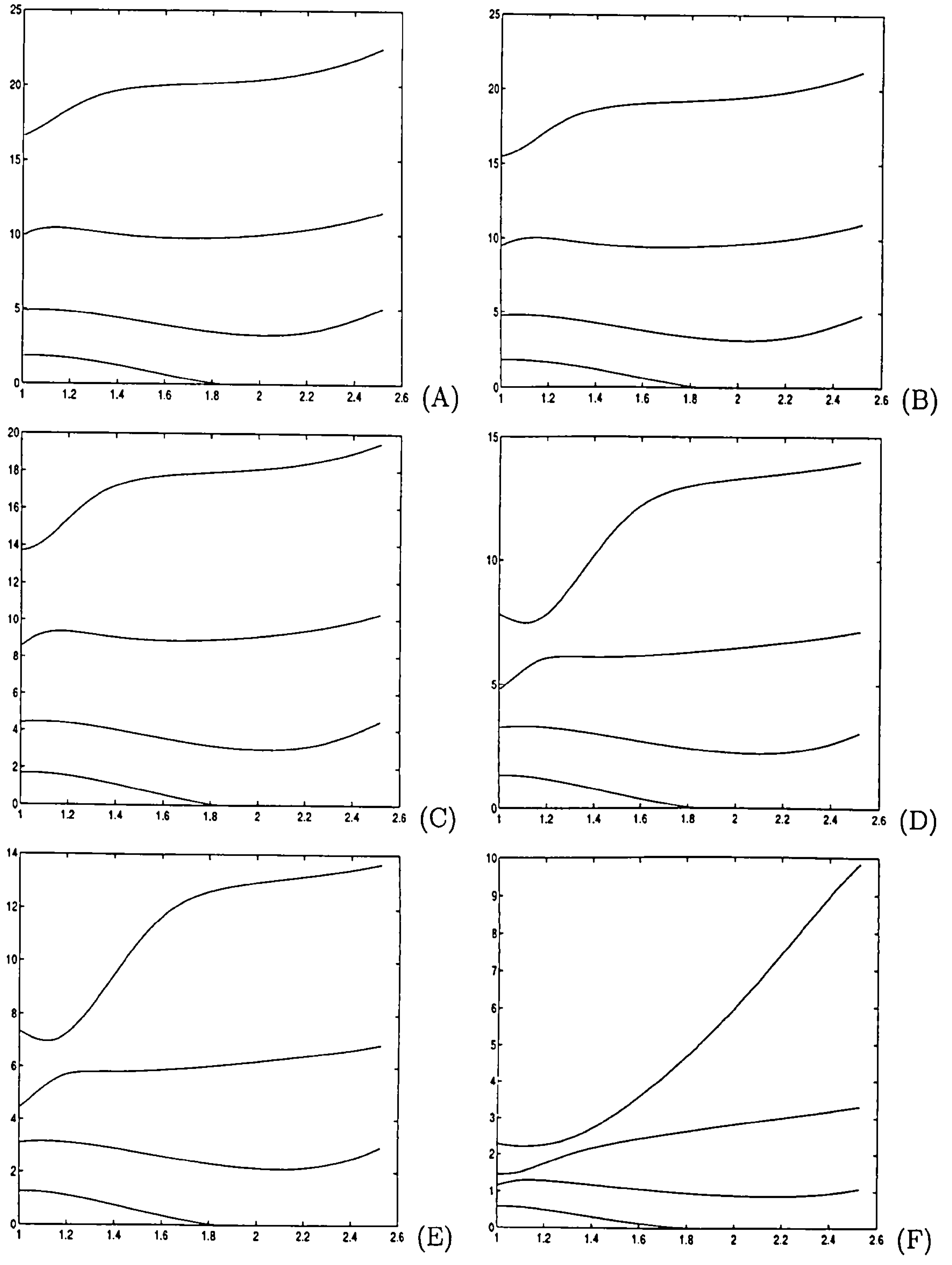

Figure 6.17: Frequency-stretch plot: coating on P^+ for $H/A = 2$ where (A) $\rho_c/\rho = 0.1$ and $\hat{n} = 0.001$, (B) $\rho_c/\rho = 0.1$ and $\hat{n} = 0.02$, (C) $\rho_c/\rho = 1$ and $\hat{n} = 0.001$, (D) $\rho_c/\rho = 1$ and $\hat{n} = 0.02$, (E) $\rho_c/\rho = 5$ and $\hat{n} = 0.001$, (F) $\rho_c/\rho = 5$ and $\hat{n} = 0.02$.

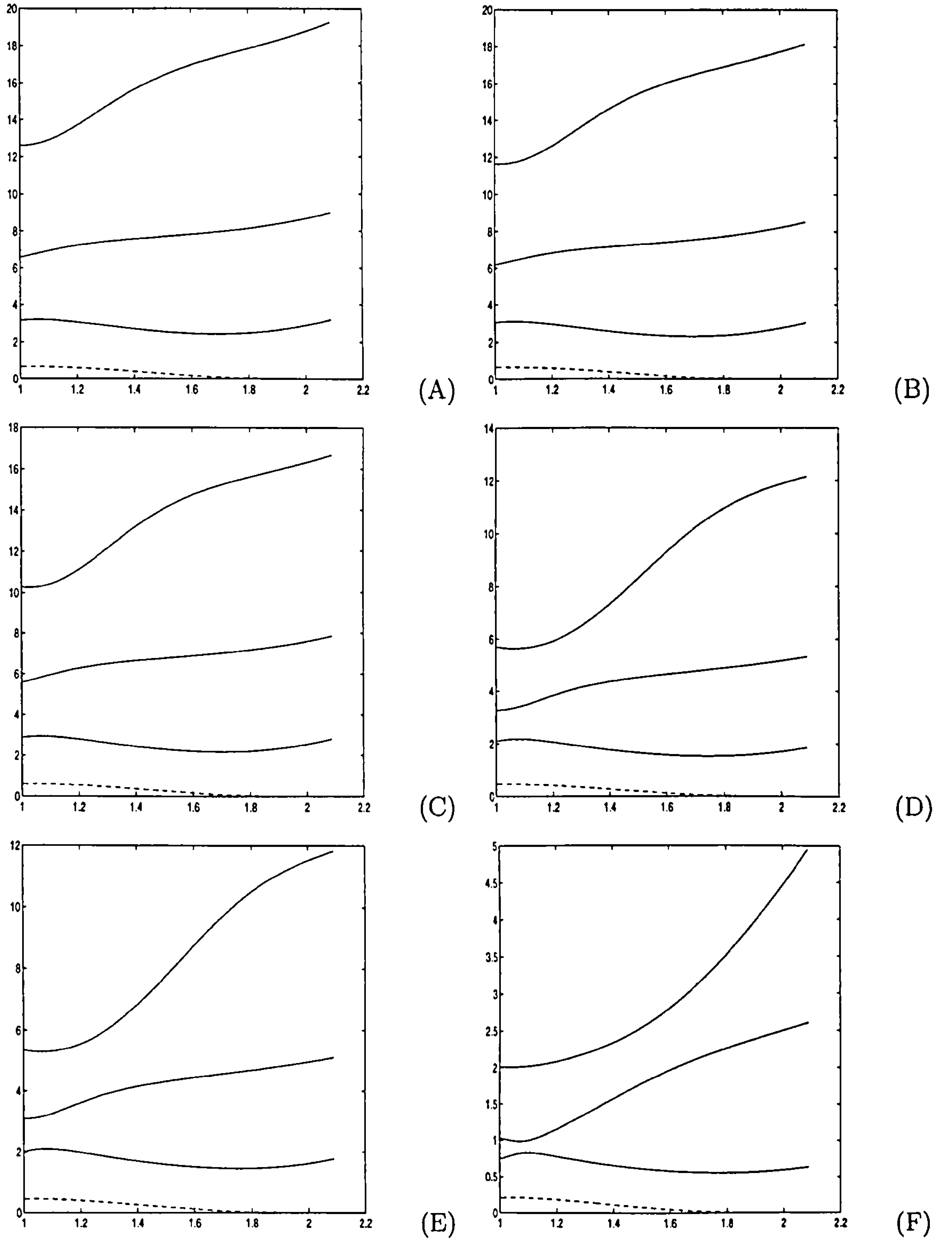

Figure 6.18: Frequency-stretch plot: coating on P^+ for $H/A = 3$ where (A) $\rho_c/\rho = 0.1$ and $\hat{n} = 0.001$, (B) $\rho_c/\rho = 0.1$ and $\hat{n} = 0.02$, (C) $\rho_c/\rho = 1$ and $\hat{n} = 0.001$, (D) $\rho_c/\rho = 1$ and $\hat{n} = 0.02$, (E) $\rho_c/\rho = 5$ and $\hat{n} = 0.001$, (F) $\rho_c/\rho = 5$ and $\hat{n} = 0.02$.

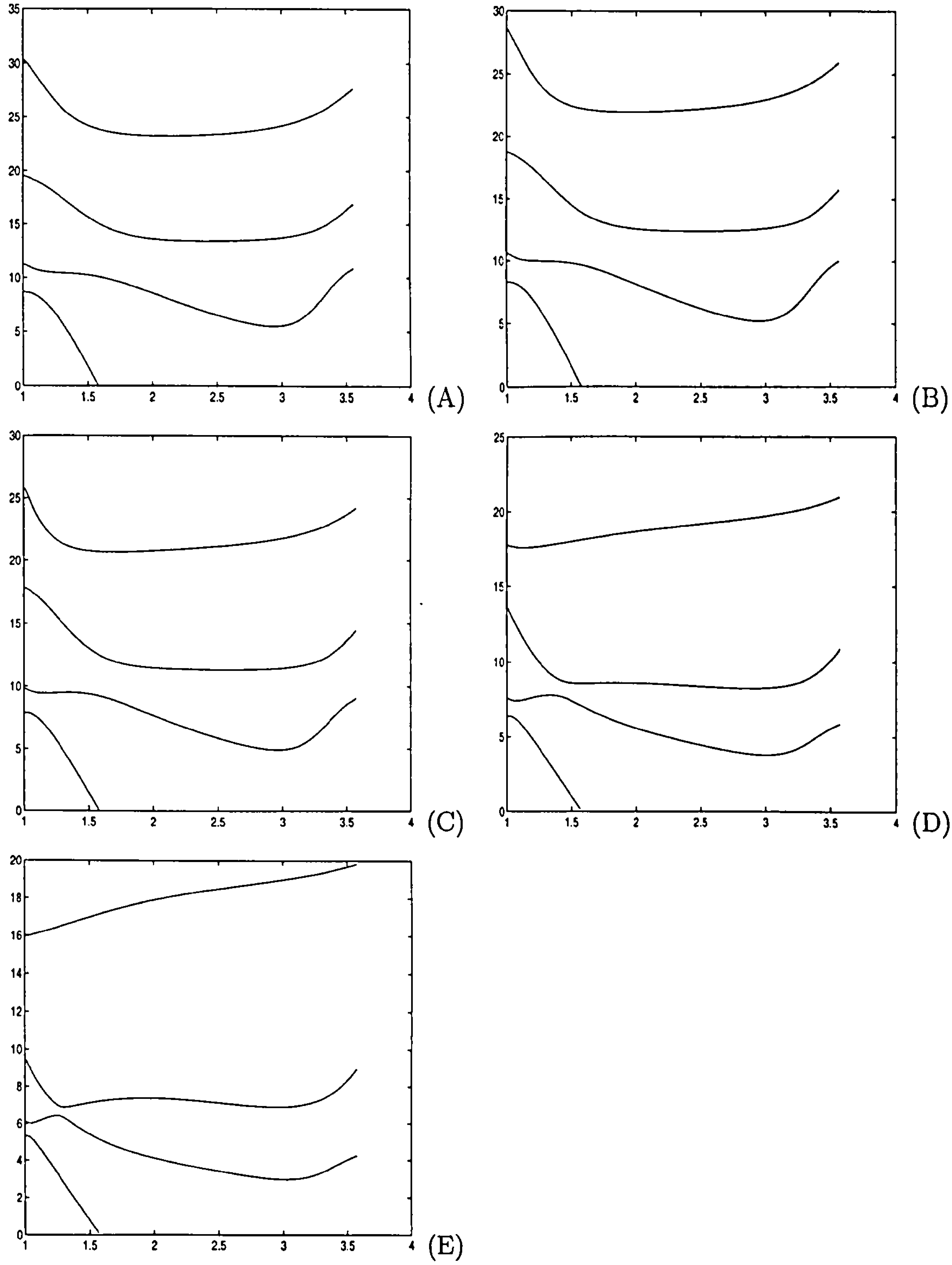

Figure 6.19: Frequency-stretch plot: coating on P^- for $H/A = 1$ and $\hat{n} = 0.001$ where (A) $\rho_c/\rho = 0.1$, (B) $\rho_c/\rho = 0.5$, (C) $\rho_c/\rho = 1$,(D) $\rho_c/\rho = 3$, (E) $\rho_c/\rho = 5$.

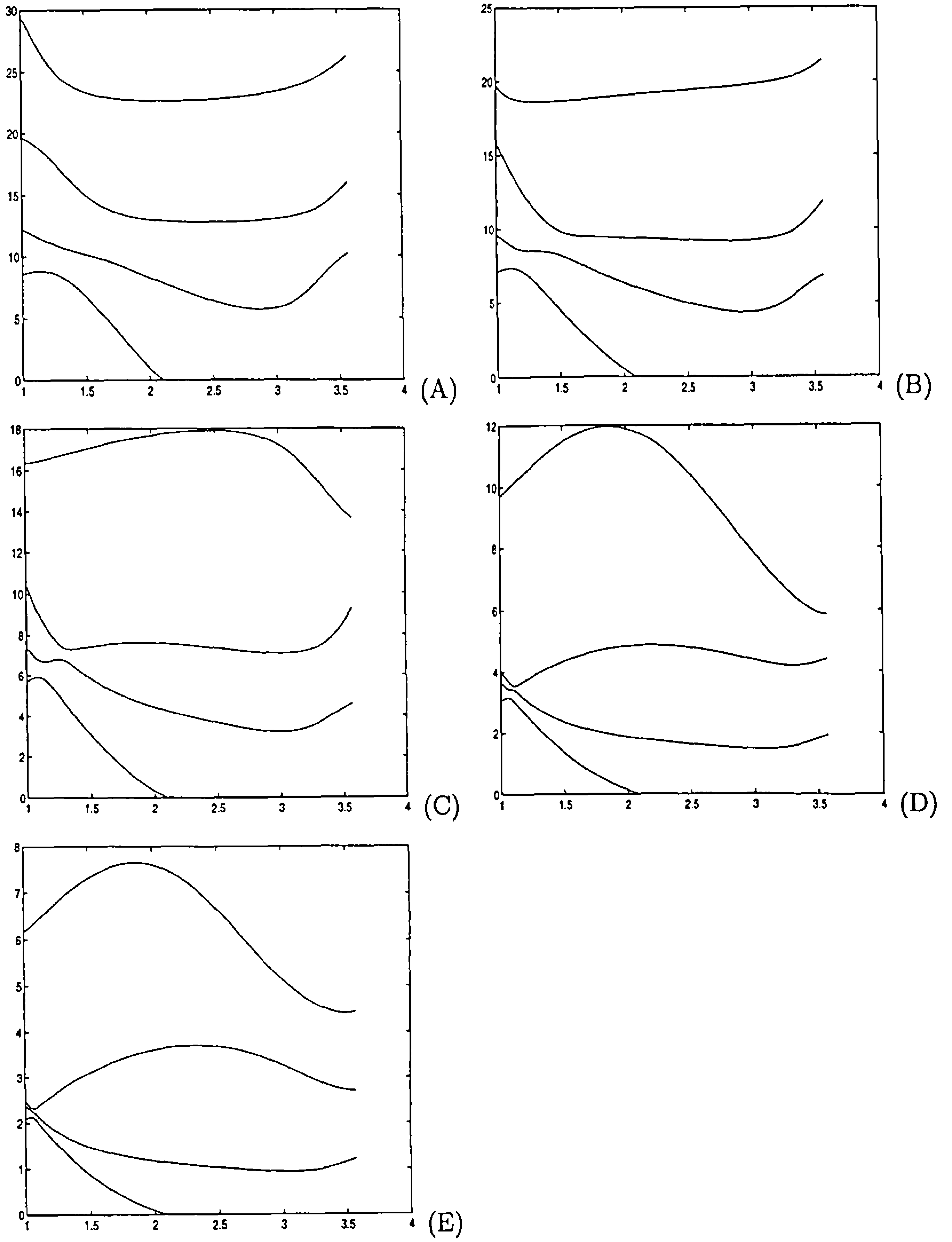

Figure 6.20: Frequency-stretch plot: coating on P^- for $H/A = 1$ and $\hat{n} = 0.02$ where (A) $\rho_c/\rho = 0.1$, (B) $\rho_c/\rho = 0.5$, (C) $\rho_c/\rho = 1$,(D) $\rho_c/\rho = 3$, (E) $\rho_c/\rho = 5$.

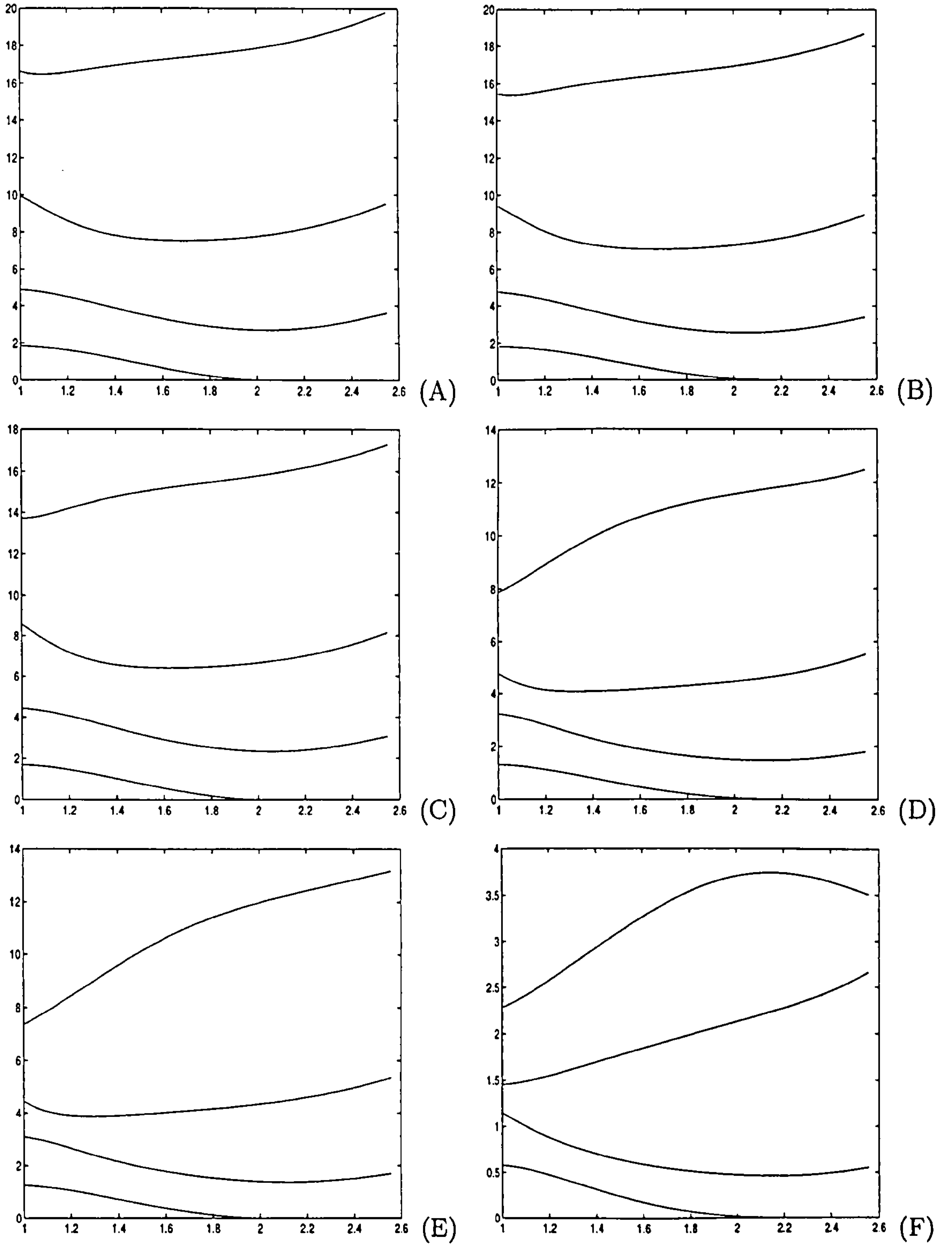

Figure 6.21: Frequency-stretch plot: coating on P^- for $H/A = 2$ where (A) $\rho_c/\rho = 0.1$ and $\hat{n} = 0.001$, (B) $\rho_c/\rho = 0.1$ and $\hat{n} = 0.02$, (C) $\rho_c/\rho = 1$ and $\hat{n} = 0.001$, (D) $\rho_c/\rho = 1$ and $\hat{n} = 0.02$, (E) $\rho_c/\rho = 5$ and $\hat{n} = 0.001$, (F) $\rho_c/\rho = 5$ and $\hat{n} = 0.02$.

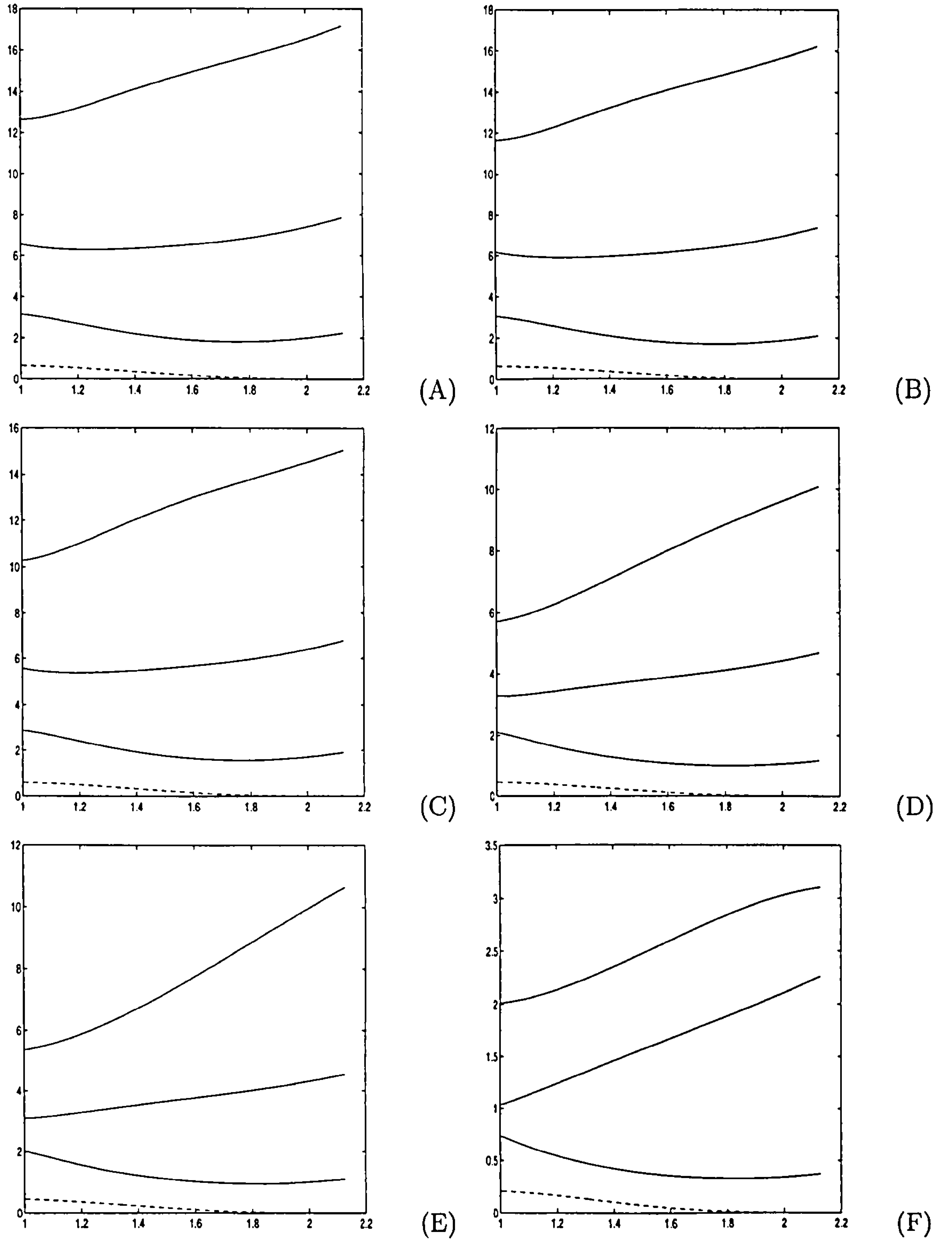

Figure 6.22: Frequency-stretch plot: coating on P^- for $H/A = 3$ where (A) $\rho_c/\rho = 0.1$ and $\hat{n} = 0.001$, (B) $\rho_c/\rho = 0.1$ and $\hat{n} = 0.02$, (C) $\rho_c/\rho = 1$ and $\hat{n} = 0.001$, (D) $\rho_c/\rho = 1$ and $\hat{n} = 0.02$, (E) $\rho_c/\rho = 5$ and $\hat{n} = 0.001$, (F) $\rho_c/\rho = 5$ and $\hat{n} = 0.02$.

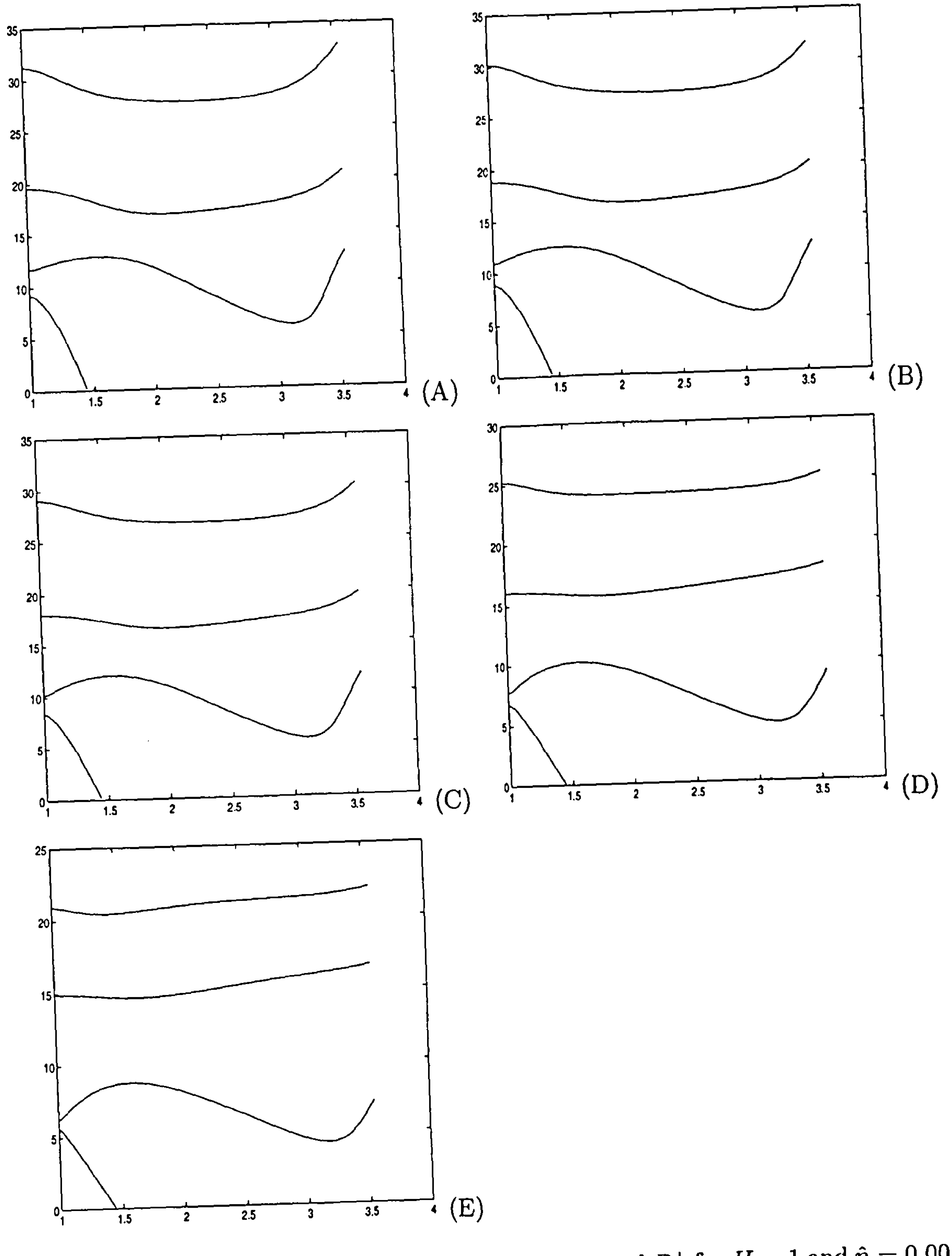

Figure 6.23: Frequency-stretch plot: coating on P^- and P^+ for $H = 1$ and $\hat{n} = 0.001$ where (A) $\rho_c/\rho = 0.1$, (B) $\rho_c/\rho = 0.5$, (C) $\rho_c/\rho = 1$, (D) $\rho_c/\rho = 3$, (E) $\rho_c/\rho = 5$.

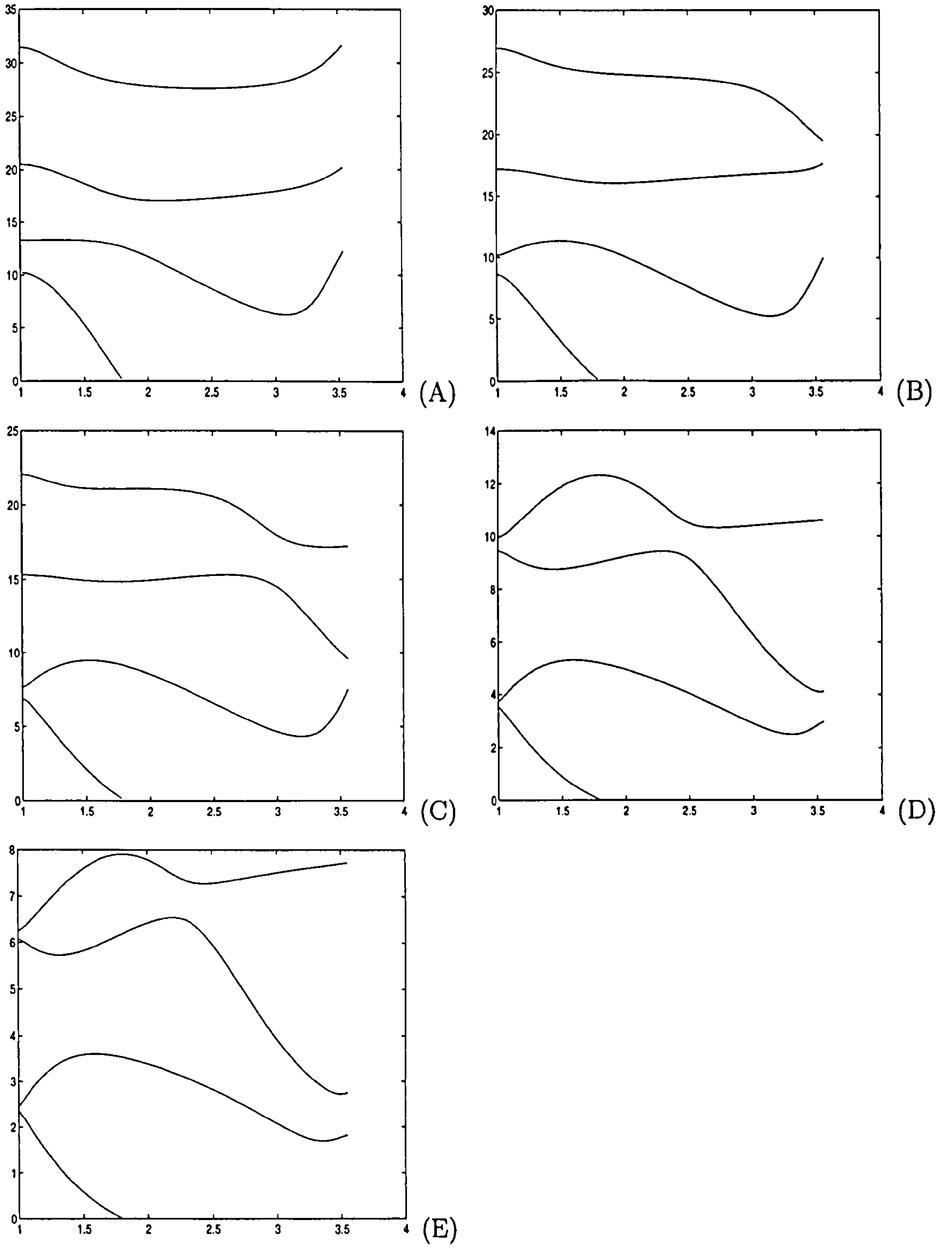

Figure 6.24: Frequency-stretch plot: coating on P^- and P^+ for $H = 1$ and $\hat{n} = 0.02$ where (A) $\rho_c/\rho = 0.1$, (B) $\rho_c/\rho = 0.5$, (C) $\rho_c/\rho = 1$, (D) $\rho_c/\rho = 3$, (E) $\rho_c/\rho = 5$.

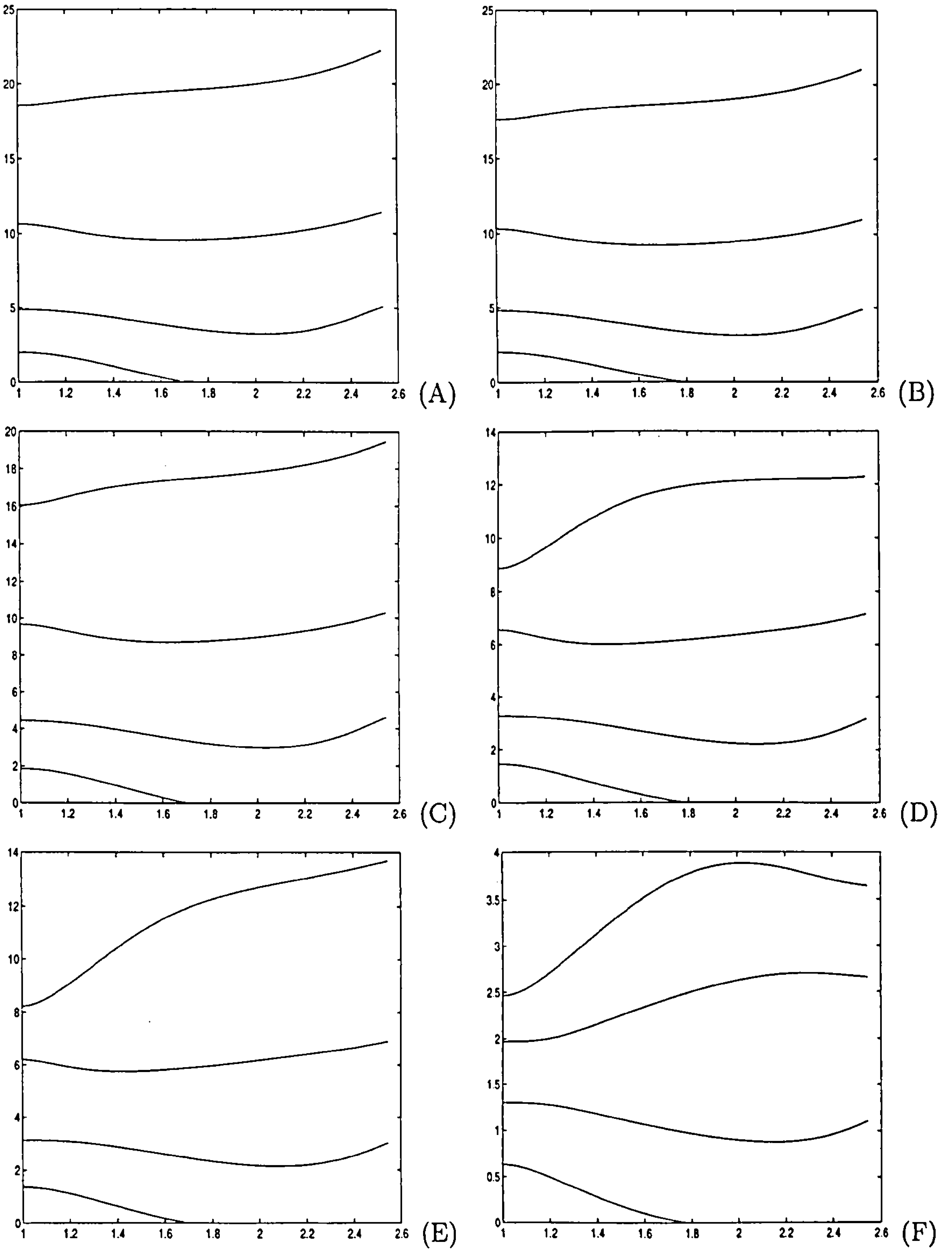

Figure 6.25: Frequency-stretch plot: coating on P^- and P^+ for $H/A = 2$ where (A) $\rho_c/\rho = 0.1$ and $\hat{n} = 0.001$, (B) $\rho_c/\rho = 0.1$ and $\hat{n} = 0.02$, (C) $\rho_c/\rho = 1$ and $\hat{n} = 0.001$, (D) $\rho_c/\rho = 1$ and $\hat{n} = 0.02$, (E) $\rho_c/\rho = 5$ and $\hat{n} = 0.001$, (F) $\rho_c/\rho = 5$ and $\hat{n} = 0.02$.

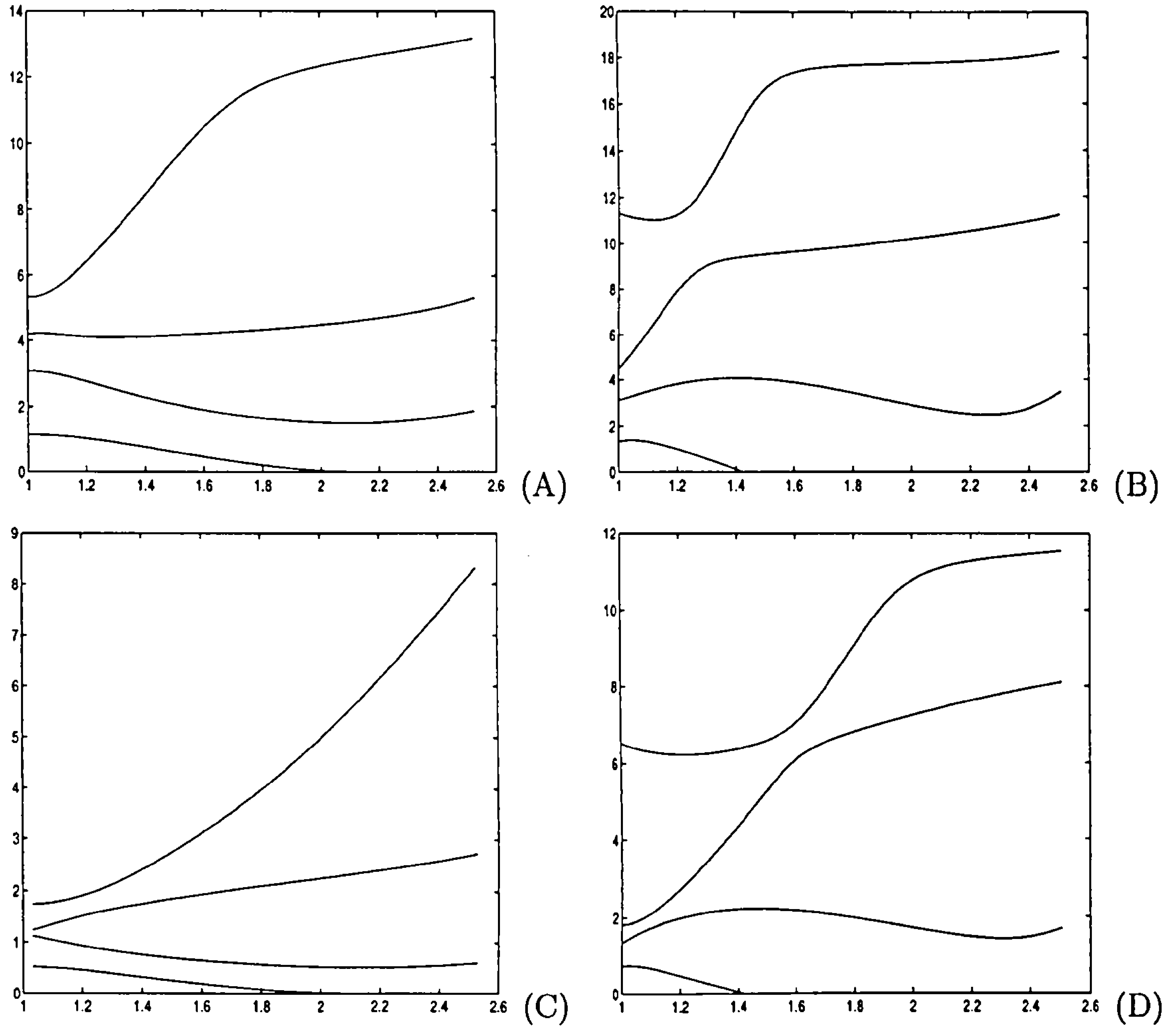

Figure 6.26: Frequency-stretch plot: coating on P^+ for $H/A = 2$ and $\rho_c/\rho = 5$ where (A) $\hat{m} = 0.1$ and $\hat{n} = 0.0001$, (B) $\hat{m} = 5$ and $\hat{n} = 0.005$, (C) $\hat{m} = 0.1$ and $\hat{n} = 0.002$, (D) $\hat{m} = 5$ and $\hat{n} = 0.07$.

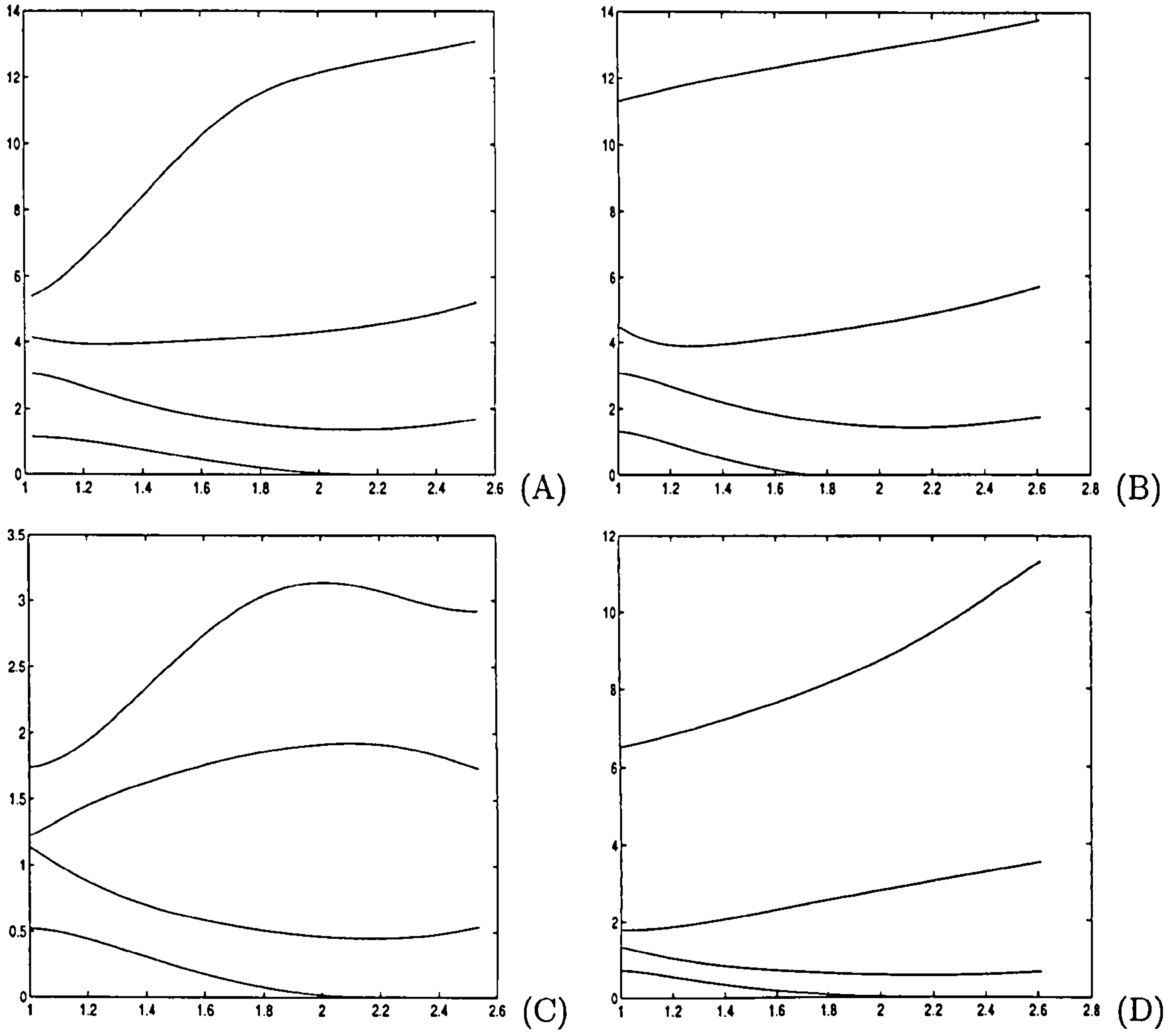

Figure 6.27: Frequency-stretch plot: coating on P^- for $H/A = 2$ and $\rho_c/\rho = 5$ where (A) $\hat{m} = 0.1$ and $\hat{n} = 0.0001$, (B) $\hat{m} = 5$ and $\hat{n} = 0.005$, (C) $\hat{m} = 0.1$ and $\hat{n} = 0.002$, (D) $\hat{m} = 5$ and $\hat{n} = 0.07$.

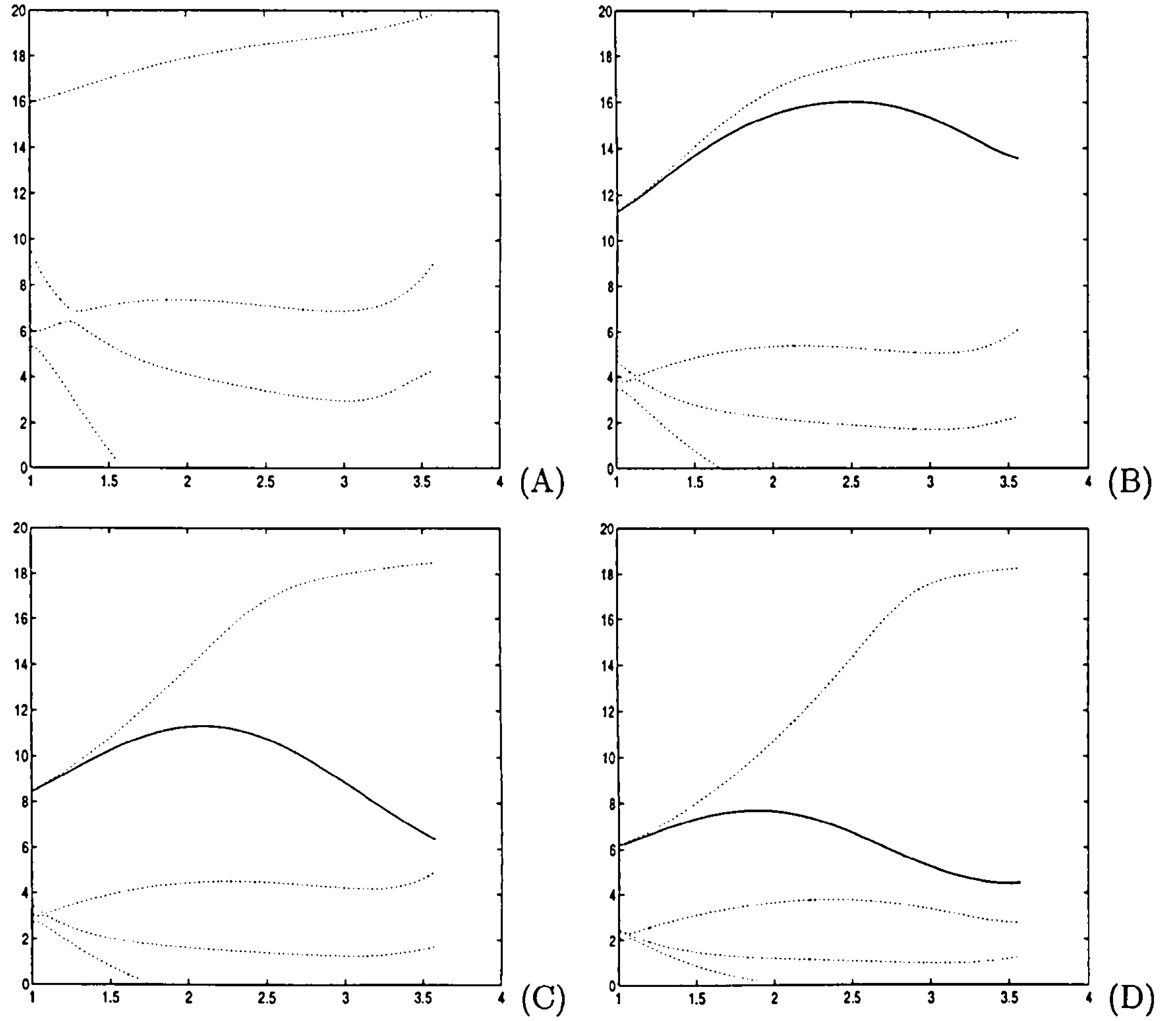

Figure 6.28: Frequency-stretch plot: coating on P^- with $\hat{I} = 0$ for $H/A = 1$ and $\rho_c/\rho = 5$ where (A) $\hat{n} = 0.001$, (B) $\hat{n} = 0.005$, (C) $\hat{n} = 0.01$, (D) $\hat{n} = 0.02$; solutions without rotatory inertia (dotted curves).

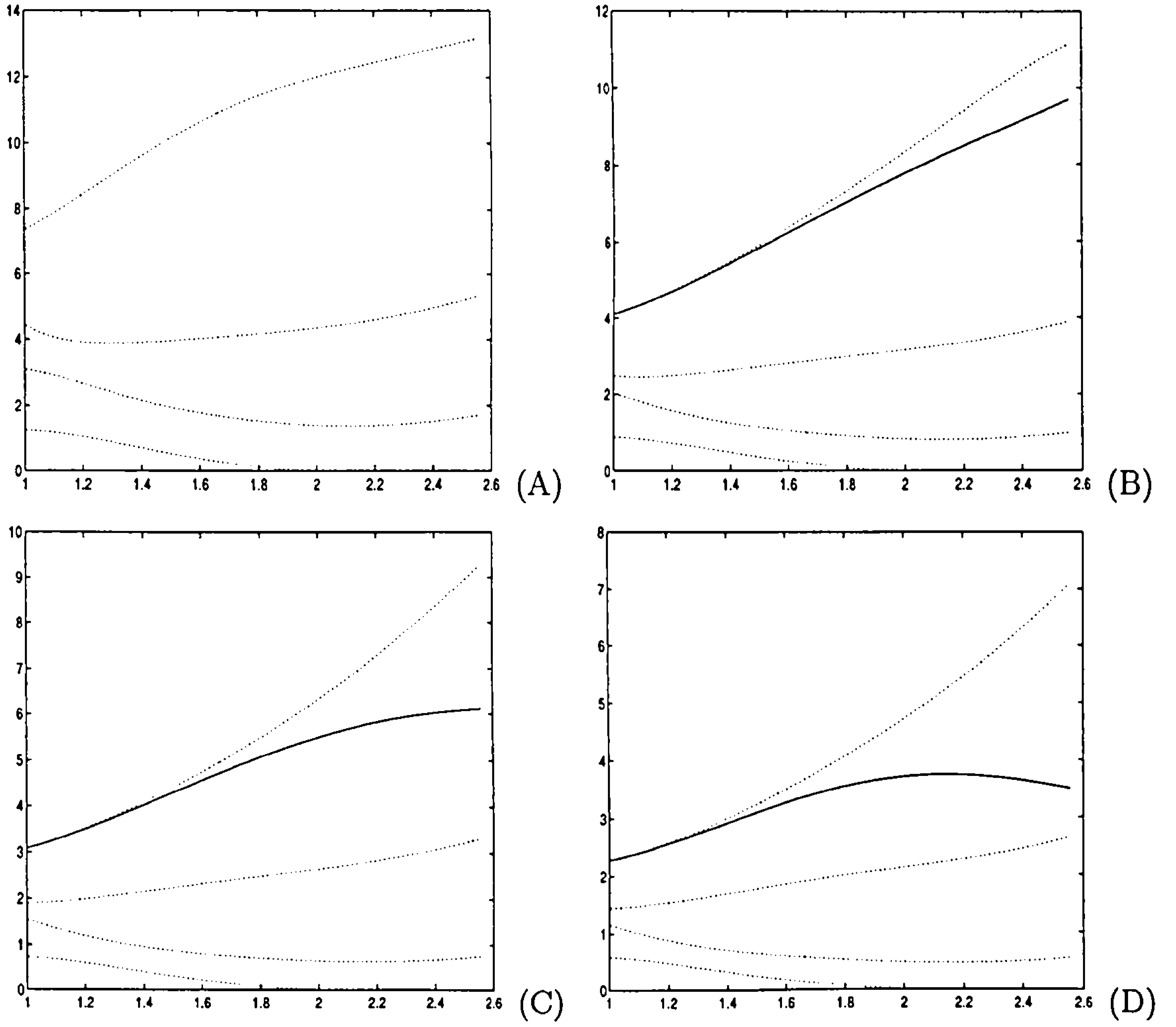

Figure 6.29: Frequency-stretch plot: coating on P^- with $\hat{I} = 0$ for $H/A = 2$ and $\rho_c/\rho = 5$ where (A) $\hat{n} = 0.001$, (B) $\hat{n} = 0.005$, (C) $\hat{n} = 0.01$, (D) $\hat{n} = 0.02$; solutions without rotatory inertia (dotted curves).

Chapter 7

Shear of a bonded elastic block

7.1 Preliminaries

In terms of the theory of non-linear elasticity, we now study the shear of compressible, isotropic elastic material bonded to two rigid parallel plates. We restrict attention to the plane-strain problem of a rectangular block with traction-free in-plane lateral surfaces, examining the behaviour of the shear force as a function of the applied strain and utilizing maximum and minimum energy arguments to construct upper and lower bounds on the energy stored in the body as a result of the deformation.

7.2 Variational and extremum principles

To set our use of extremum principles in context, we first establish some basic variational principles, more detailed discussion of which can be found in, for example, Ogden (1984), Chapter 5.

7.2.1 Variational principles

A boundary value problem is formulated for a compressible, hyperelastic body in equilibrium. This gives rise to the governing equations

$$\operatorname{Div}\mathbf{S} + \rho_r \mathbf{b} = \mathbf{0}, \tag{7.2.1}$$

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{A}}, \tag{7.2.2}$$

$$\mathbf{A} = \operatorname{Grad}\chi \tag{7.2.3}$$

for all $\mathbf{X} \in \mathcal{B}_0$ (following the notation of Chapter 2). Boundary conditions of place and traction are specified on the boundary $\partial\mathcal{B}_0$ with displacement prescribed on $\partial\mathcal{B}_0^x$ and traction on $\partial\mathcal{B}_0^t$ such that $\partial\mathcal{B}_0 = \partial\mathcal{B}_0^x \cup \partial\mathcal{B}_0^t$ and $\partial\mathcal{B}_0^x \cap \partial\mathcal{B}_0^t = \emptyset$. Then we prescribe

$$\mathbf{x} = \boldsymbol{\xi}(\mathbf{X}) \quad \text{on}\, \partial\mathcal{B}_0^x, \tag{7.2.4}$$

$$\mathbf{S}^{\mathrm{T}}\mathbf{N} = \mathbf{t}_0(\mathbf{X}) \quad \text{on}\, \partial\mathcal{B}_0^t, \tag{7.2.5}$$

where $\boldsymbol{\xi}$ is a known function of $\partial\mathcal{B}_0^x$ and, in general, $\mathbf{t}_0$ depends on both χ and $\mathbf{A}$ on $\partial\mathcal{B}_0^t$.

A *kinematically admissible deformation* is one which satisfies the displacement boundary condition (7.2.4). The stress associated with this deformation through the deformation gradient (7.2.3) and (7.2.2) will not necessarily satisfy the equilibrium equation (7.2.1) or boundary condition (7.2.5).

Similarly, a *statically admissible stress field* satisfies the traction boundary condition (7.2.5) and the equilibrium equations (7.2.1) with $\mathbf{b}$ and $\mathbf{t}_0$ fixed for a kinematically admissible deformation.

Combining the equilibrium equations and boundary conditions (7.2.4) and (7.2.5) with use of the divergence theorem, yields

$$\int_{\partial\mathcal{B}_0^t} \mathbf{t}_0 \,.\, \chi \, dA + \int_{\partial\mathcal{B}_0^x} \mathbf{S}^{\mathrm{T}}\mathbf{N} \,.\, \xi \, dA = \int_{\mathcal{B}_0} \mathrm{tr}\,(\mathbf{S}\mathbf{A}) \, dV - \int_{\mathcal{B}_0} \rho_r \mathbf{b} \,.\, \chi \, dV. \tag{7.2.6}$$

for kinematically admissible deformation fields χ and statically admissible stress fields $\mathbf{S}$.

Suppose χ^* is a kinematically admissible deformation then it is true that

$$\int_{\partial\mathcal{B}_0^t} \mathbf{t}_0 \,.\, \chi^* \, dA + \int_{\partial\mathcal{B}_0^x} \mathbf{S}^{\mathrm{T}}\mathbf{N} \,.\, \xi \, dA = \int_{\mathcal{B}_0} \mathrm{tr}\,(\mathbf{S}\mathbf{A}^*) \, dV - \int_{\mathcal{B}_0} \rho_r \mathbf{b} \,.\, \chi^* \, dV. \tag{7.2.7}$$

where $\mathbf{A}^* = \mathrm{Grad}\,\chi^*$.

By defining $\delta\chi = \chi^* - \chi$ and subtracting (7.2.6) from (7.2.7) we obtain the *principle of virtual work*

$$\int_{\partial\mathcal{B}_0^t} \mathbf{t}_0 \,.\, \delta\chi \, dA + \int_{\mathcal{B}_0} \rho_r \mathbf{b} \,.\, \delta\chi \, dV = \int_{\mathcal{B}_0} \mathrm{tr}\,[\mathbf{S}\mathrm{Grad}\delta\chi] \, dV, \tag{7.2.8}$$

which holds for a statically admissible stress field and an arbitrary virtual displacement $\delta\chi$ subject to $\delta\chi$ vanishing on $\partial\mathcal{B}_0^x$.

For *conservative* body forces we set

$$\mathbf{b} = -\mathrm{grad}\phi, \tag{7.2.9}$$

where ϕ is a scalar function of χ and $\mathbf{b} \cdot \delta\chi = -\delta\phi$. We also restrict attention to *dead-loading* surface traction so that

$$\mathbf{t}_0 = -\mathrm{grad}\psi, \tag{7.2.10}$$

where ψ is a scalar function of χ and $\psi = -\mathbf{t}_0 \cdot \chi$. Now, making use of the first order approximation

$$\delta W = \mathrm{tr}\left(\frac{\partial \mathrm{W}}{\partial \mathbf{A}} \delta\mathbf{A}\right), \tag{7.2.11}$$

together with (7.2.9) and (7.2.10), allows (7.2.8) to be rewritten as the variation of a functional of χ. Hence,

$$\delta E = \int_{\partial\mathcal{B}_0^t} (\mathbf{S}^{\mathrm{T}}\mathbf{N} - \mathbf{t}_0) \,.\, \delta\chi \, dA - \int_{\mathcal{B}_0} (\mathrm{Div}\,\mathbf{S} + \rho_r \mathbf{b}) \,.\, \delta\chi \, dV \tag{7.2.12}$$

where the potential energy, $E\{\chi\}$, takes the form

$$E\{\chi\} = \int_{\mathcal{B}_0} (W + \rho_r \phi) \, dV - \int_{\partial\mathcal{B}_0^t} \mathbf{t}_0 \cdot \chi \, dA. \tag{7.2.13}$$

Then the *principle of stationary potential energy* states that any (twice continuously differentiable) solution of the boundary value problem makes stationary the functional (7.2.13) within the class of kinematically admissible deformations.

This concept is strengthened, in the context of linear elasticity, and becomes the *principle of minimum potential energy*: of all displacements satisfying the given boundary conditions, those which satisfy the equilibrium equations make the potential energy an absolute minimum. In fact, this minimum principle is applicable in both linear and non-linear settings under certain circumstances, which we now examine.

7.2.2 Extremum principles

A χ which satisfies the above principle of stationary potential energy is a local minimizer of $E\{\cdot\}$ provided

$$\int_{\mathcal{B}_0} \mathrm{tr}\{(\boldsymbol{\mathcal{A}}\delta\mathbf{A})\delta\mathbf{A}\}\mathrm{dV} > 0$$

for $\delta\mathbf{A} \neq 0$. We examine the local version of this inequality, namely

$$\mathrm{tr}\{(\boldsymbol{\mathcal{A}}\delta\mathbf{A})\delta\mathbf{A}\}\mathrm{dV} > 0 \tag{7.2.14}$$

which, for an isotropic material referred to principal axes, has the form

$$\begin{aligned}
\operatorname{tr}\{(\mathcal{A}\delta\mathbf{A})\delta\mathbf{A}\} &= \sum_{i,j=1}^{3} \frac{\partial t_i}{\partial \lambda_j}\delta\lambda_i\delta\lambda_j \\
&+ \sum_{i\neq j}(\Omega_{ij}^{(L)} + \frac{1}{2}\Omega_{ij}^{(R)})^2(t_i - t_j)(\lambda_i - \lambda_j) \\
&+ \frac{1}{4}\sum_{i\neq j}(\Omega_{ij}^{(R)})^2(t_i + t_j)(\lambda_i + \lambda_j) > 0,
\end{aligned} \tag{7.2.15}$$

using (2.4.17)-(2.4.20) where t_i are the principal Biot stresses defined in (2.3.19), $\Omega_{ij}^{(L)}$ are the components of $\mathbf{\Omega}^L = \sum_{i=1}^{3} \delta\mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}$ (the incremental rotation of the Lagrangian axes) and $\Omega_{ij}^{(R)}$ are the components of $\mathbf{\Omega}^R = \mathbf{R}^T\delta\mathbf{R}$ on the same axes, $\mathbf{R}$ being the rotation tensor arising in the polar decomposition (2.1.9). Necessary and sufficient conditions for (7.2.15) to hold are

$$\begin{array}{lll}
\text{(i)} & \text{The Jacobian matrix } (\partial t_i/\partial\lambda_j) \text{ is positive definite,} & \\
\text{(ii)} & (t_i - t_j)/(\lambda_i - \lambda_j) > 0, \quad i \neq j, & \\
\text{(iii)} & t_i + t_j > 0, \quad i \neq j, &
\end{array} \tag{7.2.16}$$

where $i, j \in \{1, 2, 3\}$ and (ii) includes the case $\lambda_i = \lambda_j$ by an appropriate limiting process [see Ogden (1984) for further details].

For an objective W, conditions (i) and (ii) are necessary and sufficient for $\mathbf{T}(\mathbf{U}) = \partial W/\partial\mathbf{U}$ to be strictly *locally* convex. Then, if (i) holds for all $\lambda_i > 0$, it follows that (ii) holds and, hence, $W(\mathbf{U})$ is strictly *globally* convex, i.e.

$$W(\mathbf{U}^*) - W(\mathbf{U}) - \operatorname{tr}\{\mathbf{T}(\mathbf{U}^* - \mathbf{U})\} > 0 \tag{7.2.17}$$

for all positive definite symmetric second-order tensors $\mathbf{U}$ and $\mathbf{U}^* \neq \mathbf{U}$ [see Ogden (1984), Appendix 1].

The additional requirement for $W(\mathbf{A})$ to be strictly *locally* convex, over and above (i) and (ii), is (iii). If the region in λ_i-space where (iii) holds is convex, then it can be deduced that $W(\mathbf{A})$ is strictly *globally* convex in this region.

This ensures uniqueness of solution and, hence, the existence of a global minimizer χ is guaranteed. It follows that, in these circumstances, the stationary energy principle is strengthened to a minimum energy principle.

Thus, provided the conditions in (7.2.16) are met, we are justified in our use of minimum energy arguments with respect to both linear and non-linear theory.

Our use of a complementary energy argument is, likewise, justified since the fulfillment of condition (iii) in (7.2.16) ensures the stress field $\mathbf{S}$ (corresponding to the minimizer χ) is uniquely defined, as we now show.

For an actual solution χ, the variation, δE, in the energy functional may be re-expressed in the form

$$\delta\left\{\int_{\partial\mathcal{B}_0^{\tau}}(\mathbf{S}^{\mathrm{T}}\mathbf{N})\,.\,\boldsymbol{\xi}\,dA-\int_{\mathcal{B}_0}[\mathrm{tr}\,(\mathbf{SA})-W(\mathbf{A})]\,dV\right\}=0. \tag{7.2.18}$$

Since, from (2.3.15), $W(\mathbf{A}) = W(\mathbf{U})$ and $\mathrm{tr}\,(\mathbf{SA}) = \mathrm{tr}\,(\mathbf{TU})$ we can replace the integrand in the volume integral in (7.2.18) by $\mathrm{tr}\,(\mathbf{TU}) - W(\mathbf{U})$.

Now, from (7.2.17), we assume $W(\mathbf{U})$ is a strictly convex function of $\mathbf{U}$ so that the Biot stress, $\mathbf{T}$, is uniquely invertible i.e. $\mathbf{U}$ is uniquely determined by $\mathbf{T}$. This allows us to define the *complementary energy function* W_c as the Legendre transform of W with respect to $\mathbf{U}$ by

$$W_c(\mathbf{T}) = \mathrm{tr}\,(\mathbf{TU}) - W(\mathbf{U}), \tag{7.2.19}$$

then, if in the right-hand side of (7.2.19), $\mathbf{U}$ is replaced by the unique function of $\mathbf{T}$

obtained by inversion of (2.3.16), it follows that

$$\mathbf{U} = \frac{\partial W_c}{\partial \mathbf{T}}(\mathbf{T}). \tag{7.2.20}$$

Given a stress field $\mathbf{S}$, for an isotropic elastic material the polar decomposition (2.2.5) has four distinct forms, each with a different $\mathbf{T}$, $\mathbf{R}$ pair [a detailed discussion of which can be found in Ogden (1984), Chapter 6]. Because of the assumed convexity of $W(\mathbf{U})$, $\mathbf{S}$ is associated with four distinct deformation gradients $\mathbf{A}$ (thus, there are four distinct branches of the inversion of (2.2.5)).

Since $W_c(\mathbf{T})$ equates with the appropriate branch of $W_c(\mathbf{S})$, we consider (7.2.12) as a functional of $\mathbf{S}$ and write

$$E_c\{\mathbf{S}\} = \int_{\partial\mathcal{B}_0^x} (\mathbf{S}^{\mathbf{T}}\mathbf{N}) \, . \, \boldsymbol{\xi} \, dA - \int_{\mathcal{B}_0} W_c(\mathbf{T}) \, dV, \tag{7.2.21}$$

which holds for statically admissible stress fields $\mathbf{S}$ and kinematically admissible deformation fields χ and the appropriate $\mathbf{T}$ is obtained, using (2.2.5) from

$$\mathbf{S}\mathbf{S}^{\mathrm{T}} = \mathbf{T}^2. \tag{7.2.22}$$

This defines uniquely the principal axes of $\mathbf{T}^2$, and hence of $\mathbf{T}$. Consequently, the values of t_1^2, t_2^2, t_3^2 are also defined. From (2.2.5) we have $t_1 t_2 t_3 = \det \mathbf{T} = \det \mathbf{S}$, we deduce that for a given $\mathbf{S}$ there are four possible sign combinations for t_1, t_2, t_3.

It emerges that only one branch of the global inversion of $\mathbf{S}$ ensures stability: that corresponding to $t_i + t_j > 0$, $i, j \in \{1, 2, 3\}$. Hence, provided condition (iii) of (7.2.16) is satisfied the decomposition of $\mathbf{S}$ is unique.

7.3 Minimum energy and maximum complementary energy

7.3.1 Minimum energy

Recall from Section 7.2 that any (twice continuously differentiable) solution of the boundary value problem (7.2.1)-(7.2.5) makes stationary the functional

$$E\{\chi\} = \int_{\mathcal{B}_0} W(\text{Grad}\chi)\, dV - \int_{\partial\mathcal{B}_0^t} \mathbf{t}_0 \cdot \chi\, dA \qquad (7.3.1)$$

where body forces have been excluded, provided χ is within the class $\mathcal{K}$ of kinematically admissible deformations defined by

$$\mathcal{K} = \{\chi : \chi \in C^2(\mathcal{B}_0),\ \chi = \xi \text{ on } \partial\mathcal{B}_0^x\}. \qquad (7.3.2)$$

Then, if χ is a solution of the boundary value problem and $\chi^* \in \mathcal{K}$, it can be shown that

$$E\{\chi^*\} - E\{\chi\} = \int_{\mathcal{B}_0} [W(\mathbf{A}^*) - W(\mathbf{A}) - \text{tr}\,\{\mathbf{S}(\mathbf{A}^* - \mathbf{A})\}]\, dV, \qquad (7.3.3)$$

where $\mathbf{A}^* = \text{Grad}\,\chi^*$. As W is not necessarily a convex function of $\mathbf{A}$, the integrand in (7.3.3) need not be strictly positive for $A^* \neq A$, hence $E\{\chi^*\}$ is not (generally) minimized for $\chi^* = \chi$. However, in general, for locally stable solutions χ

$$E\{\chi^*\} \geq E\{\chi\}, \quad \chi^* \in \mathcal{K}. \qquad (7.3.4)$$

In terms of linear elasticity, (7.3.4) is called the *principle of minimum potential energy.* On fulfillment of the conditions in (7.2.16), the inequality in (7.3.4) becomes strict for $\chi \neq \chi^*$ and the minimum energy principle is then applicable in both the linear and non-linear theories.

We note that (7.3.3) can be rewritten in the form

$$\int_{\mathcal{B}_0} [W(\mathbf{U}^*) - W(\mathbf{U}) - \operatorname{tr}\{\mathbf{S}(\mathbf{A}^* - \mathbf{A})\}]\, dV \geq 0, \tag{7.3.5}$$

where $\mathbf{U}^*$ is the right stretch tensor associated with $\mathbf{A}^*$.

7.3.2 Maximum complementary energy

Recalling from (7.2.21), in terms of $\mathbf{S}$, the energy function takes the form

$$E_c\{\mathbf{S}\} = \int_{\partial\mathcal{B}_0^x} (\mathbf{S}^{\mathrm{T}}\mathbf{N}) \,.\, \boldsymbol{\xi}\, dA - \int_{\mathcal{B}_0} W_c(\mathbf{T})\, dV \tag{7.3.6}$$

for an actual solution χ. With reference to its definition in Section 7.2, we now define the set of statically admissible stress fields by

$$\Sigma = \{\mathbf{S} : \mathbf{S} \in C^1(\mathcal{B}_0),\ \operatorname{Div}\mathbf{S} = 0 \operatorname{in} \mathcal{B}_0,\ \mathbf{S}^{\mathrm{T}}\mathbf{N} = \mathbf{t}_0 \operatorname{on} \partial\mathcal{B}_0^t\}. \tag{7.3.7}$$

Then, for any $\mathbf{S}^* \in \Sigma$, we may write

$$E_c\{\mathbf{S}^*\} = \int_{\partial\mathcal{B}_0^x} (\mathbf{S}^{*\mathrm{T}}\mathbf{N}) \,.\, \boldsymbol{\xi}\, dA - \int_{\mathcal{B}_0} W_c(\mathbf{T}^*)\, dV, \tag{7.3.8}$$

where $\mathbf{T}^*$ is obtained from the polar decomposition of $\mathbf{S}^*$. It follows that

$$E_c\{\mathbf{S}\} - E_c\{\mathbf{S}^*\} = \int_{\mathcal{B}_0} [W_c(\mathbf{T}^*) - W_c(\mathbf{T}) - \operatorname{tr}\{\mathbf{A}(\mathbf{S}^* - \mathbf{S})\}]\, dV. \tag{7.3.9}$$

Substituting $W_c(\mathbf{T})$ into (7.2.17), using (7.2.19), it can be shown that strict convexity of $W(\mathbf{U})$ implies that of $W_c(\mathbf{T})$. Then, noting the similar structure of the integrands in (7.3.5) and (7.3.9) we assume the integral in (7.3.9) is also non-negative for $\mathbf{S}^* \in \Sigma$ where $\mathbf{S}$ is the particular stress corresponding to the global minimizer χ. Consequently, we have

$$E_c\{\mathbf{S}\} \geq E_c\{\mathbf{S}^*\}, \quad \mathbf{S}^* \in \Sigma. \tag{7.3.10}$$

In the linear theory, this inequality holds and is referred to as the *principle of maximum complementary energy*. As in the case of minimum energy, provided the conditions in (7.2.16) are satisfied, the inequality in (7.3.10) becomes strict and this maximum energy principle is justified in both linear and non-linear settings.

7.3.3 Theoretical energy bounds

Combining (7.3.4) and (7.3.10) yields a chain of inequalities in the form

$$E\{\chi^*\} \geq E\{\chi\} = E_c\{\mathrm{S}\} \geq E_c\{\mathrm{S}^*\} \quad \chi^* \in \mathcal{K},\ \mathrm{S}^* \in \Sigma. \tag{7.3.11}$$

These theoretical upper and lower bounds on the energy functional are justified in the non-linear theory assuming we restrict attention to deformation and stress fields and forms of strain-energy function which satisfy the conditions in (7.2.16). Effectively, we require that (i) and (ii) hold and that (iii) defines a convex region in λ_i-space. Note (iii) defines, quite generally, a convex region in t_i-space.

7.3.4 A particular strain-energy function

For an isotropic material, W_c is a symmetric function of the principal values of $\mathbf{T}$ such that

$$W_c(t_1,\ t_2,\ t_3) = t_1\lambda_1 + t_2\lambda_2 + t_3\lambda_3 - W(\lambda_1,\ \lambda_2,\ \lambda_3), \tag{7.3.12}$$

where

$$\lambda_i = \frac{\partial W_c}{\partial t_i} \quad i \in \{1,\ 2,\ 3\}, \tag{7.3.13}$$

Note that inversion of $t_i = \partial W/\partial\lambda_i$, defined in (2.3.19), is required to determine the form of $W_c(t_1,\ t_2,\ t_3)$. If $W(\mathrm{U})$ is strictly convex, then $W(\lambda_1,\ \lambda_2,\ \lambda_3)$ is a strictly convex function of λ_1, λ_2, λ_3 and thus the inversion of (2.3.19) is unique.

A strain-energy function which allows explicit derivation of $W_c(t_1, t_2, t_3)$ is that pertaining to the *semi-linear material*

$$W = \frac{E}{2(1+\nu)}\{(\lambda_1-1)^2+(\lambda_2-1)^2+(\lambda_3-1)^2\}+\frac{\nu E}{2(1+\nu)(1-2\nu)}(\lambda_1+\lambda_2+\lambda_3-3)^2, \tag{7.3.14}$$

where E and ν are the classical Young's modulus and Poisson's ratio respectively. Then (2.3.19) gives

$$t_i = \frac{E}{(1+\nu)}(\lambda_i-1)+\frac{\nu E}{(1+\nu)(1-2\nu)}(\lambda_1+\lambda_2+\lambda_3-3) \quad i\in\{1,\,2,\,3\} \tag{7.3.15}$$

and it follows that

$$W_c = \frac{1+\nu}{2E}(t_1^2+t_2^2+t_3^2)-\frac{\nu}{2E}(t_1+t_2+t_3)^2+t_1+t_2+t_3. \tag{7.3.16}$$

With reference to the requirements which ensure the theoretical energy bounds in (7.3.11) hold, we note that the semi-linear material does give a convex region in λ_i-space: using (7.3.15), condition (iii) in (7.2.16) becomes

$$t_i+t_j = \frac{E}{(1+\nu)}\left[(\lambda_i+\lambda_j-2)+\frac{2\nu}{(1-2\nu)}(\lambda_1+\lambda_2+\lambda_3-3)\right] > 0, \tag{7.3.17}$$

where $i,j\in\{1,2,3\}$ and t_i+t_j are then linear in λ_1, λ_2 and λ_3 with each $t_i+t_j=0$ defining a plane, with all the planes passing through the origin. The resulting region $t_i+t_j>0$ is, therefore, convex in λ_i-space.

For the shear (plane strain) problem in question, we have $\lambda_3=1$ and t_3 is then given by

$$t_3 = \frac{\partial W}{\partial \lambda_3}(\lambda_1,\,\lambda_2,\,1). \tag{7.3.18}$$

The problem can now be considered as two-dimensional with strain-energy function

$$\tilde{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, 1) \tag{7.3.19}$$

so that the (in-plane) principal Biot stresses are given by

$$t_1 = \frac{\partial \tilde{W}}{\partial \lambda_1}, \quad t_2 = \frac{\partial \tilde{W}}{\partial \lambda_2}. \tag{7.3.20}$$

In the present (two-dimensional) context, the appropriate necessary and sufficient conditions for (7.2.15) to hold are

(i) The Jacobian matrix $(\partial t_i / \partial \lambda_j)$ is positive definite,

(ii) $(t_1 - t_2)/(\lambda_1 - \lambda_2) > 0,$ (7.3.21)

(iii) $t_1 + t_2 > 0,$

where $i, j \in \{1, 2\}$ and $\lambda_3 = 1$. Now, convexity of $\tilde{W}(\lambda_1, \lambda_2)$ follows from that of $W(\lambda_1, \lambda_2, 1)$, and we may define, by analogy with (7.3.12), the two-dimensional complementary energy

$$\tilde{W}_c(t_1, t_2) = \lambda_1 t_1 + \lambda_2 t_2 - \tilde{W}(\lambda_1, \lambda_2). \tag{7.3.22}$$

Then,

$$\lambda_1 = \frac{\partial \tilde{W}_c}{\partial t_1}, \quad \lambda_2 = \frac{\partial \tilde{W}_c}{\partial t_2} \tag{7.3.23}$$

are uniquely determined by t_1 and t_2 with t_3 obtained from (7.3.18), which plays the role of maintaining $\lambda_3 = 1$. Hence, the semi-linear strain-energy function (7.3.14) has the two-dimensional form

$$\tilde{W}(\lambda_1, \lambda_2) = \frac{E}{2(1+\nu)}\{(\lambda_1 - 1)^2 + (\lambda_2 - 1)^2\} + \frac{\nu E}{2(1+\nu)(1-2\nu)}(\lambda_1 + \lambda_2 - 2)^2, \tag{7.3.24}$$

and the complementary energy function (7.3.16) becomes

$$W_c = \frac{1+\nu}{2E}\left(t_1^2 + t_2^2\right) - \frac{\nu(1+\nu)}{2E}\left(t_1 + t_2\right)^2 + t_1 + t_2. \tag{7.3.25}$$

7.4 Application to the shear of a bonded elastic block

7.4.1 Shear deformation

We consider a rectangular block of compressible elastic material under plane-strain conditions with reference configuration, $\mathcal{B}_0$, defined by

$$0 \leq X_1 \leq A, \quad 0 \leq X_2 \leq H.$$

The block, with aspect ratio $\eta = H/A$, is bonded to rigid parallel plates at $X_2 = 0,\ H$ and is deformed by fixing the base metal plate and shearing the top plate by an amount γ. The lateral surfaces $X_1 = 0,\ A$ are free of traction.

The deformation (2.1.3) is given by

$$\begin{aligned} x_1 &= \chi_1(X_1, X_2), \\ x_2 &= \chi_2(X_1, X_2), \\ x_3 &= X_3, \end{aligned} \tag{7.4.1}$$

and the equilibrium equations (2.2.9), in the absence of body forces, may be written in (two-dimensional) Cartesian rectangular coordinates as

$$\begin{aligned} \frac{\partial S_{11}}{\partial X_1} + \frac{\partial S_{21}}{\partial X_2} &= 0, \\ \frac{\partial S_{12}}{\partial X_1} + \frac{\partial S_{22}}{\partial X_2} &= 0. \end{aligned} \tag{7.4.2}$$

The (in-plane) displacement boundary conditions (7.2.4) are

$$\begin{aligned} x_1 &= X_1 && \text{on } X_2 = 0, \\ x_1 &= X_1 + \gamma X_2 && \text{on } X_2 = H, \\ x_2 &= 0 && \text{on } X_2 = 0, \\ x_2 &= H && \text{on } X_2 = H. \end{aligned} \tag{7.4.3}$$

and traction is defined on the lateral surfaces by

$$S_{11} = S_{12} = 0 \quad \text{on } X_1 = 0,\ A. \tag{7.4.4}$$

We let $d\,(\geq 0)$ denote the prescribed displacement of the top plate, and the amount of shear $\gamma = d/H$ is regarded as a measure of the strain.

The resultant shear force on each plate is found to be

$$F = \int_0^A S_{21}\, dX_1 \tag{7.4.5}$$

per unit length in the X_3 direction. On use of the equilibrium equations and traction boundary conditions it can be shown that (7.4.5) is independent of X_2, as follows

$$\frac{\partial F}{\partial X_2} = \int_0^A \frac{\partial S_{21}}{\partial X_2}\, dX_1 = -\int_0^A \frac{\partial S_{11}}{\partial X_1}\, dX_1 = [-S_{11}]_0^A = 0. \tag{7.4.6}$$

Let

$$\tau = F/A \tag{7.4.7}$$

denote the corresponding force per unit area.

At this point, we also note that the mean value of S_{11} through the plane section of the material in $\mathcal{B}_0$ is given by

$$
\begin{aligned}
\bar{S}_{11} &= \frac{1}{v(\mathcal{B}_0)} \int_0^A \int_0^H S_{11}\, dX_1 dX_2 \\
&= \frac{1}{AH} \int_0^H \left\{ [S_{11}X_1]_0^A - \int_0^A X_1 \frac{\partial S_{11}}{\partial X_1}\, dX_1 \right\} dX_2 \\
&= \frac{1}{AH} \int_0^A X_1 \left\{ \int_0^H \frac{\partial S_{21}}{\partial X_2} dX_2 \right\} dX_1 \\
&= \frac{1}{AH} \int_0^A X_1 S_{21}\, dX_1 |_{X_2=0}^{X_2=H}
\end{aligned} \tag{7.4.8}
$$

where $v(\mathcal{B}_0) = AH$ is the volume of $\mathcal{B}_0$ for the plane-strain problem.

7.4.2 General energy bounds

For the shear problem in question, the energy functional (7.3.1) reduces to

$$
E\{\chi\} = \int_{\mathcal{B}_0} \tilde{W}(\mathrm{Grad}\,\chi)\, dV, \tag{7.4.9}
$$

where $\mathcal{B}_0$ represents the two-dimensional body.

We regard (7.4.9) as a function of the prescribed displacement d. Thus,

$$
U(d) = E\{\chi(d)\} = \int_{\mathcal{B}_0} \tilde{W}(\mathrm{Grad}\,\chi(d))\, dV. \tag{7.4.10}
$$

It follows that

$$
\begin{aligned}
U'(d) &= \int_{\mathcal{B}_0} \mathrm{tr}(\mathbf{S}\mathbf{A}')\, \mathrm{dV} \\
&= \int_{\partial\mathcal{B}_0} (\mathbf{S}^T\mathbf{N}) \cdot \mathbf{x}'\, dA, && \text{since } \mathrm{Div}\mathbf{S} = \mathbf{0}, \\
&= \int_{\partial\mathcal{B}_0^x} (\mathbf{S}^T\mathbf{N}) \cdot \mathbf{x}'\, dA, && \text{since } \mathbf{S}^T\mathbf{N} = \mathbf{0} \text{ on } \partial\mathcal{B}_0^t, \\
&= \int_{X_2=H} S_{21}x_1' + S_{22}x_2'\, dA - \int_{X_2=0} S_{21}x_1' + S_{22}x_2'\, dA \\
&= \int_0^A S_{21}\, dX_1, && \text{on use of (7.4.3)} \\
&= F
\end{aligned} \tag{7.4.11}
$$

where $U'(0) = 0$. The total stored energy per unit reference volume can be denoted by

$$\hat{U}(\gamma) = U(d)/v(\mathcal{B}_0). \tag{7.4.12}$$

Differentiating (7.4.12), we establish the shear stress as a function of shear strain through

$$\tau = \hat{U}'(\gamma), \tag{7.4.13}$$

such that $\hat{U}'(0) = 0$. Further differentiation of (7.4.13), using the boundary conditions (7.4.3) and (7.4.4), yields

$$\hat{U}''(\gamma) = \frac{1}{v(\mathcal{B}_0)} \int_{\mathcal{B}_0} \mathrm{tr}\,\{(\mathcal{A}\mathbf{A}')\mathbf{A}'\}\, dV. \tag{7.4.14}$$

Recall from (7.2.15) that the integral in (7.4.14) must be strictly positive to ensure local infinitesimal stability of the body $\mathcal{B}_0$ in the configuration χ. Hence, when the conditions in (7.3.21) are satisfied

$$\hat{U}''(\gamma) > 0. \tag{7.4.15}$$

This implies τ is a monotonic increasing function of γ and we can define the Legendre transform of $\hat{U}(\gamma)$, denoted $\hat{U}_c(\tau)$, by

$$\hat{U}_c(\tau) = \gamma\tau - \hat{U}(\gamma), \tag{7.4.16}$$

where

$$\gamma = \hat{U}_c'(\tau), \tag{7.4.17}$$

such that $\hat{U}_c'(0) = 0$. Now, for subsequent use, with reference to (7.4.3), (7.4.5) and (7.4.8), we establish

$$\begin{aligned} \frac{1}{v(\mathcal{B}_0)} \int_{\mathcal{B}_0} \mathrm{tr}(\mathbf{SA})\, \mathrm{dV} &= \frac{1}{v(\mathcal{B}_0)} \int_{\partial \mathcal{B}_0^{\pm}} (\mathbf{S}^T \mathbf{N}) \cdot \mathbf{x}\, dA \\ &= \frac{1}{AH} \int_{X_2=H} \{S_{21} x_1 + S_{22} x_2\}\, dA - \frac{1}{AH} \int_{X_2=0} \{S_{21} x_1 + S_{22} x_2\}\, dA \\ &= \frac{1}{AH} \int_0^A X_1 S_{21}\, dX_1 |_{X_2=0}^{X_2=H} + \frac{\gamma F}{A} + \frac{1}{A} \int_0^A S_{22}\, dX_1 \\ &= \bar{S}_{11} + \gamma \tau + \bar{S}_{22} \\ &= \gamma \tau + \mathrm{tr}\,(\bar{\mathbf{S}}) \end{aligned} \qquad (7.4.18)$$

where $\bar{\mathbf{S}}$ is the mean nominal stress over $\mathcal{B}_0$. From (7.4.16), on substitution of (7.4.12), then (7.2.19) and (7.4.18), we obtain

$$\begin{aligned} \hat{U}_c(\tau) &= \gamma \tau - \frac{1}{v(\mathcal{B}_0)} \int_{\mathcal{B}_0} \mathrm{tr}(\mathbf{TU}) - \tilde{\mathrm{W}}_c(\mathbf{T})\, \mathrm{dV} \\ &= \frac{1}{v(\mathcal{B}_0)} \int_{\mathcal{B}_0} \tilde{W}_c(\mathbf{T})\, dV + \gamma \tau - \frac{1}{v(\mathcal{B}_0)} \int_{\mathcal{B}_0} \mathrm{tr}(\mathbf{SA})\, \mathrm{dV} \\ &= \frac{1}{v(\mathcal{B}_0)} \int_{\mathcal{B}_0} \tilde{W}_c(\mathbf{T})\, dV - \mathrm{tr}(\bar{\mathbf{S}}). \end{aligned} \qquad (7.4.19)$$

With reference to (7.4.18) and (7.2.21), using the fact that

$$\int_{\mathcal{B}_0} \mathrm{tr}(\mathbf{SA})\, \mathrm{dV} = \int_{\partial \mathcal{B}_0^{\pm}} (\mathbf{S}^{\mathrm{T}} \mathbf{N}) \cdot \boldsymbol{\xi}\, \mathrm{dA}, \qquad (7.4.20)$$

we then obtain

$$\frac{1}{v(\mathcal{B}_0)} E_c\{\mathbf{S}\} = \gamma \tau - \hat{U}_c(\tau). \qquad (7.4.21)$$

Given the dependence of any kinematically admissible deformation field $\boldsymbol{\chi}^* \in \mathcal{K}$ on the shear strain γ, we can define, analogous to (7.4.10),

$$U^*(d) = E\{\boldsymbol{\chi}^*(d)\} = \int_{\mathcal{B}_0} \tilde{W}(\mathrm{Grad}\, \boldsymbol{\chi}^*(d))\, dV, \qquad (7.4.22)$$

where the function U^* depends on the choice of χ^*. Similarly, we set

$$\hat{U}^*(\gamma) = U^*(d)/v(\mathcal{B}_0), \tag{7.4.23}$$

and consider the subset $\mathcal{K}_s$ of $\mathcal{K}$ consisting of deformation fields χ^* satisfying the required inequality $\hat{U}^{*\prime\prime}(\gamma) > 0$.

For any statically admissible stress field $\mathbf{S}^* \in \Sigma$ we define, independently of X_2,

$$F^* = \int_0^A S_{21}^*\, dX_1, \quad \tau^* = F^*/A. \tag{7.4.24}$$

Now, from (7.4.19), in terms of parameter τ^*, we have

$$\hat{U}_c^*(\tau^*) = \frac{1}{v(\mathcal{B}_0)} \int_{\mathcal{B}_0} \tilde{W}_c(\mathbf{T}^*(\tau^*))\, dV - \mathrm{tr}\,\{\bar{\mathbf{S}}^*(\tau^*)\}, \tag{7.4.25}$$

and, analogous to (7.4.21), using

$$\frac{1}{v(\mathcal{B}_0)} \int_{\mathcal{B}_0} \mathrm{tr}\,(\mathbf{S}^*\mathbf{A})\, dV = \gamma\tau^* + \mathrm{tr}\,(\bar{\mathbf{S}}^*) \tag{7.4.26}$$

which holds for any $\mathbf{S}^* \in \Sigma_s \subset \Sigma$ and any $\chi \in \mathcal{K}_s$, it follows that

$$\frac{1}{v(\mathcal{B}_0)} E_c\{\mathbf{S}^*\} = \gamma\tau^* - \hat{U}_c^*(\tau^*). \tag{7.4.27}$$

We can now re-express the chain of inequalities (7.3.11) in the form

$$\hat{U}^*(\gamma) \geq \hat{U}(\gamma) = \gamma\tau - \hat{U}_c(\tau) \geq \gamma\tau^* - \hat{U}_c^*(\tau^*), \tag{7.4.28}$$

where $\hat{U}^*$ and $\hat{U}_c^*$ are not, in general, Legendre duals. The right-hand expression in (7.4.28) is maximized, for given γ, provided τ^* satisfies

$$\gamma = \hat{U}_c^{*\prime}(\tau^*), \tag{7.4.29}$$

with the requirement that $\hat{U}_c^{*\prime\prime}(\tau^*) > 0$ for $\mathbf{S}^* \in \Sigma_s$.

Expressing (7.4.28) in terms of the parameter γ requires the right-hand expression in (7.4.28) to be recast as a function of γ, say $\hat{U}^{**}(\gamma)$. Subsequently, (7.4.28) becomes

$$\hat{U}^{*}(\gamma) \geq \hat{U}(\gamma) \geq \hat{U}^{**}(\gamma), \tag{7.4.30}$$

where the form of $\hat{U}^{**}$ depends on the choice of $\mathbf{S}^{*}$.

7.5 An example of primitive energy bounds

The choice of particular kinematically admissible and statically admissible strain and stress fields allows the construction of primitive energy bounds.

Upper bound

For the upper bound χ^{*} is chosen to correspond to *simple shear* [details of which can be found in, for example, Atkin and Fox (1980)]. Hence, in plane strain the deformation becomes

$$\begin{aligned} x_1 &= X_1 + \gamma X_2, \\ x_2 &= X_2, \\ x_3 &= X_3, \end{aligned} \tag{7.5.1}$$

with the associated principal stretches

$$\lambda_1 = \lambda, \quad \lambda_2 = \lambda^{-1}, \quad \lambda_3 = 1, \tag{7.5.2}$$

which provide the expression

$$\lambda - \lambda^{-1} = \gamma. \tag{7.5.3}$$

Taking $\hat{U}^*$ as a function of the principal stretches (and of γ through them), from (7.4.10) we obtain the upper bound

$$\hat{U}^*(\gamma) = \tilde{W}(\lambda, \lambda^{-1}) \geq \hat{U}(\gamma). \tag{7.5.4}$$

Lower bound

For the lower bound we choose a statically admissible stress field corresponding to constant $\mathbf{S}^*$. Hence, in the (X_1, X_2) plane

$$\mathbf{S}^* = \begin{bmatrix} 0 & 0 \\ S^*_{21} & S^*_{22} \end{bmatrix} \tag{7.5.5}$$

where $S^*_{21} = \tau^*$. It follows from (7.2.22) that

$$t^*_1 = 0, \quad t^{*^2}_2 = \tau^{*^2} + S^{*^2}_{22}. \tag{7.5.6}$$

On use of (7.4.25), where $\hat{U}^*_c$ is taken as a function of the principal values (7.5.6), we obtain

$$\hat{U}^*_c(\tau^*) = \tilde{W}_c(0, t^*_2) - S^*_{22}, \tag{7.5.7}$$

since $\bar{S}^*_{22} = S^*_{22}$. The lower bound in (7.4.28) now becomes

$$\gamma\tau^* - \hat{U}^*_c(\tau^*) = \gamma\tau^* - \tilde{W}_c(0, t^*_2) + S^*_{22} \tag{7.5.8}$$

where, since $t^*_1 = 0$, (7.3.22) reduces to

$$\tilde{W}_c(0, t^*_2) = t^*_2\lambda^*_2 - \tilde{W}(\lambda^*_1, \lambda^*_2), \tag{7.5.9}$$

and we recall from (7.3.23) that

$$\lambda^*_1 = \frac{\partial \tilde{W}_c}{\partial t^*_1}(0, t^*_2), \quad \lambda^*_2 = \frac{\partial \tilde{W}_c}{\partial t^*_2}(0, t^*_2). \tag{7.5.10}$$

In obtaining a lower bound we require an expression for λ_2^* in (7.5.9). To this end we optimize (7.5.8) with respect to S_{22}^*. This yields

$$\frac{\partial \tilde{W}_c}{\partial t_2^*}\frac{\partial t_2^*}{\partial S_{22}^*} = 1 \tag{7.5.11}$$

which, on use of $(7.5.6)_2$ and (7.5.10) provides

$$S_{22}^* = {\lambda_2^*}^{-1} t_2^*. \tag{7.5.12}$$

This allows $(7.5.6)_2$ to be written in the form

$${\tau^*}^2 = {t_2^*}^2(1 - {\lambda_2^*}^{-2}). \tag{7.5.13}$$

Equation (7.5.12) implies that S_{22}^* is (implicitly) a function of τ^*, with this in mind we now maximize (7.5.8) with respect to τ^* so that

$$\gamma + \frac{\partial S_{22}^*}{\partial \tau^*} - \frac{\partial \tilde{W}_c}{\partial t_2^*}\frac{\partial t_2^*}{\partial \tau^*} = 0. \tag{7.5.14}$$

Making use of $(7.5.6)_2$, (7.5.10) and (7.5.12) in (7.5.14) we find

$$\lambda_2^* = {\tau^*}^{-1} t_2^* \gamma \tag{7.5.15}$$

which, using (7.5.13), yields

$${\lambda_2^*}^2 = 1 + \gamma^2 \tag{7.5.16}$$

Combining (7.5.8) and (7.5.9), the lower bound now takes the form

$$\gamma\tau^* - \hat{U}_c^*(\tau^*) = \gamma\tau^* - \lambda_2^* t_2^* + \tilde{W}(\lambda_1^*, \lambda_2^*) + S_{22}^* \tag{7.5.17}$$

Following substitution from (7.5.12), (7.5.15) and (7.5.16) into (7.5.17), the lower bound is re-expressed as

$$\hat{U}(\gamma) \geq \hat{U}^{**}(\gamma) = \tilde{W}(\lambda_1^*, \lambda_2^*) \tag{7.5.18}$$

Primitive bounds

In light of (7.5.4) and (7.5.18) the inequalities in (7.4.30) now simplify. Thus, for a general strain-energy function, we derive the following primitive upper and lower bounds on the stored energy per unit volume

$$\tilde{W}(\lambda,\, \lambda^{-1}) \geq \hat{U}(\gamma) \geq \tilde{W}(\lambda_1^*,\, \lambda_2^*) \tag{7.5.19}$$

where λ_1^* is given in terms of λ_2^* through

$$t_1^* = \frac{\partial \tilde{W}}{\partial \lambda_1^*}(\lambda_1^*,\, \lambda_2^*) = 0 \tag{7.5.20}$$

where ${\lambda_2^*}^2 = 1 + \gamma^2$.

7.5.1 Specialization to the semi-linear material

The energy bounds in (7.5.19) can be made explicit through use of the semi-linear strain-energy function (7.3.24).

With reference to (7.5.3), the principal stretches associated with the upper bound are expressed explicitly in the form

$$\lambda_1 = \frac{1}{2}\gamma + \frac{1}{2}(\gamma^2+4)^{\frac{1}{2}}, \quad \lambda_2 = -\frac{1}{2}\gamma + \frac{1}{2}(\gamma^2+4)^{\frac{1}{2}}, \quad \lambda_3 = 1. \tag{7.5.21}$$

For the lower bound, substitution of the strain-energy function into (7.5.20), where t_1 is given by (7.3.20), provides an expression for λ_1^*, so that

$$\lambda_1^* = \frac{1-\nu(\gamma^2+1)^{\frac{1}{2}}}{1-\nu}, \quad \lambda_2^* = (\gamma^2+1)^{\frac{1}{2}}, \quad \lambda_3^* = 1. \tag{7.5.22}$$

Now substituting (7.5.21) and (7.5.22) into (7.5.19), for the semi-linear material, yields upper and lower bounds in the form

$$\frac{E}{2(1+\nu)(1-2\nu)}\left[(\gamma^2 - \nu\gamma^2 - 2(\gamma^2+4)^{\frac{1}{2}} + 4\right] \geq \hat{U}(\gamma) \geq \frac{E}{2(1-\nu^2)}\left[\gamma^2 - 2(\gamma^2+1)^{\frac{1}{2}} + 2\right]. \tag{7.5.23}$$

The explicit upper and lower bounds obtained in (7.5.23) are illustrated in Fig.7.1, where they are plotted against shear strain. It should be noted that primitive bounds offer only a rough guide to deformation behaviour and the semi-linear material only provides a good model of behaviour for moderate values of the deformation and stress.

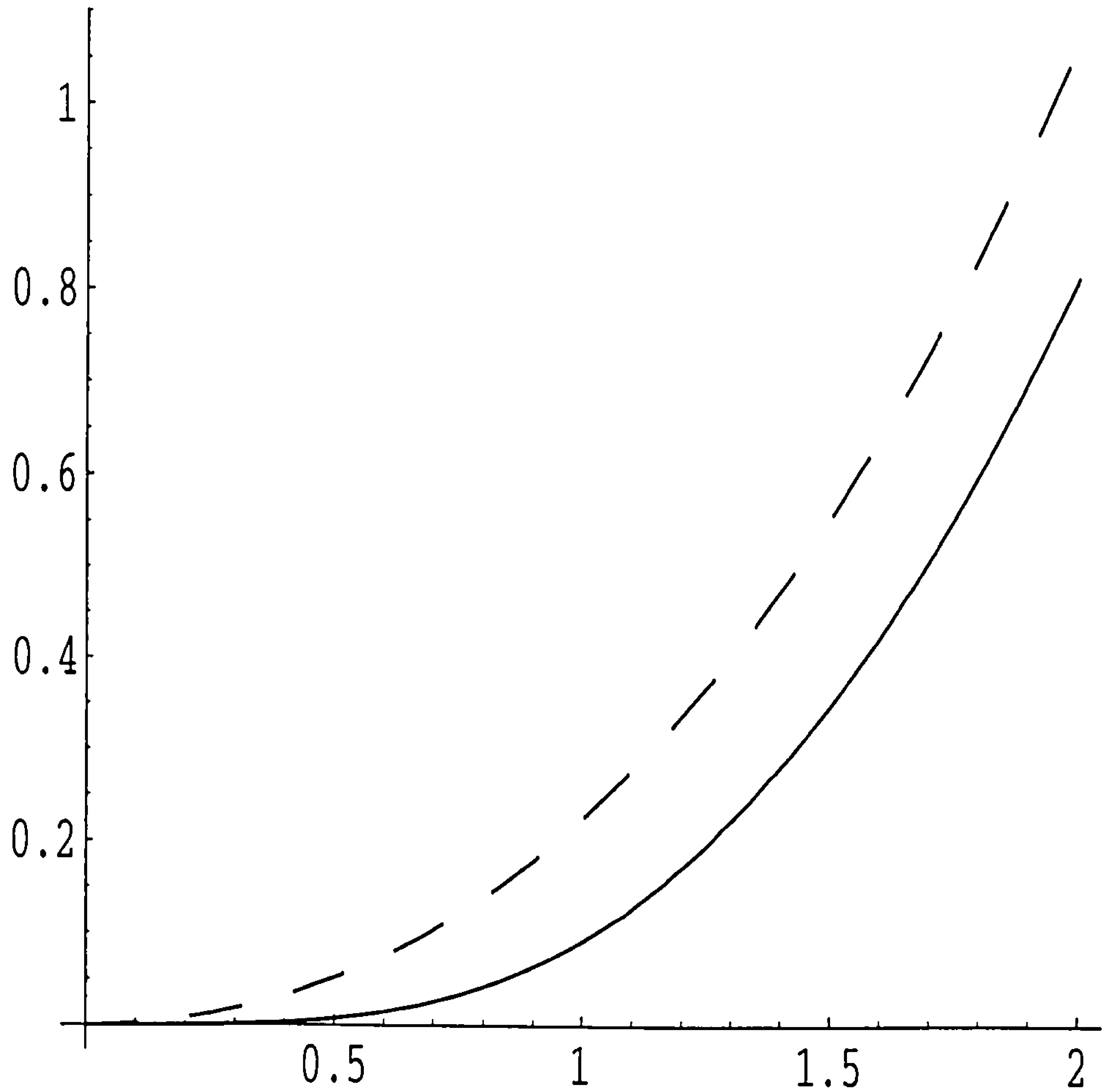

Figure 7.1: Plot of (non-dimensionalized) primitive energy bounds against shear strain for the semi-linear material, with $\nu = \frac{1}{4}$: upper bound (dashed curve); lower bound (solid curve).

Bibliography

[1] Atkin, R. J. and Fox, N. 1980 *An Introduction to the Theory of Elasticity.* Longman Group.

[2] Dryburgh, G. and Ogden, R. W. 1999 *Bifurcation of an elastic surface-coated incompressible isotropic elastic block subject to bending.* Z.A.M.P. **50**, 822-838.

[3] Green, A. E. and Zerna, W. 1968 *Theoretical Elasticity.* Oxford University Press.

[4] Haddow, J. B. and Ogden, R. W. 1988 *Compression of bonded elastic bodies.* J. Mech. Phys. Solids **36**, 551-579.

[5] Haughton, D. M. 1998 *Flexure and compression of incompressible elastic plates.* Int. J. Engng Sci. **37**, 1693-1708.

[6] Haughton, D. M. and Orr, A. 1997 *On the eversion of compressible elastic cylinders.* Int. J. Solids Struct. **34**, 1893-1914.

[7] Lindsay, K. A. and Rooney, C. E. 1992 *A note on compound matrices.* J. Comp. Phys. **133**, 472-477.

[8] Ogden, R. W. 1972 *Large deformation isotropic elasticity I: on the correlation of theory and experiment for incompressible rubberlike solids.* Proc. R. Soc. Lond. A **326**, 565-584.

[9] Ogden, R. W. 1984 *Non-linear Elastic Deformations.* Ellis Horwood.

[10] Ogden, R. W., Dryburgh, G. and Steigmann, D. J. 1998 *The effect of elastic surface coating on the bending of an incompressible elastic block.* Z.A.M.M. **78**, 145-148.

[11] Ogden, R. W., Steigmann, D. J. and Haughton, D. M. 1997 *The effect of elastic surface coating on the finite deformation and bifurcation of a pressurized circular annulus.* J. Elasticity. **47**, 121-145.

[12] Ogden, R. W. and Steigmann, D. J. 1999 *Plane strain dynamics of elastic solids with intrinsic boundary elasticity, with application to surface wave propagation.* Preprint, University of Glasgow.

[13] Prager, W. and Synge, J. L. 1947 *Approximations in elasticity based on the concept of function space.* Quart. Appl. Math. **5**, 241-269.

[14] Steigmann, D. J. and Ogden, R. W. 1997a *Plane deformations of elastic solids with intrinsic boundary elasticity.* Proc. R. Soc. Lond. A **453**, 853-877.

[15] Steigmann, D. J. and Ogden, R. W. 1997b *A necessary condition for energy-minimizing plane deformations of elastic solids with intrinsic boundary elasticity.* Math. Mech. Solids **2**, 3-16.

[16] Tadjbakhsh, I. 1966 *The variational theory of the plane motion of the extensible elastica.* Int. J. Engng Sci. **4**, 433-450.

[17] Timoshenko, S. and Goodier, J. N. 1951 *Theory of elasticity.* 2nd ed. New York: McGraw-Hill.

[18] Truesdell, C. and Noll, W. 1965 *The non-linear field theories of mechanics in Handbuch der physik.* Vol. III/3 Springer-Verlag.