Homologically arc-homogeneous ENRs

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We prove that an arc-homogeneous Euclidean neighborhood retract is a homology manifold.

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1 Introduction

The so-called Modified Bing–Borsuk Conjecture, which grew out of a question in [1], asserts that a homogeneous Euclidean neighborhood retract is a homology manifold. At this mini-workshop on exotic homology manifolds, Frank Quinn asked whether a space that satisfies a similar property, which he calls *homological arc-homogeneity*, is a homology manifold. The purpose of this note is to show that the answer to this question is yes.

2 Statement and proof of the main result

Theorem 2.1 *Suppose that X is an n–dimensional homologically arc-homogeneous ENR. Then X is a homology n–manifold.*

Definitions A *homology n–manifold* is a space X having the property that, for each $x \in X$,

$$H_k(X, X - x; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n. \end{cases}$$

A *Euclidean neighborhood retract* (ENR) is a space homeomorphic to a closed subset of Euclidean space that is a retract of some neighborhood of itself. A space X is *homologically arc-homogeneous* provided that for every path $\alpha\colon [0,1] \to X$, the inclusion induced map

$$H_*(X \times 0, X \times 0 - (\alpha(0), 0)) \to H_*(X \times I, X \times I - \Gamma(\alpha))$$

is an isomorphism, where $\Gamma(\alpha)$ denotes the graph of α. The *local homology sheaf $\mathcal{H}_k$ in dimension k* on a space X is the sheaf with stalks $H_k(X, X - x)$, $x \in X$.

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By a result of Bredon [2, Theorem 15.2], if an n–dimensional space X is cohomologically locally connected (over $\mathbb{Z}$), has finitely generated local homology groups $H_k(X, X-x)$ for each k, and if each $\mathcal{H}_k$ is locally constant, then X is a homology manifold. We shall show that an n–dimensional, homologically arc-connected ENR satisfies the hypotheses of Bredon's theorem.

Assume from now on that X represents an n–dimensional, homologically arc-homogeneous ENR. Unless otherwise specified, all homology groups are assumed to have integer coefficients. The following lemma is a straightforward application of the definition and the Mayer–Vietoris theorem.

Lemma 2.2 *Given a path $\alpha\colon [0,1] \to X$ and $t \in [0,1]$, the inclusion induced map*

$$H_*(X \times t, X \times t - (\alpha(t), t)) \to H_*(X \times I, X \times I - \Gamma(\alpha))$$

is an isomorphism.

Given points $x, y \in X$, an arc $\alpha\colon I \to X$ from x to y, and an integer $k \geq 0$, let $\alpha_*\colon H_k(X, X-x) \to H_k(X, X-y)$ be defined by the composition

$$H_k(X, X-x) \xrightarrow{\times 0} H_*(X \times I, X \times I - \Gamma(\alpha)) \xleftarrow{\times 1} H_k(X, X-y).$$

Clearly $(\alpha^{-1})_* = \alpha_*^{-1}$ and $(\alpha\beta)_* = \beta_*\alpha_*$, whenever $\alpha\beta$ is defined.

Lemma 2.3 *Given $x \in X$ and $\eta \in H_k(X, X-x)$, there is a neighborhood U of x in X such that if α and β are paths in U from x to y, then $\alpha_*(\eta) = \beta_*(\eta) \in H_k(X, X-y)$.*

Proof We will prove the equivalent statement: for each $x \in X$ and $\eta \in H_k(X, X-x)$ there is a neighborhood U of x with $\alpha_*(\eta) = \eta$ for any loop α in U based at x.

Suppose $x \in X$ and $\eta \in H_k(X, X-x)$. Since $H_k(X, X-x)$ is the direct limit of the groups $H_k(X, X-W)$, where W ranges over the (open) neighborhoods of x in X, there is a neighborhood U of x and an $\eta_U \in H_k(X, X-U)$ that goes to η under the inclusion $H_k(X, X-U) \to H_k(X, X-x)$.

Suppose α is a loop in U based at x. Let $\eta_\alpha \in H_k(X \times I, X \times I - \Gamma(\alpha))$ correspond to η under the isomorphism $H_k(X, X-x) \xrightarrow{\times 0} H_k(X \times I, X \times I - \Gamma(\alpha))$ guaranteed by homological arc-homogeneity.

Let

$$\eta_{U \times I} = \eta_U \times 0 \in H_k(X \times I, X \times I - U \times I).$$

Then the image of $\eta_{U\times I}$ in $H_k(X\times I, X\times I-\Gamma(\alpha))$ is η_α, as can be seen by chasing the following diagram around the lower square.

$$\begin{tikzcd}
H_k(X, X-U) \arrow[r] \arrow[d, "\cong"', "\times 1"] & H_k(X, X-x) \arrow[d, "\cong"', "\times 1"] \\
H_k(X\times I, X\times I-U\times I) \arrow[r] & H_k(X\times I, X\times I-\Gamma(\alpha)) \\
H_k(X, X-U) \arrow[r] \arrow[u, "\cong", "\times 0"'] & H_k(X, X-x) \arrow[u, "\cong", "\times 0"']
\end{tikzcd}$$

But from the upper square we see that η_α must also come from η after including into $X\times 1$. That is, $\alpha_*(\eta)=\eta$. □

Corollary 2.4 *Suppose the neighborhood U above is path connected and F is the cyclic subgroup of $H_k(X, X-U)$ generated by η_U. Then, for every $y\in U$, the inclusion $H_k(X, X-U)\to H_k(X, X-y)$ takes F one-to-one onto the subgroup F_y generated by $\alpha_*(\eta)$, where α is any path in U from x to y.*

Lemma 2.5 *Suppose $x, y\in X$ and α and β are path-homotopic paths in X from x to y. Then $\alpha_*=\beta_*\colon H_k(X, X-x)\to H_k(X, X-y)$.*

Proof By a standard compactness argument it suffices to show that, for a given path α from x to y and element $\eta\in H_k(X, X-x)$, there is an $\epsilon>0$ such that $\alpha_*(\eta)=\beta_*(\eta)$ for any path β from x to y ϵ–homotopic (rel $\{x,y\}$) to α.

Given a path α from x to y, $\eta\in H_k(X, X-x)$, and $t\in I$, let U_t be a path-connected neighborhood of $\alpha(t)$ associated with $(\alpha_t)_*(\eta)\in H_k(X, X-\alpha(t))$ given by Lemma 2.3, where α_t is the path $\alpha\big|[0,t]$. There is a subdivision

$$\{0=t_0<t_1<\cdots<t_m=1\}$$

of I such that $\alpha\big([t_{i-1},t_i]\big)\subseteq U_i$ for each $i=1,\dots,m$, where $U_i=U_t$ for some t. There is an $\epsilon>0$ so that if $H\colon I\times I\to X$ is an ϵ–path-homotopy from α to a path β, then $H([t_{i-1},t_i]\times I)\subseteq U_i$.

For each $i=1,\dots,m$, let $\alpha_i=\alpha\big|[t_{i-1},t_i]$ and $\beta_i=\beta\big|[t_{i-1},t_i]$, and for $i=0,\dots,m$, let $\gamma_i=H\big|t_i\times I$ and $\eta_i=(\alpha_{t_i})_*(\eta)$. By Corollary 2.4,

$$(\alpha_i)_*(\eta_{i-1})=(\gamma_{i-1}\beta_i\gamma_i^{-1})_*(\eta_{i-1})=\eta_i$$

where $\eta_0=\eta$. Since γ_0 and γ_n are the constant paths, it follows easily that

$$\alpha_*(\eta)=(\alpha_n)_*\cdots(\alpha_1)_*(\eta)=(\beta_n)_*\cdots(\beta_1)_*(\eta)=\beta_*(\eta). \qquad \square$$

Proof of Theorem 2.1 As indicated at the beginning of this note, we need only show that the hypotheses of [2, Theorem 15.2] are satisfied.

Since X is an ENR, it is locally contractible, and hence cohomologically locally connected over $\mathbb{Z}$.

Given $x \in X$, let W be a path-connected neighborhood of x such that W is contractible in X. If α and β are two paths in W from x to a point $y \in W$, then α and β are path-homotopic in X. Hence, by Lemma 2.5, $\alpha_*\colon H_k(X, X-x) \to H_k(X, X-y)$ is a well-defined isomorphism that is independent of α for every $k \geq 0$. Hence, $\mathcal{H}_k \big| W$ is the constant sheaf, and so $\mathcal{H}_k$ is locally constant.

Finally, we need to show that the local homology groups of X are finitely generated. This can be seen by working with a mapping cylinder neighborhood of X. Assume X is nicely embedded in $\mathbb{R}^{n+m}$, for some $m \geq 3$, so that X has a mapping cylinder neighborhood $N = C_\phi$ of a map $\phi\colon \partial N \to X$, with mapping cylinder projection $\pi\colon N \to X$ (see [3]). Given a subset $A \subseteq X$, let $A^* = \pi^{-1}(A)$ and $\dot{A} = \phi^{-1}(A)$.

Lemma 2.6 *If A is a closed subset of X, then $H_k(X, X-A) \cong \check{H}_c^{n+m-k}(A^*, \dot{A})$.*

Proof Suppose A is closed in X. Since $\pi\colon N \to X$ is a proper homotopy equivalence,

$$H_k(X, X-A) \cong H_k(N, N-A^*).$$

Since ∂N is collared in N,

$$H_k(N, N-A^*) \cong H_k(\operatorname{int} N, \operatorname{int} N - A^*),$$

and by Alexander duality,

$$\begin{aligned} H_k(\operatorname{int} N, \operatorname{int} N - A^*) &\cong \check{H}_c^{n+m-k}(A^* - \dot{A}) \\ &\cong \check{H}_c^{n+m-k}(A^*, \dot{A}) \end{aligned}$$

(since $\dot{A}$ is also collared in A^*). □

Since X is n–dimensional, we get the following corollary.

Corollary 2.7 *If A is a closed subset of X, then $\check{H}_c^q(A^*, \dot{A}) = 0$, if $q < m$ or $q > n+m$.*

Thus, the local homology sheaf $\mathcal{H}_k$ of X is isomorphic to the Leray sheaf $\mathcal{H}^{n+m-k}$ of the map $\pi\colon N \to X$ whose stalks are $\check{H}^{n+m-k}(x^*, \dot{x})$. For each $k \geq 0$, this sheaf is also locally constant, so there is a path-connected neighborhood U of x such that

$\mathcal{H}^q \big| U$ is constant for all $q \geq 0$. Given such a U, there is a path-connected neighborhood V of x lying in U such that the inclusion of V into U is null-homotopic. Thus, for any coefficient group G, the inclusion $H^p(U, G) \to H^p(V, G)$ is zero if $p \neq 0$ and is an isomorphism for $p = 0$.

The Leray spectral sequences of $\pi \big| \pi^{-1}(U)$ and $\pi \big| \pi^{-1}(V)$ have E_2 terms

$$E_2^{p,q}(U) \cong H^p(U; \mathcal{H}^q), \qquad E_2^{p,q}(V) \cong H^p(V; \mathcal{H}^q)$$

and converge to

$$E_\infty^{p,q}(U) \subseteq H^{p+q}(U^*, \dot{U}; \mathbb{Z}), \qquad E_\infty^{p,q}(V) \subseteq H^{p+q}(V^*, \dot{V}; \mathbb{Z}),$$

respectively (see [2, Theorem 6.1]). Since the sheaf $\mathcal{H}^q$ is constant on U and V, $H^p(U; \mathcal{H}^q)$ and $H^p(V; \mathcal{H}^q)$ represent ordinary cohomology groups with coefficients in $G_q \cong \check{H}^q(x^*, \dot{x})$.

By naturality, we have the commutative diagram

$$\begin{tikzcd} E_2^{0,q}(U) \arrow[r] \arrow[d, "\cong"] & E_2^{2,q-1}(U) \arrow[d, "0"'] \\ E_2^{0,q}(V) \arrow[r] & E_2^{2,q-1}(V) \end{tikzcd}$$

which implies that the differential $d_2\colon E_2^{0,q}(V) \to E_2^{2,q-1}(V)$ is the zero map. Hence,

$$E_3^{0,q}(V) = \ker\big(E_2^{0,q}(V) \to E_2^{2,q-1}(V)\big) / \operatorname{im}\big(E_2^{-2,q+1}(V) \to E_2^{0,q}(V)\big) = E_2^{0,q}(V),$$

and, similarly, $E_3^{0,q}(V) = E_4^{0,q}(V) = \cdots = E_\infty^{0,q}(V)$. Thus,

$$H^q(V^*, \dot{V}; \mathbb{Z}) \supseteq E_\infty^{0,q}(V) \cong E_2^{0,q}(V) \cong H^0(V; \mathcal{H}^q) \cong H^0(V; G_q) \cong G_q.$$

Applying the same argument to the inclusion $(x^*, \dot{x}) \subseteq (V^*, \dot{V})$ yields the commutative diagram

$$\begin{tikzcd} E_2^{0,q}(V) \arrow[r] \arrow[d, "\cong"] & E_2^{2,q-1}(V) \arrow[d, "0"'] \\ E_2^{0,q}(x) \arrow[r] & E_2^{2,q-1}(x) \end{tikzcd}$$

which, in turn, gives

$$\begin{array}{ccc} G_q \cong H^0(V;G_q) & \xrightarrow{\cong} & H^q(V^*,\dot{V};\mathbb{Z}) \\ \cong\downarrow & & \downarrow \\ G_q \cong H^0(x;G_q) & \xrightarrow{\cong} & H^q(x^*,\dot{x};\mathbb{Z}) \cong G_q \end{array}$$

from which it follows that the inclusion $H^q(V^*,\dot{V};\mathbb{Z}) \to H^q(x^*,\dot{x};\mathbb{Z}) \cong G_q$ is an isomorphism of G_q. Since $(x^*,\dot{x})$ is a compact pair in the manifold pair $(V^*,\dot{V})$, it has a compact manifold pair neighborhood $(W,\partial W)$. Since the inclusion $H^q(V^*,\dot{V}) \to \check{H}^q(x^*,\dot{x})$ factors through $H^q(W,\partial W)$, its image is finitely generated for each q. Hence, $H_k(X,X-x) \cong \check{H}^{n+m-k}(x^*,\dot{x})$ is finitely generated for each k. □

The following theorem, which may be of independent interest, emerges from the proof of Theorem 2.1.

Theorem 2.8 *Suppose X is an n–dimensional ENR whose local homology sheaf $\mathcal{H}_k$ is locally constant for each $k \geq 0$. Then X is a homology n–manifold.*

References

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