A dissertation submitted for the degree of Doctor of Philosophy

in the University of Glasgow

TOPICS IN THE

THEORY OF INVARIANT SUBSPACES

by

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PREFACE

This dissertation is submitted in accordance with the regulations for the degree of Doctor of Philosophy in the University of Glasgow. No part of it has been previously submitted by the author for a degree at any other University.

The results contained in this dissertation are claimed as original except where indicated in the text.

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SUMMARY

This thesis is concerned with some questions which arise in the study of the invariant subspace problem.

In Chapter One, which is not claimed as original work, we include the main results on the spectral theory of linear operators which will be required in the subsequent chapters.

In Chapter Two, we prove that every analytic Toeplitz operator and every isometry on a general Hilbert space are reflexive operators. It is shown that an operator is reflexive if its restriction to every closed separable subspace is reflexive. This both simplifies and generalizes the work of Deddens.

Chapter Three is devoted to the study of subnormal operators. It is shown that a large class of subnormal operators are reflexive in the first section. The remainder of this chapter is devoted to proving a generalization of a theorem of Naimark. It is shown how this result can be used in proving Bishop's theorem that the closure of the set of normal operators (on a separable Hilbert space) in the strong operator topology is the set of subnormal operators.

In Chapter Four we make a study of analytically compact operators. It is shown how a modified form of Ringrose's theory of superdiagonal forms (valid for a compact linear operator on a complex Banach space) holds in this more general situation.

Chapter Five is devoted to the study of compact linear operators on a real Banach space. It is shown how the use of the Hilden-Lomonosov technique very considerably simplifies the theory initiated independently by Gillespie and Meyer-Nieberg.

Finally, in Chapter Six it is shown that Ringrose's theory of superdiagonal forms can be extended to the case of a completely continuous operator on a locally convex Hausdorff topological vector space over the complex field. This satisfactorily rounds off the spectral theory of such operators initiated by Altman.

LIST OF SYMBOLS

$\underset{\sim}{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
$B(X)$	algebra of all bounded linear operators on a Banach space X
$C(X)$	algebra of all continuous complex-valued functions on a compact Hausdorff space X under the supremum norm
X^*	dual space of X
$T\|M$	restriction of a mapping T to M
$R(T)$	range of T
$N(T)$	null-space of T
$\sigma(T)$	spectrum of T
$\rho(T)$	resolvent set of T
cl A or $\overline{A}$	closure of a set A
clm A	closed linear subspace spanned by A
$A \oplus B$	the direct sum of A and B
ker f	the set $\{x : f(x) = 0\}$
dim M	the dimension of the space M
Σ_p	σ-algebra of Borel subsets of $\mathbb{C}$
$\tilde{\mathcal{F}}(T)$	functions analytic in some neighbourhood of $\sigma(T)$
Lat A	lattice of closed subspaces invariant under A

CORRIGENDA

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31	17	$= \overline{\Phi(P_1, P_2)}$	$= \overline{\Phi(P_2, P_1)}$
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CHAPTER ONE

Preliminary Concepts

1. Let X be a set. A family M of subsets of X is a σ-algebra if and only if:

   (i) $\emptyset \in M$,

   (ii) $A \in M \Rightarrow X \setminus A \in M$,

   (iii) for any sequence $\{A_n\}$ in M, $\bigcup_{n=1}^{\infty} A_n \in M$.

The class of all subsets of a given set forms a σ-algebra. The intersection of a family of σ-algebras is again a σ-algebra.

2. A topological space is an abstract set S, together with a class Γ_S of subsets of S, whose members will be called the open subsets of S, which contains the void set $\emptyset$ and the whole set S, and which is closed under the operations of forming arbitrary unions and finite intersections.

A subset of S will be called closed if its complement in S is open.

3. Let S be a topological space. The intersection of all σ-algebras of subsets of S which contain the class Γ_S of open sets, will itself be a σ-algebra (1.1), which we will denote by Σ_S. Σ_S is called the Borel family of S, and the subsets of S which are in Σ_S are called the Borel subsets of S. We shall mainly be concerned with the case S = $\mathbb{C}$ of the complex plane, in the topology induced by the metric $|\cdot|$. The Borel family of $\mathbb{C}$ is denoted by Σ_p.

4. Let X be a Banach space. If A is a subset of X, we shall

use the notation clm A to denote the intersection of all closed subspaces of X which contain A. Clm A is called the closed linear subspace generated by A.

5. Let Y be a closed subspace of the complex Banach space X. Then Y is a (complex) Banach space under the norm of X. The annihilator $Y^{\perp}$ of Y is the closed subspace of X* (dual of X) defined by:

$$Y^{\perp} = \{f \in X^* : f(y) = 0 \text{ for all } y \text{ in } Y\}.$$

(See [11]).

6. The class of all bounded linear operators mapping a Banach space X into itself will be denoted by L(X) (or B(X)). L(X) is an algebra. Three topologies will be introduced on this algebra:

The uniform operator topology in L(X) is the topology induced by the norm,

$$||T|| = \sup_{||x|| \leq 1} ||Tx||, \quad (x \in X,\ T \in L(X));$$

under this norm, L(X) is a Banach algebra.

The strong operator topology in L(X) is the topology defined by taking as a basic set of neighbourhoods of $T \in L(X)$, the sets

$$N(T;A;\varepsilon) = \{R \in L(X) : ||(T-R)x|| < \varepsilon,\ x \in A\},$$

where A is an arbitrary finite subset of X and $\varepsilon > 0$ is arbitrary. Thus a generalized sequence $\{T_\alpha\}$ converges to T in the strong operator topology if and only if $\{T_\alpha x\}$ converges to Tx, for each $x \in X$. Under this topology L(X) is a locally convex, Hausdorff, (topological) linear space.

The weak operator topology in L(X) is the topology defined by taking as a basic set of neighbourhoods

$$N(T;A,B;\varepsilon) = \{R \in L(X) : |\langle (T-R)x, x^* \rangle| < \varepsilon, x \in A, x^* \in B\},$$

where A and B are arbitrary finite subsets of X and X^* respectively and $\varepsilon > 0$ is arbitrary. Thus, a generalized sequence $\{T_\alpha\}$ converges to T in the weak operator topology if and only if $\{\langle T_\alpha x, x^*\rangle\}$ converges to $\langle Tx, x^*\rangle$ for each $x \in X$ and each $x^* \in X^*$. Under this topology, L(X) is a locally convex, Hausdorff, linear space. Between these topologies there exist the following inclusions:

$$\left\{\begin{array}{c}\text{weak}\\ \text{operator}\\ \text{topology}\end{array}\right\} \subseteq \left\{\begin{array}{c}\text{strong}\\ \text{operator}\\ \text{topology}\end{array}\right\} \subseteq \left\{\begin{array}{c}\text{operator}\\ \text{norm}\\ \text{topology}\end{array}\right\}.$$

7. Let X be a Banach space and let $T \in L(X)$. The <u>resolvent set $\rho(T)$</u> of T is the set of complex numbers λ for which $\lambda I - T$ is invertible in the Banach algebra L(X). The <u>spectrum $\sigma(T)$ of T</u> is defined to be $\mathbb{C} \setminus \rho(T)$. The function $\lambda \to (\lambda I - T)^{-1} (\lambda \in \rho(T))$ is called the <u>resolvent of T</u>.

The following are proved in [22].

(1) Let $T \in L(X)$. The resolvent set $\rho(T)$ is open. Also, the function $\lambda \to (\lambda I - T)^{-1}$ is analytic in $\rho(T)$.

(2) Let $T \in L(X)$. Then $\sigma(T)$ is compact and non-empty.

(3) If for $T \in L(X)$, the <u>spectral radius $\nu(T)$ of T</u> is defined by

$$\nu(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\},$$

then the spectral radius of T has the properties

$$\nu(T) = \lim_{n \to \infty} ||T^n||^{1/n} \leq ||T||.$$

(4) Let $T \in L(X)$. T is said to be <u>quasinilpotent</u> if and only if $\lim_{n \to \infty} ||T^n||^{1/n} = 0$. Then

(i) T is quasinilpotent if and only if $\nu(T) = 0$.

(ii) T is quasinilpotent if and only if $\sigma(T) = \{0\}$.

(5) Let $T \in L(X)$. There is an operator T^* in $L(X^*)$ called the

<u>adjoint</u> of T such that

$$\langle Tx,y\rangle = \langle x,T^*y\rangle \qquad (x\in X,\ y\in X^*).$$

The map $T \to T^*$ is an isometric linear map of L(X) into L(X*) with the additional property

$$(AB)^* = B^*A^* \quad (A,B\in L(X)).$$

The spectrum of T* is equal to the spectrum of T. Moreover

$$((\lambda I-T)^{-1})^* = (\lambda I^*-T^*)^{-1} \quad (\lambda\in\rho(T)).$$

(6) Let $T\in L(X)$. Define

$$\sigma_p(T) = \{\lambda\in\mathbb{C} : \lambda I-T \text{ is not one-to-one}\};$$

$$\sigma_c(T) = \{\lambda\in\mathbb{C} : \lambda I-T \text{ is one-to-one, } \overline{(\lambda I-T)X} = X \text{ but } (\lambda I-T)X \neq X\};$$

$$\sigma_r(T) = \{\lambda\in\mathbb{C} : \lambda I-T \text{ is one-to-one but } \overline{(\lambda I-T)X} \neq X\};$$

$\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ are called respectively the <u>point spectrum</u>, the <u>continuous spectrum</u> and the <u>residual spectrum</u> of T. Clearly $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ are disjoint and

$$\sigma(T) = \sigma_p(T)\cup\sigma_c(T)\cup\sigma_r(T).$$

(7) Let $T\in L(X)$. Define

$$\sigma_a(T) = \{\lambda\in\mathbb{C} : \text{there is a sequence } \{x_n\} \text{ in } X \text{ with } ||x_n|| = 1 \text{ and } \lim_{n\to\infty}||(\lambda I-T)x_n|| = 0\}.$$

$\sigma_a(T)$ is called the <u>approximate point spectrum</u> of T. The following result summarizes the main properties of the approximate point spectrum of $T\in L(X)$.

(i) The set $\sigma_a(T)$ is a closed non-empty subset of the spectrum of T.

(ii) The boundary of $\sigma(T)$ is contained in $\sigma_a(T)$.

(iii) $\sigma_p(T)\subseteq\sigma_a(T)$.

(iv) $\sigma_c(T) \subseteq \sigma_a(T)$.

8. A <u>Boolean algebra</u> B is an abstract set in which two binary operations, $\vee$ and $\wedge$, are defined, satisfying the following four postulates:

(i) $\vee$, $\wedge$ are commutative and associative operations.

(ii) There exist zero and identity elements, 0 and e respectively, such that for each $a \in B$,

$$a \vee 0 = a$$
$$a \wedge e = a.$$

(iii) Two distributive laws hold:

$$a \vee (b_1 \wedge b_2) = (a \vee b_1) \wedge (a \vee b_2)$$
$$b \wedge (a_1 \vee a_2) = (b \wedge a_1) \vee (b \wedge a_2),$$

for each $a, b_1, b_2, a_1, a_2, b \in B$.

(iv) For each $a \in B$, there is an element $a' \in B$ such that

$$a \vee a' = e,$$
$$a \wedge a' = 0.$$

The above constitute a minimal set of axioms which define a Boolean algebra. They are symmetric in the operations $\vee$ and $\wedge$.

An important example of a Boolean algebra is the class of all subsets of a fixed set. More precisely, let S be a fixed set, and let the binary operations be defined by

$$\tau_1 \vee \tau_2 = \tau_1 \cup \tau_2,$$
$$\tau_1 \wedge \tau_2 = \tau_1 \cap \tau_2,$$

for each $\tau_1, \tau_2 \subseteq S$. Let

$$\tau' = S \backslash \tau$$

for each $\tau \subseteq S$. In this Boolean algebra, the void set $\emptyset$ corresponds to the zero element 0 and the whole set S to the element e. Also we have:

$$\tau \vee \tau' = S,$$
$$\tau \wedge \tau' = \emptyset.$$

We can see that the Borel family Σ_p of the complex plane forms a Boolean algebra under the operations $\vee = \cup$ and $\wedge = \cap$.

9. Let $E \in L(X)$ (X be a Banach space). E is called a <u>projection</u> if and only if $E^2 = E$. If E is a projection there are closed subspaces X_1 and X_2 of X such that:

(i) X_1 is the range of E,

(ii) X_2 is the null-space of E,

(iii) $X = X_1 \oplus X_2$.

Conversely, let X_1 and X_2 be closed subspaces of X such that

$$X = X_1 \oplus X_2.$$

Then there is a projection E in L(X) whose range is X_1 and whose null-space is X_2. Moreover E is uniquely determined by these conditions (see in [22] pp. 25 and for a full discussion in [11]).

We summarise some useful properties of commuting projections. Let $E_1, E_2 \in L(X)$, $E_1E_2 = E_2E_1$, $E_1^2 = E_1$, $E_2^2 = E_2$. Then,

(a) $E_1E_2 = E_2$ if and only if $E_2X \subseteq E_1X$.

(b) $E = E_1 + E_2 - E_1E_2$ is a projection with $EX = \text{clm}(E_1X \cup E_2X)$.

(c) $E = E_1E_2$ is a projection with $EX = E_1X \cap E_2X$.

(d) $E = E_1 - E_2$ is a projection, if and only if $E_1E_2 = E_2$.

(e) If E is a projection in L(X), then E^* is a projection in $L(X^*)$, and

$$E^*X^* = \{x^* \in X^* : \langle x, x^* \rangle = 0,\ x \in (I-E)X\}.$$

The <u>natural ordering</u>, $\leq$, is defined on projections, by setting $E_1 \leq E_2$ to mean

$$E_1E_2 = E_2E_1 = E_1.$$

This is equivalent, by (a), to $E_1X \subseteq E_2X$. The natural ordering satisfies:

(i) $E_1 \leq E_1$,

(ii) if $E_1 \leq E_2$ and $E_2 \leq E_3$, then $E_1 \leq E_3$,

(iii) if $E_1 \leq E_2$ and $E_2 \leq E_1$, then $E_1 = E_2$.

Two projections E_1 and E_2, such that $E_1E_2 = E_2E_1$, will have a least upper bound,

$$E_1 \vee E_2 = E_1 + E_2 - E_1E_2$$

and a greatest lower bound

$$E_1 \wedge E_2 = E_1E_2$$

with respect to the natural order.

10. A Boolean algebra B of projections on X (X is a Banach space) is a commutative subset of L(X) such that:

(i) $E^2 = E \quad (E \in B)$;

(ii) $0 \in B$;

(iii) if $E \in B$ then $I-E \in B$;

(iv) if $E, F \in B$ then

$$E \vee F = E + F - EF \in B,$$

$$E \wedge F = EF \in B.$$

A Boolean algebra B of projections on X is said to be bounded if there is a real number M such that $||E|| \leq M \ (E \in B)$(see in [22] pp. 117).

Now, let Σ be a σ-algebra of subsets of an arbitrary set Ω. Suppose that a mapping $E(\cdot)$ from Σ into a Boolean algebra of projections on X satisfies the following conditions:

(i) $E(\delta_1) + E(\delta_2) - E(\delta_1)E(\delta_2) = E(\delta_1 \cup \delta_2) \quad (\delta_1, \delta_2 \in \Sigma)$;

(ii) $E(\delta_1)E(\delta_2) = E(\delta_1 \cap \delta_2) \quad (\delta_1, \delta_2 \in \Sigma)$;

(iii) $E(\Omega\setminus\delta) = I - E(\delta) \quad (\delta \in \Sigma)$;

(iv) $E(\Omega) = I$;

(v) there is $M > 0$ such that $||E(\delta)|| \leq M$, for all δ in Σ;

(vi) the scalar-valued set-function $\langle E(\cdot)x, x^* \rangle$ is countably additive on Σ, for each x in X and each $x^* \in X^*$.

The operator-valued set-function $E(\cdot)$ is called a <u>spectral measure</u>.

11. Let X be a Banach space. Let $T \in L(X)$. Then T is called a <u>spectral operator</u> if there is a spectral measure $E(\cdot)$ defined on Σ_p with values in $L(X)$ such that:

(i) $E(\cdot)$ is countably additive on Σ_p in the strong operator topology,

(ii) $TE(\tau) = E(\tau)T \quad (\tau \in \Sigma_p)$,

(iii) $\sigma(T|E(\tau)X) \subseteq \bar{\tau} \quad (\tau \in \Sigma_p)$.

Observe that (i) means that the vector-valued measure $E(\cdot)x$ is countably additive on Σ_p for each x in X.

The Boolean algebra of projections formed by the values of the spectral measure will be referred to as a <u>resolution of the identity</u> of the spectral operator.

12. <u>THEOREM</u> Let T be a spectral operator on X and let $E(\cdot)$ be the resolution of the identity for T. Let $A \in L(X)$ and $AT = TA$. Then

$$AE(\tau) = E(\tau)A \quad (\tau \in \Sigma_p).$$

(See [22], Theorem 6.6 pp. 161).

13. <u>THEOREM</u> Let T be a spectral operator on X. Then T has a unique resolution of the identity ([22] Theorem 6.7 pp. 162).

14. Let K be the topological space formed by the set $\sigma(T)$ $(T \in L(X)$,

X is a Banach space) in its relative topology with respect to the complex plane. Let C(K) be the algebra of continuous complex-valued functions on K, in the uniform norm, defined by

$$||f|| \doteq \sup_{\lambda \in \sigma(T)} |f(\lambda)| \qquad (f \in C(K)).$$

C(K) is a Banach algebra, and we have the following result:

<u>THEOREM</u> Let $f \in C(K)$. The Riemann-Stieltjes integral $\int_{\sigma(T)} f(J)E(dJ)$ exists, and converges in the uniform operator topology ([22] pp. 119-120).

15. <u>DEFINITION</u> Let S be a spectral operator on X with resolution of the identity $E(\cdot)$ such that

$$S = \int_{\sigma(S)} \lambda E(d\lambda).$$

Then S is called a <u>scalar-type spectral</u> operator ([22] pp. 129).

16. Now, let X be a Banach space. Let $T \in L(X)$. We introduce the functional calculus for T([22] pp. 10,11).

We denote by F(T) the family of all functions which are analytic on some neighbourhood of $\sigma(T)$. (The neighbourhood need not be connected, and can depend on the particular function in F(T)).

Let $f \in F(T)$, and let U be an open subset of $\mathbb{C}$, whose boundary B consists of a finite number of rectifiable Jordan curves. We assume throughout that B is oriented so that

$$\int_B (\lambda-\mu)^{-1} d\lambda = 2\pi i \qquad (\mu \in U)$$

$$\int_B (\lambda-\mu)^{-1} d\lambda = 0 \qquad (\mu \notin U \cup B).$$

Suppose that $U \supseteq \sigma(T)$, and that $U \cup B$ is contained in the domain of analyticity of f. Then the operator f(T) is defined by the equation

$$f(T) = \frac{1}{2\pi i} \int_B f(\lambda)(\lambda I-T)^{-1} d\lambda.$$

The integral exists as a limit of Riemann sums in the norm of L(X). It follows from the analyticity of $(\lambda I-T)^{-1}$ on $\rho(T)$, and from the Cauchy integral theorem, that f(T) depends only on the function f and not on the open set U chosen to define this operator. The above formula establishes a homomorphic map of F(T) into the algebra L(X), which maps 1 into I and λ into T.

<u>THEOREM</u> ([22] Theorem 1.19 pp. 11)

Let $T \in L(X)$. If f,g are in F(T) and a, $b \in \mathbb{C}$, then:

(i) $af+bg \in F(T)$ and $(af+bg)(T) = af(T)+bg(T)$;

(ii) $f \cdot g \in F(T)$ and $(f \cdot g)(T) = f(T) \cdot g(T)$;

(iii) if f has power series expansion $f(\lambda) = \sum_{k=o}^{\infty} a_k \lambda^k$, valid in a neighbourhood of $\sigma(T)$, then $f(T) = \sum_{k=o}^{\infty} a_k T^k$;

(iv) $f \in F(T^*)$ and $f(T^*) = (f(T))^*$.

17. A functional calculus for a much larger class of functions can be developed for a scalar type spectral operator S mapping a Banach space X into itself. In (1.14) we observed that for a function $f \in C(K)$ we may define

$$f(S) = \int_{\sigma(S)} f(\lambda) E(d\lambda).$$

This functional calculus preserves products:

$$f(S)g(S) = \int_{\sigma(S)} f(\lambda)g(\lambda)E(d\lambda) \qquad (f,g \in C(K)).$$

([22] pp. 123, 124). Moreover, we have

$$\sup_{\lambda \in \sigma(S)} |f(\lambda)| \leq \left\| \int_{\sigma(S)} f(\lambda)E(d\lambda) \right\| \leq 4M \sup_{\lambda \in \sigma(S)} |f(\lambda)| \quad (f \in C(K)),$$

where M is constant such that

$$\|E(\tau)\| \leq M < \infty \quad (\tau \in \Sigma_p).$$

(E(.) is the spectral measure for S.)

18. Let X be a Banach space. The support of a spectral measure E(•), whose values are projections in L(X), is the complement of the maximal open set on which the operator-valued set function is zero.

THEOREM Let $T \in L(X)$ be spectral and let E(•) be the resolution of the identity for T. Then the support of the spectral measure E(•) is the spectrum σ(T) of T([22] pp. 121, 122).

19. Let $T \in L(H)$ (H is a Hilbert space). Then T is said to be normal if $TT^* = T^*T$. T is said to be self-adjoint if $T = T^*$. T is said to be unitary if $TT^* = T^*T = I$.

THEOREM ([22] Theorem 7.18)

Let T be a normal operator on H. Then T is a scalar-type spectral operator. The values of the resolution of the identity of T are self-adjoint projections. (A projection $E \in L(H)$ is orthogonal if and only if $\|E\| = 1$. E is self-adjoint if and only if $E = E^*$. From [22] Proposition 7.14 we have:

For $E \in L(H)$, $E^2 = E$ and $E \neq 0$, the following statements are equivalent:

(i) E is self-adjoint,

(ii) $\|E\| = 1$,

(iii) EH and (I-E)H are orthogonal subspaces.)

THEOREM ([22] pp. 183, 184 Note 7.19)

Let A be a normal operator acting on the Hilbert space H. Then the Hilbert space adjoint A^* is given by the formula

$$A^* = \int_{\sigma(A)} \overline{\lambda}\, E(d\lambda),$$

where $E(\cdot)$ is the resolution of the identity for A.

20. The functional calculus for a normal operator S takes a special form (Lemma 7.14 of [22] pp. 181).

We have

$$||f(S)|| = \sup_{\lambda \in \sigma(S)} |f(\lambda)| \qquad (f \in C(\sigma(S))).$$

THEOREM ([22] pp. 181 Lemma 7.14)

Let T be a normal operator on H. There is an isometric algebra isomorphism ψ of $C(\sigma(T))$ into a subalgebra of L(H) consisting of normal operators such that:

(i) ψ maps the polynomial $p(\lambda,\overline{\lambda})$ into $p(T,T^*)$,

(ii) $\psi(\overline{f}) = (\psi(f))^* \qquad (f \in C(\sigma(T)))$.

(Note $\psi(f) \equiv f(T)$).

21. Let $T \in L(X)$ (X be a Banach space) and let Y be a closed subspace of X. Y is said to be <u>invariant</u> under T if and only if $TY \subseteq Y$. If this is the case we can define an operator $T|Y$ in L(Y) by

$$(T|Y)y = Ty \qquad (y \in Y).$$

$T|Y$ is called the <u>restriction</u> of T to Y.

It is an open question whether every bounded linear operator on a separable infinite-dimensional complex Banach space E has a proper closed invariant subspace; that is a closed invariant subspace other than the trivial ones $\{0\}$ and E.

Results on relationships between the spectrum of an operator and the spectrum of its restriction to a closed invariant subspace are contained in [22] pp. 19-22.

22. Let Y be a closed subspace of X. Introduce an equivalence relation on X by

$$x_1 \sim x_2 \Longleftrightarrow x_1 - x_2 \in Y.$$

The set of equivalence classes of elements of X corresponding to this equivalence relation is a complex vector space under the operations defined by

$$[x_1]_Y + [x_2]_Y = [x_1 + x_2]_Y$$

$$a[x]_Y = [ax]_Y \quad (a \in \mathbb{C}).$$

This vector space is called the quotient space of X modulo Y and is denoted by X/Y. Define

$$||[x]_Y|| = \inf\{||x + y|| : y \in Y\}.$$

This is indeed a norm on X/Y and moreover X/Y is a complex Banach space under this norm. The mapping ϕ defined by

$$\phi(x) = [x]_Y$$

is called the canonical mapping of X onto X/Y. ϕ is continuous, linear and $||\phi|| \leq 1$ (see [22]).

Let, now, $T \in L(X)$ and let Y be a closed subspace of X invariant under T. The map

$$T_Y[x]_Y = [Tx]_Y$$

is well-defined. Moreover $T_Y \in L(X/Y)$, since it is the composition $\phi \circ T$ of two continuous linear maps, and $||T_Y|| \leq ||T||$.

We will use the following propositions ([22] pp. 23, 24 Propositions 1.33, 1.34).

<u>PROPOSITION</u> Let Y be a closed subspace of X. Then there is a linear isometry J_1 of $(X/Y)^*$ onto $Y^\perp$ which is given by

$$\langle x, J_1 z\rangle = \langle [x]_Y, z\rangle$$

for all z in $(X/Y)^*$ and all x in X.

Let $T \in L(X)$ and let Y be a closed subspace of X invariant under T. The equation

$$\langle Tx, z\rangle = \langle x, T^* z\rangle \qquad (x \in X,\ z \in X^*)$$

shows that $T^* Y^\perp \subseteq Y^\perp$. In view of this and the above proposition we may and shall identify T_Y^* and $T^*|Y^\perp$.

<u>PROPOSITION</u> Let Y be a closed subspace of X. Then there is a linear isometry J_2 of $X^*/Y^\perp$ onto Y^* which is given by

$$\langle y, J_2 [z]_{Y^\perp}\rangle = \langle y, z\rangle$$

for all z in X^* and all y in Y.

Let $T \in L(X)$ and let Y be a closed subspace of X invariant under T. In view of the above proposition we may and shall identify $(T|Y)^*$ and $T^*_{Y^\perp}$.

23. A subspace M of a vector space F is said to have <u>finite codimension</u> in F if and only if the quotient space F/M has finite dimension. If M has finite codimension the dimension of F/M is called the <u>codimension</u> of M in F and is denoted by codim M.

The following lemma is in [22] pp. 82.

<u>LEMMA</u> ([22] Lemma 3.25)

A subspace M of a vector space F has finite codimension n in F if and only if there exists an n-dimensional subspace N of F such that $F = M \oplus N$.

CHAPTER TWO

Lattices of Invariant Subspaces of Analytic Toeplitz Operators and Isometries

D Sarason in [44] (see also [21]) proved that every normal (and hence every unitary) operator, on a Hilbert space H, is reflexive.

In this chapter we shall show that this property holds for analytic Toeplitz operators and for isometries.

1. Let A be a bounded linear operator on a Hilbert space H. We denote by LatA the set of all (closed) subspaces M of H such that $AM \subseteq M$.

If F is any collection of subspaces of H, then AlgF is the collection of operators $A \in B(H)$ such that $F \subset$ LatA.

Obviously the invariant subspaces of AlgF are invariant under sums and products of operators in AlgF and hence under the operator algebra generated by AlgF. Also, because a subspace is weakly closed, an invariant subspace of AlgF is invariant under every operator that lies in the closure of AlgF with respect to the weak operator topology. Then for any collection F of subspaces, AlgF is a weakly closed subalgebra of B(H) which contains I. If U is any subset of B(H), then $U \subseteq$ Alg LatU.

The algebra $U \subseteq B(H)$ is <u>reflexive</u> if U = Alg LatU; i.e., if whenever LatU $\subseteq$ LatB, then $B \in U$.

We denote by $\mathcal{A}_A$ (or $\mathcal{A}(I,A)$) the smallest weakly closed algebra containing I and A (i.e., the closure in the weak operator topology of p(A) for all polynomials p).

$A \in B(H)$ is called reflexive if, for B a bounded linear operator on H, $\mathrm{Lat}A \subseteq \mathrm{Lat}B$ implies $B \in \mathcal{U}_A$.

2. Analytic Toeplitz operators

We can see from [26] that every analytic Toeplitz operator on H^2 is reflexive; in particular the unilateral shift is reflexive.

2.1. Let H^2 denote the Hardy space of square-integrable functions on the unit circle Γ with negative Fourier coefficients zero.

Given $\phi \in H^\infty$, where $H^\infty = L^\infty \cap H^2$, the analytic Toeplitz operator S_ϕ on H^2 is defined by:

$$S_\phi f = \phi f \quad \text{a.e on } \Gamma (f \in H^2).$$

Denoting by A_2 the algebra of all analytic Toeplitz operators on H^2, it is known that $A_2 = A_2'$, where

$$A_2' = \{T \in B(H^2) : ST = TS \text{ for all } S \in A_2\}.$$

To prove this, let $z_\psi \in A_2$ for $\psi \in H^\infty$, then $z_\psi f = \psi f$ a.e on $\Gamma(f \in H^2)$, and let S_ϕ be any operator on A_2; we have

$$(z_\psi S_\phi)f = z_\psi(S_\phi f) = z_\psi(\phi f) = \psi(\phi f) = \phi(\psi f)$$
$$= S_\phi(\psi f) = S_\phi(z_\psi f) = (S_\phi z_\psi)f,$$

and $A_2 \subseteq A_2'$. The reverse inclusion is immediate from the following. If $T \in A_2'$, then

$$T_\phi = TS_\phi 1 = S_\phi T1 = \phi(T1) = (T1)\phi = S_{T1}\phi, \quad (\phi \in H^\infty)$$

and the arguments in [29] pp. 272-273. It follows that A_2, which clearly contains the identity operator, is a S-closed algebra of operators (S-closed for strong-closed).

Our aim is to show that all S-closed subalgebras of A_2 which contain I are reflexive.

2.2. LEMMA Let $f_1, f_2, \ldots, f_n \in H^2$. Then there exists $h \in H^2$ such that $|f_1| + |f_2| + \ldots + |f_n| \leq |h|$ a.e on Γ.

PROOF Let Q be the orthogonal projection of the space $L^2(\Gamma,\mu)$ of all square-integrable functions on Γ onto H^2. Let $|f_1| + |f_2|$ have Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{int}$. Since $|f_1| + |f_2|$ is non negative and

$$c_n = \int_\Gamma ke^{int} d\mu \quad (|f_1| + |f_2| = k)$$

it follows that

$$\overline{c}_{-n} = \left(\int_\Gamma ke^{-int} d\mu \right)^- = \int_\Gamma \overline{k}e^{int} d\mu = c_n$$

because k is real function. (See solution 26 pp. 199 from [21].)

Let $g = Q(|f_1| + |f_2|) - c \in H^2$. Then from

$$|f_1| + |f_2| = c_o + \sum_{n=1}^{\infty} c_n e^{in\theta} + \sum_{n=1}^{\infty} c_{-n} e^{-in\theta}$$

$$= c_o + \sum_{n=1}^{\infty} c_n e^{in\theta} + \sum_{n=1}^{\infty} \overline{c}_n e^{-in\theta},$$

we obtain

$$Q(|f_1| + |f_2|) = c_o + \sum_{n=1}^{\infty} c_n e^{in\theta}$$

and since $|f_1| + |f_2|$ is real, we have

$$|f_1| + |f_2| = c_o + 2\mathrm{Re}g$$

and so

$$|Q(|f_1| + |f_2|)| = |g + c_o| \geq |\mathrm{Re}(g + c_o)|$$

$$= \tfrac{1}{2}(|f_1| + |f_2| + c_o)$$

$$\geq \tfrac{1}{2}(|f_1| + |f_2|) \quad \text{a.e on } \Gamma.$$

Thus we can take $h = 2Q(|f_1| + |f_2|)$.

2.3. THEOREM Let $\mathcal{E}$ be a S-closed subalgebra of A_2 containing I. Then $\mathcal{E}$ is reflexive.

PROOF We have always $\check{\mathcal{E}} \subseteq \text{Alg Lat}\,\check{\mathcal{E}}$. To prove the reverse inclusion we show first that $T \in \text{Alg Lat}\,\check{\mathcal{E}}$ has the property $T \in A_2'$. Fix z in $\mathbb{C}$ with $|z| < 1$ and consider

$$M_z = \{f \in H^2 : \tilde{f}(z) = 0\},$$

where $\tilde{f}$ denotes the analytic extension of f to the interior of the unit disc. It is clear that $M_z \in \text{Lat}\,\check{\mathcal{E}}$, because $S_\phi M_z \subset M_z$ for $S_\phi \in A_2$, $\phi \in H^\infty$ ($(S_\phi f)\tilde{}(z) = \tilde{\phi}(z)\tilde{f}(z) = 0$, [29] pp. 20, problem 34). Noting that

$$f - \tilde{f}(z)1 \in M_z \quad (f \in H^2),$$

because $(f - \tilde{f}(z)1)\tilde{}(z) = \tilde{f}(z) - \tilde{f}(z) = 0$, it follows that

$$(Tf)\tilde{}(z) - \tilde{f}(z)(T1)\tilde{}(z) = [T(f - \tilde{f}(z)1)]\tilde{}(z) = 0$$

for $f \in H^2$. Hence

$$\begin{aligned}(Tf)\tilde{}(z) - \tilde{f}(z)(T1)\tilde{}(z) &= \{Tf - \tilde{f}(z)(T1)\}\tilde{}(z)\\ &= \{T(f - \tilde{f}(z)1)\}\tilde{}(z) = 0\end{aligned}$$

and so

$$\begin{aligned}(TS_\phi f)\tilde{}(z) &= (\phi f)\tilde{}(z)(T1)\tilde{}(z) = \tilde{\phi}(z)\tilde{f}(z)(T1)\tilde{}(z)\\ &= (S_\phi Tf)\tilde{}(z) \quad \text{for } \phi \in H^\infty,\ f \in H^2,\end{aligned}$$

because

$$\begin{aligned}(TS_\phi f)\tilde{}(z) &= (T_\phi f)\tilde{}(z) = (\phi f)\tilde{}(z)(T1)\tilde{}z\\ &= \tilde{\phi}(z)\tilde{f}(z)(T1)\tilde{}(z)\end{aligned}$$

and

$$\begin{aligned}(S_\phi Tf)\tilde{}(z) &= (\phi Tf)\tilde{}(z) = \tilde{\phi}(z)(Tf)\tilde{}(z)\\ &= \tilde{\phi}(z)\tilde{f}(z)(T1)\tilde{}(z).\end{aligned}$$

Hence $TS_\phi = S_\phi T$ $(\phi \in H^\infty)$, and so $T \in A_2'$. Thus $T = S_\psi$ for some $\psi \in H^\infty$, since $A_2 = A_2'$.

Let $f_1, f_2, \ldots, f_n \in H^2$. Then from the lemma 2.2.2, there exists $h \in H^2$ such that $|f_1| + |f_2| + \ldots + |f_n| \leq |h|$ a.e on Γ and from $T \in \text{Alg Lat}\,\mathcal{E}$ (i.e., if and only if given $x \in L^2(\Gamma,\mu)$ and $\varepsilon > 0$, there exists $A \in A_2$ with $|Tx - Ax| < \varepsilon$) we have that there exists $A \in \mathcal{E}$ such that

$$|Th - Ah| < \varepsilon.$$

Let $A = S_\phi | H^2$, where $\phi \in H^\infty$. Then for $i = 1, 2, \ldots, n$

$$|Tf_i - Af_i| = |\psi f_i - \phi f_i| \leq |\psi - \phi| \cdot |f_i|$$
$$\leq |\psi - \phi|\ |h| = |Th - Ah| < \varepsilon$$

and it follows that $T \in \breve{\mathcal{E}}$. Hence $\mathcal{E} = \text{Alg Lat}\,\mathcal{E}$ and so $\breve{\mathcal{E}}$ is reflexive.

3. Isometries

3.1. Our task now is to show that, in view of the next theorem, the hypothesis that the Hilbert space H is separable, in the work of J.A. Deddens "every isometry is reflexive" [14], can be omitted.

3.2. THEOREM Let H be a Hilbert space and $A \in B(H)$. The algebra $\mathcal{A}(I,A)$ (generated by I and A in the weak operator topology of B(H)) is reflexive, if for every closed separable subspace Y of H invariant under A, the algebra $\mathcal{A}(I|Y, A|Y)$ is reflexive.

PROOF Suppose B leaves invariant LatA (that is LatA $\subseteq$ LatB); we want to prove that $B \in \mathcal{A}(I,A)$. It follows from [38] pp. 118 Cor. 7.2 that it is sufficient to show that B is the strong limit of polynomials in A.

Let $G = \{T \in B(H) : ||Ty_r - By_r|| < \varepsilon, r = 1,2, \ldots, n\}$ be a strong (basic) neighbourhood of B. To show that B belongs to $\mathcal{U}(I,A)$, it is enough to show that every strong neighbourhood of B contains p(A) for some polynomial p. Let

$$Y = \mathrm{clm}\{A^m y_r : m = 0,1,2, \ldots, r = 1,2, \ldots, n\}.$$

This is a closed separable subspace of H invariant under A. Since $BY \subseteq Y$, consider

$$G_Y = \{T \in B(Y) : ||Ty_r - By_r|| < \varepsilon, r = 1,2, \ldots, n\}.$$

Then from reflexivity of $A|Y$ it follows that there exists $p(A|Y) \in G_Y$ and so there exists $p(A) \in G$.

3.3. Let now V be an isometry on H (i.e. $||Vx|| = ||x||$ for all $x \in H$). Then there exist unique reducing subspaces $M_\infty \equiv \bigcap_{n=0}^{\infty} V^n H$ and $M_+ \equiv \sum_{n=0}^{\infty} \oplus V^n(H \ominus VH)$ such that $H = M_\infty \oplus M_+$ with $U = V|M_\infty$ a unitary operator and $U_+ = V|M_+$ a unilateral shift (see [24] pp. 16 and [29] pp. 274 solution 118).

If we let $E(\cdot)$ be the resolution of the identity of U,

$$M_s \equiv \{x \in M_\infty : ||E(\cdot)x||^2 \perp \mu(\cdot)\}$$

and

$$M_a \equiv \{x \in M_\infty : ||E(\cdot)x||^2 \ll \mu(\cdot)\},$$

then $M_\infty = M_s \oplus M_a$ with $U_s = U|M_s$ and $U_a = U|M_a$ being called the singular and absolutely continuous parts of U respectively. $\mu(\cdot)$ denotes Lebesgue linear measure on the unit circle.

<u>THEOREM</u> Every isometry $V = U_s \oplus U_a \oplus U_+$ on a separable Hilbert space H is reflexive.

Before we prove this result we need some lemmas, which are of interest in their own right.

3.4. <u>LEMMA</u>

$$\mathrm{Lat}(U_S \oplus U_a) = \mathrm{Lat}(U_S) \oplus \mathrm{Lat}(U_a).$$

<u>PROOF</u> We have always $\mathrm{Lat}(U_S) \oplus \mathrm{Lat}(U_a) \subseteq \mathrm{Lat}(U_S \oplus U_a)$. We need to show that $M \in \mathrm{Lat}(U_S \oplus U_a)$ implies that $M \in \mathrm{Lat}(U_S) \oplus \mathrm{Lat}(U_a)$ and for this it is enough to have $M = P_{M_S}M \oplus P_{M_a}M$ because:

$$P_{M_S}M = M_S \in \mathrm{Lat}(U_S),\ P_{M_a}M = M_a \in \mathrm{Lat}(U_a)$$

$$U_S P_{M_S}(x_S + x_a) = U_S x_S \in M_S \subseteq P_{M_S}M$$

$$M_S = \{x_S + x_a : x_S \in M_S,\ x_a \in M_a\}.$$

Let $P_M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$. Observe that $P_M \in B(M_S \oplus M_a)$ and so

$$P_M = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

where $S_{ij} = P_i P_M P_j$ and P_i the projection on H_i where $H = H_1 \oplus H_2$ and $P_M \in B(H)$. It follows that

$$S_{12} = S_{Sa} = P_{M_S} P_M P_{M_a} = (P_{M_a} P_M P_{M_S})^* = (S_{Sa})^* = S_{21}.$$

But since $M \in \mathrm{Lat}(U_S \oplus U_a)$, $(U_S \oplus U_a)M \subseteq M$ and $M \subseteq M_S \oplus M_a$.

Let $M = \mathcal{U}_S \oplus \mathcal{U}_a$ with $\mathcal{U}_S \subseteq M_S$ and $\mathcal{U}_a \subseteq M_a$;

we have

$$P_M M_S = P_M(M_S \oplus 0) = \mathcal{U}_S \subseteq M_S$$

and so P_M is invariant under M_S and by [5] pp. 108 $S_{21} = S_{aS} = B^* = 0$ and $B = 0$ ($B^*B = 0$ iff $B = 0$). Then

$$P_M = A \oplus C = P_{M_S} P_M P_{M_S} \oplus P_{M_a} P_M P_{M_a} .$$

and so $M = P_{M_S}M \oplus P_{M_a}M$.

3.5. LEMMA

$$\mathcal{U}_{U_s \oplus U_a} = \mathcal{U}_{U_s} \oplus \mathcal{U}_{U_a} \quad (\mathcal{U}_U \equiv \mathcal{U}(I,U)).$$

PROOF See [14] pp. 510 Lemma 2.

3.6. COROLLARY

$$\mathcal{U}_{U_s \oplus U_a \oplus U_+} = \mathcal{U}_{U_s} \oplus \mathcal{U}_{U_a \oplus U_+}.$$

PROOF See [14] pp. 510 Corollary.

3.7. LEMMA

$U_a \oplus U_+$ is reflexive.

PROOF For the proof we can use Proposition 5 of [15] pp. 887 but we must prove that there is a decomposition for U_a and U_+ as in 5 of [15] pp. 887.

[A] Consider $U_a|M_a$. We have the following result.

Let U_a be a unitary operator in the separable Hilbert space

$$M_a = \{x \in M_\infty : ||E(\cdot)x||^2 << \mu(\cdot)\},$$

where $E(\cdot)$ is the resolution of the identity of U_a and $\mu(\cdot)$ is Lebesgue linear measure on the unit circle. Then from [21] pp. 160 Theorem 2.7, we have

$$M_a = M(x_1) \oplus M(x_2) \oplus \dots \oplus M(x_r) \oplus \dots,$$

where $x_1, x_2, \dots, x_r, \dots$, is a maximal family of non-zero vectors in M_a such that $||x_r|| = 1 (r = 1,2, \dots,)$ and $M(x_a) \perp M(x_b)$ if $a \neq b$. (This family is countable since M_a is separable.) Also

$$M(x_r) = \mathrm{clm}\{E(\tau)x_r : \tau \in \Sigma\}$$

with Σ the σ-algebra of Borel sets of the complex plane.

However for $x_r \in M_a$ and $\mu_r(\cdot)$ a measure on Σ defined for every τ in Σ by

$$\mu_r(\tau) = (E(\tau)x_r, x_r)$$

there is an isometric isomorphism U_r from $L^2(\mu_r)$ onto $M(x_r)$ such that

$$U_r^{-1}E(\tau)U_r f = \chi_\tau f$$

whenever $f \in L^2(\mu_r)$ and $\tau \in \Sigma$. Here, the measure μ_r has support the set $\sigma(U_a|M(x_r))$ and is absolutely continuous with respect to Lebesgue linear measure on the unit circle. It follows that U_a is unitarily equivalent to

$$\oplus \sum_r U_r \text{ on } \sum_r L^2(\mu_r).$$

[B] Consider now U_+. We know that this is a direct sum of $(\dim(VH)^\perp)$ times the unilateral shift of multiplicity one. (See [29] pp. 275, Solution 118.) We know that the unilateral shift is multiplication by z on H^2 and so U_+ is a direct sum of a family of multiplications and so is a multiplication on H^2.

After [A] and [B] we can see that $U_a \oplus U_+$ is $M_e \oplus M_e$ on $L^2(E) \oplus H^2$, where E is a closed subset of the unit circle and

$$(M_e f)(e^{i\theta}) = e^{i\theta}f(e^{i\theta}) \quad (0 \leq \theta < 2\pi)$$

for f in $L^2(E)$ or H^2. We need to prove that $M_e \oplus M_e$ is reflexive and satisfies a certain boundedness condition.

Suppose that $\operatorname{Lat}(M_e \oplus M_e) \subseteq \operatorname{Lat}B$. Then $B = B_1 \oplus B_2$ with $\operatorname{Lat}M_e \subseteq \operatorname{Lat}B_2$. Since the unilateral shift M_e on H^2 is reflexive, $B_2 \in \mathcal{A}_{M_e}$. Hence $B_2 = M_\psi$, for some ψ in H^∞. We consider the closed subspaces of the form

$$M_n = \{(M_e^{*n}\phi|E, \phi) : \phi \in H^2\}$$

invariant under $M_e \oplus M_e$. For ϕ in H^2

$$(M_e \oplus M_e)(M_e^{*n}\phi|E,\phi) = M_e M_e^{*n}\phi|E \oplus M_e\phi$$

$$= (M_e^{*n}\phi_1|E,\ \phi_1) \in M_n$$

where $\phi_1 = e^{i\theta}\phi \in H^2$. We conclude that

$$B_1 M_e^{*n}\phi|E = M_e^{*n}\psi\phi|E$$

for all ψ in H^2 and $n \geq 0$. (Note that if $\phi \in H^2$ and $\psi \in H^\infty$, then $\phi\psi \in H^2$.) Since $\{M_e^{*n}\phi|E : \phi \in H^2,\ n \geq 0\}$ is dense in $L^2(E)$, we have $B_1 = M_\psi|E$. Thus

$$B_1 \oplus B_2 = M_{\psi|E} \oplus M_\psi \quad (\psi \in H^\infty).$$

Hence $B_1 \oplus B_2 \in \mathcal{Q}_{M_e \oplus M_e}$ and so $M_e \oplus M_e$ is reflexive.

Suppose that $T \in \mathcal{Q}_{M_e \oplus M_e}$. Then $T = T_1 \oplus T_2$, where $T_2 \in \mathcal{Q}_{M_e}$ on H^2 and so $T_2 = M_\psi$ for some ψ in H^∞. Now there is a sequence $\{p_n\}$ of polynomials with $p_n \to \psi$ in the L^2 norm such that $M_{p_n} = p_n(M_e) \to M_\psi$ weakly; i.e.

$$M_{p_n} f = p_n f \to \psi f \qquad (f \in H^2)$$

or

$$M_{p_n} f = p_n(M_e) f = M_\psi f \quad (f \in H^2).$$

It follows that

$$||M_{p_n}|| = ||p_n||_\infty \leq ||\psi||_\infty$$

(See [29], Solution 33, p. 196.)

Then $p_n(M_e \oplus M_e) \to T_1 \oplus T_2$ in the weak operator topology; i.e.

$$p_n(M_e \oplus M_e) = p_n(M_e) \oplus p_n(M_e) \to T_1 \oplus T_2.$$

It follows that

$$||p_n(M_e \oplus M_e)|| = ||p_n||_\infty \leq ||\psi||_\infty$$

and the boundedness condition is satisfied.

3.8 THEOREM. Every isometry $V = U_s \oplus U_a \oplus U_+$ on a Hilbert space H is reflexive.

PROOF. Suppose $\mathrm{Lat}(V) \subseteq \mathrm{Lat}(B)$, where $V = U_s \oplus U_a \oplus U_+$. Then $B = B_1 \oplus B_2 \oplus B_3$ with $\mathrm{Lat}(U_s) \subseteq \mathrm{Lat}(B_1)$ and $\mathrm{Lat}(U_a \oplus U_+) \subseteq \mathrm{Lat}(B_2 \oplus B_3)$. Hence $B_1 \in \mathcal{A}_{U_s}$ and $B_2 \oplus B_3 \in \mathcal{A}_{U_a \oplus U_+}$. Now U_s is unitary, hence reflexive and $U_a \oplus U_+$ is reflexive by Lemma 2.3.7. But by the Corollary 2.3.6

$$B_1 \oplus B_2 \oplus B_3 \in \mathcal{A}_{U_s} \oplus \mathcal{A}_{U_a \oplus U_+} = \mathcal{A}_{U_s \oplus U_a \oplus U_+} = \mathcal{A}_V.$$

Hence V is reflexive.

CHAPTER THREE

Subnormal Operators

1. Spectral sets

1. DEFINITION. Let X be a Banach space and $T \in B(X)$. We shall say that the spectrum $\sigma(T)$ is a spectral set of order N if, for each rational function r, whose poles lie in $\rho(T)$, we have

$$||r(T)|| \leq N \sup_{z \in \sigma(T)} |r(z)|.$$

From [18] pp. 143, Lemma 2, we have that: every normal operator $A \in B(H)$ (H is a Hilbert space) has $\sigma(A)$ as a spectral set of order 1.

2. DEFINITION. Let H_1 be a Hilbert space and $A_1 \in B(H_1)$. A_1 will be called subnormal if there exists a Hilbert space $H_2 \supseteq H_1$ and a bounded normal operator $A \in B(H_2)$, such that $AH_1 \subseteq H_1$ and $A|H_1 = A_1$.

Again from [18] pp. 147, Theorem 5.2.9 we have that: if A is a subnormal operator acting on a Hilbert space H, then $\sigma(A)$ is a spectral set of order 1.

3. DEFINITION. Every operator $T \in B(H)$ whose spectrum is a spectral set for T is called a von Neumann operator.

We can see that every subnormal operator $A \in B(H)$ is a von Neumann operator (see [18] pp. 147), but the opposite does not hold; in fact, from the example in [19] pp. 671, we have that the operator T_Y induced on the quotient space X/Y by the normal operator T does not always have the single-valued extension property (see [22] pp. 141) and so is not a subnormal operator, but T_Y is a von Neumann operator. (We have T normal, $TY \subseteq Y$, so T^* is normal and $T^*Y^{\perp} \subseteq Y^{\perp}$, so $T^*|Y^{\perp}$ subnormal. Then by [22] pp. 23 we have that $T^*|Y^{\perp}$ is subnormal so $T^*|Y^{\perp}$ has its spectrum as a spectral set and so has also T_Y (see [22] pp. 23, 24)).

2. Reflexivity of subnormal operators

1. In [34] the authors have answered the following question in the affirmative.

Is every subnormal operator S on a separable Hilbert space H reflexive?

2. The purpose now is to answer the following question:

Let A be a normal operator on the separable Hilbert space H with $\sigma(A)$ contained in Γ, a simple closed rectifiable contour. Let Y be a closed subspace of H, invariant under A ($AY \subseteq Y$). There are two possibilities:

(i) $\sigma(A|Y) \subseteq \Gamma$, in which case $A|Y$ is a normal operator ([22] pp. 236 Proposition 12.21).

(ii) $\sigma(A|Y) = \Gamma \cup I(\Gamma)$, where $I(\Gamma)$ denotes the inside of Γ.

We know that in the case (i) $A|Y$ is reflexive.

The question is: In the case (ii), where we have a subnormal operator $A|Y$ (but not normal), is the operator $A|Y$ reflexive?

In view of Theorem 3 of [20] pp. 308 we have the following result.

Let ϕ map $I(\Gamma)$ conformally onto $\{z : |z| < 1\}$. Then ϕ maps $\Gamma \cup I(\Gamma)$ homeomorphically onto $\{z : |z| \leq 1\}$ ([43] pp. 273), and a Borel subset δ of Γ has $\mu(\delta) = 0$ if and only if $m(\phi(\delta)) = 0$, where $\mu(\cdot)$ is Lebesgue linear measure on Γ and $m(\cdot)$ is Lebesgue linear measure on the unit circle. Let ψ be the function inverse to ϕ.

Observe that ψ maps $\{z : |z| \leq 1\}$ homeomorphically onto $\Gamma \cup I(\Gamma)$. Define (as in Theorem 3 of [20] pp. 308)

$$U = \int_{\sigma(A)} \phi(\lambda)E(d\lambda).$$

Then, from [22] pp. 183 Note 7.19, we have that U is normal and ([22] pp. 184 Theorem 7.20) hence is unitary because $\sigma(U) \subseteq \{z : |z| = 1\}$. The resolution of the identity $F(\cdot)$ for U is given by

$$F(\phi(\tau)) = E(\tau),$$

τ Borel subset of Γ ([22] pp. 124) and so

$$\psi(U) = \int_{\sigma(A)} \psi(\phi(\lambda))E(d\lambda)$$

$$= \int_{\sigma(A)} \lambda E(d\lambda) = A$$

with $\psi \in C(\sigma(U))$ and analytic for $\{z : |z| < 1\}$. Then, from [43] pp. 386 (Mergelyan's Theorem), we have that

$$A = \psi(U) = \lim_{n \to \infty} \psi_n(U)$$

with $\psi_n(z)$ sequence of polynomials in z with complex coefficients and $|z| \leq 1$.

Hence, we have that A is the norm limit of a sequence of polynomials in I and U.

From

$$U = \int_{\sigma(A)} \phi(\lambda)E(d\lambda) = \phi(A)$$

and [43] pp. 386 (Mergelyan's Theorem) again, we have

$$U = \phi(A) = \lim_{n \to \infty} \phi_n(A),$$

where $\phi_n(z)$ is a sequence of polynomials in z with complex coefficients, $\phi \in C(\sigma(A))$ and ϕ analytic in $I(\Gamma)$.

Hence, U is the norm limit of a sequence of polynomials in I and A.

Let now $\mathcal{A}(I|Y, S|Y)$ be the closed algebra of operators generated by $S|Y$ in the weak operator topology of B(Y). Then

$$\mathcal{A}(I|Y, A|Y) = \mathcal{A}(I|Y, U|Y)$$

because $A|Y$ is a norm limit of a sequence of polynomials in $U|Y$ and $U|Y$ is the norm limit of a sequence of polynomials in $A|Y$.

Since $U|Y$ is reflexive being an isometry, then $A|Y$ is also reflexive.

3. Generalized Spectral Measures and Naimark's Theorem

1. DEFINITION. A positive operator valued measure (generalized spectral measure) $F(\cdot)$, is defined as a set function from the σ-algebra of the Borel sets of the complex plane, to the positive (and hence self-adjoint) operators on B(H), where H is a Hilbert space and B(H) the Banach algebra of all bounded linear operators on H, such that:

(i) $F(\emptyset) = 0$, $F(\mathbb{C}) = 1$

(ii) $F(\tau_1 \cup \tau_2) = F(\tau_1) + F(\tau_2)$ for τ_1, τ_2 disjoint

(iii) for each $x \in H$ the set function $(F(\cdot)x,x)$ is a measure on the σ-algebra of the Borel sets of the complex plane (see [7] pp. 5,6,9).

Now we come to the theorem of Naimark ([1] pp. 124).

2. THEOREM. Let $F(\cdot)$ be a generalized spectral measure for the Hilbert space H. Then there exists a Hilbert space H^+ which

contains H as a subspace and there exists a spectral measure $E(\cdot)^+$ for the space H^+ such that

$$F(\cdot)f = P^+E(\cdot)^+f$$

for each $f \in H$, where P^+ is the operator of projection on H (see [1] pp. 124).

PROOF. Let $\mathcal{R}$ be the set of all pairs p of the form $p = \{\tau,f\}$, where τ is an arbitrary Borel set of the complex plane, (We know that if $\tau \cap \mathrm{supp}F(\cdot) = \emptyset$ then $F(\tau) = 0$.), and f is an arbitrary vector of H. On $\mathcal{R}$ we define a function $\Phi(p_1,p_2)$ such that: if $p_1 = \{\tau_1,f_1\}$ and $p_2 = \{\tau_2,f_2\}$, then

$$\Phi(p_1,p_2) = (F(\tau_1 \cap \tau_2)f_1,f_2).$$

We show that the function $\Phi(p_1,p_2)$ is positive-definite. (For this function, see [1] pp. 122, 123).

Indeed

$$\begin{aligned}\Phi(p_1,p_2) &= (F(\tau_1 \cap \tau_2)f_1,f_2) = (f_1,F^*(\tau_1 \cap \tau_2)f_2) \\ &= (f_1,F(\tau_1 \cap \tau_2)f_2) = \overline{(F(\tau_1 \cap \tau_2)f_2,f_1)} \\ &= \overline{\Phi(p_1,p_2)}.\end{aligned}$$

and $\Phi(p,p) \geq 0$. (See [1] pp. 130, Theorem).

$$(1) \quad \sum_{i,k=1}^{n} \Phi(p_i,p_k)\xi_i\overline{\xi}_k = \sum_{i,k=1}^{n} (F(\tau_i \cap \tau_k)f_i,f_k)\xi_i\overline{\xi}_k.$$

If $\tau_i (i = 1,2, \ldots, n)$ are pairwise disjoint, then

$$(2) \quad \sum_{i,k=1}^{n} (F(\tau_i \cap \tau_k)f_i,f_k)\xi_i\overline{\xi}_k = \sum_{i=1}^{n} (F(\tau_i)f_i,f_i)|\xi_i|^2 \geq 0.$$

If, now, the $\tau_i (i = 1,2, \ldots, n)$ are pairwise disjoint and the τ_1, τ_2 coincide, then the sums in the right member of (1) fall into two parts. One part, with indices from 3 to n, is of the form (2), and the other part, with indices 1 and 2, satisfies

$$\sum_{i,k=1}^{2} (F(\tau_i \cap \tau_k)f_i, f_k)\xi_i\overline{\xi}_k = \sum_{i,k=1}^{2} (F(\tau_1)F_i, f_k)\xi_i\overline{\xi}_k$$

$$= (F(\tau_1) \sum_{i=1}^{2} \xi_i f_i, \sum_{k=1}^{2} \xi_k f_k) \geq 0.$$

In fact

$$\sum_{i,k=1}^{n} (F(\tau_i \cap \tau_k)f_i, f_k)\xi_i\overline{\xi}_k$$

$$= (F(\tau_1 \cap \tau_1)f_1, f_1)\xi_1 \cdot \overline{\xi}_1 + (F(\tau_1 \cap \tau_2)f_1, f_2)\xi_1 \cdot \overline{\xi}_2$$

$$+ (F(\tau_1 \cap \tau_3)f_1, f_3)\xi_1 \cdot \overline{\xi}_3 + \ldots + (F(\tau_1 \cap \tau_n)f_1, f_n)\xi_i \cdot \overline{\xi}_n$$

$$+ (F(\tau_2 \cap \tau_1)f_2, f_1)\xi_2 \cdot \overline{\xi}_1 + (F(\tau_2 \cap \tau_2)f_2, f_2)\xi_2 \cdot \overline{\xi}_2$$

$$+ (F(\tau_2 \cap \tau_3)f_2, f_3)\xi_2 \cdot \overline{\xi}_3 + \ldots + (F(\tau_2 \cap \tau_n)f_2, f_n)\xi_2 \cdot \overline{\xi}_n$$

$$+ \ldots$$

$$+ (F(\tau_n \cap \tau_1)f_n, f_1)\xi_n \cdot \overline{\xi}_1 + (F(\tau_n \cap \tau_2)f_n, f_2)\xi_n \cdot \overline{\xi}_2$$

$$+ (F(\tau_n \cap \tau_3)f_n, f_3)\xi_n \cdot \overline{\xi}_3 + \ldots + (F(\tau_n \cap \tau_n)f_n, f_n)\xi_n \cdot \overline{\xi}_n$$

The case with arbitrary $\tau_i (i = 1,2, \ldots, n)$ can be reduced, with the aid of additional partitions, to the cases already considered.

Hence, if $\tau_1 \cap \tau_2 = \emptyset$, then

$$(F(\tau_1 \cup \tau_2) \cap \tau_3)f, g) = (F((\tau_1 \cap \tau_3) \cup (\tau_2 \cap \tau_3))f, g)$$

$$= (F(\tau_1 \cap \tau_3)f, g) + (F(\tau_2 \cap \tau_3)f, g).$$

Thus, $\Phi(p_1,p_2)$ is a positive definite function on $\mathcal{R}$.

As in [1] pp. 122, 123, $\mathcal{R}$ can be embedded in a Hilbert space H^+.

We can consider p instead of $\mathcal{P}$, if p of $\mathcal{R}$ belongs to $\mathcal{P}$ (=elements of H^+ which are subsets of $\mathcal{R}$).

We indicate the scalar product in the space H^+ by the symbol $(\quad)_+$, and we have

$$(p_1,p_2)_+ = \Phi(p_1,p_2).$$

We now consider elements of H^+ of the form $'\{\mathbb{C},f\}$. By means of the equation

$$'(\{\mathbb{C},f\},'\{\mathbb{C},g\})_+ = \Phi'(\{\mathbb{C},f\},'\{\mathbb{C},g\})$$

$$= (F_{\mathbb{C}}f,g) = (f,g)$$

we can identify the pair $'\{\mathbb{C},f\}$ with the element f from H. The element $\sum_{k=1}^{n} \xi_k'\{\mathbb{C},f_k\}$ of the space H^+ is identified with the element $\sum_{k=1}^{n} \xi_k f_k$ of the space H. Thus, H can be considered as a subspace of H^+.

We now consider the following problem: find the projection of the element $'\{\tau,f\}$ of the space H^+ on the subspace H. We denote the projection to be found by $'\{\mathbb{C},g\}$ (this is the form of the elements of H).

For each h on H,

$$'(\{\tau,f\} - '\{\mathbb{C},g\},'\{\mathbb{C},h\})_+ = 0.$$

If $x \in H$ and M is a subspace of H, then $x = Px + (x - Px)$ with $Px \perp x - Px$, where P is the projection on M;

or

$$'(\{\tau,f\},'\{\mathbb{C},h\})_+ - '(\{\mathbb{C},g\},'\{\mathbb{C},h\})_+ = (F(\tau)f,h) - (g,h)$$

$$= (F(\tau)f - g,h) = 0,$$

so that $g = F(\tau)f$

i.e.

(3) $$P^{+}\{\tau,f\} \overset{'}{=} \{\mathbb{C},F(\tau)f\}.$$

The theorem will be proved if it is established that the operator function $E_\tau^+(\equiv E^+(\tau))$, which is defined by

(4) $$E_\tau^+\{\tau',f\} \overset{'}{=} \{\tau\wedge\tau',f\}$$

for each element of the form $\{\tau',f\}\in H^+$, is a spectral measure for the space H^+, since (3) can be expressed in the form

$$P^+E_\tau^+f = P^+E_\tau^+\{\mathbb{C},f\} = P^+\{\tau\wedge\mathbb{C},f\}$$

$$= P^+\{\tau,f\} \overset{'}{=} \{\mathbb{C},F(\tau)f\} = F(\tau)f.$$

We have identified $\{\mathbb{C},F(\tau)f\}$ of H^+ with $F(\tau)f$ of H for each $f\in H$. It is evident that E_τ^+ is an additive operator function of τ.

In fact

$$E^{+}_{\tau_1\cup\tau_2}{}'\{\tau,f\} \overset{'}{=} \{(\tau_1\cup\tau_2)\cap\tau,f\} \overset{'}{=} \{(\tau_1\cap\tau)\cup(\tau_2\cap\tau),f\}$$

$$(E^+_{\tau_1}+E^+_{\tau_2})\{\tau,f\} = E^{+'}_{\tau_1}\{\tau,f\} + E^{+'}_{\tau_2}\{\tau,f\}$$

$$\overset{'}{=} \{\tau_1\cap\tau,f\} \overset{'}{+} \{\tau_2\cap\tau,f\}.$$

Furthermore the two equations

$$(E_\tau^+)^2\{\tau',f\} = E_\tau^+\{\tau\wedge\tau',f\}$$

$$\overset{'}{=} \{\tau\wedge\tau\wedge\tau',f\} = E_\tau^+\{\tau',f\}$$

and

$$(E_\tau^+\{\tau',f\},' \{\tau'',g\})_+ = (\{\tau\wedge\tau',f\},\{\tau'',g\})_+$$

$$= (F(\tau\wedge\tau'\wedge\tau'')f,g)$$

$$= (F(\tau'\wedge\tau\wedge\tau'')f,g)$$

$$= (\{\tau',f\}, E^+\{\tau'',g\})_+,$$

where we have defined

$$'(\{\tau,f\},' \{\tau',g\})_+ = \Phi'(\{\tau,f\},' \{\tau',g\}) = (F(\tau\cap\tau')f,g),$$

imply that E^+_τ is a self-adjoint projection operator. Further, we have

$$E^+_{\mathbb{C}}\{\tau',f\} =' \{\mathbb{C}\cap\tau',f\} =' \{\tau',f\} \quad '(\{\tau',f\}\in H^+);$$

that is $E^+_{\mathbb{C}} = I$. Furthermore

$$E^+_{\tau_1\cap\tau_2}{}' \{\tau',f\} =' \{\tau_1\cap\tau_2\cap\tau',f\},$$

$$E^+_{\tau_1} E^{+\prime}_{\tau_2}\{\tau',f\} = E^{+\prime}_{\tau_1}\{\tau_2\cap\tau',f\} =' \{\tau_1\cap\tau_2\cap\tau',f\},$$

and so

$$E^+_{\tau_1\cap\tau_2} = E^+_{\tau_1} E^+_{\tau_2}.$$

Finally,

$$\begin{aligned}(E^+_{\bigcup_n \tau_n}{}' \{\tau',f\},' \{\tau'',g\})_+ &= '(\{\bigcup_n \tau_n\cap\tau',f\},' \{\tau'',g\})_+ \\ &= '(\{\bigcup_n(\tau_n\cap\tau'),f\},' \{\tau'',g\})_+ \\ &= \Phi'(\{\bigcup_n(\tau_n\cap\tau'),f\},' \{\tau'',g\}) \\ &= (F(\bigcup_n(\tau_n\cap\tau')\cap\tau'')f,g) \\ &= (\sum_n F(\tau_n\cap\tau'\cap\tau'')f,g) \\ &= \sum_n \Phi'(\{\tau_n\cap\tau',f\},' \{\tau'',g\}) \\ &= \sum_n '(\{\tau_n\cap\tau',f\},' \{\tau'',g\})_+ \\ &= \sum_n (E^{+\prime}_{\tau_n}\{\tau',f\},' \{\tau'',g\})_+\end{aligned}$$

for' $\{\tau',f\},' \{\tau'',g\}$ in H^+ and τ_n a family of disjoint Borel sets of the complex plane.

Since the family of all elements of the form $\{\tau', f\}$ is dense in H^+, the extension to H^+ by continuity of the operator E^+_τ, defined by the formula (4) is a spectral measure for the space H^+.

4. A Property of Subnormal Operators

Using Theorem 3.3.2. we have an important property of subnormal operators on a separable Hilbert space H(see [9]).

1. PROPOSITION. The subnormal operators on a separable Hilbert space H are the closure of the normal operators in the strong operator topology.

PROOF. Let T be a subnormal operator on H. Let $\tilde{T}$ be the minimal normal extension on the extended Hilbert space $\tilde{H}$. We may assume that H is infinite dimensional since, otherwise, every subnormal operator on the finite dimensional Hilbert space H, is normal (see [29] pp. 101). We may assume that $\tilde{H}$ has the same dimension as H ([30] pp. 53,54). For any finite dimensional subspace M of H, we may therefore find a unitary map U(isometric and onto) of H into $\tilde{H}$ which takes each vector of $M \cup T(M)$ onto itself. Then we can see that $U^{-1}\tilde{T}U$ is a normal operator on H.

In fact

$$(U^{-1}\tilde{T}U)(U^{-1}\tilde{T}U)^* = U^{-1}\tilde{T}UU^*\tilde{T}^*U^{-1*}$$

$$= U^{-1}\tilde{T}\tilde{T}^*U^{-1*}, \tag{1}$$

$$(U^{-1}\tilde{T}U)^*(U^{-1}\tilde{T}U) = U^*\tilde{T}^*U^{-1*}U^{-1}\tilde{T}U$$

$$= U^*\tilde{T}^*\tilde{T}U. \tag{2}$$

From (1) and (2) and from the relations

$$U^* = U^{-1},$$
$$U = U^{**} = (U^{-1})^*,$$
$$\tilde{T}^*\tilde{T} = \tilde{T}\tilde{T}^*,$$

(since $\tilde{T}$ is normal), we have

$$(U^{-1}\tilde{T}U)(U^{-1}\tilde{T}U)^* = (U^{-1}\tilde{T}U)^*(U^{-1}\tilde{T}U).$$

Further, we have that

$$U^{-1}\tilde{T}U|M = T|M.$$

In fact for $m \in M$,

$$U^{-1}\tilde{T}U(m) = U^{-1}\tilde{T}(U(m)) = U^{-1}\tilde{T}(m) = U^{-1}(\tilde{T}(m))$$
$$= U^{-1}(T(m)) = T(m).$$

The existence of such an operator for arbitrary M means that T is in the closure of the set of all normal operators on H in the strong operator topology on B(H).

To prove the converse, let $T \in \overline{G}_1$ (the closure of the set of normal operators on H). We wish to show that T is subnormal.

Let $T = \text{strong} \lim_{a \in A} N_a$, where $\{N_a : a \in A\}$ is a net of normal operators on H; that is, $Tx = \lim_{a \in A} N_a x$ $(x \in H)$.

Then Theorem 3.2 pp. 427 of [9] holds for this special case, after the following observations:

(A) If we denote the Hilbert space adjoint of T by T^o, then by [11] 9.3 we have

$$T^o = J \circ T^* \circ J^{-1},$$

where the J is a one-to-one mapping of H^* onto H with the properties:

$$x^*(x) = \langle x, Jx^* \rangle \text{ for all } x \in H,\ x^* \in H^*;$$
$$J(x^* + y^*) = Jx^* + Jy^*;$$
$$J(ax^*) = \bar{a}\, Jx^*;$$
$$||J(x^* - y^*)|| = ||x^* - y^*||;$$

that is J is isometric.

(B) We have $\tau \cdot E_a^o(\cdot) \cdot \tau^{-1} = E_a^*(\cdot)$, where $E_a^o(\cdot)$ are the Hilbert space adjoints of $E_a^*(\cdot)$, for every N_a $(a \in A)$ with resolution of the identity $E_a(\cdot)$, and τ is the inverse of J.

(C) For each $x, y \in H$, $(x,y) = \langle x, \tau y \rangle$ and $T^o = \tau^{-1} T^* \tau$ ([9] pp. 420).

(D) $E_a(\cdot)$, $a \in A$, are self-adjoint projections in H and so closed. Also

$$||E_a(\cdot)x|| \le 1 \cdot ||x|| \qquad \text{for } x \in H$$

and

$$||E_a^o(\cdot)x|| \le 1 \cdot ||x|| \qquad \text{for } x \in H.$$

(E) From Theorem 2.4 and its corollary in [9] pp. 423, we have for the bounded Borel set S,

$$m(S) = E(S)x \qquad (x \in H).$$

Hence $||m|| = ||E(\cdot)x|| \le 1 \cdot ||x||$, and so the set of T-measures m with $||m|| \le 1$ is closed in the weak operator topology of Q. (See the definitions in [9].)

In view of the above remarks (A),(B),(C),(D),(E), in the proof of Theorem 3.2 of [9] we have:

(i) The existence of $E(\cdot)$ with the property (3).

(ii) For each $x \in H$, $E^*(\cdot)x = E(\cdot)x$ is a T^*-measure for x and $||E^*(\cdot)x|| \le 1 \cdot ||x||$. This proves (1).

(iii) The proof of (2) is in pp. 429 of [9].

(iv) (4) becomes clear from the above remarks (A),(B),(C),(D) and (E).

Then with $T \in \overline{G}_1$, if $E(\cdot)$ is the operator defined in Theorem 3.2 of [9], then $E(\cdot)$ is a generalized spectral measure and satisfies (1),(2$'$),(3) of Theorem 3.3 of [9] and from pp. 432 of [9] and pp. 433, we have that T has a normal extension and so T is a subnormal operator.

CHAPTER FOUR

Analytically Compact Operators

Suppose that T is a bounded linear operator on a complex Banach space X such that, for some f in $\mathcal{F}(T)$ with f not identically zero, we have f(T) = K, where K is a compact operator and K $\neq$ 0. (See 1.1.16 for the definition of $\mathcal{F}(T)$). Such an operator T is termed analytically compact.

The purpose of this chapter is to prove the existence of a simple nest of closed invariant subspaces for T. We shall discuss the relationship between the diagonal coefficients with respect to T and the eigenvalues of T.

1. Analytically compact operators on a Banach space

We will use the terminology in [22], Chapters 1 and 2. Throughout, X is a non-zero complex Banach space.

Let $T \in L(X)$ and let τ be an open-and-closed subset of $\sigma(T)$. There is a function f in $\mathcal{F}(T)$ which is identically one on τ and which vanishes on the rest of $\sigma(T)$. We put $E(\tau; T) = f(T)$. If the operator T is understood we may write $E(\tau; T)$ simply as $E(\tau)$. It is clear from Cauchy's theorem that $E(\tau)$ depends only on τ and not on the particular f in $\mathcal{F}(T)$ chosen to define it. $E(\tau)$ is called the spectral projection corresponding to τ. If the open-and-closed set τ consists of the single point λ, the symbol $E(\lambda)$ will be used instead of $E(\{\lambda\})$. It will be convenient also to use the symbol $E(\tau)$ for any set τ of complex numbers for which $\tau \cap \sigma(T)$ is an open-and-closed subset of $\sigma(T)$. In this case we put

$$E(\tau) = E(\tau \cap \sigma(T)).$$

Thus $E(\tau) = 0$ if $\tau \cap \sigma(T)$ is void. The following result is well-known. We include a proof for completeness. See for example Theorem 19 in [23] p. 574.

1. THEOREM. Let $f \in \mathcal{J}(T)$ and let τ be an open-and-closed subset of $\sigma(f(T))$. Then $\sigma(T) \cap f^{-1}(\tau)$ is an open-and-closed subset of $\sigma(T)$ and

$$E(\tau;\, f(T)) = E(f^{-1}(\tau);\, T).$$

PROOF. Let $e_\tau(\mu) = 1$ for μ in a neighbourhood of τ, and let $e_\tau(\mu) = 0$ for μ in a neighbourhood of the rest of $\sigma(f(T))$. Then

$$e_\tau(f(T)) = E(\tau;\, f(T)).$$

If τ' is the complement of τ in $\sigma(f(T))$, then the spectral mapping theorem shows that $\sigma(f(T)) = f(\sigma(T))$ and hence that $\sigma(T)$ is the union of the disjoint sets $f^{-1}(\tau)$ and $f^{-1}(\tau')$. Since f is continuous, these two sets are both open and closed in $\sigma(T)$. It follows that $\delta = \sigma(T) \cap f^{-1}(\tau)$ is an open-and-closed subset of $\sigma(T)$. If we define $e_\delta(\lambda) = e_\tau(f(\lambda))$ for all λ in $\sigma(T)$, then

$$E(\delta;\, T) = e_\delta(T)$$

and by Theorem 1.21 of [22] p. 13 we obtain the desired conclusion

$$E(\tau;\, f(T)) = E(\delta;\, T) = E(f^{-1}(\tau);\, T).$$

For any set δ for which $E(\delta)$ is defined we define $X_\delta = E(\delta)X$. Then $TX_\delta \subseteq X_\delta$ and the restriction of T to X_δ will be denoted by T_δ.

2. DEFINITION. A point $\lambda_0 \in \sigma(T)$ is said to be an isolated point of $\sigma(T)$ if there is a neighbourhood U of λ_0 such that $\sigma(T) \cap U = \{\lambda_0\}$. An isolated point λ_0 of $\sigma(T)$ is called a pole of

T if the resolvent of T has a pole at λ_0. By the order $\nu(\lambda_0)$ of a pole λ_0 of T is meant the order of λ_0 as a pole of the resolvent of T.

3. The purpose of this section is to review spectral theory in a finite-dimensional complex Banach space Y. Let $T \in L(Y)$. In this case $\sigma(T)$ is the set of complex numbers λ such that $\lambda I-T$ is not one-to-one on Y, or equivalently $\sigma(T)$ is the set of eigenvalues of T. If the case of the zero-dimensional Banach space Y is excluded then $\sigma(T)$ is non-empty.

Let $\lambda \in \sigma(T)$. Then there exists an $x_0 \neq 0$ such that $(T-\lambda I)x_0 = 0$. The index $\nu(\lambda)$ of λ is defined as the smallest non-negative integer ν such that $(\lambda I-T)^{\nu}x = 0$ for every vector x for which $(\lambda I-T)^{\nu+1}x = 0$.

It follows that $0 \leq \nu(\lambda) \leq \dim Y$. To see this, define for each positive integer n the linear subspace

$$N_\lambda^n = \{x : (T-\lambda I)^n x = 0\}.$$

Then the index $\nu(\lambda)$ is the least integer ν such that $N_\lambda^{\nu+1} = N_\lambda^{\nu}$. Observe that

$$N_\lambda^n = N_\lambda^{\nu(\lambda)} \qquad (n \geq \nu(\lambda)).$$

Since Y has finite dimension, there can be proper inclusion for at most a finite number of terms in the sequence

$$N_\lambda^1 \subseteq N_\lambda^2 \subseteq N_\lambda^3 \quad \cdots\cdots$$

and thus $\nu(\lambda) \leq \dim Y$ for every λ in $\sigma(T)$.

We now return to the situation in which X is a non-zero complex Banach space and $T \in L(X)$. We recall some fundamental results from [22] on poles of the resolvent of T and related topics.

4. PROPOSITION. Let $T \in L(X)$ and $N(T) = \{x \in X : Tx = 0\}$. Then

(i) $N(T^n) \subseteq N(T^{n+1})$; $(n = 0,1,2, \ldots)$

(ii) if $N(T^k) = N(T^{k+1})$ for some positive integer k then

$$N(T^n) = N(T^k) \qquad (n \geq k).$$

For a proof, see Proposition 1.43 in [22].

5. DEFINITION. Let $T \in L(X)$. Suppose that there is a positive integer n such that $N(T^n) = N(T^{n+1})$. The smallest such integer is called the ascent of T and is denoted by $\alpha(T)$. If no such integer exists we put $\alpha(T) = \infty$.

6. PROPOSITION. Let $T \in L(X)$ and $R(T) = TX$. Then

(i) $R(T^{n+1}) \subseteq R(T^n)$ $(n = 0,1,2, \ldots)$;

(ii) if $R(T^k) = R(T^{k+1})$ for some positive integer k, then

$$R(T^n) = R(T^k) \qquad (n \geq k).$$

For a proof of this result, see Proposition 1.45 of [22].

7. DEFINITION. Let $T \in L(X)$. Suppose there is a positive integer n such that $R(T^n) = R(T^{n+1})$. The smallest such integer is called the descent of T and is denoted by $\delta(T)$. If no such integer exists we put $\delta(T) = \infty$.

8. PROPOSITION. Let $T \in L(X)$. Suppose that $\alpha(T)$, $\delta(T)$ are both finite and hence equal. Let $\alpha(T) = \delta(T) = p$. Then

$$X = R(T^p) \oplus N(T^p).$$

Moreover T_1, the restriction of T to $R(T^p)$ is one-to-one and onto.

For a proof of this result, see Proposition 1.51 in [22].

9. THEOREM. Let $T \in L(X)$. Let λ_0 be a pole of the resolvent

of T of order m. Let $\tau = \sigma(T)\setminus\{\lambda_0\}$. Then λ_0 is an eigenvalue of T. The ascent and descent of $\lambda_0 I-T$ are both equal to m. Also

$$E(\lambda_0)X = N((\lambda_0 I-T)^m),$$
$$E(\tau)X = R((\lambda_0 I-T)^m).$$

For a proof of this result, see Theorem 1.52 of [22].

10. THEOREM. Let $T \in L(X)$. If λ is a pole of T of order ν, then λ has index ν. Furthermore an isolated point λ in the spectrum of T is a pole of order ν if and only if the following two conditions hold:

$$(\lambda I-T)^{\nu}E(\lambda; T) = 0,$$
$$(\lambda I-T)^{\nu-1}E(\lambda; T) \neq 0.$$

The next result is crucial to the development of the theory of analytically compact operators. We give a proof for completeness. See Theorem 20 of [23] p. 524.

11. THEOREM. Let $T \in L(X)$. Let δ be an open-and-closed subset of $\sigma(T)$. Then $\sigma(T_\delta) = \delta$. If $f \in \mathcal{F}(T)$, then $f \in \mathcal{F}(T_\delta)$ and moreover $f(T)_\delta = f(T_\delta)$.

A point λ in δ is a pole of T of order ν if and only if it is a pole of T_δ of order ν.

PROOF. Suppose that $\lambda \in \delta$, but $\lambda \notin \sigma(T_\delta)$. Then there exists a bounded linear operator A on the space X_δ such that

$$(\lambda I-T)Ax = A(\lambda I-T)x = x \qquad (x \in X_\delta).$$

Let the function g be equal to zero for μ in a neighbourhood of δ and equal to $(\lambda-\mu)^{-1}$ for μ in a neighbourhood of the remaining points of $\sigma(T)$. Then

$$g(T)(\lambda I-T) = (\lambda I-T)g(T) = I-E(\delta).$$

If we define A_1 on X by $A_1x = AE(\delta)$, then

$$(\lambda I-T)(A_1+ g(T)) = (A_1+ g(T))(\lambda I-T) = I.$$

Consequently $\lambda \in \rho(T)$, contradicting $\lambda \in \delta$. This shows that $\delta \subseteq \sigma(T_\delta)$.

Conversely, suppose that $\lambda \notin \delta$. Then define h to be equal to $(\lambda-\mu)^{-1}$ for μ in a neighbourhood of δ not containing λ, and to be identically zero in a neighbourhood of the remainder of $\sigma(T)$. We have

$$h(T)(\lambda I-T) = (\lambda I-T)h(T) = E(\delta).$$

Consequently, the restriction $h(T)_\delta$ of $h(T)$ to X_δ satisfies

$$h(T)_\delta(\lambda I_\delta-T_\delta) = (\lambda I_\delta-T_\delta)h(T)_\delta = I_\delta,$$

so that $\lambda \notin \sigma(T_\delta)$. This proves that $\sigma(T_\delta) \subseteq \delta$, and that the resolvents of T and T_δ are related by $R(\lambda;\, T_\delta) = R(\lambda;\, T)_\delta$. Hence $\sigma(T_\delta) = \delta$.

Suppose now that $f \in \mathcal{F}(T)$. Let U be a neighbourhood of $\sigma(T)$ whose boundary B consists of a finite number of rectifiable Jordan contours, and such that $U \cup B$ is included in the domain of analyticity of f. Then

$$f(T)_\delta = \frac{1}{2\pi i}\left\{\int_B f(\lambda)R(\lambda;\, T)d\lambda\right\}_\delta$$

$$= \frac{1}{2\pi i}\left\{\int_B f(\lambda)R(\lambda;\, T)_\delta d\lambda\right\}$$

$$= \frac{1}{2\pi i}\left\{\int_B f(\lambda)R(\lambda;\, T_\delta)d\lambda\right\} = f(T_\delta).$$

By Theorem 10, λ is a pole of order ν for T if and only if

$$(\lambda I-T)^{\nu}E(\lambda) = 0, \ (\lambda I-T)^{\nu-1}E(\lambda) \neq 0.$$

Since $\lambda \in \delta$ we have $E(\lambda)E(\delta) = E(\lambda)$ and thus

$$(\lambda I-T)^{m}E(\lambda) = (\lambda I_{\delta}-T_{\delta})^{m}E(\lambda) \qquad (m = 1,2, \ldots).$$

Hence λ is a pole of T of order ν if and only if it is a pole of T_{δ} of order ν. This completes the proof of the theorem.

Suppose now that T, in L(X), is analytically compact. It follows that there is a function f in $\mathfrak{J}$ (T) and not identically zero and a compact operator $K \neq 0$ on X such that $f(T) = K$. Our first theorem describes the structure of the spectrum of T.

12. THEOREM. Let T, in L(X), be analytically compact.

(i) $\sigma(T)$ is countable and has only a finite number of cluster points, which form a subset of those points μ such that $f(\mu) = 0$. Every point λ in $\sigma(T)$ such that $f(\lambda) \neq 0$ is an eigenvalue of T and moreover a pole of the resolvent of T.

Let $\lambda \in \sigma(T)$ with $f(\lambda) \neq 0$, and let $\nu(\lambda)$ be the order of the pole at λ.

(ii) For each positive integer n, $(\lambda I-T)^{n}X$ is closed. Also

$$(\lambda I-T)^{m+1}X = (\lambda I-T)^{m}X \qquad (m \geq \nu(\lambda))$$

and $\nu(\lambda)$ is the smallest positive integer with this property.

(iii) For each positive integer n, $N((\lambda I-T)^{n})$ is finite-dimensional. Also

$$N((\lambda I-T)^{m}) = N((\lambda I-T)^{m+1}) \quad (m \geq \nu(\lambda))$$

and $\nu(\lambda)$ is the smallest positive integer with this property.

(iv) The spectral projection $E(\lambda)$ has a non-zero finite-dimensional range given by

$$E(\lambda)X = N((\lambda I-T)^{\nu(\lambda)}).$$

The null-space of $E(\lambda)$ is $(\lambda I-T)^{\nu(\lambda)}X$.

(v) If $d(\lambda)$ is the dimension of $E(\lambda)X$ then $1 \leq \nu(\lambda) \leq d(\lambda)$.

Note. The integers $\nu(\lambda)$ and $d(\lambda)$ are called respectively the index and the algebraic multiplicity of the eigenvalue λ.

PROOF OF THEOREM. Let $\mu \in \mathbb{C}$. We show first that there are at most a finite number of points z in $\sigma(T)$ such that $f(z) = \mu$. By hypothesis, f is analytic on some open set Ω that contains $\sigma(T)$ and so therefore is the function

$$z \to f(z) - \mu.$$

The set of zeros of this function in Ω is discrete; in other words each point of the set is isolated. For each point z in $\sigma(T)$ such that $f(z) = \mu$, there is an open disc D_z with centre z which contains no other such point. The set of zeros of $f(z) - \mu$ in $\sigma(T)$ is certainly bounded (because $\sigma(T)$ is) and is closed because it contains all its closure points. Hence this set is compact. The discs D_z cover this set and so, by the Borel covering theorem this set is finite.

We now apply the spectral mapping theorem which yields

$$\sigma(K) = \sigma(f(T)) = f(\sigma(T)),$$

and we deduce that $\sigma(T)$ is countable and has at most a finite number of cluster points. These are of course a subset of $\{z \in \sigma(T) : f(z) = 0\}$.

Let $\lambda \in \sigma(T)$ with $f(\lambda) \neq 0$. Then there is a point μ in $\sigma(K)$

with $f(\lambda) = \mu$, and $\mu \neq 0$, again by the spectral mapping theorem. Suppose that

$$G =' \{\lambda, \lambda_1, \ldots, \lambda_n\}$$

is the set of solutions in $\sigma(T)$ of the equation $f(z) = \mu$. By Theorem 4.1.1,

$$E(G; T) = E'(\{\mu\}; K).$$

By Theorem 2.21 of [22], p. 53, $E'(\{\mu\}; K)$ and hence also $E(G; T)$ has finite-dimensional range. Therefore each point z of G is a pole of the resolvent of $T|E(G; T)X$. Clearly, G is an open-and-closed subset of $\sigma(T)$ and so it follows from Theorem 4.1.11 that λ is a pole of the resolvent of T. The remaining statements of the present theorem follow from Theorem 4.1.9.

We now recall Lomonosov's theorem.

13. THEOREM. Let Y be an infinite-dimensional complex Banach space. Let $T \neq 0$ be a compact operator on Y. Then there is a proper closed subspace of Y invariant under $\mathcal{A} =' \{A \in L(Y) : AT = TA\}$; i.e. T has a hyperinvariant subspace.

For the neat proof of this result, due to Hilden and Lomonosov, see [22] p. 55. We use this result to prove an invariant subspace theorem for analytically compact operators.

14. THEOREM. Let Y be a complex Banach space of dimension at least 2. Let T be an analytically compact operator on Y. There is a proper closed subspace of Y invariant under T.

PROOF. If T = 0 the result is trivially true. If Y is finite-dimensional, $\sigma(T)$ consists of a finite set of eigenvalues. Let λ be one such. Then there is an $x \neq 0$ satisfying $Tx \doteq \lambda x$. Hence the one-dimensional subspace generated by x is a proper closed invariant subspace for T. Hence we may assume that $T \neq 0$ and is infinite-dimensional. Since T is analytically compact, there is f in $\tilde{\mathcal{J}}(T)$ not identically zero and a non-zero compact operator K such that $f(T) = K$. Observe that

$$TK = Tf(T) = f(T)T = KT.$$

By Lomonosov's theorem, K has a hyperinvariant subspace and this is a proper closed subspace of Y invariant under T. The proof is complete.

We now proceed to generalize Ringrose's theory of superdiagonal forms for compact operators to the class of analytically compact operators.

A well-known theorem (see for example [27], p. 144) asserts that every $n \times n$ matrix with complex entries may be reduced by unitary transformation to superdiagonal form. This result, together with some related theory concerning eigenvalues, may be re-formulated in the following way [27, p. 107, p. 144]. If T is a linear operator on an n-dimensional complex inner-product space X, then there exist subspaces $L_o, L_1, \ldots, L_n$ of X such that

(i) $\{0\} = L_o \subset L_1 \subset L_2 \subset \ldots \subset L_n = X$;

(ii) L_m is m-dimensional;

(iii) $TL_m \subseteq L_m \quad (m = 0, 1, \ldots, n)$;

(iv) if we choose $e_m \in L_m \setminus L_{m-1} \quad (m = 1, \ldots, n)$, then the eigenvalues of T (counted according to their algebraic multiplicities) may be specified as those numbers $\lambda_1, \ldots, \lambda_n$ such that

$$Te_m - \lambda_m e_m \in L_{m-1} \quad (m = 1, \ldots, n);$$

(v) T is nilpotent if and only if $TL_m \subseteq L_{m-1} \quad (m = 1, \ldots, n)$.

It is therefore natural to consider such a nest $\{L_m\}$ of subspaces as defining a superdiagonal form for the operator T.

We consider the extension of this concept of 'superdiagonal form' to an analytically compact operator on a non-zero complex Banach space. It is not in general possible in this case to form nests of invariant subspaces with the simple structure exhibited in the finite-dimensional case. To illustrate this point we refer to the compact operator K on the space $L^2[0,1]$ defined by the equation

$$(Kf)(x) = \int_0^x f(y)dy \qquad (f \in L^2 ;\ 0 \le x \le 1).$$

It was proved by Donoghue [17] that the only closed invariant subspaces for this operator are the subspaces $E_c (0 \le c \le 1)$ defined by

$$E_c = \{f \in L^2 : f(x) = 0 \text{ a.e. on } (0,c)\}.$$

It follows that if L_1 and L_2 are distinct invariant subspaces, then either $L_1 \subset L_2$ or $L_2 \subset L_1$, and the quotient space of the larger by the smaller is infinite-dimensional. The subspaces E_c form a continuous nest of invariant subspaces in a sense to be specified later in this chapter.

Throughout the remainder of this section, T denotes an analytically compact operator on a non-zero complex Banach space X. The term <u>subspace</u> will be used to describe a closed linear subset of X.

A family $\mathcal{J}$ of subspaces of X, which is totally ordered by the inclusion relation, will be termed a <u>nest</u> of subspaces. If in addition each subspace in $\mathcal{J}$ is invariant under T we shall describe $\mathcal{J}$ as an <u>invariant nest</u>. A trivial example of an invariant nest is the family consisting of the two subspaces $\{0\}$ and X. Non-trivial invariant nests may be constructed by means of Theorem 4.1.14.

We shall use the symbol $\subseteq$ to denote the inclusion relation, and reserve $\subset$ for proper inclusion. The norm closure of a subset S of X will be denoted by cl(S). Given a nest $\mathcal{F}$ of subspaces of X, and $M \in \mathcal{F}$, we define

$$M_- = \mathrm{cl}\left[\bigcup\{L : L \in \mathcal{F}, L \subset M\}\right].$$

If there is no L in $\mathcal{F}$ such that $L \subset M$, we define $M_- = \{0\}$. It is clear that M_- is a subspace of X, and that it will be an invariant subspace if $\mathcal{F}$ is an invariant nest. Also $M_- \subseteq M$. It should be emphasized that the definition of M_- depends on the particular nest under consideration and not merely on the subspace M. We shall say that $\mathcal{F}$ is <u>continuous at</u> M if $M = M_-$.

A nest $\mathcal{F}$ will be termed <u>simple</u>

(i) if $\{0\} \in \mathcal{F}$, $X \in \mathcal{F}$;

(ii) if $\mathcal{F}_0$ is any subfamily of $\mathcal{F}$, then the subspaces $\bigcap\{L : L \in \mathcal{F}_0\}$ and $\mathrm{cl}[\bigcup\{L : L \in \mathcal{F}_0\}]$ are in $\mathcal{F}$;

(iii) if $M \in \mathcal{F}$, then the quotient space M/M_- is at most one-dimensional.

We observe that condition (ii) implies that $M_- \in \mathcal{F}$ whenever $M \in \mathcal{F}$. Next, we establish the existence of a simple invariant nest for an analytically compact operator.

15. <u>THEOREM</u>. Let T be an analytically compact operator on X. There exists a simple nest $\mathcal{F}$, each of whose members is a subspace invariant under T.

<u>PROOF</u>. Let $\mathcal{N}_i$ denote the class of all invariant nests. Then $\mathcal{N}_i$ is not empty since it contains the trivial nest consisting of

the subspaces $\{0\}$, X. The class $\mathcal{N}_i$ may be partially ordered by inclusion; if $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{N}_i$, we say that $\mathcal{F}_1 < \mathcal{F}_2$ if every subspace in the family $\mathcal{F}_1$ is also a member of $\mathcal{F}_2$. It is easily seen that in this way, $\mathcal{N}_i$ is inductively ordered; for if $\mathcal{N}_0 \subseteq \mathcal{N}_i$ and $\mathcal{N}_0$ is totally ordered by the relation $<$, then

$$\mathcal{F}_0 = \bigcup\{\mathcal{F} : \mathcal{F} \in \mathcal{N}_0\}$$

is the least upper bound of $\mathcal{N}_0$ in $\mathcal{N}_i$. We may now deduce from Zorn's lemma the existence of at least one maximal nest of invariant subspaces. Let $\mathcal{F}$ be a maximal member of $\mathcal{N}_i$. We now establish that $\mathcal{F}$ has properties (i) and (ii) of simple nests.

Clearly $\{0\}$, $X \in \mathcal{F}$ since otherwise $\mathcal{F}$ could be enlarged by the addition of these subspaces, contrary to the assumption that $\mathcal{F}$ is maximal. Secondly, let $\mathcal{F}_0$ be a subfamily of $\mathcal{F}$, and consider

$$M_0 = \bigcap\{L : L \in \mathcal{F}_0\}$$

It is evident that M_0 is a closed subspace of X. Let $M_0 \in \mathcal{F}$. Since $\mathcal{F}$ is totally ordered by inclusion we have <u>either</u> (a) $M \subseteq L$ ($L \in \mathcal{F}_0$), and $M \subseteq M_0$, <u>or</u> (b) $L \subset M$ for some L in $\mathcal{F}_0$, and $M_0 \subset M$. It follows that the family obtained by adding M_0 to $\mathcal{F}$ remains totally ordered by inclusion and is therefore a nest. Since $\mathcal{F}$ is maximal we deduce that $M_0 \in \mathcal{F}$. A similar argument shows that

$$\mathrm{cl}\left[\bigcup\{L : L \in \mathcal{F}_0\}\right]$$

is a member of $\mathcal{F}$. Hence properties (i) and (ii) of simple nests have been established.

It remains to verify that, given any M in $\mathcal{F}$, the quotient space M/M_- is at most one-dimensional. Suppose that, for some M in $\mathcal{F}$, this is not so. When $x \in M$ we denote by $[x]$ the coset $x + M_-$.

Under the usual norm $\|\ \|_M$, defined by

$$\|[x]\|_M = \inf\{\|x-y\| : y \in M_-\},$$

M/M_- is a complex Banach space. Since M and M_- are both invariant under T, we may define a linear operator T_M on M/M_- by the equation

$$T_M[x] = [Tx] \qquad (x \in M).$$

We show next that the operator T_M is analytically compact (as an operator on M/M_-). By hypothesis, there is f in $\mathcal{J}(T)$ with f not identically zero and a non-zero compact operator K on X such that $f(T) = K$. Now, by Theorem 4.1.12, $\sigma(T)$ is countable and so $\rho(T)$ is connected. It follows from Theorem 1.29 of [22] p. 20 that $\sigma(T|M) \subseteq \sigma(T)$. Moreover, by Lemma 1.28 of [22] p. 20,

$$(\lambda I-T)^{-1}M \subseteq M \qquad (\lambda \in \rho(T)).$$

Now, by definition, for some suitable family of contours $B \subseteq \rho(T)$, we have

$$K = f(T) = \frac{1}{2\pi i} \int_B f(\lambda)(\lambda I-T)^{-1} d\lambda$$

and so it follows that $KM \subseteq M$. Similarly, K leaves invariant any other subspace (M_- say) invariant under T. We may therefore define an operator K on M/M_- by

$$K_M[x] = [Kx] \qquad (x \in M).$$

Using Theorem 1.29, Lemma 1.28, Theorem 1.36, Lemma 1.35 of [22], we deduce that $\sigma(T_M) \subseteq \sigma(T)$ and so $f \in \mathcal{J}(T_M)$. It follows that $f(T_M) = K_M$ and so T_M is analytically compact.

Since M/M_- has dimension greater than one, Theorem 4.1.14

implies the existence of a proper subspace L_M of M/M_- which is invariant under T_M. If we now set

$$L = \{x \in M : [x] \in L_M\},$$

then L is a subspace of X (being the inverse image under the continuous linear map $x \to [x]$ of the subspace L_M) such that $M_- \subset L \subset M$. We may now verify, by the method used at an earlier stage in the proof of this theorem, that $L \notin \mathcal{J}$, but that the family $\mathcal{J}_1$ consisting of L and the members of $\mathcal{J}$ is totally ordered by inclusion. Since L_M is invariant under T_M, L is invariant under T. Thus $\mathcal{J}_1$ is an invariant nest, and is a proper enlargement of the maximal invariant nest $\mathcal{J}$. This gives a contradiction. Hence for each M in $\mathcal{J}$, M/M_- is at most one-dimensional and so $\mathcal{J}$ is a simple nest. This completes the proof of the theorem.

Throughout the remainder of this section we shall use the symbols T and $\mathcal{J}$ with the meanings attributed to them in the statement of Theorem 4.1.15. If $M \in \mathcal{J}$, then either $M = M_-$ or M/M_- has dimension one. In the latter case let $z_M \in M \setminus M_-$. Then, since M is invariant under T, we have $Tz_M \in M$, and hence Tz_M can be expressed (uniquely) in the form

$$Tz_M = \alpha_M z_M + y_M, \tag{1}$$

where α_M is a scalar and $y_M \in M_-$. It is easily verified that α_M does not depend on the particular choice of z_M. When $M = M_-$, we do not define α_M. In this way we associate with certain M in $\mathcal{J}$ a scalar α_M which we call the diagonal coefficient of T at M. (In the finite-dimensional case the elements z_M form a basis of X, and with respect to this basis T has super-diagonal matrix with diagonal elements α_M.)

Let $\alpha \in \mathbb{C}$. We define the diagonal multiplicity of α to be the number (possibly infinite) of distinct subspaces M in $\mathcal{J}$ for which $\alpha_M = \alpha$.

Observe that α_M is a diagonal coefficient of T at M if and only if $\sigma(T_M) = \{\alpha_M\}$. Recall that it was shown in the course of proving the last theorem that $\mathcal{J}$ was also a simple invariant nest for the compact operator K and moreover $f(T_M) = K_M$. By the spectral mapping theorem $\sigma(K_M) = \{f(\alpha_M)\}$. It therefore follows, since T_M, K_M are operators on a one-dimensional space, that we have the following result.

16. PROPOSITION. α_M is a diagonal coefficient of T if and only if $f(\alpha_M)$ is a diagonal coefficient of K.

Note. The condition $M \neq M_-$ excludes the possibility that $f(\alpha_M) = 0$.

17. PROPOSITION. Let $\alpha \in \mathbb{C}$ and $f(\alpha) \neq 0$. The diagonal multiplicity of α is finite.

PROOF. Let M be a subspace in $\mathcal{J}$ such that α is the diagonal coefficient of T for M. Then, by Proposition 4.1.16, $f(\alpha)$ is the diagonal coefficient of K for M. The result now follows from Corollary 2.27 of [22] p. 61.

18. THEOREM. Let $\alpha \in \mathbb{C}$ and $f(\alpha) \neq 0$. Then there is a subspace M in $\mathcal{J}$ such that α is the diagonal coefficient of T for M if and only if α is an eigenvalue for T.

PROOF. Suppose that such an M exists. In order to prove that α is an eigenvalue of T, it is sufficient to prove that α is an eigenvalue of $T|M$. Observe that $M \neq M_-$. We show next that the operator $T|M$ is analytically compact. Recall that $f \in \mathcal{J}(T)$, f is not identically zero, and there is a non-zero compact operator K on X such that $f(T) = K$. Now, by Theorem 4.1.12, $\sigma(T)$ is countable and $\rho(T)$ is connected. It follows from Theorem 1.29 of [22], p. 20 that $\sigma(T|M) \subseteq \sigma(T)$. Moreover, by Lemma 1.28 of [22] p. 20,

$$(\lambda I-T)^{-1}M \subseteq M \qquad (\lambda \in \rho(T)).$$

Now, by definition, for some suitable family of contours $B \subseteq \rho(T)$, we have

$$K = f(T) = \frac{1}{2\pi i} \int_B f(\lambda)(\lambda I-T)^{-1}d\lambda$$

and so it follows that $KM \subseteq M$. Since $\sigma(T|M) \subseteq \sigma(T)$, $f \in \mathcal{J}(T|M)$ and so $K|M = f(T|M)$. Hence $T|M$ is analytically compact. From equation (1) it follows that the range of the operator $(T-\alpha I)|M$ is contained in M_- and is therefore not the whole space M. It follows that $\lambda \in \sigma(T|M)$ and, by Theorem 4.1.12, λ is an eigenvalue of $T|M$.

Conversely, if λ is an eigenvalue of T, then by the spectral mapping theorem $f(\lambda)$ is an eigenvalue of K. By Lemma 2.29 of [22] p. 62, there is M in $\mathcal{J}$ such that $M \neq M_-$ and $f(\lambda)$ is the diagonal coefficient of K for M. It now follows from Proposition 4.1.16 that λ is the diagonal coefficient of T for M and the proof of the theorem is complete.

Our next task is to establish the equality of the algebraic multiplicity and the diagonal multiplicity of a point λ in $\sigma(T)$ for which $f(\lambda) \neq 0$. This will be effected in two stages.

19. PROPOSITION. Let $\lambda \in \sigma(T)$ and $f(\lambda) \neq 0$. Let d denote the diagonal multiplicity and m the algebraic multiplicity of the eigenvalue λ of T. Then $d \leq m$.

PROOF. Let ν be the index of λ relative to T. Then

(a) ν is the least integer such that $(T-\lambda I)^{\nu+1}x = 0$ only if

$$(T-\lambda I)^{\nu}x = 0 \qquad (x \in X);$$

(b) ν is the least integer such that

$$(T-\lambda I)^{\nu+1}X = (T-\lambda I)^{\nu}X;$$

(c) the null space of the operator $(T-\lambda I)^{\nu}$ has dimension m.

Let S be the bounded linear operator defined by

$$S - \rho I = (T-\lambda I)^{\nu},$$

where $\rho = -(-\lambda)^{\nu}$. Then ρ is an eigenvalue of S which has index unity and algebraic multiplicity m. Since S is a polynomial in T, each subspace M in $\mathcal{J}$ is invariant under S. We may therefore consider the diagonal coefficients of S with respect to the nest $\mathcal{J}$.

Let $M \in \mathcal{J}$ and let α_M, σ_M denote the diagonal coefficients at M of T,S respectively, where we have assumed that $M \neq M_-$. We have

$$Tz_M = \alpha_M z_M + y_M,$$

where $y_M \in M_-$. We deduce from this equation that

$$(T-\lambda I)z_M = (\alpha_M-\lambda)z_M + y_M.$$

It easily follows that, for n = 1,2, ..., we have

$$(T-\lambda I)^n z_M = (\alpha_M-\lambda)^n z_M + y^{(n)},$$

where $y^{(n)} \in M_-$. In particular, by taking $n = \nu$, we obtain

$$Sz_M = \rho z_M + (\alpha_M - \lambda)^\nu z_M + y^{(\nu)}.$$

Thus $\sigma_M = \rho + (\alpha_M - \lambda)^\nu$. We deduce that $\sigma_M = \rho$ if and only if $\alpha_M = \lambda$. Hence the diagonal multiplicity of ρ relative to S is d. It is now sufficient to prove the present proposition under the additional hypothesis that λ has index unity relative to T, since in the general case we may reduce to this situation by replacing T, λ by S, ρ respectively.

Suppose therefore that λ has index unity relative to T and let N be the null-space of the operator $T - \lambda I$. Given $x \in N$, define

$$M = M(x) = \bigcap \{L : L \in \mathcal{J}, x \in L\}.$$

By considering $\mathcal{J}$ as a simple nest of invariant subspaces for the compact operator $f(T)$, we deduce from Lemma 2.29 of [22] p. 62 that $M \in \mathcal{J}$, $M \neq M_-$ and $\lambda = \alpha_M$. (Observe that by hypothesis $f(\lambda) \neq 0$.) Clearly $x \in M \setminus M_-$.

To complete the proof of the present proposition, we show conversely that, if $M \in \mathcal{J}$ and $\alpha_M = \lambda$, then $M = M(x)$ for some non-zero x in N. For this purpose, let T_M denote the restriction of T to M, and let W_M, N_M be the range and null-space (respectively) of the operator $T_M - \lambda I_M$. It was shown in the course of proving Theorem 4.1.18 that T_M is analytically compact. It is immediate from the definition of index in terms of null-spaces that λ has index unity relative to T_M. It follows from Theorem 4.1.12 that

$$W_M \oplus N_M = M.$$

Since, as in the proof of Theorem 4.1.18, $W_M \subseteq M_-$, it follows that N_M meets $M \setminus M_-$. If $x \in N_M \cap (M \setminus M_-)$, it is easily verified that $x \in N$, $x \neq 0$ and $M(x) = M$.

Let $M_1 \subset M_2 \subset \ldots \subset M_d$ be the distinct members of the nest $\mathcal{F}$ at which T has diagonal coefficient λ. We may choose non-zero vectors $x_1, \ldots, x_d \in N$ such that $M_i = M(x_i)$ $(i = 1, \ldots, d)$. For each $i = 1, \ldots, d$, x_i is not a linear combination of $x_1, \ldots, x_{i-1}$: for this would imply that $x_i \in M(x_{i-1}) \subseteq M_-(x_i)$, which is not so. Hence $x_1, \ldots, x_d$ are linearly independent elements of N, and since $\dim N = m$ we obtain $m \geq d$. This completes the proof.

20. PROPOSITION. Let $\lambda \in \sigma(T)$ and $f(\lambda) \neq 0$. Let $d(\lambda)$ denote the diagonal multiplicity and $m(\lambda)$ the algebraic multiplicity of the eigenvalue λ of T. Then $d(\lambda) = m(\lambda)$.

PROOF. Let $\tau = \{\lambda_1, \ldots, \lambda_n\}$ be the set of points in $\sigma(T)$ such that

$$f(\lambda_r) = f(\lambda) \qquad (r = 1, \ldots, n).$$

Then

$$\sum_{r=1}^{n} m(\lambda_r) = \dim E(\tau; T)X$$
$$= \dim E(f(\lambda); f(T))X$$

by Theorem 4.1.1. However, if d denotes the diagonal multiplicity of $f(\lambda)$ with respect to $f(T)$, then, by Proposition 4.1.16,

$$\sum_{r=1}^{n} d(\lambda_r) = d.$$

By Lemma 2.31 of [22] p. 62,

$$d = \dim E(f(\lambda); f(T))X.$$

Hence

$$\sum_{r=1}^{n} m(\lambda_r) = \sum_{r=1}^{n} d(\lambda_r)$$

and, using the preceding result that $m(\lambda_r) \geq d(\lambda_r)$ $(r = 1, \ldots, n)$, we obtain the desired conclusion.

We conclude this section by stating as a theorem the results that we have established.

20. THEOREM. Let T be an analytically compact operator on X and let $f(T)$ be compact, where $f \in \mathfrak{F}(T)$. Let $\lambda \in \mathbb{C}$ with $f(\lambda) \neq 0$. Then

(i) λ is an eigenvalue of T if and only if it is a diagonal coefficient of T;

(ii) the diagonal multiplicity of λ is equal to its algebraic multiplicity as an eigenvalue of T.

2. Analytically compact operators on Hilbert space

Henceforward we shall confine our attention to the case in which, in place of a general Banach space X, we have a Hilbert space H. The following theorem corresponds, in the finite-dimensional case, to the assertion that a square matrix is normal if and only if every unitarily equivalent super-diagonal form is in fact diagonal.

1. THEOREM. Let T be an analytically compact operator acting on a complex Hilbert space H and let $\mathfrak{F}$ be a simple invariant nest associated with T. Let

$$\mathfrak{F}_1 = \{M : M \in \mathfrak{F},\ M \neq M_-\}$$

and let α_M $(M \in \mathfrak{F}_1)$ be the diagonal coefficients of T. Also, when $M \in \mathfrak{F}_1$, let z_M be an element of M which has unit norm and is orthogonal to M_-. Then T is normal if and only if

(i) z_M is an eigenvector of T $(M \in \mathfrak{F}_1)$;

(ii) if $x \in H$ and $\langle x, z_M \rangle = 0$ $(M \in \mathcal{J}_1)$, then $Tx = 0$.

Note. If $M \in \mathcal{J}_1$, then M_- has codimension one in M. Hence the vector z_M is determined up to a scalar factor of absolute value one. Clearly, $z_M \in M \setminus M_-$.

PROOF OF THEOREM. To prove the sufficiency of the stated conditions, we note first that the elements z_M $(M \in \mathcal{J}_1)$ form an orthonormal system in H. For let L,M be distinct members of $\mathcal{J}_1$. Since $\mathcal{J}$ is totally ordered by inclusion, we may suppose that $L \subset M$ and hence that $L \subseteq M_-$. Now $z_L \in L \subseteq M_-$, while z_M is orthogonal to M_-. Hence $\langle z_L, z_M \rangle = 0$, and since z_M has unit norm for each M in $\mathcal{J}_1$, it follows that these elements form an orthonormal system. Hence, every vector x in H may be expressed in the form

$$x = \sum_{M \in \mathcal{J}_1} \langle x, z_M \rangle z_M + y,$$

where y is orthogonal to each z_M. If conditions (i) and (ii) of the theorem are satisfied, and scalars σ_M are chosen so that $Tz_M = \sigma_M z_M$ $(M \in \mathcal{J}_1)$, then

$$Tx = \sum_{M \in \mathcal{J}_1} \sigma_M \langle x, z_M \rangle z_M \qquad (x \in H).$$

It follows at once that T is a normal operator.

Suppose conversely that T is a normal operator, and that λ is a non-zero diagonal coefficient with diagonal multiplicity d. Define

$$\mathcal{J}_2 = \{M : M \in \mathcal{J}_1, \alpha_M = \rho\},$$

so that $\mathcal{J}_2 \subseteq \mathcal{J}_1$, and $\mathcal{J}_2$ contains exactly d members. Let $M \in \mathcal{J}_2$

and write T_M for the restriction of T to M. We denote by N, N_M the null-spaces of the operators $T-\lambda I$, $T_M-\lambda I_M$ (respectively) and by W, W_M, the ranges of these operators. As an eigenvalue, λ has index one relative to T. (See for example Theorem 11.14 of [22] p. 219.) It follows that λ has index one relative to T_M. If we use the symbol $\perp$ to denote orthogonal complement in H, and $\oplus$ for topological direct sum, we have

$$N = W^{\perp}, \tag{1}$$

$$M = W_M \oplus N_M, \tag{2}$$

$$M = W_M \oplus (M \cap W_M^{\perp}). \tag{3}$$

We justify these three equations as follows. The first follows from the Corollary to Theorem 1 of [47] p. 254. To prove the second observe that it was shown in the course of proving Theorem 4.1.18 that T_M is analytically compact and so the desired conclusion follows from Theorem 4.1.12. The third equation follows from an elementrary result on Hilbert space. Now

$$N_M = M \cap N = M \cap W^{\perp} \subseteq M \cap W_M^{\perp} \tag{4}$$

From (2),(3),(4) we deduce that

$$N_M = M \cap W_M^{\perp}.$$

Now, as in the proof of Theorem 4.1.18, we have $W_M \subseteq M_-$. Thus

$$z_M \in M \cap W_M^{\perp} = N_M \subseteq N.$$

Hence z_M is an eigenvector of T, corresponding to the eigenvalue λ. This proves condition (i).

Furthermore, the d vectors z_M $(M \in \mathcal{J}_2)$ form an orthonormal system in M and dim $M = d$, by Theorem 4.1.20. Hence these vectors form a basis of M. We deduce that the family z_M ($M \in \mathcal{F}_1$) is a complete set of eigenvectors of T. Property (ii) now follows from Theorem 5 of [47] p. 443, or strictly speaking a generalization of it; namely, the spectral theorem for a normal operator.

3. Simple resolutions of the identity in Hilbert space

In the context of Hilbert space, the theory developed in the first section of this chapter may be strengthened by use of the fact that, given any closed subspace M of H, there is orthogonal projection from H onto M.

Let H be a complex Hilbert space. By a simple resolution of the identity in H, we shall mean a family $\{E_\lambda : 0 \leq \lambda \leq 1\}$ of orthogonal projections in H such that

(i) $E_0 = 0$, $E_1 = I$;

(ii) $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$ $(\lambda \leq \mu)$;

(iii) if $0 \leq \mu \leq 1$ and $x \in H$, then $E_\lambda x \to E_\mu x$ as $\lambda \to \mu + 0$, convergence being in the norm topology of H;

(iv) the projection $E_\lambda - E_{\lambda-}$ has rank at most one.

Here, $E_{\lambda-}$ denotes the strong limit of E_μ as $\mu \to \lambda - 0$; the existence of this limit follows from (ii) and the fact that the projections being orthogonal are uniformly bounded in norm. (See for example Theorem 6.4 of [22] p. 159.)

1. THEOREM. Let $\{E_\lambda\}$ be a simple resolution of the identity in a complex Hilbert space H, and let $\mathcal{F}$ be the family consisting of the subspaces $E_\lambda(H)$, $E_{\lambda-}(H)$ $(0 \leq \lambda \leq 1)$. Then $\mathcal{F}$ is a simple nest.

Furthermore, the relations

$$M = E_\mu(H), \qquad M_- = E_{\mu-}(H),$$

establish a one-to-one correspondence between the sets $\{\mu : \mu \in [0,1], E_\mu \neq E_{\mu-}\}$ and $\{M : M \in \mathcal{J}, M \neq M_-\}$.

PROOF. It is apparent that $\mathcal{J}$ is a nest. We shall now show that $\mathcal{J}$ is a maximal nest. Let M be a subspace of H, and suppose that the family obtained by adding M to $\mathcal{J}$ remains totally ordered by inclusion. Denote by $L_\lambda, L_{\lambda-}$, respectively, the ranges of the projections $E_\lambda, E_{\lambda-}$, and define

$$\mu = \inf\{\lambda : \lambda \in [0,1], M \subseteq L_\lambda\}.$$

It is clear that $\mu \in [0,1]$, and that $M \subseteq L_\lambda (\lambda > \mu)$, $L_\lambda \subset M$ $(\lambda < \mu)$. Thus, by virtue of property (iii) for simple resolutions of the identity,

$$M \subseteq \bigcap_{\lambda>\mu} L_\lambda = L_\mu \ ;$$

also

$$L_{\mu-} = \mathrm{cl}\Big[\bigcup_{\lambda<\mu} L_\lambda\Big] \subseteq M.$$

Hence $L_{\mu-} \subseteq M \subseteq L_\mu$. Now $L_{\mu-}$ has codimension at most one in L_μ. It follows that M is one of the subspaces $L_\mu, L_{\mu-}$, and hence that $M \in \mathcal{J}$. This shows that $\mathcal{J}$ is maximal and therefore by Lemma 2.24 of [22] p. 58, $\mathcal{J}$ is a simple nest.

Suppose now that $M \in \mathcal{J}$ and that $M \neq M_-$. By reasoning as above, we may prove the existence of a number μ in $[0,1]$ such that $M \subseteq L_\mu$, but $L_\lambda \subset M(\lambda < \mu)$. Thus $L_\lambda \subseteq M_-(\lambda < \mu)$, and

$$L_{\mu-} = \mathrm{cl}\Big[\bigcup_{\lambda<\mu} L_\lambda\Big] \subseteq M_-.$$

We deduce that $L_{\mu-} \subseteq M_- \subset M \subseteq L_\mu$. Since $L_{\mu-}$ has codimension at most one in L_μ, it follows that $L_{\mu-} = M_-$, $L_\mu = M$. This proves the existence of a number μ with the required properties.

We now have to show that μ is uniquely determined by these conditions. Suppose that this is not so, and that ν is another such number. We may assume that $\mu < \nu$. Then $M = L_\mu = L_\nu$, and $M_- = L_{\mu-} = L_{\nu-}$; and since $L_\mu \subseteq L_{\nu-}$, we have $M \subseteq M_-$ and hence $M = M_-$, contrary to hypothesis.

Finally, let $\mu \in [0,1]$, and suppose that $E_\mu \neq E_{\mu-}$. Set $M = L_\mu$. Then the subspaces L in $\mathcal{J}$ such that $L \subset M$ are exactly $L_\lambda (\lambda < \mu)$ and $L_{\mu-} (\lambda \leq \mu)$. Since M_- is, by definition, the closed linear hull of this family of subspaces, we have $M_- = L_{\mu-}$. This completes the proof of the theorem.

In the circumstances described in the above theorem, we shall say that $\mathcal{J}$ is the simple nest associated with the simple resolution of the identity $\{E_\lambda\}$.

2. THEOREM. Let H be a complex separable Hilbert space, and let $\mathcal{J}$ be a simple nest of subspaces of H. Then there exists a simple resolution of the identity $\{E_\lambda\}$ in H, whose associated simple nest is $\mathcal{J}$.

PROOF. Let $x_1, x_2, \ldots$ be a complete orthonormal system in H. Given any subspace M in $\mathcal{J}$, we define

$$f(M) = \sum_{n=1}^{\infty} 2^{-n} ||P_M x_n||,$$

where P_M is the orthogonal projection from H onto M. We require first the following three properties of f.

(i) $0 \leq f(M) \leq 1 \quad (M \in \mathcal{J})$.

(ii) $f((0)) = 0$, $f(H) = 1$.

(iii) $L \subset M$ if and only if $f(L) < f(M)$
$L = M$ if and only if $f(L) = f(M) \quad (L, M \in \mathcal{J})$.

To see (i) observe that we have

$$0 \leq f(M) \leq \sum_{n=1}^{\infty} 2^{-n} ||x_n|| = \sum_{n=1}^{\infty} 2^{-n} = 1.$$

(ii) is obvious. Now suppose that $L \subset M$. Then $||P_L x|| \leq ||P_M x||$ $(x \in H)$, with equality only if

$$P_L x_n = P_M x_n \qquad (n = 1,2, \ldots).$$

However, the latter condition would imply that $P_L = P_M$ (since linear combinations of $x_1, x_2, \ldots$ are dense in H) and hence that $L = M$, contrary to hypothesis. Thus $f(L) < f(M)$ and (iii) is established.

Next, we establish the following property concerning sequences of subspaces in $\mathcal{J}$.

Let $M_n \in \mathcal{J}$ $(n = 1,2, \ldots)$ and let $0 \leq \mu \leq 1$. Suppose that the sequence $\{f(M_n)\}$ is monotonic and converges to μ. Then there is a subspace M in $\mathcal{J}$ such that $f(M) = \mu$. Furthermore, $P_{M_n} \to P_M$ in the strong operator topology.

For the proof we shall suppose that $f(M_n)$ increases to μ. The proof when $f(M_n)$ decreases to μ is similar. Since $f(M_n) \leq f(M_{n+1})$, we can deduce from property (iii) of f that $M_n \subseteq M_{n+1}$. We define $M = \mathrm{cl}[\bigcup M_n]$. Then $P_{M_n} \to P_M$ in the strong operator topology. (See for example Theorem 6.4 of [22] p. 159.) Since $\mathcal{J}$ is a simple nest, $M \in \mathcal{J}$. Finally we have

$$f(M) = \sum_{j=1}^{\infty} 2^{-j} ||P_M x_j||$$

$$= \lim_{n \to \infty} \sum_{j=1}^{\infty} 2^{-j} || P_{M_n} x_j ||$$

$$= \lim_{n \to \infty} f(M_n) = \mu.$$

The double limit operation may be justified by uniform convergence considerations.

Observe that the set $\Lambda = \{f(M) : M \in \mathcal{J}\}$ is a closed subset of the interval $[0,1]$.

Having established these auxiliary results, we now return to the proof of the theorem. When $\lambda \in \Lambda$, there is a unique $M = M_\lambda$ in $\mathcal{J}$ such that $f(M) = \lambda$. We define

$$E_\lambda = P_{M_\lambda} \qquad (\lambda \in \Lambda). \tag{1}$$

If (α,β) is a complementary interval of Λ, we set

$$E_\lambda = E_\alpha \qquad (\alpha < \lambda < \beta).$$

From property (iii) of f, we deduce that

$$E_\lambda(H) = \bigcup \{M : M \in \mathcal{J}, f(M) \leq \lambda\} \ (0 \leq \lambda \leq 1). \tag{2}$$

We shall now verify that the family $\{E_\lambda : 0 \leq \lambda \leq 1\}$ of orthogonal projections satisfies conditions (i), ..., (iv) for simple resolutions of the identity.

(i) We have $0 = f((0)) \in \Lambda$, $1 = f(H) \in \Lambda$. Thus

$$E_o = P_{(o)} = 0, \quad E_1 = P_H = I.$$

(ii) It is sufficient to prove that $E_\lambda(H) \subseteq E_\mu(H)$ when $\lambda \leq \mu$. This follows at once from (2).

(iii) The required result is apparent when $\mu \notin \Lambda$, since in this case E_λ is constant on some open interval containing μ. We may therefore assume that $\mu \in \Lambda$.

Suppose that λ_n decreases to μ. Define

$$\mu_n = \sup\{\lambda : \lambda \in \Lambda, \lambda \leq \lambda_n\}.$$

Then $\mu_n \in \Lambda$, $E_{\lambda_n} = E_{\mu_n}$ and μ_n decreases to μ. It is immediate from what has already been established and the definition of E_λ (see (1)) that $E_{\mu_n} \to E_\mu$ in the strong operator topology. Hence $E_{\lambda_n} \to E_\mu$ in this topology.

(iv) Let $0 \leq \lambda \leq 1$, and set $M = E_\lambda(H)$. Every subspace L in $\mathcal{J}$ such that $L \subset M$ is of the form $L = E_\mu(H)$ for some μ in Λ with $\mu < \lambda$. Hence

$$\begin{aligned} M_- &= \mathrm{cl}\left[\bigcup\{L : L \in \mathcal{J}, L \subset M\}\right] \\ &\subseteq \mathrm{cl}\left[\bigcup_{\mu<\lambda} E_\mu(H)\right] \\ &= E_{\lambda-}(H). \end{aligned}$$

Thus $M_- \subseteq E_{\lambda-}(H) \subseteq E_\lambda(H) = M$. Since $\mathcal{J}$ is a simple nest, M_- has co-dimension at most one in M. Thus the projection $E_\lambda - E_{\lambda-}$ has rank at most unity.

We have now shown that the family $\{E_\lambda\}$ is a simple resolution of the identity in H. Let $\mathcal{J}_1$ be the associated simple nest. If $M \in \mathcal{J}$, and $f(M) = \lambda$, we have $M = E_\lambda(H) \in \mathcal{J}_1$. Since the nest $\mathcal{J}$ is simple and therefore, by Lemma 2.24 of [22], p. 58, maximal, it follows that $\mathcal{J} = \mathcal{J}_1$. This completes the proof of the theorem.

Note. Examples can easily be constructed to show that the conclusions of the theorem may fail if the hypothesis of separability is removed.

3. PROPOSITION. Let H be an infinite-dimensional Hilbert space. Then there exists a simple resolution of the identity $\{E_\lambda\}$ in H such that $E_\lambda = E_{\lambda-}$ $(0 \leq \lambda \leq 1)$.

PROOF. The space H can be expressed as the direct sum of a family of pairwise orthogonal subspaces, each of countably infinite dimension, and is therefore unitarily equivalent to a space of the form

$$H_o = \bigoplus_{\alpha \in A} L^2(0,1),$$

where A is some index set. It is therefore sufficient to consider the case in which $H = H_o$.

The elements of H_o are families $(f_\alpha)_{\alpha \in A}$ such that $f_\alpha \in L^2(0,1)$ $(\alpha \in A)$ and

$$\sum_{\alpha \in A} ||f_\alpha||^2 < \infty.$$

When $0 \leq \lambda \leq 1$ and $(f_\alpha) \in H_o$, define

$$E_\lambda(f_\alpha) = (e_\lambda f_\alpha),$$

where e_λ is the characteristic function of the interval $[0,\lambda]$. It is easily verified that E_λ is an orthogonal projection in H_o, and that the family $\{E_\lambda\}$ has the required properties.

<u>DEFINITION.</u> Let T be an analytically compact operator on a complex Hilbert space H. We shall say that a simple resolution of the identity $\{E_\lambda\}$ in H <u>reduces</u> T if

$$TE_\lambda = E_\lambda TE_\lambda \qquad (0 \leq \lambda \leq 1).$$

This condition implies that $TE_{\lambda-} = E_{\lambda-}TE_{\lambda-}$ $(0 \leq \lambda \leq 1)$. Thus T is reduced by $\{E_\lambda\}$ if and only if T leaves invariant every subspace in the simple nest $\mathcal{J}$ associated with $\{E_\lambda\}$.

4. <u>THEOREM.</u> Let T be an analytically compact operator acting on a complex separable Hilbert space H. Then there exists a simple resolution of the identity $\{E_\lambda\}$ in H which reduces T.

PROOF. Let $\mathcal{J}$ be a simple nest of subspaces of H, each of whose members is invariant under T. Then Theorem 4.3.2 asserts the existence of a simple resolution of the identity $\{E_\lambda\}$ in H, whose associated simple nest is $\mathcal{J}$. Since $\mathcal{J}$ is an invariant nest, $\{E_\lambda\}$ reduces T.

In the case of a compact linear operator on a general Hilbert space, not necessarily separable, the last theorem remains true.

5. THEOREM. Let T be a compact linear operator acting on a complex Hilbert space H. Then there exists a simple resolution of the identity $\{E_\lambda\}$ in H which reduces T.

PROOF. If H is separable, the result follows from the preceding theorem. If H is not separable, we can find a separable subspace H_1 such that, if $H_2 = H_1^\perp$, then $TH_1 \subseteq H_1$, $TH_2 = \{0\}$. (See, for example, [39] p. 206, where it is shown that the space generated by the eigenvectors corresponding to the non-zero eigenvalues has the properties of H_1.) Let T_1 denote the restriction of T to H_1. Since H_1 is separable we can find a simple resolution of the identity $\{E_\lambda^{(1)}\}$ in H_1 which reduces T_1. Let $\{E_\lambda^{(2)}\}$ be any resolution of the identity in H_2 which satisfies the conclusions of Proposition 4.3.3. If we now define

$$E_\lambda = E_\lambda^{(1)}P_1 + E_\lambda^{(2)}P_2,$$

where P_i denotes the orthogonal projection from H onto H_i $(i = 1,2)$, then it is easily verified that $\{E_\lambda\}$ is a simple resolution of the identity which reduces T. This completes the proof of the theorem.

Let T be an analytically compact operator acting in a complex separable Hilbert space H, and let $\{E_\lambda\}$ be a simple resolution of the identity which reduces T. Define

$$J = \{\lambda : \lambda \in [0,1],\ E_\lambda \neq E_{\lambda-}\}. \qquad (1)$$

When $\lambda \in J$, the projection $E_\lambda - E_{\lambda-}$ has rank unity. We can therefore choose a vector z_λ in H such that

$$||z_\lambda|| = 1, \; (E_\lambda - E_{\lambda-})z_\lambda = z_\lambda \quad (\lambda \in J). \tag{2}$$

The element z_λ is determined to within a scalar factor of absolute value one. We note that z_λ is orthogonal to the range of the projection $E_{\lambda-}$. Since $z_\lambda \in E_\lambda(H)$, we have $Tz_\lambda \in E_\lambda(H)$, and

$$Tz_\lambda = E_\lambda T z_\lambda = (E_\lambda - E_{\lambda-})Tz_\lambda + E_{\lambda-}Tz_\lambda.$$

Hence

$$Tz_\lambda = \alpha_\lambda z_\lambda + E_{\lambda-}Tz_\lambda \quad (\lambda \in J), \tag{3}$$

where α_λ is a scalar. When $\lambda \notin J$, we define $\alpha_\lambda = 0$. We shall call α_λ the <u>diagonal coefficient of</u> T <u>at</u> λ (with respect to the resolution of the identity $\{E_\lambda\}$).

Let $\mathcal{J}$ be the simple nest associated with the resolution of the identity $\{E_\lambda\}$. Let $\lambda \in J$ and let M denote $E_\lambda(H)$. Then $M \in \mathcal{J}$, and from Theorem 4.3.1 it follows that $M_- = E_{\lambda-}(H) \subset M$. Furthermore, $z_\lambda \in M \setminus M_-$, and by using equation (3) we obtain $Tz_\lambda - \alpha_\lambda z_\lambda \in M_-$. Thus, the diagonal coefficient of T at M in $\mathcal{J}$ is α_λ. We may now deduce from Theorem 4.3.1 that the non-zero diagonal coefficients of T with respect to the simple resolution of the identity $\{E_\lambda\}$ and their diagonal multiplicities are the same as those with respect to the simple nest $\mathcal{J}$. (It is however possible that zero shall be a diagonal coefficient with respect to $\{E_\lambda\}$, but not with respect to $\mathcal{J}$.) It follows that, for analytically compact operators acting in separable Hilbert spaces, there is a complete analogue of the theory developed earlier in this chapter in which simple nests are replaced by simple resolutions of the identity. We confine attention to a statement of the analogue of Theorem 4.2.1 without giving detailed proofs.

6. THEOREM. Let T be an analytically compact operator on a complex separable Hilbert space H, and let $\{E_\lambda\}$ be a simple resolution of the identity which reduces T. Define

$$D = \{\lambda : \lambda \in [0,1], \alpha_\lambda \neq 0\},$$

where α_λ denotes the diagonal coefficient of T at λ. Suppose further that, when $\lambda \in D$, z_λ satisfies equation (2) in the introduction to this section. Then T is a normal operator if and only if

(i) z_λ is an eigenvector of T $(\lambda \in D)$;

(ii) if $x \in H$ and $\langle x, z_\lambda \rangle = 0$ $(\lambda \in D)$, then $Tx = 0$.

7. COROLLARY. The operator T is normal if and only if

$$Tx = \sum_{\lambda \in D} \alpha_\lambda \langle x, z_\lambda \rangle z_\lambda \quad (x \in H).$$

This conclusion remains valid if we replace D by the set J defined in equation (1) in the introduction to this section.

PROOF. The vectors z_λ $(\lambda \in D)$ form an orthonormal system in H. It is easily verified that conditions (i) and (ii) of the last theorem are satisfied if and only if there exist scalars $\sigma_\lambda (\lambda \in D)$ such that

$$Tx = \sum_{\lambda \in D} \sigma_\lambda \langle x, z_\lambda \rangle z_\lambda \quad (x \in H).$$

Direct calcualtion then gives $\alpha_\lambda = \sigma_\lambda$. The only effect of replacing D by J is the possible introduction into the summation representing Tx of a number of zero items.

This chapter represents a considerable generalization of Ringrose's theory of superdiagonal forms. We conclude with an example of an analytically compact operator which is not compact.

8. <u>EXAMPLE</u>. Let H be a separable infinite-dimensional Hilbert space with orthonormal basis $\{e_n\}$ where n is a positive integer. Define T, in L(H), by

$$Te_n = \begin{cases} e_{n+1} & (n \text{ even}) \\ \frac{1}{n} e_{n+1} & (n \text{ odd}) \end{cases}$$

Then the sequence $\{e_{2n}\}$ is bounded but the sequence $\{Te_{2n}\}$ possesses no convergent subsequence. Clearly T is not compact. However T^2 is compact since it is the norm limit of a sequence of finite-rank operators.

CHAPTER FIVE

Compact operators on a real Banach space

Gillespie [25] and Meyer-Nieberg [33] independently obtained an invariant subspace theorem for compact operators on a real Banach space. The purpose of this chapter is to show that the Hilden-Lomonosov technique referred to in the last chapter can be used to considerably simplify their work.

The spectral theory of compact linear operators on a real Banach space seems to be well known, although it does not appear explicitly in the literature. In view of this we outline a possible development of this theory.

1. The complexification of a real Banach space

Throughout this section, let X be a real Banach space. We denote by $\tilde{X}$ the complexification of X. This, by definition, is the set of formal sums x+iy, where x and y run through the set X.

Let $x,y,x_1,y_1,x_2,y_2 \in X$ and $\alpha,\beta \in \mathbb{R}$. We define equality, addition, scalar multiplication, and norm on $\tilde{X}$ by:

(i) $x_1+iy_1 = x_2+iy_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$;

(ii) $(x_1+iy_1) + (x_2+iy_2) = (x_1+x_2) + i(y_1+y_2)$;

(iii) $(\alpha+i\beta)(x+iy) = (\alpha x-\beta y) + i(\beta x+\alpha y)$;

(iv) $||x+iy|| = \sup_{\theta \in \mathbb{R}} \left[||(\cos\theta)x - (\sin\theta)y||+||(\sin\theta)x+(\cos\theta)y||\right]$.

With these definitions $\tilde{X}$ becomes a complex Banach space and we have

$$||x||+||y|| \leq ||x+iy|| \leq \sqrt{2}(||x||+||y||) \quad (x,y \in X).$$

Let M be a subspace of $\tilde{X}$. We define the <u>conjugate space</u> M^* of M as follows:

$$M^* =' \{x-iy : x \in X, y \in X, x+iy \in M\}.$$

This is also a subspace.

We say that M is <u>self-conjugate</u> if $M = M^*$. Clearly, if N is any subspace, then $N \cap N^*$ is self-conjugate and if, in addition, N is finite-dimensional, then $N + N^*$ is self conjugate.

Let M be a self-conjugate subspace of $\tilde{X}$. We define

$$\text{Re}M =' \{x \in X : x+i0 \in M\}.$$

This is also a subspace of X. The importance of this definition is that M is the complexification of ReM and so

$$\dim_{\mathbb{R}} \text{Re}M = \dim_{\mathbb{C}} M.$$

Let $T \in L(X)$. We define on $\tilde{X}$ the <u>complexification</u> $\tilde{T}$ of T by

$$\tilde{T}(x+iy) = Tx+iTy \qquad (x,y \in X).$$

Clearly, $\tilde{T} \in L(X)$ and

$$||T|| \leq \sqrt{2}\, ||\tilde{T}|| \leq 2||T||.$$

We note the following: for $\lambda \in \mathbb{C}$ and m a positive integer, we have that $(\tilde{T}-\bar{\lambda}I)^m\tilde{X}$ is the conjugate of $(\tilde{T}-\lambda I)^m\tilde{X}$ and $\ker(\tilde{T}-\bar{\lambda}I)^m$ is the conjugate of $\ker(\tilde{T}-\lambda I)^m$. (Here ker denotes the null-space of the operator.)

2. Finite-dimensional spaces

In this section the terms 'operator' and 'subspace' are used in a purely algebraic sense.

The following result may be found in [27], p. 106.

1. THEOREM. Let E be a complex vector space of dimension $n < \infty$ and let T be an operator on E. Then there exist subspaces $M_o, M_1, \ldots, M_n$ of E with:

(i) $\{0\} = M_o \subset M_1 \subset M_2 \subset \ldots \subset M_n = E$;

(ii) $\dim M_r = r \quad (r = 0,1, \ldots, n)$;

(iii) $TM_r \subseteq M_r \quad (r = 0,1, \ldots, n)$.

We need a corresponding result for real vector spaces. The following result seems to be well known although it does not appear explicitly in the literature. We shall therefore outline a proof of it.

2. THEOREM. Let E be a real vector space of dimension $n < \infty$ and let S be an operator on E. Then there exist an integer m and subspaces $N_o, N_1, \ldots, N_m$ of E with:

(i) $\frac{1}{2}n \leq m \leq n$;

(ii) $\{0\} = N_o \subset N_1 \subset N_2 \subset \ldots \subset N_m = E$;

(iii) $1 \leq \dim N_{r+1} - \dim N_r \leq 2$ for $r = 0,1, \ldots, m-1$;

(iv) $SN_r \subseteq N_r \quad (r = 0,1, \ldots, m)$.

PROOF. Let $\tilde{E}, \tilde{S}$ denote the complexifications of E,S respectively. Since $\dim_{\mathbb{C}} \tilde{E} = n$, it follows from the preceding theorem that there are subspaces $M_o, M_1, \ldots, M_n$ of $\tilde{E}$ with the following properties:

(i) $\{0\} = M_o \subset M_1 \subset M_2 \subset \ldots \subset M_n = \tilde{E}$;

(ii) $\dim M_r = r \quad (r = 0,1, \ldots, n)$;

(iii) $\tilde{S}M_r \subseteq M_r \quad (r = 0,1, \ldots, n)$.

Consider the self-conjugate subspaces:

$$M_o + M_o^*,\ M_1 + M_1^*,\ M_2 + M_2^*,\ \ldots,\ M_n + M_n^*.$$

Obviously they form a chain and

$$M_o + M_o^* = \{0\}, \qquad M_n + M_n^* = \tilde{E}.$$

A dimension argument gives

$$0 \leq \dim(M_{r+1} + M_{r+1}^*) - \dim(M_r + M_r^*) \leq 2,$$

for $r = 0,1, \ldots, n-1$. Also

$$\tilde{S}(M_r + M_r^*) \subseteq M_r + M_r^* \qquad (r = 0,1, \ldots, n).$$

Thus if we let m be the number of distinct members in the chain and let $N_o, N_1, \ldots, N_m$ be the real parts of the distinct members, retaining the same order, then these subspaces have the required properties.

The following simple example shows that a complete analogue of Theorem 5.2.1. for the case of finite-dimensional real spaces cannot be obtained.

3. EXAMPLE. Define T on $\underset{\sim}{R}^2$ by

$$T(x,y) = (-y,x) \qquad ((x,y) \in \underset{\sim}{R}^2).$$

Clearly T is linear. Also T has no proper closed invariant subspace. To see this, observe that geometrically T is rotation through an angle $\frac{\pi}{2}$.

3. Compact operators

The following result enables us to deduce results on compact operators on a real Banach space from those on compact operators on a complex Banach space.

1. LEMMA. The complexification of a compact operator on a real Banach space is also a compact operator.

PROOF. This is straightforward and is omitted.

2. THEOREM. Let S be a compact operator on a real Banach space X and let $\tilde{X}, \tilde{S}$ denote the complexifications of X,S respectively. Let $\lambda \in \sigma(\tilde{S}) \setminus \{0\}$. Then there exist subspaces M and N of X with the following properties:

(i) M is of finite codimension in X and N is finite-dimensional,

(ii) $M \oplus N = X$;

(iii) $SM \subseteq M$ and $SN \subseteq N$;

(iv) $\lambda \notin \sigma_0(\tilde{S}|\tilde{M})$, $\bar{\lambda} \notin \sigma_0(\tilde{S}|\tilde{M})$, where $\tilde{M}$ is the complexification of M, and where $\sigma_0(\tilde{S}|\tilde{M}) = \sigma(\tilde{S}|\tilde{M}) \setminus \{0\}$.

PROOF. The reader is referred to Theorem 2.21 of [22] p. 53 for a description of the spectrum of a compact operator on a complex Banach space. Suppose that m is the index of the eigenvalue λ of $\tilde{S}$. Define

$$M = \mathrm{Re}\left[(\tilde{S}-\lambda I)^m \tilde{X} \cap (\tilde{S}-\bar{\lambda} I)^m X\right]$$

and

$$N = \mathrm{Re}\left[\ker(\tilde{S}-\lambda I)^m + \ker(\tilde{S}-\bar{\lambda} I)^m\right].$$

It is not difficult to check that M and N have the required properties.

The Hilden-Lomonosov argument yields the following result in the case of a compact operator on a real Banach space.

3. THEOREM. Let Y be an infinite-dimensional real Banach space. Let $T \neq 0$ be a compact quasinilpotent operator on Y. Then there is a proper closed subspace of Y invariant under $\mathcal{U} = \{A \in L(Y) : AT = TA\}$;

i.e. T has a hyperinvariant subspace.

PROOF. Suppose that the theorem is false. Then for each $x \neq 0$

$$\mathcal{A}x = \{Ax : A \in \mathcal{A}\}$$

is dense in Y; that is $\overline{\{\mathcal{A}x\}} = Y$. We may choose a ball

$$B = \{x \in Y : ||x-x_o|| < \delta\}$$

such that $0 \notin \overline{TB}$ and $\overline{TB}$ is compact. Hence if $y \in \overline{TB}$, there is A_y in $\mathcal{A}$ and an open neighbourhood N_y of y with $A_y N_y \subseteq B$. Therefore by compactness there exists a finite subset $A_{y_1}, \ldots, A_{y_k}$ of $\mathcal{A}$ such that for each y in $\overline{TB}$ some A_{y_r} maps y into B. For brevity, write $A_{y_r} = A_r$. For each positive integer n we may find suitable indices $i_1, \ldots, i_n$ such that

$$\prod_{m=1}^{n} A_{i_m} T^n x_o = A_{i_1} T A_{i_2} T \ldots A_{i_n} T x_o \in B.$$

Now, if $M = \max\{||A_p|| : 1 \leq p \leq k\} < \infty$, then

$$||\prod_{m=1}^{n} A_{i_m} T^n x_o|| \leq M^n ||T^n|| \; ||x_o||.$$

Since T is quasinilpotent, $M||T^n||^{1/n}||x_o||^{1/n} \to 0$ as $n \to \infty$ and so we obtain

$$||\prod_{m=1}^{n} A_{i_m} T^n x_o||^{1/n} \to 0 \text{ as } n \to \infty. \qquad (1)$$

Since $0 \notin \overline{TB}$ then also $0 \notin B$ and so for some $\rho > 0$ we have

$$\rho < ||\prod_{m=1}^{n} A_{i_m} T^n x_o||,$$

$$\rho^{1/n} < ||\prod_{m=1}^{n} A_{i_m} T^n x_o||. \qquad (2)$$

Since $\lim_{n \to \infty} \rho^{1/n} = 1$, (1) and (2) give a contradiction. This proves the theorem.

4. THEOREM. Let X be a real Banach space of dimension greater than or equal to three, and let S be a compact operator on X. Then S has a proper closed invariant subspace.

PROOF. Again we refer the reader to Theorem 2.2.1 of [22] p. 53 for a description of the spectrum of a compact operator on a complex Banach space. If X is finite-dimensional, the required result follows from Theorem 5.2.2. Now suppose that X is infinite-dimensional. There are four possibilities; observe that $\tilde{S}$ must satisfy exactly one of the following:

(i) $\ker \tilde{S} \neq \{0\}$;

(ii) $\tilde{S}$ has no eigenvalues;

(iii) $\ker \tilde{S} = \{0\}$, $\sigma_0(\tilde{S})$ is finite and non-empty;

(iv) $\ker \tilde{S} = \{0\}$, $\sigma_0(\tilde{S})$ is an infinite set.

(Here we have used $\sigma_0(\tilde{S})$ to denote $\sigma(\tilde{S}) \setminus \{0\}$.)

We consider these four possibilities separately.

(i) Suppose that $x, y \in X$ are such that $x+iy \in \ker \tilde{S}$, where not both x and y are zero. Let $F = \{\alpha x + \beta y : \alpha, \beta \in \mathbb{R}\}$. Then $F \neq \{0\}$ and F is a proper closed subspace of X invariant under S.

(ii) If $\tilde{S}$ has no eigenvalues, then $\tilde{S}$ is quasinilpotent. We recall from Section 5.1 the following relationships between S and $\tilde{S}$:

(a) $\tilde{S}^n = \widetilde{(S^n)}$ $\quad (n = 1,2,3, \ldots)$,

(b) $||S|| \leq \sqrt{2}||\tilde{S}|| \leq 2||S||$.

It follows that

$$||S^n|| \leq \sqrt{2}||\tilde{S}^n|| \leq 2||S^n||,$$

and so S is also quasinilpotent. The desired conclusion in this case now follows at once from the last theorem.

(iii) Let $\sigma_0(\tilde{S}) = \{\lambda_1, \ldots, \lambda_n\}$. It follows from Theorem 5.3.2 that for each i with $1 \leq i \leq n$ we can find a closed subspace M_i of X invariant under S such that $\lambda_i \notin \sigma_0(\tilde{S}|\tilde{M}_i)$, $\bar{\lambda}_i \notin \sigma_0(\tilde{S}|\tilde{M}_i)$ and M_i is of finite codimension in X. Let $M = \bigcap_{i=1}^{n} M_i$ and observe that $\tilde{M} = \bigcap_{i=1}^{n} \tilde{M}_i$. Then M is infinite-dimensional and $\tilde{S}|\tilde{M}$ has no eigenvalues. The desired conclusion now follows by applying the result of case (ii) to $\tilde{S}|\tilde{M}$.

(iv) Let $\{\alpha_n + i\beta_n\}_{n=1}^{\infty}$ be an infinite sequence of distinct eigenvalues of $\tilde{S}$, where $\alpha_n, \beta_n \in \mathbb{R}$ for each positive integer n. By Theorem 2.2.1 of [22] p. 53,

$$\alpha_n \to 0,\ \beta_n \to 0 \text{ as } n \to \infty.$$

For each positive integer n, define

$$K_n = \ker[\tilde{S} - (\alpha_n + i\beta_n)]$$

and

$$M_n = \{x \in X : \text{there exists } y \text{ in } X \text{ such that } x + iy \in K_n\}.$$

Clearly $M_n = \mathrm{Re}[K_n + K_n]$ and M_n is a non-zero finite-dimensional invariant subspace of S.

This completes the proof of the theorem.

CHAPTER SIX

Superdiagonal forms for completely continuous linear operators on a locally convex Hausdorff topological vector space over $\mathbb{C}$

Altman [2] showed that Riesz-Schauder theory remained valid for a completely continuous linear operator on a locally convex Hausdorff topological vector space over the complex field. In a later paper [3], he proved an analogue of the Aronszajn-Smith result; specifically he showed that such a completely continuous operator possessed a proper closed invariant subspace. The purpose of this chapter is to show that Ringrose's theory of superdiagonal forms for compact linear operators [40] can be generalized to the case of such a completely continuous operator.

1. Locally convex Hausdorff topological vector spaces

1. DEFINITION. A Hausdorff topological vector space X is said to be a locally convex space if there is a basis of neighbourhoods in X consisting of convex sets.

Throughout this chapter, X is a fixed locally convex Hausdorff topological vector space over the complex field and $X \neq \{0\}$.

2. DEFINITION. A non-negative function $x \to p(x)$ on X is called a seminorm if it satisfies the following conditions: $p \not\equiv 0$ and

(i) p is subadditive; that is

$$p(x+y) \leq p(x) + p(y) \qquad (x,y \in X);$$

(ii) p is positively homogeneous of degree 1; that is

$$p(\lambda x) = |\lambda| p(x) \qquad (x \in X, \lambda \in \mathbb{C});$$

(iii) $p(0) = 0$.

Note. Property (iii) in fact follows from Property (ii).

3. DEFINITION. A seminorm on a vector space X is called a norm if

$$x \in X,\ p(x) = 0 \Longrightarrow x = 0.$$

4. DEFINITION. A family P of continuous seminorms on X will be called a basis of continuous seminorms on X if to any continuous seminorm p on X there is a seminorm g belonging to P and a constant C > 0 such that

$$p(x) \leq cg(x) \qquad (x \in X).$$

Such a basis of continuous seminorms on X certainly exists. Throughout this chapter P will denote a fixed basis of continuous seminorms on X. We shall use the following three basic facts about the concepts defined above without further comment.

(A) P determines the topology of X.

(B) A linear functional f on X is continuous if and only if there is a continuous seminorm p on X such that

$$|f(x)| \leq p(x) \qquad (x \in X).$$

(C) A closed linear subspace of X is a locally convex Hausdorff topological vector space under the topology induced by X.

The following results lie much deeper.

5. PROPOSITION. Let Y be a proper closed subspace of X and let p be a continuous seminorm on X. There exist x in X such that p(x) = 1 and

$$\inf\{p(x-y) : y \in Y\} > \tfrac{1}{2}.$$

PROOF. Consider the linear space $X/p^{-1}(0)$, which we denote by W.

Let $x, y \in X$. Observe that

$$p(x) \leq p(y) + p(x-y),$$
$$p(y) \leq p(x) + p(y-x),$$

and so $p(x) = p(y)$ if and only if $p(x-y) = 0$. It follows that if $[x]$ is the equivalence class of $X/p^{-1}(0)$ containing x, then

$$||[x]|| = p(x) \qquad (x \in X)$$

defines a norm on W. It is clear that $H = Y/p^{-1}(0)$ is a closed linear subspace of W. By the Hahn-Banach theorem, there is an f in W^* such that $f \neq 0$ and $f(u) = 0$ $(u \in H)$. We may assume without loss of generality that $||f|| = 1$. Hence there is w in W such that $||w|| = 1$ and $|f(w)| > \frac{1}{2}$. Then, if $u \in H$ we have

$$\tfrac{1}{2} < |f(w)| = |f(w-u)| \leq ||f|| \; ||w-u|| = ||w-u||.$$

If we let x be any vector in the equivalence class w, then $p(x) = ||w|| = 1$ and

$$\inf\{p(x-y) : y \in Y\} > \tfrac{1}{2}.$$

The proof is complete.

6. <u>PROPOSITION.</u> Let Y be a proper closed subspace of X and let $x \in X \setminus Y$. Then there is a continuous linear functional f on X such that $f(x) = 1$ and

$$f(y) = 0 \qquad (y \in Y).$$

<u>PROOF.</u> This result is also a consequence of the Hahn-Banach theorem. It can be deduced immediately from Corollary 2 of [42], p. 30.

Next we discuss quotient spaces of X.

7. THEOREM. Let M be a closed linear subspace of X. Let ϕ be the canonical mapping of X onto X/M. Then the following facts are true.

(i) The topology of the quotient topological vector space X/M is locally convex.

(ii) If P is a basis of continuous seminorms on X, we denote by P_M the family of seminorms on X/M consisting of the seminorms

$$[x] \to \tilde{p}([x]) = \inf\{p(y) : y \in [x]\}.$$

Then P_M is a basis of continuous seminorms on X/M.

(iii) ϕ is a continuous mapping of X onto X/M.

For a proof of this result the reader is referred to Proposition 7.9 of [46], p. 65. Whenever, in this chapter, a quotient space is introduced it will be assumed that it has been topologised in the manner specified above.

For proofs of all results described in this section and many more besides the reader is referred to the books [42] and [46].

2. Completely continuous linear operators

1. DEFINITION. A linear operator T having its domain and range in X is said to be completely continuous if there exists a neighbourhood U of 0 such that the image $T(U)$ is precompact in the sense that every infinite subset has a cluster point; equivalently the closure of $T(U)$ is compact.

2. PROPOSITION. Let Y be a closed subspace of X. Let T be a completely continuous linear operator on X with $TY \subseteq Y$. Then the restriction of T to Y is a completely continuous linear operator on Y.

PROOF. This follows immediately from the definition.

3. PROPOSITION. Let Y be a closed subspace of X. Let T be a completely continuous linear operator on X with $TY \subseteq Y$. Then the operator T_Y defined on the quotient space X/Y by

$$T_Y[x] = [Tx] \qquad (x \in X)$$

is a completely continuous operator.

PROOF. By definition, there is a neighbourhood U of 0 such that the image T(U) is precompact. Let ϕ be the canonical mapping of X onto X/Y. Then $\phi(U)$ is a neighbourhood of the zero element in X/Y and, moreover, $T_Y\phi(U) = \phi(TU)$. By [42], p. 49, the continuous image of a precompact space is precompact. It follows that T_Y is completely continuous.

We now describe Altman's generalization of Riesz-Schauder theory [2]. For an alternative presentation, see Chapter VIII of [42].

4. THEOREM. Let T be a completely continuous linear operator on X, and let λ be a non-zero complex number. There are two possibilities:

(a) $\lambda I-T$ is a homeomorphism of X onto itself;

(b) λ is an eigenvalue of T.

The set of points which satisfy (b) is countable and it has no cluster point except possibly zero. Let λ be a non-zero eigenvalue of T. Then there is a positive integer $\nu(\lambda)$ with the following properties.

(i) For each positive integer n, $(\lambda I-T)^n X$ is closed. Also

$$(\lambda I-T)^{m+1}X = (\lambda I-T)^m X \qquad (m \geq \nu(\lambda))$$

and $\nu(\lambda)$ is the smallest positive integer with this property.

(ii) For each positive integer n, $N((\lambda I-T)^n)$ is finite-dimensional. Also

$$N((\lambda I-T)^m) = N((\lambda I-T)^{m+1}) \quad (m \geq \nu(\lambda))$$

and $\nu(\lambda)$ is the smallest positive integer with this property.

(iii) $(\lambda I-T)^m X \oplus N((\lambda I-T)^m) = X \qquad (m \geq \nu(\lambda))$.

(iv) If $d(\lambda)$ is the dimension of the null-space of $(\lambda I-T)^{\nu(\lambda)}$, then

$$1 \leq \nu(\lambda) \leq d(\lambda).$$

Note. The integers $\nu(\lambda)$ and $d(\lambda)$ are called respectively the index and the algebraic multiplicity of the eigenvalue λ.

We now state Altman's generalization of the theorem of Aronszajn and Smith proved in [3].

5. THEOREM. Let T be a completely continuous linear operator on X. Then there is a proper closed subspace Y of X invariant under T, provided only that X has dimension at least two.

3. Superdiagonal forms for completely continuous linear operators

Throughout the remainder of this chapter, T denotes a completely continuous linear operator on X. The term subspace will be used to describe a closed linear subspace of X.

A family $\mathcal{J}$ of subspaces of X, which is totally ordered by the inclusion relation, will be termed a nest of subspaces. If in addition each subspace in $\mathcal{J}$ is invariant under T we shall describe $\mathcal{J}$ as an invariant nest. A trivial example of an invariant nest is the family consisting of the two subspaces $\{0\}$, X. Non-trivial invariant nests may be constructed using Altman's result, Theorem 6.2.5.

We shall use the symbol $\subseteq$ to denote the inclusion relation and reserve $\subset$ for strict inclusion. The strong closure of a subset S of X will be denoted by $cl(S)$. Given a nest $\mathcal{J}$ of subspaces of X and $M \in \mathcal{J}$ we define

$$M_- = cl[\{L : L \in \mathcal{J}, L \subset M\}].$$

If there is no L in $\mathcal{J}$ such that $L \subset M$, we define $M_- = \{0\}$. It is clear that M_- is a subspace of X, and that it will be an invariant subspace if $\mathcal{J}$ is an invariant nest. Also $M_- \subseteq M$. It should be emphasized that the definition of M_- depends on the particular nest $\mathcal{J}$ under consideration, and not merely on the subspace M. We shall say that $\mathcal{J}$ is <u>continuous</u> at M if $M = M_-$.

A nest $\mathcal{J}$ will be termed <u>simple</u> if

(i) $\{0\} \in \mathcal{J}$, $X \in \mathcal{J}$;

(ii) if $\mathcal{J}_0$ is any sub-family of $\mathcal{J}$, then the subspaces

$$\bigcap\{L : L \in \mathcal{J}_0\},\ cl[\bigcup\{L : L \in \mathcal{J}_0\}]$$

are in $\mathcal{J}$;

(iii) if $M \in \mathcal{J}$, then the quotient space M/M_- is at most one-dimensional.

We note that condition (ii) implies that $M_- \in \mathcal{J}$ whenever $M \in \mathcal{J}$.

The class $\mathcal{N}$ of all nests of subspaces of X may be partially ordered by inclusion: if $\mathcal{J}_1, \mathcal{J}_2 \in \mathcal{N}$, we say that $\mathcal{J}_1 < \mathcal{J}_2$ if every subspace in the family $\mathcal{J}_1$ is also a member of $\mathcal{J}_2$. It is easily seen that, in this way, $\mathcal{N}$ is inductively ordered; for if $\mathcal{N}_0 \subseteq \mathcal{N}$ and $\mathcal{N}_0$ is totally ordered by the relation $<$, then

$$\mathcal{J}_0 = \bigcup\{\mathcal{J} : \mathcal{J} \in \mathcal{N}_0\}$$

is the least upper bound of $\mathcal{N}_0$ in $\mathcal{N}$. We may now deduce from Zorn's lemma the existence of at least one maximal nest of subspaces.

1. <u>THEOREM</u>. Let $\mathcal{J}$ be a nest of subspaces of X. Then $\mathcal{J}$ is maximal if and only if $\mathcal{J}$ is simple.

<u>PROOF</u>. Suppose that $\mathcal{J}$ is maximal. Then in the first place it is apparent that $\{0\}, X \in \mathcal{J}$, since otherwise $\mathcal{J}$ could be enlarged by the addition of these subspaces, contrary to the assumption that $\mathcal{J}$ is maximal. Secondly, let $\mathcal{J}_0$ be a sub-family of $\mathcal{J}$ and consider

$$M_0 = \bigcap\{L : L \in \mathcal{J}_0\}.$$

It is evident that M_0 is a closed subspace of X. Let $M \in \mathcal{J}$. Since $\mathcal{J}$ is totally ordered by inclusion we have <u>either</u> (a) $M \subseteq L$ $(L \in \mathcal{J}_0)$, and $M \subseteq M_0$, <u>or</u> (b) $L \subset M$ for some L in $\mathcal{J}_0$, and $M_0 \subset M$. It follows that the family obtained by adding M_0 to $\mathcal{J}$ remains totally ordered by inclusion and is therefore a nest. Since $\mathcal{J}$ is maximal we deduce that $M_0 \in \mathcal{J}$. A similar argument shows that

$$\mathrm{cl}[\bigcup\{L : L \in \mathcal{J}_0\}]$$

is a member of $\mathcal{J}$. Hence properties (i) and (ii) of simple nests have been established.

Finally we have to show that, if $M \in \mathcal{J}$, then the quotient space M/M_- is at most one-dimensional. Suppose that, for some M in $\mathcal{J}$, this is not the case. Then we may find a subspace L of X such that $M_- \subset L \subset M$. Given any subspace N in $\mathcal{J}$ we have <u>either</u> (a) $M \subseteq N$, and $L \subset N$, <u>or</u> (b) $N \subset M$ and

$$N \subseteq \mathrm{cl}[\bigcup\{K \in \mathcal{J} : K \subset M\}] = M_- \subset L.$$

It follows that $L \notin \mathcal{J}$, and that the family obtained by adding L to

$\mathcal{J}$ is a nest. This contradicts the assumption that $\mathcal{J}$ is maximal. Hence M/M_- is at most one-dimensional, for every M in $\mathcal{J}$, and so $\mathcal{J}$ is a simple nest.

Suppose conversely that $\mathcal{J}$ is a simple nest, but that $\mathcal{J}$ is not maximal. Then we may choose a subspace L of X such that $L \notin \mathcal{J}$ and the family obtained by adding L to $\mathcal{J}$ remains totally ordered by inclusion. We shall obtain a contradiction. Let

$$M = \bigcap \{N : N \in \mathcal{J}, L \subseteq N\}, \qquad (1)$$

$$M' = \mathrm{cl}\left[\bigcup \{N : N \in \mathcal{J}, N \subset L\}\right]. \qquad (2)$$

By virtue of property (ii) of simple nests we have $M, M' \in \mathcal{J}$, and it is clear that $M' \subseteq L \subseteq M$. Since $L \notin \mathcal{J}$ we in fact have

$$M' \subset L \subset M. \qquad (3)$$

We shall now show that $M' = M_-$. It is apparent that $M' \subseteq M_-$. Suppose now that $N \in \mathcal{J}$ and $N \subset M$. It follows from (1) that $L \notin N$. Hence $N \subset L$, and by (2), $N \subseteq M'$. Since M_- is the smallest subspace containing all such N, we have $M_- \subseteq M'$. Hence $M_- = M'$. We may now deduce from (3) that the quotient space M/M_- has dimension greater than unity, contrary to hypothesis. This is the required contradiction, and the proof of the theorem is complete.

2. <u>THEOREM</u>. Let T be a completely continuous linear operator on X. Then there exists a simple nest $\mathcal{J}$, each of whose members is a subspace invariant under T.

<u>PROOF</u>. Let $\mathcal{N}_i$ denote the class of all invariant nests. Then $\mathcal{N}_i$ is not empty since it contains the trivial nest consisting of the subspaces $\{0\}$, X. The class $\mathcal{N}_i$ may be partially ordered by inclusion, and an argument based on Zorn's lemma proves the existence of at least one maximal element. Let $\mathcal{J}$ be a maximal member of $\mathcal{N}_i$. By the type

of reasoning used in proving Theorem 1, we may show that $\tilde{\mathcal{J}}$ has properties (i) and (ii) of simple nests.

It remains to verify that, given any M in $\tilde{\mathcal{J}}$, the quotient space M/M_- is at most one-dimensional. Suppose that, for some M in $\tilde{\mathcal{J}}$, this is not the case. When $x \in M$ we denote by $[x]$ the coset $x + M_-$. It follows from results stated earlier that M/M_- is a locally convex Hausdorff topological vector space. Since M and M_- are invariant under T, we may define a linear operator T_M from M/M_- into itself by the equation

$$T_M[x] = [Tx] \qquad (x \in M).$$

It follows from Propositions 6.2.2 and 6.2.3 that T_M is a completely continuous linear operator. Since M/M_- has dimension exceeding one, Theorem 6.2.5 of Altman implies the existence of a proper subspace L_M of M/M_- which is invariant under T_M. If we now set

$$L = \{x : x \in M, \ [x] \in L_M\},$$

then L is a subspace of X, being the inverse image under the continuous linear map $x \to [x]$ of the subspace L_M, such that $M_- \subset L \subset M$. We may now verify, by the method used at the corresponding stage in the proof of Theorem 1, that $L \notin \tilde{\mathcal{J}}$, but that the family $\tilde{\mathcal{J}}_1$ consisting of L and the members of $\tilde{\mathcal{J}}$ is totally ordered by inclusion. Since L_M is invariant under T_M, L is invariant under T. Thus $\tilde{\mathcal{J}}_1$ is an invariant nest, and is a proper enlargement of the maximal invariant nest $\tilde{\mathcal{J}}$. This gives a contradiction. Thus M/M_- is at most one-dimensional for every M in $\tilde{\mathcal{J}}$, and $\tilde{\mathcal{J}}$ is a simple nest.

Throughout the remainder of this section we shall use the symbols T, $\tilde{\mathcal{J}}$ with the meanings attributed to them in the statement of Theorem 2. If $M \in \tilde{\mathcal{J}}$, then either $M = M_-$ or M/M_- has dimension one. In the latter case let $z_M \in M \setminus M_-$. Then since M is invariant under T we have $Tz_M \in M$, and hence Tz_M can be expressed uniquely in the form

$$Tz_M = \alpha_M z_M + y_M, \tag{4}$$

where α_M is a scalar and $y_M \in M_-$. It is easily verified that α_M does not depend on the particular choice of z_M. When $M = M_-$ we define $\alpha_M = 0$. In this way we associate with each M in $\mathcal{J}$ a scalar α_M which we shall call the <u>diagonal coefficient of</u> T <u>at</u> M. (In the finite-dimensional case the elements z_M form a basis of X, and with respect to this basis T has superdiagonal matrix with diagonal elements α_M.)

Let α be a scalar. We define the diagonal multiplicity of α to be the number (possibly infinite) of distinct subspaces M in $\mathcal{J}$ for which $\alpha_M = \alpha$.

3. <u>LEMMA</u>. Let $\varepsilon > 0$ be given, and let $\mathcal{J}_0$ be the family consisting of those subspaces M in $\mathcal{J}$ for which $|\alpha_M| \geq \varepsilon$. Then $\mathcal{J}_0$ has only a finite number of members.

<u>PROOF</u>. Suppose that $\mathcal{J}_0$ has infinitely many members. We shall use the symbols z_M, y_M as in equation (4). Also, we may assume that the elements z_M in $M \setminus M_-$ have been chosen in such a way that there is a continuous seminorm p in P such that $p(z_M) = 1$ $(M \in \mathcal{J}_0)$ and

$$p(z_M + y) \geq \tfrac{1}{2} \qquad (y \in M_-), \tag{5}$$

equation (5) being valid for all M in $\mathcal{J}_0$. Here we are using Proposition 6.1.5. Now suppose that L,M are distinct members of $\mathcal{J}_0$. Then either $L \subset M$ or $M \subset L$. We may assume that $L \subset M$ and hence that $L \subseteq M_-$. It follows that $z_L \in M_-$ and hence that $Tz_L \in M_-$. Thus

$$\begin{aligned} Tz_L - Tz_M &= \alpha_M z_M + (y_M - Tz_L) \\ &= \alpha_M(z_M + y), \end{aligned}$$

where $y \in M_-$. From (5) we deduce that

$$p(Tz_M - Tz_L) = |\alpha_M| p(z_M + y)$$
$$\geq \tfrac{1}{2}\varepsilon \quad (L, M \in \mathcal{J}_0 : L \neq M).$$

It follows that the infinite family $\{Tz_M : M \in \mathcal{J}_0\}$ contains no cluster point. Observe that, since p is continuous, $\{x \in X : p(x) < 1\}$ is a neighbourhood of 0 and, moreover

$$\{z_M : M \in \mathcal{J}_0\} \subseteq \{x \in X : p(x) < 2\}.$$

This contradicts the assumption that T is a completely continuous linear operator.

4. <u>COROLLARY.</u> Every non-zero scalar has finite diagonal multiplicity.

<u>PROOF.</u> If $\alpha \neq 0$, we may choose ε so that $0 < \varepsilon < |\alpha|$. The preceding lemma then implies that α has finite diagonal multiplicity.

5. <u>LEMMA.</u> Let $M \in \mathcal{J}$ and $\delta > 0$ be given. Then there exists a subspace L in $\mathcal{J}$ such that $L \subset M$ and for every p in P

$$\tilde{p}_L([Tx]) \leq \delta p(x) \qquad (x \in M_-),$$

where $[y]$ denotes the coset $y + L$ $(y \in X)$ and

$$\tilde{p}_L([x]) = \inf\{p(y) : y \in [x]\}.$$

<u>REMARK.</u> The interest of this lemma lies in the case in which $M = M_-$. When $M \neq M_-$, the result is trivial since we may take $L = M_-$.

<u>PROOF OF LEMMA.</u> Suppose that the lemma is false, and denote by $\mathcal{J}_0$ the class of all L in $\mathcal{J}$ such that $L \subset M$. Since we are going to vary L, we shall not use the notation $[y]$ for cosets, but throughout

the proof will write $y + L$. It follows that there is L_0 in $\check{\mathcal{J}}$, p in P and x in L_0 such that $p(x) = 1$ and

$$\tilde{p}_{L_0}(Tx + L_0) > \delta.$$

Let $\mathcal{J}_1 =' \{L \in \check{\mathcal{J}} : L \subseteq L_0\}$. If $L \in \check{\mathcal{J}}_1$, the set

$$S_L =' \{x : x \in M_-, \tilde{p}_L(Tx + L) > \delta \text{ and } p(x) = 1\}$$

is not empty because $S_L \subseteq S_N$ if $N \subseteq L$. It is apparent that the family $\{S_L : L \in \mathcal{J}_0\}$ forms a filter base on the set U defined by

$$U =' \{x \in X : p(x) = 1\}.$$

Hence the family $\{T(S_L) : L \in \mathcal{J}_1\}$ forms a filter base on the compact set cl(T(U)). It follows that there is a point x_0 common to the strong closures of the sets $T(S_L)$ $(L \in \mathcal{J}_1)$. Since $\tilde{p}_L(y + L) > \delta$ and $p(y) = 1$ $(y \in T(S_L))$, we have

$$\tilde{p}_L(x_0 + L) \geq \delta \qquad (L \in \mathcal{J}_1) \tag{6}$$

It follows immediately that

$$\tilde{p}_L(x_0 + L) \geq \delta \qquad (L \in \mathcal{J}_0). \tag{7}$$

Furthermore we have $S_L \subseteq M_-$, $T(S_L) \subseteq M_-$, and hence

$$x_0 \in M_- = \mathrm{cl}\left[\bigcup\{L : L \in \mathcal{J}_0\}\right].$$

Thus for some L in $\mathcal{J}_0$, we may choose an element y in L such that $p(x_0 - y) < \delta$. This contradicts (7), and the lemma is proved.

6. <u>LEMMA</u>. Let ρ be a non-zero eigenvalue of T, and x a corresponding eigenvector. Let

$$M = \bigcap\{L : L \in \mathcal{J}, x \in L\}.$$

Then $M \in \mathcal{Y}$ and $\rho = \alpha_M$.

PROOF. The property (ii) of simple nests immediately implies that $M \in \mathcal{Y}$. In proving that $\rho = \alpha_M$ we shall consider separately the two cases in which respectively $M = M_-$, and $M \neq M_-$.

(a) Suppose that $M = M_-$. Choose δ so that

$$0 < \delta < \tfrac{1}{2}|\rho|,$$

and let L be chosen to satisfy the conclusions of Lemma 6.3.5. Since $L \subset M$ and $L \in \mathcal{Y}$, it is an immediate consequence of the definition of M that $x \notin L$. We may choose y in M in such a way that

$$y - x \in L, \qquad (9)$$

$$p(y) < 2\tilde{p}_L([y]) = 2\tilde{p}_L([x]), \qquad (10)$$

for some p in P. Since L is invariant under T, we deduce from (9) that $Ty - Tx \in L$, and that

$$\begin{aligned} Ty - \rho y &= Tx - \rho x + (Ty - Tx - \rho y + \rho x) \\ &= Ty - Tx - \rho(y-x) \in L. \end{aligned}$$

Hence $[Ty] = \rho[y]$, and

$$\begin{aligned} \tilde{p}_L([Ty]) &= \rho\tilde{p}_L([y]) \\ &> \tfrac{1}{2}\rho p(y) \\ &> \delta p(y). \end{aligned}$$

Here we have used (10) and (8). Since $y \in M = M_-$, this contradicts the assumption that L satisfies the conclusions of Lemma 6.3.5. Hence case (a) cannot occur.

(b) We may now suppose that $M \neq M_-$. Then $x \in M$, but $x \notin M_-$ (since M is, by definition, the smallest member of $\mathcal{Y}$ which contains x).

Let $z_M \in M \setminus M_-$, and let y_M in M_- be chosen so that $Tz_M = \alpha_M z_M + y_M$. We may set $x = \beta z_M + y$, where $y \in M_-$ and $\beta \neq 0$. Then

$$0 = Tx - \rho x = T(\beta z_M + y) - \rho(\beta z_M + y)$$

$$= \beta(\alpha_M z_M + y_M) + Ty - \rho(\beta z_M + y)$$

$$= \beta(\alpha_M - \rho)z_M + \beta y_M + Ty - \rho y.$$

Now, y, y_M and Ty, since M_- is invariant under T, are all elements of M_-, but $z_M \notin M_-$. Hence $\beta(\alpha_M - \rho) = 0$, and since $\beta \neq 0$, it follows that $\alpha_M = \rho$.

The preceding lemma asserts that a non-zero eigenvalue of T is a diagonal coefficient of T. We now have a result in the opposite direction.

7. LEMMA. Let $M \in \mathcal{J}$ and suppose that $\alpha_M \neq 0$. Then α_M is an eigenvalue of T.

PROOF. It is sufficient to show that α_M is an eigenvalue of the operator T_M obtained by restricting T to the space M. Since $\alpha_M \neq 0$ we have $M \neq M_-$. Now T_M is a completely continuous linear operator from M into itself by Proposition 6.2.2, and it is easily verified by means of equation (4) that the range of the operator $T_M - \alpha_M I_M$ is contained in M_-, and is therefore not the whole space M. It follows at once from Theorem 6.2.4 that α_M is an eigenvalue of T_M, and hence of T.

8. LEMMA. Let ρ be a non-zero eigenvalue of T. Then the diagonal multiplicity of ρ is equal to its algebraic multiplicity as an eigenvalue of T.

PROOF. Let d denote the diagonal multiplicity, m the algebraic multiplicity, and ν the index of ρ relative to T. Then

(a) ν is the least integer such that $(T-\rho I)^{\nu+1}x = 0$ only if

$$(T-\rho I)^{\nu}x = 0 \quad (x \in X);$$

(b) ν is the least integer such that

$$(T-\rho I)^{\nu+1}X = (T-\rho I)^{\nu}X;$$

(c) the null-space of the operator $(T-\rho I)^{\nu}$ has dimension m.

Let S be the completely continuous linear operator defined by

$$S-\lambda I = (T-\rho I)^{\nu},$$

where $\lambda = -(-\rho)^{\nu}$. Then λ is an eigenvalue of S which has index unity and algebraic multiplicity m. Since S is a polynomial in T, each subspace M in $\mathcal{J}$ is invariant under S. We may therefore consider the diagonal coefficients of S with respect to the nest $\mathcal{J}$.

Let $M \in \mathcal{J}$ and let α_M, σ_M denote the diagonal coefficients at M of T, S respectively. If $M = M_-$, we have $\alpha_M = \sigma_M = 0$. If $M \neq M_-$, then with the usual notation we may deduce from equation (4) that

$$(T-\rho I)z_M = (\alpha_M-\rho)z_M + y_M.$$

It easily follows that, for n = 1,2, ..., we have

$$(T-\rho I)^n z_M = (\alpha_M-\rho)^n z_M + y^{(n)},$$

where $y^{(n)} \in M_-$. In particular, by taking $n = \nu$, we obtain

$$Sz_M = \lambda z_M + (\alpha_M-\rho)^{\nu} z_M + y^{(\nu)}.$$

Thus $\sigma_M = \lambda + (\alpha_M-\rho)^{\nu}$. We deduce that $\sigma_M = \lambda$ if and only if $\alpha_M = \rho$. Hence the diagonal multiplicity of λ relative to S is d. It is now sufficient to prove the lemma under the additional hypothesis that ρ

has index unity relative to T, since in the general case we may reduce to this situation by replacing T,ρ by S,λ respectively.

Suppose therefore that ρ has index unity relative to T, and let N be the null-space of the operator $T-\rho I$. Given $x \in N$, define

$$M(x) = \bigcap \{L : L \in \mathcal{J}, x \in L\}.$$

From Lemma 6.3.6 and its proof we deduce that $M(x) \in \mathcal{J}$, $x \in M(x) \setminus M_-(x)$ and $\alpha_{M(x)} = \rho (x \in N, x \neq 0)$. The remainder of the proof is divided into three stages.

First, we show conversely that, if $M \in \mathcal{J}$ and $\alpha_M = \rho$, then $M = M(x)$ for some non-zero x in N. For this purpose, let T_M denote the restriction of T to M, and let W_M, N_M be the range and null-space respectively of the operator $T_M - \rho I_M$. Then, by Proposition 6.2.2, T_M is a completely continuous linear operator on M, and it is immediate from the definition of index in terms of null-spaces that ρ has index unity relative to T_M. Hence, by Theorem 6.2.4,

$$W_M \oplus N_M = M.$$

Since, as in the proof of Lemma 6.3.7, $W_M \subseteq M_-$, it follows that N_M meets $M \setminus M_-$. If $x \in N_M \cap (M \setminus M_-)$, it is easily verified that $x \in N$, $x \neq 0$ and $M(x) = M$.

Secondly, let $M_1 \subset M_2 \subset \ldots \subset M_d$ be the distinct members of the nest $\mathcal{J}$ at which T has diagonal coefficient ρ. We may choose non-zero vectors $x_1, \ldots, x_d \in N$ such that $M_i = M(x_i)$ $(i = 1, \ldots, d)$. For each $i = 1, \ldots, d$, x_i is not a linear combination of $x_1, \ldots, x_{i-1}$; for this would imply that $x_i \in M(x_{i-1}) \subseteq M_-(x_i)$, which is not so. Hence $x_1, \ldots, x_d$ are linearly independent elements of N, and since $\dim N = m$ we have $m \geq d$.

Thirdly, suppose that $m > d$. By Proposition 6.1.6, we can find linear functionals continuous on X, such that $\phi_i(x_i) \neq 0$, but $\phi_i(x) = 0$ $(x \in M_-(x_i))$. Then if $x \in M(x_i)$ and $\phi_i(x) = 0$, we have $x \in M_-(x_i)$. Now since dim $N > d$, we may choose a non-zero vector x in N such that $\phi_i(x) = d$ $(i = 1, \ldots, d)$. Then $\alpha_{M(x)} = \rho$, and therefore $M(x) = M(x_i)$ for some i. Thus $x \in M(x_i)$, $\phi_i(x) = 0$, and we have $x \in M_-(x_i) = M_-(x)$. However, this is impossible. Hence $m \leq d$. Since the reverse inequality has already been established we have $m = d$, and the lemma is proved.

We now state a theorem which summarizes the principal results obtained in the preceding lemmas.

9. <u>THEOREM.</u> Let T be a completely continuous linear operator on a locally convex Hausdorff topological vector space over the complex field and let $\mathcal{J}$ be a simple nest of subspaces of X, each of which is invariant under T. Then

(i) a non-zero scalar ρ is an eigenvalue of T if and only if it is a diagonal coefficient of T;

(ii) the diagonal multiplicity of ρ is equal to its algebraic multiplicity as an eigenvalue of T;

(iii) the operator T has no non-zero eigenvalues if and only if $\alpha_M = 0$ $(M \in \mathcal{J})$ or equivalently if and only if $TM \subseteq M_-$ $(M \in \mathcal{J})$.

<u>PROOF.</u> The only statement not already proved is (iii). From (i), it follows that T has no non-zero eigenvalue if and only if $\alpha_M = 0$ $(M \in \mathcal{J})$.

10. <u>COROLLARY.</u> If there is a continuous simple nest of subspaces of X, each of which is invariant under T, then T has no non-zero eigenvalue.

<u>PROOF.</u> This follows from part (iii) of the preceding theorem.

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