q-Fermionic Numbers and Their Roles in Some Physical Problems

R.Parthasarathy[1]
The Institute of Mathematical Sciences
C.I.T. Campus, Tharamani Post
Chennai, 600 113, India.

Abstract

The q-fermion numbers emerging from the q-fermion oscillator algebra are used to reproduce the q-fermionic Stirling and Bell numbers. New recurrence relations for the expansion coefficients in the 'anti-normal ordering' of the q-fermion operators are derived. The roles of the q-fermion numbers in q-stochastic point processes and the Bargmann space representation for q-fermion operators are explored.

[1]e-mail address: sarathy@imsc.res.in

q-deformed Stirling numbers were introduced by Carlitz, Gould and Milne [1]. Such numbers are encountered in [2] in the 'normal ordering' of q-deformed boson oscillator creation and annihilation operators [3]. Recently, Schork [4] considered generalized q-Stirling numbers and obtained useful properties. He further introduced unsigned q-deformed Lah numbers. Subsequently, he [5] considered q-fermionic Stirling numbers which are encountered in the 'normal ordering' of q-fermionic oscillator creation and annihilation operators introduced by the author and Viswanathan [6]. Katriel [7] has derived q-Dobinski formula for q-bosonic Bell number by using q-boson coherent states. In contrast to this semi-classical derivation, a probabilistic derivation of the q-Dobinsky formula for q-bosonic Bell number has been obtained by the author and Sridhar [8] by considering q-stochastic point processes.

The Stirling numbers are crucial in many combinatorial problems [9]. Milne [1] in his studies on generalized restricted growth functions obtained q-Stirling numbers (bosonic), q-Dobinski formula and q-Charlier polynomials. It is very intriguing that these numbers occurring in combinatorial problems, arise naturally in undeformed and q-deformed harmonic oscillator algebra of the creation and annihilation operators.

In this paper we consider q-fermionic numbers in more detail and obtain q-fermionic Stirling and Bell numbers *explicitly*. Further we consider the 'anti-normal ordering' of the q-deformed bosonic and fermionic operators and obtain new recurrence relations. Using their Fock space states, we give a meaning to these numbers as expressing powers of q-bosonic and q-fermionic numbers in terms of 'raising factorials'. This is the counterpart of the q-Stirling numbers of the second kind in expressing the powers of q-numbers in terms of 'falling factorials'.

In the subsequent part of this paper, we attempt to give a probabilistic interpretation of the q-fermionic Stirling numbers of the second kind by considering q-stochastic point processes. This introduces q-product densities, a generalization of the concept of product densities introduced in 1950 by Ramakrishnan [10] in his study of cosmic ray cascades. Then, we give a Bargmann space representation of the q-fermion operators as multiplication by and q-differentiation with respect to quasi-Grassmann variable leading to differential equations involving q-differentiation on spaces of entire functions of quasi-Grassmann variable.

2.q-fermionic numbers

Macfarlane [3] and Biedenharn [3] introduced q-boson oscillator algebra as

$$aa^\dagger - \sqrt{q}a^\dagger a = q^{-N/2} \quad ; \quad [N,a] = -a \quad , \quad [N,a^\dagger] = a^\dagger, \tag{1}$$

where $q > 0$. By making a transformation

$$A = q^{N/4}a \quad ; \quad A^\dagger = a^\dagger q^{N/4}, \tag{2}$$

one obtains

$$AA^\dagger - qA^\dagger A \;\; = \;\; 1; \quad q > 0, \tag{3}$$

and the associated q-bosonic number

$$[n]_b \;\; = \;\; \frac{1-q^n}{1-q}. \tag{4}$$

In the above q *is strictly positive*. The author and Viswanathan [6], proposed a non-trivial q-fermion oscillator algebra as

$$ff^\dagger + \sqrt{q}f^\dagger f = q^{-N/2} \quad ; \quad [N,f] = -f, [N,f^\dagger] = f^\dagger, f^2 \neq \ 0 \ , \ (f^\dagger)^2 \neq 0, \tag{5}$$

where $q > 0$. By making a transformation

$$F = q^{N/4}f \quad ; \quad F^\dagger = f^\dagger q^{N/4}, \tag{6}$$

one obtains

$$FF^\dagger + qF^\dagger F = 1 \quad ; \quad F^2 \neq 0, \ (F^\dagger)^2 \neq 0, \ q > 0, \tag{7}$$

and the associated q-fermionic number

$$[n]_f \;\; = \;\; \frac{1-(-1)^n q^n}{1+q}. \tag{8}$$

In (5) to (8), q is strictly positive. *Thus, it is not correct to replace q by $-q$ in the q-boson algebra to obtain q-fermion algebra (7), as q in (1) to (4)* *is strictly positive.* Nevertheless, for mathematical expressions such replacement may be carried out. In what follows, we shall denote (8) by q-fermion number. This definition is of fundamental importance and is quite different from $[n]_b$. In the limit $q \to 1$, $[n]_b \to n$ while $[n]_f \to \frac{1}{2}(1-(-1)^n)$ taking values 0 and 1 for n even and odd. The properties of the q-fermion numbers associated with (5) (which are different from (8)) have been studied by Narayana Swamy [11]. For $q < 1$, the q-fermion numbers (8) *never* go beyond 1 for any value of n and for $n \to \infty$, it asymptotically approches 0.5. On the other hand the q-boson numbers (4), become the usual numbers when $q = 1$, and for $q < 1$, as $n \to \infty$ we have, $[n]_b$ asymptotically goes to $1/(1-q)$. For $q > 1$, we have for q-fermion numbers, $[0]_f = 0$, $[1]_f = 1$ and $[n]_f > 0$ for n odd, < 0 for n even.

The q-fermions described by (5) or (7) with the q-numbers (8) are different from the k-fermions introduced by Daoud, Hassouni and Kibler [12]. The k-fermion algebra is a non-Hermitian realization of the q-deformed Heisenberg bosonic algebra with q being a root of unity and satisfy $f_{\pm}^k = 0$ and $f_+ \neq {f_-}^\dagger$, except for $k = 2$, for which they become ordinary fermions. On the other hand, q-fermions admit q real or complex and only when $q = 1$, they become ordinary fermions.

3.Normal Ordering of q-fermion operators

We wish to evaluate $(F^\dagger F)^r$ using (7). It is straightforward to expand,

$$(F^\dagger F)^r \;=\; \sum_{s=1}^{r} \mathcal{F}_s^r (F^\dagger)^s F^s, \tag{9}$$

and find a recurrence relation for $\mathcal{F}_s^r$. From (9), it follows

$$(F^\dagger F)^{r+1} \;=\; \sum_{s=1}^{r} \mathcal{F}_s^r (F^\dagger)^s F^s F^\dagger F. \tag{10}$$

From (7), we have

$$F^s F^\dagger \;=\; [s]_f F^{s-1} + (-1)^s q^s F^\dagger F^s. \tag{11}$$

Using (11) in (10) and noting from (9), $\mathcal{F}_0^r = 0$, $\mathcal{F}_{r+1}^r = 0$, we find

$$\mathcal{F}_s^{r+1} \;=\; (-1)^{s-1} q^{s-1} \mathcal{F}_{s-1}^r + [s]_f \mathcal{F}_s^r, \tag{12}$$

the desired recurrence relation for $\mathcal{F}_s^r$ coefficients in (9). The recurrence relation (12) with $\mathcal{F}_1^1 = 1$ is the same as that of the q-fermionic Stirling numbers of the second kind [5].

The q-fermionic Bell number introduced in [5] is

$$\mathcal{B}_r^{(f)} = \sum_{s=1}^{r} \mathcal{F}_s^r. \tag{13}$$

An attempt along the lines of [7] for q-fermionic Bell number runs into difficulty. q-fermion coherent states have been constructed in [13] using 'quasi Grassmann' variables ψ. It is to be noted that the replacement of q by $-q$ in the q-boson coherent states *will not* give q-fermion coherent states. These two coherent states are structurally very different. As $\psi^\dagger\psi + \psi\psi^\dagger = 0; \psi^2 \neq 0, (\psi^\dagger)^2 \neq 0$, it is not possible to use the analogue of $|z| = 1$ here. So, we take the matrix elements of (9) between q-fermion Fock space states [13] $|n>$ with $n > r$ and use $F|n>= \sqrt{[n]_f}|n-1>$; $F^\dagger|n>= \sqrt{[n+1]_f}|n+1>$ to arrive at

$$[n]_f{}^r = \sum_{s=1}^{r} \mathcal{F}_s^r \frac{[n]_f!}{[n-s]_f!}, \tag{14}$$

which can be verified explicitly using (8) and (12). Multiplying (14) by λ^n and summing n from 1 to ∞ and then setting $\lambda = 1$, we obtain

$$\mathcal{B}_r^{(f)} = (e_q^{(f)}(1))^{-1} \sum_{n=1}^{\infty} \frac{[n]_f^r}{[n]_f!}, \tag{15}$$

where

$$e_q^{(f)}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_f!}.$$

(15) is the q-fermionic Dobinski formula.

Some of the q-fermionic Bell numbers are:

$$\begin{aligned}
\mathcal{B}_1^{(f)} &= 1, \\
\mathcal{B}_2^{(f)} &= 1-q,
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_3^{(f)} &= 1 - q - q[2]_f - q^3, \\
\mathcal{B}_4^{(f)} &= 1 - q - q[2]_f - q[2]_f^2 - q^3 - q^3[2]_f - q^3[3]_f + q^6, \\
\mathcal{B}_5^{(f)} &= 1 - q + (-q - q[2]_f - q[2]_f^2)([2]_f + q^2) \\
&+ (-q^3 - q^3[2]_f - q^3[3]_f)([3]_f - q^3) + q^6[4]_f + q^{10}. \qquad (16)
\end{aligned}$$

In the limit $q = 1$, we have

$$\begin{aligned}
\mathcal{B}_1^{(f)} = 1 \quad ; \quad \mathcal{B}_2^{(f)} &= 0, \\
\mathcal{B}_r^{(f)} &= (-1)^r \qquad if\ r = 0\ (mod\ 3) \\
&\quad (-1)^{r+1} \quad if\ r = 1\ (mod\ 3) \\
&\quad 0 \qquad\quad if\ r = 2\ (mod\ 3). \qquad (17)
\end{aligned}$$

The results (17) have been obtained by Wagner [14] in his study of generating functions for some well known statistics on the family of partitions of a finite set.

The q-fermionic Stirling numbers of the *first* kind are introduced by expressing the inverse of (9), as

$$(F^\dagger)^r F^r = \sum_{s=1}^{r} \mathcal{S}_s^r (F^\dagger F)^s.$$

The matrix elements of the above between q-fermion Fock space states $|n>$, $(n > r)$ give

$$\frac{[n]_f!}{[n-r]_f!} = \sum_{s=1}^{r} \mathcal{S}_s^r [n]_f^s. \qquad (18)$$

Using $[n]_f - [r]_f = (-1)^r q^r [n-r]_f$, we have the recurrence relation

$$\mathcal{S}_s^{r+1} = (-1)^r q^{-r} \mathcal{S}_{s-1}^r - [r]_f (-1)^r q^{-r} \mathcal{S}_s^r. \qquad (19)$$

Similarly, the q-fermionic unsigned Lah numbers introduced as

$$\frac{[r+n-1]_f!}{[r-1]_f!} = \sum_{s=0}^{n} \mathcal{L}_s^n \frac{[r]_f!}{[n-s]_f!}, \qquad (20)$$

have the recurrence relation

$$\mathcal{L}_s^{n+1} = (-1)^{n+s-1}q^{n+s-1}\mathcal{L}_{s-1}^n + [s+n]_f\mathcal{L}_s^n. \tag{21}$$

These mathematical results reveal the feature of obtaining them from their q-bosonic counterparts by replacing q by $-q$. The unsigned q-Lah numbers are met in the normal ordering of $((A^\dagger)^r A^s)^n$ for $r=2$ and $s=1$ [15] and their q-analogues have been obtained in [4,5].

4.Anti-Normal Ordering of q-fermion operators

In this section, we seek an expansion for $(AA^\dagger)^r$. First, we consider q-bosonic operators in (3). It is straightforward to expand

$$(AA^\dagger)^r = \sum_{s=1}^{r}\mathcal{A}_s^r A^s(A^\dagger)^s, \tag{22}$$

and use

$$(A^\dagger)^s A = \frac{1}{q^s}A(A^\dagger)^s - \frac{1}{q^s}[s](A^\dagger)^{s-1},$$

to obtain a recurrence relation for $\mathcal{A}_s^r$ with $\mathcal{A}_1^1 = 1, \mathcal{A}_0^r = 0, \mathcal{A}_{r+1}^r = 0$ as

$$\mathcal{A}_s^{r+1} = q^{-(s-1)}\mathcal{A}_{s-1}^r - [s]q^{-s}\mathcal{A}_s^r. \tag{23}$$

A similar relation for q-fermionic operators anti-normal orderering, namely

$$(FF^\dagger)^r = \sum_{s=1}^{r}\mathcal{B}_s^r F^s(F^\dagger)^s, \tag{24}$$

can be obtained from (7). We have from (7)

$$(F^\dagger)^s F = (-1)^s q^{-s}F(F^\dagger)^s - (-1)^s q^{-s}[s]_f(F^\dagger)^{s-1}.$$

Using this and (24), we find ($\mathcal{B}_1^1 = 1; \mathcal{B}_0^r = 0; \mathcal{B}_{r+1}^r = 0$),

$$\mathcal{B}_s^{r+1} = (-1)^{s-1}q^{-(s-1)}\mathcal{B}_{s-1}^r - (-1)^s q^{-s}[s]_f\mathcal{B}_s^r. \tag{25}$$

These recurrence relations (23) and (25) are *different* from those of q-Stirling or Lah numbers.

In order to obtain a relationship between $\mathcal{A}_s^r$ ($\mathcal{B}_s^r$) and $[n]_b$ ($[n]_f$), we use the q-boson Fock space states for (22) and q-fermion Fock space for (24). Then using the standard action of the creation and annihilation q-operators on the Fock space states, we find

$$\begin{aligned} [n+1]_b^r &= \sum_{s=1}^{r} \mathcal{A}_s^r \frac{[n+s]_b!}{[n]_b!}, \\ [n+1]_f^r &= \sum_{s=1}^{r} \mathcal{B}_s^r \frac{[n+s]_f!}{[n]_f!}. \end{aligned} \tag{26}$$

It is interesting to observe that *while the normal ordering of operators yielded expressions for* $[n]_b^r$ ($[n]_f^r$) *in terms of q-stirling numbers of the second kind as 'falling factorials' (namely (14) and its q-bosonic analogue), the anti-normal ordering yields expressions for* $[n]_b^r([n]_f^r)$ *in terms of* $\mathcal{A}_s^r$, $\mathcal{B}_s^r$ *as 'raising factorials'.*

5.q-fermionic Stirling number of second kind - a probabilistic view

In this section we extend the theory of product densities of Ramakrishnan [10] to a q-extension of the stochastic variable $[n(E)]_f$ which depend on a continuous parameter E taken to be ordinary variable, The statistical properties of these q-fermionic stochastic variable taking values $[n(E)]_f$ will be governed by q-stochastic point processes. Recall that $[n]_f$ for $q < 1$ never exceeds unity. So the feature assumed in [10] namely atmost one particle occurs in the interval dE is maintained. Now we *define* the q-number of particles in the range E and $E + dE$ to be $[n(E + dE) - n(E)]_f$ which is just $[dn(E)]_f$. Following [10], we take the probability that there occurs $[1]_f = 1$ particle in the interval dE is proportional to dE and that for the occurrence of $[n]_f$ particles in dE is proportional to $(dE)^n$. The average number of particles in the interval dE, denoted by $\mathcal{E}([dn(E)]_f)$, is represented by a q-function $f_1^{(q)}(E)$ such that

$$\mathcal{E}([dn(E)]_f) = f_1^{(q)}(E)\, dE. \tag{27}$$

Denoting the probability that $[n]_f$ particles occuring in the interval dE by $P_q([n]_f)$, we have

$$\begin{aligned} P_q([1]_f) &\equiv P_q(1) = f_1^{(q)}(E)dE + \mathcal{O}((dE)^2), \\ P_q([0]_f) &\equiv P_q(0) = 1 - f_1^{(q)}(E)dE - \mathcal{O}((dE)^2), \\ P_q([n]_f) &= \mathcal{O}((dE)^n) \ ; \ n > 1. \end{aligned} \tag{28}$$

The average of the r^{th} moment of $[n]_f$ is then

$$\begin{aligned} \mathcal{E}([n]_f^r) &= \sum_n [n]_f^r P_q([n]), \\ &= f_1^{(q)}(E)dE = \mathcal{E}([dn(E)]), \end{aligned} \tag{29}$$

where the second step follows from (27) and (28). Thus all the moments are equal to the probability that the q-stochastic variable assumes the value $[1]_f = 1$. This feature of [10] is maintained here. $f_1^{(q)}(E)$ is the q-product density of degree 1.

Now we consider the distribution of $[n]_f$ particles in the E-axis, that is, in the intervals $dE_1, dE_2, \cdots dE_n$, with $[1]_f (= 1)$ particle in each interval. For the first interval dE_1, this can be done in ${}^{[n]_f}C_{[1]_f} = [n]_f!/[n-1]_f! = [n]_f$ *number of ways*, where ${}^{[n]_f}C_{[1]_f}$ is the q-binomial coefficient. Thus, from the average number of particles in (27), we have

$$f_1^{(q)}(E_1)dE_1 = [n]_f f_1^{(q)0}(E_1)dE_1, \tag{30}$$

with

$$\int_{whole\ range} f_1^{(q)0}(E)dE = 1. \tag{31}$$

If we now use the reamaining particles as $([n]_f - 1)$, then the number of ways of putting $[1]_f$ particle in dE_2 will be $[n]_f - 1$. Then the joint probability of putting $[1]_f$ particle *each* in dE_1 and dE_2 will be proportional to $[n]_f([n]_f - 1)$, if we were to use the same product density in [10]. But, in this way, we will not be exhausting the total number $[n]_f$ of particles. So, in dealing with q-numbers, *it is necessary to introduce q-product densities*, such that the joint probability of putting $[1]_f$ particle *each* in dE_1 and dE_2 will be taken to be proportional to $[n]_f[n-1]_f$. Then the average number in this case will be

$$\begin{aligned} \mathcal{E}([dn(E_1)]_f[dn(E_2)]_f) &\equiv f_2^{(q)}(E_1, E_2)dE_1\, dE_2, \\ &= [n]_f[n-1]_f f_1^{(q)0}(E_1) f_1^{(q)0}(E_2)\, dE_1\, dE_2. \quad (32) \end{aligned}$$

Proceeding further, the joint probability of putting $[1]_f$ particle each in $dE_1, dE_2, \cdots dE_n$ will be proprtional to $[n]_f!$, thereby exhausting the total number $[n]_f$ particles. This gives

$$\begin{aligned} f_m^{(q)}(E_1, \cdots E_m)dE_1 \cdots dE_m &= \frac{[n]_f!}{[n-m]_f!} f_1^{(q)0}(E_1) \cdots f_1^{(q)0}(E_m)dE_1 \cdots dE_m, \\ f_n^{(q)}(E_1 \cdots E_n)dE_1 \cdots dE_n &= [n]_f! f_1^{(q)0}(E_1) \cdots f_1^{(q)0}(E_n)dE_1 \cdots dE_n. \quad (33) \end{aligned}$$

In (32) and (33), the intervals do not overlap. When the intervals overlap, a degeneracy occurs [10] and then for a finite interval $\triangle E = E_u - E_t$

$$\begin{aligned} \int_{E_t}^{E_u} \int_{E_t}^{E_u} \mathcal{E}([dn(E_1)]_f[dn(E_2)]_f) &= \int_{E_t}^{E_u} f_1^{(q)}(E)dE \\ &+ \int_{E_t}^{E_u} \int_{E_t}^{E_u} f_2^{(q)}(E_1, E_2)dE_1\, dE_2. (34) \end{aligned}$$

The r^{th} moment of the q-number of particles in the finite range $\triangle E$, namely $\mathcal{E}([n]^r_{f\ \triangle E})$ can be represented, after taking the degeneracy into account, by

$$\mathcal{E}([n]^r_{f\ \triangle E}) = \sum_{s=1}^{r} \mathcal{C}_s^r \int_{E_t}^{E_u} \cdots \int_{E_t}^{E_u} f_s^{(q)}(E_1, \cdots E_s)dE_1 \cdots dE_s, \quad (35)$$

where the coefficients $\mathcal{C}_s^r$ are functions of r and s alone. For $[n]_f$ fixed, integrating over the whole range and using (31) and (33), we obtain

$$[n]_f^r = \sum_{s=1}^{r} \mathcal{C}_s^r \frac{[n]_f!}{[n-s]_f!}, \quad (36)$$

which is same as (14) upon identifying $\mathcal{C}_s^r$ with $\mathcal{F}_s^r$. This derivation of (36) gives the role of the q-fermionic Strirling number of the second kind as taking into account the degenarcaies in the joint probabilities of the distribution of q-fermionic number $[n]_f$ as a stochastic variable when the intervals overlap.

6.Bargmann Space Representation for q-fermion operators

A Bargmann space realization for q-bosons has been developed by Bracken, MacAnally, Zhang and Gould [16] and that for q-fermions has been developed by the author [17]. As the q-fermion coherent states involve quasi-Grassmann variable ψ, the space consists of monomials of ψ. In this section, we briefly recall the main results to illustrate two points. First, a naive replacement of q by $-q$ is not sufficient and is incorrect. Second, we would like to represent (9) and (24) in terms of 'differential operators'. The q-fermion coherent state [13] is given by

$$\begin{aligned} |\psi> &= \left(e_q^{\psi^\dagger\psi}\right)^{-\frac{1}{2}} e_q^{-\psi F^\dagger}|0>, \\ &= \left(e_q^{\psi^\dagger\psi}\right)^{-\frac{1}{2}} \sum_{n=0}^{\infty}(-1)^n\Big\{\frac{\psi^{2n}}{\sqrt{[2n]_f!}}|2n> \\ &- \frac{\psi^{2n+1}}{\sqrt{[2n+1]_f!}}|2n+1>\Big\}, \end{aligned} \tag{37}$$

and it can be verified $F|\psi>=\psi|\psi>$. We will first map a vector in the Hilbert space $|\phi>$ to a function $\phi(\psi)$ by

$$\begin{aligned} \phi(\psi) &\equiv <\psi^\dagger|\phi>, \\ &= \left(e_q^{\psi^\dagger\psi}\right)^{-\frac{1}{2}} \sum_{n=0}^{\infty}(-1)^n\Big\{\frac{\psi^{2n}}{\sqrt{[2n]_f!}}<2n|\phi> \\ &+ \frac{\psi^{2n+1}}{\sqrt{[2n+1]_f!}}<2n+1|\phi>\Big\}, \end{aligned} \tag{38}$$

where we have made use of the anti-commuting property of ψ with F and $F^\dagger$. In [17] it has been shown that $\phi(\psi)$ is an entire function.

Now consider the matrix element $<\psi^\dagger|F^\dagger|\phi>$. Using $F|\psi>=\psi|\psi>$, we have

$$<\psi^\dagger|F^\dagger|\phi> = \psi\phi(\psi), \tag{39}$$

and so *in the space of* $\phi(\psi)$, $F^\dagger$ *is represented by multiplication by* ψ. Consider now the expression $\psi<\psi^\dagger|F|\phi>$ which can rewritten as

$$\psi<\psi^\dagger|F|\phi> = <\psi^\dagger|F^\dagger F|\phi>. \tag{40}$$

Writing $F^\dagger F$ as $[N]_f$ (which is possible in view of the expression for $|\psi>$ in terms of expansion of q-fermion Fock space states), we have

$$\begin{aligned} \psi <\psi^\dagger|F|\phi> &= <\psi^\dagger|[N]_f|\phi>, \\ &= <\psi^\dagger|\frac{1-(-1)^N q^N}{1+q}|\phi>, \end{aligned} \tag{41}$$

where in the last step we used (8). Using (38), we have $<\psi^\dagger|q^N|\phi> = \phi(q\psi)$ with the prefactor in (38) unaltered and so,

$$\psi <\psi^\dagger|F|\phi> = \frac{\phi(\psi)-\phi(-q\psi)}{1+q}. \tag{42}$$

This suggests to introduce q-differentiation as

$$\frac{d_q}{d_q\psi}\phi(\psi) \equiv \frac{\phi(\psi)-\phi(-q\psi)}{\psi(1+q)}. \tag{43}$$

Thus *in the space of $\phi(\psi)$, F is represented by q-differentiation with respect to ψ*. It can be verified $\frac{d_q}{d_q\psi}\psi^n = [n]_f\psi^{n-1}$. In this way, we realize that the Bargmann spaces for q-boson and q-fermion are very different. The monomials of quasi-Grassman variables cannot be obtained by naive replacement of q by $-q$. Now using these results, the expression (9) can be written as

$$\Big(\psi\frac{d_q}{d_q\psi}\Big)^r\phi(\psi) = \sum_{s=1}^{r}\mathcal{F}^r_s(\psi)^s\Big(\frac{d_q}{d_q\psi}\Big)^s\phi(\psi), \tag{44}$$

and the expression (24) gives

$$\Big(\frac{d_q}{d_q\psi}\ \psi\Big)^r\phi(\psi) = \sum_{s=1}^{r}\mathcal{B}^r_s\Big(\frac{d_q}{d_q\psi}\Big)^s\psi^s\ \phi(\psi). \tag{45}$$

Expressions (44) and (45) reveal the role of q-fermion Stirling number of the second kind and 'number $\mathcal{B}^r_s$' appearing in the anti-normal ordering of q-fermion operators, in expressing powers of q-differential operators.

7.Summary

We have considered the q-fermion numbers introduced in the q-fermion oscillator algebra by Parthasarathy and Viswanathan [6] in detail and obtained q-fermionic Stirling numbers of first and second kind explicitly. The q-fermionic Bell number is obtained by means of q-fermionic Dobinsky formula. q-fermionic Lah numbers are also considered. These results agree with those of Schork [4,5]. The case of 'anti-normal ordering' of q-fermionic annihilation and creation operators is studied and expansion coefficients $\mathcal{A}_s^r$ for q-bosonic operators and $\mathcal{F}_s^r$ for q-fermionic operators are introduced. Recurrence relations for these are derived and these are very different from those encountered in q-stirling numbers or q-Lah numbers. In this sense they are new. By taking matrix elements of the defining relations for these coefficients between q-bosonic and q-fermionic Fock space states, we obtain expressions for the powers of q-bosonic and q-fermionic numbers in terms of 'raising factorials'. These compliment the role of the q-Stirling numbers of the second kind as they express powers of q-bosonic and q-fermionic numbers in terms of 'falling factorials'. The theory of product densities of Ramakrishnan [10] is extended to q-stochastic point process and the necessity of introducing q-product densities is emphasized. This leads to the identification of the effect of the degeneracy with q-fermionic Stirling numbers of the second kind. We have given a Bargmann space representation of q-fermion operators F and $F^\dagger$ using q-fermion coherent states. This representation is used to express powers of q-differential operators acting on the space of entire functions of quasi-Grassmann variable as a series.

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