A STUDY

IN HIGHER EDUCATION CALCULUS

AND

STUDENTS' LEARNING STYLES

by

HASSAN ALAMOLHODAEI

A thesis submitted for the degree of Doctor of Philosophy (Ph.D.) of Glasgow University in Mathematics Education

Centre for Science Education and Department of Mathematics
Faculty of Science

© HASSAN ALAMOLHODAEI 1996

IN THE NAME OF ALLAH

THE MERCIFUL,

THE COMPASSIONATE

TO

MY HONOURABLE AND DEVOTED WIFE

ACKNOWLEDGEMENTS

My God! to You is due all praise, always and forever, eternally, increasing, not diminishing, as You like and please.

This thesis would never have been completed without the help and encouragement I received from many people over the years of this research.

- I'm deeply indebted to Prof. A. H. Johnstone and Dr D. W. Webber, for their expert guidance and constructive comments. Their enthusiasm and patience served as a major source of inspiration and stimulus. Without their helpful suggestions, the work would have suffered significantly.

- Special thanks are due to Dr A. Shahvarani, his guidance and kindness were indispensable for the completion of this work.

- I have also benefited greatly from the help and suggestions of Dr N. R. Arghami and Mr M. Sarmad at the Department of Statistics, Ferdowsi University of Mashhad.

- My special thanks are due to all students and members of staff at the Teacher Training University of Sabzevar and the Department of Mathematics at Ferdowsi University of Mashhad for their important help during my data collecting in Iran.

- I would like to express my sincere gratitude to my colleagues in the Centre for Science Education and to members of staff and students of the Mathematics Department at Glasgow University for the help they have offered during my research.

- I'm very grateful for the financial support from the Ministry of Culture and Higher Education of the Islamic Republic of Iran and I would like also to express my appreciation to Dr H. Salar Amoli, the Scientific Adviser and Director of the Iranian students in the U.K., for his significant guidance and help during my study.

- The final thanks must go to my family, in particular, my parents, who during my studies have done so much to help me towards this achievement. It is also a pleasant duty to express my great indebtedness for unstinting help given, in compiling this work, to my wife and children, in spite of suffering a lot.

ABSTRACT

This research is devoted to focussing on the influence of different learning styles on the performance of undergraduate students in various parts of calculus.

In carrying out the study, calculus materials were classified into four main categories (Z_4,Z_5, Z_6,Cals) and, for the Iranian students, the results of their mathematical performance in the university entrance examination is labelled (En) to identify their grounding in high school mathematics at the beginning of the calculus course in higher education. Also, in the present study, students' performance (weakness) in the manipulation of mathematical notation and logical discussion is called (Z_1) category and (Cal) indicates students' total achievement in calculus examination which is, in fact, the students' performance on the combination of the categories (Z_4, Z_5, Z_6). These calculus categories are described in Chapter 5. However in short term, multi-conceptual and procedural tasks are classified as (Z_4). The (Z_5) category is defined as the translation processes between mathematical abstraction (analytic/symbolic) and (pictorial/visual) forms of calculus materials. Moreover, multi-skilled, transferable and procedural skills are labelled as (Z_6) category. It should be noted that these categories are interrelated in a scheme to exhibit activities in calculus.

572 students participated in the experimental part of this study and were selected from two Iranian universities (Sabzevar University and Mashhad University) and Glasgow University in Scotland, U.K.

During the period of the study, the samples of students were subjected to some psychological tests in order to assign their Field-dependent/Field-independent and Convergent/Divergent learning styles.

It was found throughout the study that the most effective combination of learning styles which emerged from the interacting picture of all the psychological factors used in the research, were field-independent/convergent (FI+Con) in Iran, and field-independent/divergent (FI+Div) in Scotland in performing on the calculus. On the

other hand, the combination of field-dependent and convergent styles (FD+Con) could lessen achievement in calculus by mathematics/physics students, and field-dependent and divergent styles (FD+Div) would lessen attainment in calculus by engineering students.

In addition, when the mean scores in calculus categories were calculated for various groups of students with different learning styles, the convergent thinkers (Con) were found to be best in (Z_6), while divergent thinkers (Div) exhibited higher performance in (Z_5). These findings demonstrate that the Con/Div way of thinking is the most effective in influencing performance in different areas of calculus, the FI/FD factor takes the second position. All these findings have been combined to form a model which emerges at the end of this thesis.

Moreover, in Chapters 3 and 4, a comparison is made between calculus in secondary (high school) and higher education in Iran and Scotland, focussing on content, teaching order, learning objectives and teaching methods.

Contents

Chapter 1

Cognitive Style

Chapter 2

Cognitive styles and Academic achievement

Chapter 3

Calculus and Education Policy in Iran

Chapter 4

Calculus and Education Policy in Scotland

Chapter 5

Students' Difficulties in Learning Calculus

Chapter 6

Methodology, Procedure and Samples

Chapter 7

Evaluation and Students' Performance

Chapter 8

Conclusions

Appendices

Chapter 1

Cognitive Style

1.1 Introduction

The Literature on individual differences is extensive and the consideration of individual differences in learning and teaching any subject matter is one of the most striking phenomena for cognitive psychologists and educators (see, for instance, Witkin et al., Messick, Cross, Kempa, Resnick and Johnstone).

Differentiation theory is very closely connected with the names of Witkin and his colleagues. Educational applications and implications of learning and performance differences in learners are discussed and accepted as significant concerns in the development of both curricular and instructional materials. As noted by Resnick and Ford (1984), debates such as the above among researchers arise from different responses to the same question or task on the part of individuals. This, in fact, describes the states of students' understanding and their performance, therefore there cannot be any doubt about the potential importance of individual differences as a main factor in learning processes and mental functioning (Witkin et al., 1977, 1981; Kempa, 1979). For this reason attention must be given by mathematics and science education researchers to this domain. Witkin et al. (1977) and Messick (1976) refer to individual differences as the cognitive styles of learners.

1.2 Cognitive Style

The concept of cognitive style is strongly connected with the idea of psychological differentiation. This means in a broad sense that differences exist between different individuals in relation to their cognitive structure and psychological functioning or as Witkin (1974) called it "psychological individuality". In fact, as Cross (1976) noted, each individual has his/her own styles for collecting and organizing information into beneficial knowledge. And the ways of organizing and processing information and experience is labelled by Messick (1976) as "cognitive styles".

There are considerable questions which have a significant role in the education area. Some of them are as follows:

1. What are the processes of students' perception, mental transformation, learning and problem solving?

2. What controls the students' academic choice and development?

3. How do teachers teach and how do students and teachers interact?

Cross (1976) pointed out that some students' approach to learning is analytical and systematic; others are more intuitive and global. Some students will be best in groups, while others will do better learning alone. Some students prefer abstract materials and formal discussion, while others prefer concrete materials and intuitive arguments.

It was stated by Witkin et al. (1977, 1978) and Kogan (1976) that cognitive style is the manner in which individuals acquire, store, retrieve and transform information.

1.2.1 Some Characteristics of Cognitive style

1. **Techniques for moving toward goals**

   Witkin et al. (1977, 1981) suggested that cognitive styles are ways of moving toward goals rather than goal attainment. In fact, as Kempa (1979) explained, they are concerned with the processes of mental activities, learning, and problem solving. Thus, it may be concluded that cognitive styles are independent of the subject content.

2. **Pervasive dimensions**

   Cognitive styles are pervasive dimensions (Witkin et al., 1977, 1981). This pervasiveness shows itself in the perceptual, intellectual, personality and social domains.

3. **Stability**

   Cognitive styles are stable over time. A lot of researchers, for instance Witkin et al. (1967, 1977, 1981) and Witkin (1978), noted that this stability extends not only over weeks and months, but over years. Therefore, it can be deduced that, any educational implications of cognitive styles may have long-term validity. Nonetheless, this does not mean that they are not totally unchangeable (Witkin et al., 1977) and Witkin (1978).

4. **Bipolarity**

   Cognitive style is bipolar with regard to level and its bipolarity makes the dimensions value-neutral in the sence that each pole has qualities that are adaptive in particular circumstances (Messick, 1976; Witkin, 1978 and Witkin et al., 1977, 1981). The feature of bipolarity rejected the issue of good or bad in connection with cognitive styles, but each pole has its adaptive value in different contexts. As a result, many researchers cited that this feature distinguishes cognitive style from intelligence and other ability dimensions (e.g. Messick, 1976 and Witkin et al., 1977, 1981).

In his study, Harmon (1984) suggested that cognitive styles are relatively independent of abilities and and aptitudes. Ability and aptitude represent a power to do, but cognitive styles refer to the way the power is used. Nevertheless, there is some contention between the style and ability dimension differences (for example, Messick, 1976; Kogan, 1976; Guilford, 1980; Witkin et al., 1977, 1981; McKenna, 1984 and Harmon 1984).

1.2.2 Cognitive style and Ability

As was mentioned above, many researchers have suggested some differences between cognitive style and ability and they are as follows:

1. Cognitive style is considered to be "The manner of moving toward a goal", but the concept of ability is cited to be as "competence in goal attainment".

2. Abilities are measured in terms of level of performance, whereas cognitive styles are measured by the degree of some manner of performance.

3. Abilities mainly refer to the content or the question of what, but cognitive styles in contrast connect with the way in which behaviour occurs and the question of how.

4. Abilities are suggested to be unipolar, while cognitive styles are bipolar with regard to level.

5. Values "good" or "bad" are attached to the ability dimension, but not to cognitive styles. Hence, to have more ability is better than having less of it.

In the following section of this research a review is presented of some of findings from studies into individual differences which are considered to have a direct bearing on aspects of mathematics, in particular, calculus teaching and learning.

1.3 Cognitive style and Mathematics Education

Two cognitive styles, well-established through a lot of research, are examined in the next chapters of this study in relation to mathematics and calculus education. They are:

1. Field-dependence/Field-independence.

2. Convergent/Divergent cognitive styles.

1.3.1 Field-dependence/independence Cognitive style

Many more investigations have been carried out by researchers on this dimension than any other cognitive style dimensions. The idea that field-dependence may be related to individual differences to learning and memory has been considered in many studies in recent years. For instance, Witkin et al. (1974) have shown that some students have more difficulty than others in separating relevant materials from irrelevant ones, or "signal" from "noise," in a confusing field (Johnstone and Al-Naeme, 1991). They are labelled as field-dependent people.

It should be noted that, the study of field/dependency is mainly related to the names of Witkin and his colleagues (e.g., 1954, 1974, 1977, 1981). A field-independent learner is defined by Witkin and Goodenough (1981) as a person who can easily break up an organised perceptual field and separate an item of information from its background, or can discern "signal" from "noise" (Johnstone, 1991). By contrast, a field-dependent learner is an individual who has difficulty in breaking up an organised field and separating an item of information from its background.

Witkin and Goodenough (1981) also identified two related cognitive restructuring skills that field-independent (FI) persons exhibit more than field-dependent (FD) individuals do. They are:

1. Breaking up an organised complex field into its basic elements.

2. Providing a structure to a field that lacks one, or imposing a different organization on a field which is suggested by its inherent organization.

Moreover, other research evidence indicates that FI students have greater skill in tasks which involve these cognitive restructuring skills (for instance, Frank and Nobel, 1985; Nobel and Frank, 1985; Frank and Keene 1993). On the other hand, Witkin (1978) suggested that, FI/FD cognitive styles are process variables which represent "techniques toward a goal rather than individual ability in achieving goals". In addition, he and Goodenough (1981) suggested that field-dependency could be considered "as a way of processing information from a more complicated field". Indeed this dimension style refers to a way of intellectual functioning, but has little to do with goal attainment.

More recent results have confirmed that FI/FD individuals differ not only in how they process information, but also in the effectiveness of their processing (for example, Davis and Frank, 1979; Davis and Cochran 1989; Frank and Keene 1993). Moreover, results cited by Harmon (1984) indicated that cognitive styles are information-processing habits, representing the learners typical mode of perceiving, thinking, problem solving and remembering.

1.3.2 FI/FD and Stability with Age

It was found by Witkin et al. (1967, 1974, 1977) that the field-dependence/independence cognitive style is stable with age. This means that individual differences in expressions of articulated functioning in one field are related to expressions in other field and will not change for months or years. For example, individuals who are analytical in one perceptual backgrounds tend to be analytical in other perceptual background and problem solving situation as well (Goodenough, 1976).

1.3.3 FI/FD and Short-term Memory

Many researchers including (Pascual Leone, 1970; Case, 1974; Case and Globerson, 1974 and Frank, 1983) suggested that FI/FD individuals differ in the effective use of their working memory (X-space or M-space). These studies confirmed that the efficiency of performance in short term memory tasks is related to field-dependency. In fact, FI students are better in the recall of information stored in the short term memory than FD ones. However, when information load is low the FI and FD learners can not be distinguished.

Case (1974) suggested that working memory is the information-processing function which mediates the performance superiority of FI individuals over FD individuals on complex cognitive tasks.

Literature suggested that the larger the working space of a student, the more likely he/she is to be FI (e.g., Al-Naeme, 1987 and Ziane, 1990). Ziane (1990) found that for a given working space, FI physics students generally performed better than FD students in most cases of problem solving. In another study, Johnstone et al. (1993) suggested that FD learners need more working space to compensate for their field-dependence characteristic. However, there is little variation in performance between high working space, field-dependent students and low working space, field-independent students which is suggested in a research by Johnstone and Al-Naeme (1991). They suggested that students of high working space capacity can afford to devote a small part of their space to irrelevant materials and still have enough capacity spare in tackling the problems.

1.3.4 FI/FD and Global/Analytical way of approach

Witkin et al. (1974, 1977) suggested that learners with an analytical style are more likely to analyze a field when the field is organised or to organize a field that lacks it. By contrast, learners with a global style are more likely to perceive a field as is, without analyzing and structuring it. Therefore, it should be noted that the analytical/non-analytical way of thinking may be the best criterion to distinguish the interests of FI/FD students. Essentially, the FI individuals perceive and process information analytically, while FD individuals do it in a global, holistic and passive way.

The results cited by (e.g., Witkin et al., 1974, 1976, 1977; Frank and Keen 1993) indicated that the theory of field-dependency is concerned with the preference of individuals for analytical or global information-processing. For instance, Witkin and Goodenough (1977) pointed out that FI learners show evidence of greater skills in their cognitive analysis and restructuring than FD learners. They (1981) also noted that individual differences can be conceived as an analytical field approach at one extreme and a global field approach at the other extreme. In fact, the capacity for analysis and structuring of experiences is the core of field-dependence/independence learning style (Witkin et al., 1974).

It may be concluded that, the field-independent students consistently approach backgrounds and tasks analytically, while the field-dependent students approach backgrounds and tasks in a global way, seeing the whole instead of the parts. In addition, Witkin et al. (1974) suggested that "a tendency towards an analytical or global way of experiencing characterizes a person's problem solving activities as well as his perception". It was also found by Witkin et al. (1977) that field-independent students are more likely (than field-dependent students) to give a good performance in problem solving tasks when the solution depends on using an object in an unfamiliar way. Sowder et al. (1985) suggested that the cognitive restructuring aspect of field-dependence/independence is found to be related to problem solving ability. In other words, students with a high score on cognitive restructuring tasks are normally better problem solvers than students scoring low in such tasks.

1.3.5 FI/FD and Concept attainment

In many studies (for example, Dickstein 1968; Nebelkopf and Dreyer 1973; Goodenough 1976 and Witkin et al., 1977), the field-dependent/independent students are compared on their performance in concept attainment tasks. For instance, Dickstein (1968) suggested that concept attainment is more closely related to the field-dependency dimension than to general intelligence. He found that field-independent students demonstrated significantly greater readiness to concept attainment than field-dependent ones.

Therefore, when Z-demands (thought steps in question tasks) in a task involve complex perceptual stimuli, FI students obtain information more efficiently, rely less on guessing with inadequate information and are more ready to accept the irrelevance of concept attributes than FD students. Goodenough (1976) also noted that field-independent learners are generally better than field-dependent learners in concept attainment tasks. Moreover, concepts defined in terms of more salient cues are generally easier to learn than concepts defined in terms of less salient cues and cue salience has more effects on FD than FI students in conceptual learning. Witkin et al. (1977) suggested that cue salience has more effects on field-dependent than field-independent concept learners.

In a review on the relationship between perception and learning-memory, Goodenough (1976) cited that FI students would learn concepts more rapidly when the salient cue is irrelevant to the definition of the concepts. By contrast, FD students may demonstrate greater readiness than FI when relevant cues and attributes are salient. Field-dependent individuals tend not to ignore the irrelevant attributes and nonsalient cues in concept definition.

The major finding of Elkind et al. (1963) was that, FI students score significantly higher than FD ones on a test requiring perceptual concept information. Concept learning tasks also have high demands on working memory. Review of the concept learning literature suggested that the greater effectiveness of FI students was related to their memory efficiency. For example, Davis and Frank (1979) noted that the greater effectiveness of FI students may be related to memory processes employed in concept learning. In addition, results cited by Davis and Frank indicated that, FD learners are often less efficient than FI learners in their concept learning strategies. High information load, greater interference potential and less inherent organization were suggested by Davis and Frank (1979) as factors which contribute to the less efficient memory use of FD learners.

1.3.6 FI/FD students and Abstract materials

It was found, in many investigations, that FI students are interested in the more abstract and theoretical subjects than FD ones. Witkin et al. (1977) suggested that FI learners favour areas which more abstract in their content. Moreover, in a longitudinal study by Witkin and his co-workers (1977) it was found that FI students were more concerned with formal ideas and abstract principles. In another study they (1977) suggested that FI students may be better in their cognitive and structuring skills, in nonsocial and abstract materials than FD students who are better in social and concrete contexts.

Nahinsky et al. (1979) noted that the process of abstraction can be related to the ability of a learner to extract the main features from background in which they are embedded. In addition, it was discussed in this review that field-dependency may be defined as an individual's capacity to break up a complicated field from an embedding background. Therefore, it could be concluded that FI individuals would be better than FD ones in the process of abstraction and nonvisual tasks. The other dimension of cognitive styles in this research (i.e. convergence and divergence) will be discussed in the next section of this chapter.

1.4 Convergent and Divergent Cognitive styles

Getzelts and Jackson (1962) distinguished between the groups of students; the first group was called the "high IQ" learners, who are good at intelligence tests, but relatively weak on the tests of creativity. The second group was labelled as the "high creative students" who are superior in their performance on the tests of creativity open-ended tests, but relatively weak on tests of "intelligence". The concept of convergence/divergence as a cognitive styles was explored by Hudson (1966, 1968). He suggested that the convergence/divergence dimension is a measure of bias, not a level of ability. The two ways of reasoning are called by Hudson "convergent and divergent thinking". He also noted that convergent/divergent thinking may be thought of as polar opposites. A convergent thinker is defined by Hudson (1966) as an individual whose performance on IQ tests is better than his/her performance on open-ended or "creativity tests", while a divergent thinker shows the reverse result. Guilford (1959, 1978) defined convergent thinking as thinking towards one right answer or towards a relatively uniquely determined answer, while divergent thinking is a way of thinking in which a number of ideas will be produced from a given set of information. In other words, a greater variety of answers to each question would more likely be found by divergent thinkers. Convergent thinkers see information as leading to a restricted answer or solution.

For example, in mathematics, if asked: What is the solution set of the equation

$$x^2 - 5x + 6 = 0 \tag{1.1}$$

in the set of real numbers R ? You should answer that the only solution in R is the set $\{2, 3\}$ with no other choice.

In the calculus domain we may be asked to find the value of

$$\lim_{x \to \infty} \frac{[2x^2] + \mathrm{sgn(x)}}{x^2 + |x|}. \tag{1.2}$$

The only solution is 2.

The above examples are typical of problems requiring convergent thinking. In fact, a lot of mathematical tasks are examples of convergent thinking. Although, a variety of responses to stimuli is the unique feature of divergent thinking this does not mean that this way of thinking has no positive role in the process of reaching a unique conclusion. It comes into play wherever there is trial and error thinking.

However, Hudson (1966) rejected the belief of many psychologists that divergent people are potentially creative and convergent people are potentially uncreative. He suggested that convergers are naturally attracted towards one end of spectrum and divergers to the other. Based on this, Hudson (1966) noted that convergent and divergent students use different tactics in dealing with the pressures of work and emotional experience. One tactic is not necessarily better or worse than another, hence its bipolarity makes the convergence/divergence dimension value-neutral, in the sense that each pole has its own characteristic strengths and weaknesses.

Runco (1986) suggested that a divergent learning style is of course not completely synonymous with creative ability. It is just one component of creativity despite the fact that divergent thinking tests are psychometrically reliable and widely employed as estimates of creative potential. Results of Runco's study indicated that divergent thinking and creative performance scores were moderately related in the gifted school children samples, but unrelated in the non-gifted sample.

1.4.1 Convergent/Divergent thinking and Abstract learning

The literature on the convergence/divergence cognitive style (e.g. Hudson, 1966, 1968; Guilford, 1967 and Messick, 1976) suggested that convergent thinkers prefer formal materials and logical arguments. They may show performance superior to divergent thinkers on tasks which are better structured and demand logical ability, while divergent thinkers presumably are better in the more open-ended tasks than convergent thinkers. The convergers enjoy precision and logical conclusions, which the divergers' view as a restriction. Guilford (1967) suggested that generating logical necessities is the critical feature of convergers, whereas generating the possibilities from the given information is the characteristic of divergers.

In addition, Hudson (1966, 1968) found that being highly imaginative is a striking feature of divergent thinking learners. He also suggested that convergent pupils like to keep emotion apart from studies and that divergent ones prefer studies involving emotion. As a result of Hudson's investigation it may be noted that convergent learning style superiority extends not only to the diagrammatic and numerical parts of intelligence tests, but to verbal parts as well.

Nevertheless, there is some disagreement here. For example, Sacks and Eysenck (1977) found no main effect of convergence/divergence cognitive styles on the recognition of abstract and concrete sentences. Convergers performed better than divergers with concrete sentences and vice versa with abstract sentences. Therefore, despite Hudson's findings, Sacks and Eysenck concluded that the convergent thinking process is more appropriate for learning concrete subjects. By contrast, divergent thinking is better for comprehending and learning abstract concepts. Besides, they noted that imaginal processes are more prevalent in convergent thinkers than divergent ones.

Chapter 2

Cognitive styles and Academic achievement

2.1 FI/FD Cognitive style and Science Education

As discussed in the previous chapter, FI individuals may show more interest in analytical-impersonal domains and restructuring, while FD individuals express interest in global-interpersonal fields that, in particular, require social skills.

Witkin et al. (1977) suggested that mathematicians, chemists, biologists, physicists, engineers and artists are analytical-impersonal people. In the academic activities, FI college and graduate students choose specialise in such field as mathematics, science, art, experimental psychology, engineering and architecture. On the other hand, FD students may choose sociology, humanities, languages, clinical psychology, nursing etc.

In a longitudinal study, Witkin et al. (1977) found some evidence that, in general, grades in mathematics and science courses correlated positively with the measures of field-dependency. In addition, shifts out of mathematics and science were particularly common among FD students. Results cited by Witkin and his colleagues (1977) also indicated that FI college students were significantly better than FD ones in the sciences, mathematics, architecture and engineering. Witkin (1976) also noted that FD students avoid mathematical and physical science domains where an analytical approach and skills are necessary, by contrast FI students favour such disciplines.

In another study, El-Banna (1987) found that there is a direct relationship between students' degree of field-dependence/independence and their attainment in chemistry examinations. He suggested that FI students performed better than FD students in all groups of different working space. Ziane (1990) also indicated that field-dependency was found to play an important role in the students success; FI physics students achieved higher scores in solving physics problems than those who were FD in their cognitive style. Finally, Al-Naeme (1991) found that the pupils' field-dependence/independence cognitive style is very important and may play a vital role in chemistry mini-project problem solving procedures. Moreover, he reported the superiority of FI pupils compared with FD pupils in conventional chemistry examinations and mini-projects (1989, 1991).

2.2 Cognitive styles and Mathematics learning

2.2.1 Introduction

It was shown in the previous section that there is a fine mesh of connections between psychological factors such as field-dependent/independent cognitive style and science education. In the reminder of this chapter, the researcher intends to present an outline overview of the field-dependency dimension, convergent-divergent thinking style and mathematical education in the literature. This will provide a route into further research in teaching and learning calculus.

2.2.2 FI/FD and Mathematical learning

Level of abstraction may vary from concrete representations of mathematical concepts to using only symbolic representations, and field-dependency is related to how well students can identify important attributes of concepts. Moreover, changes in the representation of concepts could also be related to differences in cognitive styles (McLeod et al., 1978). The relationship between FI/FD styles and mathematics education can be found in some research as follows:

1. As was mentioned in section (2.1) grades in the mathematics courses correlated positively with the students' scores of FI/FD test (Witkin et al., 1977).

2. In addition, Spitler (1970) found that mathematical achievements are significantly related to FI/FD. Field-independent students appeared to be more aware of the elements of complex stimuli and have more capacity rather than field-dependent students to deal with patterned stimuli in an analytical manner, certainly another capacity demanded in learning mathematics. Research evidence by Spitler also indicated that FI students behaved differently from the FD students in the mathematics class and indicated a higher performance of FI learners than FD ones in mathematical materials. Moreover, mathematics tasks involving three-dimensional questions seemed to demand the field-independent learning style.

3. Threadgill (1979) suggested that FI students achieved significantly better than FD students in mathematics post-test scores.

4. Mrosla (1983) found that low achieving students in high school algebra were mainly FD in their learning styles.

5. It was found by McLeod and Briggs (1980) that field-dependency related to some aspects of discovery learning in mathematics. They also suggested that FI students should be better in the inductive teaching of mathematical concepts, where they have opportunity for discovering concepts by themselves, and FD students may find that the deductive approach to teaching mathematics suits them better.

6. On the level of guidance of mathematics instruction, Adams and McLeod (1979) suggested that FI learners may perform best when allowed to work independently, whereas FD learners would perform best when given extra guidance.

7. McLeod et al. (1978) suggested that the differences between FI and FD cognitive styles are related to the notions of discovery learning in mathematics. They believed that FD students may respond better to the question tasks when more explanation is provided by teachers, while FI students seem to be more adept at working independently and making discovery without much assistance. Another study by Adams and McLeod (1979) confirmed the predicted interaction between field-dependency and discovery instruction in learn-

ing mathematics. They found that FI students were better in mathematics discovery learning and FD students had better performance in the expository method.

8. There is some research evidence that FI students generally appear to be better mathematics problem solvers than FD students. For example, Roberge and Flexr (1983) have shown that elementary school children who are FI scored significantly higher than FD pupils on both mathematics and problem-solving tests. Dugger's study (1984) supported that considering field-dependence/ independence cognitive style in a teaching approach, improved the students' performance in mathematical problem solving. In another investigation, Alsina (1987) found that the FI students solved more mathematics problems properly than the FD ones.

9. Finally, Talbi (1990) concluded that FI/FD learning style could influence the students' performance in mathematics examinations in a such way that FI students perform on average better than FD ones.

2.2.3 FI/FD Cognitive style and Mathematical Anxiety

Mathematical anxiety is a very common problem among students in mathematics courses and its effect on students' performance in class and examination situations can not be ignored by any educator. Therefore, it is interesting to see, if there is any relationship between FI/FD learning styles on this kind of anxiety. Some studies have been carried out by researchers in this realm. For instance, Hadfield (1986) found that cognitive styles have a link with mathematical anxiety, the higher mathematics anxiety occurrence being in the field-dependent group

So far, it can be concluded that this dimension of cognitive style does affect mathematical performance and has implications for mathematics education. The implications could be substantial for a mathematical curriculum, which allowed for the students' cognitive styles.

2.2.4 Some Disagreements

Despite the previous findings that both mathematical ability and mathematics achievement are significantly related to field-dependency there are some disagreements in this area. For instance, results by Eldersveld (1980) suggested that cognitive style of FI/FD was not a discriminator between success and failure with developmental mathematics in the community college, nevertheless the students showed a strong tendency to be field-dependent. Furthermore, major results of Webb's study (1981) in this matter could be summarized as follows:

1. The interaction between choice of academic major and cognitive styles upon the academic achievement in mathematics courses was not significant. In the other words, the matching of one's choice of academic major with cognitive style does not contribute to increased academic achievement. Perhaps the matching of student's choice of academic major with cognitive style as a counselling tool would be of more benefit if the focus were be on longer term vocational choice rather than on success in an academic context.

2. Cognitive styles did not account for any significant course grade variance in mathematics.

2.3 Con/Div style and Science Education

Support for the suggestion that science students are biased towards convergent thinking and that art students towards divergent thinking may be found in the several studies (e.g., Guilford et al., 1965; Hudson, 1966, 1968; Mackay and Cameron, 1968; Field and Poole, 1970; Richards and Bolton, 1971; Sally and Bostack, 1979; Webster and Walker, 1981 and Runco, 1986).

It was suggested by Hudson (1966, 1968) that convergent pupils tended to specialize in the sciences and classics, but divergent pupils in arts, history and modern language. He also found that between three and four times as many convergers do mathematics, physics and chemistry for every one that goes into the arts.

Results cited by Field and Poole (1970) indicated that although the majority of science specialists entering university were convergent thinkers, it is mainly the divergent thinkers among them who finally achieved the better results. In another study Runco (1986) noted that there were particular domains of performance for example art and writing, that were more strongly related to divergent thinking than other areas as music and science.

However, Hudson's views on the convergence/divergence dimension and academic choices received only partial support in a study by Sally and Bostock (1979). Their study demonstrated a clear relationship between arts orientation and divergence cognitive style as distinct from convergence one, before any specialization took place.

Despite Hudson's finding among British grammar school boys, there are however, some disagreements in the relationship between this psychological factor and students' academic development as follows:

1. The finding of an investigation by Webster and Walker (1981) casts doubt on the validity of the previous studies which indicated that arts students are better able to think divergently than science students.

2. Results cited by Field and Poole (1970) suggested that senior Australian undergraduate students who were outstanding in science were mainly divergent thinkers. Furthermore, Mackay and Cameron (1968) in an investigation among Scottish students found that, although the divergent bias was strongly associated with specialisation in arts at the first year of university there was no relationship detected between convergent bias in science specialisation.

3. In a longitudinal study at an Australian university about the relationship between convergence-divergence cognitive style and achievement in arts or science, Field and Poole (1970) noted that, while convergent bias was associated with more high level students' passes in the first year study, there was no difference in the relative success of convergent students in the second year. However, they found that there was a relationship between students choice of faculty (arts or science) and their Con/Div learning style in agreement with

Hudson's finding (1966) in this domain.

4. Al-Naeme (1991) suggested the important role of convergence/divergence style in chemistry mini-projects problem solving, with the superiority of divergent thinking over convergent thinking in such tasks. However, he found that FI/FD learning styles were better predictors than Con/Div learning styles of success in tackling the mini-project problems in chemistry.

An important question could be considered in this point. Is there any discipline in which students could cope equally well with a convergent or a divergent bias? Orton (1992) suggested that biology, geography and economics are subjects which do not fall into just one dimension of divergent thinking or convergent thinking. It seems only a minority of learners may cope well with convergent and divergent styles at the same time. It may be also reasonable to note that the nature of many mathematical tasks indicate that students should cope well with convergent and divergent thinking in the problem solving situations. In fact, at the beginning of a solution they need to think openly and converge step by step to the necessary solution.

2.3.1 Con/Div Thinking and Mathematics learning

In the domain of individual differences and mathematical learning, some significant questions should be discussed by mathematics educators. They could be as follows:

1. Are only convergent students attracted to study mathematics, or will divergent students be attracted as well?
2. Are our mathematical curricula based on a convergent or divergent bias or is a healthy balance considered?

Orton (1992) stated that if most mathematics students are predominantly convergers, does this imply that few specialists in mathematics are capable of creativity or inventiveness? If we accept that divergent thinking is completely synonymous with creative ability or just one component of creativity, do Hudson's findings suggest that the only creative mathematics students are the minority who are divergent thinkers. Or as mathematics educators, should we make allowance for both convergent and divergent learning styles in our students' performance in classroom?

Kempa and McGough (1977) suggested that students with arts bias (divergers) tend to prefer the verbal communication mode in learning mathematics, whereas students' mathematical bias are found to be strongly associated with performance in the symbolic communication mode and anti-performance for the verbal mode. On the other hand, preference for the symbolic method of presenting mathematical discussions significantly correlates with mathematical bias and mathematical achievement. And Guilford et al. (1965) noted that the abilities involving symbolic information were connected with mathematics.

From the combination of the above suggestions may emerge the conclusion that the most students with a mathematical bias seem to be predominantly convergent thinkers rather than divergent thinkers who are inclined towards the art bias and tend to prefer the verbal way of learning mathematics instead of symbolic mathematical communication.

In another study, Richards and Bolton (1971) found that teaching approaches which foster divergent thinking will produce minimal beneficial effects on children's performance on tests of mathematical ability. This means that divergent thinking style played only a minor role in mathematics achievement. However, Guilford et al. (1965) suggested positive correlation between divergent thinking and learning mathematics.

2.4 Cognitive styles and Calculus learning

Calculus is made up of a relatively sophisticated mathematical topics with many interesting and important applications in other disciplines. Because of its sophistication, it is not an easy subject to master, however due to its widespread application, an increasing number of students are expected to study this domain of mathematics (Bartle and Tulcea, 1968).

On the other hand, it has been discussed in detail earlier that cognitive styles, in general, do predict individual differences in learning mathematics and science. Therefore, it seems reasonable to conclude that, assisting students to identify their own cognitive styles may facilitate calculus understanding and enhance students' achievement.

Despite the finding of significant correlation between cognitive styles and learning mathematics, there is some evidence that in the case of learning calculus somewhat surprising results emerge. The researcher will have a brief look at the literature to see whether the cognitive styles relate to achievement in college calculus students and also to see if cognitive style is a predictor of success in calculus.

Harmon (1984) conducted a correlation study between achievement in calculus and cognitive style. Based on Harmon's findings, although students have different cognitive styles, the cognitive style neither prevents nor predicts achievement in the first year of calculus learning. Moreover, there was no statistically significant difference between measures of achievement in calculus for FI/FD learners as determined by Witkin's Group Embedded Figure Test (GEFT). The Pearson's Product Moment Coefficient (0.24) for Harmon's investigation implied that the relationship between calculus scores and measures of (GEFT) was low. However, Harmon suggested that cognitive styles are useful in different aspects of calculus study. A teacher who is knowledgeable needs to be aware of the cognitive style that best facilitates learning the particular calculus concepts being taught.

Ross (1981) also found a lack of significant correlation between field-dependency and any of Mathematics 1A (including calculus) measures. He suggested that homogeneity of the Mathematics 1A students (e.g., science, mathematics or engineering majors) accounted for this insignificant correlation. However, Ross found a weak but statistically significant correlation between FI/FD cognitive style and post-test instructor score in the predicted direction. This means that students who were relatively FI students would tend to perform better than those who were relatively FD in a particular domain of calculus which is called "calculus word problems" (CWP). The fact that field-dependency did correlate with performance in solving calculus word problems in Ross's study, but did not correlate with overall performance in (Mathematics 1A) may support the notion that CWP played a very special role in the calculus course.

More discussion will be found in later chapters of this study about students' difficulties in complex word problems in calculus and that FI/FD learning style could be an important variable in students' performance in this part of the calculus activities.

In addition to the previous investigations, Sheel (1981) also suggested that FI students were not superior to the FD students in the initial achievement and retention of the selected calculus concepts. The concepts used in Sheel's research consisted of introductory differentiation rules for finding the derivative of the product and the quotient of two differentiable functions, and the extended power rule.

2.5 Teacher-Student Interaction in Mathematics class

2.5.1 FI/FD Style and Teacher-Student Interaction

There is some evidence that a match between teachers' and students' cognitive styles has a positive role in the teaching and learning behaviour. In mathematics courses, teachers may do better with students with the same cognitive styles and students may learn effectively when taught by teachers matched to them in the cognitive style (Witkin et al., 1977). Field-independent students may exhibit superiority in mathematics and science course if teachers who teach these subjects are likely themselves to be field-independent.

It was suggested by Witkin and Goodenough (1977) that students with FI or FD cognitive style are different in their interpersonal behaviour. Field-dependent students have an interpersonal orientation. They show strong interest in others and prefer to be physically close to people. By contrast, FI students have an impersonal orientation. They are not very interested in others and avoid psychological and physical closeness to people.

On the other hand, research evidence by Witkin et al. (1977) suggested that FD teachers are more student-centered in their behaviour and teaching approach, whereas FI teachers emphasize teachers' standards. Moreover, FD teachers use questions primarily to check on students' learning following instruction, but FI teachers tend to use questions as instructional tools in introducing topics. As a result, it seems that teacher-student matching in this dimension of cognitive style may facilitate their interactions for meaningful understanding.

2.5.2 Con/Div style and Teacher-Student Interaction

In another domain of individual differences, i.e. convergence-divergence cognitive style, does matching of teacher and student matching style improve the instructional situations in a mathematics class? Do students with a convergent bias learn best from a convergent teacher and divergent students from a divergent teacher?

From a literature review on this area, the researcher concluded that further study should be done to discover more about this dimension style and teacher-student matching. However, Joyce and Hudson (1968) noted that divergent students who were taught by convergent teachers tended to be poor in examinations, but students performed significantly better in some examinations if they were taught by divergent teachers. Nevertheless, it does not always seem to be true that convergent students learn better from convergent teachers, and divergent students from divergent teachers.

Chapter 3

Calculus and Education Policy in Iran

3.1 Introduction

Formal education in Iran begins at seven years of age with children spending five years in primary school. They then spend three years in middle or guidance school which is followed by four years of secondary education.

In the current system, secondary education or high school is further divided into many branches which are mainly labelled as mathematics-physics, experimental science, human science, economic and technological skills. Other types are also available which lay emphasis on the teaching of subject-matter and teacher training. The national curriculum is common to each type of school and the same text books are used in all parts of the country.

Elementary mathematics begins immediately from the first year of schooling and the mathematical curriculum and syllabuses in each section are the same. Mathematics content and therefore calculus syllabuses in high school are prescriptive in the sense that the hours spent in each topic is predetermined.

For the first eight years of schooling, great emphasis is given overall to inquiry skills and problem solving. At this level, the method of teaching is definitely based on an intuitive mode and books are built on intuitive and informal understanding. Lessons are designed to include opportunities for pupil involvement and topics should be combined with teacher exposition and pupils' activities in the classroom (work in the class).

3.2 Calculus in High school

In the current system, calculus begins as a main part of the mathematical curriculum in the third year of high school (age 16–17) in the mathematics-physics branch. This continues for two years until the end of fourth year of high school education.

3.2.1 General Aims

It must be mentioned that pupils of the mathematics-physics branch only will be able to continue their study at higher education levels in pure and applied mathematics, physics, statistics, computer science, mathematical teacher training and engineering. Therefore, the general aims of teaching and learning calculus have to satisfy the needs of all pupils intending to specialize in a subject at the university with significant mathematical content. The calculus syllabuses in high school ensure that pupils have the necessary knowledge and skills in the calculus area that are required for the next stage.

3.2.2 General Learning objectives

1. Pupils should be able to know and work with calculus concepts and improve their mathematical skills in the scope of calculus language, notations and symbol manipulation.

2. In calculus, pupils' learning isn't always left in the intuitive state, but they learn to cope with the formal mathematical language and some logical proofs. As a result, pupils should pay attention step by step to more abstract and concise forms.

3. Pupils should become familiar with the main ideas of calculus and practise all the skills covered in the textbooks and classroom. They must acquire skills in problem solving and be able to produce them, under examination conditions.

4. Pupils should recognise the link between different parts of the two years of calculus.

3.2.3 General approach of Teaching

As a result of the learning objectives in high school, the tendency in calculus teaching is clearly going from the intuitive state to the more logical and abstract. Concepts associated with inductive and deductive reasoning should be taught and teachers must encourage pupils to develop their mathematical skills.

The Method of Teaching

The teaching is built upon an expository and deductive method and instruction is also based on a common time-table and predetermined textbooks for all the country. Teachers normally follow the same syllabus order as in the books and don't follow an individualised approach in teaching and order of materials, but they use their classroom experiences.

Pupils often engage in tasks and have a chance to receive instruction from their teachers. They work as individuals on their tasks and the teacher will check them in the class to correct the pupils' conceptual misunderstandings and skill difficulties.

Computers and calculators have no role as a tool in the teaching and learning, nonetheless there is a computer centre available in each high school.

3.2.4 Calculus in Action

In the current calculus materials, there is no emphasis on calculus applications to everyday life beyond school. Therefore, pupils will learn calculus concepts and skills almost without examples and exercises from the real world and physical reality.

Abstract concepts will be taught in the second year of calculus learning and teachers help pupils to improve their abilities at the formal level. Moreover, pupils normally are not engaged in any mathematical investigation tasks, however in private and special high schools pupils may carry out some investigations which are closely tied to the learning of calculus materials.

3.2.5 Evaluation and Assessment

Assessment is divided into internal and external examinations. In each academic year at least three internal assessments will be set up mostly in written form by the appropriate teacher. But external assessment is established for the last year of schooling, which is also in written form. Each pupils' performance will be reported as a mark from zero to 20 and the necessary score to pass examinations successfully is at least 10 out of 20.

3.3 3rd and 4th years of High school calculus

3.3.1 Section 2. Functions (Third year)

Contents and Teaching order

1. Definitions: function, domain, range.
2. Function arithmetic: sum, product and quotient; composition of two functions.
3. Odd, even and periodic functions; greatest-integer function $[x]$ and its graph.

3.3.2 Section 1. Function Review (Fourth year)

Contents and Teaching order

1. Real function (recall and supplement); surjective and injective function.
2. Monotonic function.
3. Operation on real functions; composition of more than two functions.
4. Inverse function and its properties; graph of inverse function.
5. Sketching the graphs of the curves $y = \sqrt[n]{x}$.

Aims

1. To understand the concept of a function and be familiar with elementary functions and their graphs.
2. To study more about functions in depth.

Learning objectives

After two years of function teaching pupils should be able to:

1. Find the maximal domain and image set of a given function.
2. Find the rule of composition of two or three given functions and recognize the related domain and range.
3. Determine whether or not a given function is odd, even, periodic, injective or surjective.
4. Find the inverse of a function (if it exists).
5. Sketch the graph of an inverse function with the curve of the form $y = \sqrt[n]{x}$ (without finding the criteria for an inverse function).
6. Determine whether or not a function is increasing or decreasing.

The Method of Teaching

Teaching method in the 3rd year of "function" study has an informal approach, but in the 4th year it tends to be formal and some complicated function such as the greatest integer function $[x]$ and step functions are introduced, in spite of pupils' difficulties with them.

3.3.3 Section 4. Limit and Continuity (Third year)

Contents and Teaching order

1. Variables tending to zero or a constant a; variables tending to ∞ or $-\infty$.
2. Definition of limit (informal and formal approach with ϵ, δ); definition of right hand-lefthand limit (informal approach).
3. Rules for finding limit (without proofs).
4. Infinite limit and limit at infinity (informal approach).
5. Finding limits of indeterminate forms $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \times \infty$; $\lim_{x\to 0} \frac{\sin x}{x} = 1$.
6. Continuity of a function at a point (informal approach).

3.3.4 Section 2. Limit and continuity (Fourth year)

Contents and Teaching order

1. Definition of limit (more about formal definition); limit at infinity and infinite limits (formal definition).
2. Proof of some theorems on limits of functions.
3. Right/left hand limit in the homographic function ($f(x) = ax + b/cx + d$).
4. Definition of continuity of a function at a point; right/left continuity; discontinuity cases; some theorems of continuity; continuity of a function on an interval.
5. Statement of "If f is continuous on $[a, b]$ and strictly increasing then f^{-1} is also continuous and strictly increasing".

Aims

To study and understand the basic concepts of limit of a function and continuity with informal and formal definition in two successive years.

Learning objectives

After two years of teaching limits and continuity, pupils will be expected to able to:

1. Know the informal and formal meaning of limit and continuity of a function.
2. Find by formal definition (ϵ, δ) the limit of some elementary functions.
3. Determine the limit and continuity of a function by using rules.
4. Use the result of $\lim_{x \to 0} \frac{\sin x}{x} = 1$ to evaluate such limits as $\lim_{x \to \pi/4} \frac{1 - \tan x}{4x - \pi}$ or $\lim_{x \to a} \frac{\sin(x - a)}{x^2 - a^2}$.
5. Find right/left hand limit of a function by definition.
6. Discuss the continuity and discontinuity of a real function.
7. Evaluate problems such as $\lim_{x \to \infty} x \sin \frac{1}{x}$.
8. Solve problems such as $\lim_{x \to \infty} \frac{3x + 2}{2x - 4} = 3/2$ and $\lim_{x \to 3} \frac{-1}{(x - 3)^2} = -\infty$ by formal definition.

The Method of Teaching

In the 3rd year of high school pupils will be familiar, for the first time, with the basic concepts of limit and continuity in calculus. These concepts, in particular, continuity are introduced at this stage by an informal approach and fairly intuitive methods. Nevertheless, there is a trend from informal understanding and intuitive teaching to formal learning and logical proofs. Therefore, formal definition of these concepts, will be taught in 4th year of high school. However, there is more emphasis on the development of pupils' mathematical skills.

3.3.5 Section 5. Differentiation (Third year)

Contents and Teaching order

1. Increment of a variable and a function.
2. Derivative of a function, i.e.$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$; other notation for derivative.
3. Velocity of a mobile (physical interpretation of derivative).
4. Rules for derivatives; derivatives of trigonometric functions; derivative of composite function; higher order derivatives.
5. Geometric interpretation of derivative (slope of tangent line to a curve).
6. Equation of tangent line to a curve (at a point on it); normal line to a curve (at a point on it).
7. Tangent line to a curve (from a point not on curve); normal line to a curve from a point not on curve.
8. Angle of line and curve; angle between two curves.
9. State of L'Hôpital's rule and its use.
10. Definition of increasing and decreasing functions; maximum and minimum value of a function.

11. Application of derivative in determination of functions variations and maximum/minimum points; concavity and point of inflection (a brief discussion); curve sketching.

The following items are taught in 4th year of high school as revision and a supplement to 3rd year learning of differentiation processing.

3.3.6 Section 3. Differentiable Functions (Fourth year)

Contents and Teaching order

1. Derivative at a point (recall); geometric interpretation of derivative (recall); points, at which there isn't a derivative.
2. Relationship between derivative and continuity of a function.
3. Right/left hand derivative (informal definition); derivative on an interval (function of derivative).
4. Determination of derivative by rules.
5. Derivative of $Z = (g \circ f)(x)$ and $H = (f \circ g)(x)$.
6. Derivative of inverse function and theorem $g'(y) = 1/f'(x)$, when $y = f(x)$ is a differentiable function on $[a, b]$, $f'(x) \neq 0$ and $x = g(y)$; notation $(f^{-1})' = 1/f' \circ f^{-1}$; derivative of inverse trigonometric functions.
7. Extremum points of a function; direction variations of derivative (determination of extremum in technical problems).
8. Concavity and points of inflection (state of theorem of concavity).

3.3.7 Section 4. Asymptote lines (Fourth year)

Contents and Teaching order

1. Behaviour of function at the boundaries.

2. Vertical/horizontal asymptote.

3. Inclined asymptote and the method of finding it.

4. Curves asymptotic with respect to each other.

3.3.8 Section 6. Curve Sketching (Fourth year)

Contents and Teaching order

1. Graph of functions; $y = \frac{ax+b}{cx+d}$, $(\frac{a}{c} \neq \frac{b}{d}, c \neq 0)$ and $y = \frac{ax^2+bx+c}{dx+e}$, $(a \neq 0, d \neq 0)$ and $y = \frac{ax^2+bx+c}{dx^2+ex+f}$, (at least $a \neq 0$ or $d \neq 0$).

2. Graph of irrational and trigonometric functions.

Aims

1. To understand and be familiar with the definition of the derivative of a real function and its evaluation.

2. To study the basic properties of differentiation and to improve mathematical skills in differentiation.

3. To understand the relationship between continuity and differentiation.

4. To define the asymptote lines to a curve.

5. To understand the main factors which have important roles in curve sketching.

Learning objectives

After these sections of differentiation teaching, students should be able to:

1. Find the derivative of a function (trigonometric and non-trigonometric) by definition and rules.

2. Discuss the existence of a derivative of a function at a point.

3. Determine the derivative of the inverse of a function (trigonometric and non-trigonometric).

4. Recognize the local maximum/minimum points and the points of inflection and concavity.

5. Find the higher order derivative of a given function.
6. Use the idea of extremum in some practical problems (maximisation & minimisation).
7. Determine the equation of tangent and normal lines to a given curve.
8. To follow the procedure given leading to a sketch of the graph of polynomial, rational, irrational and trigonometric functions. On the completion of this part, pupils will be expected to improve their curve-sketching abilities.
9. Recognize the relationship between continuity and differentiation of a function.
10. Determine the asymptote line for a given function curve.

The Method of Teaching

As a result of aims and objectives in this area, teaching tendency is clearly going from intuitive to formal understanding and teaching style is based on exposition and deductive method. The teaching of differentiation begins by introducing increments of a variable ($\Delta x = x_1 - x_2$) and a function ($\Delta f = f(x_2) - f(x_1) = y_2 - y_1$). Then the formal definition of derivative i.e. $f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta) - f(x)}{\Delta x}$ will be introduced.

After this, the physical interpretation of derivative and geometric interpretation of derivative will be given (derivative as a slope, gradient, of tangent line to a curve $y = f(x)$ at a point x_0). In the 4th year, pupils study the 3rd year topics in more detail and have to learn materials in depth and with formal discussion. For example, topics about the existence of derivative at a point and derivative of an inverse function are taught rigorously. There is no emphasis on application of the derivative in the real world, while the calculator and computer as tools of teaching and learning are not used.

3.3.9 Section 7. Differential and Integration (Fourth year)

Contents and Teaching order

1. Definition of the differential ($dy = f'(x)\,dx$); geometric interpretation of dif-

ferential; rules for differentials.

2. Definition of the primitive function; the indefinite integral; some rules for integration; geometric interpretation of the constant of integration.

3. Area under curve and the theorem "$S'(x) = f(x)$, $\forall x \in [a,b]$, where S is the area under the curve and f is continuous on $[a,b]$ and $f(x) > 0$". And $S = \int_a^b f(x)\,dx = F(b) - F(a)$.

4. Integration of trigonometric functions $\sin u, \cos u$ and so on, where u is a function of x.

5. Evaluation of $\int_a^b \sin^n x\,dx$, $n \in N$ and so on.

6. The area between a curve and x-axis; the area between two curves; volume of revolution; $v = \int_a^b \pi f^2(x)\,dx$.

Aims

1. To understand the meaning of the differential of a function and to know that integration is the reverse of differentiation.

2. To study the connection between integral and the area under a curve.

3. To understand the concepts of indefinite and definite integrals and some of their properties.

Learning objectives

Pupils should be able to:

1. Evaluate the primitive function (indefinite integral) of a given function.

2. Find the differential of a function.

3. Evaluate the definite integral and indefinite integral of trigonometric and non-trigonometric function.

4. Evaluate the area between a curve and the x-axis and to find the area between two curves.

5. Recall the formula $v = \int_a^b f^2(x)\,dx$ and calculate the volume of revolution.

The Method of Teaching

As a result of aims and learning objectives, a brief discussion of integration is introduced and the greater detail is left for calculus in higher education. However, the formal approach is considered rather than informal one in teaching of differential and integral and there is no discussion about application of the definite integral in the real world and in pupils' everyday life.

3.4 Calculus and Higher Education in Iran

3.4.1 Purpose of the Course

1. Calculus is a main section of the mathematical area, intended to provide one and a half years of useful mathematical education to students whose interest lies in higher mathematics alone or mathematics closely related and combined with another subject.

2. There are three courses which are called "General Mathematics (1,2,3)". The calculus content is presented as a main area of these courses such that more than 90% of topics of General Mathematics are in calculus. Therefore, in this research General mathematics are taken to be the same as Calculus (1,2,3).

3. Calculus (1,2,3) are essential for students considering honours mathematics (either as a single subject or combined with another discipline) and for mathematics teacher training.

4. For engineering students, Calculus (1,2) are compulsory including nearly the same topics as Calculus (1,2,3) in the mathematics branch, but with less emphasis on rigorous proofs.

3.4.2 General Aims

1. The overall aim of these three courses of calculus is to provide a solid foundation for topics which are explored in much greater detail and depth in the subsequent mathematical courses, in particular, in mathematical analysis, topology and so on.

2. To instil an appreciation of the beauty and depth of calculus through detailed study of proofs and key features.

3. To provide an introduction to a number of major areas of calculus rigorously and in more depth than high school calculus.

4. To proceed to provide training for those who wish to make a career either in mathematics or in a field where ability in calculus and its knowledge and techniques are required.

3.4.3 General Learning objectives

By the end of one and a half years of teaching and learning calculus, students are exposed to and are required:

1. To understand the importance of rigour in mathematical reasoning and calculus learning.

2. To be able to think more logically and analytically.

3. To become familiar in depth with all the basic calculus concepts.

4. To practise in all the mathematical skills and transferable skills (or communication skills such as clarity and succinctness in a written and oral argument) covered in these courses.

5. To be able to use their conceptual understanding and skills through problem solving.

6. To note the links between different sections of these three calculus courses.

7. To be able to reproduce, under examination, specified proofs, and to exhibit their logical abilities in various aspects of the course components.

8. To gain insight into the ways in which calculus is used in a variety of applications.

9. To be accustomed to standards of rigorous argument in General Mathematics.

3.4.4 General approach of Teaching

As a result of the overall learning objectives of calculus in higher education, naturally the teaching approach is clearly based on abstraction and more mathematical rigour. Nevertheless, there is a fair balance between the presentation of calculus in a rigorous approach and that from an intuitive and skills development point of view.

3.4.5 The Method of Teaching

The method of teaching is normally based on a didactic approach and chosen from the lecturers' experience. The predominant teaching mode has traditionally been the hour lecture—talk and chalk— associated with problem solving classes. In addition to the help provided in lectures and tutorials, assistance for students is available from lecturers during their office hours.

The teaching used in introducing a topic is combined with teacher exposition, discussion and deductive methods. Ideas of direct and indirect proof which began to develop at high school are now developed more rigorously. Teaching has more emphasis on illustrations of concepts associated with inductive and deductive reasoning, therefore lecturers should ensure that students can with them.

During the lecture, students have an opportunity to discuss materials with the lecturer and there are sometimes many discussions in the class. The lecturer often encourages students to take part in the academic discussion during the teaching.

Calculator and Computer

Calculators and computers have no role in the teaching and learning and lecturers normally don't encourage their students to use these instruments in calculus courses. However, it seems to the researcher that computers may have a considerable role to play in teaching and learning calculus concepts.

The use of computers may help students to reinforce their meaningful understanding. But, they can use these tools sensibly and under controlled conditions and lecturers should be aware that a wrong use may damage mathematical thinking in students. Therefore "how and when" are two significant conditions in this situation. Nowadays, attempts are under way at a few universities to use computers as a stimulus and a tool for reinforcing and for investigations in calculus courses.

3.4.6 Problem solving and Tutorials

The problem solving classes take nearly the same form at each university. Two hours are devoted for this purpose and students take part in the class as a whole. Question tasks are predetermined and will be solved by the lecturer or tutor and some students, however some of them do not try to find a solution before the class. Therefore, they don't think about the questions and just write the solutions from the board.

Sometimes tasks are determined for students as individuals or for small working groups. Students must solve the problems and hand them in before the class and after correction by lecturer or tutor, suggested solutions will be discussed in the tutorial. This method is extremely costly in time, and is impossible for crowded classes. Moreover, students who wish to seek additional assistance are welcome to talk to their lecturer privately.

3.4.7 Course Organization

The academic year at Iranian universities is divided into two terms each of seventeen weeks. Each calculus course includes (68–102) hours of lectures, i.e. four to six lectures per week. In addition, students will take part in two hours of tutorial per week, therefore in total they will have (6–8) hours of calculus per week. Classes are not large and normally will be between (45–55) students and, therefore class conditions are conducive to better teaching, learning and discussion.

3.4.8 Recommended books and Notes

Each mathematical department will decide the necessary textbooks for the calculus course. Nonetheless, some lecturers individually recommend their favourite book and in many cases use their own notes in the class. The following books are the most recommended for higher calculus education.

1. The Calculus with Analytic Geometry, by L. Leithold.
2. Calculus and Analytic Geometry, by G.B. Thomas and R.L. Finny.
3. Calculus with Analytic Geometry, by R.E. Johnson and F.L. Kiokemeister.
4. Calculus with Analytic Geometry, by R. Ellis and D. Gulick.
5. Calculus with Analytic Geometry, by R.A. Silverman.
6. Calculus, by J. Marsden and A. Weinstein.

3.4.9 Evaluation and Assessment

There will normally be two examinations, midterm and final examinations, which are marked from zero to 20 and students must obtain at least a score of 10 out of 20 in order to be able to pass the calculus course successfully. Some lecturers consider a percentage of the total examination marks in terms of tutorial work and problem solving, however passes will not be given to students who score less than 50% of the total and they must repeat the course. In addition, regular attendance at lectures is required.

3.4.10 Course content and Teaching order

The general guidelines and strategies of the course content are suggested by the "Mathematical Planning Committee in the Ministry of Culture and Higher Education". However, at each university and higher education institute the department of mathematics has the main responsibility for determining calculus course materials and teaching order. Syllabuses are nearly the same as shown below with a few changes which may happen in some departments. The order of teaching is based on lecturers' experiences and the recommended textbook, and are substantially as set out below with some minor changes.

3.4.11 Calculus 1

General Aims

This is a first-level course in higher education calculus, presenting fundamental topics which are of use to all students intending to specialize in mathematics to honours level. As well as learning calculus, students will be exposed to conceptual understanding and logical arguments.

3.4.12 Section 1. Real and Complex numbers

Aims

To study the basic properties of real and complex numbers.

Contents and Teaching order

1. Real number system, cartesian coordinates and modulus properties.
2. Neighbourhood and deleted neighbourhood.
3. Triangle and Cauchy-Schwarz inequality for inner product $|x.y| \leq |x||y|$.
4. Polar coordinates and complex numbers; operations on complex numbers; geometrical representation of complex numbers; root of unity; polar representation of complex numbers; De Moivre's formula.

Learning objectives

After this section students should be able to:

1. Describe what is meant by real and complex number systems.
2. Use the operation on complex numbers in problem solving.
3. Use the notation for neighbourhood and deleted neighbourhood in R.

4. To state the triangle inequalities in both geometric and algebraic form and use them to solve problems.

3.4.13 Section 2. Functions

Aims

To study in depth more about the basic concept of function and to recall some related properties.

Contents and Teaching order

1. Definition of single variable function (domain and range).
2. Operation on functions (sum, product and composition).
3. Even, odd, periodic and bounded functions.
4. Important functions (polynomial, fractional, step, sign, greatest integer and transcendental functions).
5. Injective and surjective functions (formal definition).

Learning objectives

After this section students should have ability to:

1. Find the domain and range of a given function and composition functions.
2. Determine whether or not a given function is injective or surjective.
3. Use the formal definition of function, injective or surjective functions in problem solving.

3.4.14 Section 3. Limit and continuity

Aims

To introduce a precise definition of limit and continuity of a real function and to study the main properties of these two important concepts.

Contents and Teaching order

1. Limit of a real function; problems of the type, "For a given $\epsilon > 0$, find $\delta > 0$ so that $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$"; one-sided limits (formal definition).

2. Some theorems on limits and the squeeze theorem.

3. Infinite limits; vertical asymptotes.

4. Limit at infinity; horizontal asymptotes.

5. Continuity of a function; problems of the type, "For a given $\epsilon > 0$, find $\delta > 0$ so that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$".

6. Points of discontinuity.

7. Continuity of a composition function; continuity of a trigonometric function; continuity of f on an interval.

8. Basic theorems on continuity; Bolzano theorem for continuous function; preservation property of continuous function (theorem).

9. Geometric interpretation of the mean-value theorem; mean-value theorem for a continuous function.

Learning objectives

On the completion of this section students will be expected to:

1. State rigorous definitions concerning important concepts such as limit of a function, its continuity or discontinuity.

2. Be able to determine the points at which a function has limit or is continuous by both (ϵ, δ) definition and rules.

3. State the meaning of the Bolzano theorem.

4. Know and use the mean value theorem in connection with a continuous function.

5. Reproduce the proof of theorems and rules.

6. Know the meaning of asymptote lines in connection with limit at infinity and infinite limit.

The Method of Teaching

As a result of the overall aims, course content and objectives of this section of higher education calculus, the teaching of limit and continuity and their properties tends to be formal with precise and logical proofs which are followed in the other sections. For example, rigorous definitions of limit and continuity of a real function (definition by ϵ, δ) which were introduced in high school, will now be discussed in more detail and depth. This may be a good opportunity for a lot of students who have non or misunderstanding of these calculus materials to correct themselves.

3.4.15 Section 4. Differentiation and Anti-differentiation

Aims

1. To study and understand the motivation given for the limiting process involved in differentiation.

2. To know the connection between differentiability and continuity of a function.

3. To learn various techniques for, and applications of differentiation.

4. To understand what is meant by anti-differentiation.

Contents and Teaching order

1. Definition: tangent and normal line, derivative of a real function; one-sided derivative (formal definition).

2. Differentiability and continuity.

3. Derivative on an interval $[a, b]$.

4. Chain rule and other rules on differentiation; implicit differentiation.

5. Extrema of a function (relative and global); theorem on relative max/min and its inverse; state of the theorem of extrema; critical numbers (points).

6. Monotonic functions.
7. Rolle's theorem and mean value theorem; Cauchy's mean value theorem.
8. Higher order derivatives.
9. The monotonicity theorem; first and second derivative tests for relative extremums; concavity and test for concavity; point of inflection.
10. Application of derivative; extremum and related rates problems.
11. Derivative applications in approximation of equation root.
12. Geometric and physical application of derivative.
13. Rate of change.
14. Definition of differential and related rules.
15. Inverse function and its derivative and inverse function theorem; relationship between derivative of a function and the derivative of its inverse; inverse of the trigonometric function and its derivative and graphs.
16. Asymptotes in a curve sketching.
17. Anti-differentiation, rules and techniques; indefinite integral.

Learning objectives

On the completion of this section students will be expected to:

1. Be able to determine the points at which a function is differentiable.
2. Recognize the difference between continuity and differentiability and their relationship.
3. Find the derivative of a function defined implicitly.
4. Determine higher order derivatives and prove by induction a formula for derivatives of arbitrary order.
5. Prove the chain rule and other theorems.

6. Determine the critical numbers (points) of a function and identify intervals where it is monotonic.
7. Solve problems involving maximisation, minimisation and related rates of change.
8. State and prove Rolle's theorem and the mean value theorem and apply them in a variety of contexts.
9. Find the derivative of the inverse of a given function.
10. Recognize the meaning of integration as anti-differentiation and indefinite integral.

3.4.16 Section 5. Integration

Aims

1. To define and study the basic properties of indefinite and definite integrals.
2. To introduce some numerical methods for finding roots of equations and evaluating the definite integrals.

Contents and Teaching order

1. Indefinite integral (recall).
2. Reimann sum, area under a curve and definite integral.
3. Integral of continuous and piece-wise continuous function.
4. Properties of the indefinite integral.
5. Mean-value theorem for integrals.
6. The fundamental theorems of calculus (first and second).
7. Approximation rules (Trapezoidal and Simpson's rules).
8. Some techniques of integration.

Learning objectives

After this section students must be able to:

1. Evaluate a simple definite integral using a Riemann sum.
2. Decide on a method and techniques of integration suitable for a given problem.
3. Use the mean value theorem for integrals to prove problems.

The Method of Teaching

As a result of the materials, aims and learning objectives of both sections on derivative and integral, teaching tends to be based on abstraction and mathematical rigour. Nevertheless, there is a fair balance between the presentation of the differentiation and integration processes from a rigorous approach and from an intuitive and skill development point of view, in particular, in the integration section.

3.4.17 Calculus 2

General Aims

This is a second course in higher education calculus. In this course students will study some important materials such as integrability (methods and applications) and a precise definition of sequences and series.

Prerequisites

Students will be expected to understand the Calculus 1 and a satisfactory performance is required in its examination.

3.4.18 Section 1. Integration (Method and Application)

Aims

1. To establish the connection between integration and the area under a curve.
2. To study and develop methods of integration and to use integration to find volumes of revolution areas and arc lengths.

Contents and Teaching order

1. Area of region in a plane.
2. Application: volume of a solid of revolution, circular-disk and circular ring methods, cylindrical shell revolution.
3. Arc length for graphs and parametric curves.
4. More about techniques of integration.
5. Integration of $\sin^n x$ and $\cos^n x$ etc.
6. Integrals yielding inverse trigonometric functions.
7. Hyperbolic functions and their inverses.
8. Integrals yielding inverse hyperbolic functions.
9. Improper integrals.
10. Taylor's formula, Maclaurin's formula (integral and lagrange form of remainder $R_n(x)$).
11. Taylor series and Maclaurin series.
12. Cauchy's mean value theorem.
13. L'Hôpital's rule (formal proof).

Learning objectives

After this section students should be able to:

1. Decide on a method and technique of integration suitable for a given problem.
2. Recall the formula for volume and area of revolution, and arc length, and evaluate them for a given curve.
3. Recall the definition of the hyperbolic functions, their properties and integrals.
4. Recognize an improper integral and determine whether or not the integral exists.

5. Recognize the indeterminate forms and use of L'Hôpital's rule in these situations.

6. Determine the Taylor and Maclaurin's formula of a given function and determine the Taylor and the Maclaurin series of a given function.

7. Use Taylor formula to prove L'Hôpital's rule.

8. Apply the Taylor formula in problems involving limits processing such as ($\lim_{x\to 0}\frac{\sin x}{x}=1$, $\lim_{x\to 0}\frac{e^x-1}{x}=1$, $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x$).

The Method of Teaching

It seems that the integration section of Calculus 2 is taught in depth and there is more emphasis on teaching methods and techniques of integration to improve students' skills rather than conceptual understanding and mathematical thinking.

3.4.19 Section 2. Logarithms and Exponentials

Aims

To define and study the basic properties of the logarithmic and exponential functions.

Contents and Teaching order

1. Logarithmic and the natural logarithmic functions.

2. Logarithmic differentiation and integral yielding the natural logarithmic function.

3. The natural exponential function.

4. Logarithmic and exponential function with the base of arbitrary positive number $a \neq 1$.

Learning objectives

After this section students must be able to:

1. State the definition of the logarithmic and exponential functions (domain, range and rule).

2. Use properties of logarithms and the relationship between the logarithmic and exponential functions.

3. Differentiate functions involving logarithms and exponentials.

4. Roughly sketch the graphs of logarithmic and exponential functions.

5. Recall and use the standard integrals involving logarithms.

3.4.20 Section 3. Sequences and Series

Aims

To study sequences and series and to develop techniques for studying them.

Contents and Teaching order

1. Sequences: convergent, divergent, monotonic and bounded sequences.

2. Infinite series of constant terms, harmonic and geometric series, hyperharmonic series (p series i.e.$\frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p} + \cdots$, p is a constant).

3. Series: comparison test, limit comparison test, integral test.

4. Alternating series and its test.

5. Absolutely convergent and conditionally series.

6. Ratio and root tests.

7. Power series, integral and radius of convergence.

8. Differentiation and integration of power series.

9. Taylor, Maclaurin series and the binomial series.

Learning objectives

After this students will be expected to:

1. Determine whether or not a given sequence is convergent or divergent and find its limit if it exists.

2. Be able to use the appropriate test to prove that a given series is convergent or divergent.

3. Find the interval and radius of convergence for a power series.

4. Reproduce Taylor's and Maclaurin's theorems and use them in problem solving situations.

5. Recognize under what circumstances one can differentiate or integrate a power series and reproduce the rigorous proofs.

3.4.21 Calculus 3

General aims

This is a final course of calculus in higher education. During the course, students will study more advanced materials with a rigorous approach and logical arguments. This calculus course is covered with fundamental topics such as functions of more than one variable, limit, continuity, differentiation and integration.

On the completion of the course, students will be expected to have the necessary insight in formal understanding, ability in mathematical skills and calculus applications. They must acquire the ability and enough prerequisites for their mathematical activities in the future. It should be noted, that this course of calculus is not investigated in the present study, therefore only a brief review will be presented.

Prerequisites

Students will be expected to have a satisfactory performance in both Calculus (1,2).

3.4.22 Section 1. Functions of more than one variable and their derivatives

Aims

1. To extend the concept of a function to functions of two or more independent variables.
2. To extend and study functions of two or three variables of the basic concepts, limit of a function, continuity and derivative.

Contents and Teaching order

1. Vector-valued functions and parametric equations.
2. Calculus of vector-valued functions.
3. Length of arc.
4. Unit tangent and unit normal vector.
5. A function of n-variables.
6. Graphs of a function of two or more variables.
7. Composition of functions, its domain and range.
8. Limits of functions of two variables.
9. Accumulation point.
10. Continuity of a function of two variables.
11. Partial derivatives.
12. Differentiability and total differential.
13. The chain rule for functions of many variables; the chain rule for functions defined on surfaces.
14. Higher order partial derivatives.
15. Gradients and Directional derivatives.

16. Tangent planes and normal lines to surfaces.
17. Applications of partial derivatives; critical points.
18. Sufficient conditions for differentiability.
19. Linear approximations and increment estimates.
20. Second derivative test.
21. Exact differential.
22. Obtaining of a function from its gradients and exact differentials.

Learning objectives

On the completion of this course, students must be able to do the following:

1. To find the domain and range of a given multi-variable function and draw a sketch of the graph.
2. To find the domain and rule for the composition of two given functions.
3. To use the definition of limit to determine whether or not a given function of two variables has a limit at a given point.
4. To determine that a function of two variables has a limit by use of the limit theorems.
5. To use the definition and theorems of continuity of a function of two or three variables and discuss the continuity of a given function.
6. To find a partial derivatives of a given function.
7. To find the differential and total differential of a given function.
8. To state and reproduce proof of the chain rule and to use it in problem solving.
9. To determine the higher order partial derivatives of a given function by definition.
10. To find the gradient and directional derivatives for a given function.

11. To determine an equation of tangent plane and equations of the normal line to the given surface.

12. To find extrema points for functions of two variables.

13. To determine a function from its gradient and exact differentials.

3.4.23 Section 2. Multiple integration

Aims

After this section students should be able:

1. To extend the definition of the definite integral of a function of single variable to a function of several variables.

2. To establish the properties of the double and triple integrals of a function of two variables.

3. To study and understand the physical and geometrical applications of multiple integrals similar to those for single integrals.

Contents and Teaching order

1. Definition of the double integrals and its properties; an iterated integral.

2. Evaluation of double integrals and iterated integrals; the double integral in polar coordinates.

3. Areas, moments and centres of mass.

4. Average value of a function in space.

5. Definition of the triple integral and its properties; the triple integral in cylindrical and spherical coordinates.

6. Application of double and triple integrals in physical and geometrical problems.

Learning objectives

On the completion of this section students will be expected to:

1. Know what is meant by a double integral and how to interpret one as the volume of a solid.
2. Evaluate a double integral by expressing it as iterated integral; evaluate an iterated integral by changing the order of integration.
3. Make a general change of variables in a double integral using the Jacobian.
4. Find double integral in polar coordinates.
5. Find the area of region enclosed by curves.
6. Know the meaning of a triple integral and how to evaluate, a triple integral in cylindrical and spherical polar coordinates.

3.4.24 Section 3. Introduction to the Calculus of Vector fields

Aims

To define and understand the ideas of vector fields, line integrals, surface integrals and their properties.

Contents and Teaching order

1. Definition: vector field, gradient vector field, potential function, divergence of a vector field.
2. Laplace's equation.
3. Definition of line integrals and surface integrals.
4. Smooth curves and sectionally smooth curves on an interval.
5. Line integrals, independent of the path.
6. Green's theorem in the plane.

7. Surface area and surface integrals.

8. Gauss's Divergence theorem in the plane; Stokes's theorem in the plane.

Learning objectives

On the completion of this section, students will be expected to be able to:

1. Find a potential function and prove the given vector field is conservative.

2. Know what is meant by a line integral and evaluate it.

3. Show that the line integral is independent of the path.

4. State and reproduce the proofs of Green's theorem, Stoke's theorem and Gauss's divergence theorem.

5. Use Green's theorem to evaluate a line integral.

6. Use Gauss's divergence theorem and Stoke's theorem in physical problem solving.

Chapter 4

Calculus and Education Policy in Scotland

4.1 Introduction

The heterogenous structure in secondary education in the U.K. is also present in the mathematical curriculum and its teaching methods (Porter, 1993), however the Scottish system is homogenous and centralized. In Scotland, as in all areas of the U.K., schooling is compulsory until the age of 16 but, for pupils going into higher education, the usual age at the end of secondary education is 17 or 18.

Calculus teaching, as a main part of the mathematical programme in secondary education, begins at age 16–17. It continues until the end of age 18 for those who do not leave school at the age of 17. The examination (at age 17) is known as "Higher Grade" and the Higher Grade certificate is sufficient for entrance to higher education in Scotland. A picture of the educational design for study of calculus and school examinations in the Scottish system is set out in Table 4.1 below.

Table 4.1

Calculus study & Scottish educational system

Subject	Age	Year code	Examination
Non Calculus	12-13	S_1	No exam
	13-14	S_2	No exam
	14-15	S_3	Standard Grade course
	15-16	S_4	Standard Grade exam
Calculus	16-17	S_5	Higher Grade course Higher Grade exam
Calculus	17-18	S_6	Sixth Year Studies course Sixth Year Studies exam
Calculus	17 (or 18)	S_5 (or S_6)	University Entrance

4.2 Calculus in Secondary Education

4.2.1 General Aims

At Higher Grade a different emphasis is given to particular aims compared with Standard Grade at age 16. In other words, pupils will be expected to study mathematical materials in broader and deeper ways. It is desirable for pupils to have some insight into the structure and power of mathematical thinking. They should also be familiar with the value of mathematical knowledge in the area of calculus.

On the completion of a one year calculus study, pupils will be expected to have the necessary knowledge and skills for the next stage, in particular, in higher education (university and college). This course also encourages pupils to have a thorough understanding of some basic calculus concepts and provides an opportunity for them to become familiar with the application of these ideas in the development of technology and in connection with real life.

4.2.2 Learning objectives

According to the "Scottish Examination Board (SEB) arrangements in mathematics (in and after 1987)", at Higher Grade more attention can be paid to the more abstract and concise forms, in particular, Pupils should be able:

1. To understand and work with the language and notation of calculus.
2. To have the ability to interpret a general statement or formula of calculus material to obtain information about special cases.
3. To build sequences of a greater range of algorithms to solve a problem.
4. To master concepts associated with inductive and deductive reasoning at this stage.
5. To develop ideas of proof which began at Standard Grade more rigorously, however pupils will not be expected to reproduce set proofs in the external examination.
6. To gain a broad view of the uses and applications of calculus to the world in which they live.
7. To work cooperatively with others towards a common goal in calculus content during problem solving and in investigatory tasks.
8. To develop their confidence in manipulating calculus symbols.

4.2.3 General approaches to the Teaching of Calculus

The following guidelines and strategies have been issued by the SEB for Higher Grade in mathematics and describe the general approaches to teaching calculus at age 17, upon which teachers could base their course planning and methods of teaching to meet the learning objectives. These points are as follows:

1. The teaching and learning approaches which were recommended for Standard Grade, should be continued in this calculus course.

2. Calculus topics and lessons should be designed to include opportunity for pupil involvement.

3. Pupils should be familiar with the processes of exploring new ideas under their teachers' guidance and with problem solving in small groups, which are useful to help pupils gain an understanding of new topics.

4. Pupils also should have opportunities to carry out investigations which are not so closely tied to learning new content. These investigations must provide pupils with experience of applying the mathematics they already know to solve calculus problems or problems having a context in real life or other disciplines.

5. The investigations used in introducing calculus topics should be combined with teacher exposition and discussion.

6. Illustrations should be given of "indirect proof"; in other words, teachers should place emphasis upon proof by contradiction and the use of counter-examples rather than direct proof.

7. Pupils should have opportunities to discuss the tasks, problems and their solutions in small groups, and teachers clarify their misunderstanding and guide them.

8. Calculators are recommended both for processing data and allowing pupils to explore a new situation numerically.

9. Computers also may be used both as tools and as aids to learning because computers can have a considerable role to play in teaching and learning of calculus, in particular, in sketching the graph of a function.

4.2.4 Classroom reality

Based on the researcher's interviews with some mathematics teachers and class observation the following items emerged for calculus courses in Scotland:

1. In teaching calculus there is mainly emphasis upon exposition, deductive methods and routine consolidation. Time-table limitations do not allow teachers to follow the inductive method or allow time for discussion and investigation.

2. At this stage, teachers predominantly use whole class teaching and learning, but this can mean different things in different schools and even for different teachers within the same schools. Teachers normally follow an individualised approach as a result of their knowledge and experiences. Nevertheless, they try to follow the basic guidelines of the Scottish Examination Board in the calculus content and teaching styles.

3. In calculus as well as other mathematical areas at Higher Grade, emphasis is placed upon informal and pictorial teaching. Understanding processes and many examples are clearly related to everyday life and physical reality. A greater emphasis is placed, in fact, on the relationship between pupils, calculus and environment rather than concepts, abstraction and logical arguments.

4. There is an open tendency to place materials and instruction in an intuitive position and, therefore materials are often left without the necessary emphasis on theorems and formal understanding. There is a reason behind this, in that the majority of pupils will continue their study in subjects not closely related to mathematics. However, there is a serious point that this majority and pupils who wish to proceed to a single or combined subject involving mathematics to honours level, will have some difficulty with calculus in higher education. They are afraid of abstract concepts and formal definitions. This defect is obviously appearing in the first year of learning calculus at university.
   It seems that much more emphasis on intuitive and pictorial mathematical teaching and learning could be one of the main factors contributing to pupils' difficulties in calculus at the next stage. There is, factually a weak balance between concrete learning and formal understanding in secondary calculus education policy in action.

5. Calculus in secondary education is a foundation for calculus learning in higher education and there is some correlation between topics in both stages, in particular, in Sixth year.

6. When pupils are encouraged to make use of calculators and computers, teachers believe that numerical investigations allied occasionally with graphical display, form a powerful tool for calculus experiments and for reinforcing complicated concepts. Therefore, there is much more emphasis on using these

instruments as a tool for better understanding and investigations. However, in this position it seems that, no healthy balance between the use of a machine and doing it with conceptual understanding may be the cause of pupils' weakness in mathematical thinking and skill abilities in higher education.

7. Pupils do not often engage in tasks as a whole class and usually have less chance to receive individual help from their teacher (Triadafillidis, 1994). They are encouraged to correct their tasks themselves by using answer-books.

8. Most British teachers are patient, tolerant and encouraging towards their pupils. They use all kinds of methods to make their teaching interesting (Dawei, 1992).

4.2.5 Course content, Textbook and Teaching order

Course Content

Mathematical syllabuses and their calculus content are determined in secondary education by the national guidelines set out by the Scottish Examination Board. However, schools have the freedom to shape them according to their particular points of view.

Textbooks

The majority of secondary schools and teachers prefer to use books called "Mathematics in action 5S and 6S" for Higher Grade and Sixth year calculus courses respectively. Many sections of these books have a discussion part which consists of a pertinent question to be discussed and solved by pupils while receiving guidance and aid from their teachers. The main purpose of such discussions and questions is basically to open a window to a new mathematical topic. However, a few teachers are still continuing to use old mathematics books in teaching calculus.

Teaching order

Teachers normally follow the same order as textbook topics, nonetheless they may make some changes in this order as a result of their approach and experience of teaching and learning calculus. It should be mentioned that the following materials are built into the Higher Grade calculus course.

4.2.6 Sections 1. and 3. Differentiation

Aims

1. To define and study the derivative of a function and understand the reason given for the limiting process involved in differentiation.
2. To study the role of derivative and its properties in curve sketching.

Contents and Teaching order

1. Graphs and Gradients.
2. Average rate of change.
3. Average gradient of a curve.
4. Rate of change.
5. Gradient at a point on a curve.
6. The meaning of terms; limit, differentiation at a point, derivative of $f(x)$ and notation $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.
7. Differentiation over an interval; derived function; some elementary and useful rules; other notation.
8. Equations of tangents.
9. Increasing and decreasing function; stationary points and value.
10. Curve sketching (forms of $y = ax^2 + bx + c$ and $y = ax^3 + bx^2 + cx + d$).
11. The derivatives of $\sin x$ and $\cos x$.
12. The Chain rule for differentiation.
13. Maximum turning point (value); minimum turning point (value).
14. Horizontal point of inflection.

15. Maximum and minimum values on non-closed intervals.

16. Optimization (maximisation/minimisation); mathematical models involving derivatives.

Learning objectives

After these two sections on differentiation, pupils should be able:

1. To know the motivation given for the limiting process involved in differentiation.

2. To determine the derivative of some elementary functions.

3. To use the rules concerning the derivative, know and use the chain rule.

4. To find the gradient of tangent to a curve $y = f(x)$ at $x = a$.

5. To know how to determine whether a function $f(x)$ is increasing or decreasing.

6. To find with guidance the points on a curve at which the gradient has a given value.

7. To determine the stationary points (values) of $y = f(x)$.

8. To recognize the nature of stationary points (value) of $y = f(x)$.

9. To sketch the curve $y = f(x)$ with some prompting to determine critical features.

10. To determine, with guidance, the greatest and least values of a function on a given interval.

11. To know what is meant by horizontal point of inflection and have to find one.

12. To apply the concepts of maximum/minimum to the solution of different practical (and theoretical) maximisation and minimisation problems.

The Method of Teaching

The general approach to the teaching of differentiation at Higher Grade is often based on intuitive methods with real life examples and exercises. Nevertheless, abstract and concise forms are not ignored. For instance, as a first step in the formal definition of derivative of a function, the concept of gradient at a point on a curve $y = f(x)$ is introduced. The derivative definition is taught by considering a tangent at a point P with this formula

$$f'(x) = \lim_{Q \to P}(m_{PQ}) = \lim_{Q \to P} \frac{y_Q - y_P}{x_Q - x_P} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

At the next stage, some standard rules are taught without logical proof and the material is normally introduced with a brief abstract discussion. However, topics are often left at an intuitive and elementary mathematical level. Computers and calculators are used as tools for better teaching and learning, in particular, in sketching a curve $y = f(x)$ and allowing pupils to explore a new situation.

4.2.7 Section 2. Functions

Aims

To study the concept of function, its graph and properties.

Contents and Teaching order

1. The meaning of the terms function, domain, codomain and range of a function; notation and graphs.

2. Modelling with functions.

3. Inverse of a function.

4. Composite function and its notation.

5. Formula for an inverse function.

Learning objectives

After this section pupils are expected to be able to:

1. Be fluent in associating functions with their graphs and vice versa.
2. Draw from the graph of $f(x)$, for example, the graph of $f(x+3)$ or $-f(x)$ or $f(x)+2$.
3. Know the terms domain and range of a function.
4. Know the meaning of inverse function and conditions under which a function has an inverse.
5. Find the formula for the inverse of a given function.
6. Know the meaning of composite function and determine the formula for a composite function.

The Method of Teaching

Based on aims, topics and learning objectives, the teaching method is clearly left as an informal and intuitive approach, but application of functions in the real world has been introduced in this section.

4.2.8 Section 4. Integration

Aims

1. To study the meaning of the terms integral, integrate, indefinite integral, definite integral and constant of integration.
2. To understand the link between differentiation and integration.
3. To study some applications of the definite integral.

Contents and Teaching order

1. The meaning of anti-derivative; the meaning of the terms integral and integrate.
2. If $f(x) = F'(x)$ then $\int f(x)\,dx = F(x) + c$; constant of integration.

3. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$, if $n \neq -1$.

4. Link between differentiation and integration.

5. Rough statement of the fundamental theorem of calculus, and area under a curve.

6. Definition of indefinite integral, limits of integration.

7. The area between a curve and the x-axis (calculation), area between curves.

8. The integrals of the functions defined by $f(x) = px^n$, $f(x) = (px+q)^n$ for all rational n, except $n = -1$, $f(x) = p\cos qx$ and $g(x) = p\sin qx$, and of the sum or difference of such functions.

9. Definite integral in action.

10. Modelling with differential equations.

Learning objectives

After this section pupils should be able to:

1. Recognize that if $f(x) = F'(x)$ then $\int f(x)\,dx = F(x) + c$ and $\int_a^b f(x)\,dx = F(b) - F(a)$.

2. Solve equations of the form $dy/dx = f(x)$ for suitable $f(x)$.

3. Find the area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$.

4. Determine, with guidance, the area bounded by two curves.

5. Evaluate simple indefinite and definite integrals of trigonometric and non-trigonometric functions.

The Method of Teaching

As a result of the aims and learning objectives, a brief and simple discussion of integration processing is introduced and more detail is left for the Sixth year course.

There is an obvious tendency to improve students' mathematical skills in evaluating some integrals which are not complex. In addition, more emphasis is laid on application of the definite integral, in the real world and in physical situations.

4.2.9 Section 6. Exponential and logarithmic functions

Aims

To define and study the properties of the exponential and logarithmic functions.

Contents and Teaching order

1. Growth and decay functions; the exponential and logarithmic functions.
2. Properties of logarithms; logarithms and the experimenter.

Learning objectives

On the completion of this section students are expected to be able to know:

1. What is meant by a growth and decay functions.
2. That $a^y = x$ if and only if $\log_a x = y$, $a > 0, a \neq 1$, $x > 0$.
3. That the domain of $f : x \to \log_a x$ is $\{x : x > 0, x \in R\}$.
4. That $f : x \to \log_a x$ is defined only for $a > 0$ and $a \neq 1$.
5. That $\log_a(uv) = \log_a u + \log_a v$; $\log_a(u/v) = \log_a u - \log_a v$; $\log_a u^h = h \log_a u$.
6. $\log_a u = \log_a u / \log_b a$.
7. That $\log_a a = 1$: $\log_a x = 1/\log_x a$; $\log_a 1 = 0$.
8. That $\log_a x = \log_a y$ if and only if $x = y$.

4.3 Calculus and Higher Education in Scotland

4.3.1 General purpose of the Course

It must be emphasised that, this research will cover only the calculus courses in the Mathematics Department at Glasgow University. The calculus course is a basic part of the mathematics area, essential for students considering honours mathematics (either as a single subject or combined with another). It is also intended for students who require a mathematical course for use with arts or science subjects.

The course is split up into many course components which run concurrently throughout the two years. These course components normally cover a part of Mathematics 1A and 2A or Mathematics 1B and 2B. Hence, throughout this research these courses will be labelled respectively as Calculus 1A, Advanced Calculus 2A (Calculus 2A), Calculus 1B and 2B. It should be mentioned that the present study is based only upon calculus content of Mathematics 1A at Glasgow University.

The independent first year-course Mathematics 1B, from which it may be possible in exceptional circumstances to progress to honours, covers much of the same material as Mathematics 1A in a less theoretical way and is most suitable for students who require one year of mathematics for use with arts or science subjects. It is permitted to transfer from Mathematics 1A to 1B up to the middle of the second term.

4.3.2 General approach of Teaching Calculus

As a result of the overall aims and learning objectives of the courses, the teaching tendency is obviously different in Calculus 1A and 2A from that in Calculus 1B and 2B. The teaching approach in Calculus 1A and 2A, in particular, is based on a concise and formal approach. Nevertheless, some ideas such as limit and continuity are still left at an informal level and rigorous definitions are introduced in Mathematics 2A (analysis part) in the second year. Lecturers believe that students cannot cope with them in the first year of mathematical learning.

On the other hand, in Calculus 2A, there is much more emphasis on developing students' mathematical techniques than on conceptual understanding and logical discussion. Theorems are normally taught without formal proof and are just intended for use in problem solving.

In Calculus 1B and 2B, the teaching style used in introducing the material is at an informal and intuitive level without logical discussions. There is much more emphasis on teaching techniques and skill ability rather than conceptual learning.

4.3.3 The Method of Teaching

Teaching methods normally follow lecturers' individual styles and they decide on the basis of their beliefs and experience, however the predominant teaching method has traditionally been one hour lecture—talk and chalk—with supporting tutorials. The teaching is a combination of lecturers' exposition and deductive methods. Lecturers use their own notes in the classroom and students just copy material from the blackboard without any discussion on whether they can understand or not.

For each course an essential textbook is chosen and lecturers' notes are mainly based on the topics and structure of the recommended book, but shorter.

There is little discussion, traditionally, between lecturer and students during the lectures and students do not like to ask questions in the class. However, some lecturers encourage students to ask about their non or misunderstanding. Some lecturers believe that, it is a serious problem if there is no balance between students' writing and meaningful learning in the class. All calculus lectures are crowded and large with the average number of students more than eighty, but tutorial classes are very good opportunities to support the lecturers' teaching and further the students' learning in the course. It seems that the main objectives of tutorial work are as follows:

1. The work of a week's lectures should be consolidated, therefore the one hour session would consist of discussion of new topics and the material of the week's lectures should be reviewed.

2. Students should be instructed how to apply the rules, theories and skills in a problem solving situation.

3. Students should be tested on the week's tasks and this is accomplished by completed assignments that the students should have handed into the tutor.

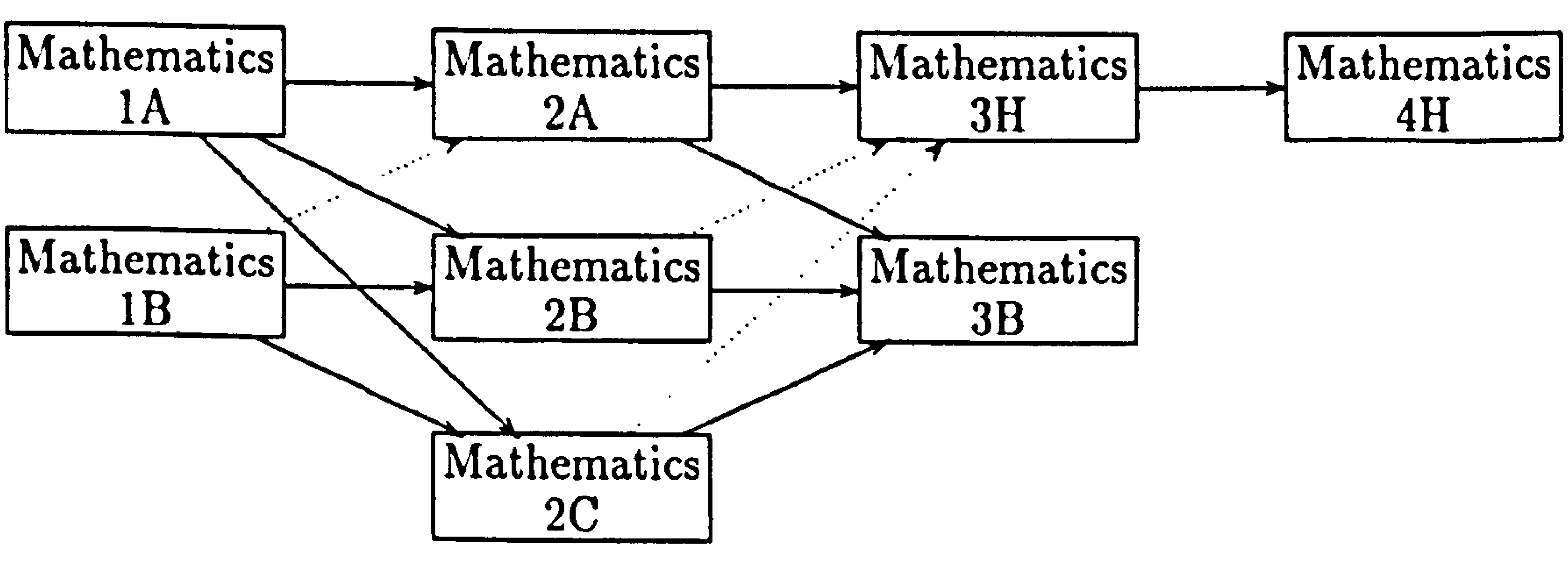
4. Correction of students' misunderstandings and difficulties in learning and problem solving.

4.3.4 Calculus 1A

General aims

This is a first course in calculus, presenting basic material which is of use to all students intending to specialize in a subject with significant mathematical content. In fact, the normal route to honours in mathematics (either as a single subject or combined with another) begins with course 1A. It is essential for students intending to study mathematics to honours level. In addition, it provides a self-contained one year course in mathematics for students intending to specialize in a non-mathematical subject. A diagram showing the progress to honours over four years is as follows (course 1A, Department of Mathematics, Glasgow University, 1995–96).

Pathways to Honours in Mathematics



KEY: ——→ normal route,
· · · · → possible route, but only under special circumstances.

General Learning objective

As well as learning mathematical techniques, students will be exposed to, and asked to reproduce, proofs of some of the results that are presented. It is for this reason that the course is both essential for intending honours students in mathematics and less suitable for students only interested in learning techniques. On the completion of this course students are expected to handle abstract concepts, mathematical skills, problem solving and communication skills.

Lectures and Tutorials

The calculus section in course 1A has two one-hour-lectures per week and, in a normal weak, there are two lectures for each course component in Mathematics 1A. The predominant teaching method is traditionally the one hour lecture—talk and chalk—associated with tutorials. Moreover assistance for students is available from lecturers during their office hours.

There is a one hour tutorial every week for all students. Work will normally be set for this by each lecturer during the preceding week. These tutorials take two forms which students have in alternate weeks as follows:

- Small group

All students will be assigned to a small group and müst hand in a small amount of the tutorial work. This will be corrected by their tutor and returned at the tutorial.

- Large group

Half of each section meets in the usual lecture room and have tutorial work corrected by the lecturers and a number of other tutors.

Computer laboratories

The computer has no role in teaching and learning Calculus 1A, however, there is a chance for students to take part in a voluntary series of computer laboratories giving an introduction to the mathematical package Maple.

Course text

An essential textbook for the course is the book:
Fundamentals of university mathematics, by C. M. McGregor, J.J.C. Nimmo and W.W. Stothers.

In addition, other books that may be used as a source of others examples include:

1. Single-variable, by Adams.
2. Schaum Outline Series—Calculus (3rd Edition), by Ayres and Mendelson.
3. Calculus, by Hunter.
4. Calculus: A First Course, by Moore.

Assessment

A student may pass Mathematics 1A (Calculus 1A) by obtaining one of the following:

1. A pass in the June degree examination or the September (resit) degree examination which consists of two papers each of two and a half hours. One paper includes "calculus topics" and another "algebra and geometry". Candidates who obtain 50% overall with at least 40% in each paper will gain a pass. In addition, there are two class exemption examinations in January and April which will be on material covered in the first and second terms respectively.

   To gain a class certificate, and so qualify to sit the degree examination at the end of session, students must fulfil all of the following:

   - Attendance at both class examination,
   - 30% of the total marks over two class examinations,
   - A satisfactory attendance at the lecturers and tutorials.

2. An Exemption.
   Students who obtain 70% or more of the total marks from both class examinations, with at least 60% in the second examination will be eligible for an exemption from sitting the degree examination. Students with a little less than these requirements may be also eligible. Eligible students must sit, and

give a satisfactory performance in, a short exemption test covering material taught in the third term so that the exemption can be confirmed.

4.3.5 Section 1. Functions (5 lectures)

Aims

To understand the basic concept of a function.

Contents and Teaching order

1. Definition: function, domain, codomain, image.
2. Operations on functions: scalar multiplication, sum, product, quotient, composition.
3. Elementary functions: zero, polynomial, rational power, modulus, trigonometric.
4. Boundednes: bounded sets and bounded functions.
5. Bijections, inverse functions, inverse trigonometric functions.

Learning objectives

After this section students should be able to:

1. Find the maximal domain and image set of a given function.
2. Find the domain, codomain and rule for the composition of two given functions.
3. Recognise bounded sets and bounded functions.
4. Determine whether or not a given function is injective or surjective.
5. Determine the inverse of a given function when it exists.
6. State the definition of the inverse trigonometric functions (including domain, codomain and rule).
7. Evaluate or simplify expressions involving inverse trigonometric functions, e.g. $\sin^{-1}(\sin 7\pi/5)$, $\sin(\cos^{-1} 4/5)$ and $\tan^{-1} 1 + \tan^{-1} 2$.

The Method of Teaching

The teaching approach is formal in introducing the concept of function, inverse function and related properties. However, some complicated functions such as the greatest integer function $[x]$ (integer part of x) and step function are not discussed in this section.

4.3.6 Section 2. Limit and Continuity (3 lectures)

Aims

To study the basic properties of limits and the concept of continuity for real functions.

Contents and Teaching order

1. Limit of function (informal definition); one-sided limits; properties of limits (some rules without proof).
2. Definition of continuity (informal definition); continuity on an interval; properties of continuous functions (rules).
3. Extreme value theorem (without proof).
4. Intermediate value theorem (without proof) and its use to prove the other theorems.
5. Approaching infinity: limit at infinity and horizontal asymptotes; infinite limit and vertical asymptotes.

Learning objectives

After this section students should be able to:

1. Determine the limit of a real function at a given point.
2. Use the result $\sin x/x \to 1$ as $x \to 0$ to evaluate the limit of e.g. $\sin 3x/5x$ as $x \to 0$.
3. State the definition of continuity for a real function.

4. Prove, using properties of limits, that a given function is continuous.

5. Determine the points where a real function, defined by cases, is continuous or discontinuous.

6. State the intermediate value theorem.

7. Use the intermediate value theorem to determine the closest integer approximation to a root of an equation.

The Method of Teaching

Despite the reality that the teaching approach is essentially based on mathematical rigour, the definition of limit and continuity are left at an informal level. Formal definition of them (ϵ, δ-definition) will be introduced in Mathematics 2A in the second year. Moreover, theorems and properties of limit and continuity are mainly left without logical proofs. The researcher, in interviews with lecturers of Calculus 1A, found that students in this stage cannot cope with formal definitions.

4.3.7 Section 3. Differentiation (13 lectures)

Aims

To define differentiability of a function and to learn various techniques for, and application of, differentiation.

Contents and Teaching order

1. Definition: first principles; one-sided derivative (informal definition).

2. Properties: product rule, quotient rule.

3. Differentiability and continuity.

4. Implicit differentiation; the chain rule.

5. Standard derivatives: polynomials, trigonometric functions and inverse trigonometric functions.

6. Higher derivatives.

7. Critical points, local and global extrema; the second derivative test.
8. Rolle's theorem and mean value theorem; monotonic functions and the monotonicity theorem.
9. Rate of change: velocity, acceleration, related rates.
10. Parametric equations $x = x(t)$ and $y = y(t)$.

Learning objectives

On the completion of this section students will be expected to be able to:

1. State the definition of differentiability for a function.
2. Determine whether or not a given function is differentiable.
3. Obtain, from first principles, the derivative of a simple function.
4. Prove that differentiability implies continuity and know that the converse is false.
5. Find the equation of a tangent or normal to a given curve.
6. Prove the chain, product and quotient rules.
7. Recall the standard derivatives of rational powers and the trigonometric and inverse trigonometric functions.
8. Determine the derivative of a given function using the chain, product and quotient rules and knowledge of the above standard derivatives.
9. Determine the derivative of a function defined implicitly.
10. Find higher order derivatives and prove by induction formula for derivatives of arbitrary order.
11. Determine the critical points on a given curve and establish their nature.
12. Determine the global extrema of a function.
13. State Rolle's theorem and the mean value theorem.

14. Use the derivative of a function to determine whether it is strictly increasing or decreasing or constant.

15. Express velocity and acceleration as derivatives and hence solve simple "dynamics" problems.

16. Solve problems involving related rates, e.g. find the rate of change of volume of a sphere given the rate of change of radius.

17. Understand the nature of a curve defined by parametric equations $x = x(t)$ and $y = y(t)$ and determine dy/dx and dy^2/d^2x.

4.3.8 Section 4. Integration—fundamental (4 lectures)

Aims

To establish the connection between the "anti-derivative" and the area under a curve.

Contents and Teaching order

1. Antiderivative/indefinite integral: some standard integrals, elementary properties.

2. Definite integral: area under a curve, Riemann sum, elementary properties, simple examples.

3. The fundamental theorem of calculus.

4. Improper integrals: definitions and simple examples.

Learning objectives

After this section students should be able to:

1. Recall the standard integrals and use them to determine or evaluate simple indefinite or definite integrals.

2. Evaluate a simple definite integral using a Riemann sum.

3. Recognize an improper integral and determine whether or not the integral exists.

The Method of Teaching

As a result of the overall aims, content and learning objectives of the both sections of differentiation and integration, teaching methods tend to be formal with fair mathematical rigour. Nonetheless, an emphasis can be seen on students' skills development, despite the presentation of differentiation and integration from an abstract approach, in particular, in the integral section.

4.3.9 Section 5. Logarithms and Exponentials (4. lectures)

Aims

To define and study the basic properties of the logarithmic and exponential functions.

1. The logarithmic function: definition as a definite integral, properties, logarithmic differentiation, standard integrals involving logarithms.
2. The exponential function: definition as $\log^{-1}$, properties.
3. General real powers: definition, properties.
4. Hyperbolic functions: definition, properties; inverse hyperbolic functions.

Learning objectives

After this section students should be able to:

1. State the definition of the logarithmic function (domain, codomain and rule).
2. Roughly sketch the graph of the logarithmic and exponential functions.
3. Use properties of logarithms and the relationship between the logarithmic and exponential functions to simplify expressions.
4. Differentiate functions involving logarithms and exponentials.
5. Recall and use the limit $\log x/x \to 0$ as $x \to \infty$ to determine other limits involving logarithms and exponentials.
6. Recognize when the use of logarithmic differentiation might be appropriate and utilize it.

7. Recall and use the standard integrals involving logarithms.

8. Solve simple problems involving exponential decay.

9. Use the definition $x^r = \exp(r \log x)$ to determine derivatives of general real powers.

10. Recall the definitions of the hyperbolic functions and their elementary properties.

11. Solve simple equations involving hyperbolic functions.

4.3.10 Section 6. Integration—Methods and Applications (8 lectures)

Aims

To develop methods of integration and to use integration to find volumes, areas and arc lengths.

Contents and Teaching order

1. Methods of integration: substitution, partial fractions, trigonometric integrals, the t-formulae, integration by parts.

2. Applications: volumes of revolution, arc length for graphs of functions and parametric curves, areas of revolution.

Learning objectives

After this section students should be able to:

1. Decide on a method of integration suitable for a given problem.

2. If appropriate, decide on a substitution and use this to determine or evaluate an indefinite or definite integral.

3. If appropriate, use partial fractions/t-formulae/integration by parts to determine or evaluate an indefinite or definite integral.

4. Recall the formulae for volume and area of revolution, and arc length and evaluate them for a given curve.

The Method of Teaching

Based on the content, aims and learning objectives, the teaching approach has more emphasis on developing skills and various techniques of integration rather than mathematical thinking and conceptual understanding.

4.3.11 Section 7. Ordinary differential Equations (4 lectures)

Aims

To define and study methods for solving simple ordinary differential equations.

Contents and Teaching order

1. First-order: separable, homogeneous and linear equations.
2. Second-order: linear equations with constant coefficients.

Learning objectives

After this section students should be able to:

1. Recognise a separable/homogeneous/linear first-order ordinary differential equation and find its general solution by the appropriate method.
2. Recognise a linear second-order equations with constant coefficients, recognise whether it is homogeneous or inhomogeneous and find its general solution.
3. Determine constant(s) in a general solution which fit initial conditions.

4.3.12 Section 8. Numerical Methods (4 lectures)

Aims

To introduce some numerical methods for finding roots of equations and evaluating definite integrals.

Contents and Teaching order

1. Roots of equations: bisection method, Newton's method.

2. Integration: Simpson's rule.

Learning objectives

After this section students should be able to:

1. Use the bisection method to find a root of an equation to within a given accuracy.

2. Determine the formula appropriate to using Newton's method for a given function and use this formula to find approximations to a zero of the function.

3. State the formula for Simpson's method on $2n + 1$ ordinates.

4. Use Simpson's rule with three or five ordinates to determine an approximate evaluation of a definite integral.

4.3.13 Sequences and Series (6 lectures)

Aims

To introduce sequences and series and to develop techniques for studying them.

Contents and Teaching order

1. Sequences: definition of real sequences, limits of sequences, properties of limits, the sandwich principle.

2. Series: partial sums, convergence, properties, comparison test, absolute convergence, alternating series test, power series, properties of power series, Taylor's theorem, Maclaurin series.

Learning objectives

After this section students should be able to:

1. Determine whether or not a simple sequence converges and find the limit if it exits.

2. Use the sandwich principle to prove that a sequence has a limit.

3. Use the linear properties of convergent sequences and series to evaluate or prove the existence of a limit or sum.

4. Use the comparison test to prove that a series is convergent or divergent.

5. Recall that $\sum(1/r^2)$ converges, $\sum(1/r)$ diverges and $\sum x^r$ converges (diverges) if $|x| < 1$ ($|x| \geq 1$).

6. Use the result (absolute convergence) $\Longrightarrow$ (convergence) to prove convergence.

7. State Taylor's theorem and use it to find bounds on function in a given interval.

8. State Maclaurin's theorem and obtain the Maclaurin series and its range of validity for a given function.

9. Use differentiation and integration of a given Maclaurin series to find the Maclaurin series of another function.

10. Determine the Maclaurin series of the sum, product or composition of functions from the Maclaurin series of the functions.

4.3.14 Advanced Calculus 2A

Introduction

The second year course for students considering honours mathematics (single or combined with another subject) is Mathematics 2A including advanced calculus, mathematical analysis and algebra. Advanced Calculus 2A (Calculus 2A) covers nearly 35% of the course content of 2A and its study begins where the calculus taught in Mathematics 1A ends.

General aims

In Calculus 1A differentiation and integration are restricted to functions of one variable, while in this course the material will extend to functions of two or more variables. The overall aim is to advance students' knowledge of calculus on a broad front and sufficiently far for much advanced work in mathematics and physical sciences (Advanced Calculus 2A, course component, 1995–96).

General Learning objectives

On the completion of this course, developed mathematical techniques, insight and care are the prime objectives rather than formal proof or pathology. At the end of one year of study of Calculus 2A, students should have acquired the procedural skills of

1. The ability to cut a problem down to the core.

2. Ability to discuss a problem in conversation.

3. Some critical appreciation of lecturing technique.

Textbook

The book "Advanced Calculus for Engineering and Science students, by Ian S. Murphy" is essential for Calculus 2A.

4.3.15 Section 1. Partial differentiation

Aims

1. To define the differentiation of a function of two or more variables and partial derivatives.

2. To understand what is meant by partial differential equations.

Contents and Teaching order

1. Partial differentiation: definition of partial derivatives and higher order partial derivatives.

2. Differentiation of functions of two or more variables, the chain rule.

3. Solution of first-and second-order partial differential equations by a given change of variables. chain rule.

Learning objectives

On the completion of these components students should be able to:

1. Perform partial differentiation on all common functions using the product rule, the chain rule and implicit partial differentiation.

2. Make a given change of variables in partial differential equations of first-or second-order and hence find general solutions and particular solutions, and particular solutions which fit certain given conditions.

3. Change the two-dimensional Laplacian to polar coordinates.

4.3.16 Section 2. Errors and Exact differentials

Aims

1. To study the form of a differential and understand the idea of errors in this situation.

2. To understand what is meant by an exact differential equation and its conditions.

Contents and Teaching order

1. The form of differential; the differential of a function of a function.

2. Taylor's theorem.

3. Test for an exact differential.

4. Integrating factor.

5. Solution of exact differential equations; use of differentials to calculate small changes and percentage errors.

Learning objectives

After this section students should be able to:

1. Find the differential of a function of several variables.

2. Use the differential to find both absolute and percentage errors in given values when the independent variables are altered by small amounts.

3. Understand how exact differentials arise and be able to find the function from which they arise and hence solve exact differential equations.

4. Find an integrating factor of a given form for forms which can be made exact and apply this to differential equations.

4.3.17 Section 3. Maxima and Minima of functions of several variables

Aims

To study what is meant by a stationary point for functions of several variables and their nature.

Contents and Teaching order

1. Identification and classification of stationary points of functions two or more variables.

2. The Hessian method and its use in settling the nature of a stationary point; cases where Hessian method fails.

Learning objectives

On the completion of this section students should be able to:

1. Find stationary points for functions of several variables and determine their nature using the Hessian method, both for the two dimensional version and also for higher dimension using the idea of positive definite quadratic forms.

2. Deal with cases where the Hessian method fails.

4.3.18 Section 4. Double and Triple integrals

Aims

To define and understand what is meant by a double and triple integral and how to interpret the double integral as the volume of a solid.

Contents and Teaching order

1. The analogy between single and double integration; the thinking behind the evaluation of a double integral by repeated integration.

2. Finding limits in a double integral; the possibility of change of order of integration in a double integral.

3. Double integration: changing the order of integration, change to polar coordinates, change of variables with Jacobian.

4. Triple integration: evaluation in x, y, z coordinates and in spherical polar coordinates, change of variables with Jacobian.

5. The mean value of a function.

Learning objectives

After this section students should be able to:

1. Evaluate double integrals in x, y and polar coordinates and change the order in a double integral.

2. Find volumes using double integration; split the field of integration; determine areas by double integration.

3. Make a general change of variables in a double integral using the Jacobian.

4. Evaluate triple integrals in x, y, z and spherical polar coordinates.

5. Calculate mean values.

6. Make a general change of variables in a triple integral using the Jacobian.

4.3.19 Section 5. Beta and Gamma functions

Aims

1. To define Beta and Gamma functions and their basic properties.

2. To study relationship between these forms and other integrals.

Contents and Teaching order

1. Beta and Gamma functions: definition and properties.

2. Reduction of other integrals to these forms.

Learning objectives

On the completion of teaching materials, students should be able to recognise and evaluate various integrals that can be reduced to Beta function or Gamma function form.

4.3.20 Section 6. Vector calculus

Aims

To study and define the ideas of scalar and vector functions.

Contents and Teaching order

1. Definition of scalar and vector functions.

2. Definition of div, grad and curl of scalar and vector function; definition of directional derivatives.

3. Standard identities.

Learning objectives

After this section student should be able to:

1. Know the idea of scalar and vector functions.

2. Calculate div, grad and curl for a given function.

3. Derive and use vector identities for div, grad and curl in examples.

4. Calculate directional derivatives.

4.3.21 Section 7. Line and surface integrals

Aims

To define and understand the ideas of line integral, surface integral and their properties.

Contents and Teaching order

1. Definition of line integrals and their evaluation.
2. Statement of Green's theorem in two dimensions.
3. The idea of a surface integral and its evaluation.
4. Statement of Gauss's divergence theorem.
5. Independence of the path and conservative fields.
6. Curvilinear line integrals in three dimensions.
7. Statement of Stokes' theorem.

Learning objectives

On the completion of this section students should be able to:

1. Evaluate a line integrals directly and where possible by Green's theorem.
2. Find the value of a surface integral directly and where possible by Gauss's divergence theorem.
3. Use the test for independence of path and understand conservative fields and potential functions.
4. Evaluate curvilinear integrals in three dimensions directly and where possible by Stokes' theorem.

4.3.22 Section 8. Fourier series

Aims

To define and study Fourier series and its properties.

Contents and Teaching order

1. Definition of full and half range series on $[-\pi, \pi]$.
2. Distinction between sum of the series and the given function.

Learning objectives

After this section students should be able to:

1. Evaluate full and half range Fourier series on $[-\pi, \pi]$.
2. Sketch the graphs of the sum function of a Fourier series on the set of all real numbers, not just on $[-\pi, \pi]$.
3. Know of the possibility of using other intervals.

4.3.23 Section 9. Summation Convention

Contents and Teaching order

Explanation and basic application to vector products etc.

Learning objective

After this section students are expected to be able to use the summation convention and make basic applications to vector products and other simple situations.

4.4 Comparison between Two Systems

4.4.1 Calculus in High school (secondary education)

At this stage, some differences between educational policy in calculus in Iran and Scotland will be discussed. The comparison is mainly considered between syllabuses, teaching order, aims, learning objectives and teaching methods of calculus in the two countries. However, some general approaches may be classified as follows:

1. The educational system in both countries is homogenous and centralized, that is a single educational policy and its own traditions determine the national curriculum in mathematics teaching and learning.

2. The calculus syllabuses, functioning, class organization and the pupils' educational experiences in two countries do not correspond.

3. In the current Iranian system, the tendency in calculus teaching is clearly going from the intuitive to a more formal level and there is no emphasis on calculus application to everyday life. As a result, pupils' beliefs about the nature of mathematics corresponds with the formal and analytic way in which the discipline is taught in high school. Moreover, pupils normally are not engaged in any investigation task.

4. In the Scottish system there is an open tendency to teach calculus materials in a pictorial and informal level rather than formal approach. Pupils are encouraged to use calculators and computers to help to improve their understanding.

   Pupils also have an opportunity to carry out some investigation tasks which may allow them to apply their mathematical knowledge in solving calculus problems in context.

5. Iranian pupils should study calculus material for two years of their schooling, therefore they learn a large amount of content in comparison with the Scottish students at Higher Grade. The specific comparison can be found in other parts as follows:

Section 1. Functions

A rapid comparison in this section shows that a lot of differences in the aims, contents, learning objectives, teaching order and teaching styles can be found. Iranian pupils should learn more materials in depth and the teaching method, in particular in the 4th year, is based on mathematical rigour from basic definitions. Some difficult concepts such as the greatest integer function $[x]$ and step function are taught, in spite of pupils' difficulty with such materials. While, Scottish pupils, at Higher Grade, are taught fewer topics with an informal approach and less mathematical rigour, but topics such as "modelling with functions" will help pupils to be familiar with the application of these concepts in the real world and their everyday experience.

Section 2. Limit and Continuity

The concepts of limit and continuity are not introduced in Scotland, even at the informal level or by the intuitive approach which may cause pupils' difficulty in learning calculus in higher education. Iranian pupils learn these concepts and their main properties with mathematical rigour. For example, pupils should be able to prove the limit and continuity of a function by the (ϵ, δ)-definition, which are not easy tasks at this stage and pupils cannot normally cope with them. Even, in the first year of teaching calculus at Glasgow University, students are not expected to learn formal definitions of limit and continuity.

Section 3. Differentiation

Despite the same of aim of teaching differentiation in both countries, Iranian pupils have to study much more material in depth than Scottish pupils. As a result they will be expected to have a better background compared to Scottish ones in higher education. But, Scottish pupils will be familiar with the application of derivative in real life and the mathematical models involving derivatives. The following topics are introduced in Iran, while they are not at the Scottish Higher Grade.

1. Normal and tangent lines to a curve (from a point not on curve).
2. Angle of line and curve and angle between two curves.
3. Statement of L'Hôpital's rule and its use.
4. Points which there is not a derivative.
5. Relationship between derivative and continuity.
6. Right hand/left hand derivative (informal definition).
7. Derivative of inverse function and related theorem.
8. Derivative of inverse trigonometric function.
9. Statement of theorem of concavity.
10. Asymptotes.

But, "mathematical models involving derivatives" which is at Higher Grade is not in the current Iranian system.

In differentiation as other sections of high school calculus in Iran, teaching and learning are not left at the intuitive level. Therefore, pupils should cope with abstract concepts and logical proofs, in particular, in the 4th year of differentiation study, however skills development is emphasised. On the other hand, in Scotland, despite the attention to abstract and concise forms in the differentiation process, material is often taught in an elementary or informal state without logical reasoning. This may cause pupils' poor background when learning the differentiation in higher education.

Section 4. Integration

In this section the following topics are taught in Iran and not at Higher Grade in Scotland.

1. Differential of a function $dy = f'(x)\,dx$.
2. Geometric interpretation of differential.
3. Rules for differentials.
4. Calculation of the volume of revolution and its formula $v = \int_a^b \pi f^2(x)\,dx$.

Two topics which are only discussed in Scotland and not in Iran are as follows:

1. Modelling with differential equations.
2. Definite integral in action (application).

It seems there is no major difference in learning aims in this section between Iran and Scotland and for the first time, pupils should study integration. Nevertheless, some formal concepts such as differential of a function, $dy = f'(x)\,dx$, are taught and Iranian pupils should be able to evaluate more complicated integrals, and the volume of revolution about the x-axis.

On the other hand in Scotland, there is more emphasis on application of definite integral to physical conditions and the real world and pupils are not expected to solve complicated integrals.

4.5 Comparison in Higher Education

A brief review of calculus education in both countries shows more similarity in class organisation, course contents, learning objectives and methods of teaching in comparison with calculus teaching and learning in secondary education (high school). For instance, Calculus 1A at Glasgow University corresponds to Calculus 1 and 2 at the Iranian Universities, but Calculus 3 and Calculus 2A have a lot of differences in syllabuses and teaching methods. In general, Iranian students learn more material with mathematical rigour compared to Scottish ones. They take part in more difficult calculus examinations compared to Scottish students who take part in examinations with the same structure and the similar questions each year. More details of differences between the two systems of teaching and learning calculus can be found in the following discussions:

4.5.1 Section 1. Functions

There are no fundamental differences in this section between the two systems in the content and teaching order, learning objectives and teaching approach. However, some small differences may be found as follows:

1. Some more complicated functions are studied in Iran and students must be able to tackle related tasks.
2. Inverse function and its properties is introduced in Calculus 1A in the present section, whereas this concept is normally discussed in the next sections of Calculus 1.

4.5.2 Section 2. Limit and Continuity

The following aspects of limit and continuity are only taught in calculus 1 (Iran)

1. Formal definition of limit, one-sided limits, infinite limit, limit at infinity and continuity of a function (ϵ, δ-definition).
2. Statement of Bolzano's theorem for continuous functions.
3. Continuity of composition functions and trigonometric functions.

There is no important difference between the teaching order, but the emphasis throughout is on rigorous argument from basic definitions of limit and continuity in Calculus 1 (Iran). While, informal study is an overall aim in Calculus 1A (Scotland). This causes some differences between learning objectives and teaching styles. On completion of this section, Iranian students will be expected to be able to discuss the points at which a real function $y = f(x)$ has limit or is continuous by the (ϵ, δ)-definition. In addition, they should be able to reproduce the proof of rules and theorems under examination conditions, whereas Scottish students in Calculus 1A are not required to do it. The teaching approach in Calculus 1 is built on mathematical precision and logical proofs and this approach is naturally followed throughout the other sections.

It seems that the fundamental differences of teaching methods between Calculus 1 and Calculus 1A have appeared in this section. Lecturers of Calculus 1A believe that students at this stage cannot cope with formal definition and a rigorous approach and hence it will be postponed until the mathematical analysis section in Mathematics 2A. On the other hand, the three lectures which are allowed for teaching limit and continuity in Calculus 1A may not be enough.

4.5.3 Section 3. Differentiation

In this section, Calculus 1 contents cover all the Calculus 1A and additional materials as follows:

1. The theorem of extrema (without proof).
2. Cauchy's mean value theorem.
3. First and second derivative tests for relative (local) extremums.

4. Definition of differential and its properties ($dy = f'(x)\,dx$).

5. Inverse function theorem.

The above differences between course contents in differentiation cause some effects in learning objectives.

Despite the relative coordination in teaching order, there are a few differences as follows:

1. The second derivative test is taught in Calculus 1A before Rolle's theorem and mean value theorem, while it is discussed after them in Calculus 1.

2. Inverse function and its properties are normally studied in Calculus 1 in this section, whereas in Calculus 1A they are taught before it.

3. Parametric equations $x = x(t)$ and $y = y(t)$ are taught in Calculus 1A at the end of this section, while they normally are introduced in Calculus 1 (Iran).

4.5.4 Section 4. Integration

It should be mentioned that integration is taught in Calculus 1 and 2 (Iran), whereas it is followed in Section 4 and 6 in Calculus 1A (Scotland). The following topics are only introduced in Calculus 1(and 2) and other materials are nearly the same:

1. Cauchy's mean value theorem for integrals.

2. Mean value theorem for integrals.

3. L'Hôpital's rule and its use.

4. Taylor's formula and Maclaurin's formula.

5. Integrals yielding inverse hyperbolic functions. In Calculus 1A these integrals yield logarithmic function.

6. Volume of a solid of revolution (cylindrical shell revolution).

Teaching Method of Differentiation and Integration

As a result of the overall aims, course content and learning objectives, the teaching approach is clearly based on mathematical rigour and formal approach in Iran. Nevertheless, skills development and teaching techniques of differentiation and integration rather than conceptual understanding are emphasised. Moreover, most frequently, we encounter these notions in the teaching approach to the differentiation and integration sections of Calculus 1A although with lower level and less depth compared to Calculus 1 and 2.

4.5.5 Section 5. Logarithms and Exponentials

A rapid comparison shows that, there are no important differences in this section between aims, contents, objectives, teaching order and teaching method in Calculus 1A (Scotland) and Calculus 2 (Iran). However, there is an exception here that in Calculus 1A hyperbolic functions are introduced, whereas in Calculus 1 these functions are taught in the next section.

4.5.6 Section 6. Sequences and Series

The aims, teaching order and methods are nearly the same in two countries, nonetheless there some differences between materials and learning objectives. This section is taught briefly in Calculus 1A in comparison with Calculus 2. Therefore Iranian students learn the materials in more detail than Scottish ones. For example, the following topics are not in Calculus 1A:

1. Series: limit comparison test, integral test and root test.
2. Alternating series and its test.

4.5.7 Section 7. Ordinary differential equations

This short section is not normally in the Iranian calculus course and students should pass this topic as an independent course in the later years of their undergraduate study.

4.5.8 Comparison between Calculus 3 and Calculus 2A

As mentioned, there some remarkable differences between Calculus 3 (Iran) and Calculus 2A, compared to Calculus 1A and Calculus (1 and 2). The materials are not taught in the same order and or with the same teaching method. Teaching methods, in Calculus 2A, are based on mathematical skills and technique development and not formal proof and logical discussion. While, formal proof and mathematical skills are both considered in teaching Calculus 3 (Iran). In addition, the following topics are only taught in Calculus 3 and not in Calculus 2A:

1. Calculus of vector valued functions.
2. Length of arc in R^3.
3. Unit tangent and unit normal vector in R^3.
4. A function of n-variables.
5. Graphs of functions of two and more variables.
6. Composition functions (with more than one variable), their domain and range.
7. Limit and continuity of function of two variables.
8. Accumulation point.
9. Tangent plane and normals to surface.
10. Sufficient condition for differentiability.
11. Second derivative test (for local maximum/minimum).

On the other hand, the following topics are only studied in calculus 2A (Scotland):

1. Solution of first-and-second order partial differential equations.
2. The Hessian method and its use in settling the nature of a stationary point.

4.5.9 Multiple integration

There are no remarkable differences in the section on double and triple integration, in contents, learning objectives, teaching order and methods.

4.6 New System of Iranian high school

4.6.1 Introduction

The importance of high school education and some difficulties with the current (old) high school curriculum led to a new reform of the high school system in 1991. In this reform, theoretical knowledge, technical and professional training are considered and pupils will begin high school education according on their abilities and interests in one of the following branches:

1. Theoretical.
2. Technical and professional.
3. Work and knowledge.

The first year of high school study in one of the above branches may be temporary, but should be fixed in the second year. The new system comprises three years of high school education and one year pre-university courses. The mathematics-physics branch in the present system is a sub-branch of the theoretical branch and each pupil in this sub-branch should study 96 units of different subjects within these three years.

The new reform required three stages to establish its new curriculum and began in 1991 by covering 10% of the pupil population and 25% in 1993. It which should be complete by the end of 1996.

4.6.2 Mathematical topics

Mathematical subjects in the three years of high school study will be 22 units (Table 4.2).

Table 4.2

Subject	Unit
Elementary Mathematics (1,2,3,4)	8
Geometry (1,2)	5
Calculus	4
Applied Mathematics	2
Computer Sciences	3

4.6.3 Pre-university course

After three years of high school study, pupils who are interested in applying for admission to higher education, should continue in the pre-university course.

According to the "Iranian Reform Committee" in the Ministry of Education (1993), the overall aim of the one year pre-university education is to develop pupils' academic background to be able to continue their study in higher education. Pupils should pass thirty two units of subjects, and after getting pre-university certification, they can take part in the higher education entrance examination. The pre-university curriculum development will be determined with the help of university departments.

4.6.4 Calculus in the New Curricula

In the new curricula, calculus will begin in the last year of high school study (third year, age 16–17) in the mathematics-physics branch as a main part of mathematical teaching four hours per week. It continues in pre-university courses in the mathematics branch to satisfy the needs of all the pupils intending to specialize in a subject in higher education with significant mathematical content. Teaching methods in high school calculus are clearly informal rather than with mathematical rigour with more application of calculus to the real world, despite the current (old) curriculum. For instance, an informal approach to limit and continuity of a function with some examples and question tasks have appeared to develop pupils' visual thinking. Formal definition of limit and continuity and a lot of materials which are studied in the third and fourth years of current (old) system, appear in the pre-university courses. Thus students should be able to cope with formal language and some logical proofs in pre-university calculus study. More details about calculus education in the new curriculum has been omitted from this research, because our samples had calculus background from the current (old) high school curricula.

4.6.5 Reform in Scotland Secondary Education

Some reports indicate that the present structure of Higher Grade and the Sixth year of secondary education is to change. It has been decided to have a one-year higher course and two-year advanced higher course, thus allowing exit points at the end of both 5th and 6th year. No great change in mathematics content is envisaged; rather, a repackaging of the present Higher Grade materials (possibly involving options). If universities continue to take Higher Grade as the standard entry, some changes will be needed to the content and teaching methods when the first schools graduates enter the universities in 1998 (Mathematics News letter, Glasgow University, 1995).

Chapter 5

Students' Difficulties in Learning Calculus

5.1 Logical and Psychological order in Calculus

5.1.1 Introduction

Can we say we have taught students by asking them to repeat words and mathematical statements or apply formulae in a rote manner? Can we make calculus ideas intelligible to be learnt? Many beginning calculus students have negative feelings about the calculus. They complain that they cannot understand their teachers and their notes. Is there a problem because we have paid too much attention to theory and too little to descriptive facts? Or we have paid too little attention to how students learn (Herron, 1996).

Many students may remember a few mathematical concepts that have been taught, but they frequently never learned them at the time. Even many mathematicians complained that they could not understand the real meaning of some significant calculus concepts at the time. Teaching calculus often caused students to memorize abstract materials with a meaningless understanding.

Halmos (1985) in an interview said that, "I remember calculus was not easy for me and I never understood the concept definition of limit by ϵ and δ when I was an undergraduate student, but as a first year graduate student suddenly I understood epsilon, when I was near the blackboard in the classroom".

Can students think and learn in the same way as their teachers? How can they communicate mathematical concepts? How can students achieve general progress in mathematics learning? Farrell (1992) suggested that how students learn, depends both on the nature of the mathematics and on the intellectual development of the student.

5.1.2 Psychological approach and Mathematics learning

Concept formation is one of the main problems in teaching and learning mathematics, but it is not the teachers who create their students mathematical concepts. In fact, as mathematics or calculus educators, we have sometimes very little insight into the formation of mathematical understanding in our students' minds. Even and Tirosh (1995) cited many studies which have shown that students often make sense of the subject-matter in their own way which is not always isomorphic or parallel to the structure of the subject or the teachers' knowledge and their methods of teaching. They also suggested that understanding of students' ways of thinking may help educators to guide students and modify their construction of mathematical knowledge. In fact, starting from students' limited conceptions in the mathematical domains would help teachers to build more sophisticated concepts based on them.

Monna (1992) noted that progress in mathematics is a subject which belongs perhaps more to the domain of psychology than to mathematics. It concerns the ways of thinking and the element of creativity. Factually, mathematical understanding, mathematical thinking and mathematical creativity are subjects in a framework of cognitive psychology. Many attempts have been made to apply cognitive psychology to mathematics instruction depending on cognitive abilities (Hiebert, 1981). And Moore (1994) suggested that the individuals' ability to study abstract mathematics and do proofs depends on a complex combination of beliefs, knowledge and cognitive skills.

In addition, Skemp (1986) noted that teaching and learning are psychological problems and students aren't expected to improve in learning mathematics until we know more about how they learn it. However, there is a confusion here between logical and psychological approaches in teaching and learning mathematics. As Skemp noted, the mathematicians try to present mathematics as a chain of logical development and the main purpose of a logical presentation is to convince doubters. While, a psychological approach in learning mathematics is to bring about understanding.

Poincaré (1924) mentioned that "a definition is satisfactory only if the students understand it". Therefore, students' understanding should be considered as a core of teaching and learning in the calculus domains. In addition, some important questions must be considered as follows:

1. How can we ensure that students understand the calculus materials?
2. Do they learn calculus concepts and mathematics ones in general through a logical order or a psychological order?
3. Furthermore, what is meant by logical and psychological order in learning mathematics and, in particular, calculus?
4. Does a logical order correspond to psychological order in mathematics instruction?

Having a clear approach to the mathematical realities and students' cognitive processes, in which these realities come to be understood, may help to find more reasonable responses to the previous questions. Monna (1992) noted that a tendency towards abstraction is one of the features of modern mathematics such that physical reality is replaced by abstract reality, that is, concepts are formed by the process of abstracting. Moreover, mathematical concepts are the result of so many abstractions and so on, such that the psychological arguments could be lost in the complexity of the mathematical discussions and examples (Skemp, 1986). Mathematics students at advanced level are confronted with more abstract ideas such as topological spaces, R^n-space, Hilbert space, Banach space, fields, rings, groups, and so many formal concept definitions.

Moreover, students are uneasy in dealing with some main mathematical concepts in calculus. For instance, students have difficulty in understanding the foundations of limit and limiting processes, piecewise and pointwise functions, the process of continuity and discontinuity of a function, interpreting of extrema and so on.

Students' difficulties with the formal concept definitions of function (Dreyfus and Eisenberg, 1982; Vinner, 1983, 1991; Vinner and Dreyfus, 1989; Leinhardt et al., 1990; Even, 1993), limit (Cornu, 1991; Tall and Vinner, 1981;), tangent lines (Vinner, 1982, 1991; Tall, 1987), rate of change (Orton, 1984; Thompson, 1994), variables in calculus word problems (White and Mitchelmore, 1996) are well documented to show a whole spectrum of the main calculus concepts which cause student disasters in learning calculus.

Although the significance of abstraction and generalization processes for development in learning mathematics can't be ignored in mathematics creation and communication, it seems that some calculus concepts may be naturally understood. For example, when you are talking about the idea of continuity in non-mathematical rigour, you may imagine a river, or a piece of curve with no gap in it. In fact, students would naturally understand the meaning of continuity of a current of water or a continuous curve by their own cognitive structures without any reference to the formal definition of continuity by ϵ and δ formula.

It seems reasonable, that at the beginning of calculus, students have a lot of cognitive conflicts with abstract definitions, in particular, when they are far from their natural understanding and experience of everyday life. Therefore, students try to learn the logical order of formal definition and mathematical statements by rote rather than by meaningful learning. As Cornu (1991) suggested, from such a starting point the conceptual obstacles could arise and cause serious student difficulties in learning mathematics. For instance, in comparison with the natural concepts of continuity, the concept of limit and limiting process may not be natural for students to understand and therefore more mental conflicts could arise in the learning of the idea of a limit of a function.

Hence, some mathematicians (e.g., Dieudonné, 1960; Pearson, 1996) prefer to discuss continuity of a function before talking about its limit at a given point and some believe that teaching the limit of sequences is more easily understood by students than teaching the limit of functions. For example, Dieudonné (1960) in his famous book "Foundation of Modern Analysis", without any argument about the limiting process of a function, defines continuous mapping in metric space and then, building on it, tries to define the concept of limit.

5.1.3 Mathematics and Satisfactory learning

Even and Tirosh (1995) noted that understanding students' ideas, and the reasoning behind them, is a key issue in making appropriate decisions for helping them in the construction of their mathematical knowledge. And Boas (1981) discussed some principles to make sense of mathematics that are frequently ignored by teachers and textbooks. He described them as abstract definition, analogy, vocabulary, symbolism, proof, enthusiasm and skills which can be adopted in teaching the higher calculus to achieve in students a deeper conceptual understanding.

The most important reward for learning, says Bruner (1966), is not praise from an adult, but "intrinsic satisfaction". Therefore, it may be concluded that in mathematics, satisfactory learning is factually an intrinsic satisfaction which should be happening in teaching and learning calculus. In Tall and Vinner (1982), the distinction is made between realizing a proof and conviction in mathematics, that mathematical truth should be established according to formal rules, while intuition, self-evidence and beliefs can play an external role.

Therefore, students' readiness for learning new mathematical concepts would guarantee, if there is a healthy integration between students' intrinsic satisfaction as an individual cognitive activity and mathematics as a formal system. In fact, this mathematical satisfaction is an idiosyncratic cognitive experience which is considered to rest on a core of psychological order, while mathematical proof achieving is placed in the domain of logical order. Therefore, it seems reasonable to say that logical order and psychological order will have many interplays in a systematic approach to the process of learning mathematics.

5.1.4 More about Psychological and Logical meaning

Reasoning and problem solving are closely related topics which are both important in mathematics education. The study of reasoning was historically related to the study of logic (Ellis and Hunt, 1993). In addition, logic as a part of mathematical foundations is the set of rules by which one can reach a valid and true conclusion in mathematical arguments. Or as Ellis and Hunt (1993) noted, logic is a formal system which attempts to specify the characteristics of good and bad arguments. Formal logic is, in fact, a prescription for correct reasoning in mathematical proofs.

Ausubel et al. (1968) suggested that logical meaning depends on the nature of materials. It refers to meaning that is inherent in certain kinds of symbolic material and does not depend on the existence of a human mind to appreciate or test it. While, psychological, actual or phenomenological meaning is a wholly idiosyncratic experience. Moreover, Ausubel (1963) made a point that the attainment of meaning is a purely idiosyncratic psychological phenomenon in a particular person. On the other hand, logical or artificial concepts are used in tasks in which subjects are presented in a form such as mathematical concepts not normally experienced in students' everyday environment (Ellis and Hunt, 1993).

5.1.5 Logical/Psychological order relationship in Calculus

Based on the previous section and Ausubel's definition, when an individual learns a mathematical concept definition or follows a proof he/she does not learn their logical meaning, but the meaning they have for him/her, i.e. what they signify to him/her. As a result, for meaningful learning to take place in calculus or any other discipline, students should be able to relate materials to their own idiosyncratic cognitive structure.

The distinction between the logical and psychological structure of knowledge justifies the difference between logical and psychological meaning and their interactions. Moreover, the possibility of transforming logical meaning into psychological meaning plays an important role in teaching and learning formal concept definitions and mathematical discussions, in particular, in the calculus course. But mathematics educators should ensure that a safe transmission has occurred in the students' mental processing.

In learning mathematics, students should create all the concept definitions (which were made by the mathematicians of the past) anew in their own mind (Skemp, 1986). But, in the framework of abstractness as a nature of concept definition and mathematical statements, there are some notions which seem to be against all common sense (in contrast to students' natural insight) and abstract to such a degree that is doubtful if they are of any use (Monna, 1992). For instance, in the calculus domain some notions may be considered as abnormal concepts for students, but they have to accept them although they don't accord with their natural insight and understanding. Existence of continuous functions without a derivative at a given point seems to be an abnormal idea for calculus students.

Students' mental processing and cognitive activities are described by Tall and Vinner (1982) as "imagination acts" which are idiosyncratic experiences and play a basic role in learning mathematics and its communication. They suggested that the imagination acts of many different students have a similar structure. This similarity has, in fact, a psychological nature which is used in conceptual learning, problem solving and mathematical arguments. In this point, there are some key questions to be discussed in calculus education as follows:

1. Are the calculus topics and our teaching order organized according to a logical or a psychological order?

2. Does our teaching order begin with materials which are closely related to the previous experience of students?

3. Do we introduce conceptual materials only when students sense a need for some way to explain what they have already observed (Herron, 1996). More likely teaching materials and most calculus textbooks are developed logically, but students can't see this logical order.

4. Maybe it does not seem at all logical to them and this could be the starting point of students' difficulties in learning calculus.

Students frequently fail to relate logical order to their idiosyncratic cognitive structure or personal imagination. As a result, formal concept definitions and mathematical proofs in calculus could be learnt in such a manner and hence memorizing occurs instead of meaningful learning. In the next section further discussion of this will be found.

5.2 Concept Definition/Concept Image in Calculus

Mathematical activity is performed by learning definitions, constructions, axioms and theorems, and mathematical concepts are normally learnt by means of their formal definitions. A defined concept which is indicated by a formula may be composed of one or more concepts that are themselves more defined than concrete (Gagné, 1985) and mathematical definitions are mainly artificial concepts that are used in the question tasks in which students express varied stimulus patterns not normally found in their experience of everyday life. Therefore, the ability to handle the defined concepts and deal with them in a flexible and changing fashion are important objectives of learning mathematics and, in particular, calculus. On the other hand, pure logical definition can't give students insight into learning mathematics, but personal mental pictures and cognitive processes have an important role in the development and refinement of evoked concepts (Tall and Vinner, 1981).

Factually, in the process of teaching and learning calculus, like the other mathematical areas, concepts are generally formed in one of two ways. Tall and Vinner (1981, 1986) called these "concept definition" and "concept image". These two categories consist of the presentation of a mathematical definition. Moreover, Moore (1994) suggested a third item in concept formation which is called "concept usage". Concept usage refers to the ways one operates with the concept in general or using examples or in doing mathematical proofs. Therefore, students' difficulties in concept understanding are discussed by Moore (1994) in terms of a concept understanding scheme involving concept definition, concept image and concept usage.

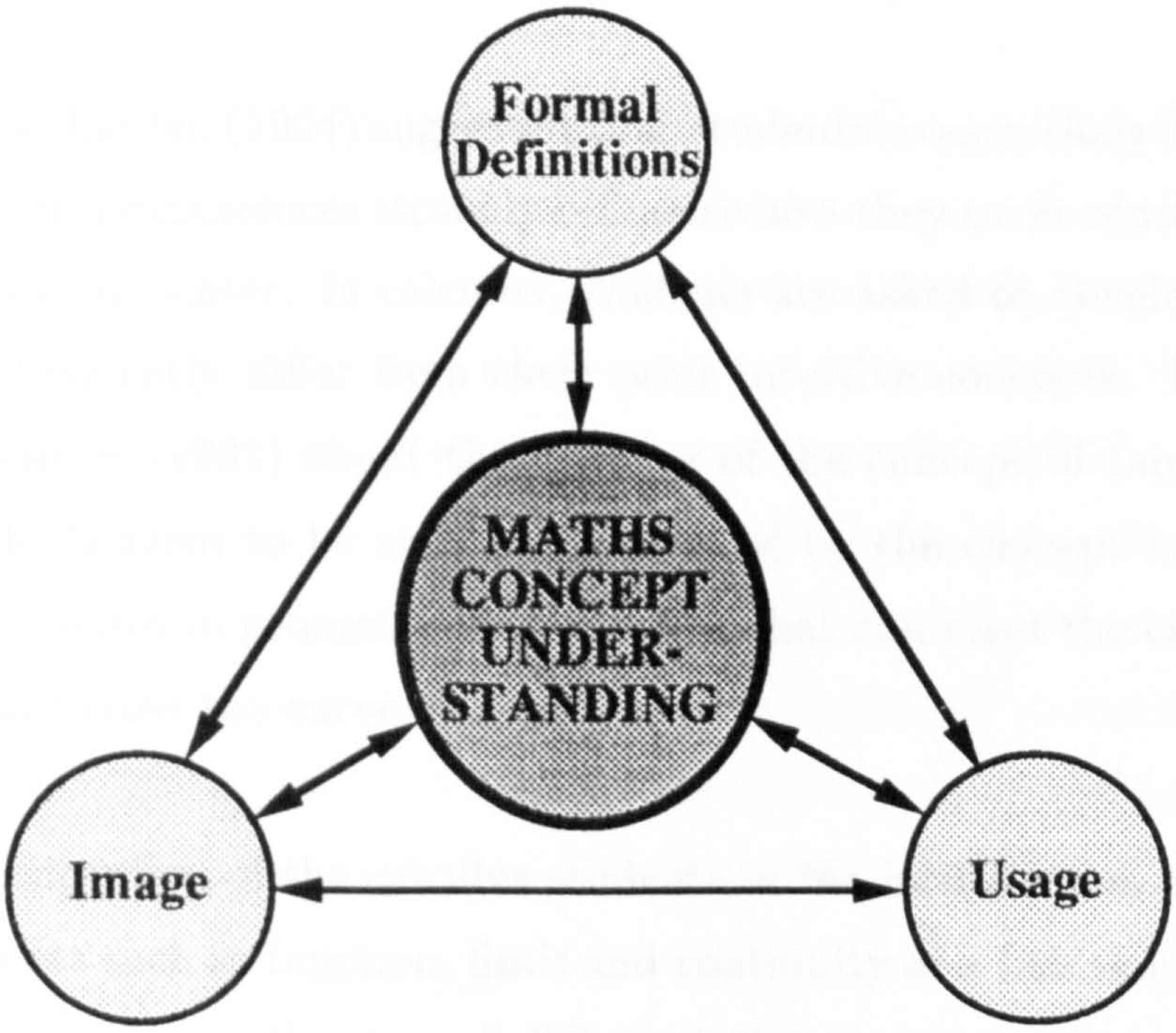
Tall and Vinner (1981) suggested a distinction should be made between formal mathematical concepts and the cognitive processes by which they are conceived. Concept definitions and concept images are discussed in detail in several papers (e.g., Tall and Vinner, 1981; Davis and Vinner, 1986, 1989; Vinner and Dreyfus, 1989; Vinner, 1991).

Davis and Vinner (1986) suggested that mathematical concept definition (as with other concepts) has a related concept image, and concept image or personal concept definition is to exhibit the way by which a concept is viewed by an individual. In other words, concept image is the set of all students' mental pictures and associated properties and cognitive processes. The concept image is derived from examples, non-examples, diagrams, graphs, symbols and other experiences a student has with the mathematical concept. Vinner (1991) noted that the concept image is a non-verbal thing associated in our mind with the concept name. It can be a visual representation of the concept, when it has visual representations; it also can be a collection of an individual's impression or experiences.

A definition may be explained in terms of a personal reconstruction by students, whether the formal definition is taught to them or constructed by themselves and they may vary the personal constructions from time to time. Therefore, in this way an individual concept definition or concept image can differ from a formal definition and the mental conflicts begin to emerge in an operational situation such as problem solving.

Moore (1994) suggested the sequence "$Images \rightarrow Definitions \rightarrow Usage$" within concept understanding, which illustrates the students' ability to use mathematics definitions in proof situations depending on their knowledge of the formal definition, which in turn depends on their informal concept images.

However, it seems to the researcher that the three items, images, definitions and usage, within mathematical conceptual understanding may interplay with each other in a system to enhance the students' performance in doing mathematics proofs and problem solving situations. Therefore, Moore's sequences could be changed into a system as follows:



But the mental process of interplay between concept image and concept definition in practice and in a teaching situation has a different story by Vinner (1991). He exhibited several different processes of interplay between the formal concept definition and its concept image.

There is some evidence that calculus students usually develop concept images which are inconsistent with formal definitions. Vinner and Dreyfus (1989) suggested that students do not necessarily use the formal definition to decide whether a given mathematical object is an example or non-example of a concept. They decide normally on the basis of their concept image which is a result of their previous experience with examples and nonexamples of a concept. Hence, the set of mathematical objects considered by students is not necessarily the same as the set of mathematical objects determined by the formal concept definition. Therefore, obstacles related to inconsistency between individual concept images and mathematical defined concepts seem to be a main source of students' difficulties in learning some basic calculus concepts.

On the other hand, some complex concepts in calculus and other complicated mathematical concepts, are not acquired in one step, but many steps have to be followed by students to complete the mastery of them.

Mundy and Lauten (1994) suggested that students' conceptions from their previous mathematical experiences strongly influence how they make sense of the calculus ideas that they encounter. In calculus, students are asked to develop their concept images that frequently differ from their prior intuitive concepts. For instance, in a study by Vinner (1991) about the learning of the concept of tangent, beginning calculus students seem to be strongly influenced by the concept image of tangent line they have learnt in geometry that is, a line that can meet the curve only at one point and can't cross the curve.

The misconception of the calculus students in the introduction of some formally defined concepts such as function, limit and continuity of a function, differentiation, relationship between continuity and differentiation, improper integral etc. could be introduced as a typical illustration of the formation of students' concept images which have inconsistencies with formal defined concepts.

For example, it is difficult for students when they are asked to find "a continuous function at a specific point, but not differentiable at that point" because, they are strongly influenced by the concept image that each continuous function should be differentiable at the point of its continuity! Or, when students hear the word "discontinous function", they may immediately visualize a graph of a continuous function and recall the functions $y = x^2, y = \cos(x)$ etc., instead of imagining the graph of a discontinuous function such as:

$$f(x) = \begin{cases} 3 + x & \text{if } x \leq 1, \\ 3 - x & \text{if } x > 1. \end{cases}$$

Indeed, not only students, but many mathematicians have mental conflicts with concept definitions and their concept images when they are teaching calculus. For instance, Hitt (1994) found that teachers have a strong tendency to think of functions in terms of continuous functions and, in many cases, they have very little skill in actually constructing them. They don't take into account the other alternative discontinuous functions, in fact, the discontinuous concept of functions has not become an active element of their mathematical thinking and concept images.

As Leinhardt et al. (1990) stated, misconceptions are features of a student's knowledge; a specific part of mathematical ideas that may or may not have been taught. They can be interpreted as incomplete formal learning or students' intuition difficulties. For example, students' tendency to recognize only one-to-one correspondences as functions would be a typical misconception in this complex part of learning calculus. Indeed, the topic of functions is one of the most central in mathematics (Dreyfus and Eisenberg, 1982). It is a basic organizing idea for the study of calculus which has both strong visual (graphical) and abstract (nongraphical) aspects. Leinhardt et al. (1990) suggested that students' misconceptions and difficulties in learning the concept of function should be discussed under the following items:

1. What is and is not a function?
2. Correspondence.
3. Linearity.
4. Continuous versus discrete graphs.
5. Representations of functions.
6. Relative reading and interpretation.
7. Concept of variable and notation.

In another study, Vinner and Dreyfus (1989) investigated some aspects of the images and formal definitions that college students and high school teachers have for the concept of function. In that study various aspects of function concept definition as conceived by students were expressed. For example:

1. One valuedness: If a correspondence assigns exactly one value to every element in its domain then it is a function, otherwise it is not a function.
2. Discontinuity: When the graph has a gap the correspondence is discontinuous at one point in its domain. Therefore, this graph can not be recognized by many students to be the graph of a function.

They concluded that sometimes a certain aspect was for some students the reason for rejecting the given relation as a function whereas, for others it was the reason to accept it.

A lot of studies confirmed that students strongly prefer functions expressed in term of algebraic formula rather than in other representations, such as graphs, tables and correspondence as a part of their concept image of the function definition . For instance, Mundy and Lauten (1994) noted that calculus students desire, frequently, for functions to be defined by a single formula and they are uneasy when dealing with piecewise functions.

In other words, students' concept image of function is merely its presentation by a single rule. It seems that the definition of function as a formula relating x and y is a wrong idea which appears to be a part of the students' concept image. Or students prefer a nonpictorial aspect of a function as their concept image, because they have difficulty with visual thinking and graphical presentation of functions.

Having a formula is only a way of function representation, whereas a function and its formula are in two different categories. There are many mathematical formulae which do not define a function (e.g., $y = \ln(\sin x - 2)$, $y = \sqrt{x-2} + \sqrt{1-x}$) and there are some mathematical functions which are not expressed in terms of an algebraic (symbolic) rule. For example, the Dirichlet function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ rational,} \\ 0 & \text{if } x \text{ irrational} \end{cases}$$

is an unusual function which has no algebraic (symbolic) formula and no graphical form.

It was noted that students are unhappy in dealing with piecewise functions when there are many multi-rule functions in calculus which may be confused by a beginning calculus student. For example, if some students are asked how many functions are defined in the following correspondence:

$$y = \begin{cases} \cos(x) & \text{if } x < 0, \\ 1 + x^2 & \text{if } 0 \leq x \leq 2, \\ \ln(x-1) & \text{if } x > 2, \end{cases}$$

they may reply "three functions (a cosine, a quadratic and a logarithmic function) are defined in this question", whereas the three rules define a unique function and no more. This type of confusion could be related to students' concept image and weakness in their visual thinking.

Tall and Vinner (1981) suggested that the concept images of limit and continuity are likely to contain factors which conflict with the formal concept definition. They found that a common concept image of $S_n \to S$ was that S_n approaches S, but never reaches it. Or one student claimed "$S_n \to S$ means S_n gets close to S as n gets large, but does not actually reach S until infinity". It was also obvious to them that if S_n gets close to S then $1/S_n$ gets close to $1/S$ (when $S_n \neq 0$ and $S \neq 0$). But a weak understanding of the limit concept definition can make the formal proof of this result very hard for them.

In addition, Tall and Vinner found that students had different aspects of concept images of the formal definition of continuous functions. A questionnaire administered by them to first year university mathematics students included a question to investigate the students' concept images of continuity (Figure 5.1). Mathematically f_1, f_2 and f_3 are continuous functions, while f_4 and f_5 are not, but the students' concept images suggested otherwise (Table 1.1, correct responses in bold).

Although all the responses for f_1 were correct, the majority were "right answers for wrong reasons", such as the idea that "f_1 is continuous, because it has a single rule". The second function, i.e. f_2, is continuous according to the (ϵ, δ)-definition in its domain. However, students' concept images suggested:

1. It is continuous, "because the function is given by a single formula".
2. It is not continuous because "the graph is not one piece". In fact, students' concept images do not allow a gap in the graph of such a function,
   "The function is not defined at the origin",

"The function becomes infinite at the origin".

The fourth function was considered discontinuous by most, with several different kinds of reasoning :

1. "It is not in one piece."

2. "There is a jump at the origin." "It is not a single formula."

The last function caused more problems. For several students it was discontinuous, because "it is impossible to draw."

Figure 5.1

Which of the following functions are continuous?

If possible, give reasons for your answer.

$f_1(x) = x^2$

$f_2(x) = 1/x \ (x \neq 0)$

$f_3(x) = \begin{cases} 0 \ (x \leq 0) \\ x \end{cases}$

$f_4(x) = \begin{cases} 0 \ (x \leq 0) \\ 1 \ (x > 0) \end{cases}$

$f_5(x) = \begin{cases} 0 \ (\text{rational}) \\ 1 \ (\text{irrational}) \end{cases}$

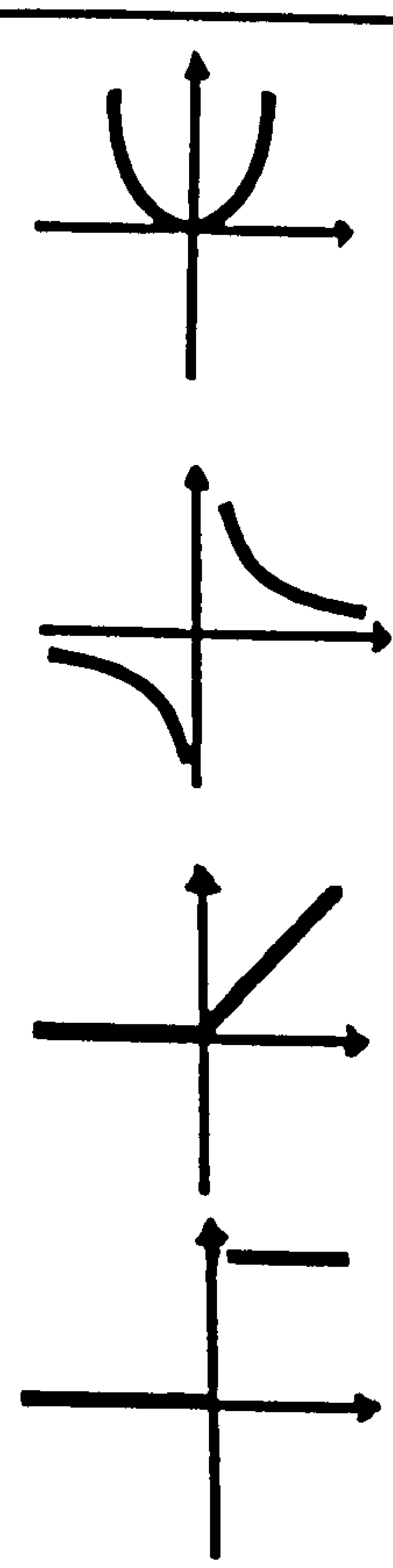


Table 1.1

students responses to the above graphs

$N = 41$	f_1	f_2	f_3	f_4	f_5
Continuous	41	6	27	1	8
Discontinuous	0	35	12	38	26
No response	0	0	2	2	7

In short, a literature review in the domain of concept image and formal defined concept indicated that students' mental pictures and their cognitive processes (as idiosyncratic phenomena) are related to every formal concept definition. But, a personal concept definition may differ in reality from a mathematical concept which is formally defined in the calculus course.

Farrell (1992) suggested that the source of students' misconceptions may be traced partially to the unique nature of mathematics materials, however they may also be traced to the students' level of intellectual development. Therefore, cognitive conflicts could occur within mental processes by which students attempt to conceive a new mathematical idea from an abstract definition. However, students may even have been taught to respond with the correct abstract definition with an inappropriate and restricted mental picture or concept image.

On the other hand, as Farrell (1992) noted, mathematics is a completely hierarchical discipline. Hence, student's misconception of one aspect of a logical chain may cause the subsequent learning blocks, in particular when the complexity of tasks is increased. Moreover, practically and informally building up concept images would cause some differences with an initial abstract definition (Tall and Vinner, 1981).

As a result, defective mental pictures and rote learning of concept definitions in the calculus course can make the mathematical proofs very hard for students. Not only in calculus learning, but in the higher mathematics courses such as mathematical analysis, they will be likely to have more difficulty in coping with rigorous definition and logical discussions. For concepts which have no pictorial aspect in the calculus learning, student's concept image includes mainly symbolic manipulations and representations as well as the set of all properties associated with the concept

(Davis and Vinner, 1986). On the other hand, Guilford (1959) suggested that semantic and symbolic information provide relating images which may contrast with the visual-figural images that come from concrete information.

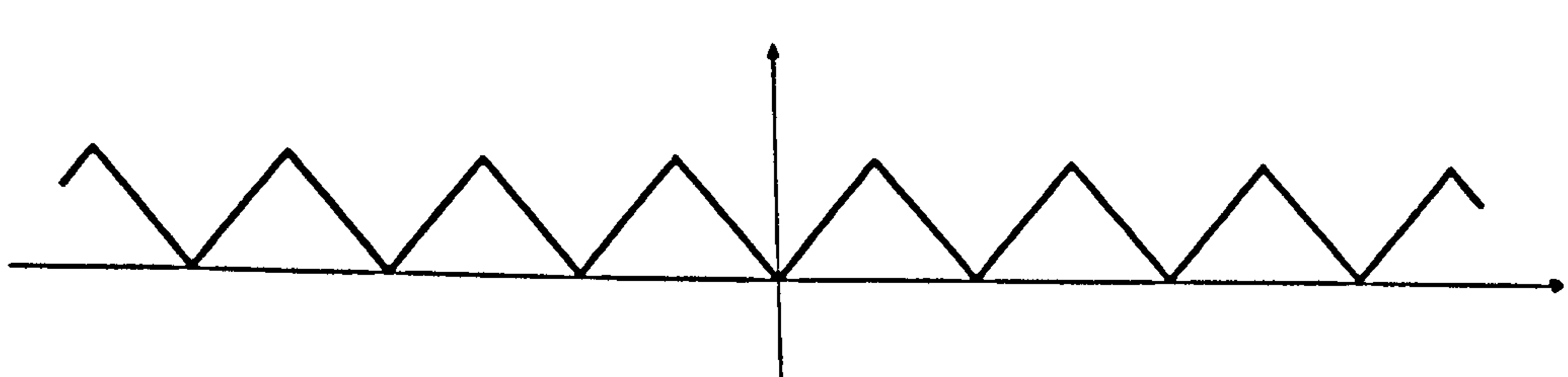
To sum up, the cognitive conflicts which have emerged in this situation cause a serious obstacle to the learning of calculus and advanced level mathematics such as real and complex analysis, topology, manifold geometry and abstract algebra. In this situation, it can be more difficult to visualise the defined concepts as proper mental pictures or images. For instance, what mental pictures may students have of topological spaces, Banach spaces or a function such as:

$$f(x) = \begin{cases} x^2 \sin 1/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

This function shows that a function may be differentiable everywhere, but its derivative fail to be continuous everywhere.

Or in advanced calculus, how may students imagine a function which is continuous everywhere in the field of real numbers and differentiable nowhere (Spivak, 1967)?! As Tall (1991) noted, exhibiting curves with corners gives inadequate intuition. Intuitively, the idea is to obtain a continuous function, the graph of which provides a "sharp edge" at every real point, but the edge should be sharp enough to confirm the absence of the derivative there (Dellio, 1982). The visualisation and discussion of such a function is not easy for students to follow (Figure 5.2).

Figure 5.2



Hence, Tall and Vinner (1981) found that the mental pictures which may help the students during early calculus learning can, at advanced mathematical levels, become an obstacle.

5.3 Students' Conceptual difficulties in Calculus

5.3.1 Introduction

Johnstone (1991) suggested that many scientific concepts are of a similar nature. These ideas are all beyond our senses and students have little or no experience in constructing such concepts. Definitions purported always to act as anchors for scientific concepts, but whether they were ever understood is open to debate. The fact that many students complain that science is difficult to learn might suggest that it is not being successfully transmitted. In another study, Johnstone (1984) noted that the nature of science concepts, the traditional method of teaching and the way of learning materials are three possibilities at least by which students' learning difficulties may arise.

One of the most important objectives of learning mathematics, per se, is to achieve deep understanding of the concepts. Hence, over the past two decades, attempting to clarify the vague dimensions of mathematical understanding has been the main aim of many researchers (e.g., Skemp, 1976, 1986). Skemp distinguished between two kinds of understanding as follows:

1. Instrumental understanding, which means knowing how without knowing why.

2. Relational understanding, which refers to knowing how and being able to elaborate why in terms of one's other mathematical knowledge. He regards this as real understanding of mathematical activities which is hard for students to achieve.

Skemp's work has been continually generating ideas on mathematical understanding including (Webb, 1979; Nesher, 1986; Pirie and Kieren, 1990; Kieren, 1992; Moore, 1994; Even and Tirosh, 1995). In particular, many of these studies have discussed the relationship between mathematical concepts and cognitive structure in an individual's mind associated with the concepts. For instance, Moore (1994) suggested three major sources of undergraduate difficulties in mathematics courses which are:

1. Concept understanding.
2. Mathematical language and notation.
3. Getting started on mathematical proofs.

5.3.2 Conceptual/Procedural Knowledge in Mathematics

The terms "conceptual knowledge" and "procedural knowledge" were used by Eisenhart et al. (1993) to denote a distinction between two forms of mathematical knowledge. The former refers to the basic structure of mathematical materials, i.e. concepts, while the latter indicates mastery of computational skills. In other words, conceptual knowledge gives and explains the meaning of mathematics procedures such that the combination of them is considered as a necessary aspect of mathematics understanding. Moreover, Webb (1979) suggested that conceptual knowledge is an important factor in mathematics problem solving.

Therefore, in the calculus domain as a basic part of mathematics learning, teaching and learning both for conceptual and procedural knowledge should be considered. However, procedural knowledge in calculus is not confined to the computational skills, as defined by Eisenhart et al. (1993), but also includes algebraic, procedural and transferable skills.

5.4 Some Main Categories of This Study

5.4.1 Introduction

Calculus, as a basic part of modern mathematics, has an important role, in particular, within the first two years of higher study of many academic branches. Rote learning and misunderstanding of the key calculus concepts can lead to student disasters in the other stages of mathematics and science learning. Mathematicians who are involved in teaching calculus believe that most undergraduate students appear to find that calculus is hard to learn. Factually, as Rosenthal (1995) noted, students don't see mathematics as the dynamic, exciting and creative discipline that it is.

Therefore, "the major purpose of this research was to study higher education calculus and students' learning styles to help educators and students towards a better calculus course and to overcome difficulties which may be experienced by educators and students in coming to understand calculus content".

In carrying out the study and after intensive investigations and interviews with mathematicians, the calculus materials were classified into three significant categories. In addition, students' performance (weakness) in the manipulation of mathematical notations and logical discussions is called as (Z_1) category. For Iranian students, the result of mathematical achievement in university entrance examination is labelled as (Z_2) or (En) to identify their grounding in high school mathematics at the beginning of the calculus course in higher education. Therefore, the six categories which are defined in the next sections, are labelled in this study as the set $D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$.The researcher found that these categories could be universal problems in teaching and learning of calculus in higher education.

It must be noted that, each $(Z_i), i = 1, 2, \cdots, 6$, may have interaction and relationship with others in a scheme. In other words, they are not isolated categories with no interaction with each other. In spite of suggesting that the same categories in the calculus domain are common factors, there is also some evidence that students' difficulties could arise from various sources at separate levels in different countries. Regardless of the ordering of the items in the set D, the more important categories, i.e. (Z_4, Z_5, Z_6, Z_1), which are involved in learning calculus and problem solving, will be discussed.

5.4.2 Misunderstanding or Nonunderstanding of Calculus concepts

Mathematical concepts are formed in the students' minds not only during the lecture time, because mathematics activities will not finish at the moment of leaving class. Rosenthal (1995) pointed out that the best way to learn mathematics is by actively doing mathematics; by discussing it with others and by synthesizing major ideas. In typical university mathematical classes in the USA, students passively watch their lecturer at a black-board and they seldom speak and discuss the materials in class. It must be emphasized that in Iran, Britain, and possibly in the other countries the situation could be the same in mathematics and calculus courses. As a result, many conceptual materials may be learnt by an individual in either a rote or meaningful fashion and the level of understanding depends more or less on the nature of mathematical concepts as a whole.

As discussed in the previous sections, personal understanding of the calculus concepts may be different from formal definitions which are accepted by mathematicians. This remarkable reality could be the starting point of students' mental conflicts and their non or misunderstanding of the calculus contents. Non or misunderstanding of concepts is more attributable to a lack of deep understanding than to technical difficulties. On many occasions a critical concept has a key role in the calculus multi-conceptual question tasks.

In a pilot study by the researcher at Glasgow University, the examination books of seventeen students, in the calculus section, were investigated. The main aim of this investigation was to find the various non or misunderstandings of some critical calculus concepts. They turned out to be injective, bijective, the inverse of a function and its domain and range, the relationship between continuity of a function and its derivative, improper integral, which appeared in questions in the second paper of Mathematics 1A in June 1994.

The following questions were selected to study the students' performance:

1. Define the term *injective* as it applies to a function $f : A \to B$.

Use the definition to show that the function $g : [0, \pi/2] \to R$ defined by,

$$g(x) = \frac{1}{2 + \sin x}$$

is injective.

2. The function $f : (-\infty, 0] \to (0, 1]$ defined by

$$f(x) = \frac{1}{1 + x^2}$$

is a bijection. Find its inverse.

3. By arguing from first principles find f' when f is the real function defined by $f(x) = \sqrt{x}$. Explain how the continuity of $\sqrt{x}$ is used in determining f'.

4. Prove that the improper integral

$$\int_1^\infty \frac{1}{x^4}\, dx$$

exists and find its value.

The analysis of the students' performance on the these questions produced some conclusions as follows:

1. Confusion between the conceptual differences among the definition of a function and some aspects of the function such as one-one (injective) or onto (surjective) functions.

2. Some mistakes as:

$$\int f(g(x))\, dx = \int f\, dx \int g(x)\, dx$$

and

$$\int f(x)g(x)\, dx = \int f(x)\, dx \int g(x)\, dx.$$

False integration: if

$$\int f(x)\, dx = F(x) + c$$

then

$$\int f(g(x))\, dx = \frac{1}{g'(x)} F(g(x)) + c.$$

3. Confusion between the definition of function and one-one function or, function and onto function by all students.

4. However, a few students could use the $(1-1)$ function in problem solving (rule learning) despite the non/misuderstanding of this kind of function.

5. Although a few students found the inverse function in question (2), all of them had a serious problem in finding the domain and range of that inverse function.

6. All of them failed to explain how the continuity of $\sqrt{x}$ is used in determining f' in question (2). Some students attempted to explain, but gave wrong explanations which could be evidence of their misunderstanding of this issue. It is interesting to have a look to some of their explanations in the examination:

   Student A: "When sketching the derivative of a function the function must be continuous over a neighbour."

   Student B: "If $\sqrt{x}$ is continuous on its max domain this means that $\sqrt{x}$ is differentiable."

   Student C: "The continuity of $\sqrt{x}$ is used in determining f' because we look at the difference quotient (D.Q.) as $h \to 0$ we could not do this if $\sqrt{x}$ was not continuous (as so would not be able) to find f' using this method."

   Student D: "$\sqrt{x}$ must be conti. so H can be evaluated as it tends to 0."

   Student E: "If function $f(x) = \sqrt{x}$ not continuous, limit would be continuous."

   Student F: "This equation $f(X) = \sqrt{x}$ is based on a graph and so function has to be continuous".

7. Most students failed to cope with the improper integral which may be an indication of their non or misunderstanding of this critical concept in calculus.

A literature review, a series of interviews with some calculus lecturers in Iran and Glasgow University in Scotland, and an analytical approach to students' performance in the calculus examinations by this researcher directed him to classify some of the important students' difficulties in the critical concepts of calculus as follows:

1. The meaning of "x approaches a" in the limit processing, the use of a formal definition of limit, that is ϵ, δ-definition, in context to find the right answer

for δ, and use of the continuity definition of a function, i.e. $\lim_{x \to a} f(x) = f(a)$, in the problem solving situation.

2. The right relationship between continuity and differentiation.

3. Students don't recognize the common nature of the 'limit of a function' and the 'limit of a sequences or series' (Ervynck, 1981) and little conception of the power of limiting processes in mathematics and, in particular in the calculus (Orton, 1983).

4. Recognizing the domain and range of an inverse function and an inverse trigonometric function, and determining the domain and range of a composite function.

5. Understanding the difference between differential and derivative concepts in

$$dy = f'(x)\, dx.$$

6. Difference between local and global maximum and minimum points.

7. Perception of the gradient of a function.

8. Being able to appreciate when an integral is improper and to be clear about what it means for an improper integral to exist.

9. The meaning of constant term in the integration process.

5.5 Multi-Conceptual and Procedural tasks

5.5.1 Definition and Illustration

Multi-conceptual and procedural calculus tasks, which are labelled in this study as (Z_4) category, are the calculus questions in which more than one critical concept come together to establish a more complicated combination. Moreover, some procedural skills are necessary in tackling such calculus problems, but skills are not so long and difficult. Hence, in this situation, students should be able to cope with both conceptual combination and conceptual articulation to follow the processing of a solution.

Having analytical thinking and restructuring skills could be beneficial for students in recognizing the critical concepts which are embedded in such type of calculus tasks. In addition, students are in need of some skills to select the more convenient procedural ways to use the separated critical concepts in context. It should be noted that the (Z_4) category is not only a feature of calculus tasks in the mathematical area, but students are expected to have more difficulties with this kind of material than with the advanced level of mathematical learning.

Trying to understand the problem, not only in learning calculus but in the other subjects as well, is the main step to be followed. Students' confusion in the conceptual accumulated tasks is a very common problem in the whole domain of mathematical and science learning. Herron (1996) suggested three possible explanations for such confusion in chemistry that may be adopted to describe students' confusion in the mathematics area, in particular, in learning calculus as follows:

1. Inadequate concept learning:

   Many difficulties which students encounter in calculus involve inadequate concept learning rather than inadequate skills ability. In a calculus examination the beginning students were asked to sketch a graph of

   $$f(x) = x[2x + 1] + |x - 1|$$

   on (0,2). The researcher found that a lot of students had difficulty with the concepts of the integral part function (the greatest integer function) $[2x + 1]$ and the absolute value function $|x - 1|$ and this misconception caused their inability to draw the graph of $f(x)$. Finding an appropriate value of δ was a problem for all students to prove that

   $$\lim_{x \to 1} \frac{2}{3x^2 - 1} = 1$$

   by the ϵ and δ definition.

2. Inadequate consolidation of concepts:

In the calculus, students know the distinction between domain and range of a function or inverse function, the difference between the definition of function and one-to-one or onto function, but the respective schemas in many new contexts seems to be insufficient to determine the related domain and range of a function or its inverse. The researcher found many examples of this inability among students' performance in calculus examinations which were discussed in the previous sections.

In addition, among 113 pre-calculus students at Sabzevar University, a few students were able to determine the range of the two real functions defined by

$$f(x) = \frac{1}{\sin x \cos x}$$

and

$$f(x) = \frac{[x^2]}{x}.$$

In another study by this researcher, 40 third-year mathematics students at Sabzevar University were asked to answer some questions about their main difficulties in calculus in high school and higher education.
All of them complained of non or misunderstanding limiting process from the ϵ and δ definition. A summary of their comments are as follows:

(a) Student A: "My main problem in learning calculus at university was the limit of a function. I never understood the real meaning of δ, and I always accepted each value for δ at the end of my solution."

(b) Student B: "No concepts are more difficult for me than limit and continuity. Essentially the concept of limit, tending to a point, evaluating a limit by (ϵ, δ)-definition have been my main problems in high school and higher calculus learning. In the limit section, rote learning of rules and memorizing formulas were my best attempts."

(c) Student C: "In limit processing, sometimes we need to choose 'one' as the radius of a neighbourhood to find the exact value of δ. The reason for this selection was not clear to me."

(d) Student D: "The concept of limit defined in terms of ϵ and δ was not digested by me, in particular when I used this definition in the Riemann Stieltjes integral. Teachers' inability to teach limit processing was the main reason for my non-understanding of limit."

(e) Student E: "The concept of limit had not been clear for me in high school, for example, I was thinking why should we write '$\forall\epsilon > 0$, $\exists\delta > 0$' and why not $\exists\epsilon$, $\forall\delta$"?

(f) Student F: "Definition of limit by (ϵ, δ) was my basic problem. What is the real meaning of ϵ and δ? Why we should use ϵ, δ and how should we choose them? What is the meaning of the radius of convergence? Why do we choose 'one' as the radius of a neighbourhood in most problems involving finding a limit? I never understood the exact meaning of ϵ and δ."

(g) Student G: "I always had difficulty in finding the radius of a neighbourhood in solving problems about limits by the formal definition."

(h) Students H: "I never understood the difference in meaning of (ϵ, δ), in particular, when I should use the radius of a neighbourhood in limit and continuity questions."

(i) Student I: "The concepts of limit and continuity have not been clear for me because the difference between limit and continuity was not explained clearly."

It can be concluded that one of the greatest difficulties for students in learning calculus lies in the concept of limit. In fact, the trouble begins in turning the statement " if x is close to x_0 then $f(x)$ is close to l"into mathematical terms. Ervynck (1981) suggested some items to identify such difficulties in understanding limit processing thoroughly as follows:

3. "The meaning of x approaches x_0".

   "The interconnection of the role of ϵ and δ".

   "The role and the order of the quantifiers $\forall$ and $\exists$".

   "The insignificance of the case $x = a$ and the value $f(a)$".

"The fact that x_0 has to be a closure point of the domain of the function.

4. Inadequate attending:

   In reading calculus questions students are sometimes not able to attend sufficiently to identify the exact meaning of the words in the questions. This can be described as students misreading. Calculus word problems, which will be discussed in the next sections, are typical examples which illustrate students' misreading of the calculus tasks.

5.5.2 Some Multi-conceptual and Procedural Calculus tasks

Let us consider some more questions which may be classified as multi-conceptual and procedural problems in calculus

Q1. (Mathematics 1A, Glasgow University, June 1994)

Evaluate

$$\lim_{n\to\infty} \tan^{-1}(n \sin \frac{1}{n}).$$

This question is in the (Z_4) category. Some critical concepts and procedures are combined and students have to recognize and separate them to be able to reach the right solution step by step as follows:

1. The concept of limit and limit at infinity.

2. The meaning of tan and sin, which are trigonometric functions.

3. The meaning of the inverse function arctan in a context which is based generally on the meaning of inverse function.

4. The concept of continuity and the continuity of arctan on R.

5. To recognize

$$\lim_{n\to\infty} \frac{\sin 1/n}{1/n}$$

and to use it in the context. In addition, more procedural skills are needed to evaluate this limit and so it seems that students must be able to think openly at the early stages and then become more and more narrow before reaching a unique answer, i.e. $\frac{\pi}{4}$.

Q2. (Calculus 1, Mashhad University, January 1995)

Suppose that $\lim_{x \to a} f(x) > 5$. Show that $\exists \delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) > 5$.

Q3. (Calculus 1, Sabzevar University, January 1995)

By using of ϵ, δ definition prove that,

$$\lim_{x \to 0} \frac{1 + x^2}{1 - x} = 1.$$

Q4. (Mathematics 2B, Glasgow University, June 1994)

A sequence $\{a_n\}$ of real numbers is defined recursively by

$$a_{n+1} = \frac{a_n^2 + 2}{3} \quad (n \geq 1).$$

where, $1 < a_1 < 2$. Prove, by induction, that $1 < a_n < 2$ for all integers $n \geq 1$. Show that $a_{n+1} < a_n$ for all integers $n \geq 1$. Deduce that $\{a_n\}$ converges and find $\lim_{n \to \infty} a_n$.

In this question, some concept definitions and procedures have to recognized and followed step by step, namely:

1. The concept of a sequence of real numbers, i.e. $\{a_n\}$.

2. Relational understanding of mathematical induction procedures and logical deductions in context.

3. The meaning of monotonic sequences (decreasing and increasing).

4. The concept of convergence/divergence of sequences and related procedural skills to determine whether a sequence is convergent or divergent.

5. The meaning of bounded below (above) and decreasing (increasing), and convergence/divergence behaviour of sequences have to adapted to the question's demands to prove that $\lim_{n\to\infty} a_n = 1$.

Q5. (Calculus 1, Mashhad and Sabzevar University, January 1995)

Evaluate

$$\lim_{n\to-\infty} \frac{[2x^2] + \text{sgn(x)}}{x^2 + |x|}.$$

Q6. (Sabzevar University, June 1995)

Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Discuss the differentiability of f at $x = 0$ and determine the asymptote lines if they exist.

Q7. (Mathematics 1A, Glasgow University, June 1994)

Evaluate the definite integral

$$\int_1^3 (x+2)^2\, dx$$

by using Riemann sums.

5.6 Mathematical Translation in Calculus

5.6.1 Definition and Illustration

This category which is labelled in this study as (Z_5), describes the process of translation between mathematical abstraction (notation) and pictorial (concrete) forms in calculus materials. Clement et al. (1981) noted that the process of translation between mathematical notations and a pictorial situation presents students with difficulty. You may encounter this with some applications of calculus to science and everyday problems. These applications appear in the form of word problems rather than in mathematical symbols.

Gagné (1983) suggested that in the process of translation between a concrete situation, or a verbally described situation, into mathematical form students should be able to translate the concrete to the abstract, and the abstract to the concrete.

In addition, there is a set of skills that enable students to identify appropriate mathematical operations. Translation skills could be critically important in learning calculus and problem solving. Going from one mathematical representation and status to another one may be a means of translation in calculus thinking. Students need to switch from one type of representation to another one for meaningful learning of calculus.

The three groups of calculus material which may have a critical role in the mathematical translation are classified in the (Z_5) category as follows:

1. Calculus word problems including:
   a) Applied maximum and minimum problems.
   b) Related rates problems.
2. Curve sketching techniques and the role of derivative.
3. Visual thinking and curve interpretation.

5.6.2 Calculus Word Problem (CWP)

Orton (1992) defined a word problem, or verbal problem, as a task which requires the application of mathematics to achieve a solution, but the required procedures are to be extracted from within sentences. All evidence suggests that learners have difficulties with the mathematical word problems in context and therefore calculus students prefer to deal with mathematical symbolism and formulae instead of verbal problems. However, as Orton (1992) suggested, the order of information, the relation between known and unknown and the transition from known variables to unknown, all influence students' understanding of a word problem.

In calculus, the idea of derivative is a powerful tool in tackling word problems. Therefore, inability to determine extreme function values, inadequate mastery of algebraic and calculus procedural skills in context could be the main students' difficulties in CWP questions. CWP requires both reading comprehension and mathematical skills which interact during the solution (Ross, 1980). In Ross's investigation, the difficulties students exhibited in CWP generally fell into one of the two categories:

1. Difficulties due to inadequate mastery of the calculus concepts and skills which are prerequisites for success in the CWP.

2. Difficulties in applying the procedures or strategies for CWP. The most significant difficulties in this category are:

   (a) getting started on a problem;

   (b) lack of an overall plan;

   (c) inappropriate use of the information given in the problem.

   Ross found that these difficulties were interrelated and cause confusion between variables and constants in maximum and minimum problems. Identifying the function to be differentiable was another common student problem.

In calculus word problems involving related rates, the hardest job may be to translate the verbal problem into mathematical terms (Marsden and Weinstein, 1985). They described many steps which should be followed to cope with the related rate problems such as:

1. Identify the variables which are changing with time.

2. Find the relationship between variables.

3. Draw a figure which could be essential to help students to spot some important relations.

4. Recognize the geometrical properties which are involved. For example, similar triangles and Pythagoras's theorem are often applicable in this type of calculus task.

5. Differentiate the relationship between variables x and y with respect to t, that is, thinking of x and y as a function of time.

Moreover, White and Mitchelmore (1996) strongly suggested that a major source of students' difficulties in calculus word problems involving rates of change lies in an underdeveloped concept of a variable. Students frequently treat variables as algebraic symbols to be manipulated without any regard to their possible contextual meaning.

5.6.3 Visual Thinking and Curve Interpretation

Introduction

Moore (1994) suggested that one of the major sources of students' difficulties in doing mathematics proofs was their poor intuitive understanding of the concepts. In many cases students were unable to do a proof because they did not understand the theorem or concepts involved. They could not produce a proof by working formally, but needed intuitive understanding before they could get started.

Intuitions are described as features of a student's knowledge that arise most commonly from his/her everyday experience. In general, they seem to exist prior to specific formal instruction (Leihardt et al., 1990). In fact, the most recent thinking in calculus and mathematics teaching views intuition and pictorial thinking as positive factors, around which to build instruction and learning (Campbell et al., 1995; Moore, 1994; Dreyfus, 1992; Lienhardt et al., 1990; Presmeg, 1986; Resnick, 1989; Vinner, 1989, 1982; Mundy, 1987; Moses, 1982; etc.).

In a review of research and theory related to the instruction of functions, graphs and graphing processes by Leinhardt et al. (1990), the discussion on students' learning is organized into two main parts: intuitions and misconceptions. Moreover, they suggested that some students' misconceptions about functions can be traced logically to intuitions and pictorial meanings. For example, students' tendency to interpret graphs iconically may be related to their intuitions regarding picture reading.

However, school and higher calculus curricula, textbooks and teaching methods favour the nonpictorial way of thinking. A visual approach to teaching and problem solving is not often valued by many calculus teachers. They are, unaware that conceptual learning in a calculus course could be easier if pictorial thinking was used. Ervynck (1981) suggested that the use of graphical representations may help to overcome the inherent difficulty of passing from a visual image to a formal definition of the limit concept.

Research into mathematics education shows that students generally are very weak visualizers in calculus course material, which in turn leads to a lack of meaning in the formalities of mathematical analysis (Tall, 1991). The researcher found that not only are calculus students naive in their visual approach to problem solving, but third-year university mathematics students, who are trained to be mathematics teachers, were uneasy in dealing with translation of a pictorial form into a mathematical formal definition and vice versa in this kind of task.

Presmeg (1986) noted that nonvisual teaching has the effect of leading students who are visual thinkers to believe that success in mathematics learning depends on rote memorisation of routine rules.

Cognitive Style and Visual Thinking

Cognitive scientists have shown an interest in investigating visual reasoning in mathematical and science problem solving (Dreyfus, 1992). In addition to external factors such as mathematics educators, curriculum and textbooks, internal factors such as preference, confidence and mathematical abilities could be possible reasons that students, even gifted ones, tend to be nonvisual thinkers (Presmeg, 1986).

In fact, students' cognitive styles would show their preference for particular ways of thinking and goal attainment. It was found in this study (see later chapters) that divergent thinkers mainly favoured a visual approach when compared with convergent thinkers. Therefore, if this natural preference is not reinforced by teaching methods this could be the starting point of students' mental conflicts in learning calculus and problem solving.

Visual/Nonvisual ways of Thinking in Mathematics

Dreyfus (1992) noted that visual thinking is generally considered helpful in supporting intuition and concept formation in mathematics learning. Visual thinking is a way of thinking and can be viewed as a nonanalytic, non-algorithmic mode (Moses, 1982). Visual and nonvisual methods of solution of mathematical problems were defined by Presmeg (1986) as follows:

1. A visual way of mathematical solution is one which involves visual imagery, with or without a diagram, as an essential part of the method of solution, even if reasoning or an algebraic way are also employed.

2. A nonvisual way of solution is one which involves no pictorial imagery as an essential part of the way of solution.

Presmeg cited Moses' (1977) definition of the visual way of mathematics problem solving that included solutions involving constructions, diagrams, drawings, tables, charts or graphs, whether written down or in the student's mind. Moreover, mathematical visual thinkers are persons who prefer to use visual ways of solutions in mathematical tasks which may be solved by both pictorial and nonpictorial methods (Presmeg, 1986). However, the most effective mathematics learning style involved the use of visual thinking together with an emphasis on abstraction and generalisation (Campbell et al., 1995). This dual emphasis could be a beneficial aid for students reducing the limitations associated with one way of visualisation or abstraction.

However, as Bennett (1988) noted, the majority of students learn to manipulate symbols and carry out procedures without acquiring insight into mathematical concepts and without insight it is hard to build conceptual knowledge. Visual representation is a significant way of gaining insight into concept relationships and algebraic statements of these relationships. Pictorial thinking of many calculus ideas may supply students with a powerful, meaningful understanding and problem solving strategy. Nonetheless, as noted by Dreyfus (1992), students' reluctance to use visual reasoning and pictorial considerations are documented widely in the literature.

Despite the the important role of visual and spatial thinking in mathematical performance, a lot of students, especially at secondary school, tend to regard mathematical thinking as being largely verbal in nature. This view is due partly to the highly algebraic form in which mathematical work is typically expressed and students often fail to develop the visual, nonverbal, component of their mathematical thinking (Shear, 1985).

Many students are unable to recognize a healthy match between their visual thinking and the answers they reach through mathematical manipulation (Mundy and Lauten, 1994). There is some evidence that a healthy balance between analytical solution and visual solution may guarantee students' meaningful understanding of the calculus course material.

Moreover, Dreyfus (1992) suggested that more than a balance in various forms of mathematics concepts, that is, the integration of algebraic, verbal and visual thinking, should be intended. Balance is to be an aim for integration and to achieve balance, visual reasoning needs to be given parity along side algebraic reasoning if calculus teachers wish to improve students' understanding.

The researcher found that calculus students are not easy in using graph sketching and graphical interpretation to recognize that a function has limit or one-handed limits, whether it is continuous at a point or on an interval, or is differentiable on that interval. Despite the fact that students tend to avoid a pictorial way of thinking, graph reading (interpretation) could be essential in reducing their difficulties in some important calculus ideas. Therefore, "working with mathematical curves without formulae supports, and inferring calculus ideas from graphical information, is to be defined in this research as curve interpretation or graphical reading".

Pictorial and geometrical interpretation of some main concepts and theorems of calculus such as Rolle's theorem, the mean value theorem, multivariable calculus (for example, evaluation of double and triple integrals) can be an important help for better understanding. As Eisenberg and Dreyfus (1986) noted, such materials are highly visual in nature and hence students fail to handle the relevant visual transformations into analytical thinking.

In addition, visual-figural information not only improves analytical manipulations, but also shows how visual orientation can provide the information that may be neither obvious nor easy to remember analytically. Visual thinking may be used effectively to achieve deeper understanding of calculus, while analytical solutions alone cannot be an indication of students' relational understanding.

As was discussed in the previous sections, in learning functions, students strongly prefer functions expressed in terms of formulae rather than in other kinds of representations such as a pictorial form. Mundy and Lauten (1994) suggested that learning about function can be promoted through the connections between functions and their graphs. And the researcher found that not only calculus students, but third-year mathematics university students have problems relating graphs of functions with their rules and matching them with the graphs of their derivative and, in general, in curve interpretation. Eisenberg and Dreyfus (1986) suggested that, in calculus, spatial visualization is commonly used for explaining the main concepts, the derivative and the integral.

It is certainly true that visualization plays a role in mathematical thinking in general and in concept acquisition and problem solving in particular. Shear (1985) stated that the theorems which are not easy to prove algebraically will often be easy to understand and prove geometrically (visually) and vice versa. And Orton (1983) suggested that graphical work has a great importance in developing "concepts of rate of change". However, pupils' graphical understanding may be limited.

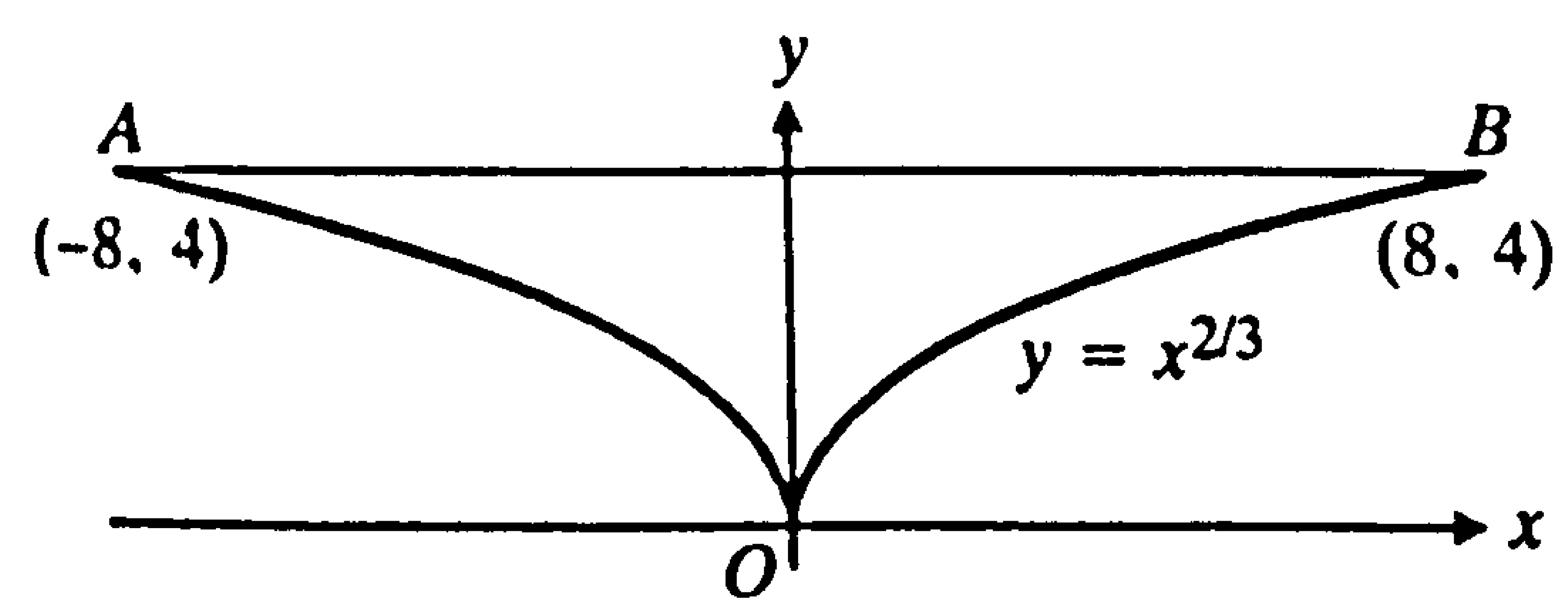
In fact, representations have a very important role in mathematical thinking and learning (Dreyfus, 1991) and mathematical concepts may be represented in various forms. Students' understanding of concepts, means having access to different ways of representation, being able to select the most appropriate form for particular uses (Schoenfeld, 1995). Pictorial or graphical, verbal, numerical and analytical forms could be used to illustrate the four various ways of representing the calculus concepts. Those should be emphasized thoroughly in teaching and learning calculus.

For instance, in the case of learning functions, Dreyfus (1991) pointed out that graphs, formulae, arrow diagrams and value tables, as various forms of function representations, should be given equal attention by teachers.

In the following question tasks, which are classified into the (Z_5) category in this study, it is not easy for students to handle the necessary mathematical transformations between visual thinking and analytical. However, they may promote and encourage students' visual thinking and its balance and in general its integration with formal and nonvisual thinking to help them in dealing with their conceptual obstacles and non/misunderstanding of calculus materials.

1. Does the mean value theorem apply to the function $y = f(x)$ on the interval $[-8, 8]$ of the Figure 5.3?

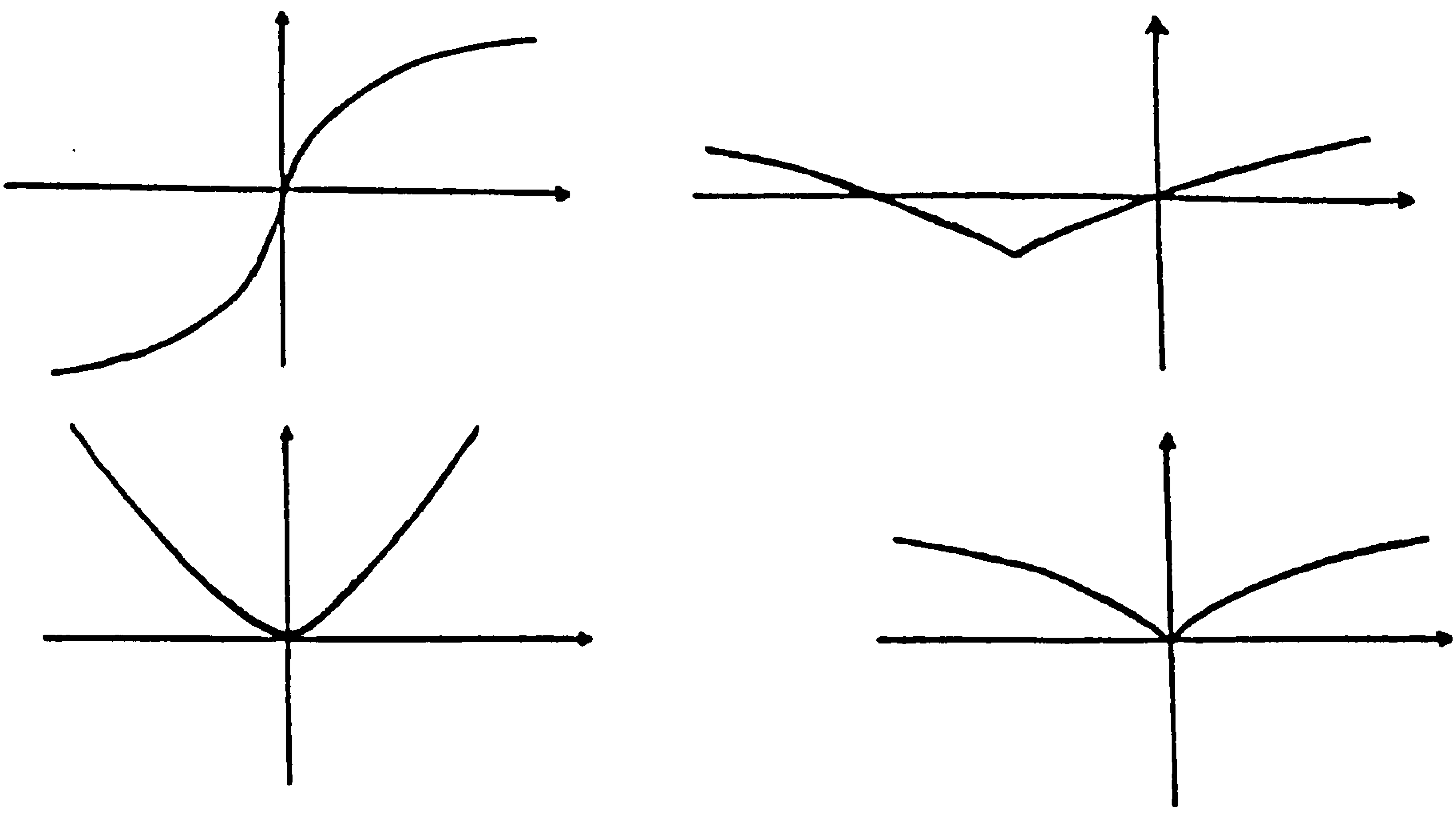
Figure 5.3



Geometrically, the mean value theorem says that somewhere between A and B the curve should have at least one tangent parallel to the secant line through A and B; that is, there are some number c in (a, b) such that $f(c)' = (f(b) - f(a))/(b - a)$. But, Having a tangent at each point of a graph does not mean having a derivative at each point and does not guarantee the conclusion of the mean value theorem. The graph of $f(x) = x^{2/3}$ has a tangent at every point (the tangent at 0 is vertical), but none of the tangent is parallel to the line segment AB. In fact, the difficulty can be traced to the f' to exist at $x = 0$ and the mean value theorem does not apply on a closed interval $[-8, 8]$ unless the function $y = x^{2/3}$ is differentiable at every point of $(-8, 8)$.

2. (Calculus exam, Sabzevar University, January and June 1995)

   Match the rules of functions: a. $x^{2/7}$, b. $x^{3/7}$, c. $(1+x)^{2/7}-1$, d. $(1+x^2)^{1/7}-1$ to the following graphs:



3. For the function in the following figure with domain $[0, 4.5]$ discuss the existence of the following items at the given points on the curve

   a) left-hand limit.

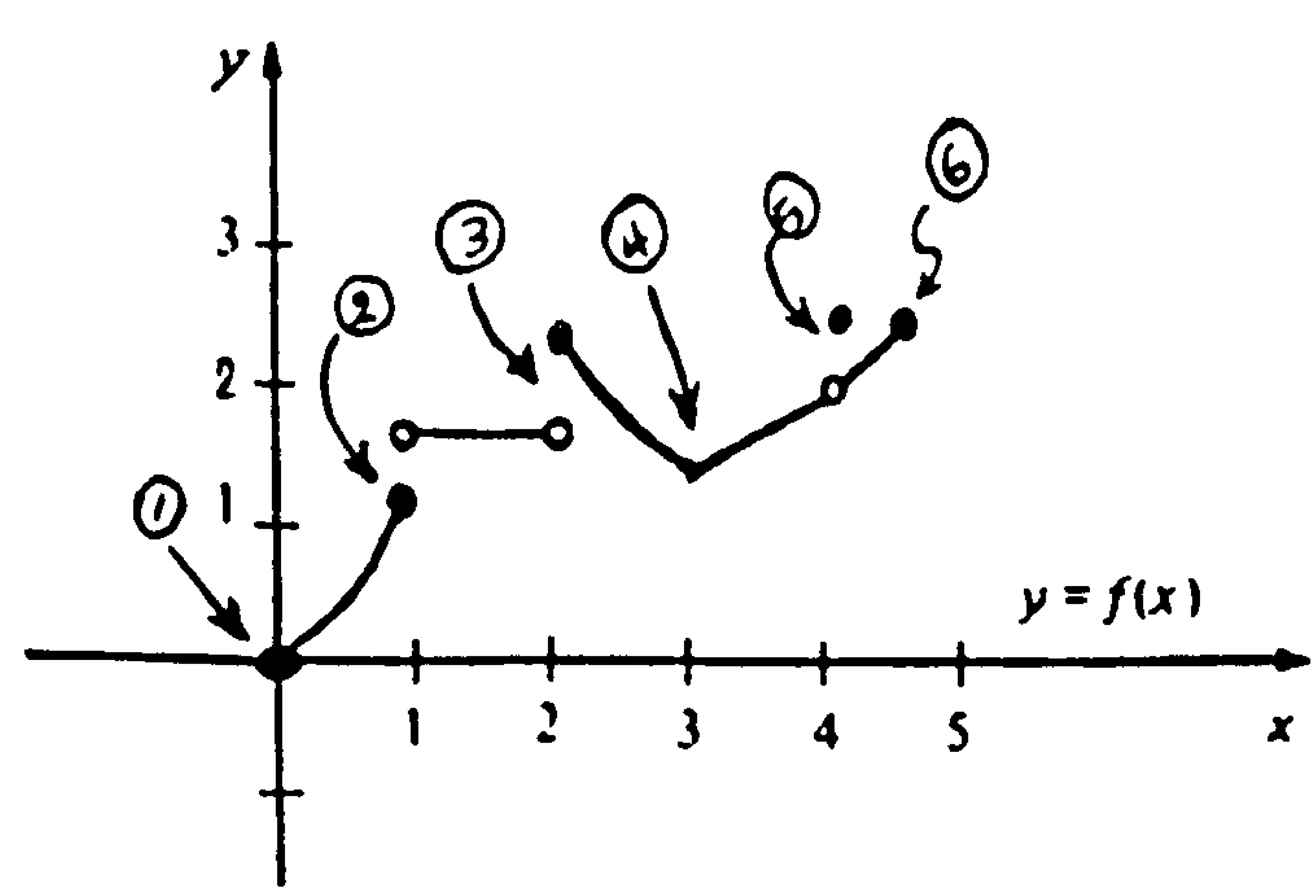
   b) right-hand limit.

   c) limit of the function.

   d) right-hand continuity.

   e) left-hand continuity.

   f) continuity of the function.

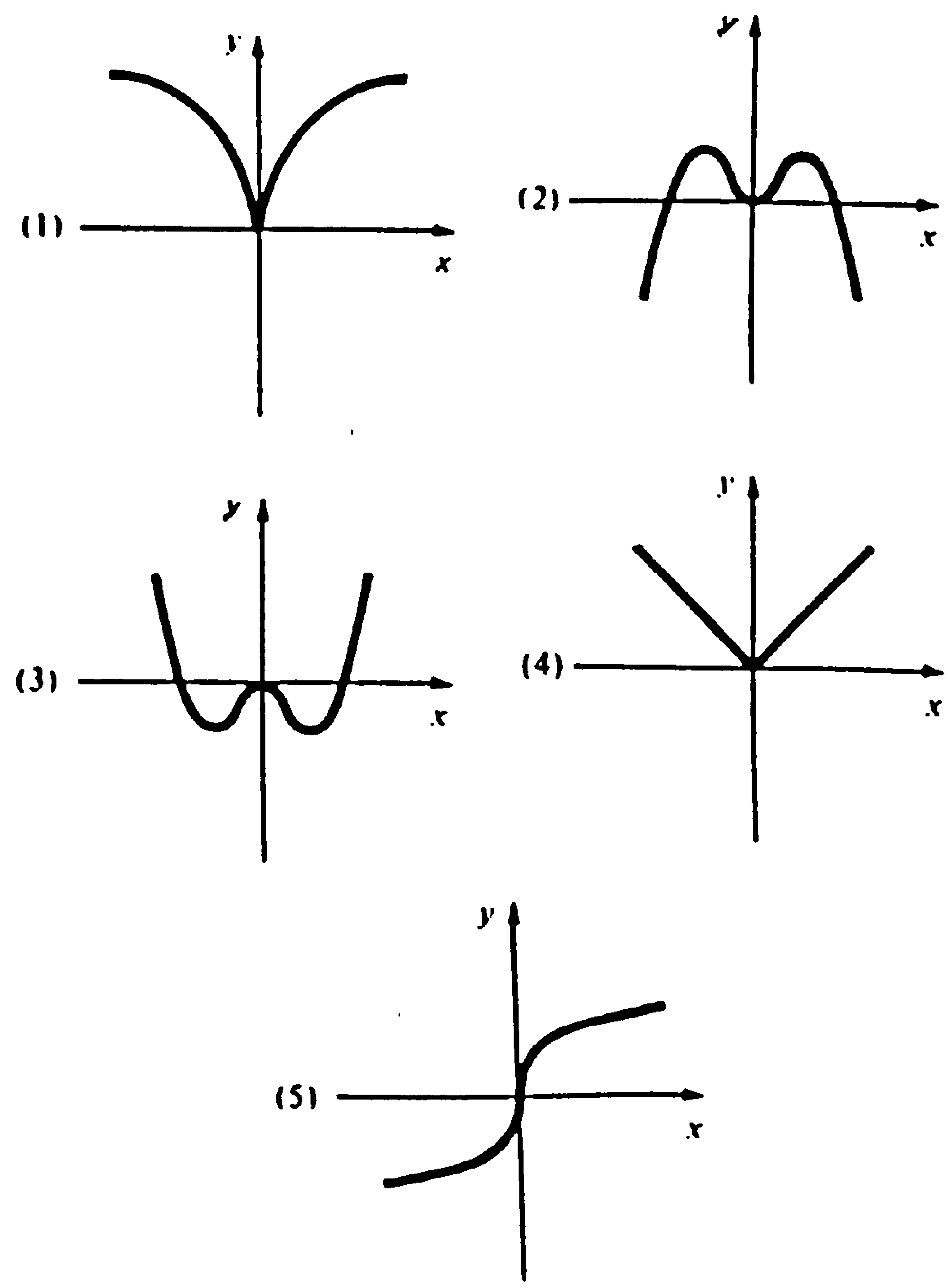


4. Sketch the curve of the function

$$f(x) = x[2x+1] + |x-1|$$

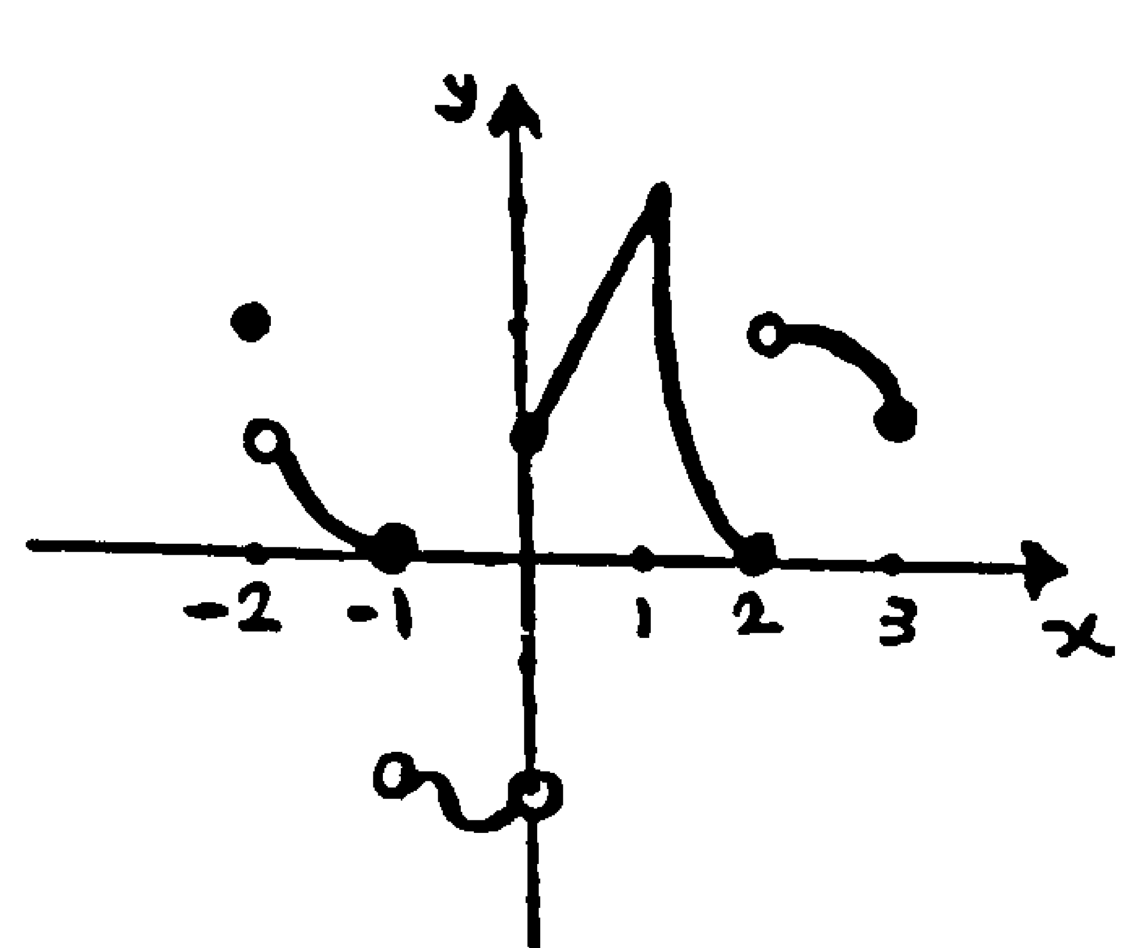
   on (0,2).

5. Sketch the derivatives whose functions are shown in the following figure:



6. (Calculus exam, Mashhad University, June 1995)

Discuss the existence of a limit (left and right hand), continuity and differentiability of the function f whose curve is given on $[-2, 3]$ in the following figure. Is f integrable on $[0, 2]$?



7. (Calculus final exam, Sabzevar University, June 1995)

   Sketch the the graph of the function

$$f(x) = \begin{cases} x + \operatorname{sgn}([e^x]) & \text{for} \quad x \leq 0, \\ 1 - x^x & \text{for} \quad x > 0. \end{cases}$$

8. (Mathematics 1A exam, Glasgow University, June 1994)

   A manufacturer, wishes to produce tins, such that each will hold 500 cm^3 of soup. The tin is a cylinder with a circular base and top. Find the radius of the tin for which the surface area is smallest.
   (For a cylinder of height h and radius r, the volume is $\pi r^2 h$ and the surface area is $2\pi r(r+h)$.)

9. (i) Sketch the curve

$$y = \frac{x^2 + 2x - 3}{(x+2)^2},$$

   showing the coordinates of any critical point, the approaches to any vertical and non-vertical asymptotes and points of intersection with asymptotes and axes.

   (ii) Show that the curve

$$y = x^2 - 2e^x$$

   has (0,2) as its only point of inflection and sketch the curve in the neighbourhood of this point. (Curve sketching question)

10. (Mathematics 1A, Glasgow University, June 1995)

    The rate of growth of a population of ants is known to be proportional to the size of its population. If there were 1000 ants in 1980 and 5000 ants in 1990, what is the size of the population in 1995?

5.7 Multi-skilled and transferable skills Tasks

5.7.1 Definition and Illustration

Multi-skills, transferable and procedural skills tasks in calculus are labelled in the present research as category (Z_6). This kind of calculus question contains material in which a lot of mathematical skills and procedural skills come together to establish a more complicated combination. In this category attention is focused on the development of transferable skills, i.e. "skills and abilities", as Kemp and Seagraves (1995) stated, "which are considered in more than one context". How to apply mathematical skills in different contexts and problem solving situations is an important matter in calculus activities. Students' ability to contextualise mathematical skills may be as significant as the skills themselves. For example, transferable skills can play important role in calculus word problems, curve sketching, application of integrals and evaluation of double and triple integrals.

In these tasks, students have to cope with some complicated skills and techniques rather than conceptual materials. Cornu (1991) stated that students are often able to deal with the many mathematical problems they are asked to perform without understanding the formalism of the concept definition at all. They may carry out the necessary procedural skills and reach to the correct answer through mathematical manipulations, despite their conceptual rote learning. In calculus, students often can produce correct solutions to a lot of limit, differentiation and integration problems, in spite of their non or misunderstanding of some involved concepts. For instance, they may reach right answer in evaluation of $\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}}$ or even $\lim_{t\to 0} \frac{\sqrt[3]{(t+a)^2} - \sqrt[3]{a^2}}{t}$ without having relational understanding of right-hand limit and limit processing or they may differentiate $f(x) = \sin 3x / \cos^3 x$ with respect to x, despite their misconception of defined concept of derivative of a real function.

The researcher found that a lot of students had difficulty in evaluating the definite integral $\int_1^3 (x+2)^2\, dx$ by using Riemann sums, while they had no problem in finding the correct solution by using the fundamental theorem of calculus. This means that there is a mismatch between their meaningful learning of Riemann integral and the answer they can reach through mathematical manipulation and procedural skills.

Moreover, the nature of this category indicates that students' ability with algebraic manipulations is a prerequisite for tackling some calculus tasks which are classified into (Z_6). Each mathematician who teaches calculus at any level would agree that algebraic skills, mastery, simplification and factorization are a prerequisite for the calculus course. In fact, they are the bases for this category of calculus task. Harmon (1984) noted that as a result of some previous study there is a strong relationship between achievement in college algebra and calculus learning.

The researcher found in this study that students' weakness in algebraic manipulations caused a lot of their blockages in (Z_6) examination tasks, in particular, in Glasgow sample of the present research. As Porter (1993) noted, students in the U.K. normally have a poor background in algebraic skills and manipulation as a result of their schooling and too much calculator use.

It was discussed in Section (1.3.2) of this chapter that teaching and learning both for conceptual and procedural knowledge should be considered in a calculus course and procedural knowledge in this situation is more than ordinary computational skills.

Students are exposed to a large number of (Z_6) questions in their calculus course, therefore they should have acquired the procedural skills of the ability to cut a (Z_6) problem down to the core. They must also be able to combine the necessary procedural skills and recall a lot of rules to find the best way of reaching the correct solution. This can sometimes be more difficult than conceptual understanding.

Indefinite integration and techniques of integration can be typical (Z_6) tasks which cause students more difficulty than others. Schoenfeld (1985) noted that the prerequisite algebraic and differentiation skills required for indefinite integration are mechanical. Students learn to perform these techniques, but they can use them well when they know which technique is helpful to use. Schoenfeld suggested that students' difficulty is basically in selecting a reasonable approach rather than in implementing that approach.

In addition, Searl (1992) stated that despite the help of more practice exercises with multi-skilled tasks such as

$$\int (x-1)\sinh x \, dx,$$

many students may be able to evaluate, under examination conditions,

$$\int \frac{(2x+2)}{x^2+2x+2)}\, dx$$

but will not be able to deal with

$$\int \frac{(2x+3)}{x^2+2x+2}\, dx.$$

He suggested that difficulties maybe arise not from the nature of students' understanding, but from the students' overload of too much to remember.

In another area, the researcher found that a large majority of calculus students could not to evaluate

$$\lim_{x\to 1} \frac{\sin \pi x^{\alpha}}{\sin \pi x^{\beta}} \qquad (\alpha, \beta \in N)$$

or

$$\lim_{x\to 1} \frac{x^1 + x^2 + \cdots + x^n - n}{x-1}.$$

He also found that students' obstacle to completing a question such as "find the volume of revolution obtained by rotating the curve

$$y = \tan x, \quad 0 \le x \le \frac{\pi}{4}$$

about the x-axis.", was mainly the evaluation of

$$\int_0^{\frac{\pi}{4}} \tan^2 x \, dx.$$

5.7.2 Some Typical (Z_6) Calculus Tasks

Some examples of questions in calculus could be classified into multi-skilled and transferable skills tasks (Z_6) in this research are as follows:

1. Let

$$f(u) = u^2 + 5u + 5$$

and

$$g(x) = \frac{x+1}{x-1}$$

Find the derivative of the function

$$h(x) = f(g(x))$$

in two ways:

a. By determining the composition of the functions and then find $(f \circ g)'$.

b. By the chain rule.

2. Evaluate

$$\lim_{x \to 1} \frac{\sin \pi x^{\alpha}}{\sin \pi x^{\beta}} \qquad (\alpha, \beta \in N).$$

(Calculus 1, Sabzevar University, January 1995)

3. Use implicit differentiation to calculate y' in terms of x and y when

$$5x^2 + 4xy + y^2 = 2.$$

(Mathematics 1A, Glasgow University, June 1995)

4. Evaluate

(i)

$$\lim_{x \to 1} \frac{x^1 + x^2 + \cdots + x^n - n}{x - 1},$$

(ii)

$$\lim_{t \to 0} \frac{\sqrt[3]{(1+a)^2} - \sqrt[3]{a^2}}{t}.$$

(Calculus 1, Sabzevar University, November 1995)

5. Evaluate

$$\int \frac{\log x^3}{x} \log(\log x^3)\, dx.$$

(Calculus 1, Mashhad University, January 1995)

6. Prove by induction that, $\forall x \in N$,

$$\frac{d^n}{dx^n}\left(\frac{1}{x+a}\right) = \frac{(-1)^n n!}{(x+a)^{n+1}} \qquad (a \in R, \quad x \neq a).$$

7. (i) Let $z = x^3 e^{y^2}$.

Show that

$$xy\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = yz.$$

(ii) Use the change of variables $u = x^2, \quad v = x + y$ to solve the partial differential equation

$$x\frac{\partial^2 z}{\partial x^2} - 2x\frac{\partial^2 z}{\partial x \partial y} + x\frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x^5.$$

8. By changing the order of integration, evaluate

$$\int_0^{\sqrt{2}} dx \int_0^{x^3} \frac{x^2}{y^2+9}\, dy.$$

(Mathematics 2B, Glasgow University, June 1994)

In fact, question 8 exhibits the combination of (Z_6) and (Z_5) categories in the single calculus task. Finding the region of integration to determine the upper and lower limits of double or triple integrals is considered as (Z_5), while the process of integration to evaluate it, is in the (Z_6) category. Therefore, both (Z_5) and (Z_6) categories interplay in such calculus tasks.

5.8 Mathematics manipulation and Logical discussion

5.8.1 Definition and Illustration

Calculus students normally have difficulty in communicating the concepts and skills in mathematical language: in other words, have difficulty in how to write mathematics statements in a formal and logical manner like those required in higher level courses such as real and complex analysis, topology and abstract algebra.

The category which is labelled in this study as (Z_1), describes students' weakness in the manipulation of mathematics notation and symbolic logic. Mathematicians would agree that mathematics is an abstract area, regardless of their views concerning the nature of mathematics and different ways of mathematical concepts representations. In mathematics one may prove or reject statements without any manipulation of the physical world. This type of independence makes mathematics a different kind subject from other realities.

Moore (1994) suggested that some major sources of students' difficulties in doing mathematics proofs are as follows:

1. The student were unable to state the definitions in a right manner.

2. The students' concept images were inadequate for doing the proofs.

3. The students were unable, or unwilling, to generate and use their own examples.

4. The students did not know how to use definitions to obtain the overall structure of proofs.

5. The students were unable to understand and use mathematical language and notation.

6. The students did not know how to begin proofs.

Therefore, students' perceptions of mathematics and proofs influenced their proof-writing performance and prevented their success in mathematical courses.

Some common students' difficulties with mathematical manipulations and logical arguments in calculus level were investigated in this research by observing the students' performance in examinations and interviews with lecturers. As a result, the researcher found some major sources of difficulties as follows:

1. Students' weakness in writing mathematical sentences in their proofs included a lack of appreciation of the use and importance of a mathematical proof. A lack of rigour is a very common problem among students. They usually write mathematical arguments with no indication of the connection between successive statements.

2. Misunderstanding of necessary and sufficient conditions, and wrong use of ($\Rightarrow$) and ($\Leftrightarrow$). In fact, the sign ($\Leftrightarrow$) is a very important logical symbol for linking successive lines of a mathematical argument and proof, because it has the force of conjunction. As Murphy (1991) said, students sometimes attempt to use ($\Rightarrow$) interchangeably with ($\Leftrightarrow$) causing confusion, because ($\Rightarrow$) has the force of a verb. In addition, confusion between ($\Rightarrow$) and ($=$) occurs in students at this stage. For instance,

   a. $x^2 - 1 \Rightarrow (x-1)(x+1) \Rightarrow x = 1 \quad \text{or} \quad x = -1$

   b.

$$\begin{aligned} \sqrt{x+1} + \sqrt{x-1} = 1 \Rightarrow x + 1 + 2\sqrt{x^2-1} + x - 1 &= 1 \\ \Leftrightarrow 2x - 1 &= -2\sqrt{x^2-1} \\ \Leftrightarrow 4x^2 - 4x + 4 &= 4(x^2-1) \\ \Leftrightarrow x &= 1. \end{aligned}$$

3.

$$\lim_{x \to -1} \frac{x^2+3x+2}{x^3+1} = \lim_{x \to -1} \frac{(x+1)(x+2)}{(x+1)(x^2-x+1)} \Rightarrow 1/3 \quad \text{as} \quad x \to -1.$$

4. Problem with mathematical notations as

$$f(x)' = \frac{f(x+h) - f(x)}{h}$$

   or $x = -1 \Leftrightarrow x^2 = 1$.

5. Logical Grammar.

Students have difficulty with the meaning of *all* and *some* and the manipulation of "quantifiers". This means that students have great problems (as is well known) with manipulating the definitions of limits and continuity (Tall and Vinner, 1981) and they are unable to understand the meaning of (ϵ, δ) in the limit processing.

Chapter 6

Methodology, Procedure and Samples

6.1 Introduction

In this chapter attention will be paid to the methodology which is employed in the present study to measure the students' cognitive styles. Several methods are used in cognitive psychology to test the Field-dependence/Field-independence and Convergence/Divergence dimensions of students. The researcher intends to apply some methods that have been developed and modified by other researchers at the Centre for Science Education at Glasgow University. The results of using such methods and measurements on the research sample will be discussed in the following chapter.

This research was conducted upon first year undergraduate students at two Iranian Universities and at Glasgow University, while they were doing the calculus course as part of higher education requirements.

6.2 Student Samples

In 1994 an attempt was made to select a sample of students from Iranian universities. The universities chosen are in Khorasan province in the north east of Iran. In 1995 the Scottish sample was selected from Mathematics 1A students at Glasgow University. Table 6.1 shows these universities with the number of participants and their subjects.

It is worth mentioning here that the original total number of the sample for psychological tests was 422 students in Iran and Scotland, but due to some procedure applied in this study, in fact, 574 individuals in Iran eventually completed the procedure for this investigation.

Table 6.1

The universities selected sample for the study

Selected university	No. of students	Subject
Teacher Training University of Sabzevar	113	Maths(N=74) Physics(N=39)
Ferdowsi University of Mashhad	254	Eng(N=200) Maths(N=54)
University of Glasgow	55	Maths 1A

The first experiment which was carried out in all universities, after selecting a sample for this study, was the measurement of the two dimensions of cognitive styles, that is, field-dependence versus field-independence and convergence versus divergence.

6.2.1 The Measurement of Field-dependence/independence

The first test applied to the sample is called the Hidden Figure Test (HFT). It is aimed to measure the learner's degree of field-dependency. A version of this test which was used in this research is to be found in the Appendix (A). Each sample was divided into three groups according to their level of field–dependence/independence measured by (HFT) from a very field-dependent to a very field-independent cognitive style.

To create the category of field-independence, students had to score at least $\frac{1}{4}$ standard deviation (SD) above the mean for their sample population (i.e.FI > Mean $+\frac{1}{4}$SD). This is a criterion used by Scardmalia (1977), Case (1974) and Case and Golberson (1974). On the other hand, students who had a score less than $\frac{1}{4}$ standard deviation below the mean (i.e.FD < Mean $-\frac{1}{4}$SD) were classified as field-dependent and between (Mean $\pm$ $\frac{1}{4}$SD) were those who may be located between the above two categories who were labelled as field-intermediate people. Table 6.2 shows the number of students in each sample in these three groups.

Table 6.2

Classification of the sample

sample	FD	FInt	FI
Sabzevar University	48	13	52
Mashhad University	121	32	98
Glasgow University	24	6	25
Total	193	51	175

6.3 Hidden Figure Test (HFT) Description

The (HFT) has been based upon Witkin and his colleagues' work (e.g., 1978, 1981). The version of this test, which is used in the Centre for Science Education at Glasgow University, comprises 20 complex figures, apart from another 2 figures used as examples. There are simple geometric and non-geometric shapes which are embedded in the 20 complex figures (only one simple shape in each complex figure and students should find these simple shapes). Two examples are used in the first two pages of the (HFT) booklet. Simple shapes (geometric/nongeometric) are located in the last page of the (HFT) test booklet as a specimen of the type to be found. Students are required to find a hidden simple shape in each of a set of complex figures within 15 minutes. They should then outline and trace it in pen or pencil on the lines of the complex figure.

6.3.1 Conditions which must be followed in the (HFT)

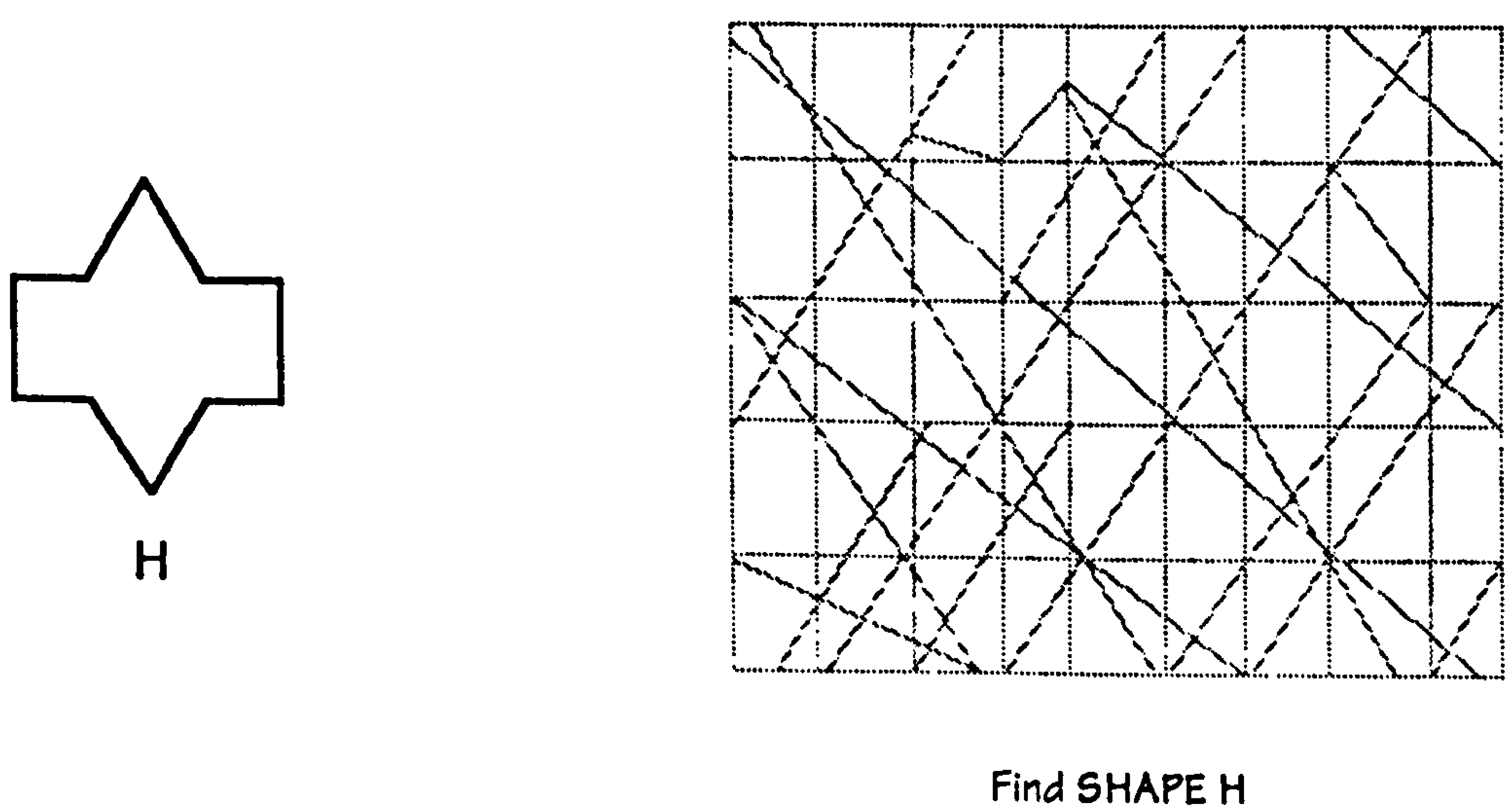
There are certain conditions in the (HFT) which should be followed.

1. The simple shape must be traced in the same size, the same proportions, and facing in the same direction within the complex figure as when it appears alone in the last page of (HFT).

2. There is more than one simple shape embedded in each complex pattern, but within each pattern, the simple shape which students have to find appears only once. Thus, they should trace the required simple shape and only that shape for each problem.

3. The students should not use any means to measure the size of the simple shape embedded in the complex figures.

4. The test booklet must be collected from the students after 20 minutes (5 minutes for instruction and 15 minutes for doing the test). An example is shown in the following figure.



6.3.2 Scoring Scheme

The main scoring scheme which is used for the (HFT) is to give one point for finding a correct simple shape embedded in a complex pattern. The over-all sum of these scores is the total mark which each student can gain. The maximum score that can be obtained is 20 since the test contains 20 complex patterns.

6.3.3 The Measurement of Convergent/Divergent thinking

The other psychological factor to be measured for this study is the convergent /divergent thinking style. The main purpose of such an assignment is to classify students into those who are divergent thinkers or convergent thinkers in their bias. Divergent thinkers would be more likely to give a greater variety of answers to each question, while convergent thinkers attempt to find a restricted answer or a unique solution for each problem.

Based upon Hudson's (1966) and, Child and Smithers' (1973) studies a convergent thinker is defined as a learner whose performance on IQ tests would be better than on open-ended tests (divergent thinking tests). A divergent thinker shows the reverse behaviour. Hudson (1966, 1968) also stated that convergence/divergence (Con/Div) dimension is a measure of bias, not the level of ability.

6.4 A Description of Con/Div tests

This research is based on Hudson's (1966) original study in this dimension of cognitive style. The researcher used a version of Con/Div tests that have been designed and applied by Al-Naeme (1991). The tests comprised of six mini tests for which a limited time for completion of each is allowed. The students are required to write as many answers as possible for every question they are given. These six mini-tests are as follows:

- Test 1

This test was designed to find out the students' ability to write as many different words as possible having the same or similar meaning to the one which is given. The time which has been allowed for this test is 5 minutes. For example, if the word was **short** student should write at least some of the words such as "brief, abbreviated, concise, compact, little, limited, deficient, abrupt, petite and crisp".

- Test 2

In this test, the students are asked to write as many sentences as possible including four given specific words in each sentence. These given words should be used in any constructive sentence in the same form in which they are written in the test. An example is provided, and 5 minutes is the time limit for such a test. For instance, students are asked to write as many sentences as they can with the words **write, words, long, often.**

- Test 3

This test is a pictorial one and may be considered as an opportunity for students to express easily their own ideas and imaginations. In this test the students are required to draw up to five different symbols for each word or given phrase. Five minutes is set as the time limit and one example is also provided at the beginning of the test to illustrate it such as "Draw as many symbols as you can think of (up to five) for the word **food.**"

- Test 4

This test is intended to see how many things students can think of that are alike in some way. For example, students may be asked to think about things that are "always round or that are round more often than any other shape". The time limit for this test is three minutes and and an example is given to describe the test.

- Test 5

This is a test of students ability to think rapidly of as many words as they can that begin with one letter and end with another. For instance, students may be asked about the words which begin with the letter "S" and end with the letter "N". Names of people or places are not allowed and four minutes is the time limit for this test.

- Test 6

This is a test to find how many ideas students can think of about a given topic. Students should be sure to list all ideas they can about a topic whether or not they seem important to them. Four minutes is set as the time limit and there is an example at the beginning of the test. For instance, students could be asked to list all the ideas they can about **"A train journey"**. In total the time allocated of these six mini Con/Div tests is 25 minutes. The tests are given in full in (Appendix B).

6.4.1 The Marking Scheme

One mark is given for every single correct response (Hudson, 1966) and the highest possible score that could be gained in these six tests is 130.

It should be mentioned at this point that the Con/Div tests were translated into Persian (Farsi) for use with the Iranian samples without any change in their structures or instructions. A copy of these translated tests is available in (Appendix C).

6.4.2 The Division of the Sample into Con/Div thinkers

Hudson (1966) divided his sample of school boys (on the basis of open-ended tests and IQ tests) into divergers (30%), who were mainly better at the open-ended tests, and the convergers (30%), who were substantially superior at the IQ tests. There were also what can be classified the all-rounder learners (40%), who were more or less equally good (or bad) at both kinds of test. Hudson's sample was again divided into: extreme convergent thinkers (10%); moderate convergent thinkers (20%); all-rounder (40%); moderate divergent thinkers (20%); and extreme divergent thinkers (10%). However, Hudson neglected the all-rounder pupils from his study to obtain two contrasting groups, which would facilitate the study.

The present study did not use the same divisions as Hudson did. Instead, the (Mean score $\pm$ $\frac{1}{4}$SD) will be regarded as a crucial point between moving from convergent thinking style into divergent thinking style or vice versa. Therefore, moving up from the (Mean score + $\frac{1}{4}$SD) score of each sample population is classified as divergent thinking, while in moving down from the (Mean score $-\frac{1}{4}$SD) is grouped as convergent thinking style. Between (Mean score $\pm$ $\frac{1}{4}$SD) are those who are called all-rounder students. Table 6.3 shows the number of students in these three classifications in each sample of this study.

Table 6.3

Classification of the samples

Sample	Con	All-R	Div
Sabzevar University	50	22	41
Mashhad University	99	48	104
Glasgow University	20	12	23
Total	169	82	168

6.5 Correlation between Psychological factors

At this point various statistical correlations can be discerned between these two dimensions of cognitive styles employed in this study. An understanding of the psychological factors involved in this research may lead to an understanding of students' performance and achievement in calculus. Based on this, there may be a significant link between students' cognitive styles and calculus learning which could lead to improved educational situations in mathematical areas.

Field-dependence/Field-independence and Convergence/Divergence cognitive styles are two psychological factors which are to be explored with the samples of the present study in order to find out any relationship between such factors and students' attainment in learning calculus. This will be discussed in the following chapters and will rely upon the results which are obtained from this chapter.

6.5.1 Pearson's Correlation between FI/FD and Con/Div

All the samples were used in this part of the research as they were participants in both FI/FD and Con/Div tests. The results of students' achievements in each sample in such tests were set out against each other. A low, but significant correlation emerged separately as the Pearson Product-Moment Correlation Coefficient. The null Hypothesis, there is no relationship between students' degree of field-dependency and Con/Div thinking style, could be rejected at 5% or 10% level

(Howell, 1992; D'Oliveria, 1982). Pearson's correlations for one Scottish and four Iranian samples are shown in the Table 6.4.

Table 6.4

The Pearson's Correlation between FI/FD and Con/Div tests

Sample	P-M	Significant
Sabzevar University	0.17	10%
Mashhad University (Sample J)	0.17	5%
Mashhad University (Sample M)	0.24	5%
Mashhad University (Sample N)	0.23	10%
Glasgow University (Sample G)	0.24	10%

6.6 The Distribution of FD/FInt/FI students over the Con/Div tests

To study the conclusion obtained in the last section between field-dependence/independence and convergence/divergence cognitive styles, the distribution of the number of FD/FI students compared with the number of Con/Div students' tests attainments for each sample was required, as it would clarify the students' percentage distribution in these tests. Tables 6.5–9 show such a distribution of percentages.

Table 6.5

The distribution of FD/FInt/FI and percentage on Con/Div tests in Sabzevar sample

Cognitive styles	FD	FInt	FI	Row Total
Con	27 24%	6 5%	17 15%	50 44%
All-R	8 7%	2 2%	12 10%	22 19%
Div	13 12%	5 4%	23 20%	41 36%
Column Total	48 42%	13 13%	52 46%	113 100%

This table indicates that convergent students tend to be more FD than FI.

Table 6.6

The distribution of FD/FInt/FI and percentage on Con/Div tests in Mashhad Sample J

Cognitive Styles	FD	FInt	FI	Row Total
Con	30 24.4%	4 3.3%	19 15.5%	53 43.1%
All-R	10 8.1%	0 0%	8 8%	18 14.6%
Div	23 18.7%	1 0.8%	28 22.8%	52 42.3%
Column Total	63 51.2%	5 4.1%	55 44.7%	123 100%

In the above sample, there were more FD students in the population than FI, and more FI than FInt. But Con and Div students roughly equinumerous, and All-R were 14.6% of the whole sample. In this sample convergent and all-rounder students were more FD than FI in their styles, however divergers were more FI than FD.

Table 6.7

The distribution of FD/FInt/FI and percentage on Con/Div tests in Mashhad Sample M

Cognitive styles	FD	FInt	FI	Row Total
Con	18 23.4%	4 5.2%	8 10.4%	30 38.9%
All-R	7 9.1%	2 2.6%	4 5.2%	13 16.9%
Div	13 16.9%	6 7.8%	15 19.5%	34 44.2%
Column Total	38 49.0%	12 15.6%	27 35.4%	77 100%

It can be seen from this table that there are more FD students in the population sample than FI, and more FI than FInt. On the other hand, divergent students had a larger population than convergent and all-rounder students in the whole sample. Most of the convergent and All-R students in this sample were FD in their cognitive styles, but divergent students had a higher population of FI cognitive style than other styles.

Table 6.8

The distribution of FD/FInt/FI and percentage on Con/Div tests in Mashhad Sample N

Cognitive styles	FD	FInt	FI	Row total
Con	10 18.5%	6 11.1%	7 12.9%	23 40.7%
All-R	7 12.9%	3 5.5%	3 5.5%	13 24.1%
Div	5 9.3%	3 5.5%	10 18.5%	18 33.3%
Column total	22 40.7%	12 22.2%	20 37%	54 100%

This table shows that convergent students are mostly FD in their cognitive styles and divergent students are more FI than FD and FInt. Again in the whole sample, FD students are more common than FI and FInt and divergent thinkers are more common than convergent and all-rounder students.

Table 6.9

The distribution of FD/FInt/FI and percentage on Con/Div tests in Glasgow Sample G

Cognitive styles	FD	FInt	FI	Row total
Con	10 18.2%	2 3.6%	8 14.5%	20 36.4%
All-R	6 10.9%	2 3.6%	4 7.3%	12 21.8%
Div	8 14.5%	2 3.6%	13 23.6%	23 41.8%
Column Total	24 43.6%	6 10.9%	25 45.5%	55 100%

Table 6.9 exhibits that Con students were more FD and Div thinkers were more FI. Moreover, in the whole sample Div students and FI were more common.

6.7 The Distribution of Cognitive styles over the Sample

A picture of the distribution of FD/FI and Con/Div students over the each sample could also be important in the understanding of such groups. Tables 6.10–14 show the groups' percentage distribution in each sample.

Table 6.10

Sabzevar sample

Group	FD	FInt	FI	Con	All-R	Div
Total	48	13	52	50	22	41
N=113	42.5%	11.5%	46%	44.2%	19.5%	36.5%

This table indicates that students of this sample were mostly FI and Con in their cognitive styles compared to other groups.

Table 6.11

Sample J at Mashhad University

Group	FD	FInt	FI	Con	All-R	Div
Total	63	5	55	53	18	52
N=123	51.2%	4.1%	44.7%	43.1%	14.6%	42.3%

It is shown in this table that FD students were more common than FI and FInt, but Con and Div students were roughly equal in this sample.

Table 6.12

Sample M at Mashhad University

Group	FD	FInt	FI	Con	All-R	Div
Total	38	12	27	30	13	34
N=77	49.4%	15.6%	35.1%	39%	16.9%	44.2%

In this sample, FD and Div students were more common than those with the other styles.

Table 6.13

Sample N at Mashhad University

Group	FD	FInt	FI	Con	All-R	Div
Total	22	12	20	23	13	18
N=54	40.7%	22.2%	37.0%	40.7%	24.1%	33.3%

This table shows that FD and Div students were more common than the other groups of thinking styles.

Table 6.14

Glasgow University sample

Group	FD	FInt	FI	Con	All-R	Div
Total	24	6	25	20	12	23
N=55	43.6%	10.9%	45.4%	36.4%	21.8%	41.8%

This table indicates that FI and Div learners were more common than others.

6.7.1 Overall-Review

The overall review of the samples of this study indicated that Con and All-R students tended to be FD in their cognitive styles, while divergent students in all the samples tended to be FI rather than FD and FInt. Furthermore, in the Sabzevar sample FI and Con students were in the majority, but, in Mashhad FD and Div students were in the majority among the engineers and FD and Con among the mathematicians. FI and Div thinkers were in the majority in the Glasgow sample.

6.8 Predictive Proposals

According to the above results the FI/FD test significantly classifies the divergent thinkers as FI students rather than FD students. Furthermore, as mentioned in the literature review of this research (Chapters 1 and 2), several studies have been devoted to this domain, in particular, by Johnstone and his co-workers since 1980. They found that field-independent learners are more able to achieve well in mathematics and science examinations when compared to field-dependent learners.

In fact, FI students have shown higher capacity (working memory) and ability in separating the "signal" from "noise" in any task in science and mathematics. Therefore, if a group of students are field-independent and divergent in their cognitive styles, it could be expected that they will perform well in mathematics and science activities and may be best in complicated tasks which require a higher ability to articulate or combine concepts, procedural skills and techniques in a new way. According to this, field-independent and convergent students would be likely perform well in conventional science and mathematics tasks.

For instance, Al-Naeme (1991) found that FI and Div pupils performed best in open-ended practical problem solving (creative tasks) in chemistry compared to other groups. Although, FD and Div pupils still performed better in chemistry creative tasks than FD and Con ones.

As was discussed, in detail, in Chapters 1 and 2 of this study, much evidence indicates that field-dependence/independence and convergence/divergence cognitive styles do affect students' mathematical behaviour and performance and therefore may have a lot of implications for mathematical education. It seems reasonable to predict that there is a significant relationship between these psychological factors and students' activities in some areas of calculus, an important part of the beginning of higher mathematics learning. It is a very general predictive proposal, which requires much more investigation (in the next chapter).

6.9 Hypotheses

6.9.1 Introduction

The main purpose of this study, was to find answers to the general question: "Is there any interaction between students' cognitive styles (FI/FD and Con/Div) and their performance in the six calculus categories?"

In order to find clear answers to this question the following "hypotheses" were formulated for this research. The overview of the previous chapters of the the present study (in particular, Chapters 1,2 and 5) are an attempt to rationalize the structure of these hypotheses.

To begin with, some definition at this point may facilitate better understanding of these study hypotheses. Let us define Witkin's cognitive style categories as (FI/FD) and Hudson's cognitive style categories as (Con/Div). Therefore, (FI,FD,Con,Div) are regarded four psychological categories in this research. The effectiveness of these cognitive styles will be investigated by the following categories, which indicate students' performance in learning calculus and problem solving activities. The six categories which were discussed in Chapter 5 are:

- Z_1

Students' performance in mathematical manipulation and logical discussion in calculus.

- Z_4

Students' performance in calculus involving multi-conceptual and procedural tasks.

- Z_5

Students' performance in mathematics translation and visual thinking in calculus.

• Z_6

Students' performance in calculus multi-skilled, transferable skills and procedural tasks.

• Z_3 or all over calculus score

Students' total performance in calculus examinations has been also considered as a separate category in this research and will be abbreviated as "Cals".

• Z_2 or En

University entrance examination in mathematics. This category is only considered for the Iranian samples to obtain more information about their background in high school mathematics at the beginning of higher calculus.

It was noted in Chapter 5 that these categories are not isolated totally, but have logical interaction with each other in a scheme.

6.10 Specific Hypotheses

6.10.1 Hypothesis One

The field-independent students would be expected to perform better than field-dependent students in all the categories in calculus.

Students with convergent cognitive styles may be better at mathematical manipulation and logical discussions in calculus, multi-skilled tasks and calculus total mark or (Z_1,Z_6,Cals); whereas, divergent students could be expected to show higher performance in calculus multi-conceptual, mathematical translation and pictorial thinking tasks and the University entrance examination on mathematics or (Z_4, Z_5,En) than others.

Rationale

The rationale of the two parts of this hypothesis was discussed in detail in this and previous chapters (1,2 and 5). Much research evidence indicates that FI students have higher performance than FD students in mathematics learning (Chapter 1 and 2), hence it can be extended to the whole of calculus learning and problem solving including the six categories of this study, which may reflect students' difficulties in a calculus course. Therefore, it seems to be reasonable to accept that, in general, FI students could be better than FD students in calculus.

For the second part of this hypothesis there is also some research evidence including Hudson (1966, 1968) which suggests that divergent students are highly imaginative compared to convergent students. This means that divergers have better performance than convergent ones in spatial and geometrical skills than others. Hence, based on definitions of the (Z_4) and (Z_5) categories, in Chapter 5, divergent thinking would thus be more appropriate for comprehending and performing calculus tasks concerned with these categories. In fact, the nature of the multi-conceptual category (Z_4) indicates that students should think more openly at the starting point and then converge step by step to the required solution and final point. This means that students must cope well with convergent and divergent thinking at the same time in the problem solving situation. And being pictorial thinkers may be a very powerful tool in tackling tasks that are classified in this research as category (Z_5). It was assumed in this research that divergent students, in Iran, may have higher performance in mathematical university entrance examination. Again, the nature and structure of this difficult and competitive examination, could be more adapted to a divergent way of thinking than a convergent one, therefore it could be reasonable to claim that divergent individuals would be better than convergent ones in the (En) category of this study.

On the other hand, convergent thought processes appear to be more appropriate compared with divergent thought processes for the precise, logical argument, conventional tasks and uniquely correct solution or conventionally best outcomes (for example, Hudson, 1966, 1968; Guilford, 1967; Messick, 1976). Moreover, as it was defined in Chapter 5, the structure of multi-skilled and procedural skills in calculus tasks indicate that, being convergers rather than divergers may be more appropriate when attempting to comprehend and reach a unique answer. In addition, Hudson and others found that convergent thinkers prefer formal problems and tasks that are presumably better structured and demand logical ability rather than the more open-ended problems favoured by divergent thinkers and calculus examinations at tertiary level are, mainly, designed in this way. In sum, it could be reasonable to suggest that convergers may show better performance and achievement than divergers in three categories (Z_1, Z_6,Cals).

It is worth stating that the other hypotheses of this study are, in fact, to be the various combinations of the two psychological factors (FI/FD and Con/Div). Therefore, the above rationale would also confirm the reasonable suggestion of the following four hypotheses of this research.

6.10.2 Hypothesis Two

The field-independent and convergent thinkers could have better performance in learning calculus and problem solving compared to field-dependent and convergent thinkers.

6.10.3 Hypothesis Three

The field-independent and divergent students would be expected to show higher performance than field-dependent and divergent students in calculus learning and problem solving.

6.10.4 Hypothesis Four

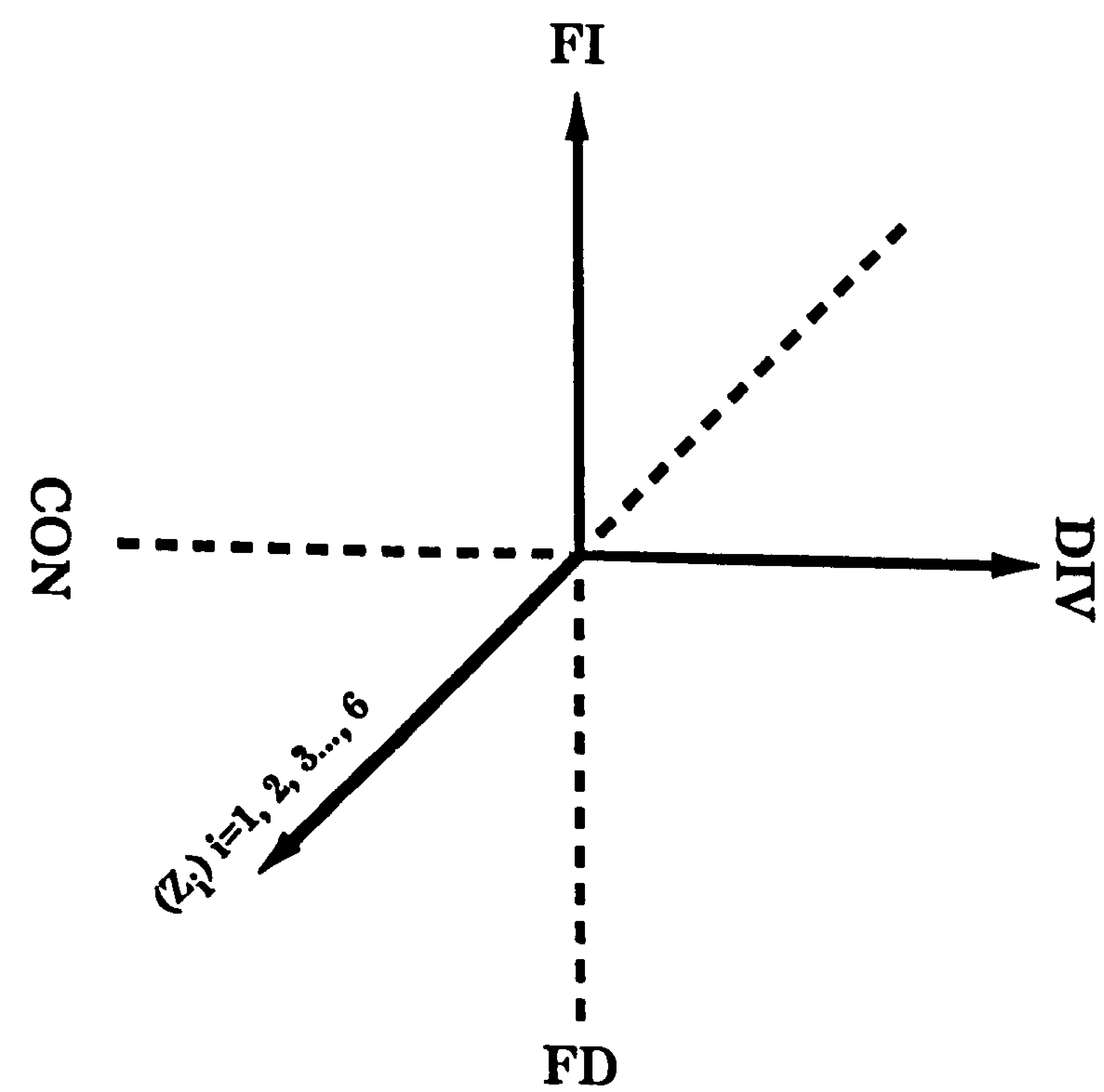
The field-independent and convergent learners may be expected to have better performance than the field-independent and divergent ones in categories (Z_1, Z_6,Cals). Whereas, field-independent and divergers may show higher performance in (Z_4, Z_5,En).

6.10.5 Hypothesis Five

The field-dependent and convergent students would be expected to exhibit better performance in some categories of the calculus domain, i.e. (Z_1, Z_6,Cals). While, the field-dependent and divergent students may have higher performance in the other areas (Z_4, Z_5,En).

6.10.6 Symbolic Hypotheses Representations

Symbolic representations have an important role in the process of learning mathematics and mathematical communication. As it was stated by Skemp (1986), 'English and mathematics have both been described by Bruner as "a calculus of thought", and it is their symbol systems which make them so.' Therefore, a symbolic exhibition which can emerge from the body of hypotheses will be used in the present study to facilitate necessary discussion about each hypothesis. A picture of this representation and a figure, which may show the relationship among cognitive styles and calculus categories D=$(Z_i)_{i=1}^{6}$ are as follows:



Specific Hypotheses

$$\textbf{Hypothesis (1)} = \begin{cases} \text{FI} > \text{FD} & \text{for} \quad (Z_i)_{i=1}^{6} \\ \text{Con} > \text{Div} & \text{for} \quad (Z_1, Z_6, \text{Cals}), \\ \text{Div} > \text{Con} & \text{for} \quad (Z_4, Z_5, \text{En}) \end{cases}$$

$$\textbf{Hypothesis (2)} = (\text{FI+Con}, Z_i) > (\text{FD+Con}, Z_i) \qquad \forall i = 1, 2, \cdots, 6$$

$$\textbf{Hypothesis (3)} = (\text{FI+Div}, Z_i) > (\text{FD+Div}, Z_i) \qquad \forall i = 1, 2, \cdots, 6$$

$$\textbf{Hypothesis (4)} = \begin{cases} (\text{FI+Con, } Z_i) > (\text{FI+Div, } Z_i) & \forall i = 1, 3, 6 \\ (\text{FI+Div, } Z_i) > (\text{FI+Con, } Z_i) & \forall i = 2, 4, 5 \end{cases}$$

$$\textbf{Hypothesis (5)} = \begin{cases} (\text{FD+Div, } Z_i) > (\text{FD+Con, } Z_i) & \forall i = 2, 4, 5 \\ (\text{FD+Con, } Z_i) > (\text{FD+Div, } Z_i) & \forall i = 1, 3, 6 \end{cases}$$

6.10.7 A Brief Summary

The final picture which emerged from the body of cognitive psychology used in this research indicated that field-dependent/independent cognitive style correlated positively, although weakly, with the convergence/divergence cognitive style. Field-independent students showed themselves as more likely to be divergent thinkers, whilst field-dependent and all-rounder students may be divergent thinkers too. Afterwards, the possible relationships obtained from the next chapter will be taken as hypotheses to be tested.

Chapter 7

Evaluation and Students' Performance

7.1 Introduction

The practical part of this study involves attempting to find how $(Z_i)_{i=1}^{6}$ categories may show how students' with different cognitive styles cope with calculus materials. Accordingly it was decided that calculus examinations should be used for this purpose.

It is intended in this chapter to display the students' performance in the $(Z_i)_{i=1}^{6}$ categories in their calculus examination by attempting to relate it with cognitive styles. The whole picture will be drawn together in order to find out what sort of calculus difficulties occur amongst various undergraduate students in Calculus 1 in Iran and Mathematics 1A (calculus topics) in Scotland.

7.2 The Assessment Method of Students' performance

In designing an appropriate and reasonable method for assessing the students' performance in calculus examinations, a lot of difficulties were found. But, the researcher started his assessment procedure of examination papers by making a number of trial assessments of some examination books. Eventually, the following design was proposed in this study for the evaluation of students performance on the categories (Z_1,Z_4,Z_5,Z_6). This method could be repeated by other mathematicians to obtain nearly the same results. The steps to be followed to complete the method are as follows:

- **First Step**

Separate the question tasks in each calculus examination into three main categories (Z_4, Z_5, Z_6).

- **Second Step**

Identify the maximum possible make for each category which was given in the marking scheme. This is, in fact, the highest mark that a student should try to obtain and is shown by $(M)_{Z_i}$, $\forall i = 4, 5, 6$ in this study.

- **Third Step**

Determine the student's total score in each category. This score is denoted by P_{Z_i}, $\forall i = 4, 5, 6$.

- **Fourth Step**

Calculate the numbers R_i from the formulae:

$$R_i = \frac{P_{Z_i}}{M_{Z_i}} \times 100 \quad (i = 4, 5, 6).$$

(The numbers R_i can be found for every student as a measure of his/her final performance in the calculus examination, and can be use to compare groups in the sample.)

- **Fifth Step: (Z_1) assessment**

Calculate the number R_1 from the formula:

$$R_{Z_1} = \frac{P_{Z_1}}{N} \times 100$$

where P_{Z_1} is the number of (Z_1) errors and N the number of questions in the examination. (The assessment of Z_1 in this study is of a different nature. As it was discussed before, in this research the category (Z_1) could occur in all the domains of calculus activity. Thus, it was not restricted to specific question tasks in calculus examinations at this level. As a result, it seems reasonable to select another method for its measurement. In the previous categories (Z_4, Z_5, Z_6) a higher R_i number indicated better students' performance, but for category (Z_1) a higher score reflected weakness rather than strength.)

- **Sixth step: (Z_3) or Calculus exam assessment**

For each students, calculate the total score (Z_3) or (Cals) obtained on the three categories (Z_4, Z_5, Z_6) and calculate the number R_3 from the formula:

$$R_3 = \frac{P_{Z_4} + P_{Z_5} + P_{Z_6}}{M_{Z_4} + M_{Z_5} + M_{Z_6}} \times 100.$$

(This gives an assessment of students' overall performance in calculus examination).

- **Seventh Step: (Z_2) or (En) assessment**

For the Iranian samples, collect the mark (En) in the university entrance examination in mathematics of each student. (This was needed to find out how strong was the students' background in high school mathematics, including pre-calculus materials, at the beginning of higher calculus.)

- **Eighth Step**

Having finished the researcher's work on students' activities in calculus examinations, set up for each student a performance (P) of category scores in order to show his/her final results in the course (a practical example will be shown in the next section).

- **Ninth Step**

Finally, based on the results of the eighth step and cognitive styles (FI/FD and Con/Div), separate the students into different groups for checking the hypotheses of this study.

It should be noted that, in some cases, the researcher had no opportunity to ask examiners to consider a healthy balance of categories (Z_4,Z_5,Z_6) in the questions' demands. Accordingly, in some calculus examinations, some items were found not to be measured, for instance, question tasks based upon pictorial thinking and mathematical translation (Z_5) could not be found in many of the calculus examinations.

7.3 A Practical Example

To understand this method and its procedure, having a typical sample could be of help to the reader. Let student A be a calculus student who has taken part in 3 examinations whose (En) score is 54 out of 100. Tables 7.1–3 show the categories in the examinations and student (A)'s performance.

Table 7.1

The distribution of maximum scores for categories (Z_4, Z_5, Z_6) in the exams

Exam Number	Z_4	Z_5	Z_6	Z_3	No. of ques.
1	45	35	28	108	14
2	50	25	30	110	12
3	25	35	46	106	10
Total	120	95	104	324	36

Table 7.2

Scores in the exams for student A

exam Number	Z_4	Z_5	Z_6	Z_1	Cals	No. of ques.
1	25	15	18	5	58	10
2	30	10	15	3	55	9
3	20	25	26	6	71	10
Total	75	50	59	14	184	29

Table 7.3

The final results of Student A on the exams

Final results	Z_4	Z_5	Z_6	Cals	Z_1
3 exams	$\frac{75}{120} \times 100$	$\frac{50}{95} \times 100$	$\frac{59}{104} \times 100$	$\frac{184}{324} \times 100$	$\frac{14}{29} \times 100$
Results	R_4=62.5	R_5=52.6	R_6=56.7	R_3= 56.8	R_1=48.3

Therefore, $P = (62.5, 52.6, 56.7, 56.8, 48.3)$ is found to show the student A's final performance in the 3 calculus examinations.

7.4 Testing Hypotheses

7.4.1 Introduction

The hypotheses which were raised in a previous section need now to be either supported or rejected. This section will deal with the hypotheses and will discuss any other results which may be obtained.

Having classified students into certain groups according to their cognitive styles (in the previous chapter), it is necessary to follow them in their performance in calculus examinations. It should enable us to understand what is the nature and the level of performance of each group of cognitive styles on the six categories of $(Z_i)_{i=1}^{6}$ of this study which require mathematical ability. The final production of such work may assist mathematics educators to place emphasis upon the needs of undergraduate students in calculus learning. It also underlines the effects of conventional calculus courses (contents, aims and objectives).

Moreover, these results may remind and stimulate the thinking of mathematics educators about what kinds of teaching are essential to develop students' mathematical thinking ability, and of the degree of attention which should be given to different groups of students in their calculus learning processes.

7.4.2 The Study's Design for Testing Hypotheses

The hypotheses of this research were tested over a period of time upon the work of first-year calculus students at the Teacher Training University of Sabzevar, Ferdowsi University of Mashhad (in Iran) and Glasgow University (in Scotland). At first, all students were tested by (HFT) and (Con/Div) tests to determine their cognitive styles (as it was discussed in the previous chapters of this thesis).

The next stage of this work is to study the relationship between the students' cognitive styles and their performance in the categories $(Z_i)_{i=1}^{6}$ in calculus.

7.4.3 Categories

1. **The psychological categories**

   The students' degree of field-dependence/independence and convergence/divergence cognitive styles as explained before, are the psychological categories of this study.

2. **The calculus categories**

   The students' achievement in $(Z_i)_{i=1}^{6}$ categories are the calculus categories in this research.

7.4.4 Data Analysis (Mean Score)

The data analysis procedure was mainly done by using the mean score. In this case the mean score (Mean) and standard deviation (SD) were calculated for each category $(Z_i)_{i=1}^{6}$ for all groups of cognitive styles. Moreover, the analysis was extended in both countries. The results are shown in the related tables which indicate, for all categories, the Mean and SD of the students' performance for different thinking styles. In addition, some non-parametric statistical tests (Mann-Whitney U and Kruskal-Wallis 1-way ANOVA) are also used in order to find out whether the difference in students' performance are statistically significant or nonsignificant. The results are given at 0.05 or 0.1 level of significance.

7.5 Sample Investigation

7.5.1 The First Investigation

An investigation was carried out on first-year calculus students at Sabzevar University over a period of two terms of the academic year (94/95) and the sample on which this work was based comprised 113 students of Calculus 1. The first term was, in fact, used to improve students' background in higher calculus learning and Calculus 1 was an overall review and extension of the first-term calculus, which was formally begun at the second term. 74 out of the 113 were mathematics students and the rest of them were physics students. In each term they had three calculus examinations and, in all, most of them had taken part in six calculus examination after one year of calculus study (Appendix D).

In many cases, question tasks during the course and examinations were conducted by this researcher to ensure that the $(Z_i)_{i=1}^6$ categories of this study would be considered. This sample is divided into three sub-samples which were labelled in this research as Sample R (113 students), Sample S (94 students) and the combination of both samples which is, in fact, to show the production of one year of study at Sabzevar University. In other words, three cases with 320 individuals were tested upon $(Z_1, Z_4, Z_5, Z_6$,Cals,En).

7.5.2 The Sample R

The population of this sample was 113 students who attempted the three calculus examinations in the first term of (94/95). The Pearson's correlation between each cognitive style and $(Z_i)_{i=1}^6$ and the mean scores and standard deviations have been calculated for each group of cognitive styles.

7.5.3 Testing Hypothesis (1)

The main aim of testing hypothesis (1) was to find out whether there was a direct relationship between the students' group of FI/FD and Con/Div and their mathematical ability to cope with the categories $(Z_i)_{i=1}^6$ of this study. Tables 7.4–5 and

7.7 will show respectively, Pearson's correlation (P-C), students' mean scores and standard deviations in calculus.

Table 7.4

The Pearson's Correlation between cognitive styles and $(Z_i)_{i=1}^{6}$

P-C	FI/FD	Con/Div	Z_1	Z_4	Z_5	Z_6	Cals	En
FI/FD	1	0.17^{*}	-0.10	0.15^{*}	0.20°	0.10	$0.24^{\star}$	$0.32^{\bullet}$
Con/Div	0.17^{*}	1	0.02	0.15^{*}	0.02	-0.10	0.06	0.20°
Z_1	-0.10	0.02	1	$-0.26^{\star}$	-0.14	$-0.28^{\star}$	-0.10	-0.12
Z_4	0.15^{*}	0.15^{*}	$-0.26^{\star}$	1	$0.31^{\bullet}$	-0.13	$0.72^{\bullet}$	$0.35^{\bullet}$
Z_5	0.20°	0.02	-0.14	$0.31^{\bullet}$	1	0.08	$0.70^{\bullet}$	$0.40^{\bullet}$
Z_6	0.10	-0.10	-0.10	-0.13	0.08	1	$0.33^{\bullet}$	-0.01
Cals	$0.24^{\star}$	0.10	$-0.28^{\star}$	$0.72^{\bullet}$	$0.70^{\bullet}$	$0.33^{\bullet}$	1	$0.40^{\bullet}$
En	$0.32^{\bullet}$	0.20°	-0.12	$0.35^{\bullet}$	$0.40^{\bullet}$	-0.01	$0.40^{\bullet}$	1

Significant at:

$\bullet$ 0.001 level; $\star$ 0.01 level;

$\circ$ 0.05 level; $*$ 0.1 level

Table 7.5

Mean and SD on different groups of Sample R in $(Z_1, Z_4, Z_5, Z_6, \text{Cals}, \text{En})$

Groups	FI (N=52)		FInt (N=13)		FD (N=48)	
	Mean	SD	Mean	SD	Mean	SD
Z_1	18.4	10.4	20.4	12.7	21.5	12.9
Z_4	52.0	15.8	42.8	16.1	45.2	15.67
Z_5	62.9	24.7	56.5	27.1	55.7	20.3
Z_6	41.0	33.7	37.1	34.3	35.9	26.9
Cals	54.7	13.8	47.5	14.6	47.7	12.2
En	31.8	9.3	32.6	7.6	27.3	9.1

Table 7.6

The significance of the difference in performance between FI/FD students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
FI&FD	Ns	S⋆	S*	Ns	S⋆	S⋆

⋆ significant at 0.05 level

* significant at 0.1 level

Table 7.7

Mean and SD on different groups of Sample R in (Z_1, Z_4, Z_5, Z_6, Cals, En)

Groups	Con (N=50)		All-R (N=22)		Div (N=41)	
	Mean	SD	Mean	SD	Mean	SD
Z_1	18.4	12.6	22.7	12.3	20.3	10.1
Z_4	45.7	16.8	44.8	15.9	52.6	14.9
Z_5	56.6	25.1	61.2	21.2	61.1	22.5
Z_6	45.8	30.3	25.7	21.5	36.1	34.1
Cals	50.3	15.9	48.0	11.8	53.3	11.1
En	28.8	8.7	30.6	8.5	33.5	7.9

Table 7.8

The significance of the difference in performance between Con/Div students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
Con&Div	Ns	S⋆	Ns	Ns	Ns	S⋆

⋆ significant at 0.05 level

7.5.4 Discussion

The results of the above tables indicate that the mean scores of (FI) students are better than FD students in all the categories $(Z_i)_{i=1}^{6}$. It is evident from this, that field-independent learners in this sample performed better in calculus activities than field-dependent learners which may support, what was predicted by the hypothesis (1) and the difference in the mean scores between both groups is significant (S) except for (Z_1, Z_6), as shown in Table 7.6.

Moreover, it can be found from Table 7.7 that convergers are better in (Z_1) and (Z_6) than divergers. This finding supports the second part of hypothesis (1) except for (Cals). But, it seems that divergent thinkers tended to be better in (Z_4,Z_5,Cals,En), which could support the third part of hypothesis (1). Again, despite the difference between means, this difference in some categories is not significant (Ns), as shown in Table 7.8. It seems that both groups of cognitive styles (Con and Div) were naive in question tasks which had been classified into (Z_5), in particular, they had no pre-practice in such calculus tasks, although the differences in the mean scores are all in the same direction.

7.5.5 The Interaction between (FI/FD) and (Con/Div) and their relationship with the Students' performance in $(Z_i)_{i=1}^6$

The researcher's attempt to understand the possible patterns which may emerge from the combination of cognitive styles in the present study versus each other and versus the students' performance in $(Z_i)_{i=1}^6$ at the same time, is the main story of the following hypotheses testing process.

7.5.6 Testing Hypothesis (2)

To examine this hypothesis, the attainment of (FI+Con) versus (FD+Con) in the $(Z_i)_{i=1}^6$ categories should be tested. The mean scores in various aspects of calculus related to (FI+Con) and (FD+Con) thinking styles in this sample are set out in Table 7.9 and the significance/nonsignificance of the linkages between the two styles and each of calculus category is also shown in Table 7.10.

Table 7.9

Mean and SD in calculus categories

Groups	FI+Con (N=17)		FD+Con (N=27)	
	Mean	SD	Mean	SD
Z_1	16.5	11.1	20.8	12.9
Z_4	50.0	20.4	44.4	13.1
Z_5	61.6	23.3	54.3	22.6
Z_6	59.6	30.3	38.8	27.0
Cals	56.9	19.4	47.7	11.7
En	29.6	8.2	24.7	8.9

Table 7.10

The significance of the difference in performance between (FI+Con) and (FD+Con) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FI+Con)&(FD+Con)	Ns	Ns	Ns	S*	Ns	Ns

⋆ significant at 0.05 level

It is evident from Tables 7.9 that (FI+Con) students performed and achieved better in all the categories $(Z_i)_{i=1}^6$ than (FD+Con) students which may support what was predicted in hypothesis (2). However, the difference in the means between both groups except for (Z_6) is not significant, but it is still an indication which may show that (FI+Con) students tend to be more capable than their (FD+Con) colleagues in calculus domain activities. In fact, the difference in mean scores between FI and FD in all the categories Table 7.5 may also explain this superiority.

7.5.7 Testing Hypothesis (3)

To test this hypothesis the performance of (FI+Div) versus (FD+Div) students in the categories $(Z_i)_{i=1}^6$ should be investigated. The mean scores, as shown in Tables 7.11–12, demonstrate that (FI+Div) learners performed better than (FD+Div) learners in (Z_4, Z_5,Cals,En) and they had nearly the same results in (Z_1, Z_6) which may support, to some extent, the hypothesis (3). Although, the difference in means between both groups is not significant in all the categories (Table 7.12). However, there is still evidence that divergent thinking could be a more important factor than FI and FD in learning calculus.

Table 7.11

Mean and SD in calculus categories

Groups	FI+Div (N=23)		FD+Div (N=13)	
	Mean	SD	Mean	SD
Z_1	19.2	10.0	19.6	9.3
Z_4	54.4	13.5	50.3	16.1
Z_5	61.0	24.5	60.4	15.8
Z_6	34.9	35.2	34.6	28
Cals	53.7	10.7	51.1	11.5
En	35.4	6.8	30.1	9.5

Table 7.12

The significance of the difference in performance between (FI+Div) and (FD+Div) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FI+Div)&(FD+Div)	Ns	Ns	Ns	Ns	Ns	Ns

7.5.8 Testing Hypothesis (4)

According to the mean scores of students in Table 7.13, the (FI+Con) students performed and achieved better than (FI+Div) students in (Z_1, Z_6), while (FI+Div) learners were better in (Z_4,En) and both groups of thinking styles had nearly the same performance on (Z_5). These results support, to some extent, the hypothesis (4). However, except for (Z_6,En) the difference in the means between these two learning styles in not significant. Table 7.14 shows these significant/nonsignificant difference.

Table 7.13

Mean and SD in calculus categories

Groups	FI+Con (N=17)		FI+Div (N=23)	
	Mean	SD	Mean	SD
Z_1	16.5	11.1	19.2	10.0
Z_4	50.0	20.4	54.4	13.5
Z_5	61.6	23.3	61.0	24.5
Z_6	59.6	30.3	34.9	35.2
Cals	56.9	19.4	53.7	10.7
En	29.6	8.2	35.4	6.8

Table 7.14

The significance of the difference in performance between (FI+Con) and (FI+Div) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FI+Con)&(FI+Div)	Ns	Ns	Ns	S*	Ns	S*

⋆ significant at 0.05 level

7.5.9 Testing Hypothesis (5)

At this stage of the testing, (FD+Div) students obtained higher means in all the categories except for (Z_6) compared to their (FD+Con) fellow-students, as shown in Table 7.15. Therefore, such a result could support the first part of hypothesis (5) and partially the second part of it. Nonetheless, the difference between the means of both groups of cognitive styles is not significant in any category (Table 7.16).

It is still evident from these findings that, for students with a FD cognitive style, being divergent in their way of thinking may be more beneficial than being convergent in tackling calculus question tasks.

Table 7.15

Mean and SD in calculus categories

Groups	FD+Con (N=27)		FD+Div (N=13)	
	Mean	SD	Mean	SD
Z_1	20.8	12.9	19.6	9.3
Z_4	44.4	13.1	50.3	16.1
Z_5	54.3	22.6	60.4	15.8
Z_6	38.8	27.0	34.6	28
Cals	47.7	11.7	51.1	11.5
En	25.4	8.9	30.1	9.5

Table 7.16

The significance of the difference in performance between (FD+Con) and (FD+Div) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FD+Con)&(FD+Div)	Ns	Ns	Ns	Ns	Ns	Ns

7.5.10 A brief Symbolic Picture of Sample R

A brief symbolic picture of the above results and discussions is shown in Table 7.17 and the significant difference between both groups of learning styles in each category is determined. In addition, more information has been found between different groups in this table which shows a significant difference between learning styles not compared within the hypotheses of this study. For instance, the performance of (FI+Con) students in (Z_6) is significantly different from others and (FI+Div) students have significantly different achievement in (En) from other groups of learning styles. Moreover, Figures 7.1–2 display the performance of students with various learning styles, based on hypotheses (1–5), in the categories (Z_4, Z_5, Z_6).

Figure 7.1

The performance of FI/FD and Con/Div students in (Z_4, Z_5, Z_6) in Sample R

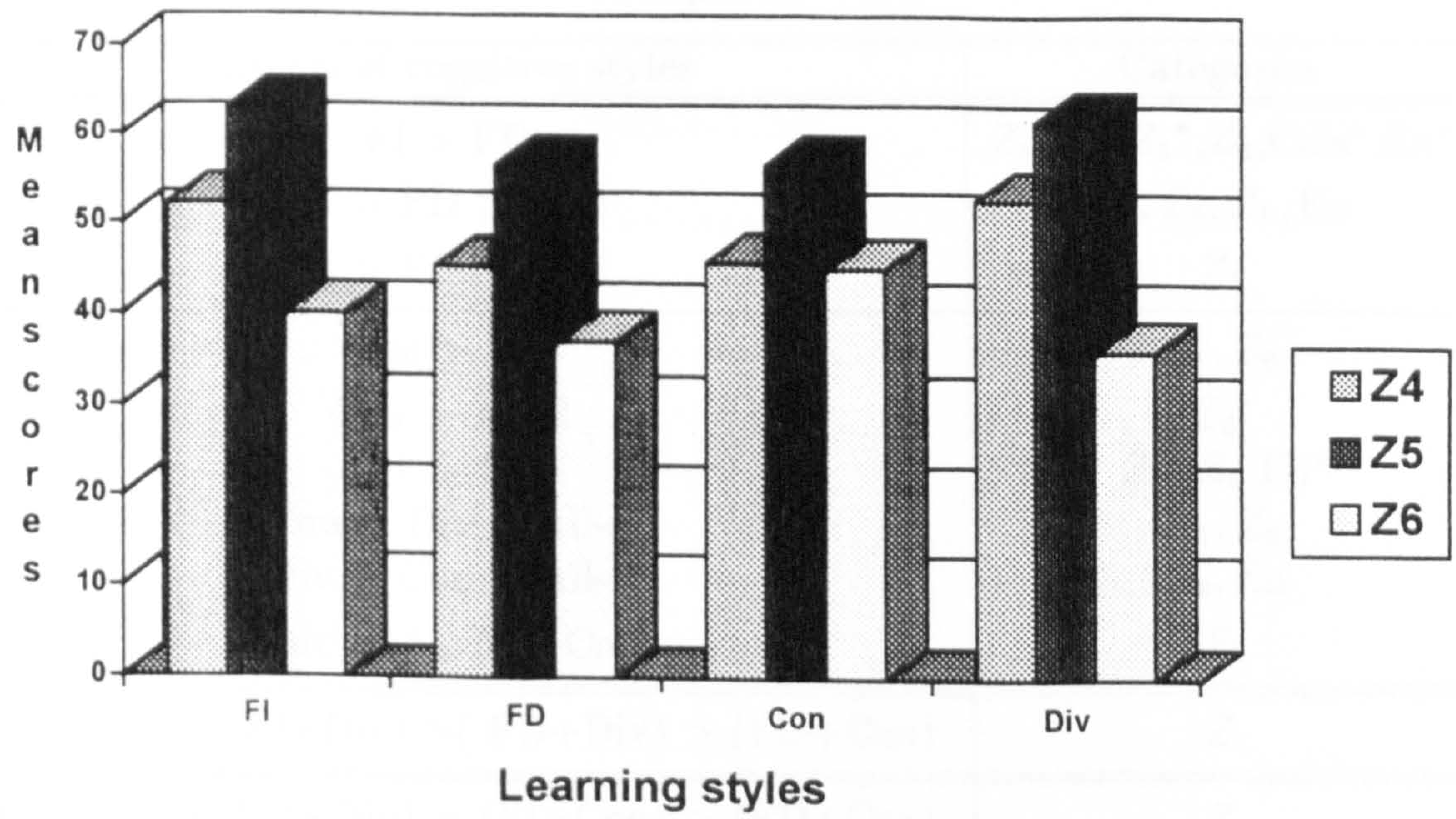


Figure 7.2

The students' performance with different learning styles in (Z_4, Z_5, Z_6) in Sample R

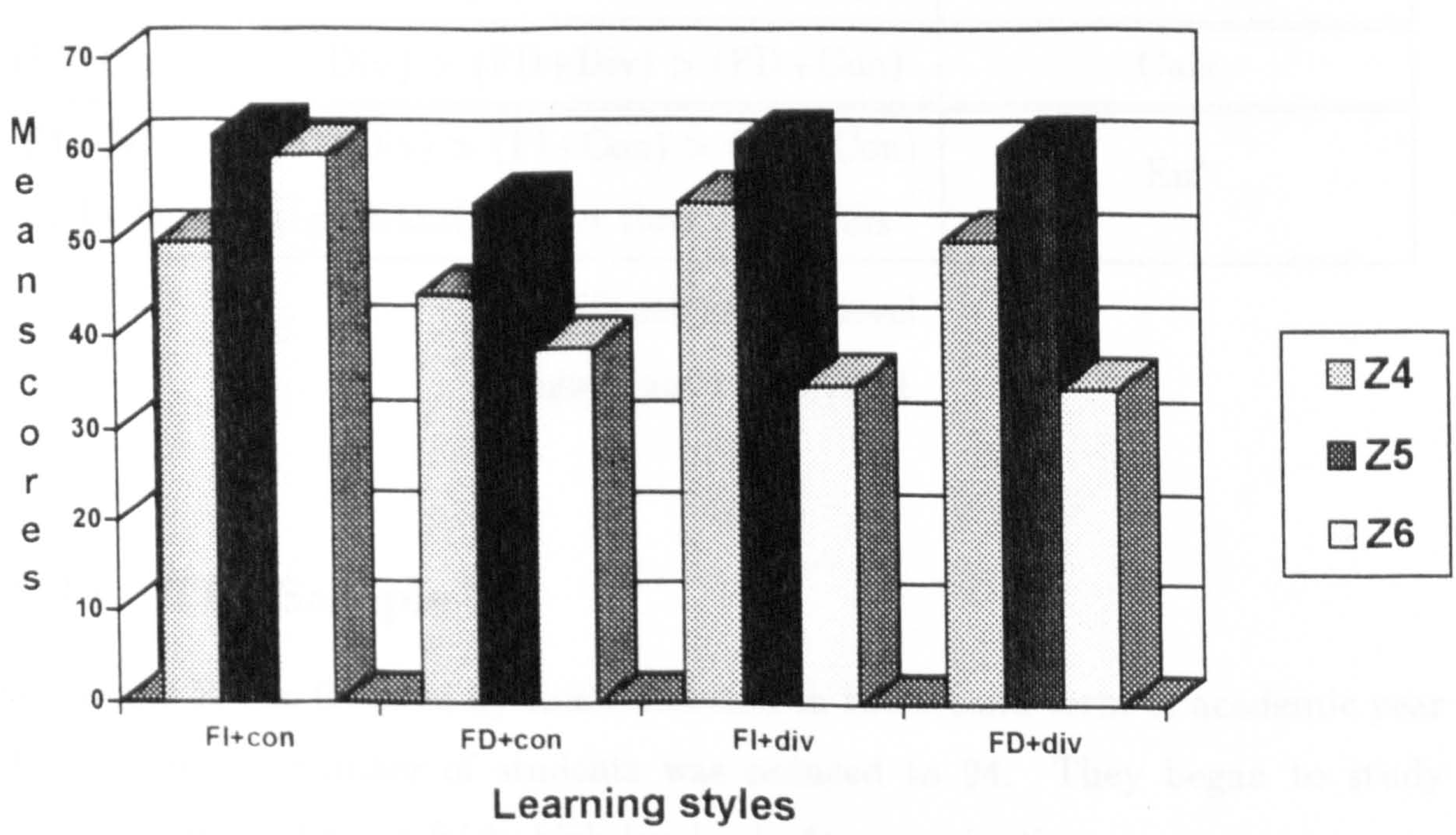


Table 7.17

The symbolic picture of students' superiority in $(Z_i)_{i=1}^{6}$ with different learning styles based on the mean scores in Sample R

Groups of cognitive styles	Categories
FI > FD FI > FD > FInt FI > FInt > FD	Z_1,$Z_4^{\star}$,Z_5^{*},Z_6,Cals*,En* Z_4, Z_5, Z_6, En Z_1
Con > Div Con > All-R Div > Con Con > Div > All-R Div > Con > All-R Div > All-R > Con	Z_1, Z_6 $Z_6^{\star}$ $Z_4^{\star}$,Z_5,En* Z_1, Z_6 Z_4, En Z_5
(FI+Con) > (FI+Div) >(FD+Div) > (FD+Con)	Z_1
(FI+Div) > (FD+Div) > (FI+Con) > (FD+Con) The significant difference between items is: **(FI+Div) > (FD+Con)**	Z_4 $Z_4^{\star}$
(FI+Con) ≥ (FI+Div) > (FD+Div) > (FD+Con)	Z_5
(FI+Con) > (FD+Con) > (FI+Div) > (FD+Div) **(FI+Con)** is significantly better than the others	$Z_6^{\star}$
(FI+Con) > (FI+Div) > (FD+Div) > (FD+Con)	Cals
(FI+Div) > (FD+Div) > (FI+Con) > (FD+Con) **(FI+Div)** is significantly better than the others	En*

⋆ significant at 0.05 level

∗ significant at 0.1 level

7.5.11 The Sample S

The Sample R was followed by this researcher in the second term of academic year (94/95), but the number of students was reduced to 94. They began to study Calculus 1 and had three fairly high level calculus examinations.

7.5.12 Testing Hypothesis (1)

Once again, the Pearson's correlations between each pair of categories, mean scores and standard deviation of students' performance are set out in Tables 7.18–21.

Table 7.18

The Pearson's Correlation between cognitive styles and $(Z_i)_{i=1}^{6}$

P-C	FI/FD	Con/Div	Z_1	Z_4	Z_5	Z_6	Cals	En
FI/FD	1	0.17^{*}	−0.02	0.01	0.20°	0.02	0.10	0.23
Con/Div	0.17^{*}	1	−0.12	0.00	$0.26^{\star}$	-0.20°	0.05	0.23°
Z_1	−0.02	−0.12	1	−0.10	−0.06	0.20°	0.00	0.01
Z_4	0.01	0.00	−0.10	1	$0.55^{\bullet}$	$0.48^{\bullet}$	$0.90^{\bullet}$	$0.26^{\star}$
Z_5	0.20°	$0.26^{\star}$	−0.06	$0.55^{\star}$	1	0.14	$0.80^{\bullet}$	$0.46^{\bullet}$
Z_6	0.02	-0.20°	0.20°	$0.48^{\star}$	0.14	1	$0.62^{\bullet}$	0.04
Cals	0.10	0.05	0.00	$0.90^{\bullet}$	$0.80^{\bullet}$	$0.62^{\bullet}$	1	$0.33^{\bullet}$
En	0.23°	0.23°	0.01	$0.26^{\star}$	$0.46^{\bullet}$	−0.03	$0.33^{\bullet}$	1

Significant at:

$\bullet$ 0.001 level; $\star$ 0.01 level;

$\circ$ 0.05 level; $*$ 0.1 level

The results of Table 7.19 indicate that FI students performed better than FD ones in all the categories which may support the prediction of hypothesis (1). However, the difference in mean scores between both groups of learning styles except for (Z_5) and (En) is not significant, but all tend to the same direction (Table 7.20).

Table 7.19

Mean and SD on different groups of Sample S in (Z_1, Z_4, Z_5, Z_6, Cals,En)

Groups	FI (N=47)		FInt (N=10)		FD (N=37)	
	Mean	SD	Mean	SD	Mean	SD
Z_1	18.2	9.7	15.9	7.2	18.7	10.5
Z_4	34.5	15.5	25.6	11.5	31.7	15.6
Z_5	32.1	15.6	23.1	18.2	25.3	14.3
Z_6	32.3	13.2	24.7	8.6	29.4	13.6
Cals	33.0	11.8	24.4	8.9	28.7	11.4
En	32.4	8.6	33.2	5.6	28.7	8.9

Table 7.20

The significance of the difference in performance between FI and FD students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
FI&FD	Ns	Ns	S*	Ns	Ns	En*

⋆ significant at 0.05 level

In addition, comparison between mean scores of both thinking styles of Con/Div in each category shows that convergers perform better than divergers in (Z_6), but divergers perform better than convergers in the other categories (Table 7.21). This means that what was predicted by hypothesis (1) is supported and the difference in mean scores between both groups of cognitive styles is significant in (Z_5,En) as shown in Table 7.22.

Table 7.21

Mean and SD on different groups of Sample S in (Z_1, Z_4, Z_5, Z_6, Cals,En)

Groups	Con (N=42)		All-R (N=19)		Div ($N = 33$)	
	Mean	SD	Mean	SD	Mean	SD
Z_1	18.8	10.3	16.8	9.0	18.0	12.8
Z_4	32.1	16.6	32.9	15.5	32.7	13.6
Z_5	24.6	14.6	27.2	15.5	34.1	15.6
Z_6	31.8	14.6	30.9	12.2	28.2	11.4
Cals	29.3	13.0	30.2	11.8	31.8	9.6
En	28.3	8.8	31.1	8.8	34.5	7.4

Table 7.22

The significance of the difference in performance between Con and Div students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
Con&Div	Ns	Ns	S*	Ns	Ns	S*

⋆ significant at 0.05 level

7.5.13 Testing Hypothesis (2)

According to Table 7.23, the mean scores of students with (FI+Con) learning styles are higher than their (FD+Con) colleagues in all the categories except for (Z_1). As a result, the final outcome supports the hypothesis (2) fairly well.

Table 7.23

Mean and SD in calculus categories

Groups	FI+Con (N=16)		FD+Con (N=21)	
	Mean	SD	Mean	SD
Z_1	19.8	8.3	18.8	11.9
Z_4	36.3	17.0	29.9	16.9
Z_5	31.4	17.9	20.9	10.5
Z_6	38.9	13.4	27.3	13.9
Cals	35.4	14.3	25.9	11.5
En	29.2	8.3	25.9	8.4

Table 7.24

The significance of the difference in performance between (FI+Con) and (FD+Con) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FI+Con)&(FD+Con)	Ns	Ns	S*	S*	S*	Ns

⋆ significant at 0.05 level

7.5.14 Testing Hypothesis (3)

The mean scores, as shown in Table 7.25 indicate, that (FI+Div) thinking style learners achieved higher results than (FD+Div) learners in (Z_5,En); by contrast (FD+Div) performed better in (Z_1, Z_4, Z_6,Cals). The former result supports hypothesis (3), the latter rejects it.

Table 7.25

Mean and SD in calculus categories

Groups	FI+Div (N=19)		FD+Div (N=10)	
	Mean	SD	Mean	SD
Z_1	18.8	11.3	17.1	7.4
Z_4	32.3	13.4	35.8	13.4
Z_5	34.2	14.2	33.5	10.7
Z_6	27.6	11.5	32.2	10.8
Cals	31.6	9.5	33.8	8.9
En	35.9	6.5	32.2	9.2

However, the difference between both groups of learning styles is not significant (Table 7.26). This, once again, indicates that being divergent in thinking style could be more beneficial than other cognitive styles such as FI or FD in most calculus activities.

Table 7.26

The significance of the difference in performance between (FI+Div) and (FD+Div) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FI+Div)&(FD+Div)	Ns	Ns	Ns	Ns	Ns	Ns

7.5.15 Testing Hypothesis (4)

According to the student's mean scores in Table 7.27, the (FI+Con) individuals performed better than the (FI+Div) individuals in (Z_4, Z_6,Cals), but (FI+Div) students have done better in (Z_1, Z_5,En) although the difference between means in (Z_1) is very small. This finding supports both parts of hypothesis (4) except for (Z_1, Z_4) and the difference in the means between both groups of learning styles is significant in (Z_6) and (En), as shown in Table 7.28.

Table 7.27

Mean and SD in calculus categories

Groups	FI+Con (N=16)		FI+Div (N=19)	
	Mean	SD	Mean	SD
Z_1	19.8	8.3	18.8	11.3
Z_4	36.3	17.0	32.3	13.4
Z_5	31.4	17.0	34.2	14.2
Z_6	38.9	13.4	27.6	11.5
Cals	35.4	14.3	31.6	9.5
En	29.2	8.3	35.9	6.5

Table 7.28

The significance of the difference in performance between (FI+Con) and (FI +Div) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FI+Con)&(FI+Div)	Ns	Ns	Ns	S*	Ns	S*

⋆ significant at 0.05 level

7.5.16 Testing Hypothesis (5)

Table 7.29 shows that (FD+Div) students achieved higher mean scores compared to their (FD+Con) fellow-students in the categories. This finding supports what was predicted by the first part of hypothesis (5) and rejects the second part.

Table 7.29

Mean and SD in calculus categories

Groups	FD+Con (N=21)		FD+Div (N=10)	
	Mean	SD	Mean	SD
Z_1	18.8	11.9	17.1	7.4
Z_4	29.9	16.9	35.8	13.4
Z_5	20.9	10.5	33.5	10.7
Z_6	27.3	13.9	32.2	10.8
Cals	25.9	11.5	33.8	8.9
En	25.9	8.4	32.2	9.2

However, the differences between the two groups of thinking styles are not significant, but they are in the same direction and exhibit the same result as seen with the previous Sample R in this domain (Table 7.30).

Table 7.30

The significance of the difference in performance between (FD+Con) and (FD+Div) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FD+Con)&(FD+Div)	Ns	Ns	Ns	Ns	Ns	Ns

7.5.17 A brief Symbolic Picture of Sample S

A symbolic picture of the students' performance in this sample with different learning styles is set out in Table 7.31 and Figures 7.3–4 exhibit the achievements of students with different cognitive styles, built on hypotheses (1–5), in (Z_4, Z_5, Z_6). The superiority displayed is based on their mean scores in each category. Moreover, some more information has been found in this table which indicates a significant difference between learning styles not compared within the hypotheses of the present research. For instance, there is a significant difference between FI and FInt in (Cals), between (FI+Div) learning styles and (FD+Con) ones in (Z_5,En), and between (FD+Div) and (FI+Con) in (Z_5).

Table 7.31

The symbolic picture of students' superiority in $(Z_i)_{i=1}^{6}$ with different learning styles displayed by the mean scores in Sample S

Groups of cognitive styles	Categories
FI > FD	Z_1, Z_4, Z_5^*, Z_6, Cals, En*
FI > FD > FInt	Z_4, Z_5, Z_6, En
In this chain the significant difference is: **FI > FInt**	Cals*
Con > Div	Z_6
Div > Con	Z_1, Z_4, Z_5^*, Cals, En*
Con > All-R > Div	Z_6
Div > Con > All-R	Z_4, Z_5, En
All-R > Div > Con	Z_1
(FD+Div) > (FI+Div) >(FD+Con) > (FI+Con)	Z_1
(FI+Con) > (FD+Div) > (FI+Div) > (FD+Con)	Z_4
(FI+Div) > (FD+Div) > (FI+Con) > (FD+Con)	Z_5
In this chain the significant differences are: **(FI+Div) > (FD+Con), (FD+Div) > (FI+Con), (FI+Con) > (FD+Con)**	Z_5^*
(FI+Con) > (FD+Div) > (FD+Con) > (FI+Div)	Z_6
In this chain the significant differences are: **(FI+Con) > (FD+Con), (FI+Con) > (FI+Div)**	Z_6^*
(FI+Con) > (FD+Div) > (FI+Div) > (FD+Con)	Cals
In this chain the significant differences are: **(FI+Con) > (FD+Con)**	Cals*
(FI+Div) > (FD+Div) > (FI+Con) > (FD+Con)	En
In this chain the significant differences are: **(FI+Div) > (FI+Con), (FI+Div) > (FD+Con)**	En*

⋆ significant at 0.05 level

Figure 7.3

The performance of FI/FD and Con/Div students in (Z_4, Z_5, Z_6) in Sample S

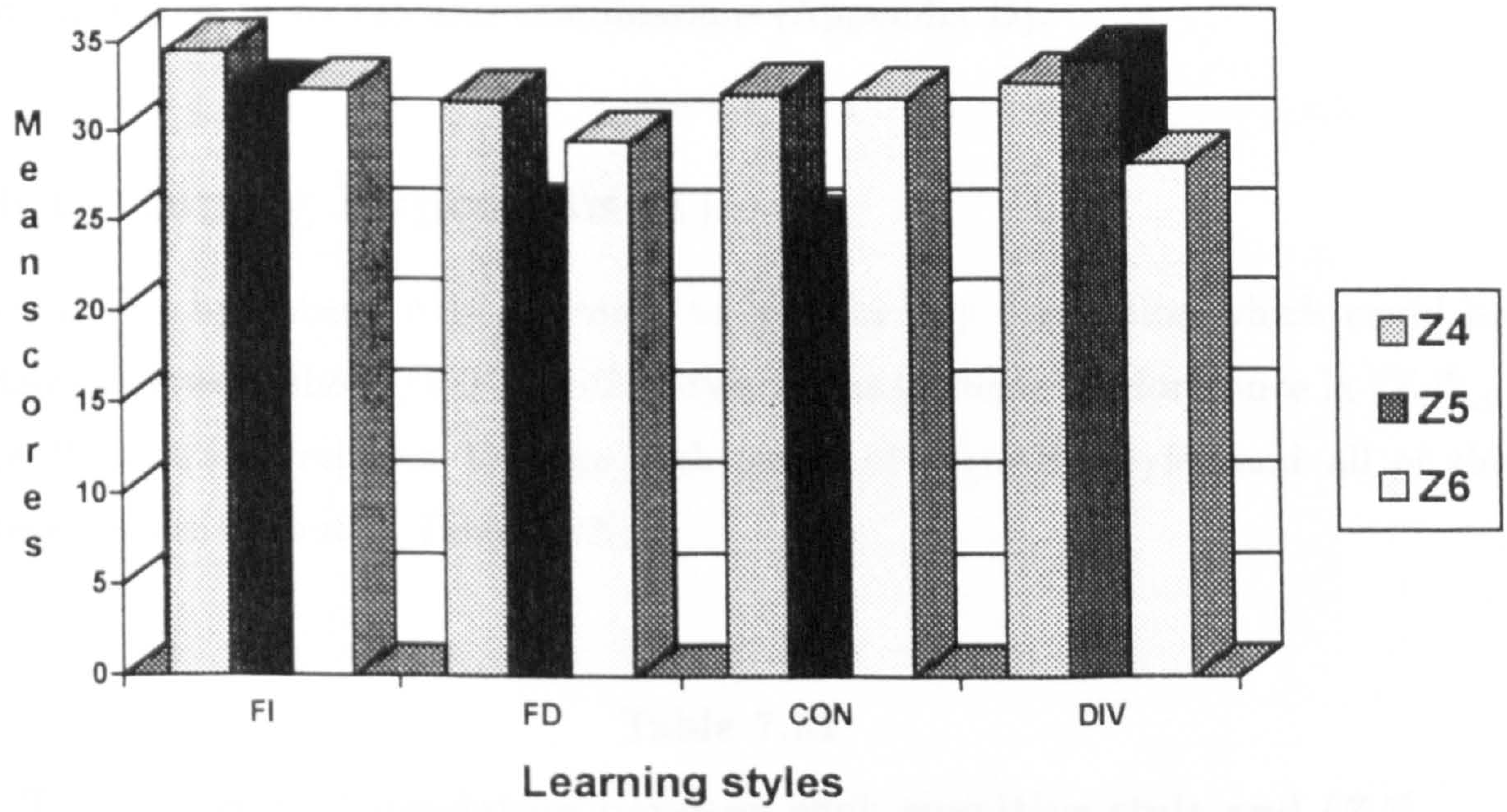
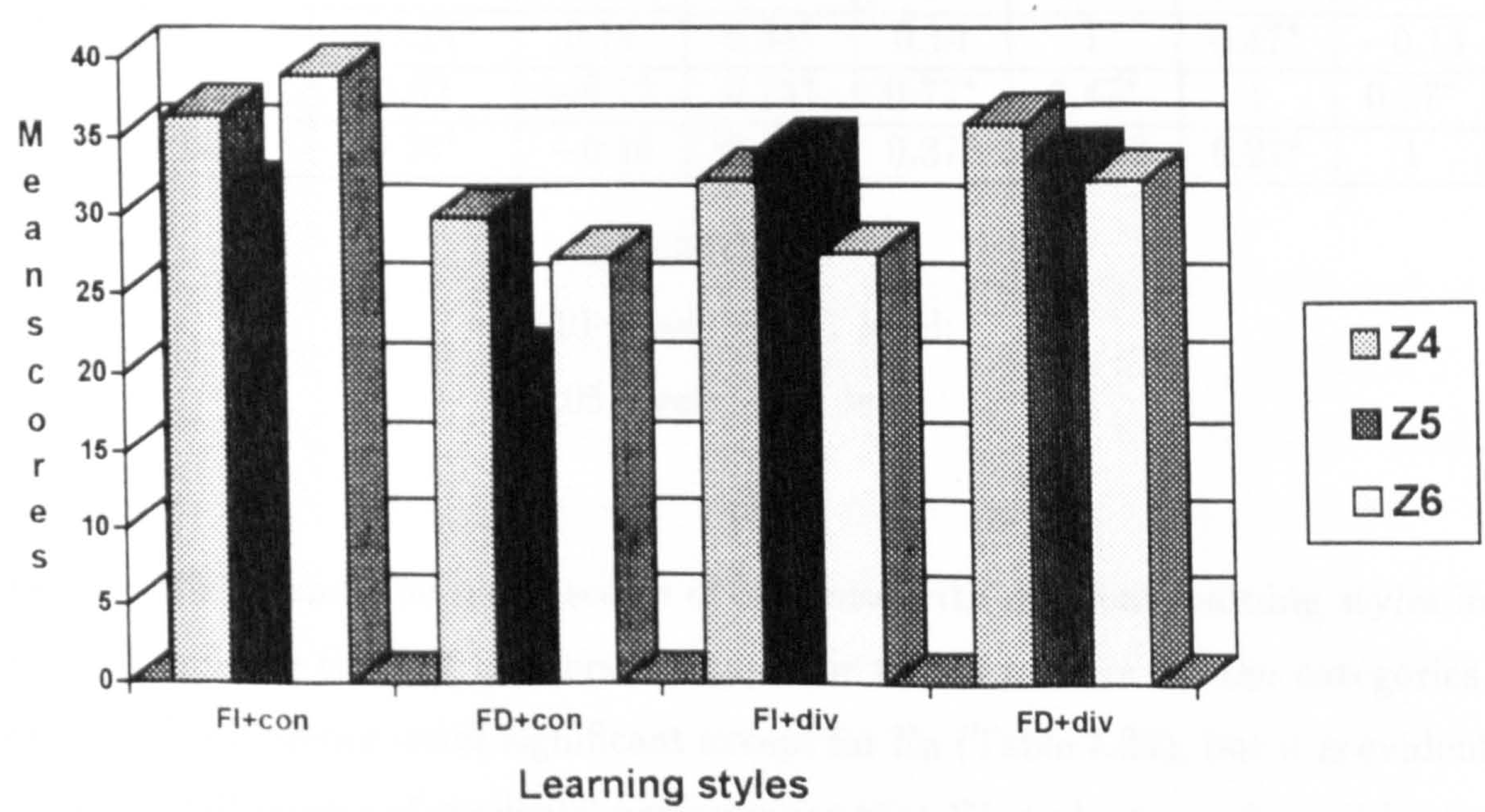


Figure 7.4

The students' performance with different learning styles in (Z_4, Z_5, Z_6) in Sample S



7.6 Overall results of Sabzevar Samples

It is interesting to see how well these calculus students performed in the two terms of their activities. It should be noted that in this academic year, 85% of students had taken part in six calculus examinations (Appendix D).

7.6.1 Testing Hypothesis (1)

To test this hypothesis it is necessary to look for any correlation which could be obtained between the FI/FD cognitive style versus students' performance in $(Z_i)_{i=1}^{6}$. The Pearson's correlation between each group of cognitive styles and all of the categories are set out in Table 7.32.

Table 7.32

The Pearson's Correlation between each cognitive style and $(Z_i)_{i=1}^{6}$

P-C	FI/FD	Con/Div	Z_1	Z_4	Z_5	Z_6	Cals	En
FI/FD	1	0.17*	−0.10	0.05	0.24°	−0.01	0.12	0.23°
Con/Div	0.17*	1	−0.13	0.02	0.23°	−0.24°	0.03	0.23°
Z_1	−0.10	−0.13	1	−0.20°	−0.16	0.15	−0.12	0.10
Z_4	0.05	0.02	−0.20°	1	0.47•	0.34•	0.75•	0.30⋆
Z_5	0.24°	0.23°	−0.16	0.47•	1	0.14	0.80•	0.37•
Z_6	−0.14	−0.24°	0.14	0.34•	0.14	1	0.47•	−0.13
Cals	0.12	0.02	−0.12	0.75•	0.77•	0.47•	1	0.27⋆
En	0.23°	0.23°	−0.10	0.28⋆	0.37•	−0.13	0.27⋆	1

Significant at:

• 0.001 level; ⋆ 0.01 level;

o 0.05 level; * 0.1 level

On the other hand, the mean scores of students with different learning styles in Table 7.33 indicate that FI learners are superior to FD ones in all the categories. However, this difference is not significant except for En (Table 7.34), but it is evident from the overall results of students' performance that FI students performed better and achieved more than their colleagues in all domains of calculus activities. Since the mean scores differences are in one direction, the results are not random and tend to reinforce each other.

Table 7.33

Mean and SD on different groups of the Sabzevar samples over one academic year in (Z_1, Z_4, Z_5, Z_6, Cals,En)

Groups	FI (N=52)		FInt (N=13)		FD (N=48)	
	Mean	SD	Mean	SD	Mean	SD
Z_1	17.9	8.3	18.8	8.8	21.8	11.2
Z_4	39.3	14.9	32.6	15.6	34.7	12.5
Z_5	39.2	17.0	34.4	21.5	34.9	17.7
Z_6	34.5	16.9	23.3	10.6	29.6	14.1
Cals	39.1	14.2	31.9	14.2	34.0	10.1
En	32.4	8.5	33.2	7.8	28.7	8.9

Table 7.34

The significance of the difference in performance between FI/FD students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
FI&FD	Ns	Ns	Ns	Ns	Ns	S*

⋆ significant at 0.05 level

The next stage of this work is to study the relationship between the convergent/divergent thinking style of students at Sabzevar university and their attainments in $(Z_i)_{i=1}^{6}$. Students' who obtained high scores in the Con/Div tests (divergent thinkers) again obtained high scores in the calculus tasks involving the categories (Z_1, Z_4, Z_5,Cals,En), while the students who obtained low scores in the Con/Div tests (convergent thinkers) obtained high scores in (Z_6). Table 7.35 shows the production of two terms of students' activities, based on their mean scores in the six calculus examinations. The difference in mean scores between both groups of cognitive styles was significant in (Z_5, Z_6,Cals,En), as shown in Table 7.36.

Table 7.35

Mean and SD on different groups of Sabzevar samples over one academic year in (Z_1, Z_4, Z_5, Z_6, Cals,En)

Groups	Con (N=50)		All-R (N=22)		Div (N=41)	
	Mean	SD	Mean	SD	Mean	SD
Z_1	19.4	10.2	21.8	11.5	18.8	8.0
Z_4	35.8	15.1	35.9	12.8	37.9	14.1
Z_5	33.6	18.3	35.1	15.5	41.8	17.8
Z_6	34.3	17.4	29.4	12.4	28.2	13.6
Cals	35.5	15.2	34.1	9.3	38.2	11.2
En	28.3	8.8	31.1	8.8	34.5	7.4

Table 7.36

The significance of the difference in performance between Con and Div students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
Con&Div	Ns	Ns	S*	S*	S*	S*

⋆ significant at 0.05 level

∗ significant at 0.1 level

To sum up, the results which have emerged from the one year study of this sample with six calculus examinations tend to confirm that field-independent students are better than field-dependent in calculus course material.

Moreover, convergent students tend to be better in (Z_6) than divergent ones. But, divergent thinkers tend to show higher performance than convergent thinkers in (Z_1, Z_4, Z_5,Cals,En). It has mainly emerged from this sample that convergers tend to be better than divergers in multi-skilled and transferable skills tasks in calculus, while divergers tend to be better than convergers in most areas of the calculus activities.

A brief review shows that the outcome of this research on the Sabzevar samples confirm the previous results by other studies that FI>FD in mathematics learning and Div>Con in imaginative tasks as suggested by Hudson (1966). Moreover, the Pearson's correlation between both dimension styles (FI/FD and Con/Div) and calculus categories was not high, but significant. This is the same direction which was found by Harmon (1984), that relationship between scores in calculus and the measures of "Group Embedded Figures Test" (GEFT) is low.

7.6.2 Testing Hypothesis (2)

According to Table 7.37, the mean scores of students with (FI+Con) learning styles performed better than (FD+Con) ones in all categories. This result confirms the prediction of hypothesis (2) that predicts the superiority of (FI+Con) students over their (FD+Con) colleagues in $(Z_i)_{i=1}^{6}$. Moreover, the difference in the mean scores between both groups of cognitive styles is significant in (Z_5, Z_6,Cals) and nonsignificant in others (Table 7.38). This indicates that (FI+Con) thinking styles exhibit statistically higher performance than (FD+Con) in the whole domain of calculus learning.

Table 7.37

Mean and SD in calculus categories

Groups	FI+Con (N=17)		FD+Con (N=27)	
	Mean	SD	Mean	SD
Z_1	17.6	7.1	21.3	11.3
Z_4	42.2	17.9	33.7	12.3
Z_5	42.6	21.2	30.8	15.4
Z_6	46.0	17.3	28.5	14.8
Cals	44.4	19.5	31.9	9.5
En	29.2	8.3	25.9	8.4

Table 7.38

The significance of the difference in performance between (FI+Con) and (FD+Con) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FI+Con)&(FD+Con)	Ns	Ns	S*	S*	S*	Ns

⋆ significant at 0.05 level

7.6.3 Testing Hypothesis (3)

The mean scores, as shown in Table 7.39 demonstrate that (FI+Div) thinkers have higher performance than (FD+Div) thinkers in (Z_1,En). By contrast, (FD+Div) students are better than (FI+div) in (Z_4, Z_5, Z_6,Cals) that is in the whole area of calculus activities. However, there is no significant difference between both groups of learning styles in all categories (Table 7.40).

Table 7.39

Mean and SD in calculus categories

Groups	FI+Div (N=23)		FD+Div (N=13)	
	Mean	SD	Mean	SD
Z_1	18.0	9.1	19.1	7.0
Z_4	37.4	12.5	38.9	12.9
Z_5	38.7	13.8	45.2	19.1
Z_6	27.0	14.5	32.2	12.5
Cals	36.8	9.9	40.2	9.3
En	38.8	6.5	32.2	9.2

Table 7.40

The significance of the difference in performance between (FI+Div) and (FD+Div) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FI+Div)&(FD+Div)	Ns	Ns	Ns	Ns	Ns	Ns

These results have rejected the prediction that (FI+Div) students are better in calculus course activities than (FD+Div) students. It emerges from the overall conclusion to this pattern that the divergent way of thinking is more effective than the FI/FD thinking style in calculus achievement, despite the higher grounding of (FI+Div) learners than (FD+Div) ones in the high school mathematics in this sample.

7.6.4 Testing Hypothesis (4)

The means of students' performance with different cognitive styles (FI+Con) and (FI+Div) in all the categories are shown in Table 7.41. These results indicate that the former learning styles are better in the calculus categories, while the latter has shown higher attainment in (En). This result, to some extent, supports hypothesis (4), however the difference between mean scores of both groups except for (Z_6,En) is not significant, but the difference in mean scores are all in the same direction.

Table 7.41

Mean and SD in calculus categories

Groups	FI+Con (N=17)		FI+Div (N=23)	
	Mean	SD	Mean	SD
Z_1	17.6	7.1	18.0	9.1
Z_4	42.2	17.9	37.4	12.5
Z_5	42.6	21.2	38.7	13.8
Z_6	46.0	17.3	27.0	14.5
Cals	44.4	19.5	36.8	9.9
En	29.2	8.3	38.8	6.5

Table 7.42

The significance of the difference in performance between (FI+Con) and (FI+Div) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FI+Con)&(FI+Div)	Ns	Ns	Ns	S*	Ns	S*

⋆ significant at 0.05 level

7.6.5 Testing Hypothesis (5)

In the final step of testing the hypotheses, (FD+Div) students obtained higher mean scores compared to their (FD+Con) colleagues in all the categories of this research (Table 7.43). Therefore, such a finding supports what was predicted by the first part of hypothesis (5) and rejected its second part. However, the difference between the mean scores is not significant except for (Z_5), as shown in Table 7.44.

Table 7.43

Mean and SD in calculus categories

Groups	FD+Con (N=27)		FD+Div (N=13)	
	Mean	SD	Mean	SD
Z_1	21.3	11.3	19.1	7.0
Z_4	33.7	12.3	38.9	12.9
Z_5	30.8	15.4	45.2	19.1
Z_6	28.5	14.8	32.2	12.5
Cals	31.9	9.5	40.2	9.3
En	25.0	8.4	32.2	9.2

Table 7.44

The significance of the difference in performance between (FD+Con) and (FD+Div) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FD+Con)&(FD+Div)	Ns	Ns	S*	Ns	Ns	Ns

⋆ significant at 0.05 level

The overall deduction indicates that, for a student who is FD, being a divergent thinker may be more helpful than being convergent in tackling calculus categories in this study.

7.6.6 An Overall symbolic Picture of Sabzevar samples

A symbolic picture of the final students' performance with different learning styles is set out in Table 7.45. The superiority displayed is based upon their mean scores in each category. Moreover, some extra information can be found in this table to indicate significant difference between learning styles not described within the hypotheses of the present research. For instance, there is a significant difference between FI and FInt in (Z_6), between (FI+Con) learning styles and (FD+Div) ones in (Z_6), and between (FI+Div) and (FD+Con) in (En). Figures 7.5–6 display the performance of students with different learning styles, based on hypotheses (1–5), in the categories (Z_4, Z_5, Z_6).

Table 7.45

The overall picture of students' superiority in $(Z_i)_{i=1}^{6}$ with different learning styles based on the mean scores in Sabzevar samples

Groups of cognitive styles	Categories
FI > FD	Z_1, Z_4, Z_5, Z_6,Cals, En*
FI > FD > FInt	Z_4, Z_6, En
In this chain the significant difference is:	
FI > FInt	Z_6*
Con > Div	Z_6*
Div > Con	Z_1, Z_4, Z_5*,Cals*,En*
Con > Div > All-R	Z_6
Div > Con > All-R	Z_1, Z_4, Z_5, En
(FI+Con) > (FI+Div) >(FD+Div) > (FD+Con)	Z_1
(FI+Con) > (FD+Div) > (FI+Div) > (FD+Con)	Z_4
(FD+Div) > (FI+Con) > (FI+Div) > (FD+Con)	Z_5
In this chain the significant differences are:	
(FD+Div) > (FD+Con), (FI+Con) > (FD+Con)	Z_5*
(FI+Con) > (FD+Div) > (FD+Con) > (FI+Div)	Z_6
In this chain the significant differences between items are:	
(FI+Con) > (FD+Div), (FI+Con) > (FD+Con), (FI+Con) > (FI+Div)	Z_6*
(FI+Con) > (FD+Div) > (FI+Div) > (FD+Con)	Cals
In this chain the significant difference is:	
(FI+Con) > (FD+Con)	Cals*
(FI+Div) > (FD+Div) > (FI+con) > (FD+Con)	En
In this chain the significant differences are:	
(FI+Div) > (FI+Con), (FI+Div) > (FD+Con)	En*

Significant at: ⋆ 0.05 level; ∗ 0.1 level

Figure 7.5

The performance of students with learning styles FI/FD and Con/Div in (Z_4, Z_5, Z_6) in Sabzevar samples

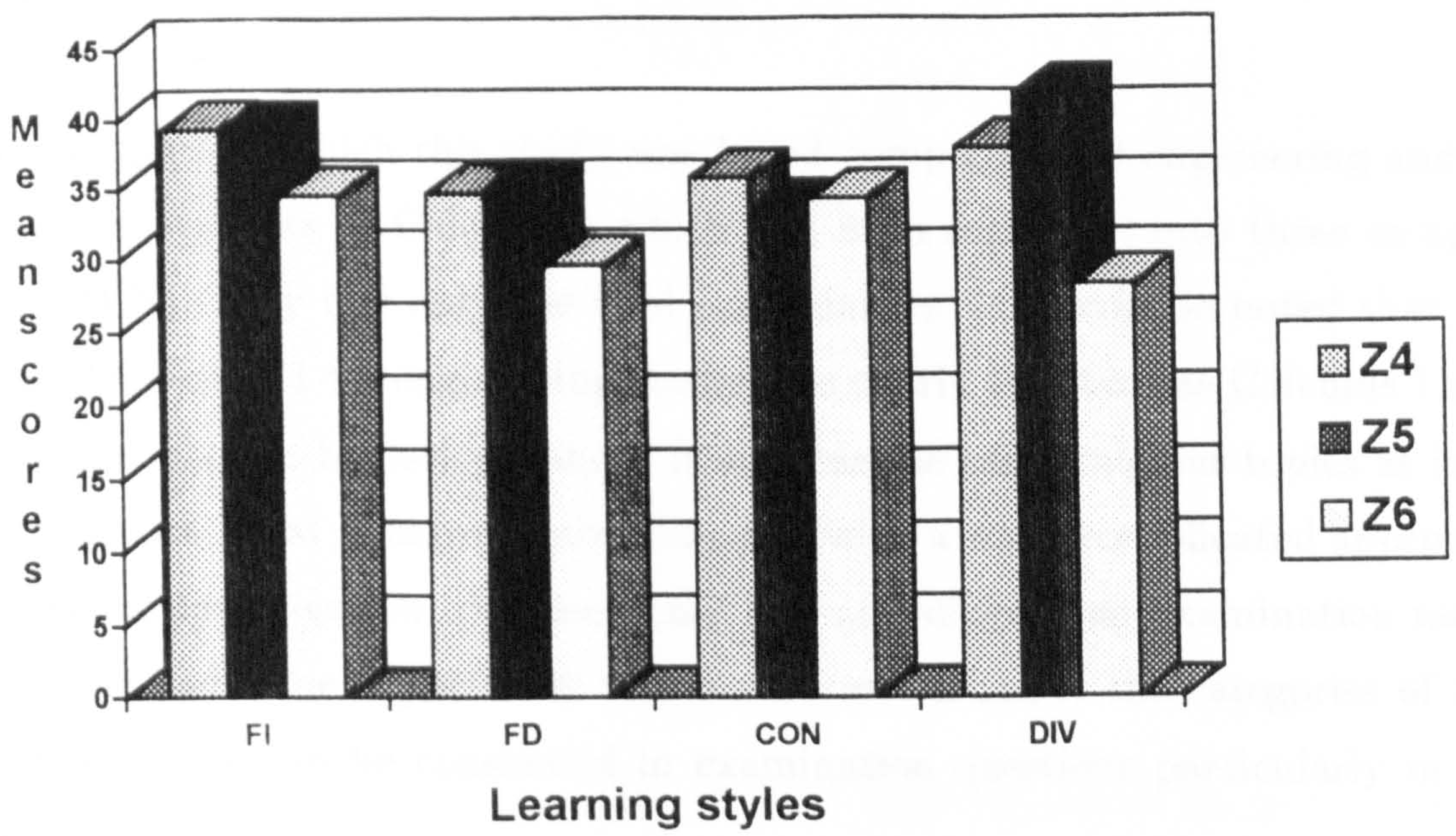
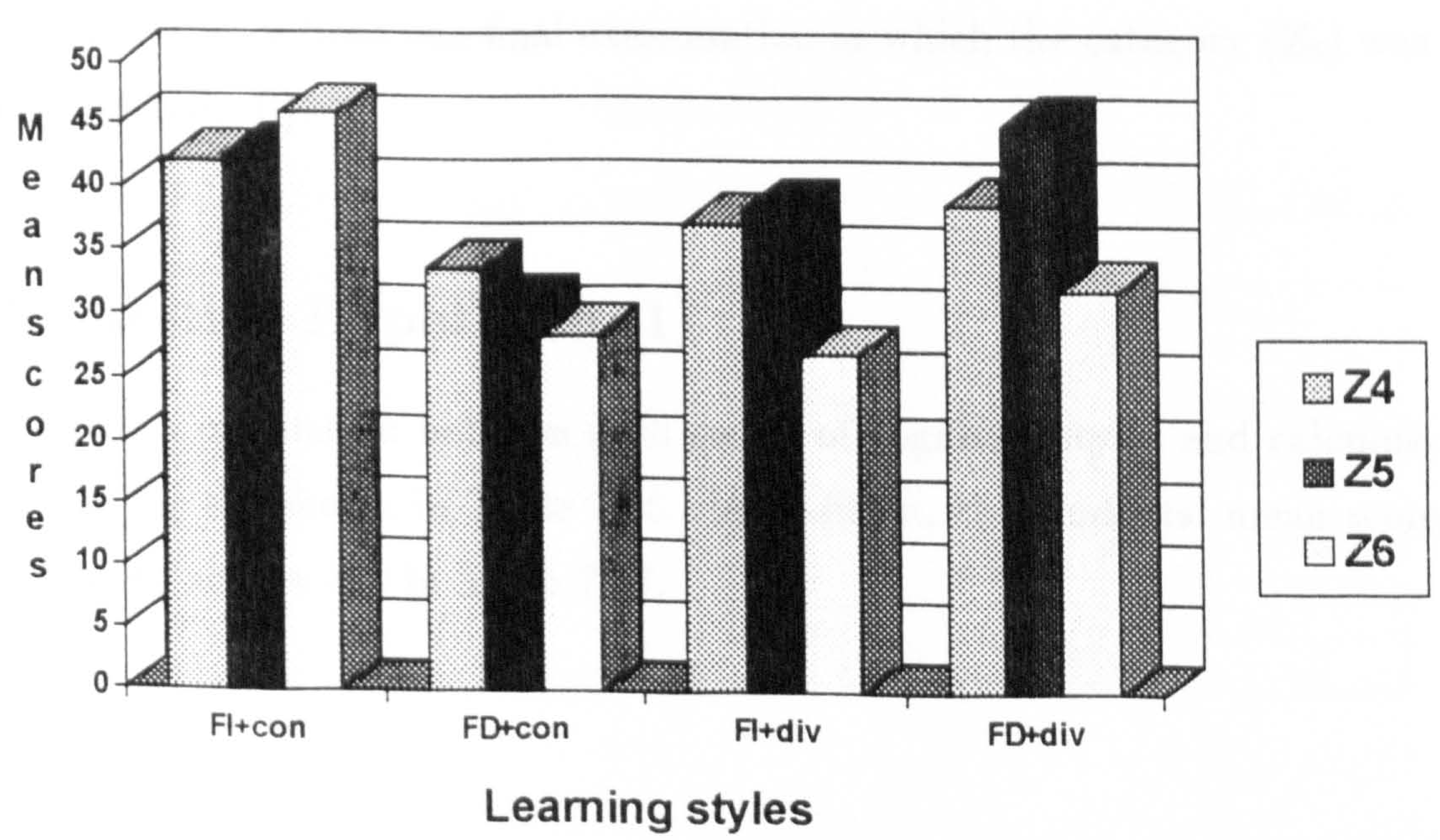


Figure 7.6

The students' performance with different learning styles in (Z_4, Z_5, Z_6) in Sabzevar samples



7.6.7 The Second Investigation

The second investigation of this research was carried out upon the students at Mashhad University. At the time of this study, all were enrolled in the first year of higher education.

The samples on which this study was based comprised 200 engineering and 54 mathematics students in Calculus 1 which had been separated into three samples called (J,M,N). They had only one final examination. It should be noted that the syllabus of Calculus 1 for engineering students is nearly the same as Calculus 1 and Calculus 2 in the mathematics branch. It also has the same calculus topics as in as in Mathematics 1A at Glasgow University, but with a more complicated approach. In the second investigation, the researcher had no influence on examination tasks. However, he had some opportunity to have discussion about the categories of this research which should be considered in examination questions particularly in the Sample M.

7.6.8 (a) Sample J

This sample was selected from three different calculus classes with a population of 123 engineering students in the first term of session 94/95. The results of students' performance emerge from one final examination in which the category (Z_5) was not tested (Appendix E).

7.6.9 Testing Hypothesis (1)

The Pearson's correlation between each group of cognitive styles and calculus categories $(Z_i)_{i=1}^{6}$ are shown in Table 7.46. In addition, the students' mean scores in each category are set out in Table 7.47.

Table 7.46

The Pearson's Correlation between each cognitive style and $(Z_i)_{i=1}^{6}$

P-C	FI/FD	Con/Div	Z_1	Z_4	Z_6	Cals	En
FI/FD	1	0.17*	−0.02	−0.10	−0.10	−0.10	0.14
Con/Div	0.17*	1	0.10	−0.10	−0.14	−0.14	−0.05
Z_1	−0.02	0.10	1	−0.10	0.10	−0.02	0.03
Z_4	−0.11	−0.10	−0.10	1	0.38•	0.84•	0.30•
Z_6	−0.10	−0.14	0.10	0.38•	1	0.75•	0.34•
Cals	−0.10	−0.14	−0.02	0.84•	0.75•	1	0.42•
En	0.14	−0.05	0.03	0.30•	0.34•	0.42•	1

Significant at:

• 0.001 level; ⋆ 0.01 level

∗ 0.05 level

Table 7.47 indicates that students who are field-dependent performed better than field-independent students in (Z_4, Z_6,Cals), but FI learners are better than FD learners in (Z_1,En). The former result did not support what was predicted by hypothesis (1). However, the difference between mean scores is not significant (Table 7.48). In this sample despite the previous findings, FD thinking styles showed higher performance compared to FI ones, but in only one calculus examination.

Table 7.47

Mean and SD on different groups of Sample J in (Z_1, Z_4, Z_6, Cals,En)

Groups	FI (N=62)		FInt (N=5)		FD (N=53)	
	Mean	SD	Mean	SD	Mean	SD
Z_1	24.2	12.2	37.0	6.9	25.8	14.7
Z_4	54.6	20.4	58.8	29.5	56.9	16.4
Z_6	67.9	24.8	86.8	13.1	75.0	18.6
Cals	59.7	18.9	67.3	23.6	62.6	14.1
En	62.3	8.3	67.1	9.9	61.5	7.7

Table 7.48

The significance of the difference in performance between FI/FD students

Groups	Z_1	Z_4	Z_6	Cals	En
FI&FD	Ns	Ns	Ns	Ns	Ns

Moreover, convergent students have shown higher performance than divergent ones in (Z_1, Z_4, Z_6,Cals,En), as shown in Table 7.49 and the difference between mean scores is significant in (Z_6,Cals,En), as exhibited in Table 7.50. This finding supports the second part of hypothesis (1) and rejects the third part.

Table 7.49

Mean and SD on different groups of Sample J in (Z_1, Z_4, Z_6, Cals,En)

Groups	Con (N=52)		All-R (N=18)		Div (N=50)	
	Mean	SD	Mean	SD	Mean	SD
Z_1	25.4	13.8	20.7	8.8	27.4	14.4
Z_4	58.7	19.7	56.9	21.0	52.7	17.1
Z_6	77.5	20.7	69.9	17.3	67.8	21.6
Cals	64.9	18.0	62.9	17.3	57.8	15.7
En	62.3	9.1	62.7	9.0	61.2	9.3

Table 7.50

The significance of the difference in performance between Con/Div students

Groups	Z_1	Z_4	Z_6	Cals	En
Con&Div	Ns	Ns	S*	S*	S*

⋆ significant at 0.05 level

The next step of this part of the research is to study the various possible patterns which appear from the combinations of all cognitive styles involved in the present investigation versus students' performance in the calculus categories. This will be found in the following sections.

7.6.10 Testing Hypothesis (2)

According to Table 7.51, the mean scores of (FD+Con) students tend to be higher than their (FI+Con) fellow-students in (Z_6,Cals,En), by contrast (FI+Con) learners are better than (FD+Con) learners in (Z_1, Z_4). The prediction of hypothesis (2) is only partially supported, however the difference between mean scores is not significant (Table 7.52).

Table 7.51

Mean and SD in calculus categories

Groups	FI+Con (N=19)		FD+Con (N=29)	
	Mean	SD	Mean	SD
Z_1	23.7	14.1	25.4	14.2
Z_4	60.6	19.1	58.8	17.7
Z_6	72.5	23.5	79.9	18.8
Cals	64.4	18.9	65.9	17.4
En	62.1	10.4	63.2	17.4

It seems that the background superiority of students with (FD+Con) learning styles in high school mathematics which is reflected in the (En) category, may lead to their higher performance in (Z_6,Cals). In addition to this, it could be concluded that being a convergent thinker is more beneficial than being FD or FI in tackling multi-skilled tasks (Z_6) in a calculus course.

Table 7.52

The significance of the difference in performance between (FI+Con) and (FD+Con) students

Groups	Z_1	Z_4	Z_6	Cals	En
(FI+Con)&(FD+Con)	Ns	Ns	Ns	Ns	Ns

7.6.11 Testing hypothesis (3)

According to the mean scores of the students in Table 7.53, (FI+Div) thinkers performed higher than (FD+Div) thinkers in (Z_1), while (FD+Div) students achieved more than (FI+Div) students in (Z_4, Z_6,Cals) and the both groups had nearly the same result in (En). This also indicates that the hypothesis prediction has been rejected partially, and being a divergent thinker is better indicator than being a convergent thinker in calculus achievement of the engineering course.

Table 7.53

Mean and SD in calculus categories

Groups	FI+Div (N=26)		FD+Div (N=23)	
	Mean	SD	Mean	SD
Z_1	24.3	12.2	29.9	15.6
Z_4	51.5	17.3	52.2	16.4
Z_6	65.1	24.6	69.7	18.8
Cals	55.8	16.8	58.0	11.8
En	60.9	4.9	60.4	6.3

However, the difference between mean scores is not significant (Table 7.54), but the difference in means in the calculus area (Z_4, Z_5,Cals) are all in the same direction.

Table 7.54

The significance of the difference in performance between (FI+Div) and (FD+Div) students

Groups	Z_1	Z_4	Z_6	Cals	En
(FI+Div)&(FD+Div)	Ns	Ns	Ns	Ns	Ns

7.6.12 Testing Hypothesis (4)

How is the interaction of (FI+Con) thinking style versus the (FI+Div) style in this sample? According to research produced in this domain (Table 7.55), it seems more likely that (FI+Con) students have shown higher behaviour than (FI+Div) students in all the categories. This means that the first part of hypothesis (4) has been supported, while the second part is rejected by this sample. Nonetheless, the difference between means in both groups of learning styles is not significant, but they are all in the same direction (Table 7.56).

Table 7.55

Mean and SD in calculus categories

Groups	FI+Con (N=19)		FI+Div (N=26)	
	Mean	SD	Mean	SD
Z_1	23.7	14.1	24.3	12.2
Z_4	60.6	19.1	51.5	17.3
Z_6	72.5	23.5	65.1	24.6
Cals	64.4	18.9	55.8	16.8
En	62.1	10.4	60.9	4.9

Table 7.56

The significance of the difference in performance between (FI+Con) and (FI+Div) students

Groups	Z_1	Z_4	Z_6	Cals	En
(FI+Con)&(FI+Div)	Ns	Ns	Ns	Ns	Ns

7.6.13 Testing Hypothesis (5)

The final stage of hypothesis testing in this sample (Table 7.57) shows that (FD+Con) learning styles exhibit higher results than (FD+Div) ones in all the categories. This finding supports the second part of hypothesis (5) and rejects the first part. However, the difference in the means was not significant (Table 7.58). On this point, once again, the same justification as has been demonstrated in hypothesis (4), could be exhibited.

Table 7.57

Mean and SD in calculus categories

Groups	FD+Con (N=29)		FD+Div (N=23)	
	Mean	SD	Mean	SD
Z_1	25.4	14.2	29.9	15.6
Z_4	58.8	17.7	52.2	16.4
Z_6	79.9	18.8	69.7	18.8
Cals	65.9	17.4	58.0	11.8
En	63.2	17.4	60.4	6.3

Table 7.58

The significance of the difference in performance between (FD+Con) and (FD+Div) students

Groups	Z_1	Z_4	Z_6	Cals	En
(FD+Con)&(FD+Div)	Ns	Ns	Ns	Ns	Ns

7.6.14 A brief Symbolic Picture of Sample J

A symbolic picture of the final students' performance with different learning styles are set out in Table 7.59 and its pictorial forms, based on hypotheses (1–5), are displayed by Figures 7.7–8. The superiority displayed is built from students' mean scores in each category. Moreover, some more information can be found in this table to indicate significant differences between learning styles not described within hypotheses of the present research. For instance, there is a significant difference between FI and FInt in (Z_6), between (FI+Con) learning styles and (FD+Div) ones in (Z_6), and between (FI+Div) and (FD+Con) in (En).

Figure 7.7
The performance of FI/FD and Con/Div students in (Z_4, Z_6,Cals) in Sample J

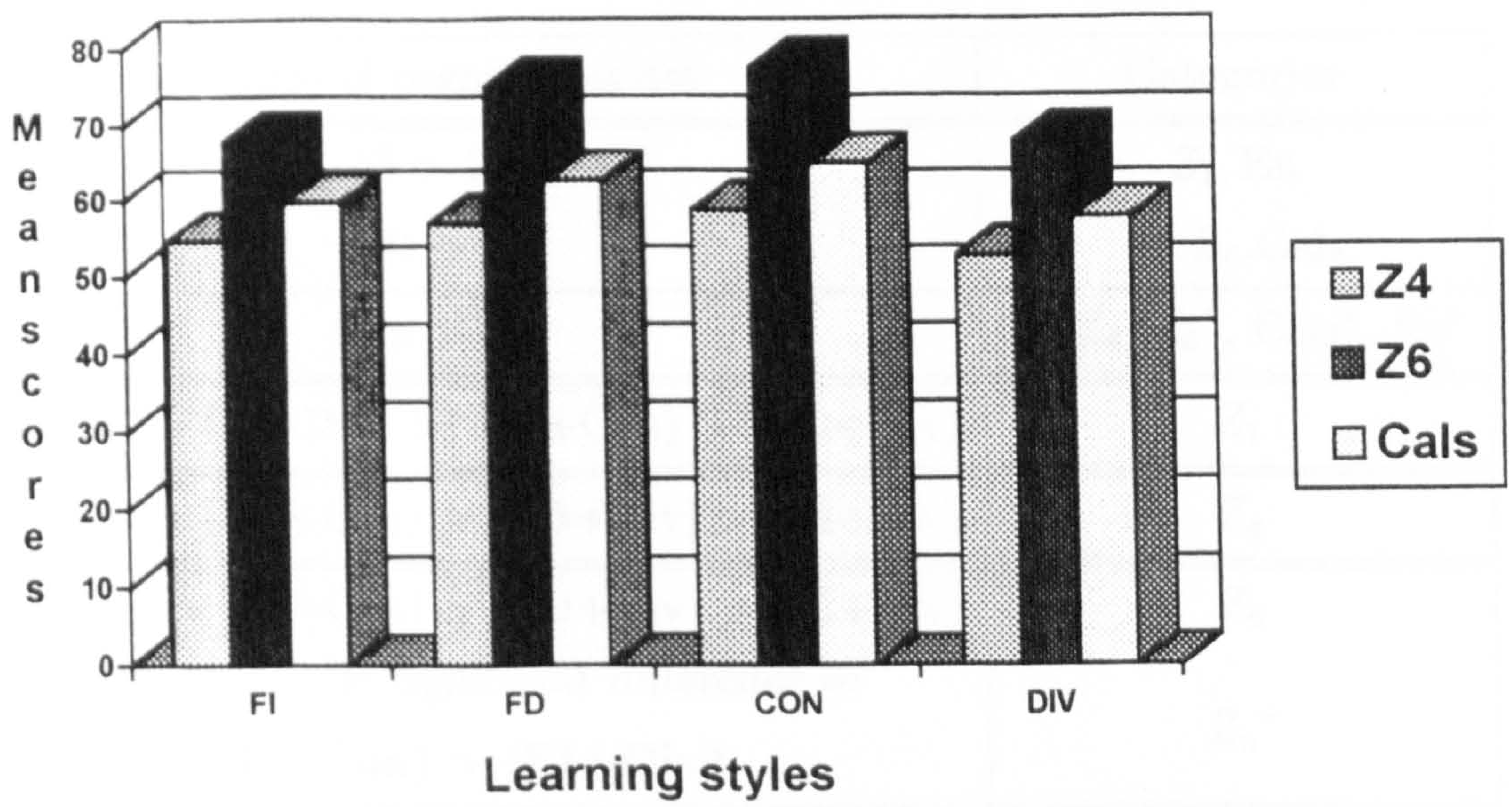


Figure 7.8
The students' performance with different learning styles in (Z_4, Z_6,Cals) in Sample J

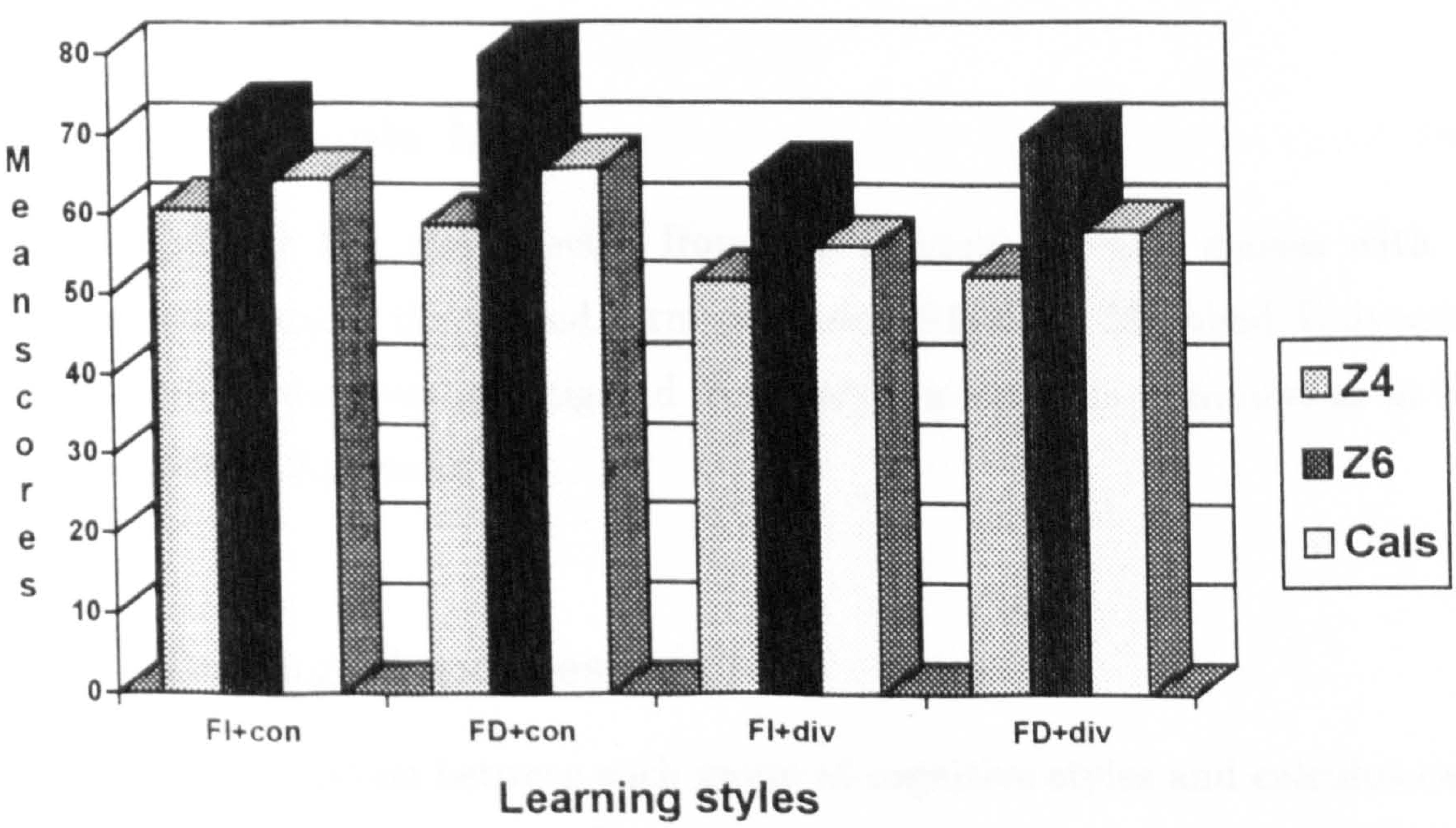


Table 7.59

The overall picture of students' superiority in $(Z_i)_{i=1}^6$ with different learning styles based on the mean scores in Sample J

Groups of cognitive styles	Categories
FI > FD FD > FI	Z_1, En Z_4,Z_6,Cals
Con > Div	Z_1,Z_4, $Z_6^\star$, Cals*, En*
(FI+Con) > (FI+Div) >(FD+Con) > (FD+Div)	Z_1
(FI+Con) > (FD+Con) > (FD+Div) > (FI+Div)	Z_4
(FD+Con) > (FI+Con) > (FD+Div) > (FI+Div) In this chain the significant difference is: **(FD+Con) > (FI+Div)**	Z_6 $Z_6^\star$
(FD+Con) > (FI+Cov) > (FD+Div) > (FI+Div) In this chain the significant difference is: **(FD+Con) > (FI+Div)**	Cals Cals*
(FD+Con) > (FI+Con) > (FI+Div) > (FD+Div)	En

⋆ significant at 0.05 level

7.6.15 (b) Sample M

The second sample was also selected from two different calculus classes with 77 engineering students in the second term of session 94/95 at Mashhad University. Students' performance was investigated from only one calculus examination at the end of the course (Appendix F).

7.6.16 Testing Hypothesis (1)

The Pearson's correlation between each group of cognitive styles and calculus categories $(Z_i)_{i=1}^6$ are exhibited in the Table 7.60. Moreover, mean scores and standard deviation of students' performance are shown in Table 7.61.

Table 7.60

The Pearson's Correlation between each cognitive style and $(Z_i)_{i=1}^{6}$

P-C	FI/FD	Con/Div	Z_1	Z_4	Z_5	Z_6	Cals	En
FI/FD	1	0.23*	−0.22°	−0.10	0.10	−0.10	−0.10	0.13
Con/Div	0.23*	1	−0.10	−0.11	−0.07	−0.14	−0.13	−0.02
Z_1	−0.22°	−0.01	1	−0.12	−0.06	0.06	−0.10	−0.02
Z_4	−0.10	−0.11	−0.12	1	0.10	0.43•	0.90•	0.42•
Z_5	0.10	−0.07	−0.06	0.10	1	0.03	0.40•	0.06
Z_6	−0.11	−0.14	0.06	0.43•	0.03	1	0.67•	0.35⋆
Cals	−0.10	−0.13	−0.10	0.90•	0.40•	0.67•	1	0.44•
En	0.13	−0.02	−0.02	0.42•	0.05	0.35⋆	1	0.44•

Significant at:

• 0.001 level; ⋆ 0.01 level;

∗ 0.05 level; ∘ 0.1 level

According to the Table 7.61 the mean scores of FI students tend to be better than FD ones in all the categories. Nonetheless, the difference between mean scores in both groups except in (Z_1) is not significant (Table 7.62), but they are all in the same direction. This results supports what was predicted by hypothesis (1) and, once again, confirms what was found in Sabzevar samples.

Table 7.61

Mean and SD on different groups of Sample M in $(Z_1, Z_4, Z_5, Z_6,$ Cals,En)

Groups	FI (N=27)		FInt (N=12)		FD (N=38)	
	Mean	SD	Mean	SD	Mean	SD
Z_1	19.9	11.7	23.4	16.4	30.1	23.0
Z_4	63.2	23.2	51.2	20.9	54.6	27.1
Z_5	66.7	19.6	62.1	16.0	62.0	21.0
Z_6	69.8	26.3	60.4	14.9	62.0	25.4
Cals	13.1	3.4	11.2	2.5	11.7	3.7
En	41.0	13.7	42.0	8.9	39.6	10.9

Table 7.62

The significance of the difference in performance between FI/FD students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
FI&FD	S*	Ns	Ns	Ns	Ns	Ns

⋆ significant at 0.05 level

Moreover, based upon Table 7.63 convergent thinkers tended to have higher performance than divergent ones in all categories except for (En), whereas the two groups had the same result in (En). Therefore, the second part of hypothesis (1) was supported, but the third part rejected by this sample. However, the mean difference between both groups of learning styles is not significant (Table 7.64), but the differences in mean scores are all in the same direction.

Table 7.63

Mean and SD on different groups of Sample M in $(Z_i)_{i=1}^{6}$

Groups	Con (N=30)		All-R (N=13)		Div (N=34)	
	Mean	SD	Mean	SD	Mean	SD
Z_1	25.2	15.4	20.5	13.2	27.6	23.8
Z_4	60.3	22.9	56.5	27.8	54.6	23.8
Z_5	63.3	21.1	67.3	22.1	62.7	17.7
Z_6	67.8	21.9	68.9	24.6	60.6	26.2
Cals	12.5	3.3	11.2	4.0	11.7	3.5
En	40.1	9.4	41.7	9.1	40.3	14.3

Table 7.64

The significance of the difference in performance between Con/Div students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
Con&Div	Ns	Ns	Ns	Ns	Ns	Ns

7.6.17 Testing Hypothesis (2)

The mean scores and standard deviations of (FI+Con) and (FD+Con) thinking styles are shown in Table 7.65. According to these results, (FI+Con) students tend to have higher performance than (FD+Con) students in all the categories in learning calculus, however (FD+Con) learners have a better results in (En). Therefore, this may demonstrate the superiority of the (FI+Con) learning style compared to the (FD+Con) one, as it is predicted by this hypothesis, in the calculus course, despite the higher results of (FD+Con) students in high school mathematics. This finding, once again, confirms the results of Sabzevar samples. Nonetheless, the sample size was small and no difference was found to be significant between the groups in any category (Table 7.66).

Table 7.65

Mean and SD in calculus categories

Groups	FI+Con (N=8)		FD+Con (N=17)	
	Mean	SD	Mean	SD
Z_1	18.3	7.5	29.1	17.7
Z_4	67.8	22.1	55.9	22.8
Z_5	65.0	22.3	63.8	21.2
Z_6	75.6	22.0	64.7	23.1
Cals	13.7	3.4	12.0	3.4
En	35.9	10.5	41.1	7.5

Table 7.66

The significance of the difference in performance between (FI+Con) and (FD+Con) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FI+Con)&(FD+Con)	Ns	Ns	Ns	Ns	Ns	Ns

7.6.18 Testing Hypothesis (3)

In this sample, (FI+Div) learners exhibited a higher performance than (FD+Div) in all the categories. Table 7.67 shows their mean scores and standard deviations which indicate this predominance. This finding supports what was predicted by the hypothesis (3), however the difference between the means in the groups is not significant, but they are in the same direction (Table 7.68). It seems from this result that being field-independent in cognitive style is more helpful than being a divergent thinker in high school mathematics as a whole and in the calculus area.

Table 7.67

Mean and SD in calculus categories

Groups	FI+Div (N=15)		FD+Div (N=13)	
	Mean	SD	Mean	SD
Z_1	20.3	13.5	35.5	31.4
Z_4	62.0	23.3	50.9	23.8
Z_5	66.7	18.1	57.7	18.3
Z_6	64.3	29.7	59.2	24.7
Cals	12.8	3.3	11.1	3.5
En	44.0	15.2	35.3	13.8

Table 7.68

The significance of the difference in performance between (FI+Div) and (FD+Div) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FI+Div)&(FD+Div)	Ns	Ns	Ns	Ns	Ns	Ns

7.6.19 Testing Hypothesis (4)

From the students' mean scores in Table 7.69, it emerged that (FI+Con) learning styles performed better than the (FI+Div) learning styles in (Z_1, Z_4, Z_6,Cals), while (FI+Div) students achieved more than (FI+Con) ones in (Z_5,En). These results support the first part of hypothesis (4) and the second part of it except for the category (Z_4). However, no difference was found to be significant between the groups of cognitive style in any category (Table 7.70).

Table 7.69

Mean and SD in calculus categories

Groups	FI+Con (N=8)		FI+Div (N=15)	
	Mean	SD	Mean	SD
Z_1	18.3	7.5	20.3	13.5
Z_4	67.8	22.1	62.0	23.3
Z_5	65.0	22.3	66.7	18.1
Z_6	75.6	22.0	64.3	29.7
Cals	13.7	3.4	12.8	3.3
En	35.9	10.5	44.0	15.2

Table 7.70

The significance of the difference in performance between (FI+Con) and (FI+Div) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FI+Con)&(FI+Div)	Ns	Ns	Ns	Ns	Ns	Ns

7.6.20 Testing Hypothesis (5)

In the final stage of this sample investigation, the mean scores and standard deviations of (FD+Con) and (FD+Div) students are shown in Table 7.71. This table indicates that (FD+Con) thinkers performed better and achieved more in all the cate6gories. Hence the first part of hypothesis (5) is rejected and the second part of it is supported by this sample. It seems that being a convergent thinker was more helpful than being divergent thinker for this sample of engineering students and the higher background of (FD+Con) in high school mathematics could be also considered as a factor of their superiority in the calculus course materials. Nonetheless, there is no significant difference between the means of both groups of students in each category (Table 7.72), but all are in the same direction.

Table 7.71

Mean and SD in calculus categories

Groups	FD+Con (N=17)		FD+Div (N=13)	
	Mean	SD	Mean	SD
Z_1	29.1	17.7	35.5	31.4
Z_4	55.9	22.8	50.9	23.8
Z_5	63.8	21.2	57.7	18.3
Z_6	64.7	23.1	59.2	24.7
Cals	12.0	3.4	11.1	3.5
En	41.1	7.5	35.3	13.8

Table 7.72

The significance of the difference in performance between (FD+Con) and (FD+Div) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FD+Con)&(FD+Div)	Ns	Ns	Ns	Ns	Ns	Ns

7.6.21 A brief Symbolic Picture of Sample M

A symbolic picture of the final students' performance with different learning styles are set out in Table 7.73. And Figures 7.9–10 display the attainments of students, according to hypotheses (1–5), in (Z_4, Z_5, Z_6). The superiority displayed is based upon their mean scores in each category. In addition, some more information can be found in this table to indicate a difference between thinking styles not described within hypotheses of the present research.

Figure 7.9

The performance of FI/FD and Con/Div students in (Z_4, Z_5, Z_6) in Sample M

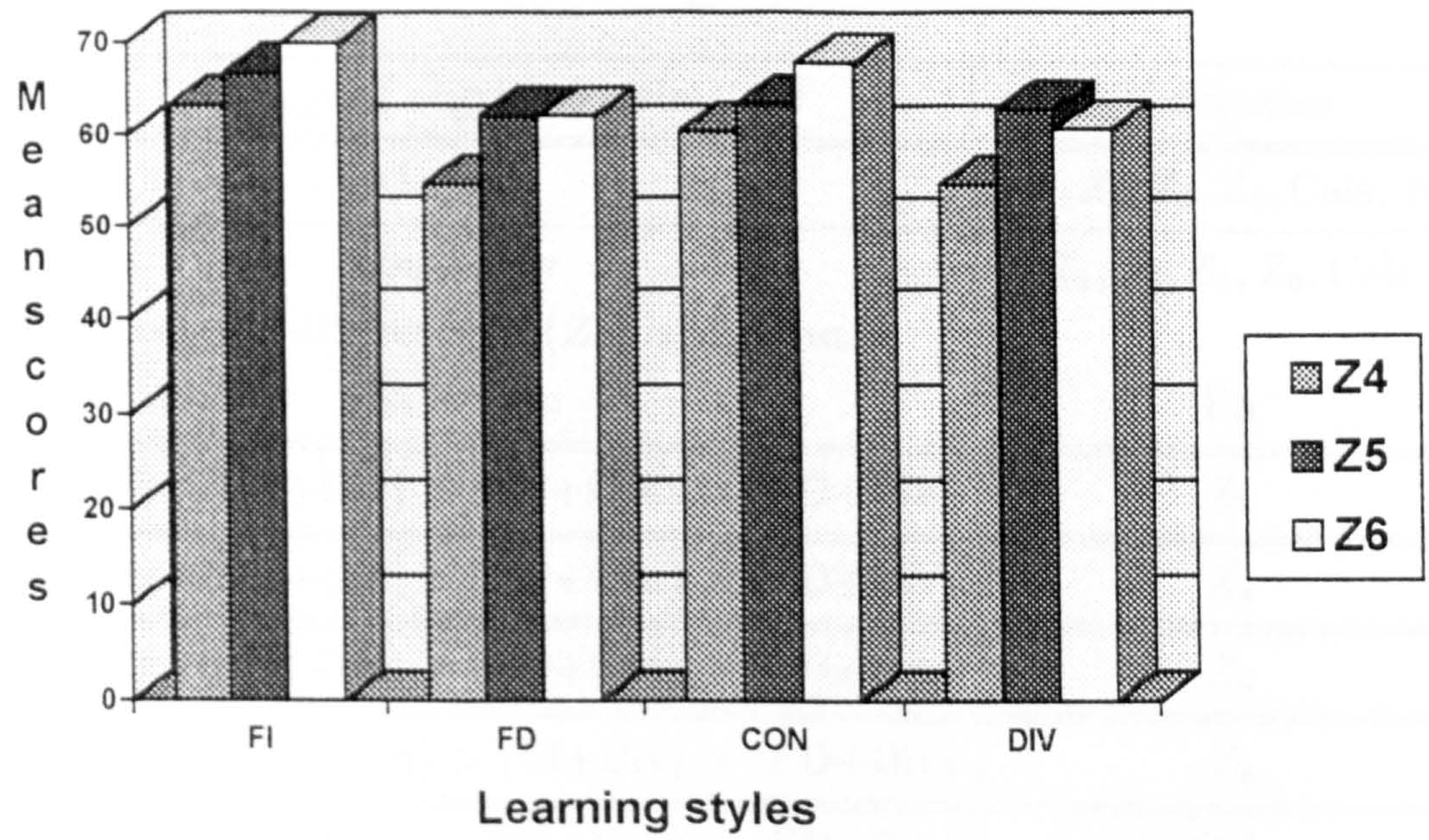


Figure 7.10

The students' performance with different learning styles in (Z_4, Z_5, Z_6) in Sample M

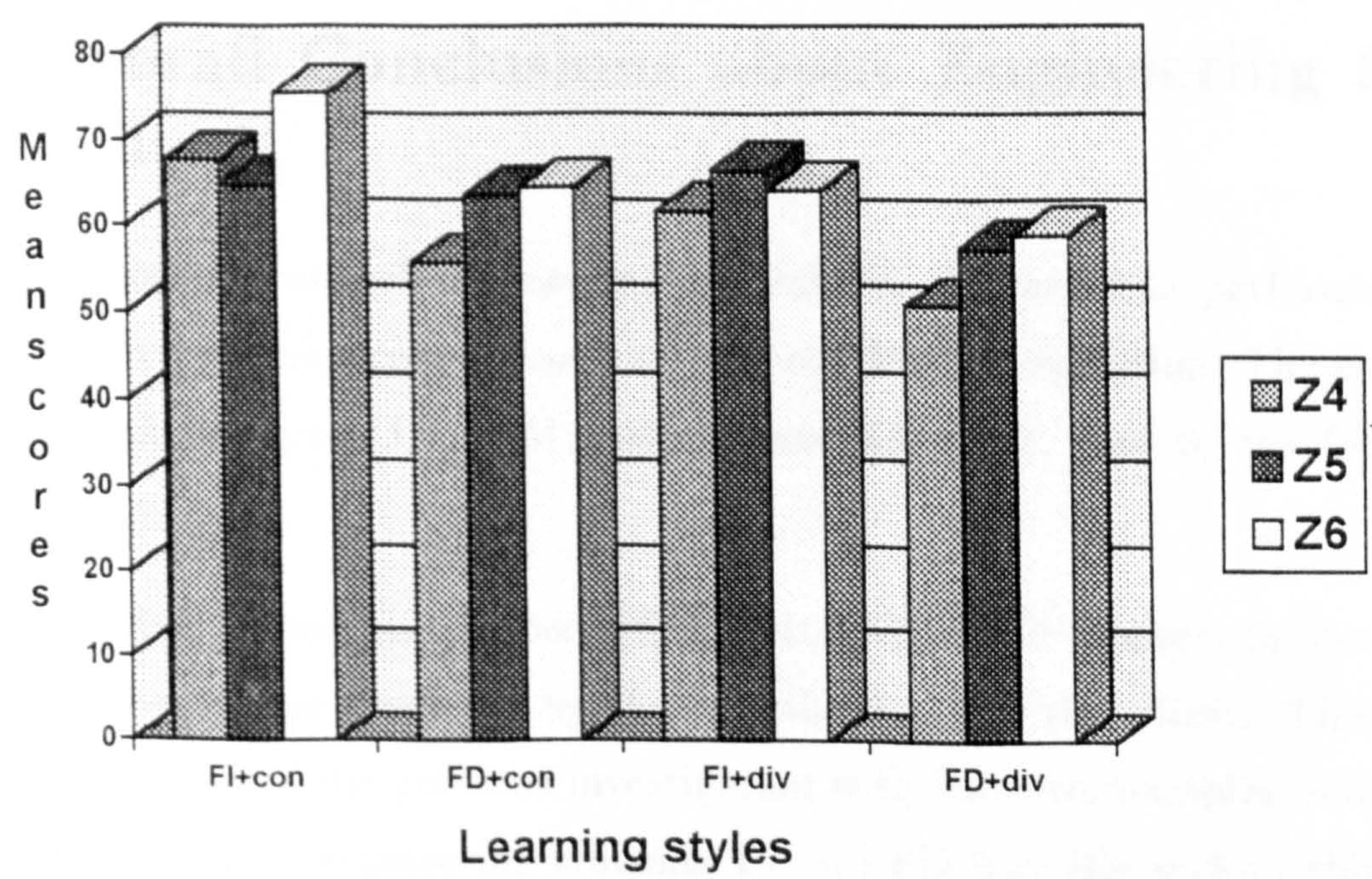


Table 7.73

The overall picture of students' superiority in $(Z_i)_{i=1}^{6}$ with different learning styles based on the mean scores in Sample M

Groups of cognitive styles	Categories
FI > FD	$Z_1{}^*$, Z_4, Z_5, Z_6, Cals, En
Con > Div However, the difference in (Z_5) is very small Div = Con	Z_1, Z_4, Z_5, Z_6, Cals En
(FI+Con) > (FI+Div) >(FD+Con) > (FD+Div)	Z_1
(FI+Con) > (FI+Div) > (FD+Con) > (FD+Div)	Z_4
(FI+Div) > (FI+Con) > (FD+Con) > (FD+Div)	Z_5
(FI+Con) > (FD+Con) > (FI+Div) > (FD+Div)	Z_6
(FI+Con) > (FI+Div) > (FD+Con) > (FD+Div)	Cals
(FI+Div) > (FD+Con) > (FI+Con) > (FD+Div)	En

⋆ significant at 0.05 level

7.7 Overall Conclusions about Engineering Students

It is worth noting how well engineering students in this research performed and achieved in their calculus course based on only one final examination. The previous results of both Samples (J and M), which were discussed, lead to the following conclusions:

1. FI thinkers in one sample performed better than FD thinkers in the calculus course as was predicted by the hypothesis (1) of this study. This result was the same as the previous investigation with Sabzevar samples, but in the other sample of engineering students FI and FD learning styles exhibited a behaviour different from what was found before in this research. In this sample, FD students just had a better performance than FI ones in the calculus

course. There are some points here, to be noted:

• This finding is mainly based on "one final calculus examination". Therefore, it seems to the researcher that the students' performance in only one mathematics examination could not be a confident predictor of their capacity to cope with the core materials and, hence some important factors of students' performance may have been lost.

• However, the researcher asked the relevant lecturers to describe the capacity of some FI students who had unsatisfactory results in the calculus examination. They believed that many of them were serious and talented students in class activities, but most likely the students' effort and perseverance during the course, family and economic problems, social and cultural situations, health problems during the examination time and etc. could be considered more likely as important factors having effects on students' results, despite having favourable cognitive style for a domain such as learning mathematics.

• The third point is that FD students had nearly the same results as FI students in (En). Hence, both groups of cognitive styles may have had the same background in high school mathematics which affected their calculus performance.

2. Convergent learners have shown superiority over divergent learners in all categories except for (Z_5), in which both groups of learning styles have nearly the same result. This may support what was predicted by the second part of hypothesis (1) and reject the third part of it The findings, once again, confirm the previous result of the present research that Con>Div in (Z_6) and Div>Con in (Z_5).

3. (FI+Con) learners tend to show higher performance than (FD+Con) colleagues in the categories (Z_1,Z_4,Z_5) which, to some extent, supports hypothesis (2) and, once again, confirms what was found in the Sabzevar samples. But, (FD+Con) students had higher results than (FI+Con) students in (En) which rejected hypothesis (2).

In one sample of engineering students (FI+Con) thinkers achieved more than (FD+Con) thinkers in (Z_6,Cals), while the opposite result emerged in another sample. It could be noted that nearly the same background of high school mathematics as (FI+Con) students, helped (FD+Con) ones to this superiority.

4. (FI+Div) students have shown better results than (FD+Div) in (Z_1,Z_5) and (Cals,En), which emphasize the previous findings of this study. Moreover, the same performance in (Z_4) has been exhibited by engineering students in this research.

5. (FI+Con) learning styles have done better than (FI+Div) ones in (Z_1, Z_4,Z_6) and (Cals), while (FI+Div) show higher performance in (Z_5). In addition, in one sample (FI+Div) students were better than (FI+Con) in (En). These results confirm the previous findings of this study.

6. (FD+Con) learners are better than (FD+Div) learners in all the categories except for (Z_5), while (FD+Div) students have shown higher performance than their colleagues (FD+Con) in (Z_5). These findings support what is predicted by the second part of hypothesis (4) and, to some extent support the first part of it. However, the results which emerged from the engineering samples may reject the previous findings based on the Sabzevar samples, except for the superiority of (FD+Div) students to (FD+Con) ones in (Z_5).

7.7.1 A brief Symbolic Picture of the Engineering samples

A symbolic picture of the final performance of engineering students with different learning styles is set out in Table 7.74. The superiority is displayed based upon their mean scores in each category.

Table 7.74

The overall picture of Engineering students' superiority in $(Z_i)_{i=1}^{6}$ with different learning styles based on the mean scores

Groups of cognitive styles	Categories
FI > FD (in one sample) FD > FI (in the other sample) However the difference is small	Z_1, Z_4, Z_5, Z_6, Cals,En Z_4, Z_6,Cals
Con > Div However, the difference in the means in (Z_5) was very small	Z_1, Z_4, Z_5, Z_6, Cals,En
(FI+Con) > (FD+Con) (FI+Con) > (FD+Con) (Supported by one sample and rejected by another) (FD+Con) > (FI+Con)	Z_1, Z_4, Z_5 Z_6 En
(FI+Div) > (FD+Div) (FI+Div) > (FD+Div) (Supported by one sample rejected by another)	Z_1, Z_5, En Z_4, Z_6,Cals
(FI+Con) > (FI+Div) (FI+Div) > (FI+Con) (FI+Div) > (FI+Con) (Supported by one sample and rejected by another)	Z_1, Z_4, Z_6 Z_5 En
(FD+Con) > (FD+Div) (FD+Div) > (FD+Con)	Z_1, Z_4, Z_6,Cals,En Z_5

7.7.2 (c) The Sample N

The third sample from Mashhad University and the last sample of Iranian students comprised 54 mathematics students in Calculus 1 in the second term of session 94/95. Students' results were investigated from only one final calculus examination and the researcher had no opportunity to explain his research calculus categories to the lecturer of this class. Moreover, the question tasks were unbalanced such that they were mostly categorised in (Z_4) and the appearance of (Z_5, Z_6) were slight, in fact, no question of the category (Z_5), pictorial thinking and etc., was included in the examination except for one question on curve sketching.

7.7.3 Testing Hypothesis (1)

The results that emerged as the Pearson's correlation between each group of learning styles and calculus categories are set out in Table 7.75.

Table 7.75

The Pearson's Correlation between each cognitive style and $(Z_i)_{i=1}^{6}$

P-C	FI/FD	Con/Div	Z_1	Z_4	Z_5	Z_6	Cals	En
FI/FD	1	0.23*	0.08	−0.20	0.12	−0.25*	−0.18	0.12
Con/Div	0.23*	1	−0.10	0.17	0.02	−0.10	0.04	0.0
Z_1	0.08	−0.10	1	−0.26*	−0.07	−0.20	−0.27*	−0.11
Z_4	−0.20	0.17	−0.26*	1	−0.05	0.38⋆	0.80•	−0.10
Z_5	0.12	0.02	−0.07	−0.05	1	0.16	0.43⋆	0.26*
Z_6	−0.26*	−0.10	−0.20	0.38⋆	0.16	1	0.68•	0.14
Cals	−0.18	0.04	−0.27*	0.80•	0.43⋆	0.68•	1	0.02
En	0.12	0.0	−0.11	−0.10	0.26*	0.14	0.02	1

Significant at:
• 0.001 level; ⋆ 0.01 level;
* 0.1 level

On the other hand, the mean scores and standard deviations of students with different learning styles are shown in Table 7.76. According to this table, the mean scores of FI students tend to be better than FD ones in (Z_5,En), while FD learning styles are better than FI ones in (Z_1, Z_4, Z_6,Cals). This means that the first part of hypothesis (1) was partially supported. Nonetheless, the difference between mean scores in both groups is not significant (Table 7.77).

Table 7.76

Mean and SD on different groups of Sample N in (Z_1, Z_4, Z_5, Z_6, Cals, En)

Groups	FI (N=20)		FInt (N=12)		FD ($N = 22$)	
	Mean	SD	Mean	SD	Mean	SD
Z_1	43.5	20.3	44.7	18.4	40.6	16.5
Z_4	59.7	25.1	71.5	15.4	68.4	18.9
Z_5	67.4	29.7	72.2	21.7	60.2	25.3
Z_6	55.4	35.4	67.3	26.5	67.0	30.7
Cals	12.1	4.7	14.1	3.1	13.3	3.1
En	46.6	10.1	37.4	7.5	42.6	5.2

Table 7.77

The significance of the difference in performance between FI/FD students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
FI&FD	Ns	Ns	Ns	Ns	Ns	Ns

The next stage is to study the relationship between the convergent/divergent learning style of students and their attainment in calculus. As a response to it, the researcher set out the students' mean scores and standard deviations in the Con/Div tests versus their scores in all the calculus categories in Table 7.78. According to this table convergers have shown better results than divergers in (Z_5, Z_6,Cals,En), while divergers are better than convergers in (Z_4) and the same in (Z_1). However, no difference was found to be significant in any the categories (Table 7.79). These findings, to some extent, supported the second and third parts of hypothesis (2) of the present study and previous results. In addition, convergent thinkers tend to be better in calculus activities as a whole.

Table 7.78

Mean and SD on different groups of Sample N in (Z_1, Z_4, Z_5, Z_6, Cals,En)

Groups	Con ($N = 23$)		All-R ($N = 13$)		Div ($N = 18$)	
	Mean	SD	Mean	SD	Mean	SD
Z_1	44.4	15.7	36.9	21.4	44.4	20.3
Z_4	62.7	24.1	69.4	16.1	67.8	25.1
Z_5	66.2	26.3	68.4	19.8	61.7	29.7
Z_6	66.1	32.6	65.0	29.2	57.0	31.9
Cals	13.1	3.9	13.4	3.1	12.7	5.2
En	42.9	9.7	45.0	6.2	40.8	8.2

Table 7.79

The significance of the difference in performance between Con/Div students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
Con&Div	Ns	Ns	Ns	Ns	Ns	Ns

7.7.4 Testing Hypothesis (2)

Table 7.80 demonstrates that the performance of (FD+Con) students is better than their (FI+Con) fellow-students in (Z_1,Z_4,Z_6,Cals), while the (FI+Con) learners have higher results than (FD+Con) in (Z_5,En). This means that hypothesis (2) is, to some extent, supported by the Sample N. Nonetheless, there is no significant difference between the mean scores of both groups in all the categories except for (Z_5). The sample size of the two groups (FI+Con) and (FD+Con) is very small (Table 7.81).

Table 7.80

Mean and SD in calculus categories

Groups	FI+Con ($N = 7$)		FD+Con ($N = 10$)	
	Mean	SD	Mean	SD
Z_1	50.9	19.8	44.6	9.5
Z_4	50.7	28.9	66.8	22.1
Z_5	83.0	20.8	50.0	25.7
Z_6	60.0	34.3	71.0	32.8
Cals	12.1	4.8	13.3	3.7
En	47.9	13.7	41.5	5.4

Table 7.81

The significance of the difference in performance between (FI+Con) and (FD+Con) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FI+Con)&(FD+Con)	Ns	Ns	S*	Ns	Ns	Ns

⋆ significant at 0.05 level

7.7.5 Testing Hypothesis (3)

The mean scores and standard deviations of (FI+Div) and (FD+Div) students in $(Z_6)_{i=1}^{6}$ are set out in Table 7.82. According to their mean scores, except for (En), (FD+Div) learning styles performed better than (FI+Div) in all the domains of calculus activity. However, the difference between the means of the two groups in each category is not significant (Table 7.83). This result does not support hypothesis (2) except for (En) and may indicate that being divergent in thinking style is more helpful than being FI in this sample, in addition the small sample size could be another problem at this point.

Table 7.82

Mean and SD in calculus categories

Groups	FI+Div ($N = 10$)		FD+Div ($N = 5$)	
	Mean	SD	Mean	SD
Z_1	40.8	19.2	38.5	11.3
Z_4	62.7	20.7	63.8	17.2
Z_5	50.8	32.9	63.7	24.8
Z_6	47.2	37.9	70.0	23.2
Cals	11.2	4.9	12.7	1.6
En	43.9	5.5	42.2	3.3

Table 7.83

The significance of the difference in performance between (FI+Div) and (FD+Div) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FI+Div)&(FD+Div)	Ns	Ns	Ns	Ns	Ns	Ns

7.7.6 Testing Hypothesis (4)

According to Table 7.84, (FI+Con) students performed better than (FI+Div) in (Z_5, Z_6,Cals,En), while (FI+Div) thinkers achieved more than (FI+Con) thinkers in (Z_1, Z_4). Therefore, hypothesis (4) is partially supported, however no difference was found to be significant between the groups of thinkers except for (Z_5) in Table 7.85. The higher performance of (FI+Con) students than (FI+Div) ones in (Z_6), once again, was confirmed by this sample. Moreover, the results of this sample of mathematics students indicate that (FI+Con) thinkers have better grounding in high school mathematics and tend to exhibit higher achievement than (FI+Div) in the calculus course except for (Z_4) tasks.

Table 7.84

Mean and SD in calculus categories

Groups	FI+Con ($N = 7$)		FI+Div ($N = 10$)	
	Mean	SD	Mean	SD
Z_1	50.9	19.8	40.8	19.2
Z_4	50.7	28.9	62.7	20.7
Z_5	83.0	20.8	50.8	32.9
Z_6	60.0	34.3	47.2	37.9
Cals	12.1	4.8	11.2	4.9
En	44.3	13.7	43.9	5.5

Table 7.85

The significance of the difference in performance between (FI+Con) and (FI+Div) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FI+Con)&(FI+Div)	Ns	Ns	S*	Ns	Ns	Ns

⋆ significant at 0.05 level

7.7.7 Testing Hypothesis (5)

The students' mean scores and standard deviations in both groups of cognitive styles are shown in Table 7.86. Based on these results, (FD+Con) learners have higher mean scores than (FD+Div) ones in (Z_4, Z_6, Cals), but the (FD+Div) students are better than their (FD+Con) colleagues in (Z_1, Z_5,En). The findings support, to some extent, what was predicted by hypothesis (5), however there is no significant difference between the two groups in any category (Table 7.87).

Table 7.86

Mean and SD in calculus categories

Groups	FD+Con ($N = 10$)		FD+Div ($N = 5$)	
	Mean	SD	Mean	SD
Z_1	44.6	9.5	38.5	11.3
Z_4	66.8	22.1	63.8	17.2
Z_5	50.0	25.7	63.7	24.8
Z_6	71.0	32.8	70.0	23.2
Cals	13.3	3.7	12.7	1.6
En	41.5	5.4	42.2	3.3

Table 7.87

The significance of the difference in performance between (FD+Con) and (FD+Div) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals	En
(FD+Con)&(FD+Div)	Ns	Ns	Ns	Ns	Ns	Ns

These findings support, to some extent, what was predicted by hypothesis (5) and the previous results that (FD+Div) thinking styles have predominantly achieved better than (FD+Con) in mathematical translation and pictorial calculus tasks (Z_5). While, (FD+Con) produced higher results than (FD+Div) in multi-skilled problems (Z_6) in this study. In addition, the overall symbolic picture of the above results and more information about significant relationships between learning styles are set out in Table 7.88.

7.7.8 A brief Symbolic Picture of Sample N

A symbolic picture of the final students' performance with different learning styles is set out in Table 7.88 and its pictorial forms, based on hypotheses (1–5), in the categories (Z_4, Z_5, Z_6) are exhibited by Figures 7.11–12. The superiority displayed is based upon their mean scores in each category. Moreover, some more information can be found in this table to indicate significant difference between learning styles not described within hypotheses of the present research. For instance, there is a significant differences between FI and FInt in (En), and between (FI+Con) learning styles and (FD+Div) ones in (Z_1).

Table 7.88

The overall symbolic pictures of students' superiority in $(Z_i)_{i=1}^{6}$ with different learning styles based on the mean scores in Sample N

Groups of cognitive styles	Categories
FI > FD FD > FI FI > FInt	Z_5,En Z_1, Z_4, Z_6,Cals,En En*
Con > Div Div > Con Con = Div	Z_5, Z_6,En,Cals Z_4 Z_1
(FD+Div) > (FI+Div) > (FD+Con) > (FI+Con)	Z_1
(FD+Con) > (FD+Div) > (FI+Div) > (FI+Con)	Z_4
(FI+Con) > (FD+Div) > (FI+Div) > (FD+Con) Here the significant difference is: **(FI+Con) > (FD+Con), (FI+Con) > (FI+Div)**	Z_5 Z_5*
(FD+Con) > (FD+Div) > (FI+Con) > (FI+Div)	Z_6
(FD+Con) > (FD+Div) > (FI+Con) > (FI+Div)	Cals
(FI+Div) > (FI+Con) > (FD+Div) = (FD+Con)	En

⋆ significant at 0.05 level

Figure 7.11

The performance of FI/FD and Con/Div students in (Z_4, Z_5, Z_6) in Sample N

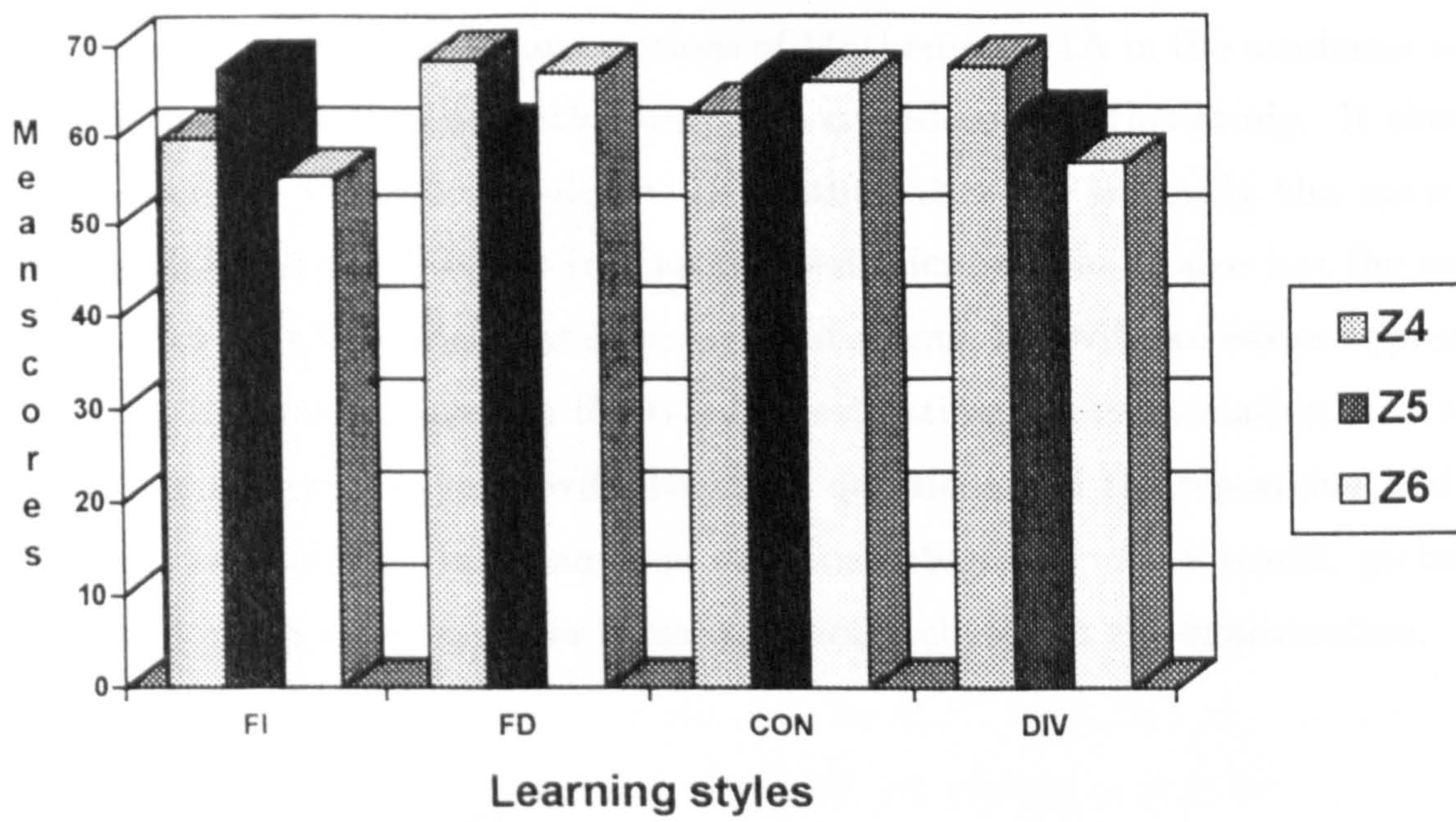
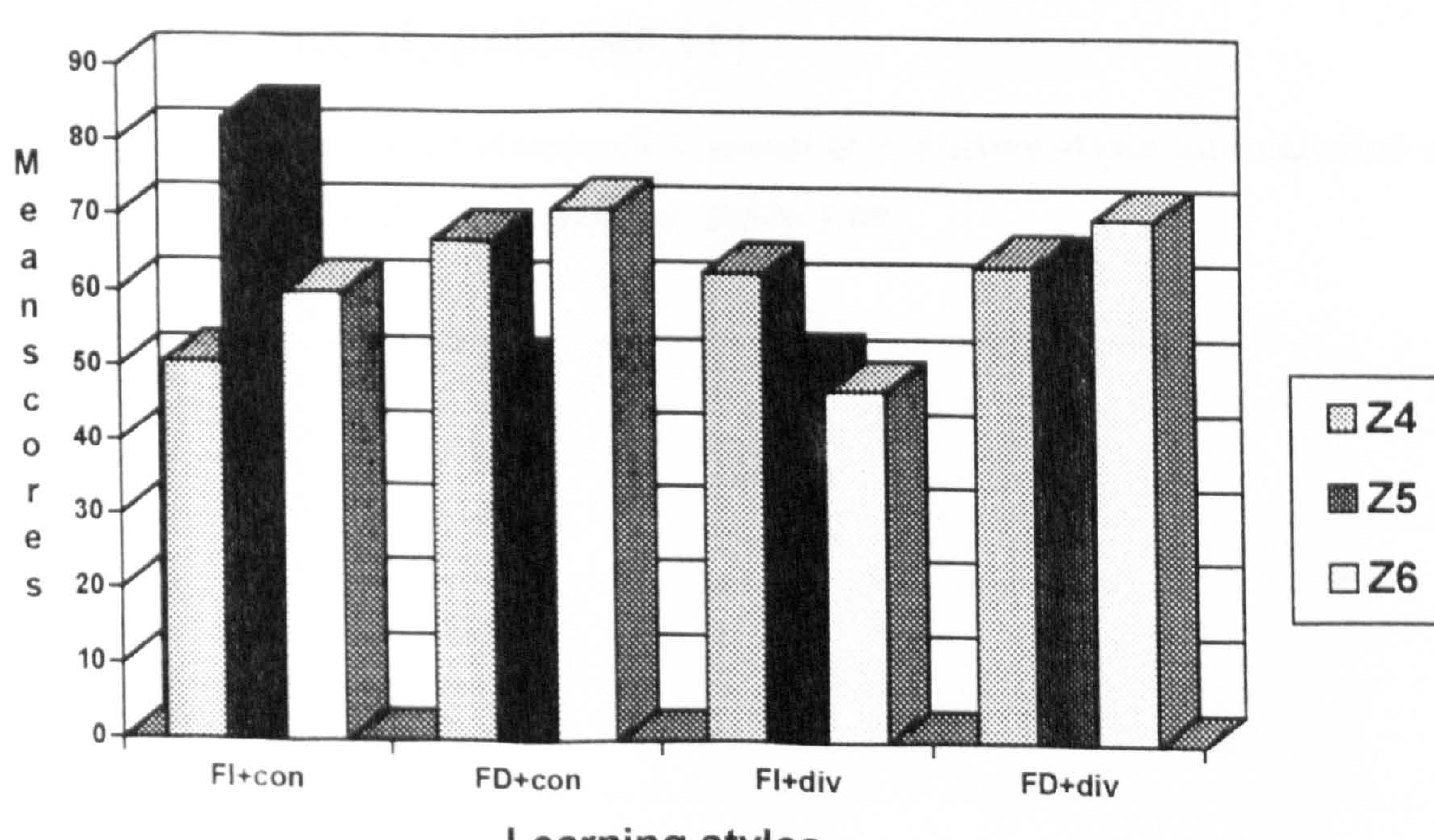


Figure 7.12

The students' performance with different learning styles in (Z_4, Z_5, Z_6) in Sample N



7.7.9 The Third Investigation

The third investigation of this research was carried out upon the students at Glasgow University in Scotland. At the time of this study, all were currently enrolled in the first year of higher education. The samples on which this study was based comprised 55 mathematics students in various sections of Mathematics 1A in the academic year (94/95), and their final examination only was considered for this study. It should be noted that the syllabus of calculus in Mathematics 1A is nearly the same as Calculus 1 and Calculus 2 in the Iranian mathematics branch. It also has the same calculus topics as in Calculus 1 for engineering students, but with an easier approach compared to the Iranian ones. In the third investigation, the examination structure was the same as previous years with standard questions and the researcher had no control over examination tasks and his calculus categories. As a result, pictorial thinking and graph interpretation tasks were not included in the examination.

7.7.10 (g) Sample G

This sample was selected from different calculus sections of Mathematics 1A with a population of 55 mathematics students in the academic year (94/95). The results of students' performance emerge from one final examination (Appendix H).

7.7.11 Testing Hypothesis (1)

The Pearson's correlation between each group of cognitive styles and calculus categories (Z_1, Z_4, Z_5, Z_6,Cals) are shown in Table 7.89.

Table 7.89

The Pearson's Correlation between each cognitive style and $(Z_i)_{i=1}^{6}$

P-C	FI/FD	Con/Div	Z_1	Z_4	Z_5	Z_6	Cals
FI/FD	1	0.24*	−0.10	0.15	−0.10	−0.10	0.02
Con/Div	0.24*	1	−0.10	0.25*	0.42•	0.20	0.39*
Z_1	−0.10	−0.10	1	0.01	−0.04	0.01	0.05
Z_4	0.15	0.25*	0.10	1	0.39*	0.41•	0.76•
Z_5	−0.10	0.42•	−0.04	0.39•	1	0.32*	0.72•
Z_6	−0.10	0.20	0.01	0.41•	0.32*	1	0.75•
Cals	0.02	0.39*	0.05	0.76•	0.72•	0.75•	1

Significant at:

• 0.001 level; 0.01 level;

∗ 0.1 level

On the other hand, the mean scores and standard deviation of students with different learning styles are shown in Table 7.90. According to this table, the mean scores of FI students tend to be better than FD ones in (Z_4,Cals), while FD learning styles have a higher performance than FI ones in (Z_5). In addition, both groups of learning styles achieved and performed the same in (Z_1, Z_6). This means that the first part of hypothesis (1) was partially supported and indicated that FI students tend to achieve better results than FD ones in learning calculus overall. Once again, the results confirmed the previous findings of this research. Nonetheless, the difference between means in the two groups is not significant (Table 7.91).

Table 7.90

Mean and SD on different groups of Sample G in (Z_1, Z_4, Z_5, Z_6,Cals)

Groups	FI ($N=25$)		FInt ($N=6$)		FD ($N=24$)	
	Mean	SD	Mean	SD	Mean	SD
Z_1	36.4	20.4	24.1	13.6	36.3	23.1
Z_4	55.8	18.3	52.6	16.2	48.3	16.3
Z_5	35.6	24.4	34.5	29.6	38.6	26.4
Z_6	51.1	12.9	46.9	11.9	51.1	16.3
Cals	49.7	12.2	45.3	16.3	47.4	13.5

Table 7.91

The significance of the difference in performance between FI/FD students

Groups	Z_1	Z_4	Z_5	Z_6	Cals
FI&FD	Ns	Ns	Ns	Ns	Ns

Moreover, the mean scores of students with different learning styles in Table 7.92 indicate that divergent learners have superiority over convergent learners in all the categories of calculus learning in this study. The results which have emerged from this sample reject the second part of hypothesis (1) and support the third part and the difference in means between the two groups of thinking styles is significant in (Z_5,Cals) and nonsignificant in the other categories (Table 7.93).

The above results confirm the previous finding of all the Iranian samples of the present study that Div>Con in (Z_5) and conflict with the predominant result that Con>Div in (Z_6).

Table 7.92

Mean and SD on different groups of Sample G in (Z_1, Z_4, Z_5, Z_6,Cals)

Groups	Con ($N = 20$)		All-R ($N = 12$)		Div ($N = 23$)	
	Mean	SD	Mean	SD	Mean	SD
Z_1	39.0	26.9	30.4	16.9	33.9	16.9
Z_4	49.8	18.7	48.7	16.3	56.0	16.8
Z_5	26.5	23.4	35.3	27.8	46.6	23.3
Z_6	50.0	16.9	47.7	12.2	52.8	12.9
Cals	44.4	12.9	45.9	12.3	52.7	13.0

Table 7.93

The significance of the difference in performance between Con/Div students

Groups	Z_1	Z_4	Z_5	Z_6	Cals
Con&Div	Ns	Ns	S*	Ns	S*

⋆ significant at 0.05 level

7.7.12 Testing Hypothesis (2)

The mean scores in various aspects of calculus related to (FI+Con) and (FD+Con) learning styles are exhibited in Table 7.94 and the significance/nonsignificance of the linkages between the two styles and each calculus category is also shown in Table 7.95. It is evident from Table 7.94 that (FD+Con) students performed better than (FI+Con) in all categories except for (Z_1, Z_4). It was shown, in fact, being convergent in learning style could be more beneficial that being FI or FD. However, this does not support hypothesis (2) and most of the previous samples. In addition, there is no significant difference in means between the two groups of learning styles.

Table 7.94

Mean and SD in calculus categories

Groups	FI+Con ($N = 8$)		FD+Con ($N = 10$)	
	Mean	SD	Mean	SD
Z_1	41.2	17.8	42.2	35.4
Z_4	49.2	21.9	48.7	16.7
Z_5	17.4	14.3	36.7	26.1
Z_6	46.9	16.6	54.2	17.7
Cals	41.4	9.4	48.2	13.8

Table 7.95

The significance of the difference in performance between (FI+Con) and (FD+Con) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals
(FI+Con)&(FD+Con)	Ns	Ns	Ns	Ns	Ns

7.7.13 Testing Hypothesis (3)

The students' mean scores, as shown in Table 7.96, demonstrate that (FI+Div) thinking styles achieved better than (FD+Div) in all the calculus categories except for (Z_1). These results support what is predicted by hypothesis (3). However, the differences in the means are not significant in any category (Table 7.97). This finding is, once again, in the same direction as the previous result that, in many samples of this research, (FI+Div) > (FD+Div) in (Z_5).

Table 7.96

Mean and SD in calculus categories

Groups	FI+Div ($N = 13$)		FD+Div ($N = 8$)	
	Mean	SD	Mean	SD
Z_1	36.8	20.9	29.7	7.6
Z_4	59.7	17.1	47.3	13.4
Z_5	46.7	23.9	40.0	19.9
Z_6	55.6	9.8	46.4	14.6
Cals	55.5	11.7	44.7	11.9

Table 7.97

The significance of the difference in performance between (FI+Div) and (FD+Div) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals
(FI+Div)&(FD+Div)	Ns	Ns	Ns	Ns	Ns

7.7.14 Testing Hypothesis (4)

The means and standard deviations of students' performance with different cognitive styles are set out in Table 7.98. These results indicate that (FI+Div) learning styles have done better in all the calculus domains and the difference in mean scores between the two groups of thinking styles is significant in (Z_5,Cals), as shown in Table 7.99. This finding supports the second part of hypothesis (4) and rejects the first part of it, and also confirms the former result that (FI+Div) learners tend to perform better than (FI+Con) in (Z_5).

Table 7.98

Mean and SD in calculus categories

Groups	FI+Con ($N = 8$)		FI+Div ($N = 13$)	
	Mean	SD	Mean	SD
Z_1	41.2	17.8	36.8	20.9
Z_4	49.2	21.9	59.7	17.1
Z_5	17.4	14.3	46.7	23.9
Z_6	46.9	16.6	55.6	9.8
Cals	41.4	9.4	55.5	11.7

Table 7.99

The significance of the difference in performance between (FI+Con) and (FI+Div) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals
(FI+Con)&(FI+Div)	Ns	Ns	S*	Ns	S*

⋆ significant at 0.05 level

7.7.15 Testing Hypothesis (5)

In the final stage of testing the hypotheses, students' mean scores with different cognitive styles (FD+Con) and (FD+Div) are exhibited in Table 7.100. Based on these results, (FD+Con) students obtained higher means compared to their colleagues in (Z_4, Z_6,Cals), while (FD+Div) learners were better in (Z_1, Z_5). This finding, to some extent, supports hypothesis (5), however there is no significant difference in students' mean scores between the two groups of learning styles (Table 7.101). The results of this section, once again, support the previous findings of the present study that predominantly (FD+Con) tend to achieve a higher performance in (Z_6), and vice versa in (Z_5).

Table 7.100

Mean and SD in calculus categories

Groups	FD+Con ($N = 10$)		FD+Div ($N = 8$)	
	Mean	SD	Mean	SD
Z_1	42.2	35.4	29.7	7.6
Z_4	48.7	16.7	47.3	13.4
Z_5	36.7	26.1	40.0	19.9
Z_6	54.2	17.7	46.4	14.6
Cals	48.2	13.8	44.7	11.9

Table 7.101

The significance of the difference in performance between (FD+Con) and (FD+Div) students

Groups	Z_1	Z_4	Z_5	Z_6	Cals
(FD+Con)&(FD+Div)	Ns	Ns	Ns	Ns	Ns

7.7.16 A brief Symbolic Picture of Sample G

A symbolic picture of the final students' performance with different thinking styles is set out in Table 7.102. Moreover, Figures 7.13-14 display their achievements, based on hypotheses (1-5), in the categories (Z_4, Z_5, Z_6). The superiority displayed is based upon their mean scores in each category and some more information can be found in this table differences between learning styles in this sample.

Table 7.102

The overall symbolic picture of students' superiority in (Z_1, Z_4, Z_5, Z_6Cals) with different learning styles based on the mean scores in Sample G

Groups of cognitive styles	Categories
FI > FD	Z_4,Cals
FI = FD	Z_1, Z_6
FD > FI	Z_5
Div > Con	$Z_1, Z_4, Z_5^{\star}, Z_6$, Cals*
(FD+Div) > (FI+Div) > (FI+Con) > (FD+Con)	Z_1
(FI+Div) > (FI+Con) > (FD+Con) > (FD+Div)	Z_4
(FI+Div) > (FD+Div) > (FD+Con) > (FI+Con)	Z_5
Here the significant difference is: **(FI+Div) > (FI+Con)**	$Z_5^{\star}$
(FI+Div) > (FD+Con) > (FI+Con) > (FD+Div)	Z_6
(FI+Div) > (FD+Con) > (FD+Div) > (FI+Con)	Cals
Here the significant difference is: **(FI+Div) > (FI+Con)**	Cals*

⋆ significant at 0.05 level

Figure 7.13

The attainments of FI/FD and Con/Div students in $(Z_4, Z_5, Z6)$ in Sample G

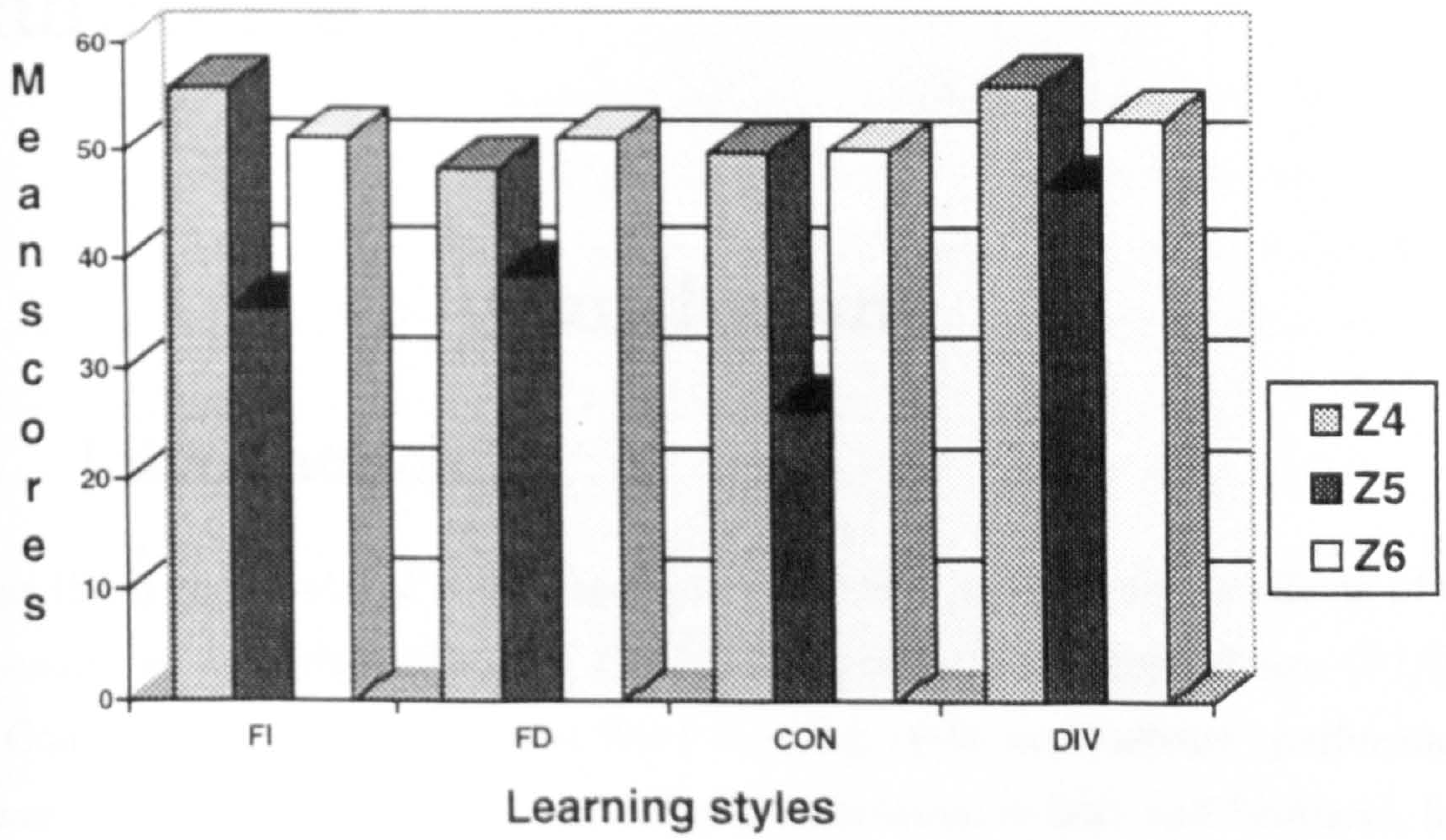
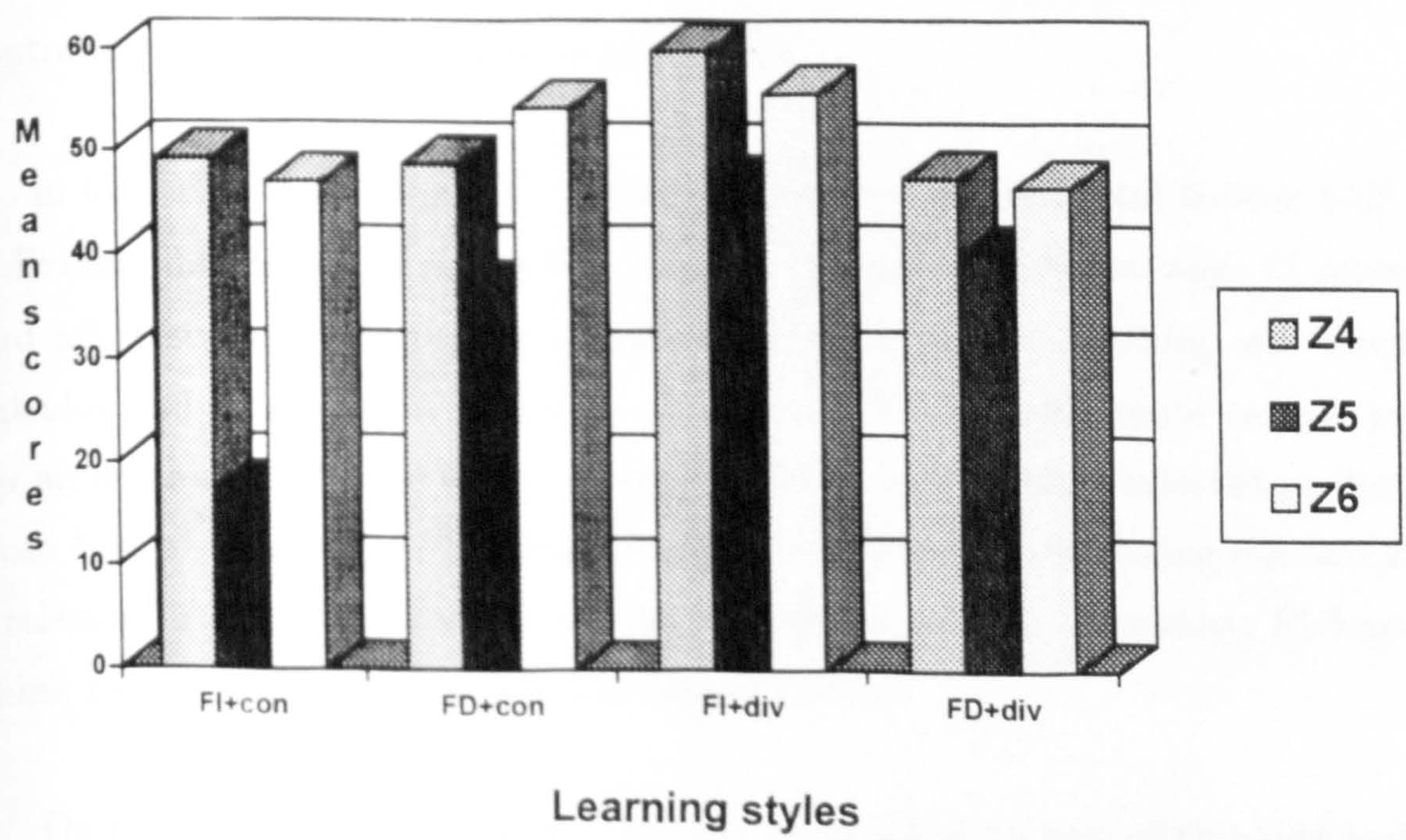


Figure 7.14

The performance of students with different learning styles in (Z_4, Z_5, Z_6) in Sample G



Chapter 8

Conclusions

8.1 Introduction

In this thesis the results of seven experimental studies concerning the effects of two dimensions of cognitive style, i.e. Field-independence/Field-dependence (FI/FD) and Convergence/Divergence (Con/Div) thinking style, on students' performance in learning calculus in the first year of higher education, in Iran and Scotland, have been reported and discussed.

In summary, students' cognitive styles are based on individual differences and would show in their performance of particular ways of thinking and goal attainments. Harmon (1984) cited Gagné (1977) that cognitive styles operate analogously to an executive program in a computer thus the reaction of an individual to a context is controlled by a sequence of routine procedures.

In fact, FI/FD and Con/Div cognitive styles are psychological factors built on individual differences in perceptual behaviours which determine ways of learning and affect students' performance and have implications for teaching and learning calculus and mathematics education as a whole. FI students would readily break up an organised field and easily separate relevant items from irrelevant or "noise" from "voice" in context. FD students may find difficulty in separating relevant from irrelevant in context and would accept the context as it is. Moreover, FI learners think analytically, while FD learners think globally.

On the other hand, convergent thinking is stated as "a way of thinking looking for one right solution or towards a uniquely determined answer. While, divergent thinking style is a way of thinking in which a number of ideas will be produced from a given set of information. In other words, divergers would find a greater variety of answers to a question compared to convergers.

In this research, insight has been gained into the effect of learning styles (FI/FD and Con/Div) on students' difficulties in learning calculus and problem solving. Therefore, after intensive investigations and face to face interviews with a lot of mathematicians who were involved in teaching calculus for undergraduates, calculus materials were classified into four main categories of (Z_4,Z_5, Z_6,Cals). In fact, (Cals) indicates the combination of (Z_4,Z_5,Z_6) which is considered as students' total performance in a calculus examination and, for the Iranian students, results of their mathematical attainment in the university entrance examination (En) is also surveyed to understand their grounding in high school mathematics at the beginning of a calculus course at tertiary level.

Multi-conceptual and procedural tasks, as described in Chapter 5, are labelled in this research as category (Z_4). They are the calculus questions in which more than one critical mathematical concept comes together to establish a more complicated combination. Category (Z_5) describes the translation process between mathematical abstraction (symbolic) and pictorial (visual) forms in calculus material including; calculus word problems, curve sketching techniques, visual thinking and curve interpretation. Moreover, multi-skilled, transferable and procedural skills tasks in calculus are classified as category (Z_6). In this category attention focused on the development of transferable skills, that is, mathematical skills which are considered in more than one context. In addition, students' performance (weakness) in the manipulation of mathematical notation and symbolic logic is called category (Z_1).

The researcher has used two psychological tests to measure the students' leaning styles. (These tests have been designed in the Centre for Science Education at Glasgow University). Also, he has designed a method for assessment of the above-mentioned categories for the purpose of the present research.

Sample populations were mainly divided into first year engineering students, mathematics and physics students and students intending (or considering) single or combined honours in mathematics at Ferdowsi University of Mashhad, Teacher Training University of Sabzevar (Iran) and the University of Glasgow (Scotland).

Rather than merely providing a summary repeating the findings from different samples of this study, it was decided that a more effective conclusion of the present research would be to offer recommendations to people who are involved in providing calculus curricula, teaching and examining in high school and higher education and to offer some suggestion for further research.

8.2 The Findings of this Study

The aspects which have emerged in the present research are summarised in the following conclusions:

- Pearson's correlation between FI/FD scores and Con/Div scores yielded low values of (0.17–0.24) in different samples, indicating that these two dimensions of learning style were fairly independent, but FI students tended to be divergent thinkers and FD students tended to be convergent thinkers.

- This finding could indicate that students with various ways of learning and preference, who appear to have different FI/FD and Con/Div learning styles, should be considered in calculus education.

- Five hypotheses were proposed and then tested in this research related to the influence of various learning styles on the students' performance in the categories $(Z_i)_{i=1}^{6}$.

- The findings of hypothesis (1): As was expected in this study, FI students tend to perform better than FD students in calculus courses. Moreover, FI Iranian students tend to show a better grounding compared to FD colleagues in high school mathematics at the beginning of their calculus course.

• The convergent/divergent learning style had a remarkable role in the students' achievements in learning calculus and problem solving; divergent thinkers, in general, exhibited higher performance than convergent thinkers in (Z_5). While, convergers did better than divergers in (Z_6) as predicted in hypothesis (1).

• Science students (maths, etc.) who were divergent thinkers tended to show higher performance in (Z_1, Z_4,Z_5,Cals,En) when compared with convergers who were better in (Z_6) except for the Glasgow sample.

• It is safe to suggest that FI/FD and, in particular, Con/Div ways of thinking play important roles and do have an effect in calculus learning and problem solving. However, the research findings indicate that Con/Div cognitive style was more important than FI/FD in learning calculus particularly in tackling calculus tasks such as (Z_5, Z_6). In contrast with Al-Naeme (1991), that FI/FD style was more effective than Con/Div learning style in chemistry mini-projects. But chemistry mini-projects may be using other scales or looking at other skills in another context.

• The findings of hypothesis (2): In this study (FI+Con) students tended to exhibit higher results than (FD+Con) in calculus areas which confirms the research prediction.

• The findings of hypothesis (3): Based on findings of this research, students with a (FI+Div) learning style tend to show better performance than (FD+Div) in (Z_1, Z_5,En), whereas (FD+Div) students achieved more in (Z_4, Z_6,Cals). The findings indicate that being divergent in learning style could be more helpful than being FI/FD in calculus activities. Moreover, students with (FI+Div) learning styles, in particular in science (mathematics, etc.) were mainly better than all others in the mathematical entrance examination for higher education as predicted in the study.

• The findings of hypothesis (4): It was found that (FI+Con) students tend to perform better in (Z_1, Z_4,Z_6,Cals) compared to (FI+Div) ones, while (FI+Div) students tend to attain higher results than (FI+Con) in (Z_5,En) as expected in this study, except for (Z_4).

• The findings of hypothesis (5): The overall findings suggested that science students who were (FD+Div) in learning style tend to produce better performance than (FD+Con) in (Z_1, Z_4,Z_5), while (FD+Con) learning styles, in general, achieved more in (Z_6,Cals,En). As was expected in the present research, (FD+Con) > (FD+Div) in (Z_6,Cals).

8.2.1 Conclusions for Engineering Students

• The performance profile of the Iranian engineering students exhibited a somewhat different picture in some situations. For example:

• Engineering students with convergent learning style tended to show higher attainments than divergent ones in all the categories and nearly the same result in (Z_5). This could be an indication that convergent thinking is an advantageous way of learning calculus in engineering branches.

• It was found that (FD+Con) > (FD+Div) in all categories $(Z_i)_{i=1}^{6}$ which, once again, suggested that being convergent in learning style could be more beneficial than being divergent in the engineering first course calculus.

• Except for (Z_5), it was found that (FI+Con) > (FI+Div) in the other categories of this research and confirmed the important role of convergent thinking compared to divergent or field-independent styles in calculus course. The present findings, based on two engineering samples, may suggest that calculus materials and teaching approach favour the convergent rather than the divergent, (FD+Div) and (FI+Div) learners.

8.3 The Overall Symbolic Picture

It is worth having a symbolic picture of the overall findings of this study which were discussed in previous sections of this chapter. Table 8.1 shows the tendency of superiority of different learning styles in all categories of the present research as follows:

Table 8.1

The tendency of students' superiority in $(Z_i)_{i=1}^{6}$ with different learning styles, based on the mean scores

Groups of learning styles	Categories	Subject	Hypothesis
FI > FD	$(Z_i)_{i=1}^{6}$	Nearly All	One
Con > Div	Z_6	All	One
Div > Con	Z_5	All	
Div > Con	Z_1, Z_4, Cals,En	Maths	
Con > Div	Z_1, Z_4, Cals,En	Engineering	
FI+Con > FD+Con	$(Z_i)_{i=1}^{6}$	Nearly All	Two
FI+Div > FD+Div	Z_1, Z_5, En	Nearly All	Three
FD+Div > FI+Div	Z_4, Z_6, Cals	Nearly All	
FI+Con > FI+Div	Z_1, Z_4, Z_6, Cals	All	Four
FI+Div > FI+Con	Z_5, En	All•	
FD+Div > FD+Con	Z_1, Z_4, Z_5	Maths	Five
FD+Con > FD+Div	Z_6, Cals,En	Maths	
FD+Con > FD+Div	$(Z_i)_{i=1}^{6}$	Engineering	

• In the Glasgow sample, mainly (FI+Div)>(FI+Con) in all categories.
In Engineering samples, mostly (FI+Con)>(FI+Div) in all the categories except for (Z_5).

8.4 Implications for Calculus Education

A model of expected performance of students with various learning styles in all the categories $(Z_i)_{i=1}^{6}$, emerged from the present study (Table 8.1). It seems that some dimensions of cognitive style investigated in this research would enhance the achievement in learning calculus and problem solving, while others would lessen it and the situation in the science and engineering branches is not the same. The Iranian (FI+Con) thinkers would all be considered capable of doing well in a calculus course, while the Scottish (FI+Div) thinkers performed substantially better compared to the other styles.

• On the other hand, science students with (FD+Con) learning styles and engineering ones who were (FD+Div) thinkers would all be considered weakers in calculus than others in the samples.

• As was discussed, FI/FD and Con/Div dimensions of cognitive styles are psychological constructs which are built upon individual differences in specific contexts. Therefore, the findings of this research indicate that these dimensions of cognitive style do affect mathematical performance and achievement and do have implications for calculus teaching and learning and mathematical education as a whole.

• These implications seem substantial enough to suggest that calculus curricula should be planned which are adapted to the learning styles of learners. It was found in the present study that learning various aspects of calculus demands different dimensions of cognitive style on the part of learners. For instance, divergent thinkers favour pictorial thinking, curve interpretation and calculus word problems, which together are called mathematical translation i.e. (Z_5), in the research, and multi-conceptual tasks (Z_4) which demand divergent bias and then step by step converge to a deduction and solution. Whereas, convergent thinkers favour multi-skilled tasks (Z_6). FI students also tend to perform better than FD in calculus course activities. In the science samples, (FD+Con) students have shown worse results than the others, and similarly (FD+Div) students in engineering branches.

• What can we do for those people in science and engineering courses? Can we ignore these students' problems or see them as failures or should we, in the light of this research, do something about the courses to accommodate them? It is safe to suggest that we could modify the students' learning by modifying the learning demands of the calculus curricula and changing our teaching methods to help students' with different ways of thinking have meaningful understanding. Modification, at this point, does not mean changing students' cognitive styles, but trying to make a healthy balance between different approaches in teaching calculus to help to make the necessary accommodation.

• However, textbooks and our teaching methods in calculus in secondary and, in particular, in higher education favour analytical and non-pictorial ways of thinking and the balance between them is not often valued in teaching and learning calculus by a lot of calculus educators. They are unaware that conceptual learning could be easier if there was a healthy balance of analytical, visual, verbal and skills in teaching methods of calculus. This researcher found that not only calculus students were naive in visual thinking and curve interpretation, but third-year university mathematics students, in Iran who are trained to be mathematics teachers in high school, are uneasy in tackling translation of pictorial forms into formal definition (analytical) and vice versa, in calculus tasks.

• With respect to the curriculum, the results of the present study favour a parallel development of calculus that is visual in nature and as well analytical. As Dreyfus (1992) noted, "theories and analyses from cognitive science clearly show the potential for and an extremely powerful role for visual reasoning in learning many mathematical concepts and processes". Moreover, it could be possible to develop curricular topics and teaching methods to match FI/FD and Con/Div learning styles. Such changes can all be emphasized throughout calculus education such that various aspects of calculus concepts, i.e. analytical, pictorial, verbal and manipulative skills, are introduced into teaching and problem solving. This approach may encourage students with different ways of learning to grasp complicated materials.

8.5 Recommendations for Calculus Education

• The researcher would stress that the recommendations in this chapter are not criticisms of existing lecturing and examination policy. Some of the recommendations may already be in operation. However, if lecturers (teachers) wish to instruct their students so as to attain higher results in calculus activities and examinations, the recommendations should assist.

• For several reasons, including the sophistication of calculus, its key role in mathematical learning and widespread applications, students' difficulties in learning calculus and the rapid advance of the use of technology in teaching have encouraged mathematical educators to think about pedagogical difficulty and the necessity for universal reform in teaching and learning calculus. Foley and Ruch (1995) suggested that emphasis on laboratory courses, group work and projects as aspects of reform, rather than individual working.

• The researcher considers that there is sufficient evidence in the findings of the present study to recommend that mathematicians, from most parts of mathematics, in particular the calculus area, who are involved in teaching and curriculum development pay attention to students' various learning styles based on individual differences which do affect their mathematical performance and achievements. The wise lecturers (teachers) need to be aware of the students' ways of thinking that could make easier the conceptual and mathematical skills currently being taught. Harmon (1984) noted that perhaps the most important use of cognitive styles is in alerting calculus educators to the possibility that some learners may understand the mathematical concepts differently from the way the teacher (lecturer) does. In addition, he stated that teachers are more important than the subjects for students. Therefore, the lecturer (teacher) who is able to relate to individual differences in the ways of thinking may be better able to assist learners in improving their achievement in a calculus course.

• It is a reality that students come from secondary education (high school) to undergraduate calculus courses with a strong dependency on lecturers and a high degree of anxiety about their ability to cope with the course. Keith (1995) suggested that encouraging students to become independent learners, should be a main objective of calculus reform.

Students who are normally conditioned to memorize formal definitions, theorems and learn rules, as a product of their schooling, may encounter more emphasis on depth and mathematical rigour in higher education calculus. The reality that most pupils arrive at university with mis or non-understanding of such calculus material as 'limit' and 'continuity processing' and their insufficient grounding, are university problems and can not be ignored. As Johnstone (1988) suggested, "what we already know and understand controls how we interpret, process and even store new information".

• Some lecturers may consider that teaching students how to interpret question tasks in calculus is not a university-level activity, whereas this approach could reduce students' mis/non-understanding of their previous instruction in secondary education. It may be considered a deviation from teaching calculus into more legitimate use of lecture time in an already overloaded curriculum. This can damage students' meaningful understanding in a calculus course and advanced mathematical subjects such as analysis, topology and so on. Therefore, it seems reasonable to suggest a reduction of course content avoiding students' overload for the benefit of learners with insufficient working memory capacity, particularly in Iranian universities.

• Assisting students to perform well in a calculus course is the primary duty of calculus lecturers (teachers): the purpose of education is not to teach students how to pass examinations. Calculus educators should help students to see the connections within calculus and the other disciplines and its applications in their everyday life, in particular, in Iranian secondary education.

• Teaching methods and textbooks should favour developing calculus material and students' knowledge with analytical as well as visual and verbal aspects to make allowance for students with different learning styles to grasp materials in the way which is matched to their thinking style.

• Three categories of calculus tasks which are described in this study as (Z_4), (Z_5) and (Z_6), should be considered in a healthy balanced way, in courses and examinations, to help students to understand better the different aspects of calculus materials. The researcher found that nearly all conventional calculus examinations did not measure students' pictorial thinking and curve interpretation, that is (Z_5). The calculus examination is usually built on multi-conceptual or multi-skilled tasks or both of them. Therefore, success in a calculus course, even in engineering branches, would be measured by routine questions which do not require pictorial thinking and interpretation. Despite the previous finding, Vinner (1989) suggested that visual considerations and graphical interpretations have a crucial role in learning calculus. In fact, this approach can explain some algebraic and analytic manipulation which, otherwise, look artificial and meaningless.

Visual ways of learning should be emphasized in calculus teaching and learning and the belief that pictorial proofs are not mathematical proofs must be modified, at least, in teaching calculus. The researcher found that the use of curve interpretation could help students to improve their meaningful understanding of, for example, 'limit' and 'one-sided limit processing', 'continuity and discontinuity', 'differentiation at a point or over an interval', 'finding the rule of a function $f(x)$, its derived function $f'(x)$ and inverse function $f^{-1}(x)$' and so on.

• Lecturers (teachers) should match question tasks within the course with those in examinations to the specific outcomes being assessed and formulate questions so that students are presented with clearly-defined tasks showing what is expected of them based on learning objectives (Ellington, 1987). However, a lot of tasks a in calculus course and examinations very often assess repeated questions with the same demands, but in different forms. Therefore, they are unable to measure fully the students' knowledge and achievement in the course.

• Scottish students who are coming from Higher Grade have not sufficient knowledge to begin the calculus course at university. It seems that Higher Grade material cannot provide the necessary grounding for first-year university calculus. If universities continue to take the 5th year Higher Grade as the standard entry, some changes will be needed to materials and teaching methods.

- It was found in this research that calculus students very often exhibited weakness in logical argument and manipulation of mathematical symbols, particularly in Scotland. Moreover, Scottish students have shown more difficulties compared to Iranian ones in algebraic manipulations. As a result, remedial work is necessary before students begin to use such skills in curricula.

- It is safe to state that the Scottish mathematical education at secondary level, clearly has a divergent bias, while calculus (for the mathematics specialist) at university tends to favour a convergent rather than a divergent approach. Therefore, convergent thinkers are more likely to lose their chance to develop their mathematical knowledge both in secondary and higher education. The researcher found that Scottish students who were divergent thinkers performed significantly better than convergent thinkers in calculus activities at university level.

- On the contrary, mathematical curricula (including calculus) in Iran are built with a convergent bias rather than a divergent bias in high school and higher education. As a result, convergent thinkers are more likely to have a better chance to develop their mathematical grounding compared to divergent ones. They can also achieve higher performance in the mathematics university entrance examination (En), because question tasks in this important examination favour more convergent thinking rather than divergent thinking. If we accept that divergent thinkers are those people who show higher performance in the open-ended questions and creative tasks, they would be more likely to miss higher education entry in comparison with convergent learners. Hence people who are involved in mathematical curricula and university entrance examinations, in Iran and Scotland, should modify curriculum development for the benefit of both groups of cognitive styles.

- The tendency in teaching calculus in Iranian high schools, even with the new curricula, is clearly going from the intuitive to more formal mathematical learning with a weak emphasis on modelling and calculus applications to everyday life. However, rigour and formal teaching of such concepts as "limit and continuity" are being replaced with an informal approach. In Scottish secondary education, there is an open tendency towards teaching and learning calculus in a more pictorial and informal approaches with more emphasis on mathematical modelling and applications in the real world. As a result, a rethink is necessary, to adapt Iranian calculus education in high school and pre-university courses to tasks involving the use of calculus in action, and the Scottish calculus, in particular in Higher Grade, needs more emphasis on analytical and mathematical rigour.

- The researcher strongly recommends to his colleagues in secondary and higher education that our teaching methods as a transmission of passive knowledge should be shifted to an acquisition of knowledge by students through an active involvement in the subject-matter, recognizing students' individual differences in the ways of thinking and goal attainment in learning calculus. As mathematics educators, we should pay attention to how students think and learn and, therefore make the necessary opportunity for FI/FD and Con/Div cognitive styles to be equally involvement in classroom activities, in spite of some difficulties. We should know that our ways of thinking and cognitive styles are not always the same as those of our students.

- The results of this study strongly suggested that calculus assessment built on one final examination would not be a reasonable way of measuring students' knowledge and degree of meaningful understanding. Science students are normally assessed largely by a written form of examination. The researcher considers it to be at least part of the lecturer's duty to provide students with skills to cope with this type of examination adequately. But, he recommends that, in spite of some difficulties, students' involvement in the course by individual investigation tasks, considering the tutorial class as a workshop, and verbal examination could be more beneficial compared to conventional methods. In sum, as Keith (1995) suggested, we need better assessment measures of calculus courses.

Based on the conventional system and inflexible methods of lectures, tutorials and inexperienced mathematicians, no possible change would happen to increase students' interest or to develop new thinking in calculus courses. As Schoenfeld (1995) suggested, radical shifts in teaching mathematics (calculus) from a lecture course to laboratory course with a discovery component are needed. The recommendations of the present research could be a significant step towards radical shifts in calculus as a laboratory activity with more teachers and students' involvement. Students should learn and work cooperatively, and be able to use calculus to formulate problems, solve them, and use the solutions in a different context.

• Limited use of calculators and computers, in particular in secondary education, under supervision of educators could be an appropriate way to incorporate modern technology in teaching and learning calculus. However, as Smith (1995) noted, use of computers is not a solution to pedagogical problems, but rather an opportunity to think about and solve calculus tasks in new ways particularly for rational understanding of limit processing and so on.

• The researcher recommends that sessions of one-hour, instead of two hours in one session for teaching calculus as is usual in Iranian universities, would seem to be the maximum exposure if those students who are (FD+Con) in the mathematics branch, and (FD+Div) in engineering, with low capacity of working memory are to be assisted. It is most likely that students in two-hour lectures will become overloaded and rote learning could be the result of such instruction.

8.6 Recommendations for further Research

As is usual with pioneering research, many questions could arise from this study, each of which may become a point of departure for further research. Some suggestions are now offered which might yield even more understanding about how students with different ways of learning should be instructed in calculus courses and mathematics as whole and how educators should assess their achievement.

• The findings of the present research are based upon four samples at two Iranian universities and and a sample with a small population at Glasgow University. Consequently, further experiments are necessary perhaps under more specific conditions for finding more information, in particular, in Scotland.

• As already mentioned, some results which are reported in this thesis show students' performance in only one final calculus examination. It seems reasonable that one mathematics examination, very often with unbalanced questions, cannot measure students' achievement and basic knowledge in a calculus course. Therefore, finding more information about the performance of students' with various learning styles, by means of more and different methods of assessing would be necessary.

• It was discussed in this research that, for meaningful learning to take place in calculus and mathematics as whole or any other discipline, students should be able to relate materials to their own idiosyncratic cognitive structure. In other words, students learning conceptual definitions or following a proof in calculus (mathematics) cannot necessarily learn the involved logical meaning, but rather the meaning which they have worked out for themselves, that is the psychological meaning. On this point, a lot of questions should be investigated. For example:

1. Are the calculus topics and our teaching order and methods organised according to the logical or psychological order?

2. How can we introduce calculus material and textbooks which are mainly developed logically to students who cannot see this logical order, to obtain a balance between logical and psychological order?

• It could be interesting to follow this study in high school to find pupils' performance in calculus and other mathematical areas and the other subjects such as physics and so on.

• It would also be valuable to continue this study in more advanced mathematical courses such as calculus with more than one variable, mathematical analysis, complex analysis, and so on at undergraduate level.

Further research may be pertinent on the involvement of the other leaning styles and other psychological factors such as working memory capacity. This could lead a researcher to more findings about any selected sample. Could the nature of mathematical ability and creativity favour divergent thinking or convergent thinking, or does a balance between them need to be considered. Are mathematically talented students (pupils) more divergers than convergers? Are divergent thinkers attracted to mathematics learning more than convergent thinkers or are there other psychological factors involved in performing mathematical courses?

• The present research has often used conventional calculus examinations to exhibit students' achievement with different thinking styles. Creative tasks in calculus courses are generally not involved in showing students' performance.

• Match-mismatch between educators' cognitive styles and that of students in calculus courses could be investigated to improve teaching and learning. In sum, in teaching calculus, knowledgeable mathematicians may go through three distinct phases. They grow in their teaching approaching that "I teach calculus" might become "I teach students calculus", and finally they would arrive at the position "I help students to learn calculus by finding their ways of learning". However, as Sa'di (A.D. 1184–1292) the famous poet of Persia, said:

"When raindrops from the heavens fall,
Tenderly and slow,
They nourish garden lawns and make
The desert thistles grow."

References

Adams, V. M. and McLeod, D. B. (1979). The interaction of field-dependence/independence and the level of guidance of mathematics instruction. Journal for Research in Mathematics Education, *10*(5), 347–355.

Al-Naeme, F. F. A. (1991). The influence of various learning styles on Practical problem-solving in chemistry in Scottish secondary schools. Ph.D. Thesis, University of Glasgow.

Al-Naeme, F. F. A. (1989). A study of some psychological factors affecting performance in chemistry at secondary and tertiary levels, M.Sc. Thesis, University of Glasgow.

Alsina, I. M. (1987). Problem-solving processes of field-independent & field-dependent Mexican-American girls. Ph.D. Thesis, The University of Texas at Austin (DAI 48/10A, P. 2565, Publication No.:AAC872850)

Ausubel, D. P.; Novak, J. D. and Hanesian, H. (1968). Educational Psychology: A cognitive view, second edition, New York: Holt, Rinehart & Winston.

Ausubel, D. P. (1963). The psychology of meaningful verbal learning. New York: Grune & Stratton.

Boas, R. P. (1981). Can we make mathematics intelligible? The American Mathematical Monthly, *88*, 727–731.

Bartle, R. G. and Tulcea, C. I. (1968). Calculus. New York: Scott, Foresman & Company.

Bennelt, A. B. Jr. (1988). Visual thinking and number relationships. Mathematics Teacher, *81*(4), 267–272.

Bruner, J. S. (1966). Toward a Theory of Instruction. Cambridge, MA: Harvard University Press.

Case, R. and Globerson, T. (1974). Field-independence and central computing space. Child Development, *45*, 772–778.

Case, R. (1974). Structures and strictures, some functional limitations on the course of cognitive growth. Cognitive Psychology, *6*, 544–574.

Child, D. and Smithers, A. (1973). An attempted validation of the Joyce-Hudson scale of convergence and divergence. British Journal of Educational Psychology, *43*, 57–61.

Clement, J.; Lochhead, J. and Monk, G. S. (1981). Translation difficulties in learning mathematics. The American Mathematical Monthly, *88*, 286–290.

Campbell, R. J.; Collis, K. F. and Watson, J. M. (1995). Visual processing during mathematical problem solving. Educational Studies in Mathematics, *28*, 177–194.

Cornu, B. (1991). Limit. in Advanced Mathematical Thinking, edited by Tall. D., The Netherlands: Kluwer Academic Publishers.

Cross, K. L. (1976). Accent on Learning. New York: Jossey-Bass Publishers.

Davis, R. B. and Vinner, S. (1986). The notion of limit: some seemingly unavoidable misconception. Journal of Mathematical Behaviour, *5*(3), 281–303.

Davis, J. K. and Frank, B. M. (1979). Learning and memory of field-independent/dependent individuals. Journal of Research in Personality, *13*, 469–479.

Davis, J. K. and Cochran, K. F. (1989). An information processing view of field-dependence. Early Child Development and Care, *51*, 31–47.

Dawei, Wu (1992). Mathematics education in Britain and China. Mathematics Teaching, *141*, 23–27.

Dellio, N. J. (1982). Advanced calculus with applications. New York: Macmillan Publishing Co. Inc.

Dickstein, L. S. (1968). Field-independence in concept attainment. Perceptual and Motor Skills, *27*, 635–642.

Dieudonné, J. A. (1960). Foundation of Modern Analysis. Vol. I. New York and London: Academic Press Inc.

Dreyfus, T. and Eisenberg, T. (1982). Intuitive functional concepts: A baseline study on intuitions. Journal for Research in Mathematics Education, *13*(5), 360–

380.

Dreyfus, T. (1991). Advanced mathematical processes. in Advanced Mathematical Thinking, edited by Tall, D., 25–41, The Netherlands: Kluwer Academic Publishers.

Dreyfus, T. (1992). Imagery and reasoning in mathematics and mathematics education. Selected Lectures from the 7th international Congress on Mathematics Education, 107–122, Québec, Canada, Université de Laval.

Dugger, C. R. V. (1984). A study of the effectiveness of field-dependent/field-independent instruction on mathematics problem solving ability of Hispanic students. Ed.D. dissertation, Rutgers University (The State University of New Jersey, New Brunswick, DAI 45/08A, P.2388, Publication No.:AAC842404.)

Eisenberg, T. and Dreyfus, T. (1986). On visual versus analytical thinking in mathematics. Proceedings of the tenth international conference of the International Group for the Psychology of Mathematics Education, 153–158. London:IGPME.

Eisenhart, M.; Borko, H.; Underhill, R.; Brown, C.; Jones, D. and Agard, P. (1993). Conceptual knowledge falls through the cracks: complexities of learning to teach mathematics for understanding. Journal for Research in Mathematics Education, *4*(1), 8–40.

Eldersveld, P. J. (1980). The relationship of field-dependence/independence and failure in developmental mathematics at selected community colleges in Northern Illinois. Ed.D. Thesis, Northern Illinois University (DAI 42/02A, P.517, Publication No.:AAC811698).

El-Banna, H. A. A. M. (1987). The development of a predictive theory of science education based upon information processing hypothesis. Ph.D. Thesis, University of Glasgow.

Elkind, D.; Koegler, R. R.; Go, E. (1963). Field-independence and concept formation. Perceptual and Motor Skills, *17*, 383–386.

Ellington, H. (1987). Essay-type questions. The Scottish Institutions Committee for Educational Development.

Ellis, H. C. and Hunt, R. R. (1993). Fundamentals of Cognitive Psychology, fifth edition, U.S.: Brown & Benchmark (WCB) Publishers.

Ervynck, G. (1981). Conceptual difficulties for first year university students in the acquisition of the notion of limit of a function. Proceedings of the 5th International Conference of the Psychology of Mathematics Education, Grenoble, France, 330–333.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. Journal for Research in Mathematics Education, *24*(2), 94–116.

Even, R. and Tirosh, D. (1995). Subject-matter knowledge and knowledge about students as source of teacher presentation of subject-matter. Educational Studies in Mathematics, *29*, 1–20.

Farrell, M. A. (1992). Learn from your students. The Mathematics Teacher, *85*(8), 656–659.

Field, T. W. and Poole, M. E. (1970). Intellectual style and achievement of arts and science undergraduates. British Journal of Education, *40*, 338–341.

Frank, B. M. and Keene, D. (1993). The effect of learners' field-independence, cognitive strategy instruction, and inherent word-list organization on free-recall memory and strategy use. Journal of Experimental Education, *62*(1), 14–25.

Frank, B. M. and Noble, J. P. (1985). Field-independence/dependence and cognitive restructuring. Journal of Personality and Social Psychology, *5*, 1129–1135.

Frank, B. M. (1983). Flexibility of information processing and the memory of field-independent and field-dependent learners. Journal of Research in Personality, *17*, 89–96.

Foley, G. D. and Ruch, D. K. (1995). Calculus reform at comprehensive universities and two-year colleges, UME Trends (News and Report on Undergraduate Mathematics Education), *6*(6), 8–9.

Gagné, R. M. (1985). The Condition of Learning and Theory of Instruction. New York: Holt, Rinehart & Winston.

Gagné, R. M. (1983). Some issues in the psychology of mathematics instruction.

Journal for Research in Mathematics Education, *14*(1), 7–18.

Getzelts, S. W. and Jackson, P. W. (1962). Creativity and Intelligence. New York and London, Wiley & Son Inc.

Green, J. and D'Oliveira, M. (1982). Learning to use statistical tests in psychology. Milton Keynes and Philadelphia: Open University Press.

Goodenough, D. R. (1976). The role of individual differences in field-dependence as a factor in learning and memory. Psychological Bulletin, *83*(4), 675–694.

Guilford, J. P.; Hoepfner, R. and Petersen, H. (1965). Predicting achievement in ninth-grade mathematics from measures of intellectual aptitude factors. Educational and Psychological Measurement, Vol. XXV, No.3.

Guilford, J. P. (1959). Three faces of intellect. American Psychologist, *14*, 469–479.

Guilford, J. P. (1967). The nature of human intelligence. New York: McGraw-Hill Book Company.

Guilford, J. P. (1978). Traits of creativity. in P. E. Vernon (Ed.) "Creativity". England: Harmondsworth, Penguin Books.

Guilford, J. P. (1980). Cognitive styles: what are they? Educational and Psychological Measurement, *40*, 715–735.

Hadfield, O. D. (1986). Cognitive style and mathematics anxiety among high school students (Field-dependence/independence). Ed.D. Dissertation, Northern Arizona University (DAI 47/05A, P.1590, Publication No.:AAC861739).

Halmos, P. (1985). Mathematical people, profiles and interviews. Edited by Albers, D. J. and Alexanderson, G. L., Boston: Birkhauser.

Harmon, H. B. J. (1984). A correlational study of correspondence between achievement in calculus and complementary cognitive style (Learning). Ph.D. Thesis, Wayne State University (DAI 45/03A, P.738, Publication No.:AAC8414502).

Herron, J. D. (1996). The chemistry classroom: formulas for successful teaching. Washington, DC: American Chemical Society.

Hiebert, J. (1891). Cognitive development and learning measurement. Journal for

Research in Mathematics Education, *12*(3), 197–211.

Hitt, F. (1994). Teachers' difficulties with the construction of continuous and discontinuous functions. Focus on Learning Problems in Mathematics, *16*(4), 10–20.

Howell, D. C. (1992). Statistical methods for psychology, third edition, Boston, U.S.: PWS-KENT Publishing Company, Boston, U.S.

Hudson, L. (1966). Contrary imagination. London: Penguin Books.

Hudson, L. (1968). Frames of mind. London: Methuen & Co Ltd.

Johnstone, A. H. (1984). New stars for the teacher to steer by? Journal for Chemical Education *61*(10), 847–849.

Johnstone, A. H. (1988). Meaning beyond readability. Guildford, Southern Examining Group.

Johnstone. A. H. (1993). "Why is science difficult to learn? Things are seldom what they seem". in Teaching Learning and Assessment in Science Education, ES821, Reader for Open University (eds D. Edwards, E. Scanlon, R. West), London: Paul Chapman Publishing Ltd, 115–123.

Johnstone, A. H.; Hogg, W. R. and Ziane, M. (1993). A working memory model applied to physics problem solving. International Journal of Science Education, *15*(6), 663–672.

Johnstone, A. H. and Al-Naeme, F. F. (1991). Room for scientific thought. International Journal of Science Education, *13*(2), 187–192.

Joyce, C. R. B. and Hudson, L. (1968). Student style and teacher style: An experimental study. British Journal of Medical Education, *2*, 28-32.

Keith, S. Z. (1995). How do students feel about calculus reform, and how can we tell? UME Trends (News and Reports on Undergraduate Mathematics Education, *6*(6), 6.

Kieren, T. E. (1992). Bonuses of understanding mathematical understanding. Selected Lectures from the 7th International Congress on Mathematical Education, 211–228, Québec, Canada.

Kemp, I. J. and Seagraves, L. (1995). Transferable skills; can higher education deliver? Studies in Higher Education, *20*(3), 315–328.

Kempa, R. (1979). The future. in Cognitive development research in science and mathematics, edited by Archenhold, W. F. et al., Proceedings of an International Seminar, The University of Leeds, 423.

Kempa, R. F. and McGough, J. M. (1977). A study of attitudes towards mathematics in relation to selected students characteristics. British Journal of Educational Psychology, *47*, 296–304.

Kogan, N. (1976). Individuality in learning. New York: Jossey-Bass Publishers.

Leinhardt, G.; Zaslavsky O. and Stein, M.K. (1990). Functions, graphs and graphing: Tasks learning, and teaching. Review of Educational Research, *60*(1), 1–64.

Mackay, C. K. and Cameron, M. B. (1968). Cognitive bias in Scottish first year science and arts undergraduates. British Journal of Educational Psychology, *38*, 315–318.

Marsden, J. and Weinstein, A. (1985). Calculus. Second edition, New York: Springer-Verlag.

McKenna, F. P. (1984). Measures of field-dependence: Cognitive style or cognitive ability? Journal of Personality and Social Psychology, *47*(3), 593–603.

McLeod, D. B.; Carpenter, T. P.; McCornack, R. L. and Skvarcius, R. (1978). Cognitive styles and mathematics learning: The interaction of field-independence and instructional treatment in numeration system. Journal for Research in Mathematics Education, *9*, 163–174.

McLeod, D. B. and Briggs, J. T. (1980). Interactions of field-independence and general reasoning with inductive instruction in mathematics. Journal for Research in Mathematics Education, *11*(2), 94–103.

Messick, S. (1976). Individuality in learning. New York: Jossey-Bass publishers.

Monna, A. F. (1992). The way of mathematics and mathematicians, from reality towards fiction. CWI Tract (87), Stichting Mathematisch centrum, Amsterdam, The Netherlands.

Mrosla, H. P. (1983). Difference between field-dependence/field-independence cognitive styles of low and high achievement mathematics students. Ph.D. Thesis, University of North Texas (DAI 44/08A, P.2397, Publication No.:AAC832705).

Moore, R. C. (1994). Making the transition to formal proof. Educational Studies in Mathematics, *27*, 249–266.

Moses, B. (1982). Visualisation: A different approach to problem solving. School Science and Mathematics, *82*, 141–147.

Mundy, J. F. (1987). Spatial training for calculus students: Sex differences in achievement and in visualization ability. Journal for Research in Mathematics Education, *18*(2), 126–140.

Mundy, J. F. and Lauten, D. (1994). Learning about calculus learning. The Mathematics Teacher, *87*(2), 115–121.

Murphy, I. S. (1991). Basic mathematical analysis: The facts. Third edition, Stirling, Scotland: Arklay Publishers.

Nahinsky, I. D.; Morgan, M. S. and Oeschger, D. E. (1979). Cognitive strategies, field-dependence, and the abstraction process. Journal of Research in Personality, *13*, 490–504.

Nebelkopf, E. B. and Dreyer, A. S. (1973). Continuous-discontinuous concept attainment as a function of individual differences in cognitive style. Perceptual and Motor Skills, *36*, 655–662.

Nesher, P. (1986). Are mathematical understanding and algorithmic performance related? For the Learning of Mathematics, *6*(3), 2–9.

Nobel, J. P. and Frank, B. M. (1985). Field-independece/dependence and verbal restructuring. The Journal of Experimental Education, *54*(1), 28–33.

Orton, A. (1983). Students' understanding of integration. Educational Studies in Mathematics, *14*, 1–18.

Orton, A. (1984). Understanding rate of change. Mathematics in School, *13*(5), 23-26.

Thesis, University of California, Berkeley (DAI 41/08A, P.3465, Publication No.: AAC802956).

Runco, M. A. (1986). Divergent thinking and creative performance in gifted and nongifted children. Educational and Psychological Measurement, *46*, 375–383.

Sacks, H. V. and Eysenck, M. W. (1977). Convergence-divergence and the learning of concrete and abstract sentences. British Journal of Psychology, *68*, 215–221.

Sally, M. A. and Bostock, L. (1979). Convergent-divergent thinking and arts-science orientation. British Journal of Psychology, *70*, 155–163.

Schoenfeld, A. H. (1985). Mathematical problem solving. New York: Academic Press Inc.

Schoenfeld, A. H. (1995). A brief biography of calculus reform. UME Trends (News and Report on Undergraduate Mathemativs Education), *6*(6), 3–5.

Searl, J. (1992). Developing research into mathematical education in higher education. A paper presented in the 7th International Congress on Mathematical Education, Québec, Canada.

Scardamalia, M. (1977). Information processing capacity and the problem of Horizontal Décalage: A demonstration using combinatorial reasoning tasks. Child Development, *48*(1), 28–37.

Shear, J. (1985). Visual thinking, algebraic thinking and a full unit-circle diagram. Mathematics Teacher, *78*(7), 518–522.

Sheel, S. T. (1981). The effect of cognitive style on the acquisition of mathematical concepts presented through emphasis on positive and negative instances. Ph.D. Thesis, The University of Oklahama (DAI 42/01A, P.122, Publication No.:AAC8113249).

Skemp, R. R. (1976). Relational understanding and instructional understanding. Mathematics Teaching, *77*, 20–26.

Skemp, R. R. (1985). The psychology of learning mathematics. Second edition, London: Penguin Books.

Smith, D. A. (1995). Learning software (computers in calculus reform). UME Trends

(News and Reports on Undergraduate Mathematics Education), *6*(6), 14–15.

Sowder, J. T.; Sowder, L.; Moyer, J. C. and Moyer M. B. (1985). Cognitive variables and performance on mathematical story problems. The Journal of Experimental Education, *54*(1), 56-62.

Spitler, G. J. (1970). An investigation of various cognitive styles and the implication for mathematics education. Ph.D. Thesis, Wayne State University (DAI 32/105A, University Microfilms No.:71-17317).

Spivak, M. (1967). Calculus. New York: W. M. Benjamin, Inc.

Talbi, M. T. (1990). An information processing approach to the investigation of mathematical problem solving at secondary and university levels. Ph.D. Thesis, University of Glasgow.

Tall, D. O. and Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. Educational Studies in Mathematics, *12*, 151–159.

Tall, D. O. (1987). Constructing the concept image of a tangent. in J. Bergeron, N. Herscovics and C. Kieran (Eds.), Proceedings of 11th International Conference for the Psychology of Mathematics Education, *3*, 69–75. Montréal, Canada, Université de Montréal.

Tall, D. O. (1991). Intuition and rigour: The role of visualization in the calculus. in W. Zimmermann and S. Cunningham (Eds.), Visualisation in teaching and learning mathematics. Washington: The Mathematical Association of America, 105–119.

Threadgill, J. A. (1979). The relationship of field-independent/dependent cognitive style and two methods of instruction in mathematics learning. Journal for Research in Mathematics Education, *10*(3), 219–222.

Thompson, P. W. (1994). Images of rate and operational understanding of the Fundamental Theorem of Calculus. Educational Studies in Mathematics, *26*, 229–274.

Triadafillidis, T. A. (1994). Practical activities in mathematics learning. Ph.D. Thesis, University of Edinburgh.

Vinner, S. and Tall, D. O. (1982). Existence statements and constructions in mathematics and some consequences to mathematics learning. The American Mathematical Monthly, *89*, 752–756.

Vinner, S. (1982). Conflicts between definitions and intuitions: The case of tangent. in A. Vermandel (Ed.), Proceedings of the 6th International Conference on the Psychology of Mathematics Education (PME 6), 24–28, Antwerp, Belgium.

Vinner, S. (1983). Concept definition, concept image and notion of function. International Journal of Mathematics Education in Science and Technology, *14*, 293–305.

Vinner, S. (1989). The avoidance of visual consideration in calculus students. Focus on Learning Problems in Mathematics, *11*(2), 149–156.

Vinner, S. (1991). The role of definitions in teaching and learning mathematics. in "Advanced Mathematical Thinking", edited by Tall, D., 65–81, The Netherlands: Kluwer Academic Publishers.

Vinner, S. and Dreyfus, T. (1989). Images and definitions for the concept of function. Journal for Research in Mathematics Education, *20*(4), 356–366.

Webb, N. L. (1979). Processes, conceptual knowledge, and mathematical problem-solving ability. Journal for Research in Mathematics Education, *10*(2), 83–93.

Webb, L. V. (1981). The relationship of cognitive styles and academic majors with university level academic achievement. Ph.D. Thesis, University of California (DAI 42/04A P.1563, Publication No.:AAC0535761.

Webster, M. A. and Walker, M. B. (1981). Divergent thinking in arts and science students: The effect of item content. British Journal of Psychology, *72*, 331–338.

White, P. and Mitchelmore, M. (1996). Conceptual knowledge in introductory calculus. Journal for Research in Mathematics Education, *27*(1), 79–95.

Witkin, H. A.; Lewis, H. B.; Hertzman, M.; Machover, K.; Meissner, P. B. and Wapner, S. (1954). Personality through perception. New York: Harper.

Witkin, H. A. (1976). Cognitive style in academic performance and in teacher-student relations. in "Individuality in learning". New York: Jossey-Bass Publishers.

Witkin, H. A. and Goodenough, D. R. (1977). Field-dependence and interpersonal behavior. Psychological Bulletin, *84*(4), 661-689.

Witkin, H. A.; Moore, C. A.; Goodenough, D. R. and Cox, P . W. (1977). Field-dependent and field-independent cognitive styles and their educational implications. Review of Educational Research, *47*(1), 1-64.

Witkin, A. H. and Goodenough, D. R. (1981). Cognitive styles: Essence and origins. *New York: International Universities* Press.

Witkin, A. H.; Moore, C. A.; Oltman, P. K.; Goodenough, *D. R.; Friedman, F.;* Donald, R.; Owen, D. R. and Raskin, E. (1977). Role of the field-dependent and field-independendt cognitive styles in academic evolution: A longitudinal study. Journal of Educational Psychology *69*(3), 197-211.

Witkin, H. A.; Dyk, R. B.; Faterson, H. F.; Goodenough, D. R. and Karp, S. A. (1974). Psychological differentiation. New York: John Wiley and Sons.

Witkin, H. A.; Goodenough, D. R. and Karp, S. A. (1967). Stability of cognitive style from childhood to young adulthood. *7*(3), 291-300.

Witkin, A. H. (1978). Cognitive styles in personal and cultural adaptation. Worcester, Mass: Clark University Press.

Webb, N. L. (1979). Processes, conceptual knowledge, and mathematical problem-solving ability. Journal for Research in Mathematics Education, *10*(2), 83-93.

Webb, L. V. (1981). The relationship of cognitive styles and academic majors with university level academic achievement. Ph.D. Thesis, University of California (DAI 42/04A P.1563, Publication No.:AAC0535761).

Webster, M. A. and Walker, M. B. (1981). Divergent thinking in arts and science students: The effect of item content. British Journal of Psychology, *72*, 331-338.

White, P. and Mitchelmore, M. (1996). Conceptual knowledge in introductory calculus. Journal for Research in Mathematics Education, *27*(1), 79-95.

Ziane, M. (1990). The application of information processing theory to the learning of physics. Ph.D. Thesis, University of Glasgow.

APPENDICES

Appendix A

Hidden Figure Test (HFT)

SHAPES

NAME: SEX:

MATRICULATION NUMBER: DATE OF BIRTH:

This is a test of your ability to recognise simple SHAPES, and to pick out and trace HIDDEN SHAPES within complex patterns..

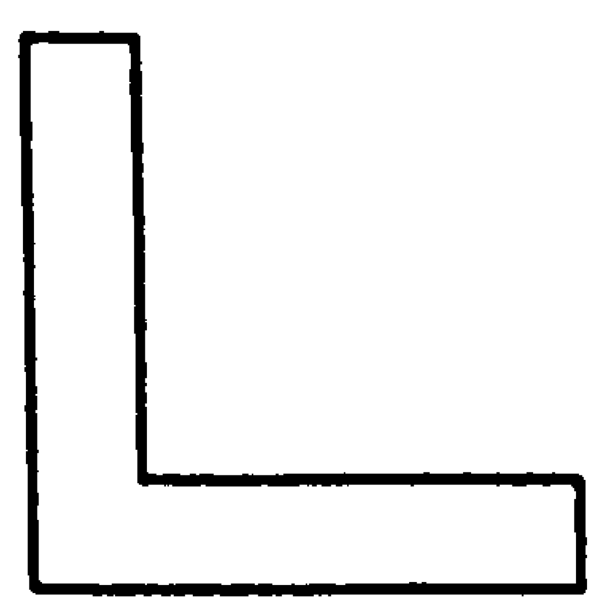
YOU ARE ALLOWED ONLY 15 MINUTES TO ANSWER ALL THE ITEMS.
TRY TO ANSWER EVERY ITEM, BUT DON'T WORRY IF YOU CAN'T.
DO AS MUCH AS YOU CAN IN THE TIME ALLOWED.
DON'T SPEND TOO MUCH TIME ON ANY ONE ITEM.

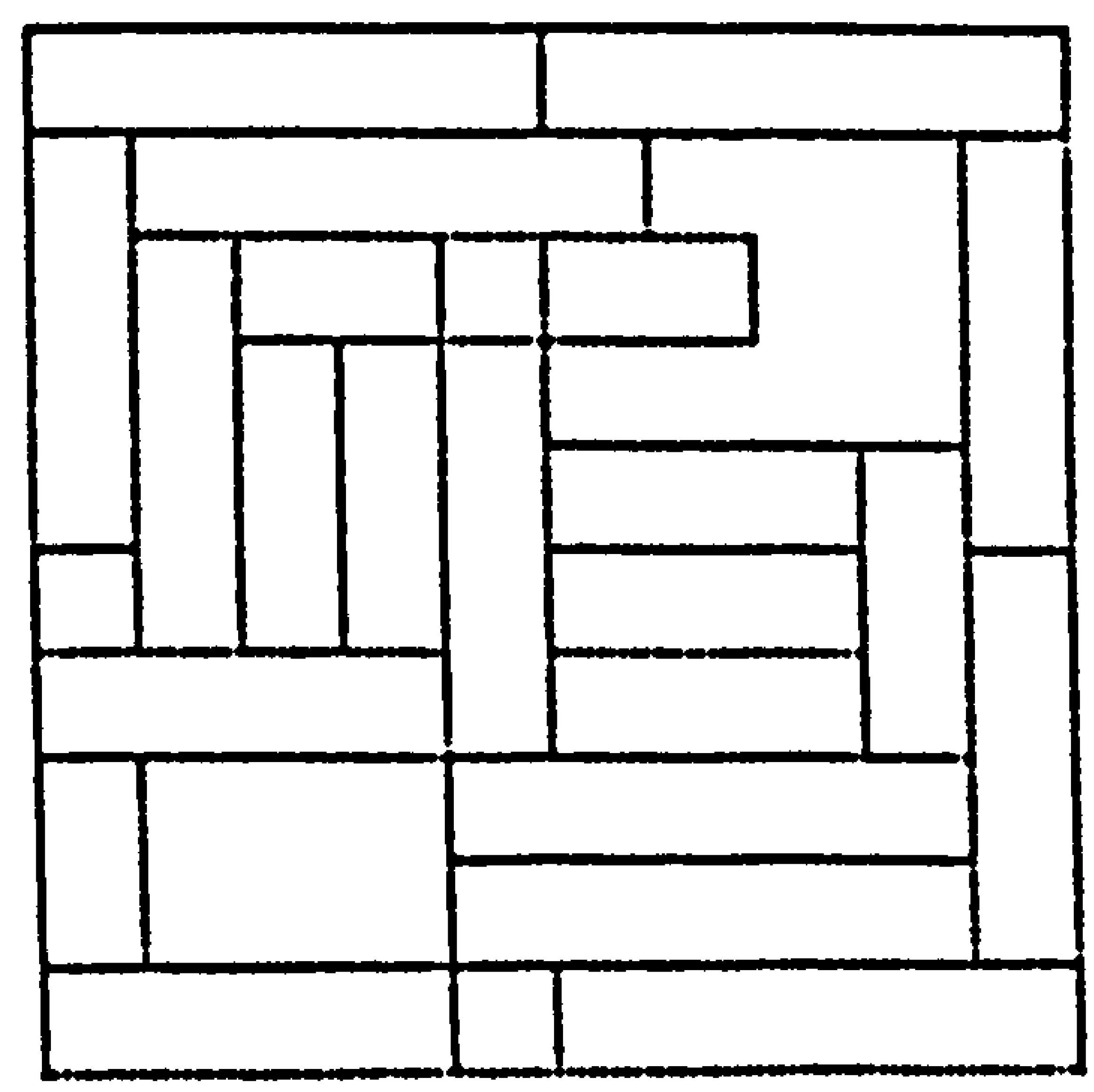
DO NOT START UNTIL YOU ARE TOLD

Appendix A (cont'd)

LOOKING FOR HIDDEN SHAPES

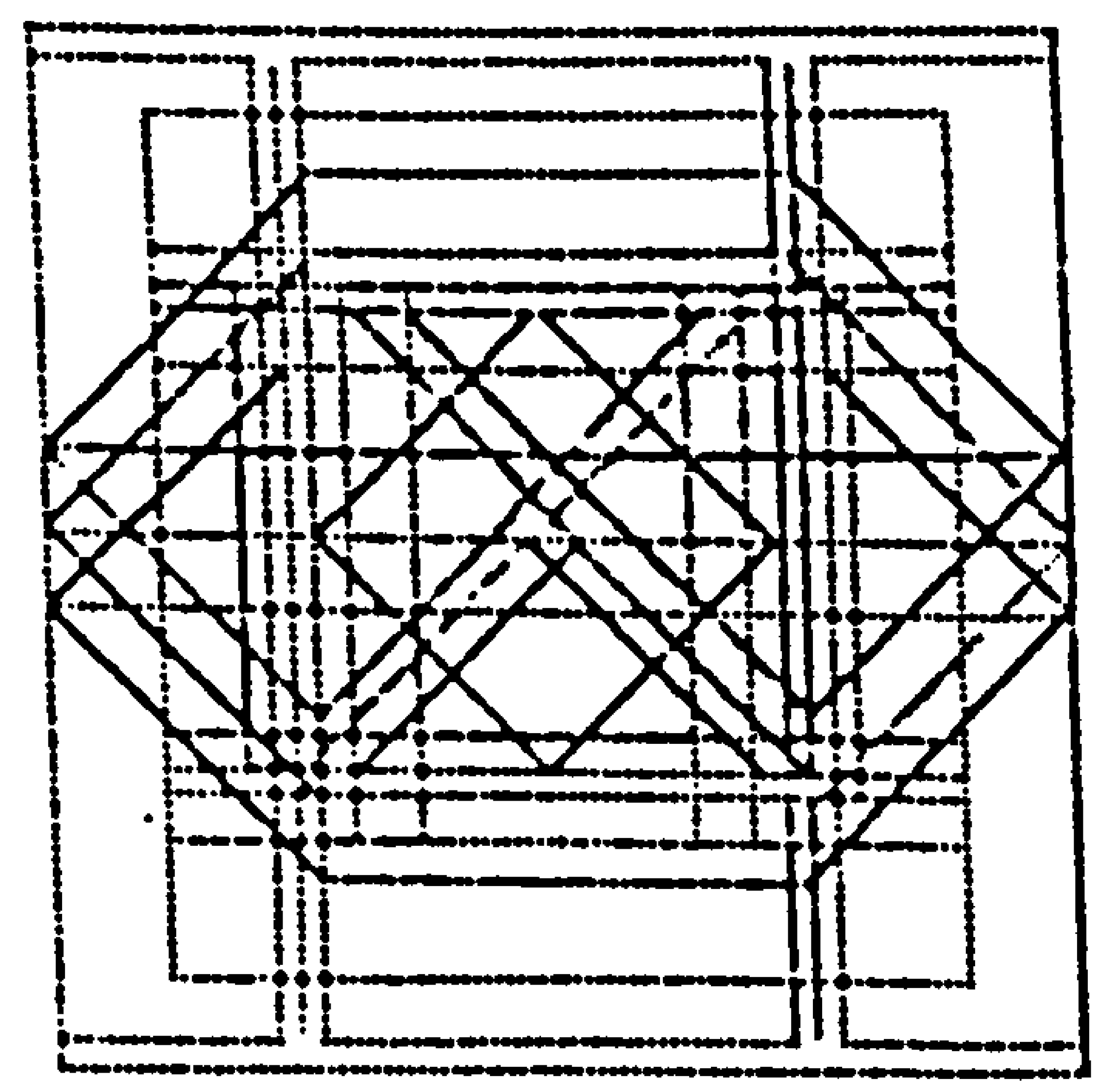
A simple geometrical figure can be 'hidden' by embedding it in a complex pattern of lines. For example, the simple L-shaped figure on the left has been hidden in the pattern of lines on the right. Can you pick it out?





Using a pen, trace round the outline of the L-shaped figure to mark its position.

The same L-shaped figure is also hidden within the more complex pattern below. It is the same size, the same shape and faces in the same direction as when it appears alone. Mark its position by tracing round its outline using a pen.



(To check your answers, consult the last page of this document.)

Appendix A (cont'd)

More problems of this type appear on the following pages. In each case, you are required to find a simple shape 'hidden' within a complex pattern of lines, and then, using a pen, to record the shape's position by tracing its outine.

There are 4 patterns on each page. Below each pattern there is a code letter (A, or B, or C etc.) to identify which shape is hidden in that pattern.

You will see all the shapes you have to find, along with their corresponding code letters, on the last page of this booklet. You may tear off this page and refer to it as often as you wish while you are doing the problems.

Note these points:

(1) You can refer to the page of simple shapes as often as necessary.

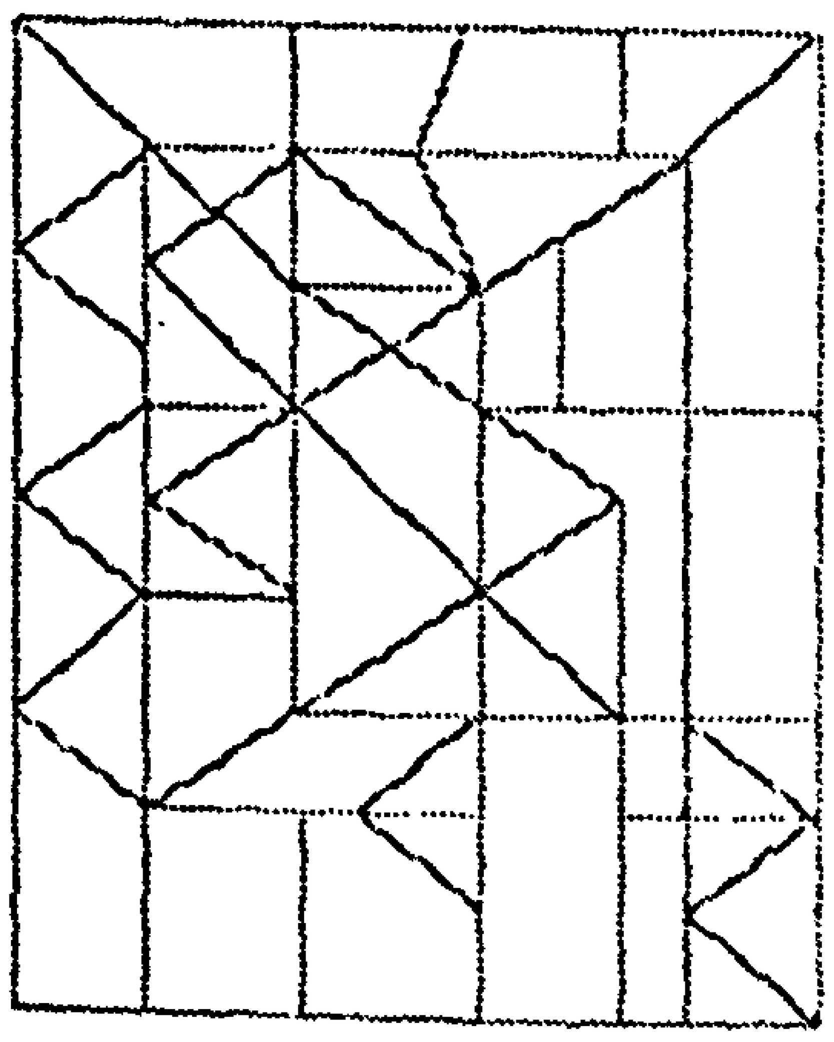
(2) When it appears within a complex pattern, the required shape is always
the same size,
has the same proportions,
and faces in the same direction
as when it appears alone.

(3) Within each pattern, the shape you have to find appears only once. Trace the required shape **and only that shape** for each problem.

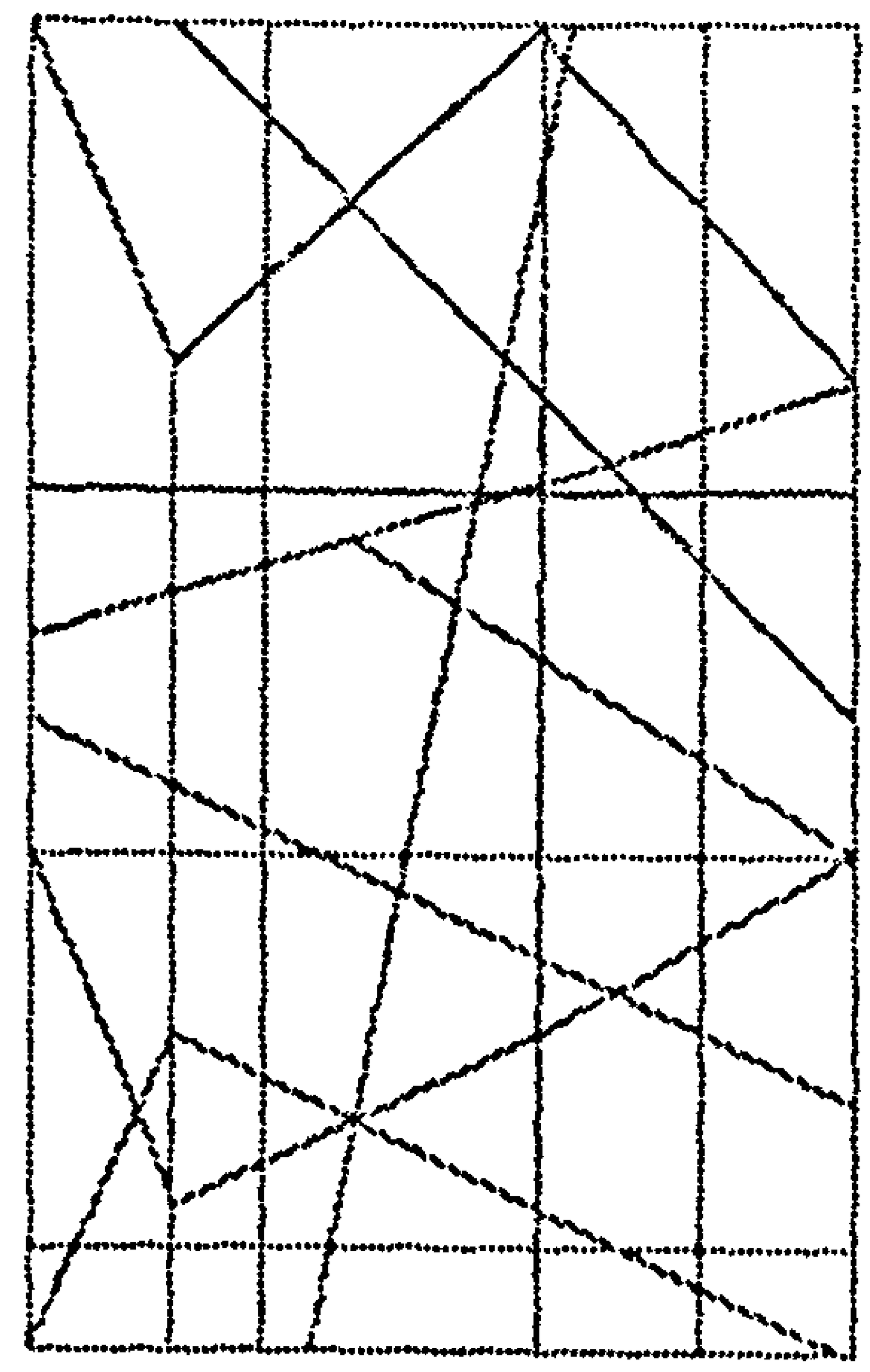
(4) Do the problems in order — don't skip one unless you are absolutely stuck.

START NOW

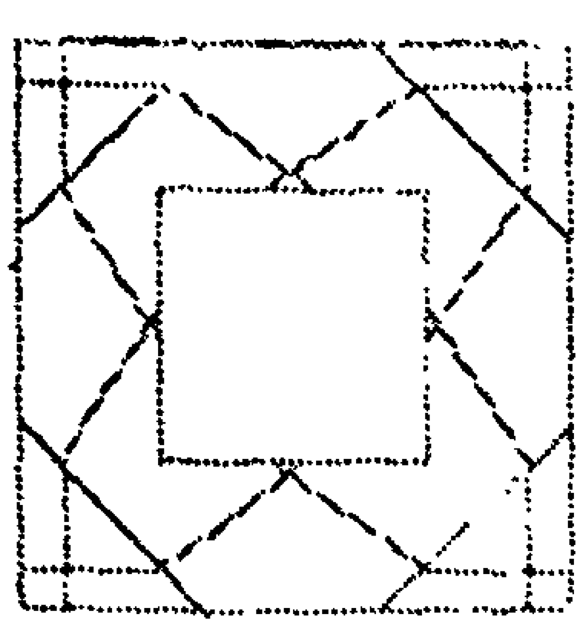
Appendix A (cont'd)



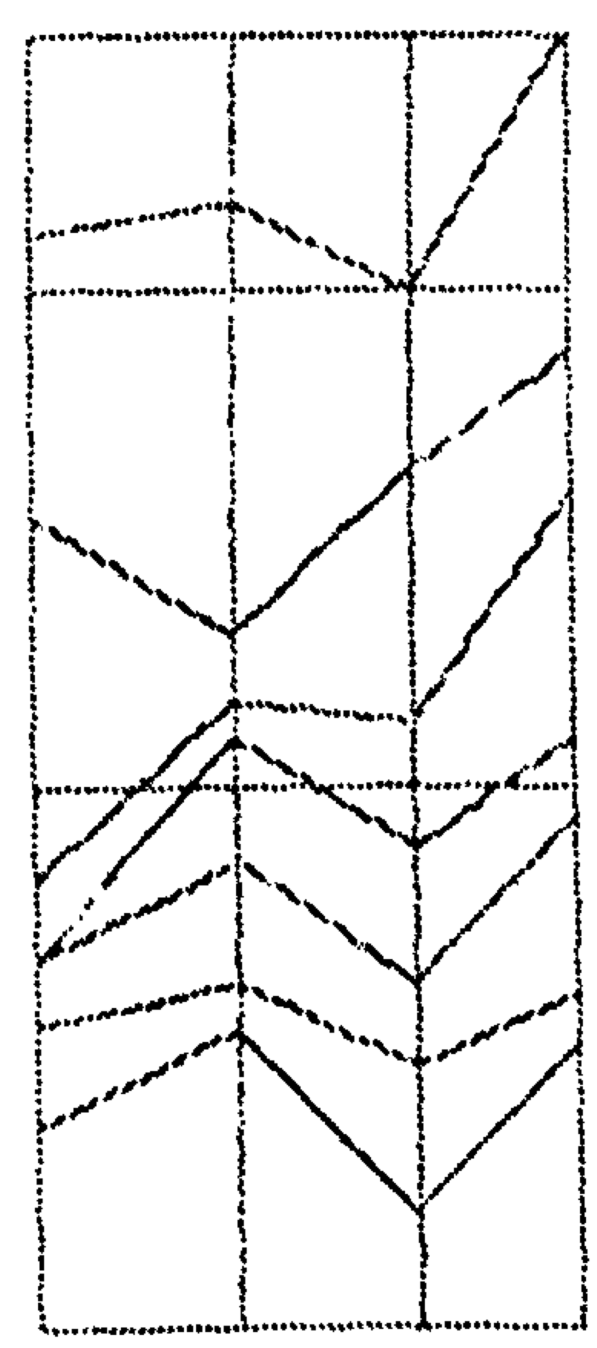
Find SHAPE H



Find SHAPE E

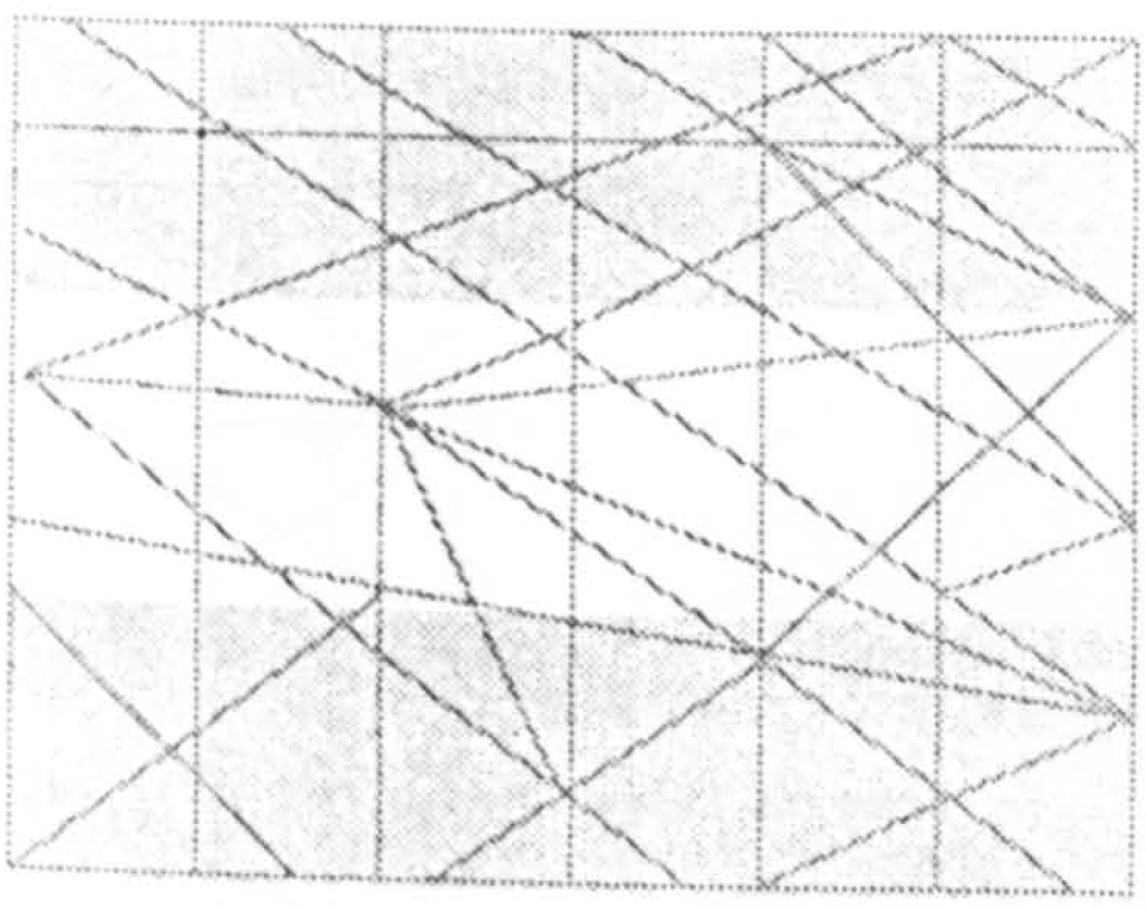


Find SHAPE B

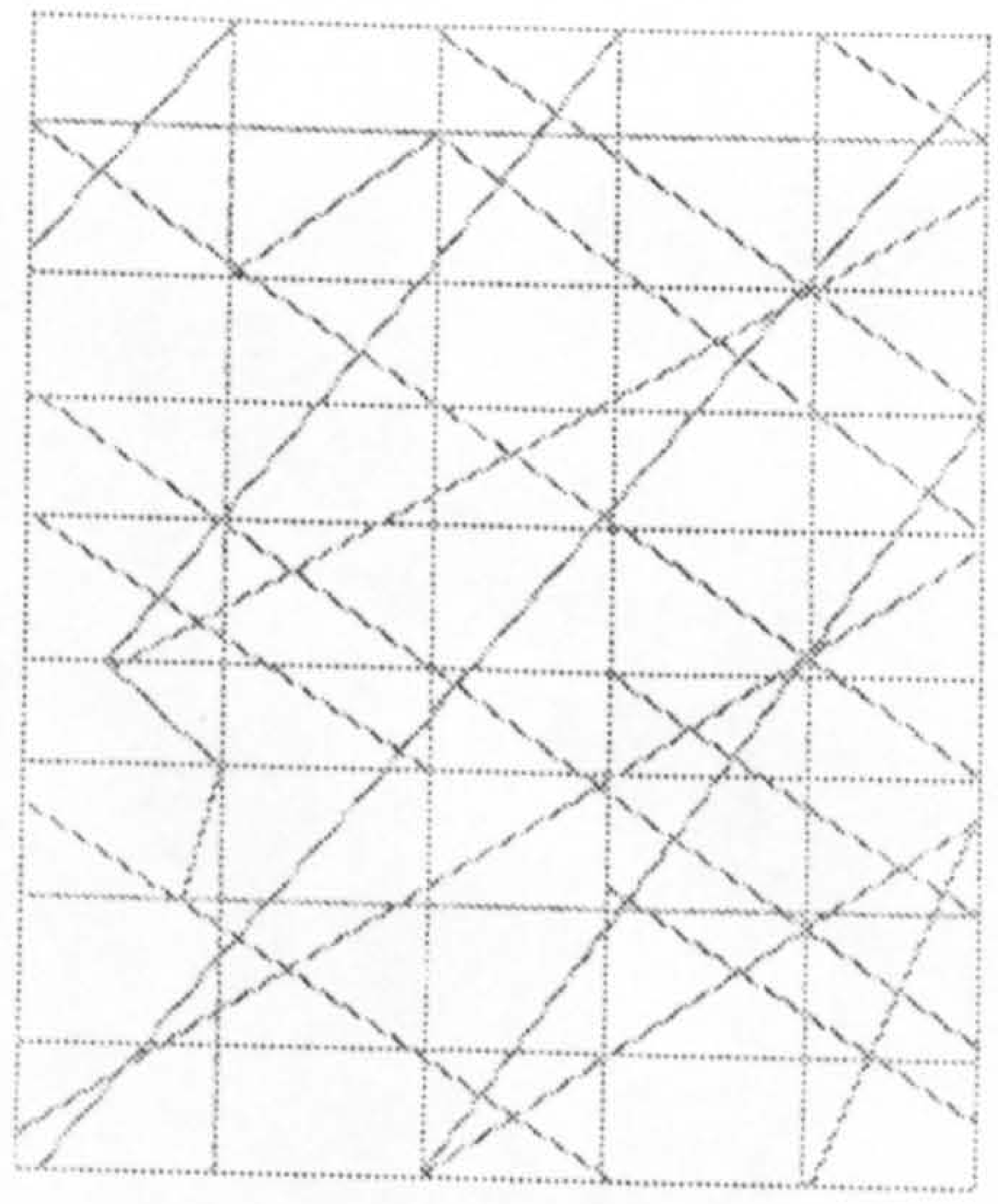


Find SHAPE D

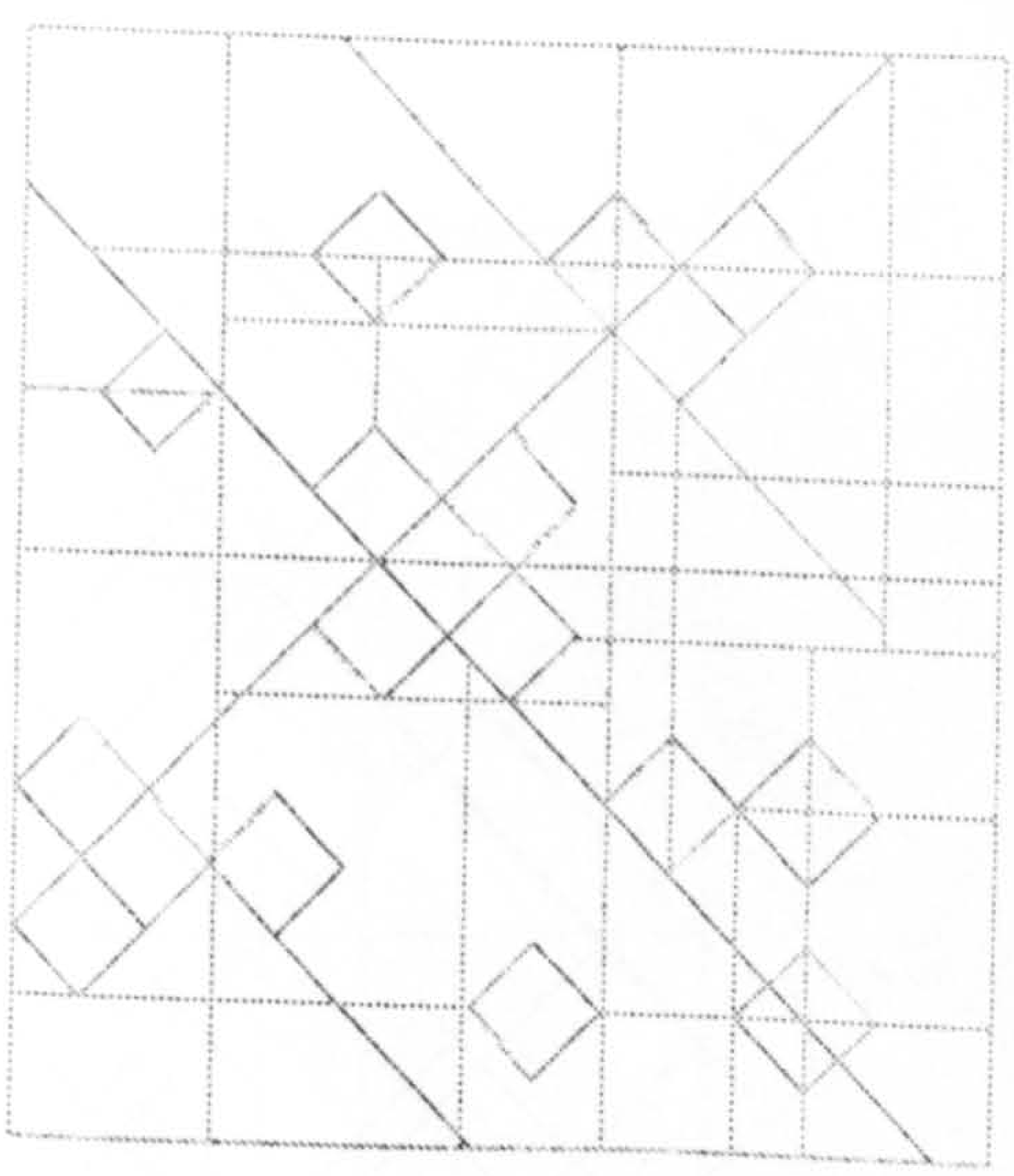
Appendix A (cont'd)



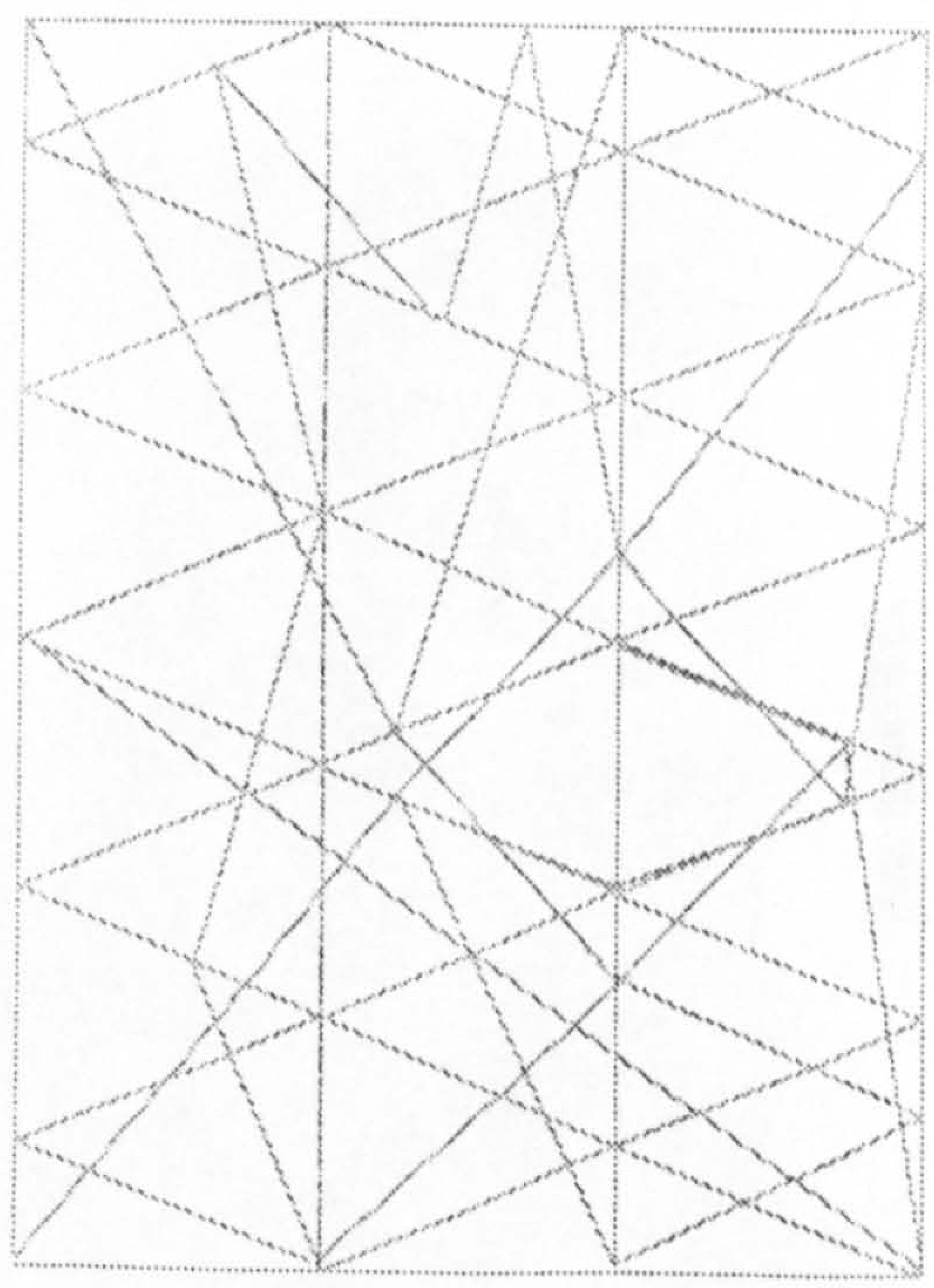
Find SHAPE E



Find SHAPE H

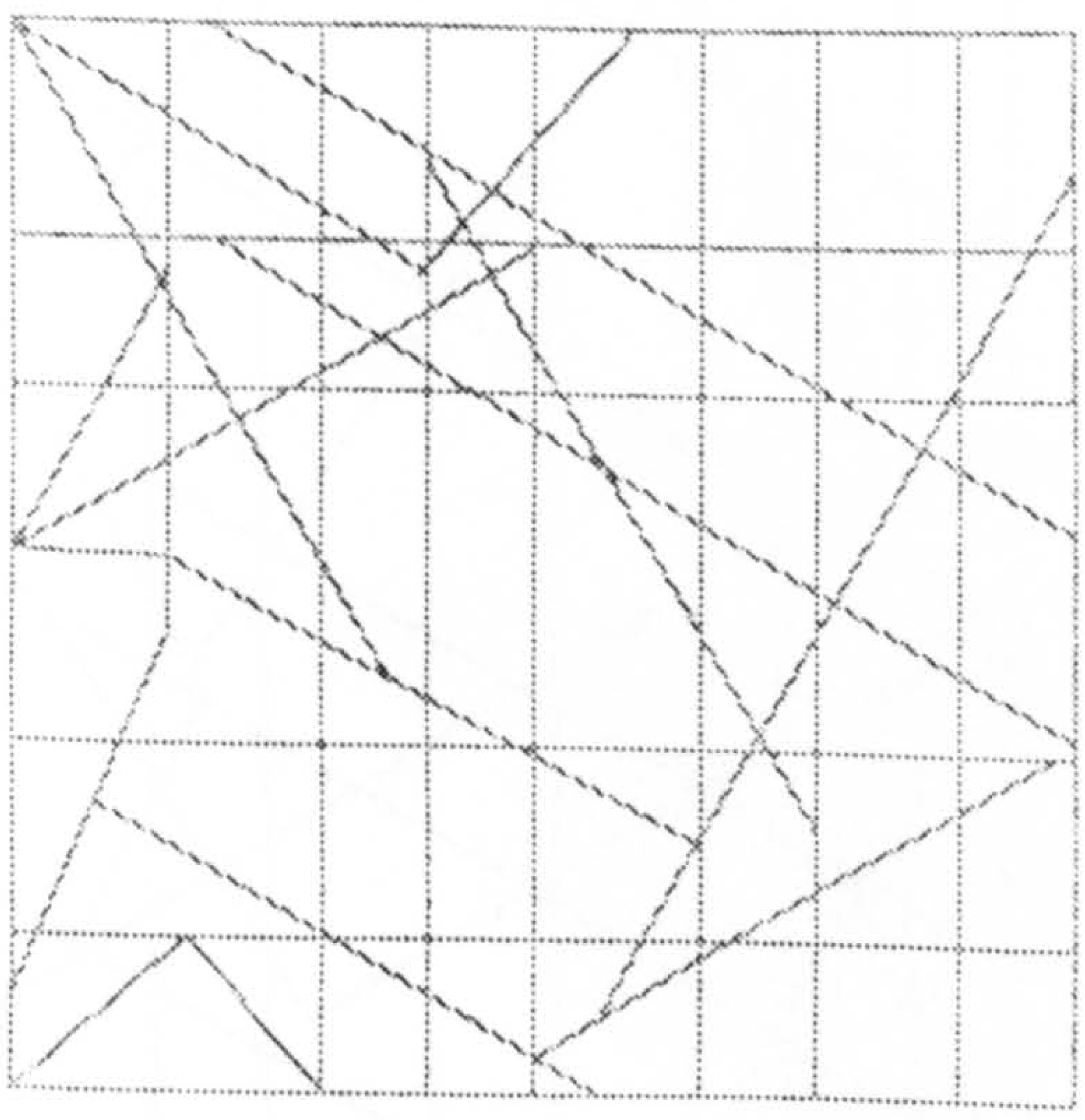


Find SHAPE F

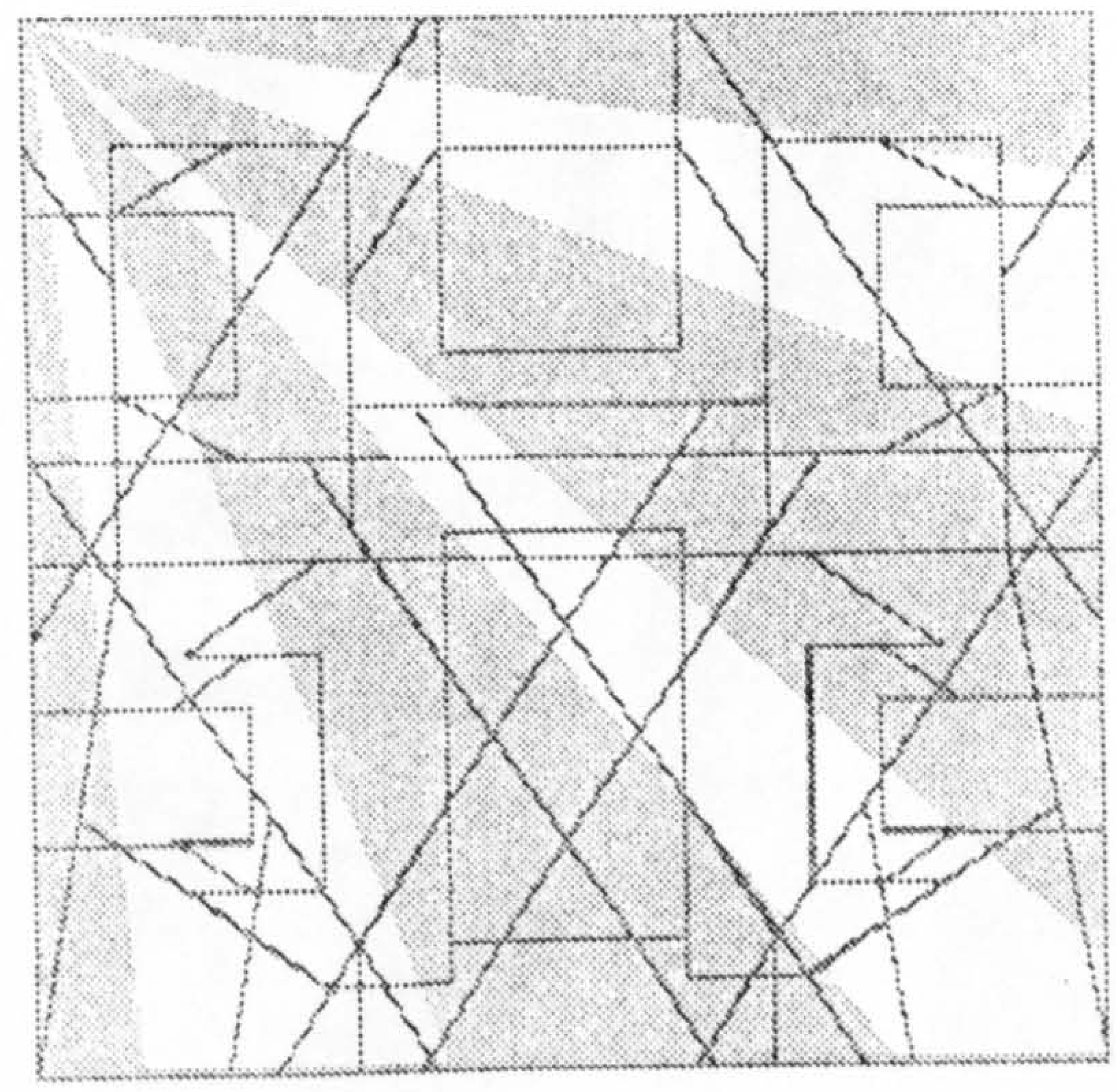


Find SHAPE A

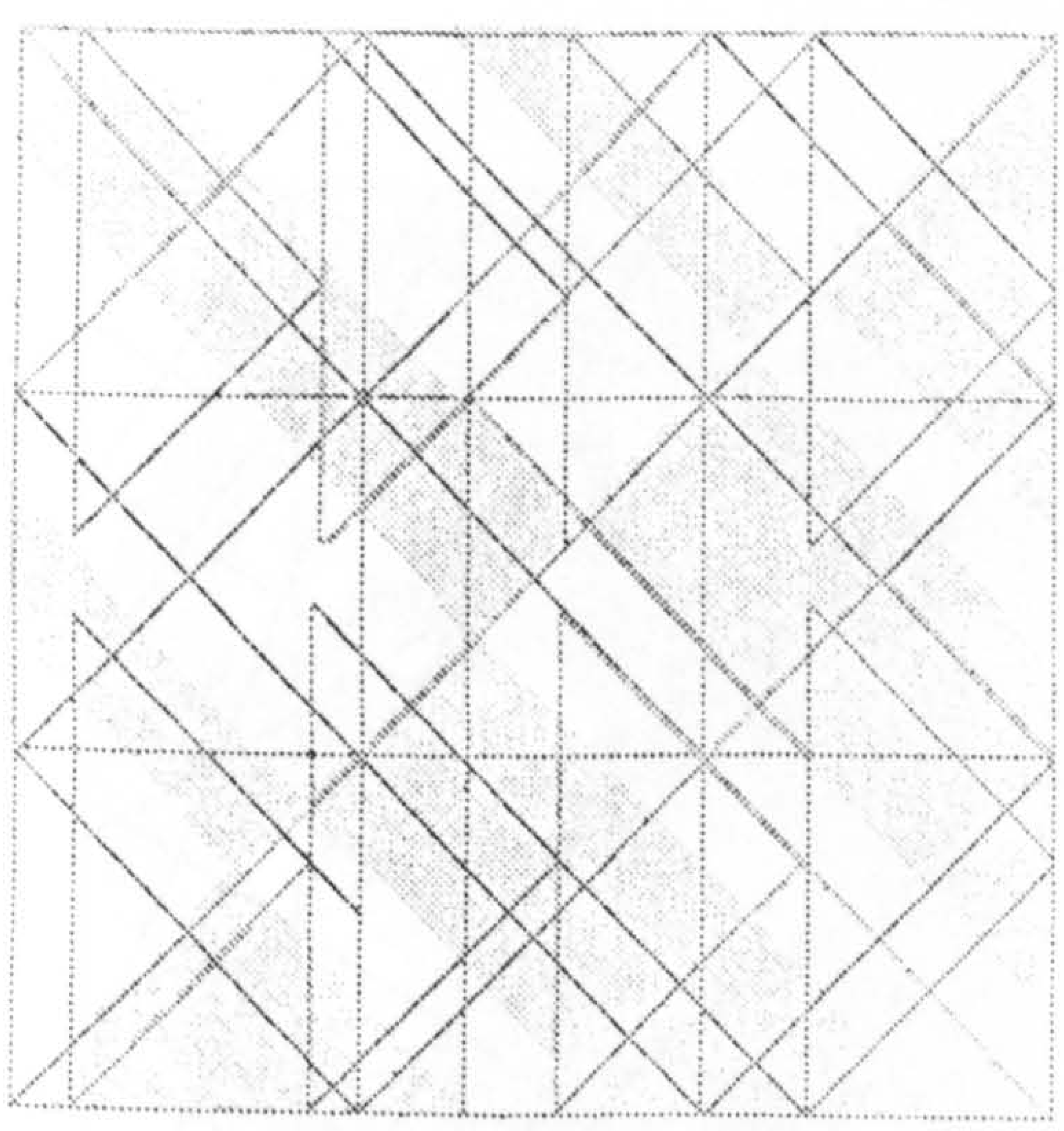
Appendix A (cont'd)



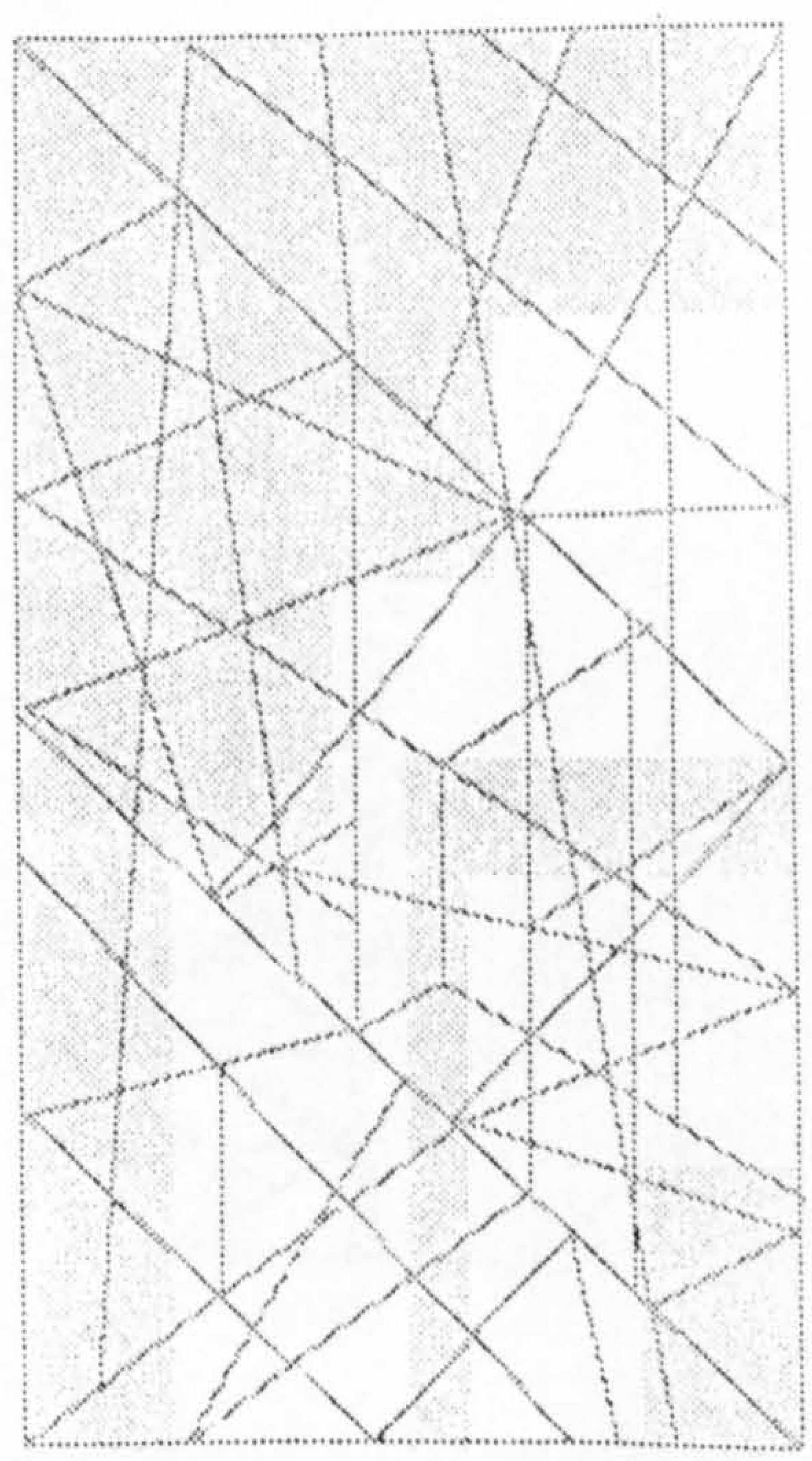
Find SHAPE C



Find SHAPE B

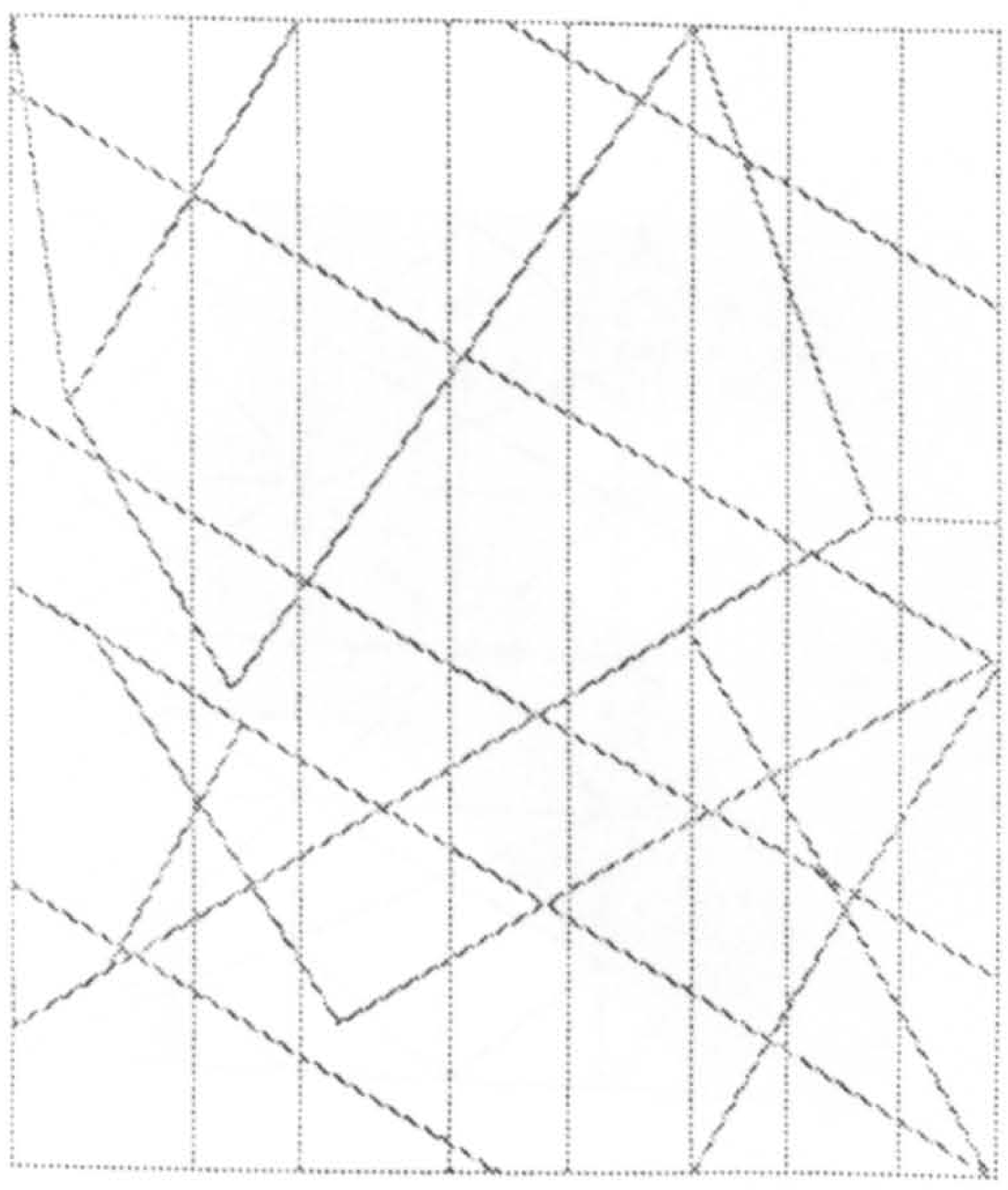


Find SHAPE D

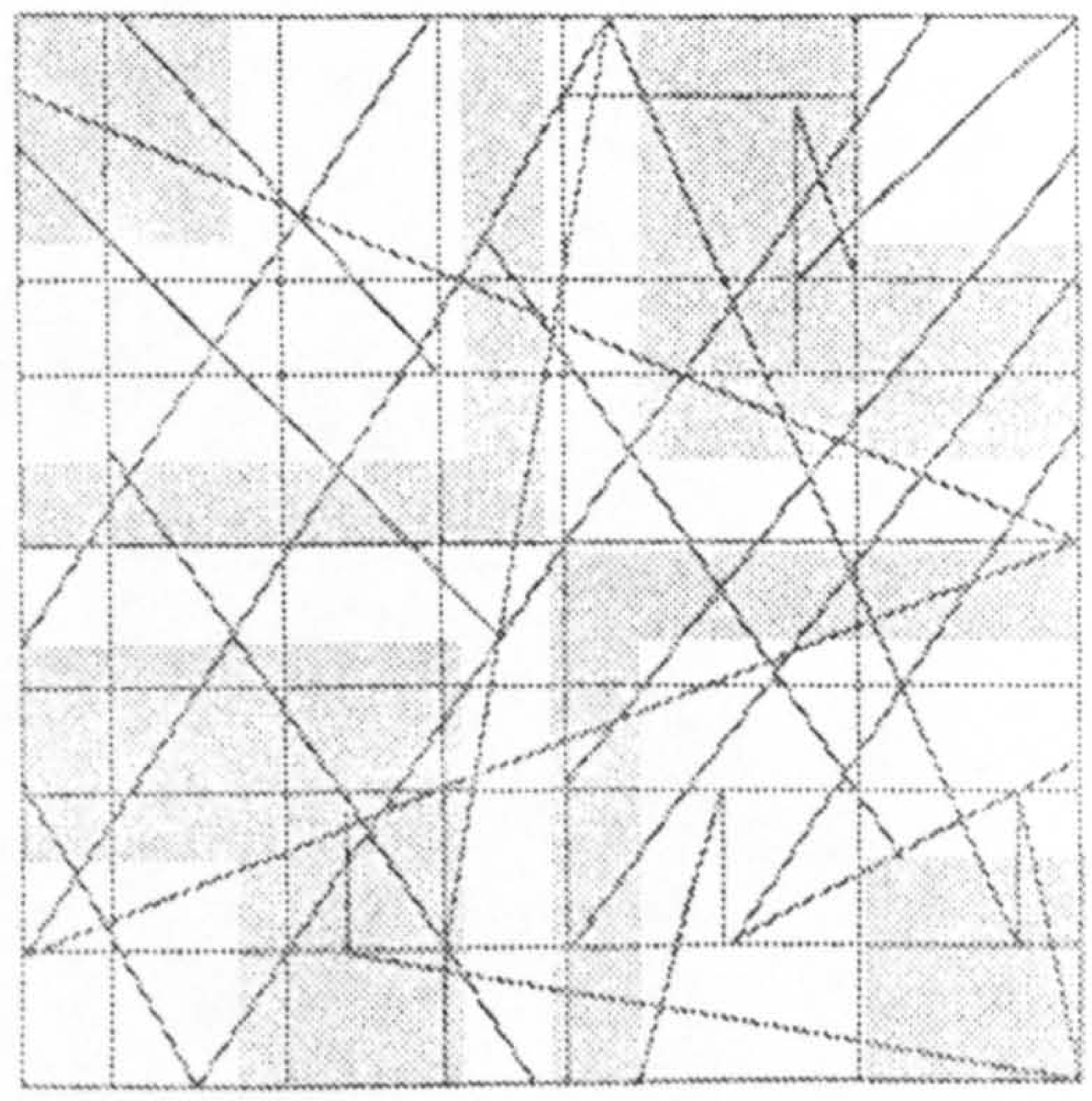


Find SHAPE G

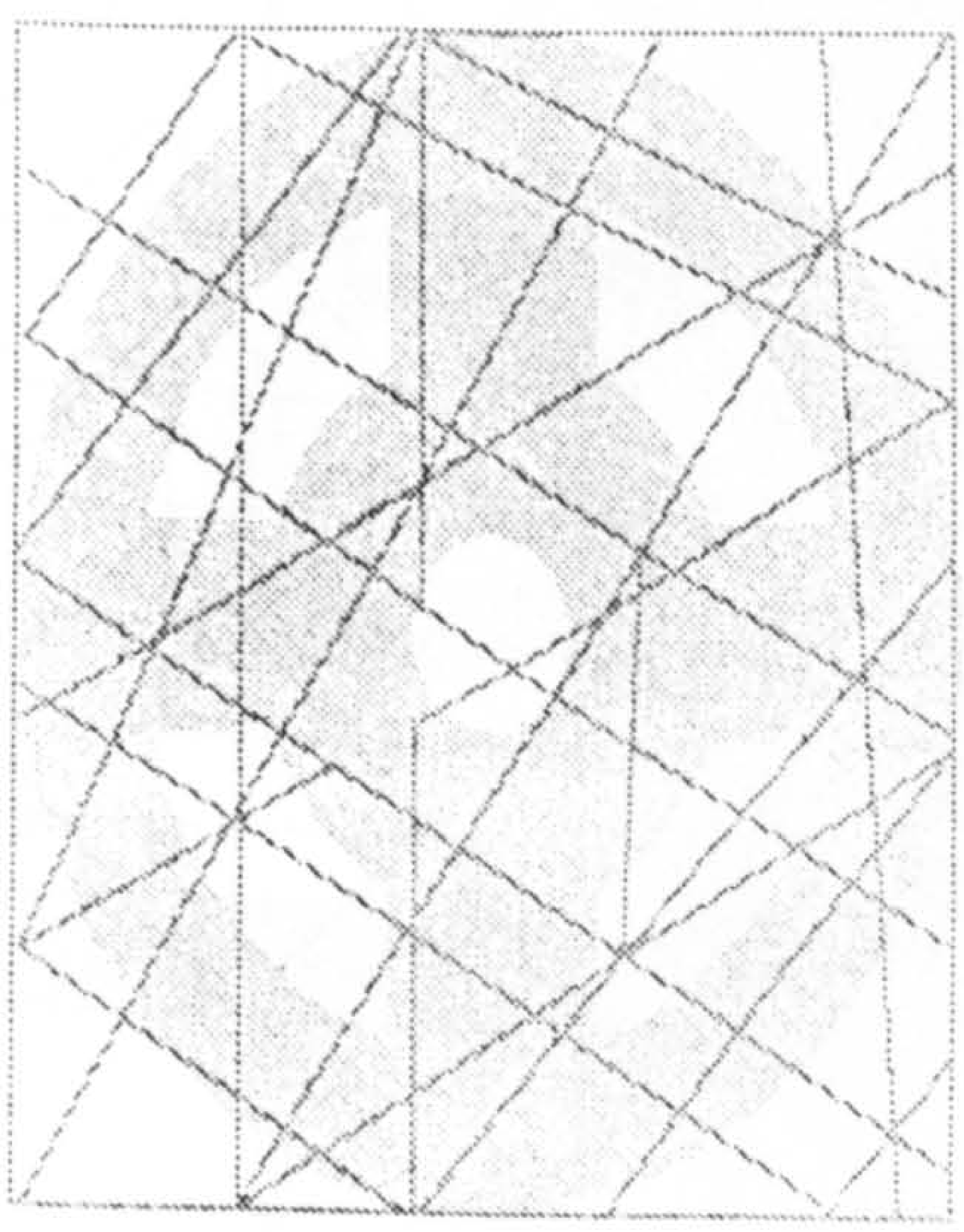
Appendix A (cont'd)



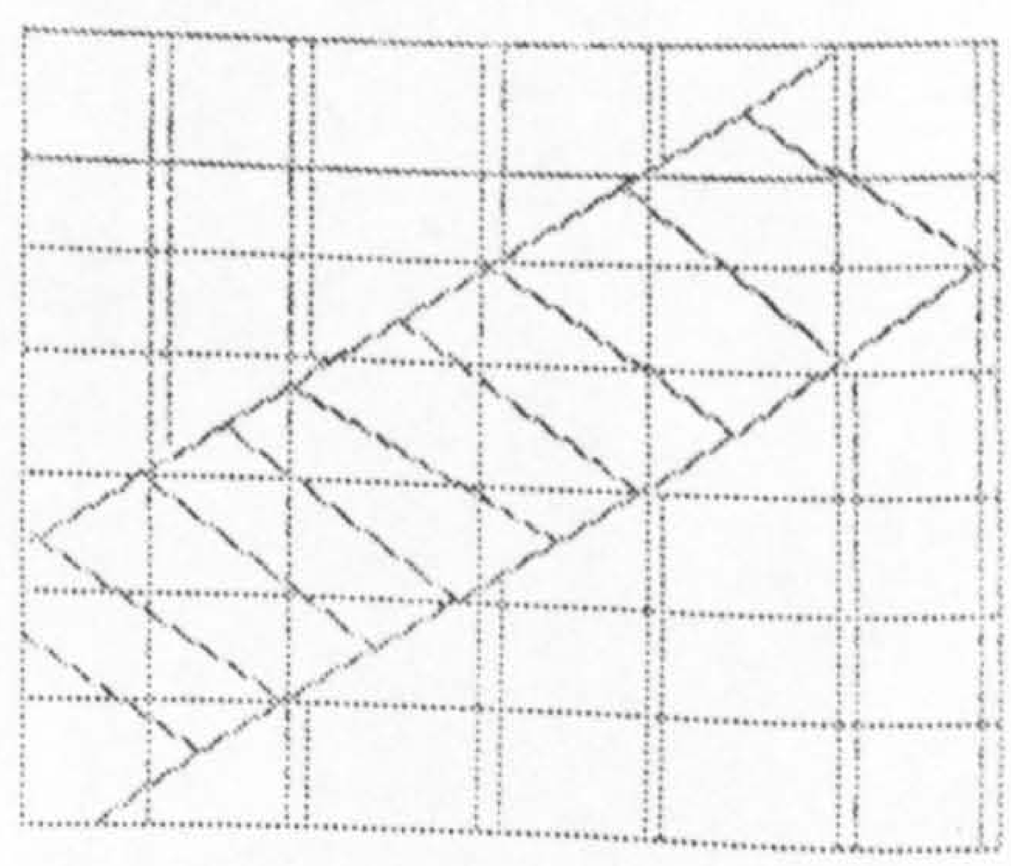
Find SHAPE C



Find SHAPE B

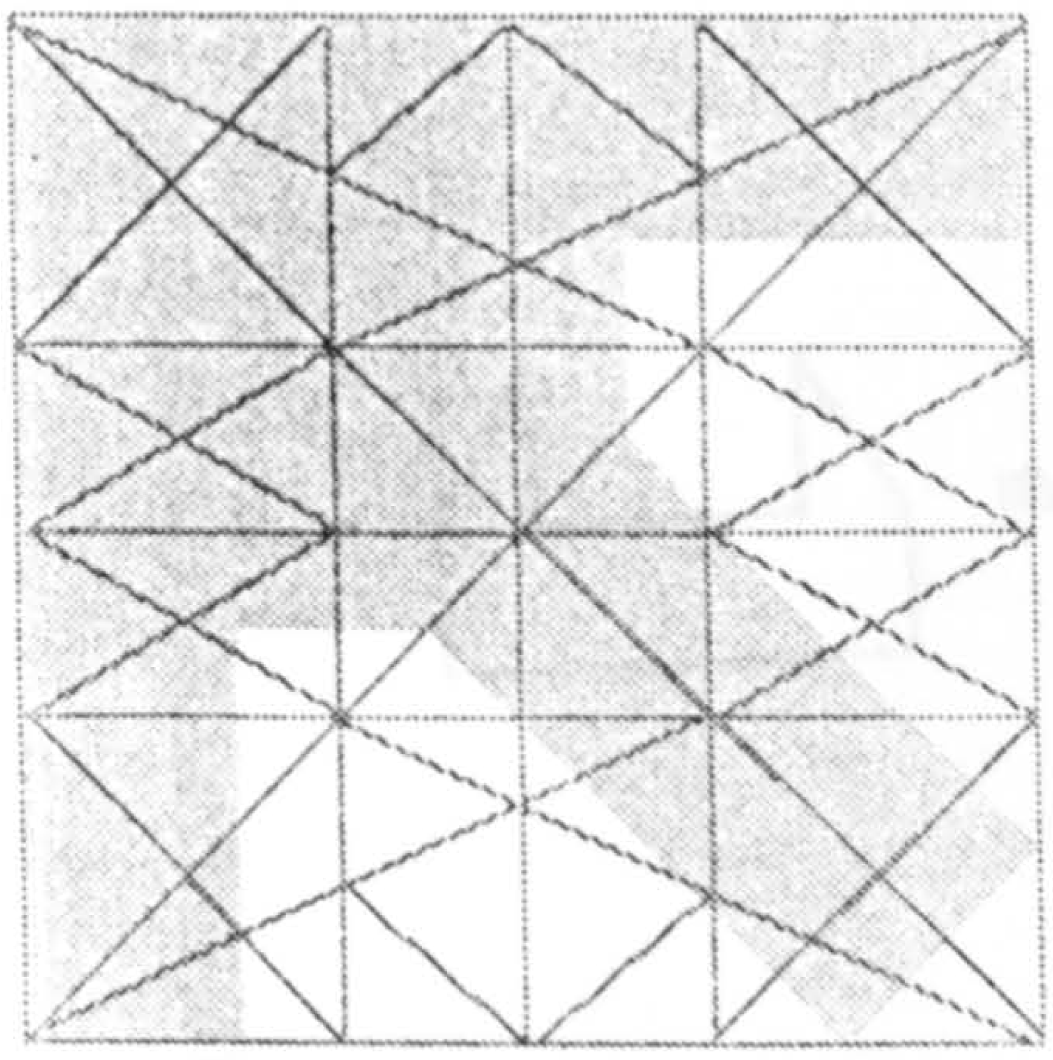


Find SHAPE G

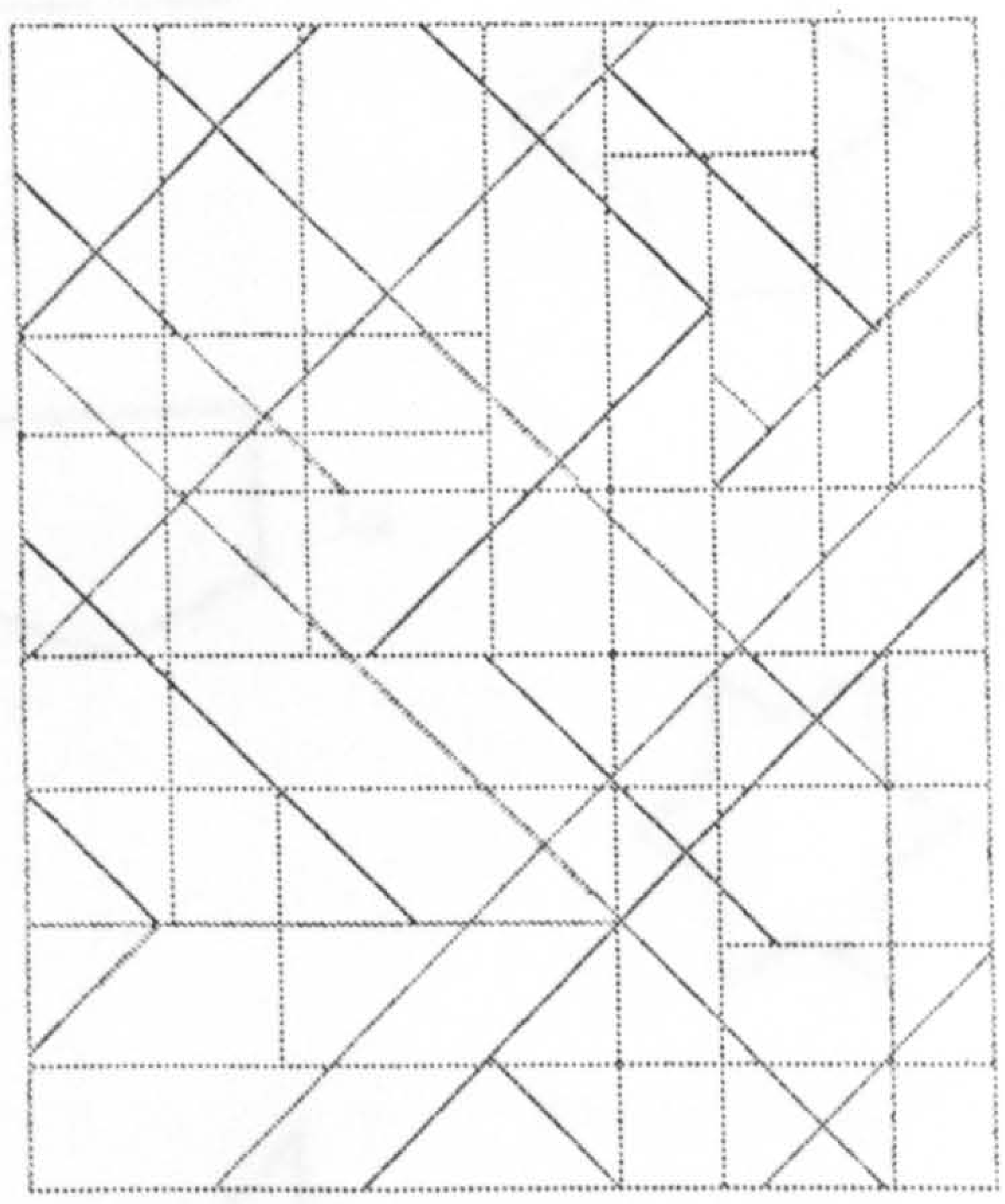


Find SHAPE H

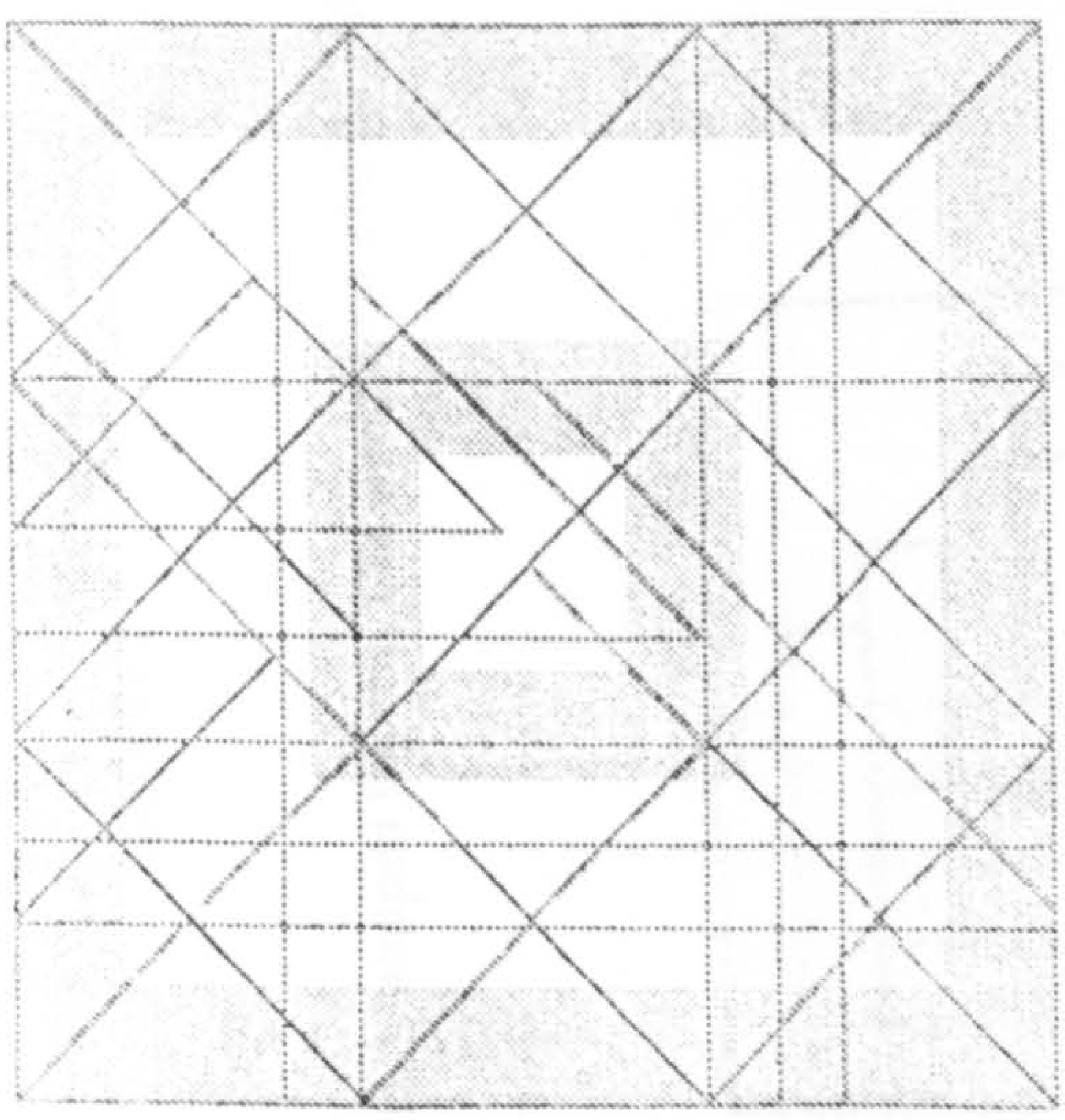
Appendix A (cont'd)



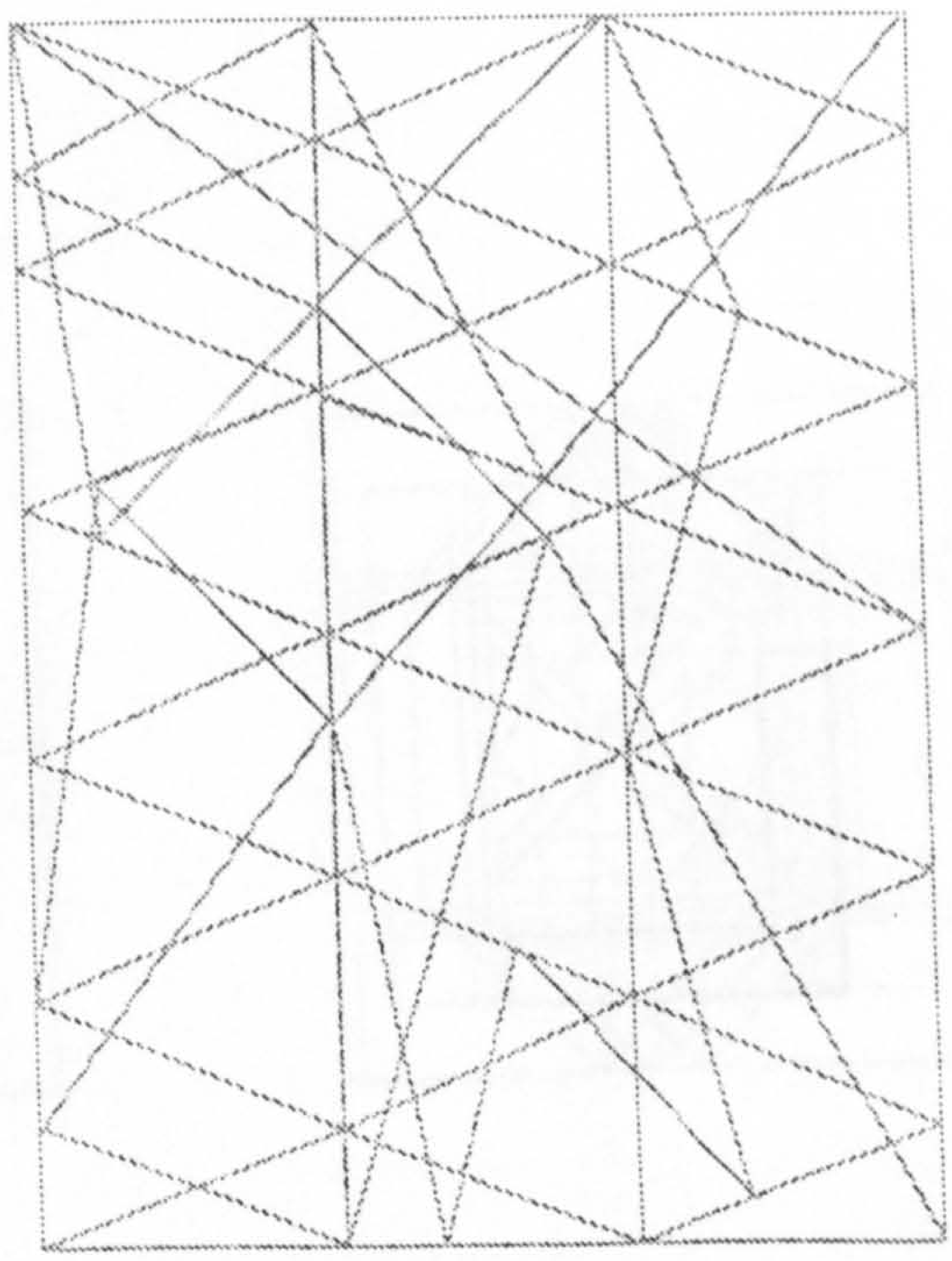
Find SHAPE E



Find SHAPE F



Find SHAPE D

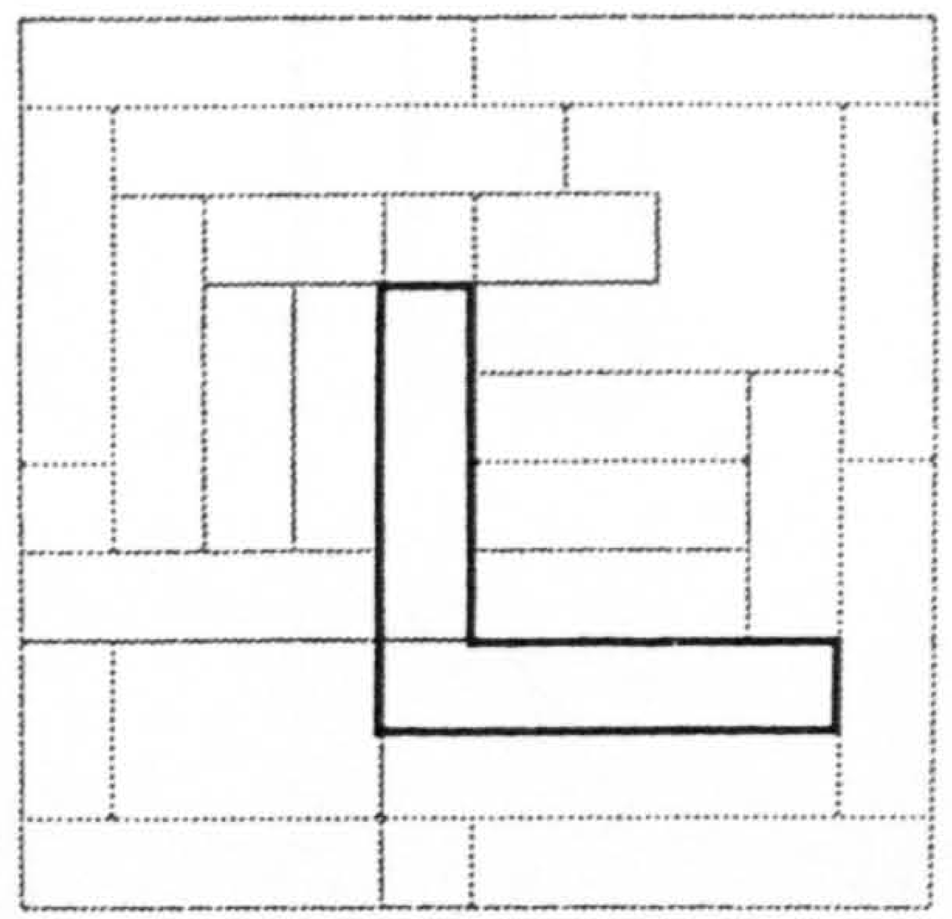


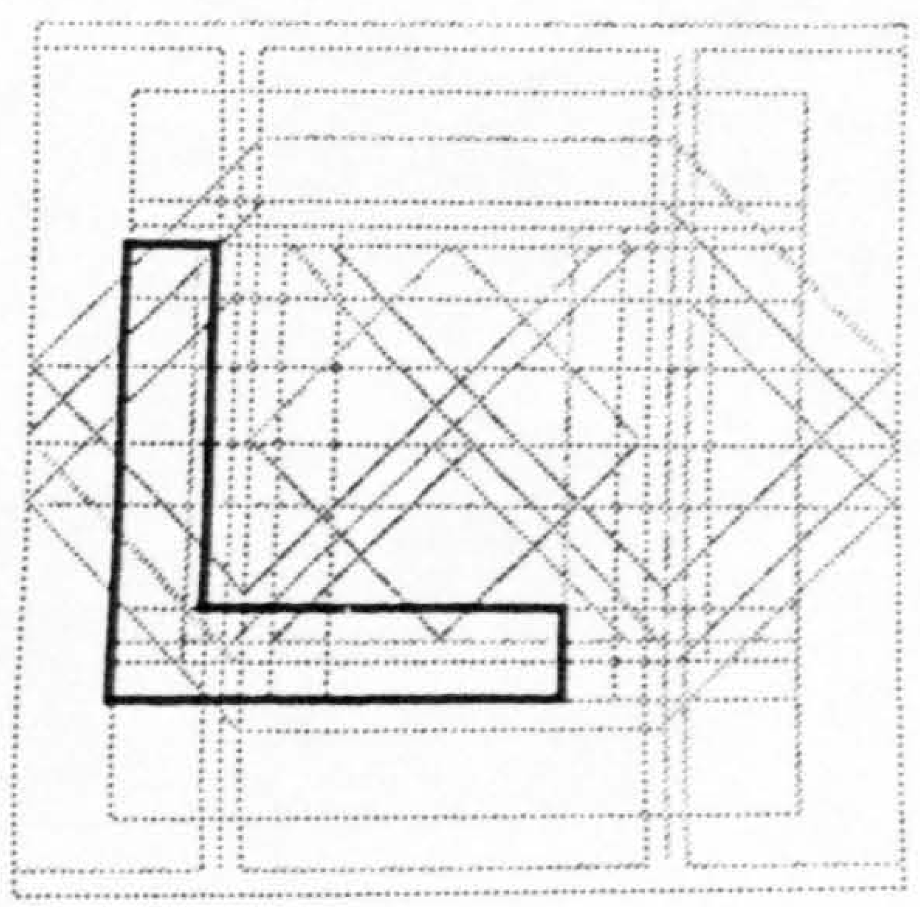
Find SHAPE A

Appendix A (cont'd)

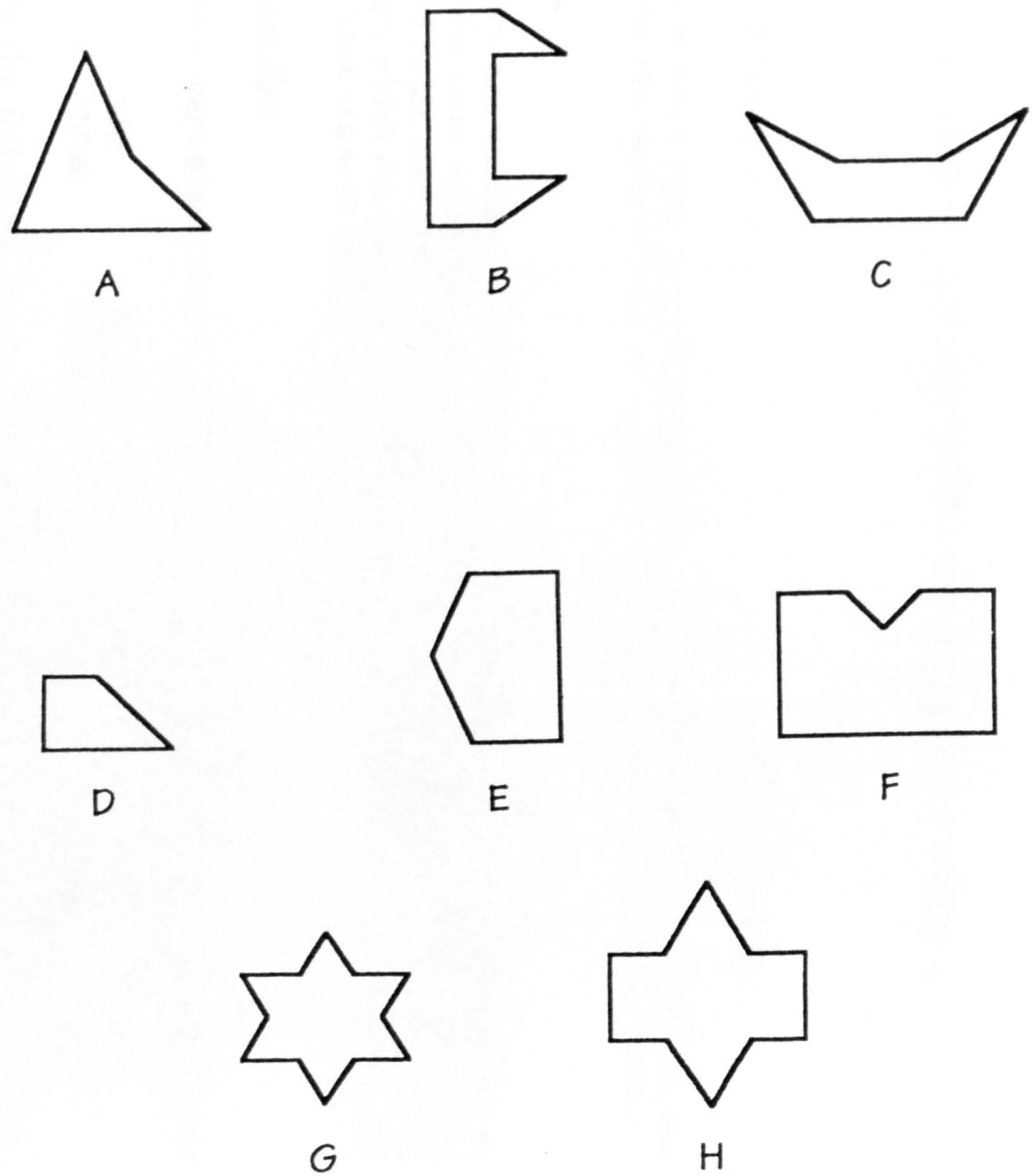
ANSWERS

When you have traced both L-shaped figures, the diagrams should look like this :





THE SHAPES YOU HAVE TO FIND



Glasgow University Science Education Research Group

Appendix B

((THE CONVERGENT / DIVERGENT TEST))

NAME: SEX:

MATRICULATION NO.: DATE OF BIRTH:

These are some tests to measure the way you think. There is no limit to the responses you can give and your answers will not affect any part of your course.

TEST 1

When you are writing, it is often necessary to think of several different words having the same meaning or similar meanings, so that you do not have to repeat one word again and again. In this test you will be asked to think of words having meanings which are the same as or similar to a given word. The given words will be ones that are well known to you.

For example:

If the word were **short** you would write at least some of the words written below:

Short:	brief	abbreviated	concise	compact	little
	limited	deficient	abrupt	petite	crisp

Now try the following words. You probably will not be able to fill in all the spaces, but write as many as you can think of.

1- Strong:

_______________ _______________ _______________ _______________
_______________ _______________ _______________ _______________
_______________ _______________ _______________ _______________
_______________ _______________ _______________ _______________

2- Dark:

_______________ _______________ _______________ _______________
_______________ _______________ _______________ _______________
_______________ _______________ _______________ _______________
_______________ _______________ _______________ _______________

3- Clear:

_______________ _______________ _______________ _______________
_______________ _______________ _______________ _______________
_______________ _______________ _______________ _______________
_______________ _______________ _______________ _______________

5 Minutes

Appendix B (cont'd)

TEST 2

In this test you will be asked to write as many sentences as you can. Each sentence should contain the four special words mentioned and any other words you choose:

For example:

TAKE FEW LAND LITTLE

1- Few crops take little land.
2- A few little boats take supplies to land.
3- Take a few little boys with you to see the green land.

All the four words are used in each sentence. The words must be used in the form that is given; for example, you cannot use "taking" instead of "**take**". Notice that the sentences may be of any length. All sentences must differ from one another by more than merely one or two changed words, such as different pronouns or adjectives.

Now try the following words. Remember to number each new sentence as was done in the example above.

1- WRITE WORDS LONG OFTEN

2- SISTER MAN YEAR CATCH

----

5 Minutes

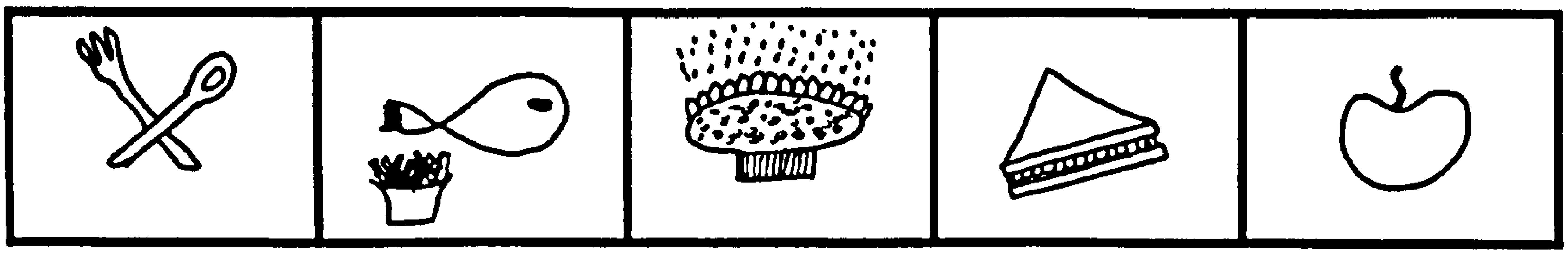
Appendix B (cont'd)

TEST 3

This is a test of your ability to think up a number of different symbols that could be used to stand for certain words or ideas.

For example:

The word is **food**. A sketch has been made to represent a fork and spoon. Can you think of other symbols that could represent food? Draw them in the boxes. Each drawing can be as complicated as you choose.



Now draw as many symbols as you can think of (up to five) for each word or phrase below.

1- Quiet

2- Keep off the grass

3- Happy

4- Post office

5 Minutes

Appendix B (cont'd)

TEST 4

This is a test to see how many things you can think of that are alike in some way.

For example:

What things are always red or that are red more often than any other colour? You may use one word or several words to describe each thing.

tomatoes bricks watermelon

Go ahead and write all the things **that are round or that are round more often than any other shape.**

---------------	---------------	---------------	---------------
---------------	---------------	---------------	---------------
---------------	---------------	---------------	---------------
---------------	---------------	---------------	---------------

3 Minutes

TEST 5

This is a test of your ability to think rapidly of as many words as you can that begin with one letter and end with another.

For example:

The words in the following list all begin with S and end with N.

sun spin stain solution

Now try thinking of words beginning with G and ending with T. Write them on the lines below. **Names of people or places are not allowed.**

---------------	---------------	---------------	---------------
---------------	---------------	---------------	---------------
---------------	---------------	---------------	---------------
---------------	---------------	---------------	---------------

3 Minutes

Appendix B (cont'd)

TEST 6

This is a test to see how many ideas you can think of about a topic. Be sure to list all the ideas you can about a topic whether or not they seem important to you. You are not limited to one word. Instead you may use a word or a phrase to express each idea.

For example:

"A train journey". examples are given below of ideas about a topic like this.

number of miles catching the train the train stations people in the train

Now list all the ideas you can about "crossing the stream".

------------------------------ ------------------------------ ------------------------------

------------------------------ ------------------------------ ------------------------------

------------------------------ ------------------------------ ------------------------------

------------------------------ ------------------------------ ------------------------------

------------------------------ ------------------------------ ------------------------------

4 Minutes

END OF TESTS

Appendix C

باسمه تعالی

آزمون‌های تفکر همگرا یا واگرا

نام و نام خانوادگی : تاریخ تولد :
کلاس و دانشگاه جنسیّت :

آزمونهایی راکه ملاحظه می‌کنید شیوهٔ تفکّر شما را خواهد سنجید. نتایج مورد نظر به هیچ عنوان تأثیری در نمره‌های امتحانی‌تان ندارد و صرفاً در جهت یک کار تحقیقاتی مهم در امر آموزش ریاضی عمومی به کار گرفته خواهد شد.

آزمون ۱

هنگام نگارش در بارهٔ یک موضوع اغلب لازم است به واژه‌های گوناگونی بیندیشید که دارای معنایی واحد یا معانی نزدیک به هم هستند تا از به کارگیری مکرر یک کلمه خودداری شود. در این آزمون از شما خواسته شده است تا به واژه‌های هم معنا یا دارای معانی مشابه فکر کنید در اینجا واژه‌هایی خواهد آمد که شما کاملاً با آنها آشنا هستید.

مثال:

اگر کلمهٔ مورد نظر واژهٔ کوتاه باشد شما می‌توانید حداقل برخی از واژه‌های زیر را بنویسید.

کوتاه: مختصر، مجمل، موجز، زودگذر، کم، محدود، ناقص، سریع الانقطاع، کوچک، اجمالی، فشرده، مخفف

حال سعی کنید واژه‌های هم معنا یا دارای معنای مشابه کلمات زیر را بیابید. احتمالاً نمی‌توانید همهٔ جاهای خالی را پرکنید، ولی تلاش کنید هر تعداد کلمه راکه به ذهنتان می‌رسد، بنویسید.

۱ـ قوی:

....
....
....

۲ـ تاریک:

....
....
....

۳ـ روشن:

....
....
....

(۵ دقیقه)

Appendix C (cont'd)

آزمون ۲

در این آزمون از شما خواسته شده است هر تعداد جمله که می‌توانید بسازید . هر جمله باید شامل ۴ کلمهٔ داده شده و احتمالاً کلمات دیگری باشد که خود برای تکمیل جمله انتخاب خواهید کرد .

مثال :   زمین   کوچک   کمی   می‌بردند

۱ ـ تعداد کمی قایق کوچک تدارکات لازم را به زمین دشمن می‌بردند.
۲ ـ مربیان مدرسه ماه گذشته همه روزه شمار کمی از پسران کوچک دبستان را برای بازی به زمین سبز زیبا می‌بردند .
۳ ـ دیروز خودروهای ادارهٔ کشاورزی تعداد کمی از درختان کوچک را به طرف زمین شما می‌بردند

هر چهار واژه بالا باید در هر جمله با همان شکل داده شد به کار برده شود .
مثلاً به جای کلمهٔ " می‌بردند " از واژه‌هایی مانند "برده" "می‌برد" یا "بردند" استفاده نشود .
توجه داشته باشید که طولانی شدن جمله‌ها مسأله‌ای نیست امّا آنها باید با یکدیگر در بیش از یک یا دو کلمه مانند صفات یا ضمایر متفاوت باشند .
اینک با کلمه‌های زیر جمله‌های مورد نظر خود را بسازید و هر جمله را مانند مثال‌های بالا با شمارهٔ خاصی مشخص نمایید .

الف :

می‌نویسد   طولانی   اغلب   کلمات

....

....

....

....

....

ب :

خواهر   مرد   سال   گرفت

....

....

....

....

....

(۵ دقیقه)

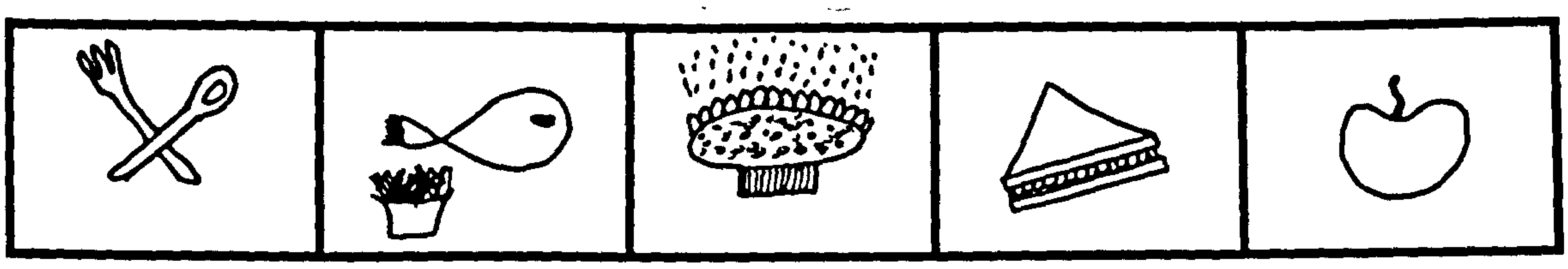
Appendix C (cont'd)

آزمون ۳

آزمون زیر توانایی شما را در ابداع و طراحی نمادهای متنوعی می‌سنجد که این نمادها، نشانگر واژه‌ها یا مفاهیم هستند.

مثال:

چنانچه کلمه مورد نظر غذا باشد طرح کلی ارائه شده می‌تواند نمایشگر یک قاشق و چنگال باشد. آیا می‌توانید درباره شکلهای نمادین دیگری بیندیشید که به غذا اشاره داشته باشد؟ شکلهای مورد نظرمی‌توانند به قدر کافی پیچیده باشند.



حال برای هر واژه یا عبارت زیر نمادهای مربوط را رسم کنید. (حداکثر تا ۵ نماد)

۱ - سکوت

۲ - حمل و نقل

۳ - خوشحالی

۴ - ادارهٔ پست

(۵ دقیقه)

Appendix C (cont'd)

آزمون ۴

این آزمون در واقع تواناییهای شما رادرباره شناخت اشیایی می‌سنجد که به طریقی با یکدیگر شباهت دارند .

مثال :

چه اشیایی همیشه یا اغلب قرمز رنگ هستند ؟ شما ممکن است واژه یا واژه‌هایی را برای توصیف هر یک از اشیای مورد نظر خود به کار برید .

گوجه‌فرنگی　　　انار　　　هندوانه

اکنون این کار را ادامه داده و نام اشیایی را بنویسید که گرد هستند یا لااقل گردتر از اشیاء دیگر می‌باشند .

....

....

....

....

....

(۳ دقیقه)

Appendix C (cont'd)

آزمون ۵

این آزمون سرعت تفکر و انتقال شما را در مورد یادآوری کلماتی می‌سنجد که با حرف خاصی شروع شده و به حرف معین دیگری ختم می‌گردد.

به عنوان مثال:

واژه‌های زیر هر کدام با حرف "گ" شروع شده و به حرف "ش" ختم می‌گردد.

گوش　　گزارش　　گنجایش　　گردش

اینک به کلماتی بیندیشید که با حرف "س" شروع شده و به حرف "ن" ختم گردد.
آنها را روی خطهای زیر بنویسید

توجّه! *نام اشخاص و مکانها مورد قبول نمی‌باشد.*

....	....	....	....
....	....	....	....
....	....	....	....
....	....	....	....
....	....	....	....

(۳ دقیقه)

Appendix C (cont'd)

آزمون ٦

در این آزمون می‌خواهیم بدانیم که وقتی دربارهٔ موضوعی فکر می‌کنید ، چه ایده‌هایی به ذهنتان می‌رسد ؟
اطمینان حاصل کنید که همهٔ ایده‌های خود را دربارهٔ موضوع مورد بحث ، اعم از این که این ایده‌ها برای شما مهم باشند یا نباشند ، یادداشت کرده باشید . محدودیتی در تعداد کلمات وجود ندارد .
بنابراین می‌توانید از یک کلمه یا یک عبارت که بیانگر نقطه نظرات‌تان باشد استفاده نمائید .

مثال :

" مسافرت با قطار "

نمونه‌هایی از ایده‌هایی که در این مورد به ذهن می‌رسد در زیر آمده است .

مسافت به کیلومتر　　سوارشدن به قطار　　ایستگاههای قطار　　مسافران قطار

اکنون تمام ایده‌هایی راکه در رابطه با *" عبور از رودخانه "* به ذهن شما می‌رسد ، بنویسید .

....　　................................　　................................

....　　................................　　................................

....　　................................　　................................

....　　................................　　................................

....　　................................　　................................

(۴ دقیقه)

پایان آزمونها

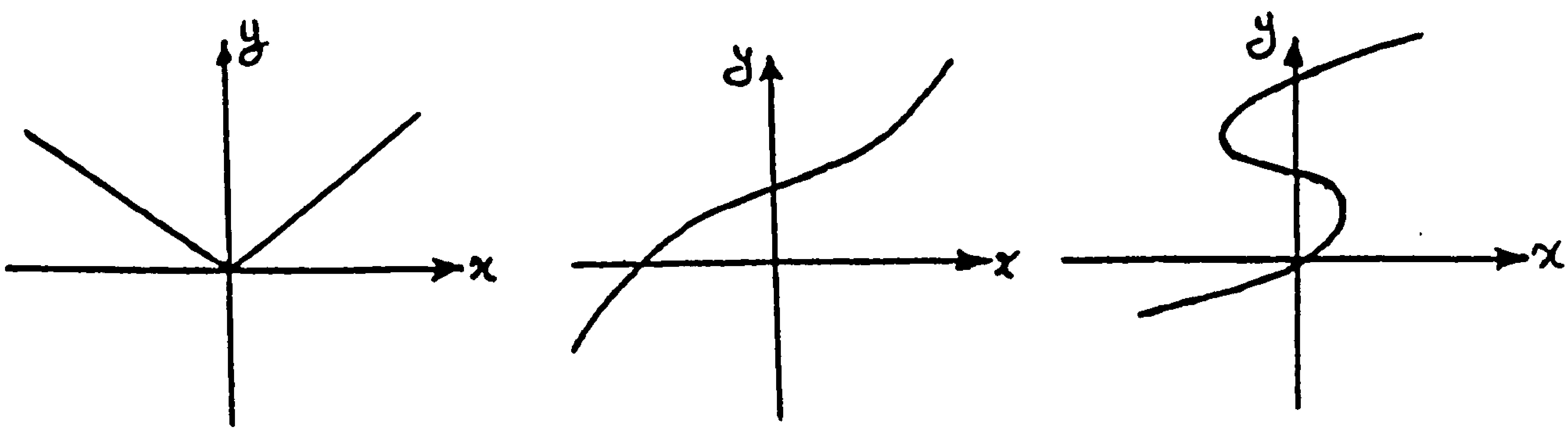
Appendix D

Aban, 1373 (November 1994)

Sabzevar University

Calculus Examination (1)

1. Which of the following curves are a graph of a function?



2. Determine the domain and range of the given functions as follows:

a. $y = \frac{(x^2+3x-4)(x^2-5x+6)}{(x^2-3x+2)(x-3)}$.

b. $y = \frac{[x^2]}{|x|}$.

c. $y = \sqrt{6x^2 - 5x - 4}$.

d. $y = \frac{1}{\sin(x)\cos(x)}$.

3. Let function f be defined by $f(x) = x[2x+1] + |x-1|$. Draw a sketch of the graph of f on $(0,2)$.

4. Let $f(x) = g(x)$ for all x except for $x = a$ and suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Prove that

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x).$$

5. Use the (ϵ, δ)-definition to prove that

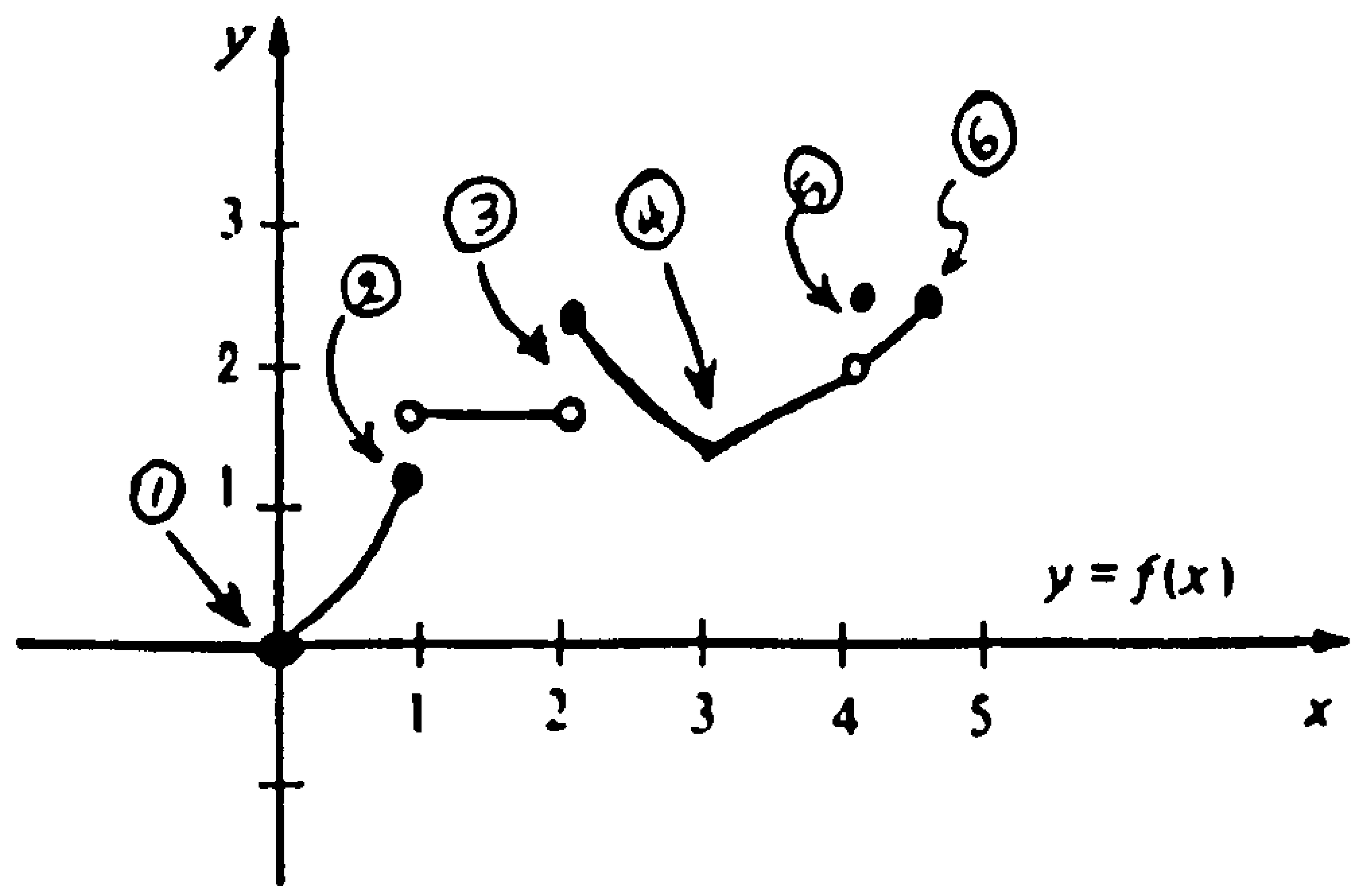
$$\lim_{x\to 1} \frac{2}{3x^2-1} = 1.$$

Appendix D (cont'd)

Azar, 1373 (December 1994)

Sabzevar University
Calculus Examination 2

1. For the function $y = f(x)$ in the following figure with domain $[0, 4.5]$, discuss the existence of the following items at the given points on the curve:
   a. left-hand limit,
   b. right-hand limit,
   c. limit of the function,
   d. right-hand continuity,
   e. left-hand continuity,
   f. continuity of the function.



2. Evaluate:
   a. $\lim\limits_{x\to 1} \dfrac{x^1 + x^2 + \cdots + x^n - n}{x - 1}$,
   b. $\lim\limits_{t\to 0} \dfrac{\sqrt[3]{(t+a)^2} - \sqrt[3]{a^2}}{t}$.

3. Let M be a constant and suppose that, for all x, $|f(x)| \le M$ and $\lim\limits_{x\to a} |g(x)| = 0$. Then $\lim\limits_{x\to a} f(x)g(x) = 0$. By use of this theorem, evaluate the

$$\lim_{x\to\infty} \frac{x - \sin x}{x + \sin x}.$$

4. State and prove the squeeze theorem and then evaluate

$$\lim_{x\to 0} x\left[\frac{1}{x}\right].$$

Appendix D (cont'd)

Day, 1373 (January 1995)

Sabzevar University
Calculus Final Examination

1. Use the (ϵ, δ)-definition to prove the following:

a. $\lim_{x\to 0} \frac{1+x^2}{1-x} = 1$,

b. $\lim_{x\to 0} x[1/x] = 1$.

2. Evaluate:

a. $\lim_{x\to 1} \frac{\sin \pi x^\alpha}{\sin \pi x^\beta}$, when $(\alpha, \beta \in N)$,

b. $\lim_{x\to 2} \frac{x[x]}{1+x^2}$.

3. Let f be a differentiable function at a. Prove that

$$f'(a) = \lim_{\Delta x\to 0} \frac{f(a+\Delta x) - f(a-\Delta x)}{2\Delta x}.$$

4. Discuss whether the function $f(x)$ is continuous or differentiable at $x = 0$:

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

5. Let f be continuous on $[a, b]$ and differentiable on (a, b) and suppose that, for all $x \in (a, b)$, $f'(x) < 0$. Then prove that f is decreasing on $[a, b]$.

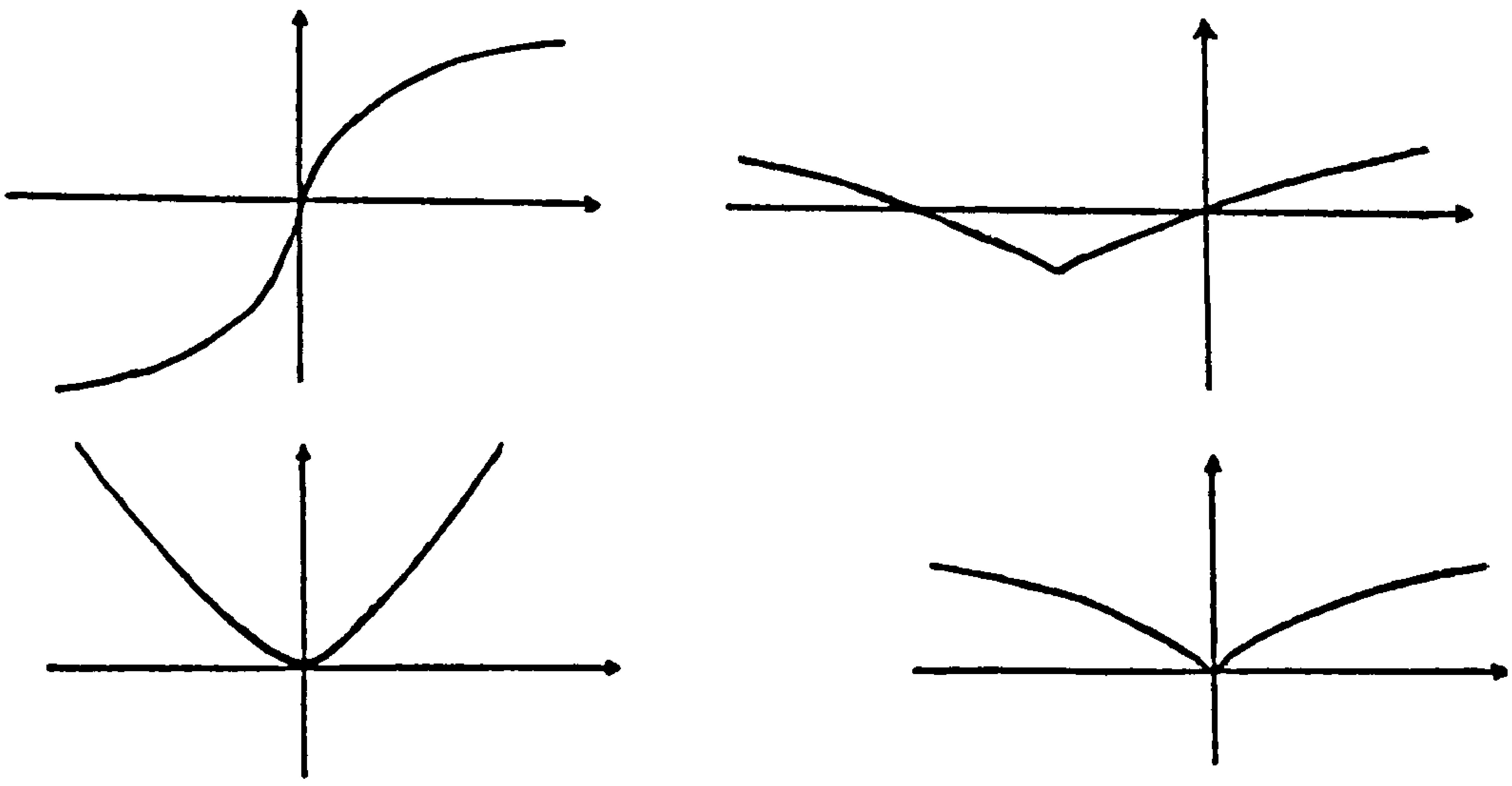
6. A wire of length L is available for making a circle and a square. How should the wire be divided between the two shapes to minimize the sum of the enclosed areas?

Appendix D (cont'd)

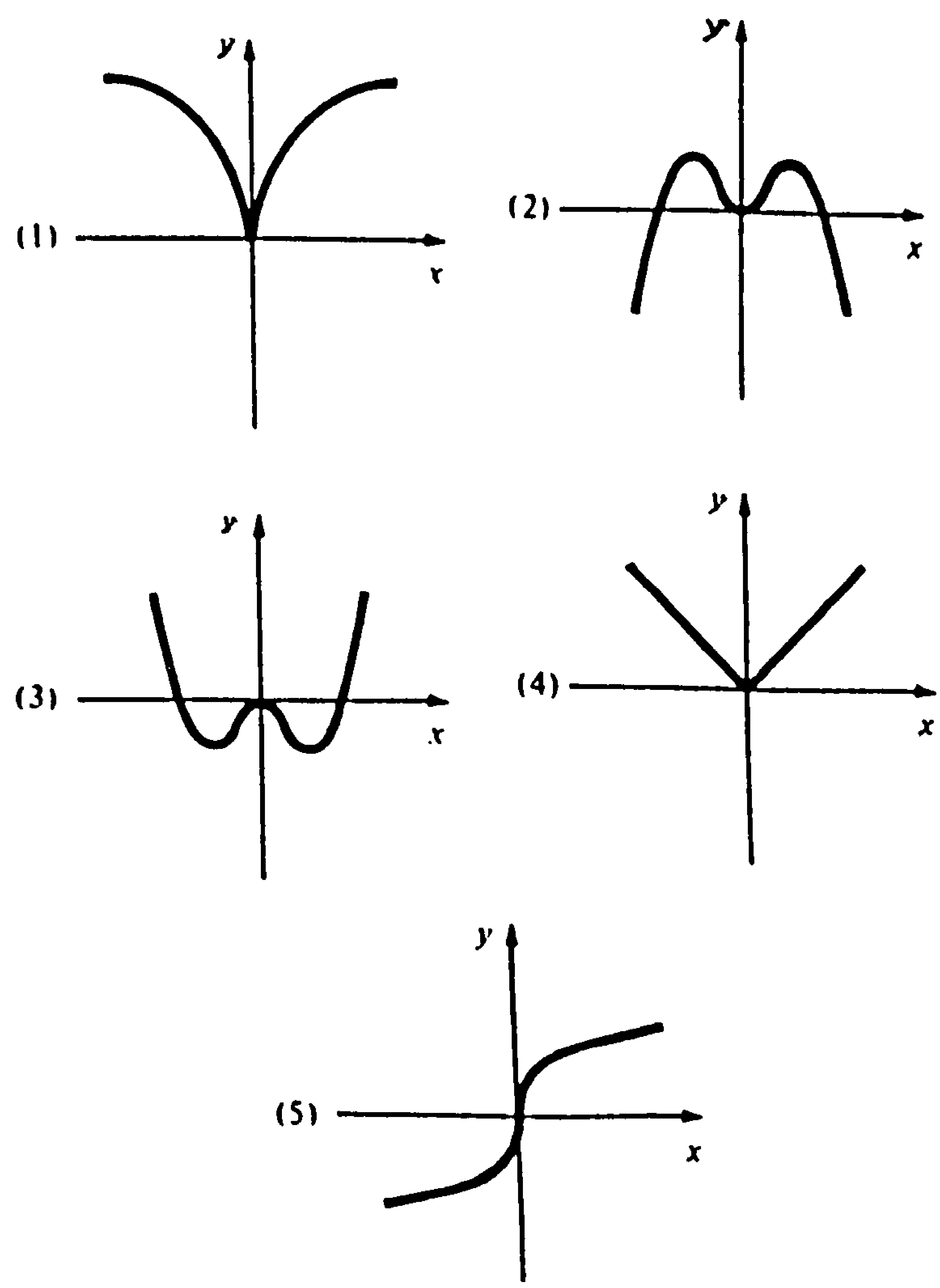
Day, 1373 (January 1995)

Sabzevar University
Calculus Final Exam, Physics group

1. Match the rules of functions: a. $x^{2/7}$; b. $x^{3/7}$; c. $(1+x)^{2/7}-1$; d. $(1+x^2)^{1/7}-1$ to the graphs as follows:



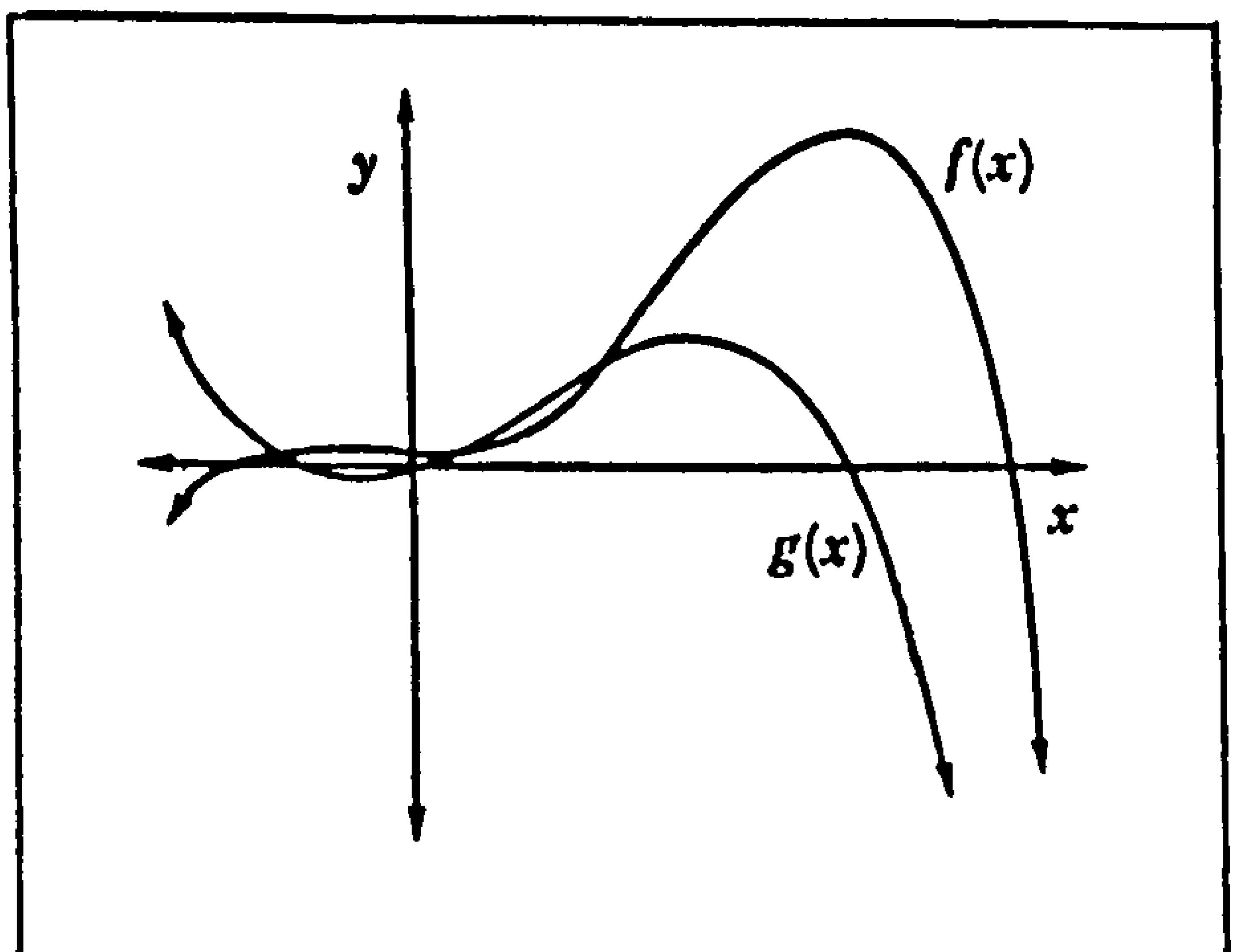
2. Sketch the derivatives whose functions are shown as follows:



3. (i) Discuss the right-left hand continuity of the function $y = \text{sgn x}$ at $x = 0$ and then prove that $|x| = x\text{sgn x}$.

(ii) By use of part (i), evaluate $\lim\limits_{x \to -\infty} \dfrac{[2x^2] + \text{sgn x}}{x^2 + |x|}$.

4. In the following figure, which curve is the function and which one is the derivative?



Which is the function and which is the derivative?

5. Differentiate

a. $f(x) = \sqrt{\frac{\sqrt[3]{x}}{\sqrt{3x^2+1+x}}}$

b. $f(x) = \frac{x^4 - \tan x^2}{x^2+1}$.

6. Let $f(u) = u^2 + 5u + 5$ and $g(x) = \frac{x+1}{x-1}$. Find the derivative of $h = f \circ g$ by:

(i) finding the rule of function h and the determining $(f \circ g)'(x)$.

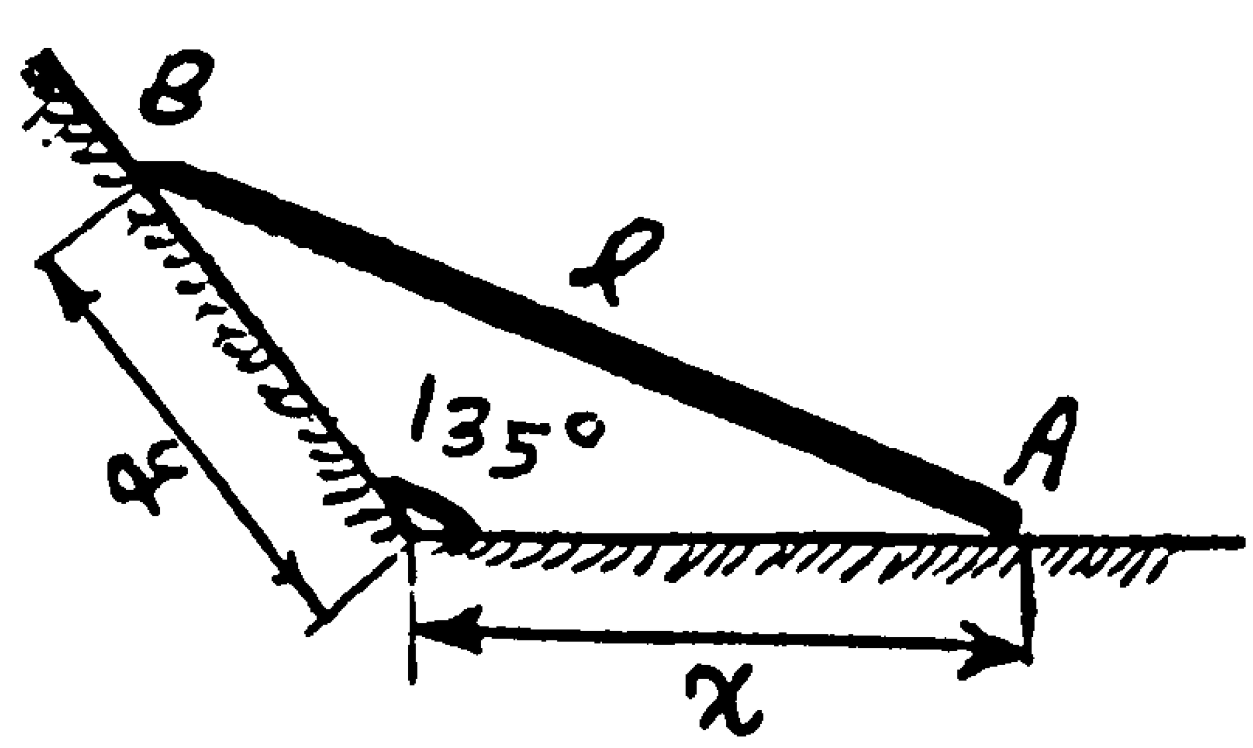
(ii) using the chain rule.

7. State Rolle's theorem and use it for the function

$$f(x) = \begin{cases} x^2 \cos \frac{1}{x} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0 \end{cases}$$

on $[-2/\pi, 2/\pi]$ to find a suitable value of $C \in (-2/\pi, 2/\pi)$ satisfying the result of Rolle's theorem.

8. Find the velocity and acceleration of the particle B in terms of the velocity and acceleration of the particle A in the following figure:



Appendix D (cont'd)

Farvardin, 1374 (April 1995).

Sabzevar University

Calculus 1, Exam (1)

1. (i) If $y = t - t^3$ and $x = t - t^2$, evaluate $\frac{d^2y}{dx^2}$ at the point $t = 1$.

(ii) Evaluate $\frac{d\sqrt{x^2+16}}{d(x/x-1)}$ at $x = 3$.

(iii) Prove that, for $t \in (0, \pi/2)$, $t\sec^2 t - \tan t > 0$ and deduce that $f(x) = \tan x/x$ is increasing on $(0, \pi/2)$.

2. (i) We wish to make a garden with the shape of a sector of a circle with central angle θ and radius r. Find r and θ to minimize the circumference of the sector, given that the area of the sector is constant.

(ii) Determine the absolute and relative extrema of the function $y = x - [x]$ if they exist, and find the set of discontinuities of y and its range.

3. (i) Let f and its first and second derivative be continuous in a neighbourhood of the point a and let $L_a(x)$ be a linear approximation of f at a. Prove, without using L'Hôpital's rule, that

$$\lim_{x\to a} \frac{f(x) - L_a(x)}{(x-a)^2} = \lim_{x\to a} \frac{f'(x) - f'(a)}{2(x-a)}.$$

(ii) Let functions f and g be such that:

a. $\forall x > 0$, $f'(x) = 1/x$ and $\lim_{x\to\infty} f(x) = \infty$,

b. $\forall x > 0$, $g'(x) = g(x)\left((1 - 2f(x))/x^3\right)$ and $\lim_{x\to\infty} g(x) = 1$.

By using a. and b., evaluate the following limits if they exist:

(i) $\lim_{x\to\infty} f(x)/x^2$,

(ii) $\lim_{x\to\infty} x^2 f(x)/(g(x) - 1)$.

4. (i) A function f is uniformly continuous on the interval I if $I \subseteq D_f$ and, $\forall \epsilon > 0, \exists \delta > 0$ such that, $\forall x, x' \in I$, $|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \epsilon$.

Prove that $f(x) = \sin x$ is uniformly continuous on the set of real numbers R.

(ii) Prove that if $f'(x) > 0$ on the interval I, then f is increasing on I.

Appendix D (cont'd)

74/3/1 (May 1995)

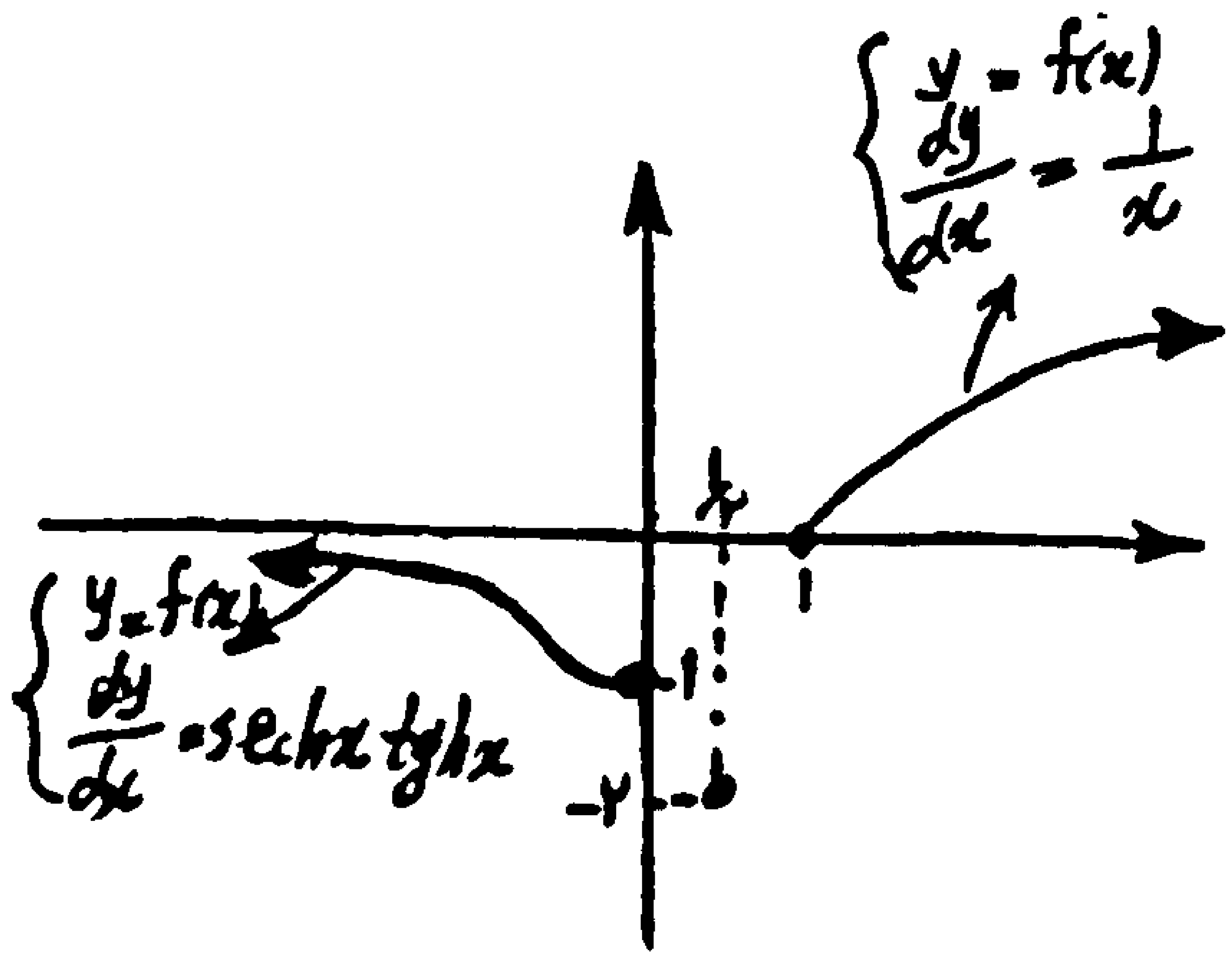
Sabzevar University
Calculus 1, Exam (2)

Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Investigate differentiability of f at $x = 0$ and find the asymptotes of the graph of the function if they exist.

2. By use of the following figure, which shows the graph of the function f:



(i) find the rule of the function f,

(ii) determine if the given function has an inverse.
If the inverse function exists:

a. find the rule of the function $f^{-1}(x)$.

b. draw a sketch of the graph of the function $f^{-1}(x)$.

3. Evaluate:

a. $\lim_{x \to \infty} \frac{e^x - x}{e^{2x} + 2^x}$,

b. $\lim_{x \to \infty} x^{\left(\frac{1}{1-x}\right)}$.

c. $\lim_{n \to \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{4n}\right)$.

4. Evaluate:

(i) $\int \frac{xe^x}{(x+1)^2}\,dx.$

(ii) $\int \frac{dx}{e^x(2+e^{3x})^{5/3}}.$

(iii) $\int \frac{\sin^2 x}{\sin x + 2\cos x}\,dx.$

5. Draw a sketch of the graph of the function

$$f(x) = \begin{cases} 1 + x\operatorname{sgn}(\ln x + 1) & \text{if} \quad x > 0, \\ \tan^{-1} x & \text{if} \quad x \leq 0. \end{cases}$$

Tir, 1374 (July 1995)

Sabzevar University

Calculus 1, Exam 3

1. Let $y = f(x)$ be a function differentiable at $x = a$. Explain the geometric interpretation of Δy and dy and discuss the difference between them.

2. State and prove the mean value theorem.

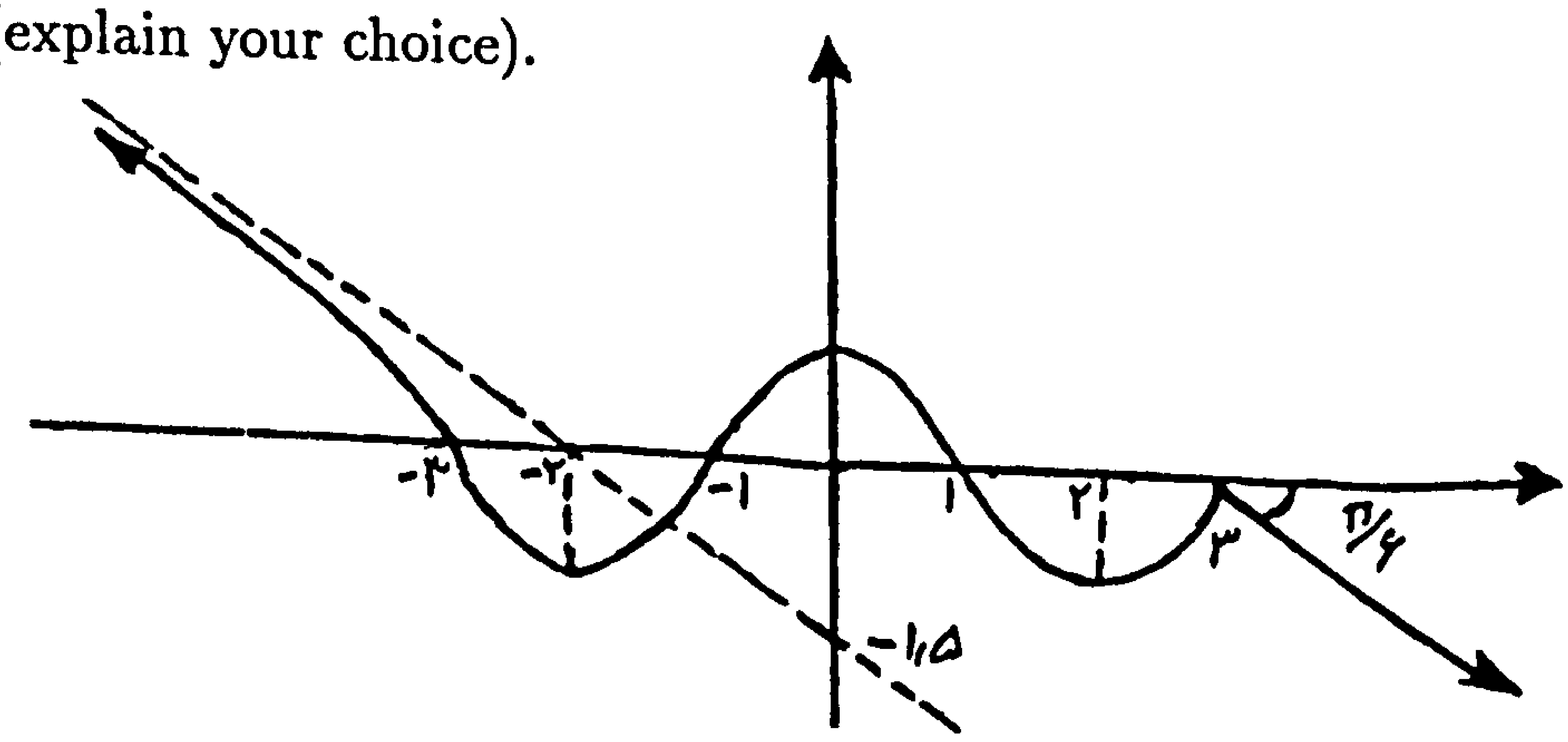
3. Let

$$f(x) = \begin{cases} x & \text{if} \quad x \in Q, \\ \sin x & \text{if} \quad x \in R - Q. \end{cases}$$

Investigate the differentiability of f at $x = 0$.

4. A ladder 26 m long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 4 m/s. If at $t = 0$ ladder is against the wall, when would the area of triangle enclosed by the ladder, the ground and the wall be maximum.

5. A sketch of the graph of the function $y = f(x)$ is shown in the following figure: Draw the graph of f' (explain your choice).



6. Suppose that $f''(c) = 0$ and $f(c)''' \neq 0$. Prove that $(c, f(c))$ is a point of inflection of the graph of $y = f(x)$.

7. Draw a sketch of the graph of the function

$$f(x) = \begin{cases} x + \operatorname{sgn}([e^x]) & \text{if} \quad x \leq 0, \\ 1 - x^x & \text{if} \quad x > 0. \end{cases}$$

8. The region bounded by the curve $y = \sec x$ and $y = \tan x$, for $0 \leq x \leq \pi/2$, is revolved about the x axis. Let A and V be the area and volume of the solid generated.

(i) Is A finite?

(i) Is V finite?

If they are finite find their values.

9. Find a curve which passes through the origin of the cartesian coordinate system such that the distance from the origin to the point (x, y) on the curve is $S = e^x + y - 1$.

10. (i) Prove that if $x > 0$, then $e^x > 1 + x$.

(ii) Use the (ϵ, δ)-definition to prove that $\lim_{x \to \infty} e^x = \infty$.

11. A triangle is made with an area of 36 cm^2 by cutting a corner from a square with side of length 12 cm. If the distance of the centre of the remaining shape from one side of the original square is 7 cm, find its distance from the other sides. (Use Pappus' theorem).

12. Evaluate:

(i)

$$\int e^x \sqrt{1 + e^{2x}}\, dx.$$

(ii)

$$\int_0^\pi \frac{x \sin^2 x}{1 + \cosh^2 x}\, dx.$$

Good luck!

Appendix E

Khordad, 1374 (June 1995)

Mashhad University
Calculus 1 exam (Engineering)

1. If the limit of a function exists, prove that it is unique.

2. Suppose that $\lim_{x\to a} f(x) > 5$. Show that $\exists \delta > 0$ such that if $0 < |x-a| < \delta$ then $f(x) > 5$.

3.Show that $|\tan^{-1} a - \tan^{-1} b| \leq |a-b|, \quad \forall a, b \in R$.

4. Evaluate:

a. $\lim\limits_{x\to-\infty} \dfrac{[2x^2] + \text{sgn x}}{x^2 + |x|}$,

b. $\lim\limits_{x\to+\infty} \left(xe^{-x^2} \int_0^x e^{t^2}\, dx\right)$,

c. $\lim\limits_{n\to\infty} n(\sqrt[n]{a} - 1), \quad a > 0$.

5. Find the length of the arc of the curve $f(x) = \ln(\sec x)$, $x \in [0, \pi/3]$.

6. Evaluate:

(i) $\displaystyle\int \frac{dx}{x(x^3+1)^2}$,

(ii) $\displaystyle\int \frac{\ln x^3}{x} \ln(\ln x^3)\, dx$.

7.

(i) Find the sum of series

$$\sum_{i=1}^{\infty} \left(\int_{n-1}^{n} e^{-x}\, dx\right).$$

(ii) Discuss the behaviour of the following series:

$$\sum_{n=1}^{\infty} (\cos \frac{2}{n})^{n^3}.$$

8. Show that the function $f(x) = \ln(x^2 - 5x + 6)$ as the sum of one or more power series.

Appendix F

Khordad, 1374 (June 1995)

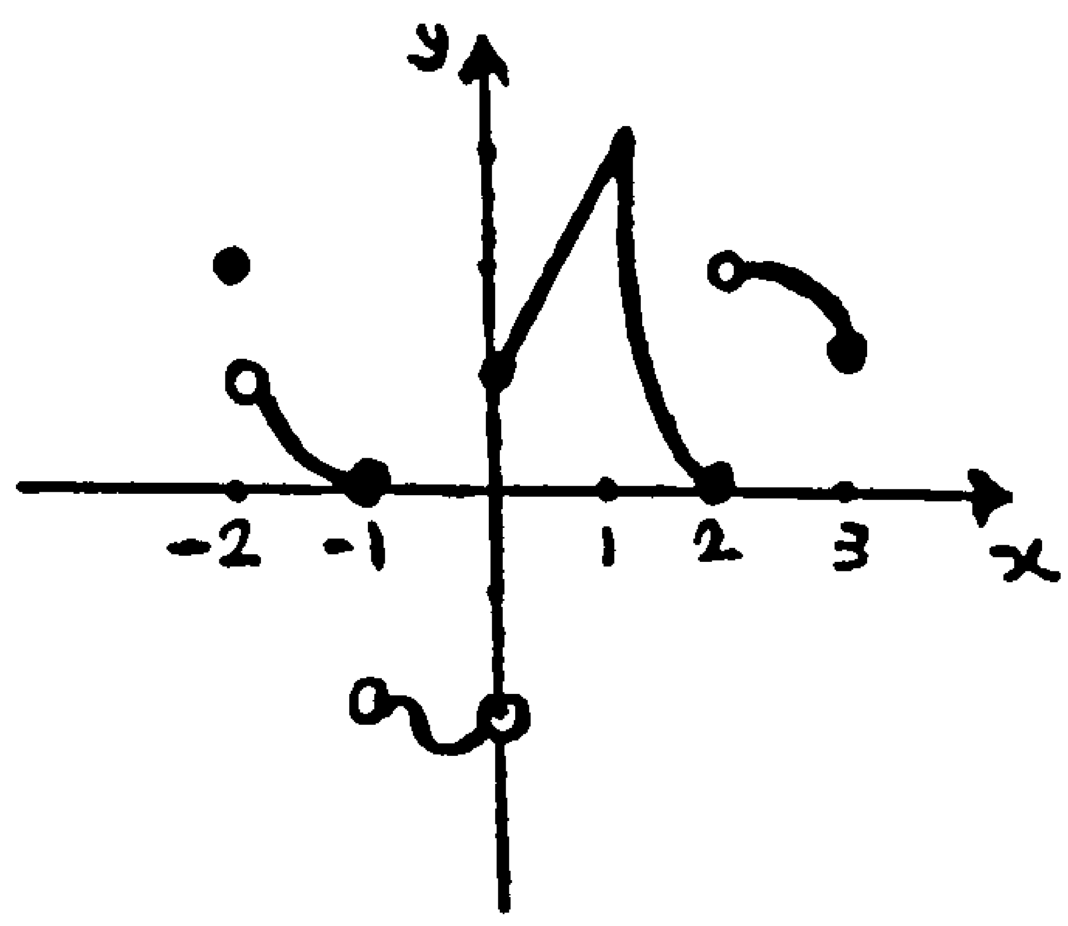
Mashhad University
Calculus 1 exam (Engineering)

1. Attempt one of the following questions:

(i) Use the (ϵ, δ)-definition to prove that $\lim_{x \to a} x^2 = a^2$.

(i) Evaluate $\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sqrt[n]{2^{k+22n}}$.

2. Discuss the existence of limits (left and right-hand), continuity and differentiability of the function f, whose curve is given on $[-2, 3]$ in the following figure. Is f integrable on $[0, 2]$?.



3. Attempt one of the following questions:

(i) Let a be a relative maximum of the function f. Show that $f'(a) = 0$ (if it exists). What can we say, in general, about the converse of this statement? Is it true?

(ii) Let $f : R \to R$ satisfy the condition $|f(x) - f(y)| \leq (x - y)^2 \quad (x, y \in R)$. Show that f is a constant function.

4. (i) Find

$$\int \frac{dx}{(x+1)(x^3-1)}.$$

(ii) Suppose f' is continuous on $[a, b]$. Prove that

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_a^b xf'(x)\, dx.$$

5. Evaluate the area A of the region bounded by the curve $y = -x^2 + 2x$ and lines $y = 1$ and $x = 0$, and find the volume of the solid generated by revolving of the region about the x axis.

6. Determine the behaviour of the series

$$\sum_{i=1}^{\infty} \frac{|\cos n| n^{47}}{1 + n^{74}}$$

and then discuss the convergence or divergence of the sequence

$$\{\int_1^n \frac{x^{47}}{1 + x^{74}} dx\}.$$

7. Exhibit the function $f(x) = \ln(x^4 + 1)$ as a power series and determine its radius of convergence, and show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)2^{4n+4}} = \ln(\frac{17}{16}).$$

Appendix G

Khordad, 1374 (June 1995)

Mashhad University

Calculus 1 exam (Maths)

1. (i) Draw a sketch of the graph of the function $\sqrt{x} + \sqrt{y} = 1$.

(ii) Evaluate the area of the region bounded by the curve, the x-axis and the y-axis. Find the volume of the solid generated revolving the region about the x-axis.

2. Define the integral of a function f on $[a, b]$ in three different ways and evaluate $\int_0^1 x^2\,dx$ by using one of those definitions.

3. Evaluate $\lim_{n\to\infty} \dfrac{1^p + 2^p + \cdots + n^p}{n^{p+1}}$, where p is a positive constant number.

Attempt only one of the following questions:

4. (i) State the formal definition of $\lim_{x\to a} f(x) = L$, where $a, L \in R$, and then use it to prove that $\lim_{x\to -1} \dfrac{1}{2x-3} = -1$.

(ii) State the formal definition of $\lim_{x\to\infty} f(x) = \infty$ and then, by using it, prove that $\lim_{x\to\infty} \dfrac{x^3}{100 + x^2\sqrt{x}} = \infty$.

5. (i) Prove that if $\lim_{x\to a} g(x) = l > k$, there exists a deleted neighbourhood of a such that for all x of that neighbourhood, $g(x) > k$.

(ii) Let f be a function differentiable at a and such that $f(a) \neq 0$. Prove that $1/f$ also is differentiable at a and

$$\left(\frac{1}{f}\right)'(a) = -\frac{f'(a)}{(f(a))^2}.$$

(Attention! $1/f$ is meaningful in a neighbourhood of a).

6. (i) State and prove the fundamental theorem of calculus (FTC).

(ii) Find the centre of mass of a homogenous quadrant.

Appendix H

Monday, 5th June, 1995 2.30 *p.m. to* 5 *p.m.*

University of Glasgow

EXAMINATION FOR THE DEGREES OF
M.A. AND B.Sc.

MATHEMATICS 1A

Second Paper

NOTE. *Candidates must not attempt more than* SIX *of the following questions.*

1. (i) Evaluate

$$\sin(\cos^{-1}(-3/5))$$ 3

(ii) Prove that the function $f : [0,1] \to [0,3]$, where

$$f(x) = x^2 + 2x,$$

is bijective. 9

(iii) Evaluate

$$\text{(a)} \lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}}, \qquad \text{(b)} \lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$$ 5

2. (i) Let f and g be real functions, each differentiable at x. Prove that fg is differentiable at x with

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$ 5

(ii) Differentiate $\dfrac{\sin 3x}{\cos^3 x}$ with respect to x, simplifying your answer as far as possible. 7

(iii) Differentiate $\tan^{-1}(e^{2x})$ with respect to x, and express your answer in terms of $\cosh 2x$. 5

[OVER

Appendix H (cont'd)

3. (i) Use implicit differentiation to calculate y' in terms of x and y when
$$5x^2 + 4xy + y^2 = 2.$$ 4

(ii) A rectangle is to have two vertices on the x-axis and the other two vertices *above* the x-axis on the curve $y = 9 - x^2$. Find the maximum possible area of the rectangle. 7

(iii) For the curve C with parametric equations
$$x = t^3 + 1, \quad y = t^2 + 1 \quad (t > 0),$$
find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t. 5

Show that
$$\frac{d^2y}{dx^2} \Big/ \left(\frac{dy}{dx}\right)^4$$
is constant on C. 1

4. (i) Show that the derivative of
$$f(x) = 2\log x + (x-1)(x-3)$$
is positive when $x > 1$. 4

Deduce that $\log(1.1) > 0.095$. 4

(ii) The rate of growth of a population of ants is known to be proportional to the size of its population. If there were 1000 ants in 1980 and 5000 ants in 1990, what is the size of the population in 1995? 9

5. (i) Find
$$\int \frac{x\,dx}{x^2 - 2x + 1}.$$ 7

(ii) Evaluate
$$\int_0^{\pi/4} \sin^3 x\,dx.$$ 6

(iii) Find the volume of revolution for the curve
$$y = \sin x, \qquad \frac{\pi}{2} \le x \le \pi.$$ 4

[OVER

Appendix H (cont'd)

6. (i) Find the general solution of the ODE

$$(\cos x)y' + (\sin x)y = \cos^3 x,$$

and the particular solution satisfying $y(0) = -1$. 8

(ii) Find the general solution of

$$y'' + 3y' + 2y = \cos 2x + x.$$ 9

7. (i) Show that

$$x^3 + 4x - 2 = 0$$

has exactly one real root and find the best integer approximation to this root. 6
Taking 1 as a first approximation to the root, use Newton's method to find further approximations. The calculation should be continued until successive approximations differ by at most 0.01. 6

(ii) Use Simpson's rule with 3 ordinates to approximate

$$\int_0^1 \sin^{-1} \sqrt{x}\, dx,$$

expressing your answer as a rational multiple of π. 5

8. (i) Find the sum of each of the following series

$$\text{(a)} \sum_{i=0}^{\infty} \frac{3^i}{4^{i+1}}, \qquad \text{(b)} \sum_{i=1}^{\infty} \frac{2}{i(i+1)}$$ 3,5

(ii) Write down the Maclaurin series for $\cos x$ and $\log(1 + x)$. 4
Deduce the Maclaurin series for

$$\log(\cos x)$$

as far as the term in x^4. 5

GLASGOW UNIVERSITY LIBRARY

END]